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THE EXPONENTIAL SPACE OF AN L^2-STOCHASTIC PROCESS WITH INDEPENDENT INCREMENTS*

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1. INTRODUCTION

The Exponential or Fock space associated with a Gaussian process has been a useful concept in both theory and applications of multiple Wiener integrals (see Taqqu (1986) for up to date references). In recent years, the works on Malliavin Calculus of Zakai (1985), probability on Fock spaces of Meyer (1985) and invariance principle for symmetric statistics of Mandelbaum and Taqqu (1984) have stimulated even more the subject. It has been customary to study the Exponential space of a process using multiple Wiener integrals. This has been the approach taken by Itô (1951) for the Wiener process and by Surgailis (1984) for the Poisson random measure. However, there are situations where one is first interested in studying the exponential space and only after that define multiple Wiener integrals through symmetric tensor product techniques. In this direction Neveu (1968) has identified the exponential spaces associated with a general Gaussian system of random variables and with a Poisson random measure having finite control measure. Following the approach of Neveu, in this note we identify the exponential spaces associated with a Poisson random measure having σ-finite control measure and with a general L^2-stochastic process with independent increments. Our approach uses discrete martingales techniques and the monotone class lemma rather than multiple Wiener integrals.

2. MAIN RESULTS

Let H be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$, for $n \geq 0$ $H^{\odot n}$ be the n-fold symmetric tensor product Hilbert space of H with inner product $\langle\circ,\circ\rangle_{H^{\odot n}}$ and EXP(H) be the Hilbert space orthogonal direct sum of the subspaces $H^{\odot n}$ $n \geq 0$ with inner product $\langle\circ,\circ\rangle_e$. This space is called the Exponential space of H and in the mathematical physics literature it is known as the Fock space. The elements of EXP(H) are interpreted as sequences $\underline{h}=(h_0,h_1,h_2,...)$ where h_0 is a constant, h_1 belongs to $H^1=H$, $h_n \in H^{\odot n}$ $n \geq 2$ and their inner product is given by

$$(1)\qquad \langle\underline{h},\underline{k}\rangle_e = \sum_{n\geq 0} \langle h_n,k_n\rangle_{H^{\odot n}}.$$

Of special interest are the *exponential* elements

$$(2)\qquad \exp\odot(h) = (1,h,(2)^{-1/2}h^{\odot 2},(3!)^{-1/2}h^{\odot 3},...)\quad h \in H$$

which generate EXP(H) and whose inner product is given by

$$(3)\qquad \langle\exp\odot(h),\exp\odot(k)\rangle_e = \exp(\langle h,k\rangle_H).$$

Let (Ω,F,P) be a complete probability space and H_g be a Gaussian Hilbert space of random variables defined on (Ω,F,P). Without using multiple Wiener integrals Neveu (1968) has shown (Proposition 7. 3) that

$$(4)\qquad \mathrm{EXP}(H_g) \overset{\psi}{\cong} L^2(\Omega,F^g,P)$$

where $F^g = \sigma(H_g)$ and for $h \in H_g$

(5) $\psi(\exp\odot(h)) = \exp(h-(1/2)E(h^2))$

and $\{\psi(\exp\odot(h)) : h\in H_g\}$ generates $L^2(\Omega,\mathcal{F}^g,P)$, where $E(\cdot)$ denotes expected value.

A similar result is possible for the Poisson case: Let q be a centered Poisson random measure on an arbitrary measurable space $(S, \mathcal{E})$ with control measure ν and let

(6) $H_q = \{I_q(f) : f \in L^2(S,\mathcal{E},\nu)\}$

where $I_q(\cdot)$ denotes the isometric integral with respect to the random measure q. The Hilbert space of random variables H_q is called the *generalized Poisson space* associated with q. It is also shown in Neveu (1968) (Proposition 7.13) that if ν is a finite measure on $(S,\mathcal{E})$ then

(7) $EXP(H_q) \overset{\eta}{\cong} L^2(\Omega,\mathcal{F}^q,P)$

where $\mathcal{F}^q = \sigma(H_q)$ and for $f \in L^2(S,\mathcal{E},\nu)$

(8) $\eta(\exp\odot(I_q(f))) = \{ \prod_{j=1}^{N(S)} (1+f(Z_j))\}\exp(-\int_S f(s)d\nu(s))$

where $\{Z_j\}_{j\geq 1}$ is a sequence of independent random elements, independent of $N(S) = q(S) + \nu(S)$, each Z_j taking values in S and having distribution $\{\nu(S)\}^{-1}\nu(\cdot)$.

The next theorem extends the above result to the case where ν is a σ-finite measure on (S,E). For this situation Surgailis (1984) has shown an isometry between $EXP(H_q)$ and $L^2(\Omega,F^q,P)$. However, Surgailis uses multiple Poisson integrals techniques to prove it. We give a proof using the martingale convergence theorem.

<u>THEOREM</u> <u>1</u>. Let ν be a σ-finite measure on (S,E). Then

$$(9) \qquad EXP(H_q) \overset{\phi}{\cong} L^2(\Omega,F^q,P)$$

where $F^q = \sigma(H_q)$ and for $f \in L^2(S,E,\nu)$

$$(10) \qquad \phi(\exp\odot I_q(f)) = \{\prod_{i=1}^{\infty} \prod_{j=1}^{N(S_i)} (1+f(Z_j^{(i)}))\exp(-\int_{S_i} f(s)d\nu(s))\}$$

where: (i) S_i $i \geq 1$ are disjoint sets in E, $0<\nu(S_i)<\infty$ and $\bigcup_{i\geq 1}^{\infty} S_i=S$; (ii) for each $i=1,2,\ldots$ and $j=1,2,\ldots$ $Z_j^{(i)}$ is an S_i-valued random element with distribution given by the measure $\nu(S_i)^{-1}\nu(\circ)$, and for each $i=1,2,\ldots,N(S_i)$ follows a Poisson distribution with parameter $\nu(S_i)$; (iii) $Z_j^{(i)}$, $N(S_i)$ $i=1,2,\ldots$, $j=1,2,\ldots$ are mutually independent.

In order to prove this theorem we use the following technical result:

<u>LEMMA</u> <u>1</u>. Let ν and S_i, $N(S_i)$, $Z_j^{(i)}$ $j = 1,2,\ldots,i=1,2,\ldots$ be as in (i)-(iii) of the above theorem. If for some $i\geq 1$ $g \in L^1(S_i,E\cap S_i,\nu)$ then

$$E\left(\prod_{j=1}^{N(S_i)} g(Z_j^{(i)})\right) = \exp\left(\int_{S_i}(g-1)d\nu\right).$$

<u>Proof</u>: Since $N(S_i)$ follows the Poisson distribution with parameter $\nu(S_i)<\infty$ and for each $j=1,2,\ldots$ $Z_j^{(i)}$ has distribution $\nu(S_i)^{-1}\nu(\cdot)$, using the independence of $N(S_i)$, $Z_1^{(i)}$, $Z_2^{(i)}$, ... we obtain:

$$E\left[\prod_{j=1}^{N(S_i)} g(Z_j^{(i)})\right] = \sum_{n=0}^{\infty} \frac{e^{-\nu(S_i)}}{n!} \int_{S_i^n} \prod_{j=1}^{n} g(t_j) \prod_{j=1}^{n} d\nu(t_j)$$

$$= e^{-\nu(S_i)} e^{\int_{S_i} g d\nu} = e^{\int_{S_i}(g-1)d\nu}$$

<u>Proof</u> <u>of</u> <u>Theorem</u> <u>1</u>. We have to prove the following three conditions:

a) For each $f \in L^2(S,E,\nu)$ $\phi(\exp\circ(I_q(f))) \in L^2(\Omega,F^q,P)$.

b) For f_1, $f_2 \in L^2(S,E,\nu)$

$$E(\phi(\exp\circ(I_q(f_1)))\phi(\exp\circ(I_q(f_2)))) = \exp\left(\int_S f_1 f_2 d\nu\right).$$

c) $\{\phi(\exp\circ(I_q(f))) : f \in L^2(S,E,\nu)\}$ generates $L^2(\Omega,F^q,P)$.

Since ν is a σ-finite measure on (S,E) there exists a sequence of sets $\{S_i\}_{i\geq 1}$ in E such that $0<\nu(S_i)<\infty$ and $\bigcup_{i=1}^{\infty} S_i=S$. The existence of the random elements $Z_j^{(i)}$ $j=1,2,\ldots,$ $i=1,2,\ldots$ satisfying (ii) and (iii) follows from the construction of a Poisson random measure N with control measure ν (see for example Theorem 8.1 in Ikeda and Watanabe (1981)).

Let $f \in L^2(S,\mathcal{E},\nu)$, then for each $i\geq 1$ f belongs to $L^2(S_i,\mathcal{E}\cap S_i,\nu)$ and $L^1(S_i,\mathcal{E}\cap S_i,\nu)$. Then taking $g=(1+f)$ in Lemma 1 we obtain

$$E\left[\prod_{j=1}^{N(S_i)}(1+f(Z_j^{(i)}))\exp\left(-\int_{S_i} f d\nu\right)\right] = 1 \text{ for all } i=1,2,\ldots$$

Then using (iii) $G_i = \prod_{j=1}^{N(S_i)}(1+f(Z_j^{(i)}))\exp(-\int_{S_i} f d\nu)$ is a sequence of independent random variables with $E(G_i) = 1$, $i\geq 1$ and therefore $D_n = \prod_{i=1}^{n} G_i$ is a martingale. Next, using Lemma 1 with $g=(1+f)^2$ and the independece of $Z_j^{(i)}$, $N(S_i)$ $j\geq 1$, $i\geq 1$ we have:

$$E(D_n^2) = \prod_{i=1}^{n} E\left[\prod_{j=1}^{N(S_i)}(1+f(Z_j^{(i)}))\exp\left(-\int_{S_i} f d\nu\right)\right]^2$$

$$= \exp\left(\int_{US_i} f^2 d\nu\right) \leq \exp\left(\int_S f^2 d\nu\right) < \infty .$$

Then by the martingale convergence theorem D_n converges a.s. and in mean square to $\phi(\exp\odot(I_q(f)))$. Therefore

$$E\left[\prod_{i=1}^{\infty}\prod_{j=1}^{N(S_i)}(1+f(Z_k^{(i)}))\exp\left(-\int_{S_i} f d\nu\right)\right]^2 = \lim_{n\to\infty} E\, D_n^2$$

$$= \lim_{n\to\infty} \exp\left(\int_{\bigcup^n S_i} f^2 d\nu\right) = \exp\left(\int_S f^2 d\nu\right) < \infty$$

which shows (a).

Next let $f_1, f_2 \in L^2(S,\mathcal{E},\nu)$. Applying Lemma 1 to $g=(1+f_1)(1+f_2)$ one shows in a similar manner as above that

$$E(\exp\circ(I_q(f_1))\exp\circ(I_q(f_2)))$$

$$=\lim_{n\to\infty} \prod_{i=1}^{n} E\left[\prod_{j=1}^{N(S_i)} (1+f_1)(1+f_2)(Z_j^{(i)})\exp(\int_{S_i}(f_1^2+f_2^2)d\nu)\right]$$

$$=\lim_{n\to\infty} \prod_{i=1}^{n} \exp(\int_{S_i} f_1f_2d\nu) = \exp(\int_S f_1f_2d\nu)$$

proving (b).

Finally, to prove (c) let $G \in L^2(\Omega,F^q,P)$ and assume that

$$E(G\exp\circ(I_q(f))) = 0 \text{ for all } f \in L^2(S,E,\nu).$$

We have to show that G=0 a. e. dP_{F^q}. Using (10) we have that for each $f \in L^2(S,E,\nu)L^2(S,E,\nu)$

$$E\left[G \prod_{j=1}^{\infty}\left\{\prod_{j=1}^{N(S_i)} (1+f(Z_j^{(i)}))\exp(-\int_{S_i} fd\nu)\right\}\right] = 0.$$

Next let $i\geq1$ be fixed and for $g \in L^2(S_i,E\cap S_i,\nu)$ define $f: S \to \mathbb{R}$ by $f(t) = g(t)$ $t \in S_i$ and zero if $t \notin S_i$. Then $f \in L^2(S,E,\nu)$ and

$$E\left[G\prod_{j=1}^{N(S_i)} (1+g(Z_j^{(i)}))\exp(-\int_{S_i} gd\nu)\right]$$

$$= 0 \text{ for all } g \in L^2(S_i,E\cap S_i,\nu).$$

Hence, by Proposition 7.13 in Neveu (1968) we obtain that $E(G|F_i^q) = 0$ a.s. where $F_i^q = \sigma(I_q(g): g \in L^2(S_i,E\cap S_i,\nu))$ and $F_i^q\subset F^q$ $i\geq1$. Thus for each $n\geq1$ $E(G|\bigvee_{i=1}^{n} F_1^q)=0$ a.s. since $F_1^q, \ldots, F_n^q$ are independent σ-fields. Let $F_n = \bigvee_{i=1}^{n} F_i^q$ then $F^q = \bigvee_{n=1}^{\infty} F_n$, and since $E(G^2) < \infty$ it follows by the martingale convergence theorem that G=0 a.s. dP_{F^q}. The proof of the theorem is completed.

Finally we prove the following general result that identifies the exponential space of any Hilbert space H which is a direct sum of an arbitrary Gaussian space H_g and an arbitrary generalized Poisson space H_q, where H_g and H_q are stochastically independent. From this result and the Lévy-Itô representation we obtain the exponential space associated with an L^2-stochastic process with independent increments.

<u>THEOREM</u> <u>2</u>. Let $(\Omega,\mathcal{F},P)$ be a complete probability space and q be a centered Poisson random measure on a measurable space $(S,\mathcal{E})$ defined on $(\Omega,\mathcal{F},P)$, with σ-finite control measure ν and generating the Poisson space H_q given by (6). Let H_g be a Gaussian space on $(\Omega,\mathcal{F},P)$ stochastically independent of the system of random variables H_q. Define the σ-fields $\mathcal{F}^g = \sigma(H_g)$, $\mathcal{F}^q = \sigma(H_q)$ and the Hilbert space $H = H_g \oplus H_q$. Then

$$\text{(11)} \qquad \mathrm{EXP}(H) \overset{\gamma}{\cong} L^2(\Omega,\mathcal{F}^g \vee \mathcal{F}^q,P)$$

where for $h \in H, h=h_g+h_q, h_g \in H_g, h_q \in H_q, \gamma\colon \mathrm{EXP}(H) \to L^2(\Omega,\mathcal{F}^g\vee\mathcal{F}^q,P)$ is defined by

$$\text{(12)} \qquad \gamma(\exp\odot(h)) = \psi(\exp\odot(h_g))\phi(\exp\odot(h_q))$$

where ψ and ϕ are the isometries given in (5) and (10) respectively.

<u>Proof</u>. It follows by the independence of H_g and H_q that for all $h \in H$ $\gamma(\exp\odot(h))$ is an element of $L^2(\Omega,\mathcal{F}^g\vee\mathcal{F}^q,P)$ and that $E(\exp\odot(h))^2 = \exp(Eh^2)$.

Next we shall prove that $\{\gamma(\exp\odot(h))\colon h \in H\}$ generates $L^2(\Omega,\mathcal{F}^g\vee\mathcal{F}^q,P)$. Let $Z \in L^2(\Omega,\mathcal{F}^g\vee\mathcal{F}^q,P)$ and assume that

$$E(Z\gamma(\exp\odot(h))) = 0 \quad \text{for all } h \in H.$$

Then for each $h_g \in H_g$ and $h_q \in H_q$ we have

$$E(Z\{\psi(\exp\circ(h_g))\phi(\exp\circ(h_q))\})=0.$$

But $\{\psi(\exp\circ(h_g)):h_g \in H_g\}$ and $\{\phi(\exp\circ(h_q):h_q \in H_q\}$ generate $L^2(\Omega,F^g,P)$ and $L^2(\Omega,F^q,P)$ respectively. Then for each $A_1 \in F^g$ and $A_2 \in F^q$ $\int_{A_1 \cap A_2} ZdP=0$. But $F^g \vee F^q$ is generated by the field C_o of all finite disjoint unions of sets $A_1 \cap A_2$ $A_1 \in F^g$, $A_2 \in F^q$. Then, since Z is P-integrable, $C=\{A \in F: \int_A ZdP=0\}$ is a monotone class, and by the monotone class theorem

$$\int_A ZdP = 0 \quad \forall A \in F^g \vee F^q$$

for $C_o \subset C$. That is, Z=0 a.s. $dP_{F^g \vee F^q}$ and the theorem is proved.

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