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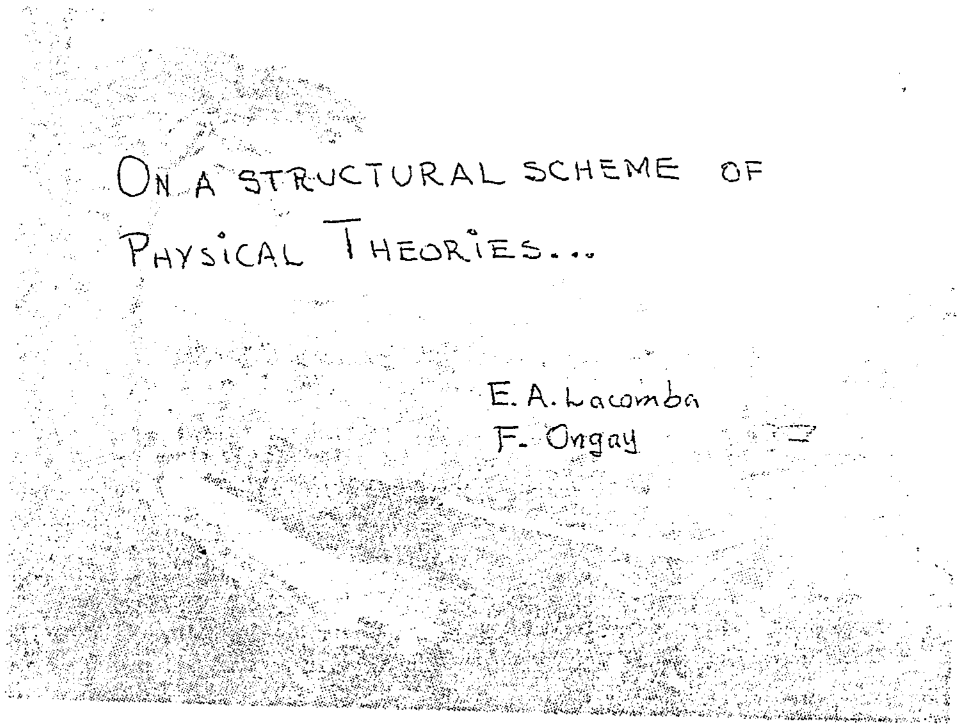

ON A STRUCTURAL SCHEME OF PHYSICAL THEORIES...

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On a structural scheme of physical theories proposed by E. Tonti.[a]

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A structural scheme of physical theories proposed by E. Tonti is analyzed, using modern mathematical concepts but avoiding extreme generality or unnecessary abstractions. The merit of this scheme is that it emphasizes a systematic aspect of methodology in physics; this way some essential properties and equations of a given theory are easily read off. A few basic examples are studied, mainly electromagnetism and the linearized Einstein equation, and then the scheme is used to gain some insight into the structure of these theories.

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I. INTRODUCTION.

In a series of papers published around 1975 ([1]), E. Tonti developed a structural scheme trying to formalize the analogies among diverse physical theories, using some ideas of algebraic topology and differential geometry. These articles were preceded by that of Branin ([2]), published in 1966 and motivated by general ideas on electrical networks.

Tonti's work is essentially a wide recollection of laws or equations of physics, that he tries to reduce following several basic patterns in the form of commutative diagrams. The merit of his viewpoint lies mainly in the fact that it emphasizes a systematic aspect of methodolgy in physics.

The purpose of this work is to discuss this scheme in a not so wide context, but with a more modern mathematical presentation, hence throughout this work we use the language of modern differential geometry (manifolds, differential forms and fiber bundles). These notions are now an important tool in theoretical physics, e. g. in gauge theories, and there are several good accounts on them written for physicists ([3], [4], [5]). This analysis is carried out in section II, and then some examples are carefully discussed in section III.

A more original contribution is our discussion of reductions and extensions of theories in section IV, in the context of our main representative examples: classical electromagnetism and the linear approximation to Einstein's equation. This way we describe some possible extensions and consequences of the method. In the last section we give a critique of the scheme.

Several other works related to this sort of ideas, although with different viewpoints, can be found in the literature ([6], [7], [8]).

II. DESCRIPTION OF THE SCHEME.

The starting point of this scheme is the following seemingly trivial observation, which in fact has very deep implications:

"Every measurement of a physical quantity is related to some region of space-time."

If we assume that we are dealing with a differentiable theory, that is, if we assume that space-time is a differentiable (C^∞) manifold, then the above remark says that the physical quantities are differentiable functions defined on submanifolds, possibly with boundary, of this manifold.

A further assumption, which is implicit in Tonti's works, is the superposition principle. Using the language of differential geometry, these hypotheses may be expressed as a mathematically precise axiom as follows:

Physical quantities are sections of a vector bundle over a submanifold, possibly with boundary, of the space-time manifold.

Later on we will state two postulates for the structure scheme, but for the time being we accept the above remark as the mathematical description of physical quantities. Readers not familiar with the theory of vector bundles may think of sections as "generalized vector-valued functions".

According to Tonti and following his terminology, the equations of physics may be classified in three basic categories:

a) Topological equations.

b) Constitutive or phenomenological equations.

c) Equations of interaction between two theories.

For reasons described below, types b) and c) are also called metric equations. In this work however we shall restrict ourselves to topological and constitutive equations, because the equations of interaction are of a rather different nature.

To describe equations of class a) within the framework described above we must add a new hypothesis. Indeed, these equations are systematically obtained by a process of differentiation, which may be described as follows: if we let $B \longrightarrow M$ denote the vector bundle whose sections are the physical quantities; then this differentiation is of one of the following two types:

i) B posseses a differential operator; that is an operator d, acting on the sections of B, and such that $d^2 = 0$. In general one has to deal with vector-valued differential forms or even vector-bundle-valued formsand d is the usual exterior differential in B; to simplify the discussion we shall assume only vector valued forms, that is forms whose coefficients are vectors rather than scalars. For those familiar with the theory of vector bundles, formally this means that B is of the form $B = \Lambda(M) \otimes V$, where $\Lambda(M)$ denotes the exterior bundle of T^*M , V is a trivial vector bundle whose rank (i. e. its fiber dimension) is not necessarily finite .

ii) B has an affine connection ∇ and differentiation is covariant derivation with respect to this connection. One may see [3] or [5] for a discussion of this concept. However we shall not make use of this notion in the description of the scheme, restricting ourselves to the case described in *i)*.

Thus, these topological equations are in fact differential equations although of a special type. It is also pertinent to point out that Tonti calls coboundary this differentiation process, but the term seems inappropiate, specially with the second kind, which is not in general a coboundary process. Since differentiation is a linear operation, these equations are always linear in the observable, that is, the section of the bundle, which is differentiated (although in some cases they may be equivalent to nonlinear equations in other variables).

Furthermore, the manifolds appearing in physics posses pseudo-riemannian metrics; this metric is the additional mathematical structure required for equations type b), hence the name metric equations. As a rule this metric is linked to the physical properties of the media, so that it must be determined from experiment and moreover, the constitutive equations can also be nonlinear in contrast with the topological equations.

We also remark that if the manifold posses a metric both differentiation processes are related, since the exterior differential may be computed using the standard Levi-Civita connection (see [5], appendix B). This mathematical fact should also have physical implications.

Finally all the manifolds are assumed orientable, which means that the manifold admits a global volume form, but, as Tonti points out, one must not impose an a priori fixed orientation in M . Thus we are led to distinguish between what de Rham ([9]) called odd forms, which change sign under an orientation reversing change of coordinates, and even forms which do not change sign; the quantities related to even forms are called configuration variables and those related to odd forms are called source variables. Recall however that in an oriented manifold, that is, a manifold with a fixed orientation both types of forms may be identified. We shall return to this point later on.

Now in any orientable n-dimensional manifold endowed with a pseudo-riemannian metric there exists an operator * , called Hodge's star operator (see for instance [10] or [11]), giving a duality between even p-forms and odd (n-p)-forms; this operator plays a crucial role in the construction of constitutive equations.

We may now state the two basic postulates of the scheme:

P1: The domain of physical quantities are orientable submanifolds, possibly with boundary and endowed with pseudo-riemannian metrics, of the space-time manifold.

P2: The physical quantities are vector-valued, possibly odd, differential forms and the differentiation is exterior differential on this forms.

These two postulates are more or less explicit in Tonti's works (see [1,(3)], p. 228 for a statement of P2). However, although P1 appears as a natural hypothesis, P2 is a subtler requirement. This is more so when we want to incorporate the general covariance principle. We will discuss this with more detail in the last section of this paper.

With the above conventions, the structure scheme can be summarized in the form of commutative diagrams of the sort described in fig. 1 . To fix ideas and simplify the writing, we will assume that M is an orientable 3 dimensional manifold endowed with a riemannian metric g ; thus the $*$ operator relates even p-forms and odd (3-p)-forms.

It should be emphasized that once the above postulates are accepted, the form of the diagrams is a direct consequence of them, and this has already some consequences: From the relation $d^2 = 0$ satisfied by the exterior differential it turns out that certain "physical quantities" appearing in Tonti's diagrams are either trivial or do not have a direct physical interpretation in a given theory; the importance of this observation will become evident when discussing concrete examples.

A key role in the field equations of a given theory is played by the Laplace-Beltrami operator, which generalizes the laplacian and which is defined as follows: recall that the adjoint δ of d is the operator defined on p-forms as $\delta = (-1)^{p(n-p)} {*d*}$, then the Laplace-Beltrami operator is defined as $\Delta = d\,\delta + \delta\, d$. Since the $*$ operator is used, the Laplace-Beltrami operator depends on the metric of the manifold. We also remark that linear constitutive equations (or their linear approximations) automatically yield a field equation which is a Poisson equation.

III. BASIC EXAMPLES.

In this section we discuss the application of the method to some representative examples. The notation for the different physical magnitudes is as much as possible the one used by Tonti in [1,(2)] , to make comparisons easier.

a) Dynamics of a classical particle in a conservative force field.

In this example the equation to deduce is Newton's second law $f = ma$. The base manifold here is the time axis, so that the the diagram is in dimension 1, and associated to position, velocity, linear momentum and force we have vector valued differential forms of rank 3 . The exterior differential is then identified with the usual derivative of curves in $\mathbb{R}^3$ by the choice of a basis for the time axis (which in this case is equivalent to choosing a metric and an orientation).

The constitutive equations are the one relating momentum with velocity plus the force law; the resulting field equation is precisely Newton's law. So, Tonti's diagram related with this equation is that of fig. 2 .

b) Classical electromagnetism.

Tonti's method is perhaps best shown in this case so, although there are several excellent accounts of it ([12], pp. 44 - 48 or [11]), we will give a rather complete discussion.

The manifold in question here is $M = \mathbb{R}^4$ with the Minkowski metric g_1, which in the global coordinates (t,x,y,z) is given by

$$ds^2 = dt^2 - (dx^2 + dy^2 + dz^2) \quad . \tag{1}$$

The vector bundle is just $B = \Lambda(M)$, so that physical quantities are usual differential forms and if $\chi : M \longrightarrow \mathbb{R}$ is a differentiable function recall that the differential of χ is

$$d\chi = \partial_t\chi\, dt + \partial_x\chi\, dx + \partial_y\chi\, dy + \partial_z\chi\, dz \quad , \tag{2}$$

where $\partial_t = \partial/_{\partial t}$, etc.

The Hodge $*$ operator is not the same as in the euclidean case since g_1 is different from the euclidean metric. For our purposes it is enough to recall that this operator, associated to the metric g_1 and the orientation given by $dt \wedge dx \wedge dy \wedge dz$, takes on 2-forms the following expression:

$$*(dx \wedge dy) = - dz \wedge dt \quad , \quad *(dx \wedge dt) = dy \wedge dz \tag{3}$$

$$*(dx \wedge dz) = dy \wedge dt \quad \text{and} \quad *^2 = -\text{identity}$$

The electric field is a (even) 1-form $E = E_1dx + E_2dy + E_3dz$, while the magnetic field is a 2-form $B = B_1dy\wedge dz + B_2dz\wedge dx + B_3dx\wedge dy$. This allows us to define the four dimensional Maxwell 2-form

$$F = E \wedge dt + B \tag{4}$$

representing the classical electromagnetic field. Applying the exterior differential we get

$$dF = \text{rot}(E) \wedge dt + \text{div}(B) \wedge dx \wedge dx \wedge dz + \partial_t B \wedge dt \tag{5}$$

where

$$\text{rot}(E) = (\partial_x E_2 - \partial_y E_1)\, dx\wedge dy + (\partial_x E_3 - \partial_z E_1)\, dx\wedge dz + \tag{6}$$

$$+ (\partial_y E_3 - \partial_z E_2)\, dy \wedge dz$$

and

$$\mathrm{div}(B) = \partial_x B_1 + \partial_y B_2 + \partial_z B_3. \qquad (7)$$

Hence, Faraday's law together with the absence of magnetic monopoles are equivalent to $dF = 0$, which is known as homogeneous Maxwell equation. Since M is contractible, which means that it may be smoothly deformed into a point, this equation is equivalent to $F = dA$, for some 1-form , A, known as the Maxwell 4-potential.

The electric displacement D and the magnetic field strength H are identified with 2- and 1-forms respectively. If we denote by $*$ the Hodge star operator associated to the euclidean metric in dimension 3, and assuming the medium to be isotropic, these fields are simply described by $D = \varepsilon * E$ and $H = \mu^{-1} * B$, where the scalars dielectric permitivity ε and magnetic permeability μ are functions of the medium.

These constitutive equations may be summarized in a single dimension 4 equation, by a slight change of the metric on M with the corresponding change in the operator $*$. This can be described as follows: Let us write $c = (\varepsilon\mu)^{-1/2}$ and modify the metric g_1 by taking instead g_c defined by

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2) \qquad (8)$$

We now define the Maxwell dual field G , an odd 2-form which in coordinates collects together D and H as

$$G = D - H \wedge dt \qquad (9)$$

Then the constitutive equation of classical electromagnetism becomes

$$G = (\varepsilon/\mu)^{1/2} * F \quad . \tag{10}$$

The above defined function c of the medium has units of speed: it is just the speed of propagation of electromagnetic signals in the medium. This can be concluded from the field equations we will briefly discuss below.

The charge density ρ and the current density j are put together in the 1-form J, which we call 4-current:

$$J = -\rho\, dt + j_1\, dx + j_2\, dy + j_3\, dz \tag{11}$$

Ampère's and Gauss' laws can be written as $dG = 4\pi * J$. This is the so-called inhomogeneous Maxwell equation.

In this case the operator Δ applied to functions is modulo constants equivalent to the D'Alembert operator, denoted by $\square$, and given in coordinates by

$$\square\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial z^2} \tag{12}$$

The inhomogeneous Maxwelll equation can be rewritten as $\delta F = (\mu/\varepsilon)^{1/2}\, 4\pi\, J$, and hence

$$\Delta F = (d\delta + \delta d)F = d\delta\, F = (\mu/\varepsilon)^{1/2}\, 4\pi\, dJ \tag{13}$$

so that the electromagnetic field satisfies a Poisson equation. If charges and currents are absent (or more generally if J is a closed form, i. e. $dJ = 0$), we obtain the Laplace equation associated to g_c. If c is a constant this is equivalent to a wave equation for each component of F, with propagation speed equal to c. It is interesting to remark that if A is chosen so that $\delta A = 0$ (null divergence or Lorentz gauge condition), then this 4-potential also satisfies the wave equation, with the same propagation speed c, provided that $J = 0$.

We have seen that F is a closed form, while $G = (\varepsilon/\mu)^{1/2} * F$ is not; however, the equation $d^2G = 0$ is also important in physics: indeed, $d^2 G = 0$ is equivalent to $d{*} J = 0$, which in coordinates becomes

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0 \tag{14}$$

which is the so-called continuity equation, expressing charge conservation. This continuity equation is the archetype of conservation laws stemming from the relationship $d^2 = 0$.

Tonti's diagram for classical electromagnetism is as in figure 3 .
Recall that not all the indicated magnitudes have a sound physical interpretation.

c) Linearized Einstein equation.

The linear approximation to Einstein's equation is quite similar to, although somewhat more complicated than, classical electromagnetism.

Once again the base manifold is $M = \mathbb{R}^4$ with a pseudo-metric g of index 3, referred to as a Lorentz metric, where the index of the metric may be defined as the number of minuses appearing in the coordinte expression of the metric when it is written in diagonal form; for instance the euclidean metric has index 0 while the metrics g_c have index 3 .The explicit expression of the metric is however unknown since it is determined through Einstein's equation.

Before showing how postulates P1 and P2 are satisfied, let us recall Einstein's linearized equation in the more familiar tensor notation; we follow here the presentation of Wald [5].

Denote by g_{ab} the components of the metric tensor, and assume they have the form

$$g_{ab} = h_{ab} + \gamma_{ab} \qquad (15)$$

Here h_{ab} are the components of the metric g_c and γ_{ab} is a small peturbation. We will discuss the meaning of smallness in this context in somewhat greater length in the next section, where more explicit calculations are done, but a more complete account of this point may be found in [5]. Einstein's equation can then be obtained directly in terms of the perturbation: indeed computing the Christoffel symbols and Ricci tensor to first order, we get as Einstein tensor

$$G_{ab} = \partial^c \partial_{(b} \gamma_{a)c} - {}^1/_2 \partial_a \partial_b \gamma - {}^1/_2 h_{ab} (\partial^c \partial^d \gamma_{cd} - \partial^c \partial_c \gamma) \qquad (16)$$

where $\gamma = h^{ac}\gamma_{ca}$. Parentheses enclosing some indices indicate a symmetrical sum with respect to those indices. Notice also that in the computations the flat metric tensor h_{ab} is used to raise and lower indices.

The above expression can be simplified by a judicious choice of the perturbation: first of all we can rewrite the Einstein tensor in terms of a new tensor defined by

$$\phi_{ab} = \gamma_{ab} - {}^1/_2 h_{ab} \gamma \qquad (17)$$

Indeed, γ_{ab} can be recovered from ϕ_{ab} by contracting with h^{ab} (recall that this tensor is defined by the condition $h^{ab}h_{bc} = \delta^a{}_c$) to get $\phi = {}^1/_2 \gamma$. Then Einstein's equation becomes

$$G_{ab} = -{}^1/_2 \, \partial^c \partial_c \phi_{ab} + \partial^c \partial_{(b} \phi_{a)c} - h_{ab} \partial^c \partial^d \phi_{cd} = 8\pi \, T_{ab} \qquad (18)$$

where T_{ab} denotes the energy-momentum tensor.

On the other hand, we can make a gauge transformation by means of an "infinitesimal transformation" of the base manifold M , corresponding to the fact that it has no natural coordinate system, which is indeed the mathematical statement of Einstein's equivalence principle. More precisely, we can perform on ϕ_{ab} a transformation of the type

$$\phi_{ab} \longrightarrow \phi_{ab} + \partial_{(a}\xi_{b)} \tag{19}$$

without changing Einstein's equations: here ξ_a is just a vector field on M and we must symmetrize to preserve the symmetric nature of the metric. We can then choose a tensor ϕ_{ab} satisfying $\partial^b\phi_{ab} = 0$, which is the equivalent of the previously described Lorentz gauge condition of the electromagnetism. With these choices a straightforward computation shows that only one term of the the Einstein tensor remains, so that Einstein's equation simply becomes

$$\partial^c\partial_c\phi_{ab} = -16\pi\, T_{ab} \tag{20}$$

which is a Poisson equation.

We can now easily describe this equation by a Tonti diagram as follows:

Since the base manifold is contractible any fiber bundle is trivial; this allows us to consider tensors as tensor-valued differential forms by raising one of the indices with the metric tensor h^{ab} . In particular the tensors ϕ_{ab} and T_{ab} are identified with vector-valued 1-forms denoted by Φ and T respectively.

Moreover, associated to the metric g_c there is a well defined exterior differential and a Hodge $*$ operator as described in the previous example. In this terms the operator ∂^a is the adjoint of ∂_a , so that in intrinsic notation and due to our choice of gauge, the form Φ satisfies $\delta\Phi = 0$. Einstein's equation then becomes

$$\delta d\,\Phi = -16\pi\, T \qquad (21)$$

or more briefly

$$\Delta\,\Phi = -16\pi\, T\,. \qquad (22)$$

Taking into account all of the above, Tonti's diagram for this case is depicted in fig. 4 .
Once again, not all the quantities have a natural physical interpretation. We also remark that this analysis of the linear case does not apply directly to the general Einstein equation (see [4]).

IV. EXTENSION AND REDUCTION OF THEORIES.

We may summarize the discussion of section II, by saying that a physical theory is determined by a base manifold M , a vector bundle $B \longrightarrow M$ and the constitutive relations of the theory. This formal framework will be used in this section to show how "formal manipulations" of Tonti's diagrams, suggested by usual techniques of algebraic topology and differential geometry, furnish a more or less systematic method to analyze related physical theories. We shall discuss two processes, which in some sense are dual to each other, which will be called reduction and coupling of physical theories.

IV.1. Reductions of a Physical Theory.

We may heuristically define a reduction of a physical theory as a combination of some of the following steps:

i) Choice of a submanifold or a quotient manifold of the base manifold. Recall that a quotient manifold is a manifold obtained by identifying some parts of the original manifold to points. In the cases we shall consider the base manifold is $\mathbb{R}^4$ and by identifing the time axis to points we get a manifold which is "identical" to $\mathbb{R}^3$. This new manifold will be the base manifold of the restricted theory.

ii) Determination of a subbundle or a quotient bundle of the bundle B. This will determine the physical quantities of the restricted theory.

iii) Performing some approximations, either on the geometry of the manifolds or on the constitutive equations.

As a rule, these choices are dictated by physical considerations.

Perhaps the best example of this process is the passage from electromagnetic theory to electrostatics. As already mentioned, in this example the base manifold is $M = \mathbb{R}^4$ with the Lorentz metric g_c and the bundle $B = \Lambda(M)$. M has a natural global system of coordinates (t, x, y, z) where the metric is diagonal, and a natural foliation, that is, a partition of the manifold into disjoint but isomorphic submanifolds, called the leaves of the foliation, given by $t =$ constant. As pointed above, it is easy to see that both, the leaves of the foliation and the quotient manifold of M by t may be identified with $\mathbb{R}^3$ furnished with the euclidean metric; we shall denote these manifolds by $\hat{M}$.

Electrostatics results from a projection of electromagnetism to $\hat{M}$ for even forms and a restriction to a leaf of the foliation for odd forms. (recall that both manifolds may be identified). The electric differential forms, E, D, ρ and ϕ, described before will then become identified with differential forms on $\hat{M}$.

That we must combine a projection and a restriction to recover electrostatics can be seen directly from the expressions of the electrostatic fields; however, we can get an idea as to why this is so if we consider the definitions of E and D in terms of line and surface integrals respectively: physically a line integral is associated to a displacement (i. e. motion along a trajectory) but the process is quite different for computing a surface integral.

Also, in order to project the forms to $\hat{M}$, we must assume that the electric forms are static, that is, that the time derivatives of their coefficients vanish. Then the absence of currents ($\partial_t \rho = 0$) allows us to neglect the magnetic fields and, by choosing an appropiate gauge, the magnetic potential may be taken as 0 . In this gauge the 4-potential reduces to $A = \phi\, dt$, where now ϕ is time independent and, upon passing to the quotient, A becomes identified with the three dimensional 0-form ϕ , the electrostatic potential. A similar reasoning applied to the electric field E yields the electrostatic field form, also written E .

It is interesting to remark that this projection to a quotient manifold may be interpreted as taking a time average of the forms, which may be written symbolically as

$$\phi = \int A \left[\int dt \right]^{-1} \qquad (23)$$

This average process describes a physical way to relate both potentials but also shows that a similar approximation argument may be applied to quasi-static forms, that is forms with negligible time derivatives.

For odd forms the situation is simpler: since we are considering submanifolds in this case, every form that does not involve dt is automatically identified with a form in three dimensional space $\hat{M}$. Also remark that there is no inconsistency with the interpretation of $\hat{M}$ as a quotient manifold, because all the coeficients are assumed to be time independent.

The previous remarks are summarized in diagram 5 , where $\mathcal{A}$ denotes the approximations performed and $*$ denotes the usual $*$-operator in $\mathbb{R}^3$. The middle columns are precisely those of Tonti's diagram of electrostatics.

A similar treatment will yield the diagram of magnetostatics. This theory is somehow dual to the previous one because we must now perform a projection down to a quotient for even forms and a restriction to a leaf for odd forms. Making approximations akin to those of the electrostatic case we get diagram 6, where the middle columns are those of magnetostatics.

We recall that not all the forms in these diagrams are physically meaningful, but also obviously some of the approximations appearing in the diagrams have no intepretation.

In both these examples the restriction of the theory is obtained by combining a restriction to a submanifold and a projection down to a quotient manifold, but other possibilities appear in practice; for instance, to recover the diagram of linear electric conduction ([1,(2)], p. 160) one must perform projections in both even and odd forms. We will not go into details of this case here, but instead we shall exploit the analogy between electromagnetism and linear gravitation to show how the use of Tonti's diagrams allows us to easily recover newtonian gravitation from Einstein's theory (see [13], p. 26-29, also [5] or [10]).

Let us consider the diagram of linear gravitation and make the newtonian approximations, namely a static universe, weak gravitational fields and $c \gg 1$. These hypotheses imply ([5]) that the metric tensor g may be written in the form

$$g_{oo}\, dt^2 + \sum_{i,j=1}^{3} g_{ij}\, dx^i dx^j \qquad (24)$$

where the metric coefficients are independent of t and $|g_{ij}|$ are of the order of 1, for i, j = 1,2,3 and negligible with respect to $g_{oo} \simeq c^2$.

According to the discussion of the previous section we can assume the metric tensor to first order can be written as

$$(c^2 + \phi_{oo})\, dt^2 + \sum_{i,j=1}^{3} (\delta_{ij} + \phi_{ij})\, dx^i dx^j \qquad (25)$$

where the components of the perturbation ϕ_{ab} are much smaller than those of the flat metric g_c. Then Einstein's equation reduces to a partial differential equation involving only spatial coordinates for the component ϕ_{oo} of the perturbation.

Since we already know that Newton's law of gravitation is similar in structure to Coulomb's law of electrostatics, we propose for the reduction a diagram similar to that of the reduction from electromagnetism to electrostatics (but we must point out that other possibilities do exist). With this in mind and using the same conventions as before, the diagram for the reduction is as in figure 7 .

In this diagram the middle columns represent the restricted theory where the base manifold is $\mathbb{R}^3$ equipped with a metric that is approximately the euclidean one. In this way the structure of the diagram suggests as field equation of this restricted theory

$$\nabla^2 \phi_{oo} = 16\pi \, T_{oo} \quad . \tag{26}$$

where ∇^2 denotes the usual laplacian in $\mathbb{R}^3$. This equation coincides with the usual field equation of newtonian gravitation

$$\nabla^2 \varphi = 4\pi \rho \tag{27}$$

where φ is the newtonian gravitational potential and ρ the newtonian mass density, provided we make the identifications

$$\varphi = \frac{\alpha}{4} \phi_{oo} \quad ; \quad \rho = \frac{1}{\alpha} T_{oo} \quad ; \tag{28}$$

and the usual choice for α is c^2 ([1,(2)], p. 177) .

This also justifies a posteriori our construction of the diagram of the reduction, since we actually know that configuration variables are measured in practice by taking time averages (for instance dropping a stone to measure the strength of gravity). However it would be interesting to analyze other reductions of Einstein's equation, as was done for electromagnetic theory.

On the other hand, in Newton's theory we have $\nabla\varphi = a$ as the equation of gravitational acceleration, where ∇ denotes the usual gradient. Reversing the approximation argument, the analog of this equation in Einstein's theory is

$$\frac{1}{2}\,\partial^i \phi_{oo} = -\,\Gamma^i_{oo} \quad , \qquad (29)$$

where Γ^a_{bc} denotes the Christoffel symbols of the metric g , and in this way one easily recovers the well known fact that gravitational acceleration is due to the fact that the metric is curved by the presence of massive objects.

IV.2. Coupling of Theories.

We shall now describe how to obtain electromagnetism as a "coupling" of electrostatics and magnetostatics, when we consider time-dependent fields. In this sense, electromagnetism may be considered as a very special extension of electrostatics or magnetostatics.

Keeping in mind the diagrams of these theories and also the dual nature of electrostatics and magnetostatics, the coupling is obtained by making a shift in the diagrams, whose effect is to put both constitutive equations at the same level. The key to perform this operation is given by the expressions of the 4-dimensional electromagnetic forms, $F = E \wedge dt + B$ and $G = D - H \wedge dt$. These equations show that the electric form E and the magnetic form H , which are defined in the 3-dimensional quotient manifold $\hat{M}$, must be multiplied by dt to become forms defined in Minkowski space M . In technical language, we must lift the forms from the quotient manifold to the original manifold, but this may be intuitively understood as an "infinitesimal integration" of the forms, which is in a sense the inverse process to the time average used to pass to the quotient.

Diagram 8 summarizes these conclusions; it is clearly shown there how to couple the columns of electrostatics and magnetostatics to get the corresponding columns of electromagnetism. The diagonal arrows marked ∂_t correspond to the additional part of the exterior differential added because of the time dependence of the fields. This coupling of the columns has a well defined physical counterpart, namely, that time-varying fields produce new fields that get coupled in a physical way with the previous fields.

V. CONCLUSIONS.

As mentioned the key idea of Tonti's method is to exhibit in a formal way the analogies between different physical theories. This gives an algorithmic strategy to analyze the equations of physics. For instance, by using Tonti's diagrams one may gain understanding of a difficult theory by comparing it with a simpler or better known one. Likewise, some new aspects of a physical theory may be studied by trying to fill in the stages of the corresponding diagram.

Also, although this is not a new idea, the introduction of some notions of algebraic topology and differential geometry provides a unified language for the various physical theories, and also means to exploit in the study of physics some well-known techniques of these mathematical branches. Such is the case of the reduction and coupling of theories discussed in this paper.

However the most important point about these diagrams is the fact that they point towards a systematic aspect of the methodology of physics, at both the theoretical and experimental levels, stemming from the mere fact that we are using mathematical models for physical theories. As a consequence, certain procedures and conservation laws are automatic within a given theory. Certainly it will be impossible to reduce physics to a formal theory but realizing the existence of formal or formalizable aspects in a science is an important step towards a better understanding of it.

We must point out some deficiencies of Tonti's work. A most obvious critique is the old-fashioned mathematical language employed in his works which causes some notational inconveniences, but also some unnecessary limitations in both the comprehension of the ideas and the application of mathematical results. A good example of these shortcomings is the confusion arising in the definition of connection as a result of parallel transport without explaining that although neither parallelism nor connection are intrinsic notions in an arbitrary manifold, there is a canonical way to perform these operations once a riemannian metric is given, using the Levi-Civita connection.

The problem of orientation, while fundamental because of the difference between configuration and source variables, is also unclear in Tonti's work. In particular the role of Hodge's *-operator is not mentioned, although it is systematically employed. (See for instance [1,(4)], p. 455, where a complete description of this operator is given, but regarding it as little more than a notational trick.)

On the other hand, since Tonti's papers deal only with local theories ([1,(2)], p. 57), the question arises as to the necessity of concepts which are only relevant when global problems are considered, as is the case of orientation. Recall that locally every manifold is orientable and as was mentioned, in an orientable manifold odd and even forms can always be identified, but not so in a nonorientable manifold; that is, nonorientability is a global problem. The intrinsic description of Tonti's diagrams given in this paper shows that global aspects may be incorporated into the scheme in a natural way, but the question still remains. However, there are at least two reasons for introducing global matters in this type of study:

On the one hand, there exist physical theories such as gravitation or analytical mechanics, which make considerations about the global structure of space-time. On the other hand, there are some experiments which show the existence of physical phenomena linked to a non trivial topology of the physical space where the experiment is performed: such is the case of the Aharanov-Bohm effect where one detects magnetic perturbations in electron scattering, in the absence of measurable magnetic fields, due to conductors which make the topology of space non trivial. See [14], for a theoretical discussion of some consequences of this experiment.

Of course the analysis of physical theories in full generality is much more complicated and it remains to verify whether Tonti's scheme is still valid with this generality. For instance, in some cases it is not clear whether physical observables may be represented by vector-valued differential forms; for example, if the base space is not-contractible, vector-bundle-valued differential forms may not be identified with vector-valued forms, so this more general theory may be required, and so on.

In a sense this shows why a postulate like P2 may not be enough to describe all physical quantities. But moreover, if we take into account the symmetry principles of modern physics, such as the general covariance principle of general relativity, we are forced to require invariance or equivariance properties of physical magnitudes which may not be satisfied by differential forms. While this type of restrictions may be incorporated into some theories, e. g. gauge theories, the formal structure of the full theory of gravitation seems to be of a more complicated nature.

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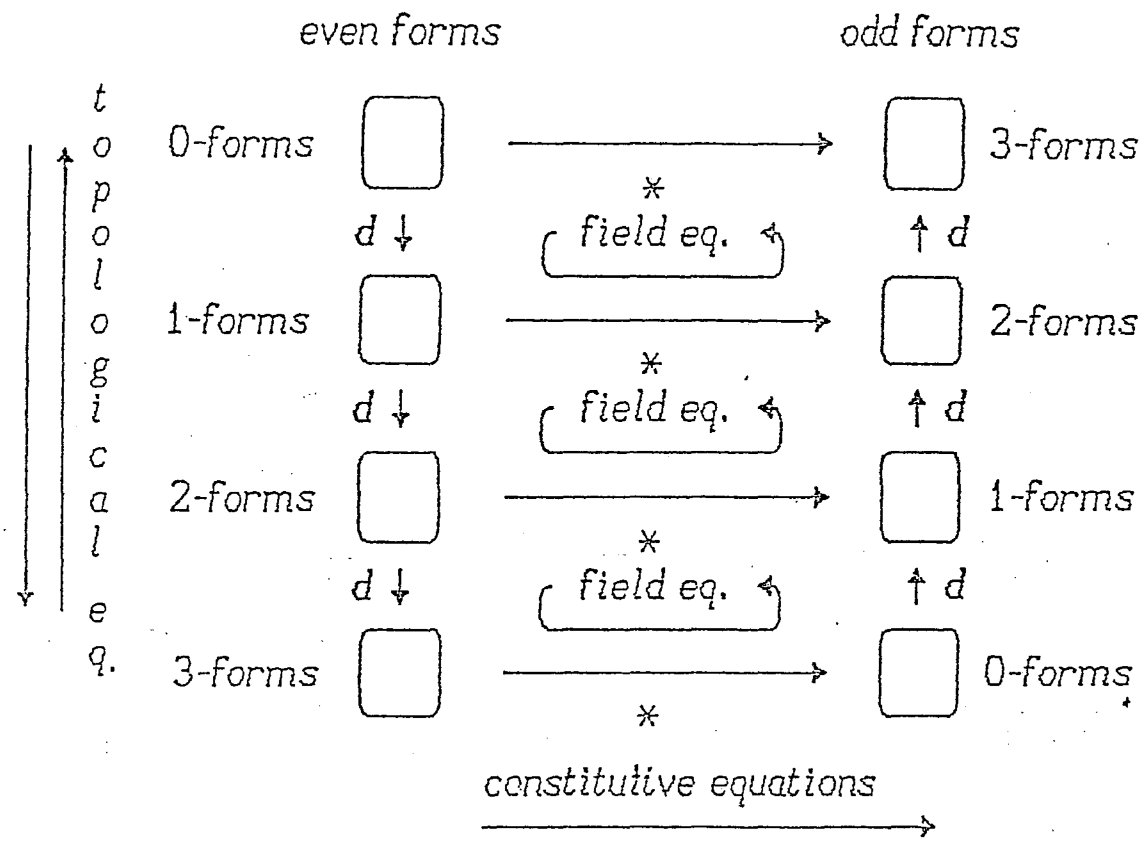

Fig. 1 . Standard diagram in 3 dimensions.

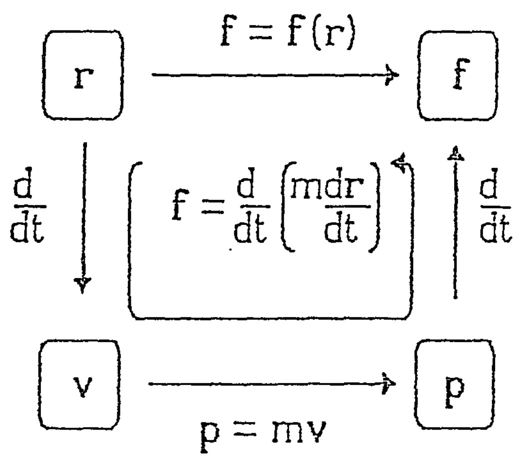

Fig. 2 . Tonti's diagram for Newton's second law.

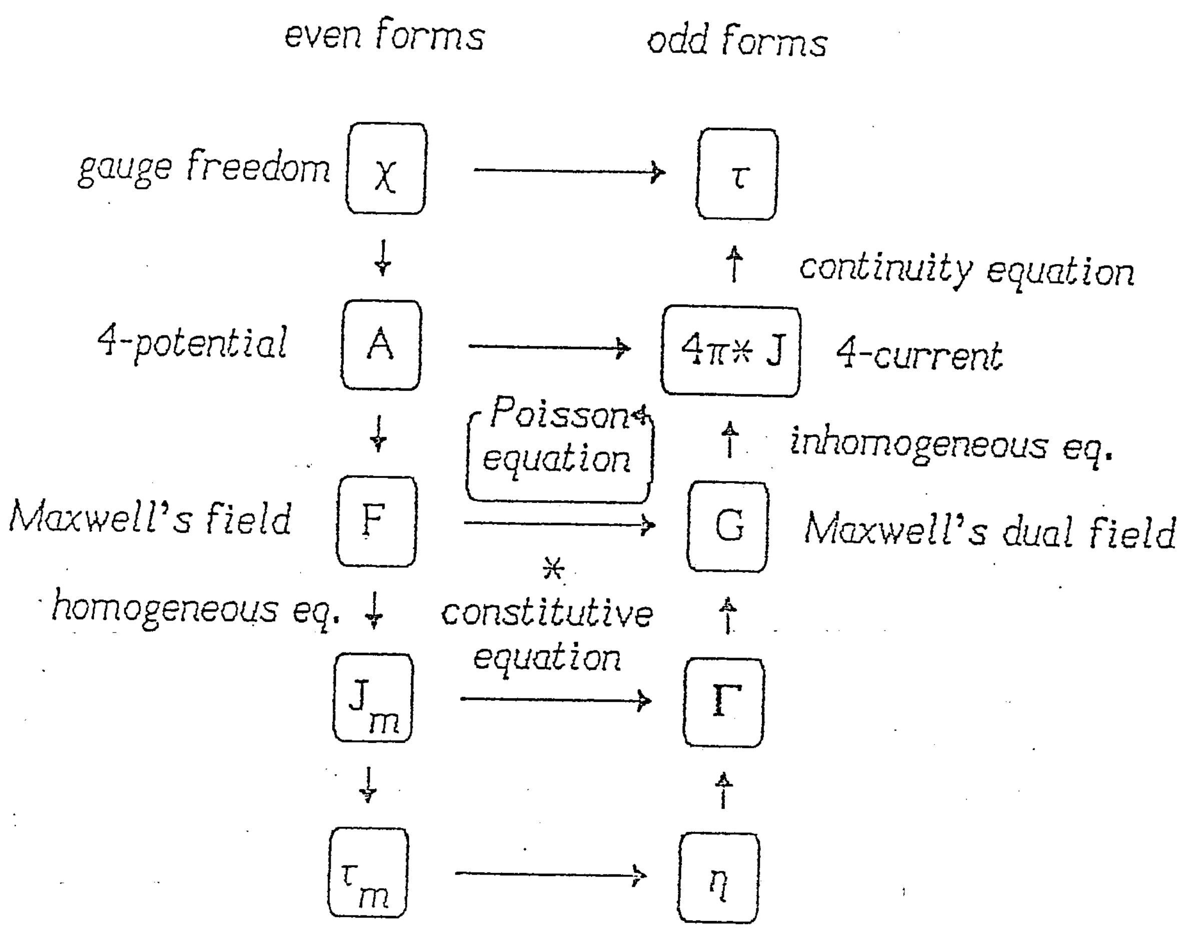

Fig. 3 . Tonti's diagram for Classical Electromagnetism.

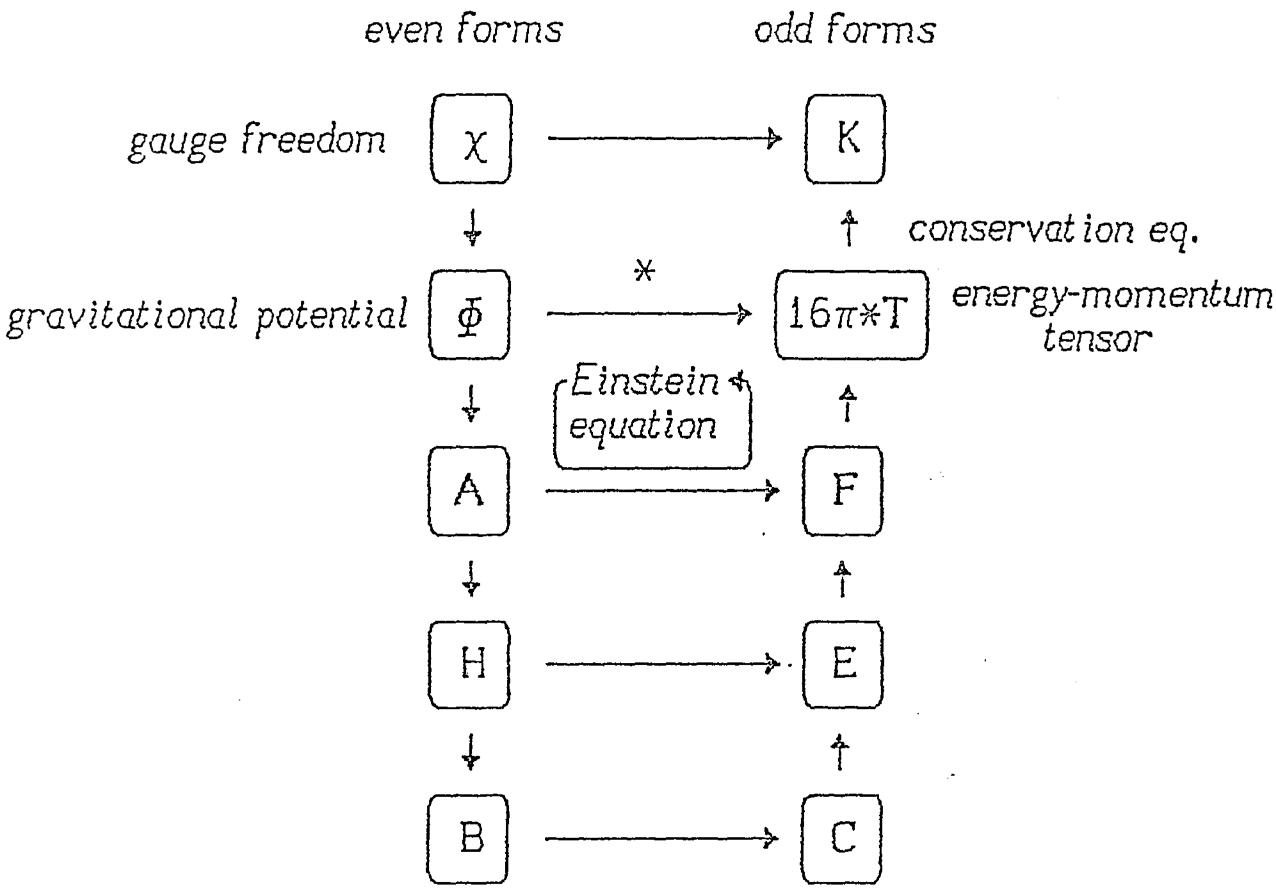

Fig. 4 . Tonti's diagram for the linearized Einstein equation..

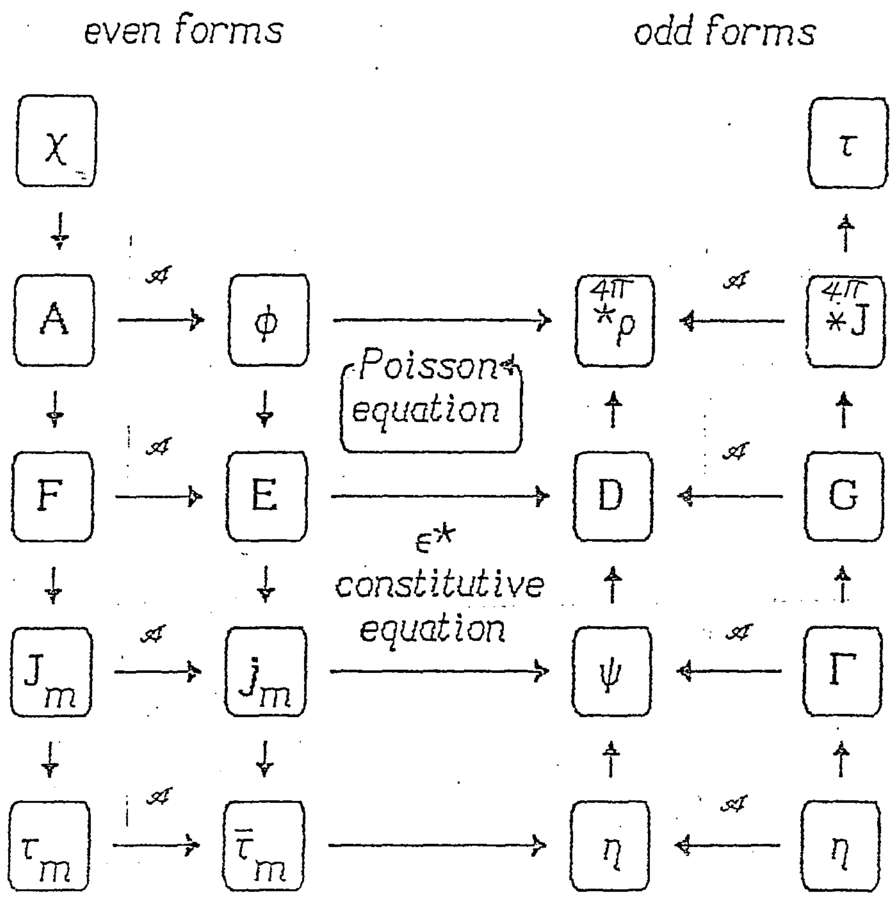

Fig. 5 . Reduction of Electrostatics.

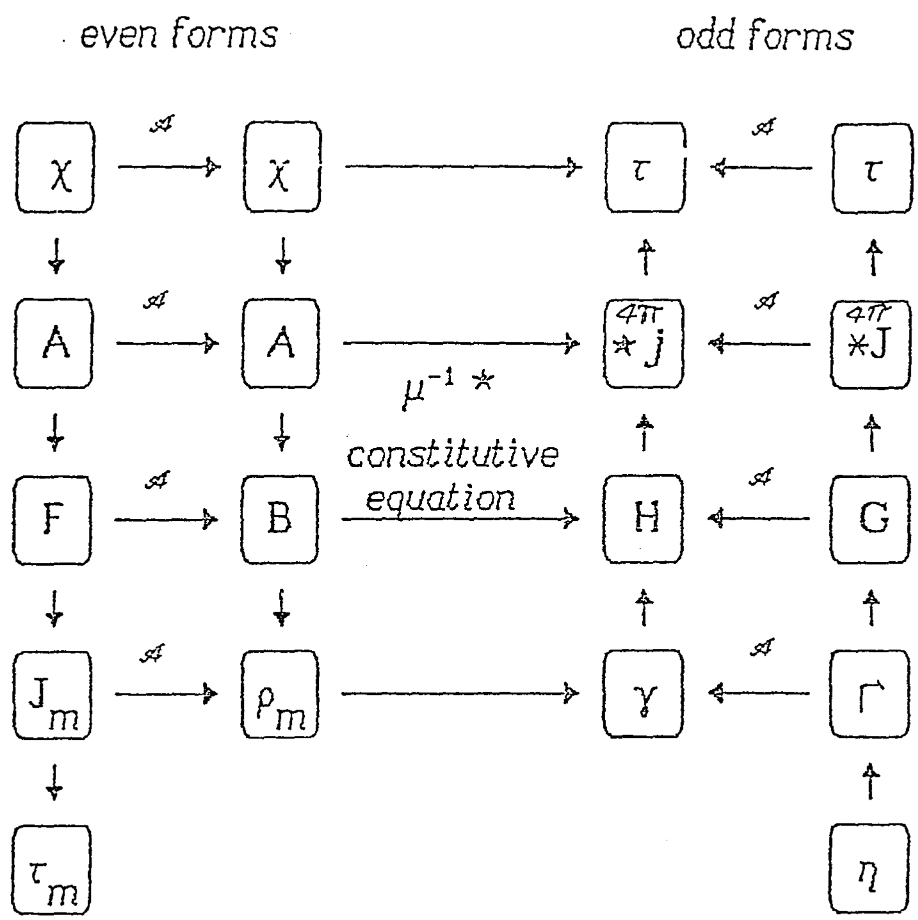

Fig. 6 . Reduction of Magnetostatics.

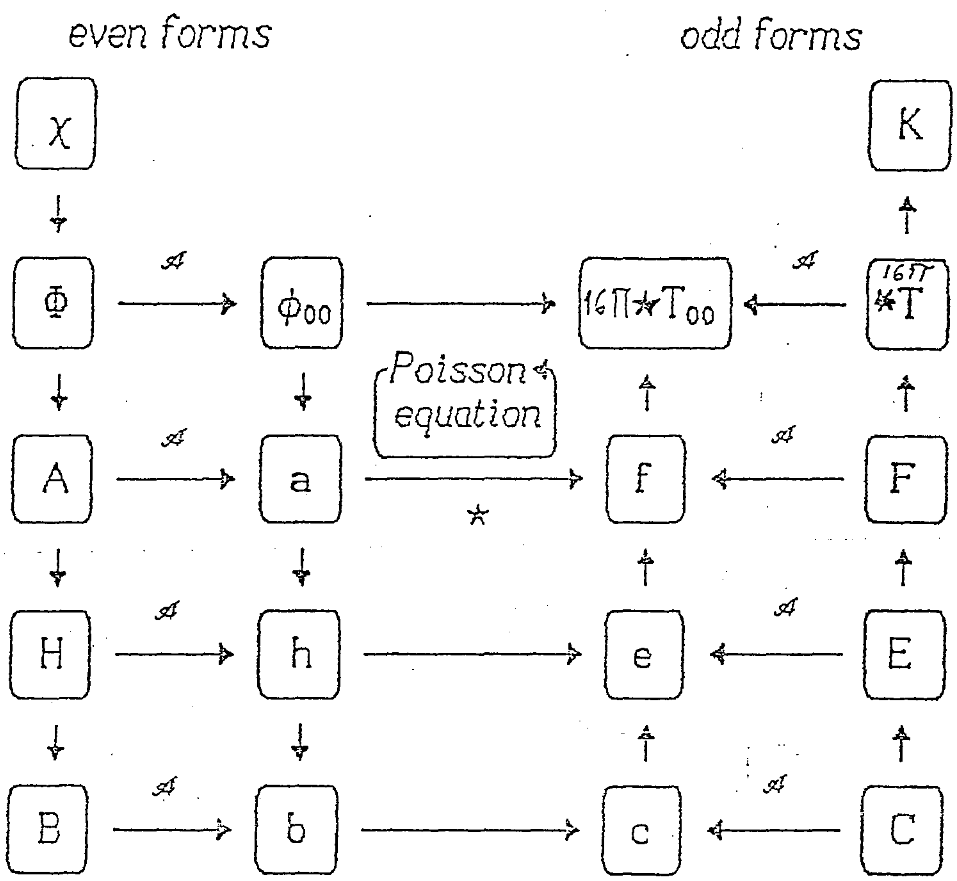

Fig. 7. Newtonian Gravitation as a reduction of Einstein's equation.

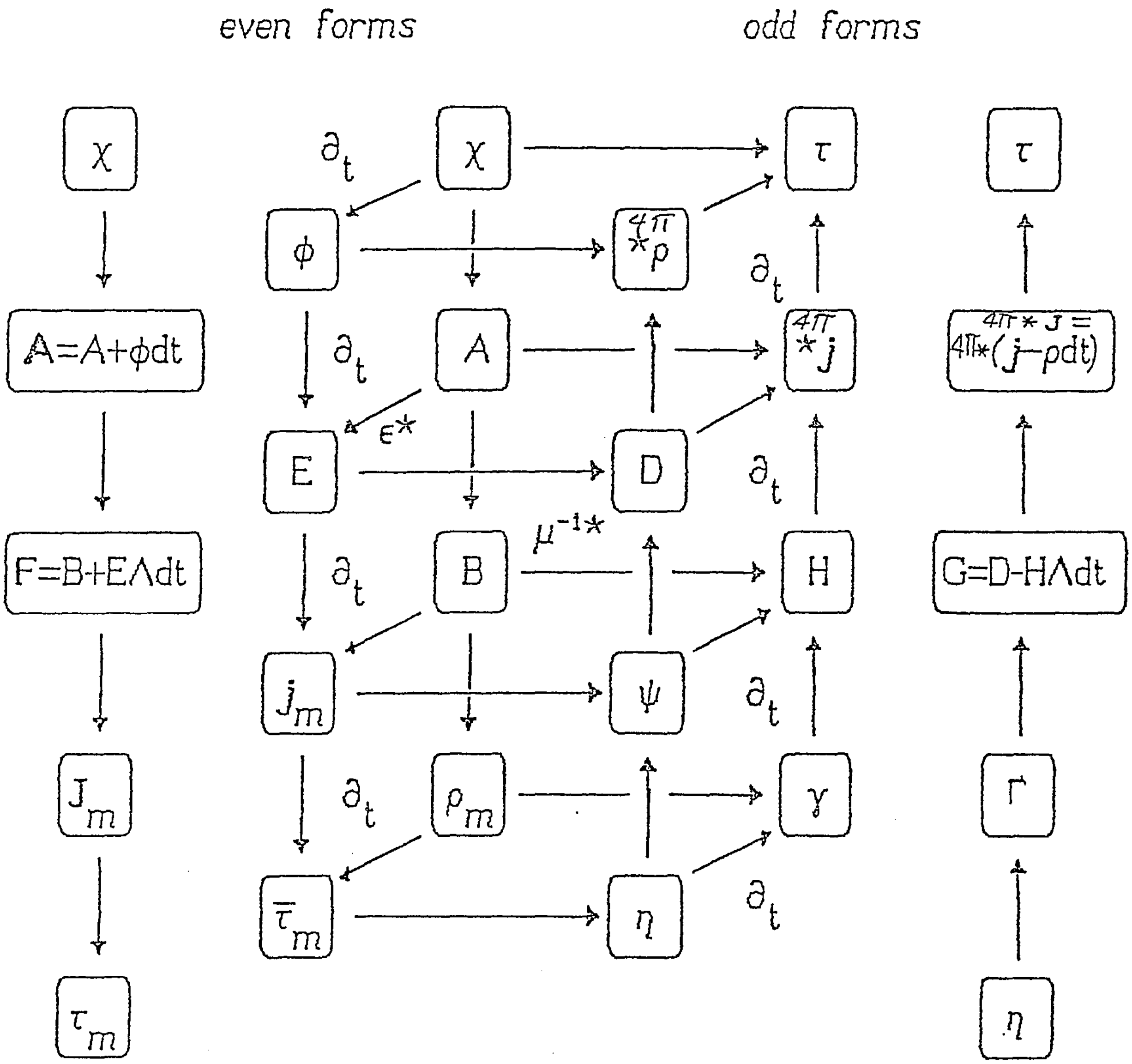

Fig. 8 . Coupling of Electrostatics and Magnetostatics.