CIMAT

Centro de Investigación en Matemáticas, A.C.

SOME RESULTS IN
QUOTIENT STACKS

T E S I S

Que para obtener el grado de

Doctor en Ciencias

con Orientación en

Matemáticas Básicas

Presenta

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Dedication

To my parents

Patricia Álvarez Villada

Héctor Jaime Acosta Vélez

Acknowledgements

I must acknowledge a lot of people and institutions who significantly contributed to the realization of this work and myself. First, and foremost to my advisor, Pedro Luis, without whose support, patience and understanding I would have been lost in half way. My family, who have accompanied me throughout my life, their affection and support have been fundamental in the path. CIMAT, an incredible research center full of lovely people who welcomed me and gave me the opportunity to achieve this academic goal. An special recognition to my teachers, classmates and the staff of the School Services department. CONACyT, magnificent institution that gave me sufficient financial support to devote my efforts exclusively to my studies. Last, but not least to México, Mexican are the kindest people I have known in my life, during all these years I felt at home. I keep in my heart a very special place for this country. ¡Viva México!

Contents

Introduction

Many interesting kind of geometric-algebraic stacks are defined as the quotient stack associated to a groupoid in algebraic spaces. More precisely, if (U, R, s, t, c) is a groupoid in algebraic spaces we have the quotient stack $[U/R]$. Some of them are quotients by the action of a group space into an algebraic space. When we have 1-morphisms $\mathcal{X} \longrightarrow \mathcal{Z}$ and $\mathcal{Y} \longrightarrow \mathcal{Z}$ of quotient stacks, some natural questions arise: if a fibre or a 2-fibre product exists, is this also a quotient stack? If the answer is affirmative, what is the associated groupoid? What if one of those 1-morphisms has an special property like to be an open immersion? In this work we are going to give some results about these questions, trying to solve it in the most general possible case.

While in categories over a fixed category $\mathcal{C}$ there are always a fibre product and a 2-fibre product, when we are working with fibred categories we have found that a fibre product does not always exist. However, we found a condition about the fibred product as a category over $\mathcal{C}$ and a class of fibred categories and 1-morphisms for which fibre product can always be constructed. We say that the fibre product has componentwise pullbacks if it satisfies that condition. In particular, when we define a quotient stack as the stackification of the fibred category associated to a functor induced by a groupoid in algebraic spaces, the fibred category belongs to this class and we can take the fibre product.

In groupoid categories, we have shown that a fibre product always exists and the obtained groupoid category is easy to compute. Then we have proved that the functoriality related to the associated functor of a groupoid category is compatible with fibre product. However, we have shown that the fibre product and the 2-fibre product on fibred categories are not always isomorphic, even worse they may not be equivalent as categories, so we can not conclude that the 2-fibre product has also the simple form obtained for the fibre product.

Stackification process is compatible with 2-fibre products, but we did not find a similar result in the literature about fibre products and stackification when the fibre product exists. Here we find the following problem: when is stackification compatible with fibre product? If it is not always the case, in which instances can we ensure it? We conclude that if the fibred categories are such that fibre product has componentwise pullbacks, then stackification is compatible with fibre product. In particular the result is true for fibred categories associated to functors, then for quotient stacks, and so this makes possible to determine the form of the fibre product of quotient stacks in a very simple way. Furthermore, in the case that one of the 1-morphisms is an open immersion, although we have not proved the same for 2-fibre product, we can compute the "strong" change of base via any 1-morphism and we have proved that this is also an open immersion. Here the word strong is used in order to emphasize that we are taking fibre product and not 2-fibre product. The last statement is not a direct result from the theory of stacks, because in fibred categories and therefore in stacks, change of base is made by taking 2-fibre product, not fibre product. Then we needed to give a different argument in order to conclude what we have done.

We have found a close relation between the fibre product and the 2-fibre product of categories over $\mathcal{C}$ which has interesting desirable properties. More precisely, there is a canonical fully faithful functor $H : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \longrightarrow \mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y}$, which is compatible with stackification. We have also found some results that ensure when fibre and 2-fibre products are isomorphic, which are interesting in some algebro-geometrical constructions of stacks. In those cases the results found in this work are less interesting, but they have the advantage of being applied directly considering fibre products instead 2-fibre products.

Some questions arise at the end: what about "strong" change of base via another kind of 1-morphisms different from open immersions? Also, what kind of properties has the fully faithful functor from the fibre product into the 2-fibre product? This questions are not solved here, but leaves open a project which the author is interested in and will deal with later.

The literature is extensive and each of the references has its own notation and style. In order to keep the things simple, the theory as exposed here and its principal results are taken from Stacks Project, which in many of the cases has the same references than the cited in the bibliography at the end of this work. Since this is such an extensive book (more than 5000 pages and growing) many of the results are not completely proved there, so we include a proof of most of them when we consider it necessary, especially when they are constructive. In some cases we made original proofs and discover some other results not mentioned there, which later we use in the results we are pursuing. In order to indicate to the reader which results we have taken we use the tagging system proposed in the Stacks Project web site, which consist of a chain of four alpha-numerical characters. This is good because the project is still open, is not suitable to keep a traditional numeration for chapters, definitions, lemmas, theorems, etc., as done by default in LaTeX. Those tags never change, they are determined once the part is added and remains in the servers even if it is wrong or removed from the project. Then, at the right side of every definition, lemma, theorem, example, etc., that we take from Stacks Project, there is the corresponding tag which is a hyperlink to the page where the specific tag is located, for those who read this work in a digital device. For those who read the physical version of this work, if it is necessary to find a tag, in the web page http://stacks.math.columbia.edu/tag there is a browser created for this purpose.

The examples, different from those that illustrate definitions, were built after trying to prove unsuccessfully some results. Then, when we saw some obstacle to continue, we decide to create instances where they were unavoidable. They are original from the author.

At the beginning of the work this was supposed to be a thesis in algebraic geometry. However, when we saw a possible way and we designed a plan of realization, turned out to be more categorical than geometric. The principal advantage of this is that the scope is extended beyond geometry, since we can replace $\mathcal{C}$ for any category with a Grothendieck topology, not only schemes or algebraic spaces with some specific nice topology like the étale site. Then the results can be applied to many categories similar to those which are of interest in algebraic geometry. Also, the categorical approaching helps us to understand what are the essential characteristics of the objects we are studying.

Chapter

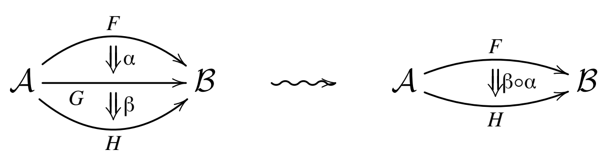
1 Fibred categories

In this chapter we are going to introduce fibred categories as the basic 2-categorical concept in order to define stacks, which will be introduced in a separated chapter. Here we are interested in more general facts like fibre and 2-fibre products and strongly cartesian morphisms or pullbacks, which allow us to consider "restrictions" of morphisms and therefore locally defined morphisms when we are working on categories with Grothendieck topologies. Many subsequent developments are based in what is in this chapter, so we encourage the reader to see at least the results. In particular, the fibred category associated to a functor is important in order to consider quotient stacks and we give some functorial properties which will be used later.

We consider categories over a fixed category and later fibred categories through the concept of strongly cartesian morphism. Then we show some results when the fibre categories are groupoids, in particular setoids or sets.

1.1 2-categories

Given categories $\mathcal{A}, \mathcal{B}$ we have the category $\mathbf{Fun}(\mathcal{A}, \mathcal{B})$, whose objects are functors from $\mathcal{A}$ to $\mathcal{B}$ and its morphisms are natural transformations of functors, i.e. if $F, G, H : \mathcal{A} \longrightarrow \mathcal{B}$ are functors and $\alpha : F \Longrightarrow G$, $\beta : G \Longrightarrow H$ are natural transformations, the composition rule $\beta \circ \alpha : F \Longrightarrow H$, called vertical composition, is defined for an object x of $\mathcal{A}$ by $(\beta \circ \alpha)_x = \beta_x \circ \alpha_x$. We illustrate this composition in the following diagram:



Given categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ there is a composition law $\circ : \mathrm{Ob}(\mathbf{Fun}(\mathcal{A}, \mathcal{B})) \times \mathrm{Ob}(\mathbf{Fun}(\mathcal{B}, \mathcal{C})) \longrightarrow \mathrm{Ob}(\mathbf{Fun}(\mathcal{A}, \mathcal{C}))$ defined by composition of functors $\circ(f, g) = g \circ f$. This law is associative and the identity functors act as units. Then we have a category whose objects are categories and morphisms are functors which is denoted **Cat**.

If $F, G : \mathcal{A} \longrightarrow \mathcal{B}$ are functors and $\alpha : F \Longrightarrow G$ is a natural transformation, then for any functor $H : \mathcal{B} \longrightarrow \mathcal{C}$ it can be defined a natural transformation ${}_H\alpha : H \circ F \Longrightarrow H \circ G$ by $({}_H\alpha)_x = H(\alpha_x)$ for all $x \in \mathrm{Ob}(\mathcal{A})$. In this way ${}_H(\_)$ is a functor $\mathbf{Fun}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathbf{Fun}(\mathcal{A}, \mathcal{C})$. In order to see it, we need to check that ${}_H(id_F) = id_{H \circ F}$ and ${}_H(\beta \circ \alpha) = {}_H\beta \circ {}_H\alpha$. If $id_F : F \Longrightarrow F$ is the identity transformation, then for all $x \in \mathrm{Ob}(\mathcal{A})$ it follows $({}_H(id_F))_x = H((id_F)_x) = H(id_{F(x)}) = id_{H(F(x))} = id_{H \circ F(x)} = (id_H \circ F)_x$ and $({}_H(\beta \circ \alpha))_x = H((\beta \circ \alpha)_x) = H(\beta_x \circ \alpha_x) = H(\beta_x) \circ H(\alpha_x) = ({}_H\beta)_x \circ ({}_H\alpha)_x = ({}_H\beta \circ {}_H\alpha)_x$. In particular, ${}_{id_{\mathcal{B}}}\alpha = \alpha$.

Similarly, given functors $R, S : \mathcal{B} \longrightarrow \mathcal{C}$ and a natural transformation $\gamma : R \Longrightarrow S$, then for all functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ we define $\gamma_F : R \circ F \Longrightarrow S \circ F$ by $(\gamma_F)_x = \gamma_{F(x)}$ for $x \in \mathrm{Ob}(\mathcal{A})$ and

$(\_)_F : \mathbf{Fun}(\mathcal{B},\mathcal{C}) \longrightarrow \mathbf{Fun}(\mathcal{A},\mathcal{C})$ is a functor with $(id_R)_F = id_{R\circ F}$ and $(\gamma \circ \delta)_F = \gamma_F \circ \delta_F$. Also we see that $\gamma_{id_{\mathcal{B}}} = \gamma$.

The preceding constructions satisfy the further properties ${}_{H_1}({}_{H_2}\alpha) = {}_{(H_1\circ H_2)}\alpha$, $(\gamma_{F_1})_{F_2} = \gamma_{(F_1\circ F_2)}$ and ${}_H(\varepsilon_F) = ({}_H\varepsilon)_F$, provided that those compositions can be defined. Finally, given functors $F, G : \mathcal{A} \longrightarrow \mathcal{B}$ and $R, S : \mathcal{B} \longrightarrow \mathcal{C}$ and natural transformations $\alpha : F \Longrightarrow G$ and $\gamma : R \Longrightarrow S$, the following diagram commutes:

$$\begin{array}{ccc} R\circ F & \overset{{}_R\alpha}{\Longrightarrow} & R\circ G \\ {\scriptstyle\gamma_F}\Downarrow & & \Downarrow{\scriptstyle\gamma_G} \\ S\circ F & \underset{{}_S\alpha}{\Longrightarrow} & S\circ G \end{array}$$

that is to say, $\gamma_G \circ {}_R\alpha = {}_S\alpha \circ \gamma_F$. In order to proof this, we shall see what happens at each object x of $\mathcal{A}$. We have a morphism $\alpha_x : F(x) \longrightarrow G(x)$ in $\mathcal{B}$ and, since $R, S : \mathcal{B} \longrightarrow \mathcal{C}$ are functors, they induce the morphisms $R(\alpha_x)$ and $S(\alpha_x)$. But $R(\alpha_x) = ({}_R\alpha)_x$ and $S(\alpha_x) = ({}_S\alpha)_x$. Also $\gamma_{F(x)} = (\gamma_F)_x$ and $\gamma_{G(x)} = (\beta_G)_x$. Since γ is a natural transformation of functors, considering the morphism α_x in $\mathcal{B}$, we have the following commutative squares:

$$\begin{array}{ccc} R(F(x)) & \xrightarrow{R(\alpha_x)} & R(G(x)) \\ {\scriptstyle\gamma_{F(x)}}\downarrow & & \downarrow{\scriptstyle\gamma_{G(x)}} \\ S(F(x)) & \xrightarrow[S(\alpha_x)]{} & S(G(x)) \end{array} \quad \leftrightsquigarrow \quad \begin{array}{ccc} R(F(x)) & \xrightarrow{({}_R\alpha)_x} & R(G(x)) \\ {\scriptstyle(\gamma_F)_x}\downarrow & & \downarrow{\scriptstyle(\gamma_G)_x} \\ S(F(x)) & \xrightarrow[({}_S\alpha)_x]{} & S(G(x)) \end{array}$$

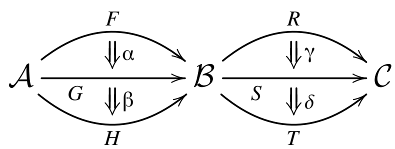
This compatibility relation allow us to define the horizontal composition of natural transformation as:

$$\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B} \underset{S}{\overset{R}{\rightrightarrows}} \mathcal{C} \;\;(\Downarrow\alpha,\ \Downarrow\gamma) \;=\; \mathcal{A} \underset{S\circ G}{\overset{R\circ F}{\rightrightarrows}} \mathcal{C} \;\;(\Downarrow\gamma\star\alpha)$$

where $\gamma \star \alpha = \gamma_G \circ {}_R\alpha = {}_S\alpha \circ \gamma_F$. Hence we may recover the transformations ${}_R\alpha$ and γ_F as ${}_R\alpha = id_R \star \alpha$ and $\gamma_F = \gamma \star id_F$. Furthermore, all of the rules above are consequences of the properties stated in the next lemma.

Lemma 1.1.1. *(003F) Horizontal and vertical compositions satisfy the following properties:*

1. *$\circ$ and $\star$ are associative.*
2. *The identity transformations id_F are units for $\circ$.*
3. *For any category $\mathcal{A}$, the identity transformations of the identity functors, i.e. the transformations $id_{id_{\mathcal{A}}}$, are units for $\circ$ and $\star$.*
4. *Given a diagram*



we have $(\delta \circ \gamma) \star (\beta \circ \alpha) = (\delta \star \beta) \circ (\gamma \star \alpha)$

Proof. Properties (1) – (3) are immediate from the definitions. To see (4), by using the previous notation we have

$$(\delta \circ \gamma) \star (\beta \circ \alpha) = (\delta \circ \gamma)_H \circ {}_R(\beta \circ \alpha) = \delta_H \circ \gamma_H \circ {}_R\beta \circ {}_R\alpha$$

$$(\delta \star \beta) \circ (\gamma \star \alpha) = (\delta_H \circ {}_S\beta) \circ (\gamma_G \circ {}_R\alpha) = \delta_H \circ ({}_S\beta \circ \gamma_G) \circ {}_R\alpha$$

and by definition $\gamma \star \beta = \gamma_H \circ {}_R\beta = {}_S\beta \circ \gamma_G$ and this conclude the proof. □

Another way of formulating (4) is that the composition of functors and the horizontal composition of natural transformations induces a functor

$$(\circ, \star) : \mathbf{Fun}(\mathcal{A}, \mathcal{B}) \times \mathbf{Fun}(\mathcal{B}, \mathcal{C}) \longrightarrow \mathbf{Fun}(\mathcal{A}, \mathcal{C})$$

whose source is the product category. Then **Cat** is a category where for every pair of categories, the morphisms $\mathbf{Fun}(\mathcal{A}, \mathcal{B})$ are itself a category, in which the morphisms have, together with the usual composition (vertical), one horizontal composition satisfying the properties in the previous lemma. It gives rise to the concept of 2-category in the next definition.

Definition 1.1.1 (2-category). *(003H) A 2-category $\mathcal{C}$, consists of the following data:*

1. *A class of objects* $\mathrm{Ob}(\mathcal{C})$.

2. *For each pair* $x, y \in \mathrm{Ob}(\mathcal{C})$ *a category* $\mathrm{Mor}_{\mathcal{C}}(x, y)$. *The objects of this category are called 1-morphisms and denoted* $F : x \longrightarrow y$. *Given another 1-morphism* $G : x \longrightarrow y$, *a morphism* α *from* F *to* G *in* $\mathrm{Mor}_{\mathcal{C}}(x, y)$ *are called 2-morphisms and denoted* $\alpha : F \Longrightarrow G$. *The composition of 2-morphisms on* $\mathrm{Mor}_{\mathcal{C}}(x, y)$ *will be named vertical composition and denoted* $\beta \circ \alpha$, *where* $\beta : G \Longrightarrow H$.

3. *For all* $x, y, z \in \mathrm{Ob}(\mathcal{C})$ *a functor*

$$(\circ, \star) : \mathrm{Mor}_{\mathcal{C}}(x, y) \times \mathrm{Mor}_{\mathcal{C}}(y, z) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(x, z)$$

   The image of a pair of 1-morphisms (F, S) *is calle composition of* F *and* S *which is denoted* $S \circ F$. *The image of a pair* $(\alpha, \gamma) : (F, G) \Longrightarrow (R, S)$ *of 2-morphisms, where* $\alpha : F \Longrightarrow G$ *and* $\gamma : R \Longrightarrow S$ *will be called horizontal composition and be denoted* $\gamma \star \alpha$.

These data satisfies the following rules:

i. *The class of objects with the set of 1-morphisms and the composition of 1-morphisms forms a category.*

ii. *Horizontal composition of 2-morphisms is associative.*

iii. *For each* $x \in \mathrm{Ob}(\mathcal{C})$, *the identity 2-morphism of the identity 1-morphism* id_{id_x} *is a unit for horizontal composition.*

Remark. Since $(\circ, \star)$ is a functor, and the pair (id_F, id_R) is the identity of (F, R) in the product category $\mathrm{Mor}_{\mathcal{C}}(x, y) \times \mathrm{Mor}_{\mathcal{C}}(x, y)$, then we have

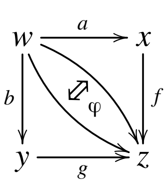
$$id_R \star id_F := (\circ, \star)(id_F, id_R) = id_{(\circ,\star)(F,R)} =: id_{R \circ F}$$

Example 1. (003J) The following are examples of 2-categories. The last two will be constructed after.

1. The 2-category of categories **Cat**
2. The 2-category of grupoids.
3. The 2-category of fibre categories over a fixed category.

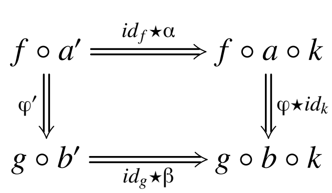
Example 2. Let $\mathcal{C}$ be a 2-category and $g : y \longrightarrow z$, $f : x \longrightarrow z$ two 1-morphisms on $\mathcal{C}$. The 2-category of 2-commutative diagrams relative to the pair (f, g) is defined as follows:

1. Objects are quadruples (w, a, b, φ), where $w \in \mathrm{Ob}(\mathcal{C})$ and $a : w \longrightarrow x$ and $b : w \longrightarrow y$ are 1-morphisms and $\varphi : f \circ a \Longrightarrow g \circ b$ is a 2-isomorphism. Hence, objects correspond to diagrams with the form:

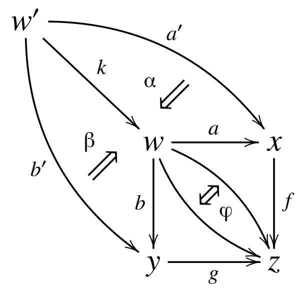


   We say such diagrams are 2-commutative.

2. The 1-morphisms from (w', a', b', φ') to (w, a, b, φ) are triples $(k : w' \longrightarrow w, \alpha : a' \Longrightarrow a \circ k, \beta : b' \Longrightarrow b \circ k)$ such that the next diagram is commutative



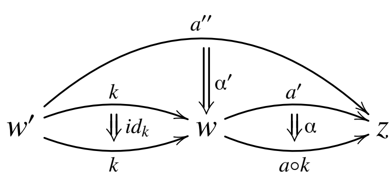
In this case we say the following diagram is 2-commutative

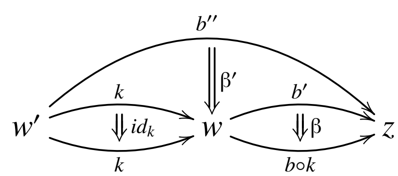


3. Given a second 1-morphims$(k', \alpha', \beta') : (w'', a'', b'', \varphi'') \longrightarrow (w', a', b', \varphi')$, the composition $(k, \alpha, \beta) \circ (k', \alpha', \beta') : (w'', a'', b'', \varphi'') \longrightarrow (w, a, b, \varphi)$ is defined by

$$(k'', \alpha'', \beta'', \varphi'') = (k \circ k', (\alpha \star id_{k'}) \circ \alpha', (\beta \star id_{k'}) \circ \beta')$$

This triple makes sense because $k'' : w'' \longrightarrow w$, $\alpha'' : a'' \longrightarrow a \circ (k \circ k')$ and $\beta'' : b'' \longrightarrow b \circ (k \circ k')$ are well defined as we may see at the diagrams





Moreover, in the diagrama below the two sub-rectangles are commutative and therefore is the external one

$$\begin{array}{ccccc} f \circ a'' & \xRightarrow{id_f \star \alpha''} & f \circ a' \circ k' & \xRightarrow{id_f \star \alpha \star id_{k'}} & f \circ a \circ (k \circ k') \\ {\scriptstyle \varphi''}\Downarrow & & {\scriptstyle \varphi' \star id_{k'}}\Downarrow & & {\scriptstyle \varphi \star id_{k \circ k'}}\Downarrow{\scriptstyle \varphi \star id_k \star id_k} \\ g \circ b'' & \xRightarrow[id_g \star \beta]{} & g \circ b' \circ k' & \xRightarrow[id_g \star \beta \star id_{k'}]{} & g \circ b \circ (k \circ k') \end{array}$$

Here we have used $id_{k \circ k'} = id_k \star id_{k'}$ and the associativity of horizontal composition to see that the right one is commutative and $(id_f \star \alpha \star id_{k'}) \circ (id_f \star \alpha') = (id_f \circ id_f) \star ((\alpha \star id_{k'}) \circ \alpha) = id_f \star \alpha''$. Similarly $(id_g \star \alpha \star id_{k'}) \circ (id_g \star \beta') = id_g \star \beta''$.

4. A 2-morphism from a 1-morphism (k_1, α_1, β_1) to a 1-morphism (k_2, α_2, β_2) is given by a 2-morphism $\delta : k_1 \Longrightarrow k_2$ in $\mathcal{C}$ such that the next diagrams are commutative

$$\begin{array}{ccc} a' & \overset{\alpha_1}{\Longrightarrow} & a \circ k_1 \\ & {}_{\alpha_2}\searrow & \Downarrow {\scriptstyle id_a \star \delta} \\ & & a \circ k_2 \end{array} \qquad \begin{array}{ccc} b' & \overset{\beta_1}{\Longrightarrow} & a \circ k_1 \\ & {}_{\beta_2}\searrow & \Downarrow {\scriptstyle id_b \star \delta} \\ & & b \circ k_2 \end{array}$$

5. Vertical and horizontal compositions of 2-morphisms are the inherited from $\mathcal{C}$.

Definition 1.1.2 (Sub 2-category). *(02X7) Let $\mathcal{C}$ be a 2-category. A sub 2-category $\mathcal{C}'$ of $\mathcal{C}$ is given by a sub class* $\mathrm{Ob}(\mathcal{C}')$ *of* $\mathrm{Ob}(\mathcal{C})$ *and subcategories* $\mathrm{Mor}_{\mathcal{C}'}(x, y)$ *of the categories* $\mathrm{Mor}_{\mathcal{C}}(x, y)$ *for every $x, y \in \mathrm{Ob}(\mathcal{C}')$, such that with the operations $\circ$ (composition of 1-morphisms), $\circ$ (vertical composition of 2-morphisms) and $\star$ (horizontal composition of 2-morphims) forms a 2-category.*

Some 2-categories has the property that all the 2-morphisms are actually 2-isomorphims. This class of 2-categories are important and of course it is easier to work with these.

Definition 1.1.3 ((2,1)-category). *(003I) A (2,1)-category is a 2-category in which every 2-morphism is a 2-isomorphism.*

Example 3. For any 2-category there is a (2,1)-category obtained by allowing only 2-isomorphisms. This (2,1)-category is also a sub 2-category. As a concrete examples we have:

1. The (2,1)-category of categories.

2. The (2,1)-category of groupoids.

3. The (2,1)-category of fibre categories over a fixed category.

Equivalence of objects (isomorphism) has a generalization in 2-categories as follows.

Definition 1.1.4 (Equivalence of objects). *(003L) Two objects x, y of a 2-category are equivalents if there exists 1-morphisms $F : x \longrightarrow y$ and $G : y \longrightarrow x$ such that $F \circ G$ is 2-isomorphic to id_y and $G \circ F$ is 2-isomorphic to id_x.*

Here, $F \circ G$ 2-isomorphic to id_y means that exists $\alpha : F \circ G \Longrightarrow id_y$ and $\beta : id_y \Longrightarrow F \circ G$ such that $\beta \circ \alpha = id_{F \circ G}$ and $\alpha \circ \beta = id_{id_y}$. The concept of functor has also a generalization.

Definition 1.1.5 (Functor and pseudo-functor in a 2-category). *(003N) Let $\mathcal{A}$ be a category and $\mathcal{C}$ be a 2-category.*

1. *A functor from $\mathcal{A}$ to $\mathcal{C}$ is a functor (in the usual sense) from $\mathcal{A}$ to the category formed out of $\mathrm{Ob}(\mathcal{C})$ and the 1-morphisms.*

2. *A pseudofunctor φ from $\mathcal{A}$ to the 2-category $\mathcal{C}$ is given by the following data:*

   a. *For each $x \in \mathrm{Ob}(\mathcal{A})$ an object $\varphi(x) \in \mathrm{Ob}(\mathcal{C})$.*

   b. *For every pair $x, y \in \mathrm{Ob}(\mathcal{A})$ and any morphism $f : x \longrightarrow y$, a 1-morphism $\varphi(f) : \varphi(x) \longrightarrow \varphi(y)$.*

   c. *For every $x \in \mathrm{Ob}(\mathcal{A})$ a 2-morphism $\alpha_x : id_{\varphi(x)} \Longrightarrow \varphi(id_x)$.*

   d. *For every pair of composable morphisms $f : x \longrightarrow y$ and $g : y \longrightarrow z$ on $\mathcal{A}$ a 2-morphism $\alpha_{g,f} : \varphi(g \circ f) \Longrightarrow \varphi(g) \circ \varphi(f)$.*

   These data are subject to the following conditions:

   i. *The 2-morphisms α_x and $\alpha_{g,f}$ are 2-isomorphims.*

ii. *For any morphism* $f : x \longrightarrow y$ *in* $\mathcal{A}$ *we have* $\alpha_{id_y,f} = \alpha_y \star id_{\varphi(f)}$.

$$\begin{tikzcd} \varphi(x) \arrow[r, bend left, "\varphi(f)"] \arrow[r, bend right, "\varphi(f)"'] \arrow[r, phantom, "\Downarrow id_{\varphi(f)}"] & \varphi(y) \arrow[r, bend left, "id_{\varphi(y)}"] \arrow[r, bend right, "\varphi(id_y)"'] \arrow[r, phantom, "\Downarrow \alpha_y"] & \varphi(y) \end{tikzcd} \;=\; \begin{tikzcd} \varphi(x) \arrow[r, bend left, "\varphi(f)"] \arrow[r, bend right, "\varphi(id_y)\circ\varphi(f)"'] \arrow[r, phantom, "\Downarrow \alpha_{id_y,f}"] & \varphi(y) \end{tikzcd}$$

and also for any morphism $f : x \longrightarrow y$ *in* $\mathcal{A}$ *we have* $\alpha_{f,id_x} = id_{\varphi(f)} \star \alpha_x$

iii. *For any triple of morphisms* $f : w \longrightarrow x$, $g : x \longrightarrow y$ *and* $h : y \longrightarrow z$ *in* $\mathcal{A}$ *we have* $(id_{\varphi(h)} \star \alpha_{g,f}) \circ \alpha_{h,g\circ f} = (\alpha_{h,g} \star id_{\varphi(f)}) \circ \alpha_{h\circ g,f}$. *In other words the following diagram commutes*

$$\begin{tikzcd} \varphi(h \circ g \circ f) \arrow[r, Rightarrow, "\alpha_{h\circ g,f}"] \arrow[d, Rightarrow, "\alpha_{h,g\circ f}"'] & \varphi(h \circ g) \circ \varphi(f) \arrow[d, Rightarrow, "\alpha_{h,g}\star id_{\varphi(f)}"] \\ \varphi(h) \circ \varphi(g \circ f) \arrow[r, Rightarrow, "id_{\varphi(h)}\star\alpha_{g,f}"'] & \varphi(h) \circ \varphi(g) \circ \varphi(f) \end{tikzcd}$$

Now we introduce the concept of 2-final object, which allow us to define what a 2-fibre product is.

Definition 1.1.6 (2-final object). *(003P) A 2-final object in a 2-category* $\mathcal{C}$ *is an object* x *of* $\mathcal{C}$ *such that:*

1. *For all* $y \in \mathrm{Ob}(\mathcal{C})$ *exists a 1-morphism* $y \longrightarrow x$.

2. *Each pair of 1-morphisms* $y \longrightarrow x$ *are 2-isomorphic by a unique 2-morphism.*

Definition 1.1.7 (2-fibre product). *(003Q) Let* $\mathcal{C}$ *be a 2-category and* $f : x \longrightarrow z$, $g : y \longrightarrow z$ *two 1-morphisms on* $\mathcal{C}$. *A 2-fibre product on* $\mathcal{C}$ *of* f *and* g *is a 2-final object in the 2-category of 2-commutative diagrams on* $\mathcal{C}$ *related to the pair* (f, g).

Then, a 2-fibre product of f and g on $\mathcal{C}$ is given by a quadruple (w, a, b, φ) in the 2-category o 2-commutative diagrams related to (f, g) such that for any other quadruple (w', a', b', φ')

there exist a triple (k_1, α_1, β_1) which is a 1-morphism in the 2-category of 2-commutative diagrams related to (f, g) such that if there ir another triple (k_2, α_2, β_2), then exists a unique 2-isomorphism $\delta : k_1 \Longrightarrow k_2$ on $\mathcal{C}$, satisfying $(id_a \star \delta) \circ \alpha_1 = \alpha_2$ and $(id_b \star \delta) \circ \beta_1 = \beta_2$.

1.2 Categories over a category

If $p : \mathcal{S} \longrightarrow \mathcal{C}$ a functor between categories, we say that $p : \mathcal{S} \longrightarrow \mathcal{C}$ is a category over $\mathcal{C}$. In the following definition we are going to introduce the 2-category of categories over $\mathcal{C}$.

Definition 1.2.1 (2-Category of categories over $\mathcal{C}$). *(003Y) Let $\mathcal{C}$ be a category. The 2-category of categories over $\mathcal{C}$ is the 2-category constructed as follows:*

1. *Objects are pairs $(\mathcal{S}, p)$, where $\mathcal{S}$ is a category and $p : \mathcal{S} \longrightarrow \mathcal{C}$ is a functor.*
2. *The 1-morphisms $(\mathcal{S}, p) \longrightarrow (\mathcal{S}', p')$ are functors $G : \mathcal{S} \longrightarrow \mathcal{S}'$ such that $p' \circ G = p$.*
3. *Given $G, H : (\mathcal{S}, p) \longrightarrow (\mathcal{S}', p')$, the 2-morphisms from G to H are natural transformations $\alpha : G \Longrightarrow H$ which satisfy $p'(\alpha_x) = id_{p(x)}$, for all $x \in \mathrm{Ob}(\mathcal{S})$.*
4. *Compositions are defined as before on functors and natural transformations of* **Cat**.

Then is clear that the 2-category of categories over $\mathcal{C}$ is a sub 2-category of **Cat**. *This is denoted as* **Cat** $/\mathcal{C}$.

Definition 1.2.2 (Fibre category and lift). *(02XH) Let $\mathcal{C}$ be a category and $p : \mathcal{S} \longrightarrow \mathcal{C}$ a category over $\mathcal{C}$*

1. *The fibre category over an object $U \in \mathrm{Ob}(\mathcal{C})$ is the category $\mathcal{S}_U$ whose objects are $\mathrm{Ob}(\mathcal{S}_U) = \{x \in \mathrm{Ob}(\mathcal{S}) \mid p(x) = U\}$ and for $x, y \in \mathrm{Ob}(\mathcal{S}_U)$, $\mathrm{Hom}_{\mathcal{S}_U}(x, y) = \{\varphi \in \mathrm{Mor}_{\mathcal{S}}(x, y) \mid p(\varphi) = id_U\}$.*

2. *A lift of an object* $U \in \mathrm{Ob}(\mathcal{C})$ *is an object* $x \in \mathrm{Ob}(\mathcal{S})$ *such that* $p(x) = U$. *Namely* $x \in \mathcal{S}_U$. *We say that* x *is over* U.

3. *Similarly a lift of a morphism* $f : V \longrightarrow U$ *of* $\mathcal{C}$ *is a morphism* $\varphi : y \longrightarrow x$ *of* $\mathcal{S}$ *such that* $p(\varphi) = f$.

If $F : (\mathcal{S}, p) \longrightarrow (\mathcal{S}', p')$ is a 1-morphism of categories over $\mathcal{C}$, then the restriction of F to $\mathcal{S}_U$ is a functor between fibre categories $F|_U : \mathcal{S}_U \longrightarrow \mathcal{S}'_U$. In fact, to see that is enough to show that for all $x \in \mathcal{S}_U$ we have $F(x) \in \mathcal{S}'_U$ and for any $x \xrightarrow{\varphi} y \in \mathrm{Mor}_{\mathcal{S}_U}(x, y)$ it follows $F(\varphi) \in \mathrm{Mor}_{\mathcal{S}_U}(F(x), F(y))$, which is true because $p'(F(x)) = p(x) = U$ and since $p(\varphi) = id_U$, then $p'(F(\varphi)) = p(\varphi) = id_U$.

Now we will see that in the (2,1)-category of categories over a fixed category $\mathcal{C}$, there exists a 2-fibre product. This is a constructive proof, which is very useful in the following results related to fibred categories. In order to keep things simple, we do not include all the details, they are straightforward and easy to do. Also, there is a difference here respect to Stacks Project, since we don't ask for the condition $p(f) = q(g)$, because this is a consequence of the commutativity of the diagram below.

Lemma 1.2.1 (Existence of 2-fibre product). *(0040) Let* $\mathcal{C}$ *be a category with fibre product. The (2,1)-category of categories over* $\mathcal{C}$ *has 2-fibre product.*

Proof. Let $p : \mathcal{X} \longrightarrow \mathcal{C}$, $q : \mathcal{Y} \longrightarrow \mathcal{C}$, $r : \mathcal{Z} \longrightarrow \mathcal{C}$ be categories over $\mathcal{C}$ and $F : \mathcal{X} \longrightarrow \mathcal{Z}$, $G : \mathcal{Y} \longrightarrow \mathcal{Z}$ two 1-morphisms of categories over $\mathcal{C}$. An explicit 2-fibre product $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}, pr_{\mathcal{X}}, pr_{\mathcal{Y}}, \psi)$ of F and G on **Cat** $/\mathcal{C}$ is given by:

1. An object $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is a quadruple (U, x, y, α) where $U \in \mathrm{Ob}(\mathcal{C})$, $x \in \mathcal{X}_U$, $y \in \mathcal{Y}_U$ and $\alpha : F(x) \longrightarrow G(y)$ is an isomorphism in Z_U.

2. A morphism $(U, x, y, \alpha) \longrightarrow (U', x', y', \alpha')$ is a pair (f, g) with $f : x \longrightarrow x'$ a morphism in $\mathcal{X}$ and $g : y \longrightarrow y'$ is a morphism in $\mathcal{Y}$ such that the following diagram commutes

$$\begin{array}{ccc} F(x) & \xrightarrow{\alpha} & G(y) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(g)} \\ F(x') & \xrightarrow[\alpha']{} & G(y') \end{array}$$

In particular we have:

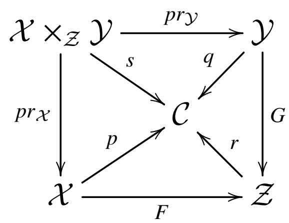
$$\begin{aligned} p(f) &= id_{U'} \circ p(f) \\ &= r(\alpha') \circ r(F(f)) \\ &= r(\alpha' \circ F(f)) \\ &= r(G(g) \circ \alpha) \\ &= r(G(g)) \circ r(\alpha) \\ &= q(g) \circ id_U \\ &= q(g) \end{aligned}$$

3. The 1-morphisms $pr_{\mathcal{X}} : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \longrightarrow \mathcal{X}$ and $pr_{\mathcal{Y}} : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \longrightarrow \mathcal{Y}$ are the forgetful functors, that is to say, if $(U, x, y, \alpha) \in \mathrm{Ob}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$, $pr_{\mathcal{X}}(U, x, y, \alpha) = x$, $pr_{\mathcal{Y}}(U, x, y, \alpha) = y$ and if $(U, x, y, \alpha) \xrightarrow{(f,g)} (U', x', y', \alpha')$ is a 1-morphism in $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$, $pr_{\mathcal{X}}(f, g) = f$ and $pr_{\mathcal{Y}}(f, g) = g$. The 2-isomorphism $\psi : F \circ pr_{\mathcal{X}} \Longrightarrow G \circ pr_{\mathcal{Y}}$ is the natural transformation defined by $\psi_{(U,x,y,\alpha)} : F(pr_{\mathcal{R}}(U, x, y, \alpha)) = F(r) \xrightarrow{\alpha} G(t) = G(pr_{\mathcal{T}}(U, x, y, \alpha))$, which is invertible because α is an isomorphism in $\mathcal{S}$. □

Lemma 1.2.2. *(02XI) Let $\mathcal{C}$ be a category, $F : \mathcal{X} \longrightarrow \mathcal{Z}$ and $G : \mathcal{Y} \longrightarrow \mathcal{Z}$ 1-morphisms of categories over $\mathcal{C}$. For all $U \in \mathrm{Ob}(\mathcal{C})$ we have the following equality of fibre categories:*

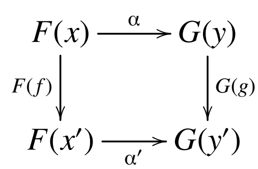
$$(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})_U = \mathcal{X}_U \times_{\mathcal{Z}_U} \mathcal{Y}_U$$

Proof. In the construction of $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ we have the following diagram, in which the triangles are commutative



Now, by definition of fibre category and 2-fibre product in **Cat** we have $\mathrm{Ob}((\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})_U)$ is the class of quadruples (U, x, y, α), with $x \in \mathrm{Ob}(\mathcal{X}_U)$, $y \in \mathrm{Ob}(\mathcal{Y}_U)$ and $\alpha : F(x) \longrightarrow G(y)$ an isomorphism in $\mathcal{Z}_U$.

A morphism from (U, x, y, α) to (U, x', y', α') is a pair (a, b) such that the diagram



is commutative, since U is fixed, the objects of $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})_U$ may be seen as triples (x, y, α) with the previous features. It is easy to see that this defines an isomorphism between these categories. □

In the study of 2-categories, the concept of fibre product is replaced by the most suitable concept of 2-fibre product and the first one is not considered anymore. We will give an example that shows why the 2-fibre product is more suitable than fibre product. However, the concept of fibre product also makes sense and indeed in categories over $\mathcal{C}$ there is always a fibre product, which will be constructed in the next lemma. As we shall show, we can be working on categories and build a natural category which is a fibre product, but is not a 2-fibre product. Therefore we are going to make some considerations about fibre products in categories over $\mathcal{C}$.

Lemma 1.2.3 (Existence of fibred product). *Let* $p : \mathcal{X} \longrightarrow \mathcal{C}$, $q : \mathcal{Y} \longrightarrow \mathcal{C}$ *and* $r : \mathcal{Z} \longrightarrow \mathcal{C}$ *categories over* $\mathcal{C}$ *and* $F : \mathcal{X} \longrightarrow \mathcal{Z}$, $G : \mathcal{Y} \longrightarrow \mathcal{Z}$ *1-morphisms of categories over* $\mathcal{C}$. *Then there exists* $s : \mathcal{W} \longrightarrow \mathcal{C}$, *a category over* $\mathcal{C}$, *which is a fibre product of* F *and* G.

Proof. We are going to define $\mathcal{W}$ in a similar way as the fibre product is built in the category of sets. More precisely, let

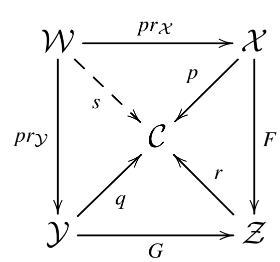
$$\mathrm{Ob}(\mathcal{W}) = \{(x, y) \mid x \in \mathrm{Ob}(\mathcal{X}), y \in \mathrm{Ob}(\mathcal{Y}), F(x) = G(y)\}$$

If $(x, y), (x', y') \in \mathrm{Ob}(\mathcal{W})$ we define

$$\mathrm{Hom}_{\mathcal{W}}((x, y), (x', y')) = \{(f, g) \mid f \in \mathrm{Hom}_{\mathcal{X}}(x, x'), g \in \mathrm{Hom}_{\mathcal{Y}}(y, y'), F(f) = G(g)\}$$

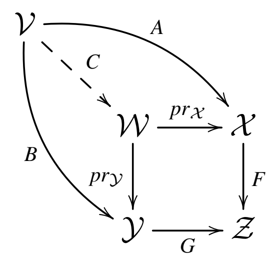
Composition in $\mathcal{W}$ is defined with the formula $(f', g') \circ_{\mathcal{W}} (f, g) := (f' \circ_{\mathcal{X}} f, g' \circ_{\mathcal{Y}} g)$ provided the composition makes sense. Then is easy to see that composition is well defined, associative and (id_x, id_y) is the identity of (x, y).

The projections $pr_{\mathcal{X}} : \mathcal{W} \longrightarrow \mathcal{X}$ and $pr_{\mathcal{Y}} : \mathcal{W} \longrightarrow \mathcal{Y}$ are defined by $pr_{\mathcal{X}}(x, y) = x$ and $pr_{\mathcal{Y}}(x, y) = y$ in the objects and by $pr_{\mathcal{X}}(f, g) = f$ and $pr_{\mathcal{Y}}(f, g) = g$ in the morphisms. Then is trivial to show that $F \circ pr_{\mathcal{X}} = G \circ pr_{\mathcal{Y}}$. We define $s : \mathcal{W} \longrightarrow \mathcal{C}$ as any of the compositions $r \circ (F \circ pr_{\mathcal{X}})$, $r \circ (G \circ pr_{\mathcal{Y}})$, $p \circ pr_{\mathcal{X}}$ or $q \circ pr_{\mathcal{Y}}$ wich are equal because F and G are 1-morphisms of categories over $\mathcal{C}$ and the commutativity of the square as is showed in the next diagram



Hence $s : \mathcal{W} \longrightarrow \mathcal{C}$ is a category over $\mathcal{C}$ and by construction is clear that $pr_{\mathcal{X}}$ and $pr_{\mathcal{Y}}$ are 1-morphisms of categories over $\mathcal{C}$.

In order to prove the universal property, let $t : \mathcal{V} \longrightarrow \mathcal{C}$ a category over $\mathcal{C}$ and $A : \mathcal{V} \longrightarrow \mathcal{X}$, $B : \mathcal{V} \longrightarrow \mathcal{Y}$ 1-morphisms of categories over $\mathcal{C}$ such that $F \circ A = G \circ B$. Consider the following diagram



If there is a 1-morphism $C : \mathcal{V} \longrightarrow \mathcal{W}$ such that $pr_{\mathcal{X}} \circ C = A$ and $pr_{\mathcal{Y}} \circ C = B$, then given $v \in \mathrm{Ob}(\mathcal{V})$, let $c(v) = (x, y)$. Hence $x \in \mathrm{Ob}(\mathcal{X})$, $y \in \mathrm{Ob}(\mathcal{Y})$ and $F(x) = G(y)$. Therefore $pr_{\mathcal{X}} \circ C(v) = x$ and $pr_{\mathcal{Y}} \circ C(v) = y$, that is to say, $A(v) = x$ and $B(v) = y$. So, if there is a such C, it must be defined for $v \in \mathrm{Ob}(V)$ by $C(v) = (A(v), B(v))$. In the same way, if $h \in \mathrm{Hom}_{\mathcal{V}}(v, v')$ it must satisfy $C(h) = (A(h), B(h))$. Lets see that C defined in this manner is a functor. For $v \in \mathrm{Ob}(\mathcal{V})$ we have $(A(v), B(v)) \in \mathrm{Ob}(\mathcal{W})$, because $F \circ A = G \circ B$. In the same way if $f \in \mathrm{Hom}_{\mathcal{V}}(v, v')$, we have $(A(f), B(f)) \in \mathrm{Hom}_{\mathcal{W}}(C(v), C(v'))$, and so C is well defined. Now,

$$\begin{aligned} C(h' \circ h) &= (A(h' \circ h), B(h' \circ h)) \\ &= (A(h') \circ A(h), B(h') \circ B(h)) \\ &= (A(h'), B(h')) \circ (A(h), B(h)) \\ &= C(h') \circ C(h) \end{aligned}$$

provided the composition $h' \circ h$ makes sense. Also

$$\begin{aligned} C(id_v) &= (A(id_v), B(id_v)) \\ &= (id_{A(v)}, id_{B(v)}) \\ &= id_{(A(v),B(v))} \end{aligned}$$

$$= id_{C(v)}$$

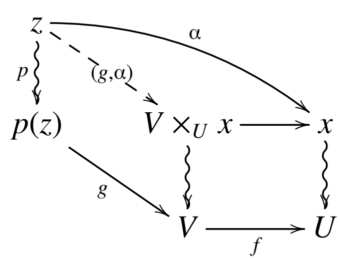
Then we have the universal property for $(\mathcal{W}, pr_{\mathcal{X}}, pr_{\mathcal{Y}})$ and so, this is a fibre product of F and G in the category of categories over $\mathcal{C}$. □

Remark. When there is possibility of confusion, we denote the fibre product as $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ and the 2-fibre product as $\mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y}$ in order to avoid ambiguity.

1.3 Fibred categories

Stacks are fibred categories with two more features that we are going to introduce later. The most important properties of fibred categories in groupoids , setoids or sets are satisfied by a more ample class of categories, then we will study those before.

Let $p : \mathcal{S} \longrightarrow \mathcal{C}$ a category over $\mathcal{C}$. Given an object $x \in \text{Ob}(\mathcal{S})$ with $p(x) = U$ and a morphism $f : V \longrightarrow U$, we want to give some sense to a "fibre product" $V \times_U x$" or pullback of x via $V \xrightarrow{f} U$. This is drawn in the diagram below. That fibre product is not really defined because x and U are not in the same category, the arrow $x \rightsquigarrow U$ means $p(x) = U$. For $z \in \text{Ob}(\mathcal{S})$ a morphism from z to $V \times_U x$ must be a pair (g, α) where $g : p(z) \longrightarrow V$ and $\alpha : z \longrightarrow x$ morphisms such that $p(\alpha) = f \circ g$.



If there exists a morphism $V \times_U x \longrightarrow x$ in $\mathcal{S}$ as before, we say this is a strongly cartesian morphism. The following definition that appears here is different from the one in Stacks

Project, but we actually proved that both are equivalent. The reason why to prefer this one is because in the subsequent results is easier to include diagrams and so is better to work with this. When we prove that some morphism is strongly cartesian, we show that the morphism satisfy the condition below.

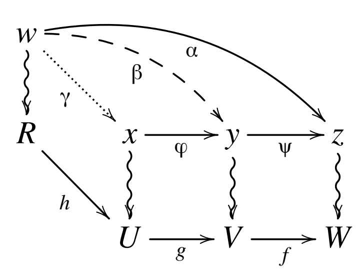
Definition 1.3.1 (Strongly cartesian morphism). *Let $p : \mathcal{S} \longrightarrow \mathcal{C}$ be a category over $\mathcal{C}$. A morphism $\varphi : y \longrightarrow x$ in $\mathcal{S}$ is strongly cartesian if and only if for all $z \in \mathrm{Ob}(\mathcal{S})$, given a pair $(g, \alpha) \in \mathrm{Mor}_{\mathcal{C}}(p(z), p(y)) \times \mathrm{Mor}_{\mathcal{S}}(z, x)$ such that $p(\alpha) = p(\varphi) \circ g$ there exists a morphism $\gamma : z \longrightarrow y$ which is unique such that $p(\gamma) = g$ and $\varphi \circ \gamma = \alpha$.*

Lemma 1.3.1. *(02XL) Let $p : \mathcal{S} \longrightarrow \mathcal{C}$ be a category over $\mathcal{C}$.*

1. *The composition of strongly cartesian morphisms is a strongly cartesian morphism.*
2. *Any isomorphism in $\mathcal{S}$ is strongly cartesian.*
3. *If φ is a morphism with $p(\varphi)$ an isomorphism in $\mathcal{C}$, then φ is an isomorphism in $\mathcal{S}$.*

Proof.

1. Let $\varphi : x \longrightarrow y$ and $\psi : y \longrightarrow z$ be strongly cartesian morphisms on $\mathcal{S}$ over g and f respectively. We want to prove that $\psi \circ \varphi$ is strongly cartesian over $f \circ g$. For this, let $w \in \mathrm{Ob}(\mathcal{S})$ over R and (α, h) as shown in the next diagram



Since ψ is strongly cartesian, there exists a unique $\beta : w \longrightarrow y$ over $g \circ h$ such that $\psi \circ \beta = \alpha$. In the same way, using now the pair (β, h) there exists a unique $\gamma : w \longrightarrow x$ over h such that $\varphi \circ \gamma = \beta$. Hence $(\psi \circ \varphi) \circ \gamma = \alpha$. If there is another morphism $\gamma' : w \longrightarrow x$ such that $(\psi \circ \varphi) \circ \gamma' = \alpha$, then $\beta' = \varphi \circ \gamma'$ is a morphism from w to y such that $\psi \circ \beta' = \alpha$ and therefore $\beta' = \beta$. But γ is the only such that $\varphi \circ \gamma = \beta$ and so $\gamma' = \gamma$.

2. Let $\varphi : x \longrightarrow y$ be an isomorphism over f, then f is also an isomorphism. Given z over w and a pair (α, g) such that $\alpha : z \longrightarrow y$ is over $f \circ g$, then $\gamma = \alpha \circ \varphi^{-1} : z \longrightarrow x$ is the only morphism over g such that $\varphi \circ \gamma = \alpha$.

3. If $\varphi : y \longrightarrow x$ is a strongly cartesian morphism and $p(\varphi)$ is an isomorphism in $\mathcal{C}$ we shall see that φ is an isomorphism. The pair $(p(\varphi)^{-1}, id_x)$ satisfy $p(id_x) = id_{p(x)} = p(\varphi) \circ p(\varphi)^{-1}$. Since φ is strongly cartesian there exists a unique morphism $\psi \in \mathrm{Mor}_{\mathcal{S}}(x, y)$ such that $p(\psi) = p(\varphi)^{-1}$ and $\varphi \circ \psi = id_x$. Likewise, the pair $(id_{p(y)}, \varphi)$ is such that $p(\varphi) = p(\varphi) \circ id_{p(y)}$ and therefore there is a unique $\eta \in \mathrm{Mor}_{\mathcal{S}}(y, y)$ such that $p(\eta) = id_{p(y)}$ and $\varphi \circ \eta = \varphi$. But id_y has these properties as also has $\psi \circ \varphi$, because $p(\psi \circ \varphi) = p(\psi) \circ p(\varphi) = p(\varphi)^{-1} \circ p(\varphi) = id_{p(y)}$ and $\varphi \circ (\psi \circ \varphi) = id_x \circ \varphi = \varphi$. Unicity says $\psi \circ \varphi = \eta = id_y$. Then ψ is the inverse morphism of φ. □

Lemma 1.3.2. *(06N4) Let $p : \mathcal{S} \longrightarrow \mathcal{C}$ be a category over $\mathcal{C}$, $x \xrightarrow{\gamma} y$ and $z \xrightarrow{\beta} y$ morphisms in $\mathcal{S}$. Suppose that:*

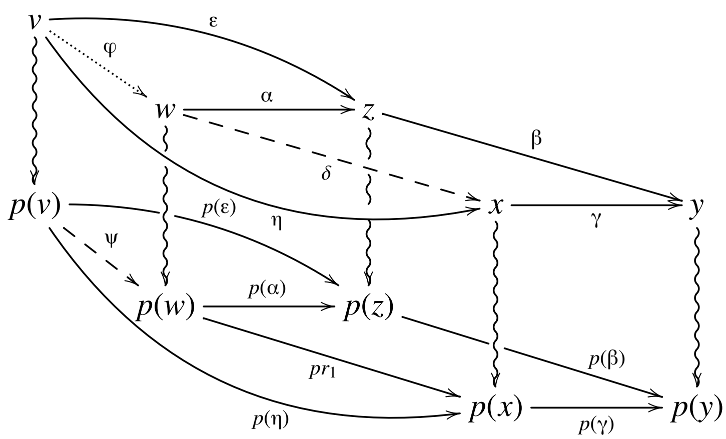
1. *$x \longrightarrow y$ is strongly cartesian.*

2. *$p(x) \times_{p(y)} p(z)$ exists in $\mathcal{C}$.*

3. *There is a strongly cartesian morphism $\alpha : w \longrightarrow z$ in $\mathcal{S}$ satisfying $p(w) = p(x) \times_{p(y)} p(z)$ and $p(\alpha) = pr_2 : p(x) \times_{p(y)} p(z) \longrightarrow p(z)$.*

Then exists $x \times_y z$ and is isomorphic to w.

Proof. Consider the following diagram

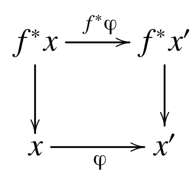


Since α is strongly cartesian, the pair $(pr_1, \beta\circ\alpha)$ is such that $p(\gamma)\circ pr_1 = p(\beta)\circ p(\alpha) = p(\beta\circ\alpha)$. Hence exists a unique morphism $w \xrightarrow{\delta} x$ with $p(\delta) = pr_1$ and $\gamma\circ\delta = \beta\circ\alpha$. If $v \in \mathrm{Ob}\mathcal{S}$ and we have morphisms $V \xrightarrow{\eta} x$ and $v \xrightarrow{\varepsilon} z$ such that $\gamma\circ\eta = \beta\circ\varepsilon$, then $p(\gamma)\circ p(\eta) = p(\beta)\circ p(\varepsilon)$ and because $p(w) = p(x) \times_{p(y)} p(z)$, there exists a unique morphism $p(v) \xrightarrow{g} p(w)$ such that $pr_1 \circ g = p(\eta)$ and $p(\alpha)\circ g = p(\varepsilon)$. Since $w \xrightarrow{\alpha} z$ is strongly cartesian, there is a morphism $v \xrightarrow{\varphi} w$ which is unique such that $p(\varphi) = g$ and $\alpha\circ\varphi = \varepsilon$. Thus $\eta = \delta\circ\varphi$ because both are morphism $v \longrightarrow x$ satisfying $p(\delta\circ\varphi) = p(\delta)\circ p(\varphi) = pr_1\circ g = p(\eta)$ and $\gamma\circ\delta\circ\varphi = \beta\circ\alpha\circ\varphi = \beta\circ\varepsilon = \gamma\circ\eta$ and since $x \xrightarrow{\gamma} y$ is strongly cartesian there exists a unique morphism with these properties. Then by unicity of g, φ is the unique morphism such that $\alpha\circ\varphi = \varepsilon$ and $\delta\circ\varphi = \eta$. Therefore $w = x \times_y z$. □

Definition 1.3.2 (Fibred category). (02XM) *A category $p : \mathcal{S} \longrightarrow \mathcal{C}$ over $\mathcal{C}$ is said to be fibred over $\mathcal{C}$, if given $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in \mathrm{Ob}(\mathcal{S}_U)$, then for every morphism $f : V \longrightarrow U$ in $\mathcal{C}$ there exists a strongly cartesian morphism in $\mathcal{S}$ which is a lift of f.*

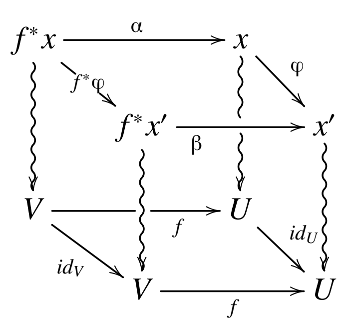
If $p : \mathcal{S} \longrightarrow \mathcal{C}$ is a fibred category. For any $f : V \longrightarrow U$ and every $x \in \mathrm{Ob}(\mathcal{S}_U)$ exists a strongly cartesian morphism $y \longrightarrow x$ over f. Moreover, given another strongly cartesian morphism $z \longrightarrow x$ over f, is clear that there is a unique isomorphism $z \longrightarrow y$ such that $z \longrightarrow y \longrightarrow x = z \longrightarrow y$. Then for all $f : V \longrightarrow U = p(x)$, we can choose a strongly cartesian morphism in $\mathcal{S}$ over f, which we denote $f^*x \longrightarrow x$ and is called a pullback of x over f.

Lemma 1.3.3 (Pullback functor). *Let $p : \mathcal{S} \longrightarrow \mathcal{C}$ be a fibred category. Then for any morphism $\varphi : x \longrightarrow x'$ in $\mathcal{S}_U$ and every $f : V \longrightarrow U$ there exists a unique morphism $f^*\varphi : f^*x \longrightarrow f^*x'$ in $\mathcal{S}_V$ such that the following diagram commutes*



Furthermore, this define a functor $f^* : \mathcal{S}_U \longrightarrow \mathcal{S}_V$.

Proof. In fact, consider the diagram



By definition of the fibre category $\mathcal{S}_U$, the morphism $\varphi : x \longrightarrow x'$ is such that $p(\varphi) = id_U$ and therefore the pair $(id_V, \varphi \circ \alpha)$ satisft $p(\varphi \circ \alpha) = p(\varphi) \circ p(\alpha) = p(\alpha) = f = f \circ id_U$. Since $f^*x' \longrightarrow x'$ is strongly cartesian, there is a unique morphism $f^*\varphi : f^*x \longrightarrow f^*x'$ such that $p(f^*\varphi) = id_V$, that is to say $f^*\varphi$ is a morphism in $\mathcal{S}_V$ and $\beta \circ f^*\varphi = \varphi \circ \alpha$ as we wanted.

Now, f^*id_x is the unique morphism $f^*x \longrightarrow f^*x$ such that $p(f^*id_x) = id_V = id_{p(f^*x)}$ and $\alpha \circ f^*id_x = id_x \circ \alpha = \alpha$. But id_{f^*x} satisfy $p(id_{f^*x}) = id_{p(f^*x)}$ and $\alpha \circ id_{f^*x} = \alpha$. Hence $f^*id_x = id_{f^*x}$. Let $x'' \in \mathrm{Ob}(\mathcal{S}_U)$ and $\gamma : f^*x'' \longrightarrow x''$ its chosen lift. If $\psi : x' \longrightarrow x''$ is a morphism in $\mathcal{S}_U$, then $f^*(\psi \circ \varphi)$ and $f^*\psi \circ f^*\varphi$ are morphisms from f^*x to f^*x'' such that $p(f^*(\psi\circ\varphi)) = id_V = id_V \circ id_V = p(f^*\psi)\circ p(f^*\varphi) = p(f^*\psi\circ f^*\varphi)$ and $\gamma\circ f^*(\psi\circ\varphi) = \psi\circ\varphi\circ\alpha = \psi \circ \beta \circ f^*\varphi = \gamma \circ f^*\psi \circ f^*\varphi$. Since γ is strongly cartesian there exists a unique morphism with such properties and therefore $f^*(\psi \circ \varphi) = f^*\psi \circ f^*\varphi$. This show that $f^* : \mathcal{S}_U \longrightarrow \mathcal{S}_V$ is indeed a functor. □

Definition 1.3.3 (Choice of pullbacks). (02XN) *Let $p : \mathcal{S} \longrightarrow \mathcal{C}$ be a fibred category.*

1. *A choice of pullbacks for $p : \mathcal{S} \longrightarrow \mathcal{C}$ is given by a choice of a strongly cartesian morphism $f^*x \longrightarrow x$ over f, for any $x \in \mathrm{Ob}(\mathcal{S}_U)$ and every morphism $f : V \longrightarrow U$ in $\mathcal{C}$.*

2. *Given a choice of pullbacks, then for any morphism $f : V \longrightarrow U$ in $\mathcal{C}$ the functor $f^* : \mathcal{S}_U \longrightarrow \mathcal{S}_V$ constructed in the previous lemma is called pullback functor associated to the choices $f^*x \longrightarrow x$.*

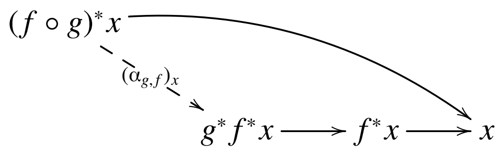
As id_x is strongly cartesian, then clearly $id_U^*x \cong x$ by a unique isomorphism. Then we can make a choice of pullbacks with $id_U^*x = x$ and $id_U^*x \longrightarrow x$ the identity morphism.

Lemma 1.3.4 (Pseudo-functoriality). (02XO) *Assume $p : \mathcal{S} \longrightarrow \mathcal{C}$ is a fibred category with a choice of pullbacks.*

1. *For every pair of composable morphisms $f : V \longrightarrow U$ and $g : W \longrightarrow V$ in $\mathcal{C}$ there exists a unique invertible natural transformation*

$$\alpha_{g,f} : (f \circ g)^* \longrightarrow g^* \circ f^*$$

of functors $\mathcal{S}_U \longrightarrow \mathcal{S}_V$ *such that for all* $x \in \mathrm{Ob}(\mathcal{S}_U)$ *the following diagram commutes*



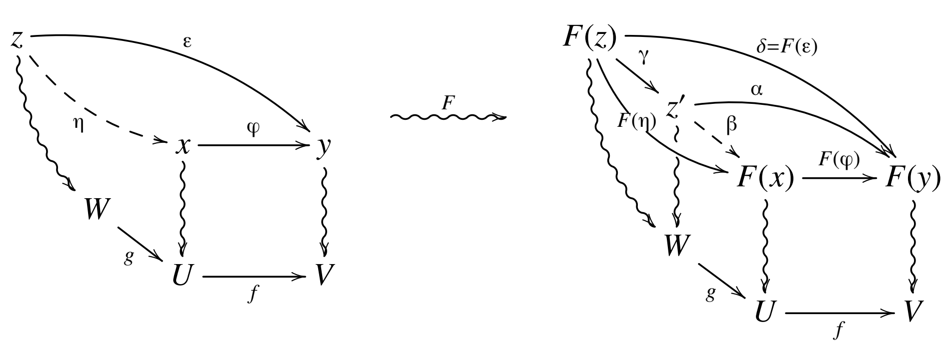
2. *For all* $U \in \mathrm{Ob}(\mathcal{C})$ *there is an invertible natural transformation* $\alpha_U : id_{\mathcal{S}_U} \longrightarrow (id_U)^*$ *of functors* $\mathcal{S}_U \longrightarrow \mathcal{S}_U$.

3. *The quadruple* $(U \longmapsto \mathcal{S}_U, f \longmapsto f^*, \alpha_{g,f}, \alpha_U)$ *defines a pseudo functor from* $\mathcal{C}^{op}$ *to the (2,1)-category of categories.*

The next lemma shows that equivalence of categories is a good notion in order to identify categories over $\mathcal{C}$.

Lemma 1.3.5. *(042G) Let* $\mathcal{C}$ *be a category and* $\mathcal{S}_1, \mathcal{S}_2$ *categories over* $\mathcal{C}$ *which are equivalentes as categories over* $\mathcal{C}$*. Then,* $\mathcal{S}_1$ *is fibred over* $\mathcal{C}$ *if and only if* $\mathcal{S}_2$ *is fibred over* $\mathcal{C}$.

Proof. Since $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent as categories over $\mathcal{C}$, there are functors $F : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ and $G : \mathcal{S}_2 \longrightarrow \mathcal{S}_1$ over $\mathcal{C}$ and invertible natural transformations $i : F \circ G \longrightarrow id_{\mathcal{S}_2}$ and $j : G \circ F \longrightarrow id_{\mathcal{S}_1}$. We shall see that F preserves strongly cartesian morphisms. If $\varphi : x \longrightarrow y$ is a strongly cartesian morphism in $\mathcal{S}_1$ over f, then $F(\varphi) : F(x) \longrightarrow F(y)$ is a strongly cartesian morphism in $\mathcal{S}_2$ over f. Given $z' \in \mathrm{Ob}(\mathcal{S}_2)$ over W, $g : W \longrightarrow U$ a morphism in $\mathcal{C}$ and $\alpha : z' \longrightarrow F(y)$ a morphism in $\mathcal{S}_2$ over $f \circ g$ we need to show that exists $\beta : z' \longrightarrow F(x)$ over g which is unique such that $F(\varphi) \circ \beta = \alpha$. Consider the following diagrams:



Since F is essentially surjective, there is $z \in \mathrm{Ob}(\mathcal{S}_{1,W})$ and an isomorphism $\gamma : F(z) \longrightarrow z'$ over id_W. Let $\delta = \alpha \circ \gamma$. Then $\delta : F(z) \longrightarrow F(y)$ is a morphism over $f \circ g$ and since F is fully faithful there exists ε over $f \circ g$ which is unique such that $F(\varepsilon) = \gamma$. But φ is strongly cartesian and so there exists an unique $\eta : z \longrightarrow x$ satisfying $\varphi \circ \eta = \varepsilon$. Hence $F(\eta)$ is a morphism from $F(z)$ to $F(x)$ such that $F(\varphi) \circ F(\eta) = F(\varepsilon) = \delta$ and again since F is fully faithful, $F(\eta)$ is unique with this property. Let $\beta = F(\eta) \circ \gamma^{-1}$. We will show that β is unique satisfying $F(\varphi) \circ \beta = \alpha$. Indeed,

$$
\begin{aligned}
F(\varphi) \circ \beta &= F(\varphi) \circ F(\eta) \circ \gamma^{-1} \\
&= F(\varepsilon) \circ \gamma^{-1} \\
&= \delta \circ \gamma^{-1} \\
&= \alpha \circ \gamma \circ \gamma^{-1} \\
&= \alpha
\end{aligned}
$$

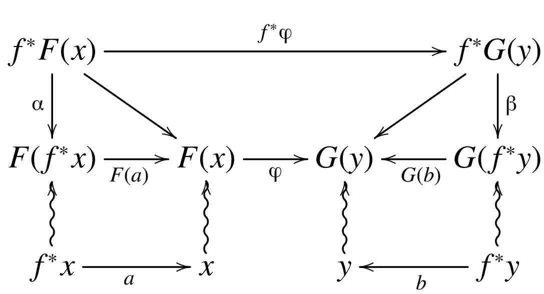
If $\beta' : z' \longrightarrow F(x)$ is another morphism with that property, then $F(\varphi) \circ \beta' \circ \gamma = \alpha \circ \gamma = \delta = F(\varepsilon)$ and therefore $\beta' \circ \gamma = F(\eta)$ and so $\beta' = F(\eta) \circ \gamma^{-1} = \beta$. Hence $F(\varphi)$ is strongly cartesian. In this way any equivalence preserves strongly cartesian morphisms and so $\mathcal{S}_1$ is fibred if and only if $\mathcal{S}_2$ is fibred. □

Definition 1.3.4 (2-category of fibred categories). *(02XP) Let $\mathcal{C}$ be a category. The 2-category of fibred categories over $\mathcal{C}$ is the sub 2-category of the 2-category of categories over $\mathcal{C}$ defined as follows:*

1. *Objects are fibred categories $(\mathcal{S}, p)$ over $\mathcal{C}$.*
2. *The 1-morphisms $(\mathcal{S}, p) \longrightarrow (\mathcal{S}', p')$ are functors $G : \mathcal{S} \longrightarrow \mathcal{S}'$ such that $p' \circ G = p$ and such that G maps strongly cartesian morphisms to strongly cartesian morphisms.*
3. *Given 1-morphisms $G, H : (\mathcal{S}, p) \longrightarrow (\mathcal{S}', p')$, the 2-morphisms $t : G \longrightarrow H$ are natural transformations such that $p'(t_x) = id_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.*

Lemma 1.3.6. *(02XQ) Let $\mathcal{C}$ be a category. The (2,1)-category of fibred categories over $\mathcal{C}$ has 2-fibre products and they are described as in categories over $\mathcal{C}$.*

Proof. For this, given $F : \mathcal{X} \longrightarrow \mathcal{Z}$ and $G : \mathcal{Y} \longrightarrow \mathcal{Z}$ 1-morphisms of fibred categories over $\mathcal{C}$ we need to prove that $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is a fibred category over $\mathcal{C}$. Let $(U, x, y, \varphi) \in \mathrm{Ob}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$ and $f : V \longrightarrow U$ a morphism in $\mathcal{C}$. Then $x \in \mathcal{X}_U$, $y \in \mathcal{Y}_U$ and $\varphi : F(x) \longrightarrow G(y)$ is an isomorphism in $\mathcal{Z}_U$. Since $\mathcal{X}$ and $\mathcal{Y}$ are fibred, there exist strongly cartesian morphism $a : f^*x \longrightarrow x$ and $b : f^*y \longrightarrow y$ over f. Because F and G are 1-morphisms of fibred categories, $F(a)$ and $F(b)$ are strongly cartesian morphisms in $\mathcal{Z}$. Since $\varphi : F(x) \longrightarrow G(y)$ is an isomorphism, then $f^*\varphi : f^*F(x) \longrightarrow f^*G(y)$ is isomorphism. Moreover, the following diagram is commutative, being α and β the only morphisms making commute the respective triangles



Therefore $\beta \circ f^*\varphi \circ \alpha^{-1}$ is an isomorphism such that the rectangle

$$\begin{array}{ccc} F(f^*x) & \longrightarrow & G(f^*y) \\ {\scriptstyle F(a)}\downarrow & & \downarrow{\scriptstyle F(b)} \\ F(x) & \xrightarrow[\varphi]{} & G(y) \end{array}$$

is commutative, that is to say, $(V, f^*x, f^*y, \beta \circ f^*\varphi \circ \alpha^{-1})$ is an object of $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ and (a, b) is a morphism from $(V, f^*x, f^*y, \beta \circ f^*\varphi \circ \alpha^{1})$ to (U, x, y, φ). It remains to show that (a, b) is strongly cartesian. Given (W, x', y', φ'), an object of $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$, $(a', b') : (W, x', y', \varphi') \longrightarrow (U, x, y, \varphi)$ and $g : W \longrightarrow V$ such that $s(a', b') = f \circ g$. Then $p(a) = f = q(b)$ and $p(a') = f \circ g = q(b')$. Since $a : f^*x \longrightarrow x$ and $b : f^*y \longrightarrow y$ are strongly cartesian morphisms, there are morphisms $c : x' \longrightarrow f^*x$ and $d : y' \longrightarrow f^*y$ which are unique satisfying $p(c) = g = q(d)$, $a' = a \circ c$ and $b' = b \circ d$. Hence, in the following diagram the side triangles, and the lower and the external rectangles are commutative.

$$\begin{array}{ccc} F(x') & \xrightarrow{\varphi'} & G(y') \\ {\scriptstyle F(c)}\downarrow & & \downarrow{\scriptstyle G(d)} \\ F(f^*x) & \xrightarrow{\beta\circ f^*\varphi\circ\alpha^{-1}} & G(f^*y) \\ {\scriptstyle F(a)}\downarrow & & \downarrow{\scriptstyle G(b)} \\ F(x) & \xrightarrow[\varphi]{} & G(y) \end{array} \qquad F(a') : F(x') \to F(x), \quad G(b') : G(y') \to G(y)$$

We want to show that the upper rectangle is commutative. We have $F(b) \circ (\beta \circ f^*\varphi \circ \alpha^{-1}) \circ F(c) = F(b) \circ (F(d) \circ \varphi')$ and commutativity follows since $F(b)$ is strongly cartesian. Then (c, d) is a morphism from (W, x', y', φ') to $(V, f^*x, f^*y, \beta \circ f^*\varphi \circ \alpha^{-1})$ and by construction $r(c, d) = p(c) = q$ and $(a, b) \circ (c, d) = (a \circ c, b \circ d) = (a', b')$. Furthermore, it is clear that is the unique with these properties and therefore (a, b) is strongly cartesian over f. □

Fibre product in fibred categories

As we have seen, the (2,1)-category of fibred categories has 2-fibre products and is the same constructed in categories over $\mathcal{C}$. The analogue statement relating fibre product is not true in general, that is to say, in general the fibre product of fibred categories over $\mathcal{C}$ constructed before does not work in fibred categories. Furthermore, in some cases there is no possible construction of a fibre product. Before we enter into this this, we will analyze what could be the trouble that does not allows us to determine the existence of fibre product in the 2-category of fibred categories over $\mathcal{C}$ and what conditions are required in order to prove that the fibre product of categories over $\mathcal{C}$ works also in the 2-category of fibred categories over $\mathcal{C}$.

Suppose $p : \mathcal{X} \longrightarrow \mathcal{C}$, $q : \mathcal{Y} \longrightarrow \mathcal{C}$ and $r : \mathcal{Z} \longrightarrow \mathcal{C}$ are fibred categories over $\mathcal{C}$ and F and G are 1-morphisms of categories over $\mathcal{C}$. We wish $s : \mathcal{W} \longrightarrow \mathcal{C}$ to be a fibred category over $\mathcal{C}$. Let $U \in \mathrm{Ob}(\mathcal{C})$, $(x, y) \in \mathrm{Ob}(\mathcal{W}_U)$ and $f : V \longrightarrow U$ be a morphisms of $\mathcal{C}$. We need to prove that there is a strongly cartesian lift of f in $\mathcal{W}$. Since $\mathcal{X}$ and $\mathcal{Y}$ are fibred categories there are $\alpha : f^*x \longrightarrow x$ and $\beta : f^*y \longrightarrow y$ strongly cartesian lifts of f in $\mathcal{X}$ and $\mathcal{Y}$ respectively. The pairs (f^*x, f^*y) and (α, β) are good candidates for being a pullback in the fibred product. We have $p(f^*x) = V = q(f^*y)$, but as we will see, in general there is not enough information to conclude $F(f^*x) = G(f^*y)$ and therefore we are not sure whether or not $(f^*x, f^*y) \in \mathrm{Ob}(\mathcal{W})$.

Example 4. Consider $\mathcal{C}$ one category with $\mathrm{Ob}(\mathcal{C}) = \{U, V\}$ and the class of morphisms is $\{id_U, id_V, V \xrightarrow{f} U\}$, where f is an abstract arrow. We can picture this category as is shown in the following diagram:

$$id_V \circlearrowright V \xrightarrow{f} U \circlearrowleft id_U$$

Then $\mathcal{C}$ is a category with fibre products and there is a natural topology in $\mathcal{C}$ where $\{f : V \longrightarrow U\}$ is a covering. In fact, the only four possible fibre products are:

$$
\begin{array}{ccc} V & \overset{id_V}{\dashrightarrow} & V \\ {\scriptstyle id_V}\downarrow & & \downarrow{\scriptstyle id_V} \\ V & \underset{id_V}{\longrightarrow} & V \end{array}
\qquad
\begin{array}{ccc} U & \overset{id_U}{\dashrightarrow} & U \\ {\scriptstyle id_U}\downarrow & & \downarrow{\scriptstyle id_U} \\ U & \underset{id_U}{\longrightarrow} & U \end{array}
\qquad
\begin{array}{ccc} V & \overset{f}{\dashrightarrow} & U \\ {\scriptstyle id_V}\downarrow & & \downarrow{\scriptstyle id_U} \\ V & \underset{f}{\longrightarrow} & U \end{array}
\qquad
\begin{array}{ccc} V & \overset{id_V}{\dashrightarrow} & V \\ {\scriptstyle id_V}\downarrow & & \downarrow{\scriptstyle f} \\ V & \underset{f}{\longrightarrow} & U \end{array}
$$

1 2 3 4

In the first two cases the fibre product is clear since all arrows are identities. In the third one, in order to verify the universal property there is only one option for an object in $\mathcal{C}$ with morphisms to U and V, namely V and the morphisms id_V and f respectively. Then id_V is a morphism such that $f \circ id_V = f$ and $id_V \circ id_V = id_V$ and is the only one with this property. In the fourth case there is also only one option V with id_V as arrow in both cases. Then $\mathcal{C}$ has fibre products.

We shall see $\mathbf{Cov}\,\tau = \{\{id_U\}, \{id_V\}, \{f\}\}$ is a topology in $\mathcal{C}$. The only isomorphisms in $\mathcal{C}$ are id_U and id_V and $\{id_U\}, \{id_V\} \in \mathbf{Cov}\,\tau$. Given $\{V \xrightarrow{f} U\} \in \mathbf{Cov}\,\tau$ and any covering of V, that is to say $\{id_V\}$, then $\{V \xrightarrow{id_V} V \xrightarrow{f} U\} = \{f\} \in \mathbf{Cov}\,\tau$. In the same way, given $\{U \xrightarrow{id_U} U\}$ and any covering of U, that is to say $\{f\}$ or id_U itself, then $\{V \xrightarrow{f} U \xrightarrow{id_U} U\} = \{f\} \in \mathbf{Cov}\,\tau$ or $\{U \xrightarrow{id_U} U \xrightarrow{id_U} U\} = \{id_U\} \in \mathbf{Cov}\,\tau$. With respect to change of base, the result follows from the analysis done before with the fibre products. Therefore $\mathcal{C}$ is a site.

Consider also $\mathcal{X} = {}_{id_{x_1}}\circlearrowleft x_1 \xrightarrow{\alpha} x_2 \circlearrowright_{id_{x_2}}$ and $\mathcal{Y} = {}_{id_{y_1}}\circlearrowleft y_1 \xrightarrow{\beta} y_2 \circlearrowright_{id_{y_2}}$ as before and $\mathcal{Z}$ the category whose graph is pictured in the diagram

$$
\begin{array}{ccc} \circlearrowleft z_1 & \overset{\varphi}{\longleftrightarrow} & z_2 \circlearrowright \\ & {\scriptstyle \gamma}\searrow \quad \swarrow{\scriptstyle \delta} & \\ & z_3 & \\ & \circlearrowleft & \end{array}
$$

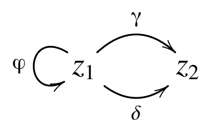
where loop arrows means identity and bidirectional arrows means isomorphism. We ask also the diagram to be commutative. We define $p : \mathcal{X} \longrightarrow \mathcal{C}$ and $q : \mathcal{Y} \longrightarrow \mathcal{C}$ by $p(x_1) = V = q(y_1)$, $p(x_2) = U = q(y_2)$ and $p(\alpha) = f = q(\beta)$ we can see $\mathcal{X}$ and $\mathcal{Y}$ are fibred over $\mathcal{C}$. We define also $r : \mathcal{Z} \longrightarrow \mathcal{C}$ by $r(z_1) = r(z_2) = V$ and $r(z_3) = U$ in the objects and $r(\gamma) = r(\delta) = f$ and $r(\varphi) = id_V$ in the morphism which are not identities. Then γ and δ are strongly cartesian lifts of f and we may choose any as pullback. Hence $\mathcal{Z}$ is a fibred category over $\mathcal{C}$.

Now, consider $F : \mathcal{X} \longrightarrow \mathcal{Z}$ and $G : \mathcal{Y} \longrightarrow \mathcal{Z}$ functors defined by $F(x_1) = z_1$, $G(y_1) = z_2$ and $F(x_2) = z_3 = G(y_2)$ in objects and $F(\alpha) = \gamma$ and $G(\beta) = \delta$ in the not identity morphisms. Then F and G are trivially 1-morphisms of categories over $\mathcal{C}$ and send strongly cartesian morphisms to strongly cartesian morphisms, so they are 1-morphisms of fibred categories over $\mathcal{C}$. Then $(x_2, y_2) \in \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$, but $(x_1, y_1) \notin (\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$. Indeed (x_2, y_2) is the only object of $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ and its identity is the only morphism. Hence $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is the trivial category and therefore the projections are also trivial. The natural functor $t : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is given by $t(x_2, y_2) = V$ and $t(id_{(x_2,y_2)}) = id_V$. Thus $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is not fibred over $\mathcal{C}$ because f has not even a lift, and much less a strongly cartesian one.

The handicap here is that in general it can not be proved that (f^*x, f^*y) is an object of $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ and therefore the strongly cartesian lifting may not exist. This does not imply that a fibre product doesn't exists, but the fibre product constructed in categories over $\mathcal{C}$, in general doesn't work in fibred categories. However we can prove that in this example none construction can be done. Suppose there exists a fibred category $t : \mathcal{W} \longrightarrow \mathcal{C}$ which is a fibre product. Hence there are 1-morphisms $pr_{\mathcal{X}} : \mathcal{W} \longrightarrow \mathcal{X}$ and $pr_{\mathcal{Y}} : \mathcal{W} \longrightarrow \mathcal{Y}$ such that $F \circ pr_{\mathcal{X}} = G \circ pr_{\mathcal{Y}}$. There are two classes of objects in $\mathcal{W}$, those $\{w_i^1\}_{i \in I}$ who are sent to x_1 via pr_X and therefore are sent to U via the functor t and $\{w_j^2\}_{j \in J}$ which are sent to x_2 via pr_X and to V via t. Then $\{w_i^1\}_{i \in I}$ must be mapped to $\{y_1\}$ and $\{w_j^2\}_{j \in J}$ must be mapped to $\{y_2\}$. However, when we apply F and G, the w_i^1's are sent to z_1 and z_2 respectively, and commutativity implies they must be equal, which is not true. Therefore there is only one family of objects, namely the w_j^2's. This means all objects of $\mathcal{W}$ are sent to V via s and the same argument as before shows that $\mathcal{W}$ can not be fibred.

Then, we need to ask $(f^*x, f^*y) \in \mathrm{Ob}(\mathcal{W})$ for every $U \in \mathrm{Ob}(\mathcal{C})$, $(x, y) \in \mathrm{Ob}(\mathcal{W})$ and $f : V \longrightarrow U$ a morphism in $\mathcal{C}$. We need to ask also $(\alpha, \beta) \in \mathrm{Hom}_{\mathcal{W}}((f^*x, f^*y), (x, y))$ which does not follows directly from $(f^*x, f^*y) \in \mathrm{Ob}(\mathcal{W})$.

Example 5. Consider $\mathcal{C}$, $\mathcal{X}$ and $\mathcal{Y}$ as in the previous example, but this time $\mathcal{Z}$ is the category whose graph is



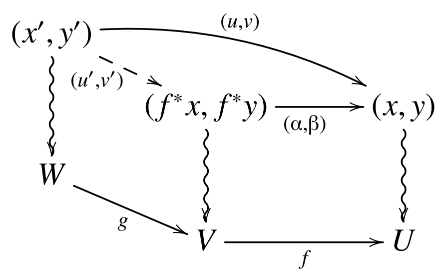
Where φ is a non trivial automorphism such that $\gamma = \delta \circ \varphi$. Define $r(z_1) = V$, $r(z_2) = V$ and $r(\gamma) = r(\delta) = f$. Then $r : \mathcal{Z} \longrightarrow \mathcal{C}$ is a fibred category over $\mathcal{C}$. Define $F : \mathcal{X} \longrightarrow \mathcal{Z}$ and $G : \mathcal{Y} \longrightarrow \mathcal{Z}$ by $F(x_i) = z_i = G(y_i)$ for $i = 1, 2$, but $F(\alpha) = \gamma$ and $G(\beta) = \delta$. Then $(x_1, y_1) \in \mathcal{W}$, but (α, β) is not a morphism of $\mathcal{W}$.

However, if we are in the case $(f^*x, f^*y) \in \mathrm{Ob}(\mathcal{W})$ for every $U \in \mathrm{Ob}(\mathcal{C})$, $(x, y) \in \mathrm{Ob}(\mathcal{W})$ and $f : V \longrightarrow U$ a morphism in $\mathcal{C}$ and (α, β) is a morphism in $\mathcal{W}$ from (f^*x, f^*y) to (x, y) then we can prove that $\mathcal{W}$ is also a fibre product in the 2-category of fibred categories over $\mathcal{C}$. Before that we present the following definition.

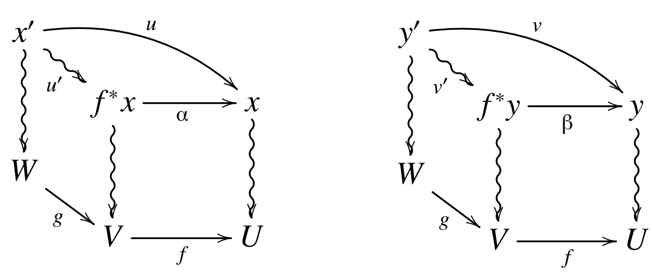
Definition 1.3.5 (Componentwise pullbacks). *Given 1-morphisms of fibred categories $F : \mathcal{X} \longrightarrow \mathcal{Z}$ and $G : \mathcal{Y} \longrightarrow \mathcal{Z}$ over $\mathcal{C}$, we say that $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$, considered as category over $\mathcal{C}$, has componentwise pullbacks if given $(x, y) \in \mathrm{Ob}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$ over V and $f : U \longrightarrow V$ a morphism in $\mathcal{C}$ we have (f^*x, f^*y) and $(f^*x \longrightarrow x, f^*y \longrightarrow y)$ are respectively an object and a morphism of $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$.*

Theorem 1. *If $F : \mathcal{X} \longrightarrow \mathcal{Z}$ and $G : \mathcal{Y} \longrightarrow \mathcal{Z}$ are 1-morphisms of fibred categories over $\mathcal{C}$ such that the $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ has componentwise pullbacks, then $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is a fibred category over $\mathcal{C}$.*

Proof. We shall prove that (α, β) is the strongly cartesian lift of f in $\mathcal{W}$, so we can write $f^*(x, y) = (f^*x, f^*y)$. For construction we have $s(f^*x, f^*y) = p(f^*x) = V$ and $s(\alpha, \beta) = p(\alpha) = f$, so $f^*(x, y) \longrightarrow (x, y)$ is a morphism in $\mathcal{W}$ over f. Let (x', y') be an object of $\mathcal{W}$ over W, $g : W \longrightarrow V$ a morphism of $\mathcal{C}$ and $(u, v) : (x', y') \longrightarrow (x, y)$ a morphism in $\mathcal{W}$ over $f \circ g$. Let us show that there exists a morphism $(u', v') : (x', y') \longrightarrow (x, y)$ in $\mathcal{W}$ over g such that in the following diagram the upper triangle is commutative



Then $f \circ g = s(u, v)$ which is equal to both $p(u)$ and $q(v)$. The diagram can be splitted in two diagrams one of them in $\mathcal{X}$ and the other one in $\mathcal{Y}$ respectively, as follows:



Since $f^*x \longrightarrow x$ and $f^*y \longrightarrow y$ are strongly cartesian, there are morphisms $u' : x' \longrightarrow f^*x$ and $v' : y' \longrightarrow f^*y$ over g which are unique making the respective triangles commutative. Now, $\alpha \circ u' = u$ and $\beta \circ v' = v$, then $F(\alpha) \circ F(u') = F(u)$ and $G(\beta) \circ G(v') = G(v)$. Since $F(u) = G(v)$ and $F(\alpha) = G(\beta)$, then $F(\alpha) \circ F(u') = F(\alpha) \circ G(v')$ in $\mathcal{Z}$. But F is a 1-morphism of fibred categories and therefore $F(\alpha)$ is a strongly cartesian morphism and so, $F(u') = G(v')$. Hence (u', v') is a morphism in $\mathcal{W}$ and by construction is clear that this is the only one making the triangle commutative. Then $s : \mathcal{W} \longrightarrow \mathcal{C}$ is fibred.

We must prove that $pr_{\mathcal{X}} : \mathcal{W} \longrightarrow \mathcal{X}$ and $pr_{\mathcal{Y}} : \mathcal{W} \longrightarrow \mathcal{Y}$ are 1-morphisms of fibred categories over $\mathcal{C}$, that is to say, they preserve strongly cartesian morphism. Let $(\gamma, \delta) : (x', y') \longrightarrow (x, y)$ be a strongly cartesian morphism in $\mathcal{W}$ over f and $(\alpha, \beta) : (f^*x, f^*y) \longrightarrow (x, y)$ the strongly cartesian lift built before. Since $\mathcal{W}$ is fibred, there is an isomorphism $(\varphi, \psi) : (x', y') \longrightarrow (f^*x, f^*y)$ unique such that $(\alpha, \beta) \circ (\varphi, \psi) = (\gamma, \delta)$. Then $\alpha \circ \varphi = \gamma$ and $\beta \circ \psi = \delta$ and therefore γ and δ are each one composition of two strongly cartesian morphism in $\mathcal{X}$ and $\mathcal{Y}$ respectively. Hence γ and δ are also strongly cartesian morphisms. The conclusion follows from the fact $pr_{\mathcal{X}}(\gamma, \delta) = \gamma$ and $pr_{\mathcal{Y}}(\gamma, \delta) = \delta$. A similar argument shows that if $\mathcal{V}$ is a fibred category and $A : \mathcal{V} \longrightarrow \mathcal{X}$ and $B : \mathcal{V} \longrightarrow \mathcal{Y}$ are 1-morphisms of fibred categories over $\mathcal{C}$, the 1-morphism $\gamma : \mathcal{V} \longrightarrow \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ of categories over $\mathcal{C}$ given by the universal property is also a 1-morphism of fibred categories over $\mathcal{C}$. □

Relation between $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ and $\mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y}$

There is a close relation between the fibre product and the 2-fibre product, more precisely we have the following statement

Proposition 1.3.1. *Let $p : \mathcal{X} \longrightarrow \mathcal{C}$, $q : \mathcal{Y} \longrightarrow \mathcal{C}$ and $r : \mathcal{Z} \longrightarrow \mathcal{C}$ be categories over $\mathcal{C}$ and $F : \mathcal{X} \longrightarrow \mathcal{Z}$ and $G : \mathcal{Y} \longrightarrow \mathcal{Z}$ 1-morphisms of categories over $\mathcal{C}$. There is a canonical 1-morphism*

$$H : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \longrightarrow \mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y}$$

of categories over $\mathcal{C}$. Furthermore, this is a fully faithful functor. If $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are fibred categories, F, G are 1-morphisms of fibred categories and $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ has componentwise pullbacks, then H is a 1-morphism of fibred categories.

Proof. The fibre product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ and the 2-fibre product $\mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y}$ are the categories with the following data:

1. $\mathrm{Ob}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}) = \{(x, y) \mid x \in \mathrm{Ob}(\mathcal{X}), y \in \mathrm{Ob}(\mathcal{Y}), F(x) = G(y)\}$

   $\mathrm{Hom}_{\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y}}((x, y), (x', y')) = \{(f, g) \mid f \in \mathrm{Hom}_{\mathcal{X}}(x, x'), g \in \mathrm{Hom}_{\mathcal{Y}}(y, y'), F(f) = G(g)\}$

2. $\mathrm{Ob}(\mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y}) = \{(U, x, y, \alpha) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in \mathrm{Ob}(\mathcal{X}_U), y \in \mathrm{Ob}(\mathcal{Y}_U), \alpha : F(x) \longleftrightarrow G(y)\}$

   $$\mathrm{Hom}_{\mathcal{X}\times^2_{\mathcal{Z}}\mathcal{Y}}((U, x, y, \alpha), (U', x', y', \alpha')) = \{(f, g) \mid f \in \mathrm{Hom}_{\mathcal{X}}(x, x'), g \in \mathrm{Hom}_{\mathcal{Y}}(y, y'), \alpha' \circ F(f) = G(g) \circ \alpha\}$$

Define $H : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \longrightarrow \mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y}$ as follows: given $(x, y) \in \mathrm{Ob}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$ let $H(x, y) = (U, x, y, id_{F(x)})$, where $U = p(x) = q(y)$ and since $F(x) = G(y)$, then $id_{F(x)}$ is a morphism from $F(x)$ to $G(y)$ which is actually an isomorphism in $\mathcal{Z}_U$ and therefore $(U, x, y, id_{F(x)}) \in \mathrm{Ob}(\mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y})$. If $(f, g) \in \mathrm{Hom}_{\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y}}((x, y), (x', y'))$, then $F(f) = G(g)$ and so, the diagram

$$\begin{array}{ccc} F(x) & \xrightarrow{id_{F(x)}} & G(y) = F(x) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(g)=F(f)} \\ F(x') & \xrightarrow[id_{F(x')}]{} & G(y') = F(x') \end{array}$$

is trivially commutative. Hence $(f, g) \in \mathrm{Hom}_{\mathcal{X}\times^2_{\mathcal{Z}}\mathcal{Y}}(H(x, y), H(x', y'))$. Moreover, the composition rule and identities are the same, so H is a functor. This functor is fully faithful.

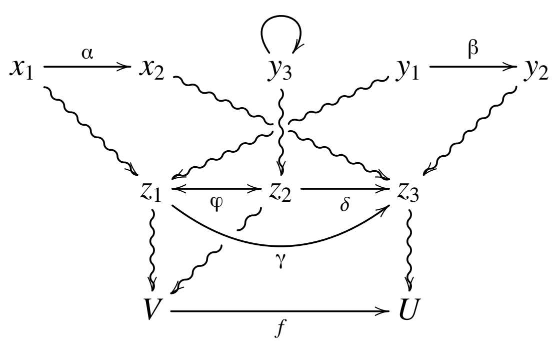
Now, if $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ has componentwise pullbacks, we have proved before that $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is a fibred category over $\mathcal{C}$. Let $(f, g) : (x', y') \longrightarrow (x, y)$ be a strongly cartesian morphism in $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ and $h = p(f) = q(g)$. Consider $h^*(x, y) = (h^*x, h^*y)$ and the pullback $(h^*x \longrightarrow x, h^*y \longrightarrow y)$ of (x, y) over h. Hence there is an isomorphism $(x', y') \longrightarrow (h^*x, h^*y)$ such that the following triangle commutes

$$\begin{array}{ccc} (x', y') & \longleftrightarrow & (h^*x, h^*y) \\ & \searrow \quad \swarrow & \\ & (x, y) & \end{array}$$

Since isomorphisms are strongly cartesian morphisms and H is a functor, is enough to show that $H(h^*x \longrightarrow x, h^*y \longrightarrow y) := (h^*x \longrightarrow x, h^*y \longrightarrow y)$ is a strongly cartesian morphism in $\mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y}$. But it follows from this pair is exactly the pullback of (x, y) over h. Hence H is a 1-morphism of fibred categories over $\mathcal{C}$. □

As we will see in the next examples, in general this functor is not essentially surjective and therefore H is not an equivalence of categories, even if we are in the case $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are fibred categories and $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ has componentwise pullbacks.

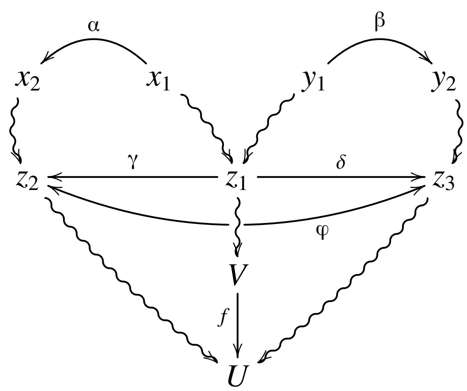
Example 6. Let $id_V \circlearrowleft V \xrightarrow{f} U \circlearrowright id_U$ be the site with the same topology as in previous examples. Consider the following graph, where $\text{Ob}(\mathcal{X}) = \{x_1, x_2\}$, $\text{Ob}(\mathcal{Y}) = \{y_1, y_2, y_3\}$ and $\text{Ob}(\mathcal{Z}) = \{z_1, z_2, z_3\}$. The curly arrows means 1-morphisms of fibred categories.



Then $\mathcal{Y}$ has an additional object z_3 with only the identity morphism, and p, r and F are defined as in the first example, so $p : \mathcal{X} \longrightarrow \mathcal{C}$ and $r : \mathcal{Z} \longrightarrow \mathcal{C}$ are fibred categories over $\mathcal{C}$ and $F : \mathcal{X} \longrightarrow \mathcal{Z}$ is a 1-morphisms of fibred categories over $\mathcal{C}$. Defining $q(y_1) = q(y_3) = V$, $q(y_2) = U$ and $q(\beta) = f$ we have $q : \mathcal{Y} \longrightarrow \mathcal{C}$ is a fibred category in which β is a strongly cartesian lift of f. The 1-morphism $G : \mathcal{Y} \longrightarrow \mathcal{Z}$ is defined by $G(y_1) = z_1$, $G(y_2) = z_3$, $G(y_3) =$ and $G(\beta) = \gamma$. This time we have (x_1, y_1) and (α, β) are in $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ and therefore $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is a fibred category over $\mathcal{C}$. Moreover, $\text{Ob}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}) = \{(x_1, y_1), (x_2, y_2)\}$. On the other hand, $\text{Ob}(\mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y}) = \{(V, x_1, y_1, id_{z_1}), (U, x_2, y_2, id_{z_3}), (V, x_1, y_3, \varphi)\}$, but since there are no morphisms between y_1 and y_3 nor y_2 and y_3, then (V, x_1, y_3, φ) is neither isomorphic to $H(x_1, y_1)$ nor to $H(x_2, y_2)$. Hence $H : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \longrightarrow \mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y}$ is not essentially surjective.

In the previous example the problem is that the fibre category $\mathcal{Z}_V$ was disconnected. However, even if every fibre category is a connected groupoid we can not conclude H is essentially surjective.

Example 7. Consider the following graph,[1] where $\mathrm{Ob}(\mathcal{X}) = \{x_1, x_2\}$, $\mathrm{Ob}(\mathcal{Y}) = \{y_1, y_2\}$, $\mathrm{Ob}(\mathcal{Z}) = \{z_1, z_2, z_3\}$ and $\mathcal{C}$ is the same as before. The curly arrows means 1-morphisms of fibred categories.



Then $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are fibred categories over $\mathcal{C}$, whose fibre categories are always connected groupoids. We have $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} = \{(x_1, y_1)\}$ and $\mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y} = \{(V, x_1, y_1, id_{z_1}), (U, x_2, y_2, \varphi)\}$. The pair (α, β) is a morphism from (V, x_1, y_1, id_{z_1}) to (U, x_2, y_2, φ) because $F(\alpha) = \gamma$, $G(\beta) = \delta$ and therefore the diagram

$$\begin{array}{ccc} F(x_1) & \xrightarrow{id_{Z_1}} & G(y_1) \\ {\scriptstyle F(\alpha)}\downarrow & & \downarrow{\scriptstyle G(\beta)} \\ F(x_2) & \xrightarrow[\varphi]{} & G(y_2) \end{array}$$

[1]The shape of this diagram was inspired in Amelia, my newborn daughter.

is commutative. However we don't have $(V, x_1, y_1, id_{z_1}) \cong (U, x_2, y_2, \varphi)$ because there is not a morphism in the other direction. Note that in this example $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is still fibred, since there is no object over U it is not necessary that exists a strongly cartesian morphism over f.

1.4 Fibred category associated to a functor

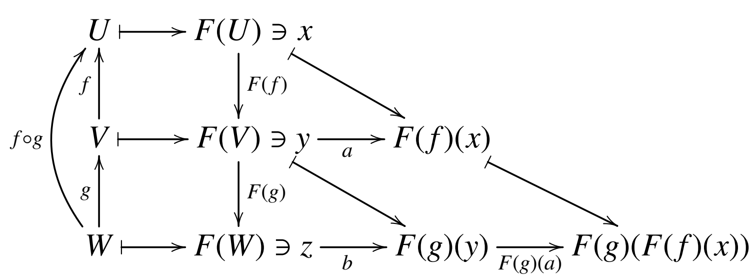
Suppose that $\mathcal{C}$ is a category and $F : \mathcal{C}^{op} \longrightarrow \mathbf{Cat}$ is a functor from $\mathcal{C}$ to the 2-category of categories. We will construct a fibred category $\mathcal{S}_F$ over $\mathcal{C}$ as follows:

$$\mathrm{Ob}(\mathcal{S}_F) = \{(U, x) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in \mathrm{Ob}(F(U))\}$$

For a pair (U, x) and (V, y) in $\mathrm{Ob}(\mathcal{S}_F)$ we define

$$\mathrm{Hom}_{\mathcal{S}_F}((V, y), (U, x)) = \{(f, a) \mid f \in \mathrm{Hom}_{\mathcal{C}}(V, U), a \in \mathrm{Hom}_{F(V)}(y, F(f)(x))\}$$

In order to define the composition of morphisms we we use $F(g) \circ F(f) = F(f \circ g)$ for a pair of composable morphisms in $\mathcal{C}$. More precisely, composition of $(V, y) \stackrel{(f,a)}{\longrightarrow} (U, x)$ and $(W, z) \stackrel{(g,b)}{\longrightarrow} (V, y)$ is defined by $(f \circ g, F(g)(a) \circ b)$, which is showed in the next diagram



It is easy to see that $(f \circ g, F(g)(a) \circ b)$ is in $\mathrm{Hom}_{\mathcal{S}_F}((W, z), (U, x)$, since $f \circ g : W \longrightarrow U$ and $F(g)(a) \circ b : z \longrightarrow F(f \circ g)(x)$. If (h, c) is a morphism such that the composition $(h, c) \circ (g, b)$ is well defined, then

$$(h, c) \circ ((g, b) \circ (f, a)) = (h, c) \circ (g \circ f, F(f)(b) \circ a)$$

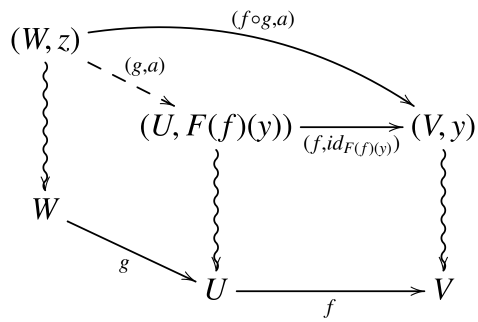
$$= (h \circ (g \circ f), F(g \circ f)(c) \circ F(f)(b) \circ a)$$

and on the other hand

$$\begin{aligned}((h, c) \circ (g, b)) \circ (f, a) &= (h \circ g, F(g)(c) \circ b) \circ (f, a)\\ &= ((h \circ g) \circ f, F(f)(F(g)(c) \circ b) \circ a)\\ &= (h \circ (g \circ f), (F(f) \circ F(g))(c) \circ F(f)(b) \circ a)\end{aligned}$$

and so the composition is associative. Moreover, given $(V, y) \in \mathrm{Ob}(\mathcal{S}_F)$, the pair (id_V, id_y) is in $\mathrm{Hom}_{\mathcal{S}_F}((V, y), (V, y))$, since $id_V : V \longrightarrow V$ e $id_y : y \longrightarrow (id_V)^* y$, inasmuch as $F(id_V) = id_{F(V)}$. Thus we have $(id_V, id_y) \circ (g, b) = (id_V \circ g, F(g)(id_y) \circ b) = (g, b)$, because $F(g)$ is a functor and so $F(g)(id_y) = id_{F(g)(y)}$. Also $(f, a) \circ (id_V, id_y) = (f \circ id_V, F(id_V)(a) \circ id_x) = (f, a)$, insomuch as $F(id_V)(a) = id_{F(V)}(a) = a$. Then $\mathcal{S}_F$ is a category.

The functor $p_F : \mathcal{S}_F \longrightarrow \mathcal{C}$ is defined by $(U, x) \longmapsto U$ and for $(V, y) \stackrel{(f,a)}{\longrightarrow} (U, x)$, $p_F(f, a) = f$. The fibre category over $U \in \mathrm{Ob}(\mathcal{C})$ is clearly $\mathcal{S}_{F,U} = F(U)$. We shall show that the pair $(\mathcal{S}_F, p_F)$ is a fibred category over $\mathcal{C}$. If $(V, y) \in \mathrm{Ob}(\mathcal{S}_F)$, then $p_F(V, y) = V$. Let $f : U \longrightarrow V$ a morphism in $\mathcal{C}$. We will build an object $f^*(V, y) \in \mathcal{S}_F$ over U and a morphism $F(f)(V, y) \longrightarrow (V, y)$ strongly cartesian over f. Since $F(f) : F(V) \longrightarrow F(U)$ is a functor and $y \in \mathrm{Ob}(F(V))$, then $F(f)(y) \in F(U)$ and so $(U, F(f)(y) \in \mathrm{Ob}(\mathcal{S}_F)$. Thus define $f^*(V, y) := (U, F(f)y)$. The pair $(f, id_{F(f)(y)})$ is clearly a morphism from $(U, F(f)(y))$ to (V, y). Let $(W, z) \in \mathrm{Ob}(\mathcal{S}_F)$, $g : W \longrightarrow U$ morphism in $\mathcal{C}$ and $(f \circ g, a) : (W, z) \longrightarrow (V, y)$ a morphism in $\mathcal{S}_F$ over $f \circ g$ as showed in the next diagram



Hence $a : z \longrightarrow F(f \circ g)(y)$ or equivalently $a : z \longrightarrow (F(f) \circ F(g))(y)$ and therefore (g,a) is a morphism in $\mathcal{S}_F$ from (W,z) to $(U, F(f)(y))$ and satisfy $(f, id_{F(f)(y)}) \circ (g,a) = (f \circ g, F(g)(id_{F(f)(y)}) \circ a) = (f \circ g, id_{(F(g) \circ F(f))(y)} \circ a) = (f \circ g, a)$. If there exists (g,b) from (W,z) to $(U, F(f)(y))$ such that $(f, id_{F(f)(y)}) \circ (g,b) = (f \circ g, a)$, then $F(g)(id_{F(f)(y)}) \circ b = a$ and so $b = a$. Consequently $(f, id_{F(f)(y)})$ is strongly cartesian and therefore $p_F : \mathcal{S}_F \longrightarrow \mathcal{C}$ is a fibred category.

Summarizing, if $(V,y) \in \mathcal{S}_F$ and $f : U \longrightarrow V$ is a morphisms in $\mathcal{C}$, then we can define $f^*(V,y) = (U, F(f)(y))$ and $(f, id_{F(f)(y)})$ is the strongly cartesian lift of f. Let $g : W \longrightarrow U$ be another morphism, then $g^*(U, F(f)(y)) = (W, F(g)(F(f)(y)))$ and $(g, id_{F(g)(F(f)(y))})$ is the strongly cartesian lift of g. Then we have

$$\begin{aligned} g^*(f^*(V,y)) &= g^*(U, F(f)(y)) \\ &= (W, F(g)(F(f)(y))) \\ &= (W, (F(g) \circ F(f))(y)) \\ &= (W, F(f \circ g)(y)) \\ &= (f \circ g)^*(V,y) \end{aligned}$$

and so $(f \circ g)^* = g^* \circ f^*$ in the objects. Furthermore

$$(f, id_{F(f)(y)}) \circ (g, id_{F(g)(F(f)(y))}) = (f \circ g, F(g)(id_{F(f)(y)}) \circ id_{F(g)(F(f)(y))})$$

$$= (f \circ g, id_{F(g)(F(f)(y))} \circ id_{F(g)(F(f)(y))})$$
$$= (f \circ g, id_{F(g)(F(f)(y))})$$
$$= (f \circ g, id_{F(f \circ g)(y)})$$

Therefore strongly cartesian liftings are compatible with composition in $\mathcal{C}$ and we may conclude $(f \circ g)^* = g^* \circ f^*$.

Functoriality

In this subsection we are going to see some functorial properties about the fibred category associated to a functor. The most important in this research is the related to the compatibility with fibre products when those exist. At the end we put together these results in a useful theorem.

If $\alpha : F \longrightarrow G$ is a natural transformation of functors, there is a canonical 1-morphism $\widetilde{\alpha}$ of fibred categories over $\mathcal{C}$ from $\mathcal{S}_F$ to $\mathcal{S}_G$. Indeed, given $U \in \mathcal{C}$, there is a functor[2] $\alpha_U : F(U) \longrightarrow G(U)$ and for $U \xrightarrow{f} V$ a morphism in $\mathcal{C}$ the rectangle

$$\begin{array}{ccc} F(U) & \xrightarrow{\alpha_U} & G(U) \\ {\scriptstyle F(f)}\uparrow & & \uparrow{\scriptstyle G(f)} \\ F(V) & \xrightarrow[\alpha_V]{} & G(V) \end{array}$$

is commutative. If $(U, x) \in \text{Ob}(\mathcal{S}_F)$, then $x \in \text{Ob}(F(U))$ and so $\alpha_U(x) \in G(U)$. Hence $\widetilde{\alpha}(U, x) := (U, \alpha_U(x)) \in \text{Ob}(\mathcal{S}_G)$. If $(f, a) : (U, x) \longrightarrow (V, y)$ is a morphism in $\mathcal{S}_F$, then $f : U \longrightarrow V$ and $a : x \longrightarrow F(f)(y)$. Therefore $\alpha_U(a) : \alpha_U(x) \longrightarrow \alpha_U(F(f)(y))$ and $\alpha_U(F(f)(y)) = G(f)(\alpha_U(y))$. Hence $\widetilde{\alpha}(f, a) := (f, \alpha_U(a))$ is a morphism in $\mathcal{S}_G$ from $(U, \alpha_U(x))$ to $(V, \alpha_U(y))$.

[2]In **Cat** the morphisms are functors

Let us show compatibility with composition and identity. Given $(f, a) : (U, x) \longrightarrow (V, y)$ and $(g, b) : (V, y) \longrightarrow (W, z)$ we have

$$\begin{aligned}
\widetilde{\alpha}((g, b) \circ (f, a)) &= \widetilde{\alpha}(g \circ f, F(f)(b) \circ a) \\
&= (g \circ f, \alpha_U(F(f)(b) \circ a)) \\
&= (g \circ f, \alpha_U(F(f)(b)) \circ \alpha_U(a)) \\
&= (g \circ f, G(f)(\alpha_V(b)) \circ \alpha_U(a)) \\
&= (g, \alpha_V(b)) \circ (f, \alpha_U(a)) \\
&= \widetilde{\alpha}(g, b) \circ \widetilde{\alpha}(f, a)
\end{aligned}$$

and so we have compatibility with composition. On the other hand if $(V, y) \in \mathcal{S}_F$, then

$$\begin{aligned}
\widetilde{\alpha}(id_{(V,y)}) &= \widetilde{\alpha}(id_V, id_y) \\
&= (id_V, \alpha_V(id_y)) \\
&= (id_V, id_{\alpha_V(y)}) \\
&= id_{(V,\alpha_V(y))} \\
&= id_{\widetilde{\alpha}(V,y)}
\end{aligned}$$

and so we have compatibility with identity. Then $\widetilde{\alpha} : \mathcal{S}_F \longrightarrow \mathcal{S}_G$ is a functor. Moreover, $p_G \circ \widetilde{\alpha} = p_F$ and therefore is a 1-morphism of categories over $\mathcal{C}$. In order to see that this is a 1-morphism of fibred categories over $\mathcal{C}$ we need to prove that this functor preserves strongly cartesian morphisms. Let $(f, a) : (U, x) \longrightarrow (V, y)$ be a strongly cartesian morphism in $\mathcal{S}_F$, we shall see $\widetilde{\alpha}(f, a) : \widetilde{\alpha}(U, x) \longrightarrow \widetilde{\alpha}(V, y)$ is a strongly cartesian morphism in $\mathcal{S}_G$. It is enough to see that for the strongly cartesian lift $(f, id_{F(f)(y)}) : (U, F(f)(y)) \longrightarrow (V, y)$ of f with target (V, y) in $\mathcal{S}_F$, the morphism $(f, id_{F(f)(y)})$ is strongly cartesian in $\mathcal{S}_G$. This is immediate since

$$\widetilde{\alpha}(f, id_{F(f)(y)}) = (f, \alpha_U(id_{F(f)(y)}))$$

$$= (f, id_{\alpha_U(F(f)(y))})$$
$$= (f, id_{G(f)(\alpha_V(y))})$$

which is the strongly cartesian lift of f with target $(V, \alpha_V(y))$ in $\mathcal{S}_G$.

If $\eta : P \longrightarrow R$ and $\theta : R \longrightarrow T$ are natural transformations of functors $R, S, T : \mathcal{C}^{op} \longrightarrow \mathbf{Cat}$, we claim $\widetilde{\theta \circ \eta} = \widetilde{\theta} \circ \widetilde{\eta}$. In fact, given $(U, x) \in \mathrm{Ob}(\mathcal{S}_P)$ we have $\widetilde{\eta}(U, x) = (U, \eta_U(x)) \in \mathrm{Ob}(\mathcal{S}_R)$ and then

$$\begin{aligned}(\widetilde{\theta} \circ \widetilde{\eta})(U, x) &= \widetilde{\theta}(\widetilde{\eta}(U, x)) \\ &= \widetilde{\theta}(U, \eta_U(x))) \\ &= (U, \theta_U(\eta_U(x))) \\ &= (U, (\theta_U \circ \eta_U)(x)) \\ &= (U, (\theta \circ \eta)_U(x)) \\ &= \widetilde{\theta \circ \eta}(U, x)\end{aligned}$$

In the same way if (f, a) is a morphisms from (U, x) to (V, y), then $\widetilde{\eta}(f, a) = (f, \eta_U(a))$ and we have $\widetilde{\theta}(f, \eta_U(a)) = (f, (\theta \circ \eta)_U(a)) = \widetilde{\theta \circ \eta}(f, a)$.

Fibred categories of the form $\mathcal{S}_F$ forms a sub 2-category of the 2-category of fibred categories. We want to see if they are a complete sub 2-category. Given $\mathcal{X} = \mathcal{S}_F$, $\mathcal{Y} = \mathcal{S}_G$ and $A = \mathcal{X} \longrightarrow \mathcal{Y}$ a 1-morphism of fibred categories over $\mathcal{C}$, we have this question: there is a unique natural transformation $\alpha : F \longrightarrow G$ such that $\widetilde{\alpha} = A$? Unicity is true, since if $\alpha, \beta : F \longrightarrow G$ are natural transformations such that $\widetilde{\alpha} = \widetilde{\beta}$, then for any $U \in \mathrm{Ob}(\mathcal{C})$ and every $x \in F(U)$ we have $(U, x) \in \mathrm{Ob}(\mathcal{S}_F)$ and so $\widetilde{\alpha}(U, x) = \widetilde{\beta}(U, x)$ which means $(U, \alpha_U(x)) = (U, \beta_U(x))$ and therefore $\alpha_U(x) = \beta_U(x)$. Hence $\alpha = \beta$. In other words, the functorial assignation $F \longmapsto \mathcal{S}_F$ and $(\alpha : F \longrightarrow G) \longmapsto (\widetilde{\alpha} : \mathcal{S}_F \longrightarrow \mathcal{S}_G)$ is faithful. However, we will see this is fully faithful if and only if A commutes with choice of pullbacks.

As before, given $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in \mathrm{Ob}(F(U))$, $(U,x) \in \mathrm{Ob}(\mathcal{S}_F)$ and so $A(U,x) \in \mathcal{S}_G$. Since $q(A(U,x)) = p(U,x) = U$ we have $A(U,x) = (U,y)$ for some $y \in \mathrm{Ob}(G(U))$. If we have $A(U,x) = \widetilde{\alpha}(U,x) := (U, \alpha_U(x))$ then we must to define $\alpha_U(x) = y$. If $a : x_1 \longrightarrow x_2$ is a morphism in $F(U)$, then (id_U, a) is a morphism in $\mathcal{S}_F$ from (U,x_1) to (U,x_2). Therefore $A(id_U, a)$ is a morphism in $\mathcal{S}_G$ from $A(U,x_1) = (U, \alpha_U(x_1))$ to $A(U,x_2) = (U, \alpha_U(x_2))$. Since $q(A(id_U,a)) = p(id_U,a) = id_U$ we have $A(id_U,a) = (id_U,b)$. Again, if $A(id_U,a) = \widetilde{\alpha}(id_U,a) := (id_U, \alpha_U(a))$, then we must define $\alpha_U(a) = b$. In particular, since $(id_U, id_x) = id_{(U,x)}$ in $\mathcal{S}_F$ and A is a functor, then

$$\begin{aligned}
(id_U, \alpha_U(id_x)) &= A(id_U, id_x) \\
&= A(id_{(U,x)}) \\
&= id_{A(U,x)} \\
&= id_{(U,alpha_U(x))} \\
&= (id_U, id_{\alpha_U(x)})
\end{aligned}$$

and so $\alpha_U(id_x) = id_{\alpha_u(x)}$. If $a_1 : x_1 \longrightarrow x_2$ and $a_2 : x_2 \longrightarrow x_3$ are morphisms in $F(U)$ and then

$$\begin{aligned}
(id_U, \alpha_U(a_2 \circ a_1)) &= A(id_U, a_2 \circ a_1) \\
&= A((id_U, a_2) \circ (id_U, a_1)) \\
&= A(id_U, a_2) \circ A(id_U, a_1) \\
&= (id_U, \alpha_U(a_2)) \circ (id_U, \alpha_U(a_1)) \\
&= (id_U, \alpha_U(a_2) \circ \alpha_U(a_1))
\end{aligned}$$

and therefore $\alpha_U : F(U) \longrightarrow G(U)$ is a functor. Now, these functors define a natural transformation $\{a_U\}_{U \in \mathrm{Ob}(\mathcal{C})}$ if and only if for every morphism $U \xrightarrow{f} V$ in $\mathcal{C}$ the following square is commutative

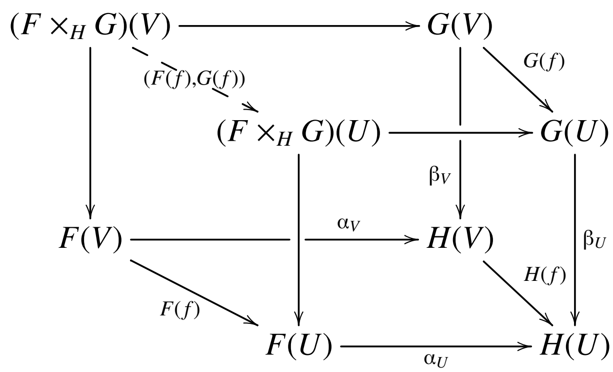
$$\begin{array}{ccc} F(U) & \xrightarrow{\alpha_U} & G(U) \\ {\scriptstyle F(f)}\uparrow & & \uparrow{\scriptstyle G(f)} \\ F(V) & \xrightarrow[\alpha_V]{} & G(V) \end{array}$$

That is to say $\alpha_U(F(f)(z)) = G(f)(\alpha_V(z))$ for all $z \in F(V)$. Given $(V, z) \in \mathcal{S}_F$ we have $f^*(V, z) = (U, F(f)(z))$ and $(f, id_{F(f)(z)}) : (U, F(f)(z)) \longrightarrow (V, z)$ is the strongly cartesian lift of f in $\mathcal{S}_F$. Hence $A(f^*(V, z)) = A(U, F(f)(z)) = (U, \alpha_U(F(f)(z)))$. On the other hand $A(V, z) = (V, \alpha_V(z))$, and so $f^*A(V, z) = (U, G(f)(\alpha_V(z)))$. Therefore, $f^*A(V, z) = A(f^*(V, z))$ if and only if $\alpha_U(F(f)(z)) = G(f)(\alpha_V(z))$.

Since $\widetilde{\alpha} : \mathcal{S}_F \longrightarrow \mathcal{S}_G$ comes from a natural transformation, then $\widetilde{\alpha}$ send the strongly cartesian lift of f in $\mathcal{S}_F$ into the strongly cartesian lift of f in $\mathcal{S}_G$.

Now we are going to show that this construction preserves fibre products. Let $F, G, H : \mathcal{C}^{op} \longrightarrow$ **Cat** functors and $\alpha : F \longrightarrow H$ and $\beta : G \longrightarrow H$ are natural transformations. Hence there exists the fibre product $F \times_H G : \mathcal{C}^{op} \longrightarrow$ **Cat** defined as follows:

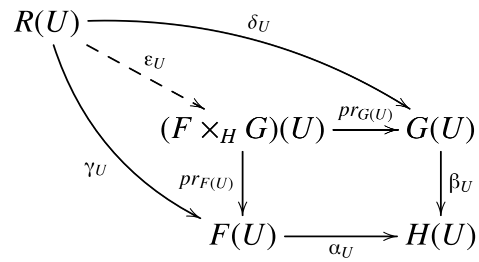
Given $U \in \mathrm{Ob}(\mathcal{C})$, we have morphisms $\alpha_U : F(U) \longrightarrow H(U)$ and $\beta_U : G(U) \longrightarrow H(U)$, and so we can consider the fibre product $F(U) \times_{H(U)} G(U)$ wich is the category whose objects are pairs (x, y) where $x \in \mathrm{Ob}(F(U))$, $y \in \mathrm{Ob}(G(U))$ and $\alpha_U(x) = \beta_U(y)$ and the morphisms from (x, y) to (x', y') are pairs (a, b) with $a : x \longrightarrow x'$ and $b : y \longrightarrow y'$ morphisms in $F(U)$ and $G(U)$ respectively such that $\alpha_U(a) = \beta_U(b)$. The composition law is $(a, b) \circ (a', b') := (a \circ a', b \circ b')$ which is well defined because of α_U and β_U are functors. Projections $pr_{F(U)}$ and $pr_{G(U)}$ are defined as . Define $(F \times_H G)(U) := F(U) \times_{H(U)} G(U)$ and for $f : U \longrightarrow V$ a morphism in $\mathcal{C}$, $(F \times_H G)(f) : (F \times_H G)(V) \longrightarrow (F \times_H G)(U)$ is the only morphism such that in the following cube the upper and left faces are commutative



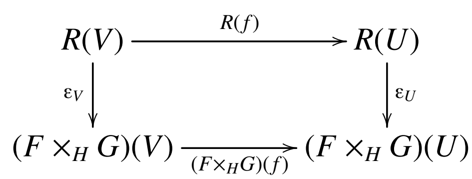
which is given by the universal property of the fibre product $F(U) \times_{H(U)} G(U)$. More precisely, if α and β are natural transformations, then the lower and right faces are commutative and front and back are commutative because are indeed cartesian. Hence we can use this and the universal property of fibre product in order to prove the existence and unicity of such a morphism. This morphism is indeed $(F(f), G(f))$, which is defined for $(x, y) \in (F \times_H G)(V)$ by $(F(f), G(f))(x, y) := (F(f)(x), G(f)(y))$. If $g : V \longrightarrow W$ is another morphism, the same argument shows that $(F \times_H G)(g \circ f) = (F \times_H G)(f) \circ (F \times_H G)(g)$. Also, considering $id_U : U \longrightarrow U$, $F(id_U) = id_{F(U)}$, $G(id_U) = id_{G(U)}$ and $H(id_U) = id_{H(U)}$, and so $(F \times_H G)(id_U) = id_{(F \times_H G)(U)}$. Therefore $F \times_H G : \mathcal{C}^{op} \longrightarrow \mathbf{Cat}$ is a functor.

We will define now the projections. Let $pr_F : F \times_H G \longrightarrow F$ defined for $U \in \mathcal{C}$ by $pr_{F,U} : (F \times_H G)(U) \longrightarrow F(U)$ as the projection $pr_{F(U)} : F(U) \times_{H(U)} G(U) \longrightarrow F(U)$. Hence, given $U \xrightarrow{f} V$, since the cube's left face is commutative, p_F is a natural transformation. Analogously $pr_G : F \times_H G \longrightarrow G$ is defined. Clearly $\alpha \circ pr_F = \beta \circ pr_G$. In order to prove the universal property, let $R : \mathcal{C}^{op} \longrightarrow \mathbf{Cat}$ be a functor with natural transformations $\gamma : R \longrightarrow F$

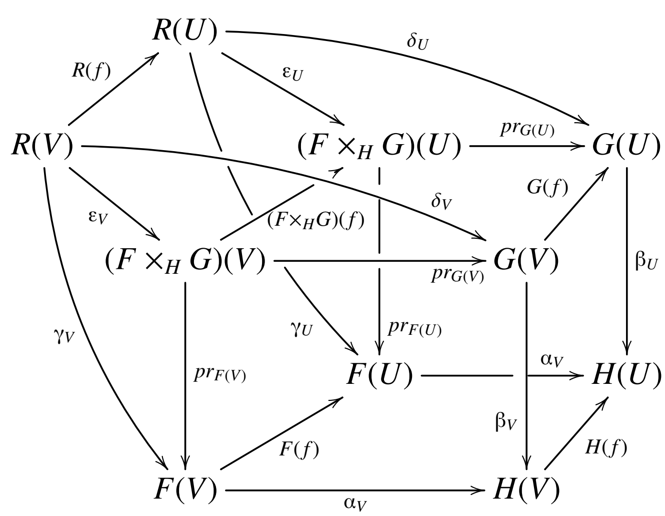
and $\delta : R \longrightarrow G$ such that $\alpha \circ \gamma = \beta \circ \delta$. Then for $U \in \mathrm{Ob}(\mathcal{C})$ we have a commutative diagram



Lets show $\{\varepsilon_U\}_{U \in \mathrm{Ob}(\mathcal{C})}$ defines a natural transformation $\varepsilon : R \longrightarrow F \times_H G$. If $f : U \longrightarrow V$ is a morphism in $\mathcal{C}$, we want to see the next square is commutative:



For this look at the following diagram



Consider the two morphisms $(F \times_H G)(f) \circ \varepsilon_V$ and $\varepsilon_U \circ R(f)$ and compose them with both pr_F and pr_G. Then we have:

$$pr_{F(U)} \circ ((F \times_H G)(f) \circ \varepsilon_V) = F(f) \circ pr_{F(V)} \circ \varepsilon_V$$

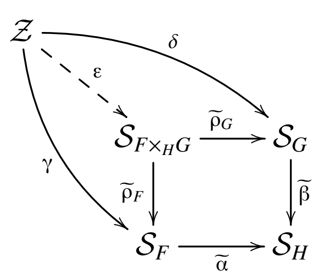
$$\begin{aligned} &= F(f) \circ \gamma_V \\ &= \gamma_U \circ R(f) \\ &= pr_{F(U)} \circ (\varepsilon_U \circ R(f)) \end{aligned}$$

and

$$\begin{aligned} pr_{G(U)} \circ ((F \times_H G)(f) \circ \varepsilon_V) &= G(f) \circ pr_{G(V)} \circ \varepsilon_V \\ &= G(f) \circ \delta_V \\ &= \delta_U \circ R(f) \\ &= pr_{G(U)} \circ (\varepsilon_U \circ R(f)) \end{aligned}$$

But, for the universal property of the fibre product $(F \times_H G)(U)$ there is only a morphism with this property and so they are equal as we wanted. Then ε is a natural transformation and the construction shows this is the unique such that $pr_F \circ \varepsilon = \gamma$ and $pr_G \circ \varepsilon = \delta$. Then $F \times_H G$ is a fibre product in the **Cat** and we are done.

Considering the corresponding associated fibred categories $p_F : \mathcal{S}_F \longrightarrow \mathcal{C}$, $p_G : \mathcal{S}_G \longrightarrow \mathcal{C}$, $p_H : \mathcal{S}_H \longrightarrow \mathcal{C}$ and $p_{F \times_H G} \longrightarrow \mathcal{C}$ we have 1-morphisms of fibred categories $\widetilde{\alpha} : \mathcal{S}_F \longrightarrow \mathcal{S}_H$, $\widetilde{\beta} : \mathcal{S}_G \longrightarrow \mathcal{S}_H$, $\widetilde{\rho}_F : \mathcal{S}_{F \times_H G} \longrightarrow \mathcal{S}_F$ and $\widetilde{\rho}_G$ such that $\widetilde{\beta} \circ \widetilde{\rho}_G = \widetilde{\alpha} \circ \widetilde{\rho}_F$, where $\widetilde{\rho}_F$ is $\widetilde{pr_F}$. Furthermore, this is more then simple commutativity, the associated square is actually cartesian. We need only to show the universal property is satisfied. Let $q : \mathcal{Z} \longrightarrow \mathcal{C}$ be a fibred category over and $\gamma : \mathcal{Z} \longrightarrow \mathcal{S}_F$ and $\delta : \mathcal{Z} \longrightarrow \mathcal{S}_G$, 1-morphisms of fibred categories over $\mathcal{C}$ such that the following diagram commutes



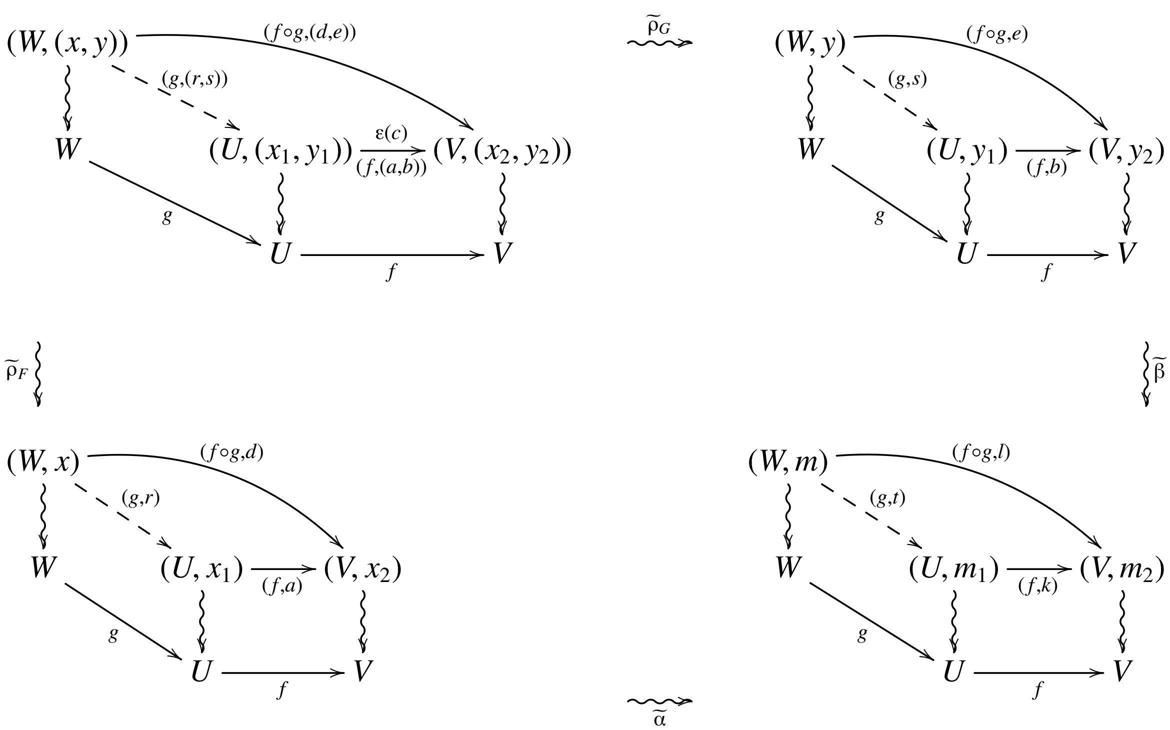
We will construct a 1-morphism $\varepsilon : \mathcal{Z} \longrightarrow \mathcal{S}_{F\times_H G}$ which is unique making the triangles in the diagram commutative. If $z \in \mathrm{Ob}(\mathcal{Z})$, then $\gamma(z) \in \mathrm{Ob}(\mathcal{S}_F)$ and so $\gamma(z) = (U, x)$, with $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in \mathrm{Ob}(F(U))$. Since γ is a morphism of categories over $\mathcal{C}$, then $p_F \circ \gamma = q$ and then $q(z) = p_F \circ \gamma(z) = p_F(U, x) = U$. Therefore $\delta(z) = (U, y)$ with $y \in \mathrm{Ob}(G(U))$. Moreover, $\widetilde{\alpha} \circ \gamma = \widetilde{\beta} \circ \delta$ and so $\widetilde{\alpha}(U, x) = \widetilde{\beta}(U, y)$, which means $(U, \alpha_U(x)) = (U, \beta_U(y))$ and we have $\alpha_U(x) = \beta_U(y)$. Then $(U, (x, y)) \in \mathrm{Ob}(\mathcal{S}_{F\times_H G})$. We define $\varepsilon(z) = (U, (x, y))$, where $U = q(z)$, $(U, x) = \gamma(z)$ and $(U, y) = \delta(z)$. If $c : z_1 \longrightarrow z_2$ is a morphism in $\mathcal{Z}$, then $\gamma(c); \gamma(z_1) \longrightarrow \gamma(z_2)$ is a morphisms in $\mathcal{S}_F$. Writing $\gamma(z_1) = (U, x_1)$ and $\gamma(z_2) = (V, x_2)$, where $U = q(z_1)$ and $V = q(z_2)$, we have $\gamma(c) = (f, a)$ with $f : U \longrightarrow V$ a morphism in $\mathcal{C}$ and $a : x_1 \longrightarrow F(f)(x_2)$ a morphism in $F(U)$. Moreover, $q(c) = p_F \circ \gamma(c) = f$. Therefore $\delta(c) : \delta(z_1) \longrightarrow \delta(z_2)$ is a morphism in $\mathcal{S}_G$. Writing $\delta(z_1) = (U, y_1)$ and $\delta(z_2) = (V, y_2)$, we have $\gamma(c) = (f, b)$ with $b : y_1 \longrightarrow G(f)(y_2)$ a morphism in $G(U)$. Since $\widetilde{\alpha} \circ \gamma = \widetilde{\beta} \circ \delta$ and by definition $\widetilde{\alpha} \circ \gamma(c) = \widetilde{\alpha}(f, a) = (f, \alpha_U(a))$ and $\widetilde{\beta} \circ \delta(c) = \widetilde{\beta}(f, b) = (f, \beta_U(b))$, then $(f, \alpha_U(a)) = (f, \beta_U(b))$ and so $\alpha_U(a) = \beta_U(b)$. Hence $(a, b) \in \mathrm{Hom}_{(F\times_H G)}((x_1, F(f)(x_2)), (y_1, G(f)(y_2)))$ and therefore $(f, (a, b)) \in \mathrm{Hom}_{\mathcal{S}_{F\times_H G}}((U, (x_1, y_1)), (V, (x_2, y_2)))$. Define $\varepsilon(c) = (f, (a, b))$. then $\widetilde{\rho}_F \circ \varepsilon(z) = \widetilde{\rho}(U, (x, y)) = (U, x) = \gamma(z)$ and $\widetilde{\rho}_F \circ \varepsilon(c) = \widetilde{\rho}_F(f, (a, b)) = (f, a) = \gamma(c)$, and so $\widetilde{\rho}_F \circ \varepsilon = \gamma$. In the same way $\widetilde{\rho}_G \circ \varepsilon = \delta$. If there exists $\eta : \mathcal{Z} \longrightarrow \mathcal{S}_{F\times_H G}$ with these properties, if $\eta(z) = (U'(x', y'))$, then $\widetilde{\rho}_F \circ \eta(z) = (U', x')$, but $\widetilde{\rho}_F \circ \eta = \gamma$ and it follows $(U', x') = (U, x)$ and so $\eta(z) = \varepsilon(z)$. Similarly, $\eta(c) = \varepsilon(c)$ for any morphism c in $\mathcal{Z}$.

The proof ends once it is stablished that $\varepsilon : \mathcal{Z} \longrightarrow \mathcal{S}_{F\times_H G}$ is a 1-morphism of fibred categories over $\mathcal{C}$. For this only rest to show $p_{F\times_H G} \circ \varepsilon = q$ and ε preserves strongly cartesian morphisms. If $z \in \mathrm{Ob}(\mathcal{Z})$, then

$$
\begin{aligned}
p_{F\times_H G} \circ \varepsilon(z) &= p_{F\times_H G}(U, (x, y)) \\
&= p_F(U, x) \\
&= p_F(\gamma(z))
\end{aligned}
$$

$$= q(z)$$

Now, if $z_2 \in \mathrm{Ob}(\mathcal{Z})$ and $f : U \longrightarrow q(z_2)$ a morphism in $\mathcal{C}$, let $c : z_1 \longrightarrow z_2$ be the strongly cartesian morphism lift of f in $\mathcal{Z}$. Is enough to show that $\varepsilon(c)$ is strongly cartesian in $\mathcal{S}_{F\times_H G}$. By hypothesis $\gamma(c)$ and $\delta(c)$ are strongly cartesian in $\mathcal{S}_F$ and $\mathcal{S}_G$ respectively. Writing $\varepsilon(z_1) = (U,(x_1,y_1))$, $\varepsilon(z_2) = (V,(x_2,y_2))$ and $\varepsilon(c) = (f,(a,b))$ we have $U = q(z_1)$, $V = q(z_2)$, $(U,x_1) = \gamma(z_1)$, $(U,y_1) = \delta(z_1)$, $(V,x_2) = \gamma(z_2)$, $(V,y_2) = \delta(z_2)$, $(f,a) = \gamma(c)$ and $(f,b) = \delta(c)$. If $(W,(x,y))$ is an object of $\mathcal{S}_{F\times_H G}$, $g : W \longrightarrow U$ is a morphism in $\mathcal{C}$ and $(f\circ g,(d,e)) : (W,(x,y)) \longrightarrow (V,(x_2,y_2))$ is a morphism, then $\gamma(f\circ g,(d,e)) = (f\circ g,d)$, $\delta(f\circ g,(d,e)) = (f\circ g,e)$ and $\widetilde{\alpha}\circ\gamma(f\circ g,(d,e)) = \widetilde{\beta}\circ\delta(f\circ g,(d,e))$. Hence $\widetilde{\alpha}(f\circ g,d) = \widetilde{\beta}(f\circ g,e)$. Consider the following array of diagrams where arrows connecting them means the respective functor transforms one diagram in another, using the notation as above



The morphisms (f, a) and (f, b) are strongly cartesian in $\mathcal{S}_F$ and $\mathcal{S}_G$ respectively, and so there are morphisms (g, r) and (g, s) which are unique such that the triangles in the respective diagrams are commutative. We shall see $(g, (r, s))$ is a morphisms in $\mathcal{S}_{F\times_H G}$. We already know $\widetilde{\alpha}(U, x) = \widetilde{\beta}(U, y) =: (U, m)$, $\widetilde{\alpha}(U, x_1) = \widetilde{\beta}(U, y_1) =: (U, m_1)$, $\widetilde{\alpha}(V, x_2) = \widetilde{\beta}(V, y_2) =: (V, m_2)$ and also $\widetilde{\alpha}(f, a) = \widetilde{\beta}(f, b) =: (f, k)$ is strongly cartesian. Since $\widetilde{\alpha}(f \circ g, d) = \widetilde{\beta}(f \circ g, e)$ we have $\alpha_U(d) = \beta_U(e) := l$ and so $(f \circ g, l)$ is a morphism in $\mathcal{S}_H$. Therefore there exists a morphism $(g, t) : (W, m) \longrightarrow (U, m_1)$ unique such that the respective triangle commutes. But $\widetilde{\alpha}(g, r)$ and $\widetilde{\beta}(g, s)$ satisfy that property and so are equal. Hence $\alpha_U(r) = \beta_U(s)$ and it follows $(g, (r, s))$ is a morphism in $\mathcal{S}_{F\times_H G}$ and is such that

$$\begin{aligned}(f, (a, b)) \circ (g, (r, s)) &= (f \circ g, (F \times_H G)(a, b) \circ (r, s)) \\ &= (f \circ g, (F(g)(a), G(g)(b)) \circ (r, s)) \\ &= (f \circ g, (F(g)(a) \circ r, G(g)(b) \circ s)) \\ &= (f \circ g, (d, e))\end{aligned}$$

where we use $(f \circ g, d) = (f, a) \circ (g, r) := (f \circ g, F(g)(a) \circ r)$ and $(f \circ g, e) = (f, b) \circ (g, s) := (f \circ g, G(g)(b) \circ s)$. Is easy to see that $(g, (r, s))$ is unique satisfying the equality above. Therefore there exists $\mathcal{S}_F \times_{\mathcal{S}_H} \mathcal{S}_G$ and is given by $\mathcal{S}_{F\times_H G}$, that is to say, there exists fibre product in the category of fibred categories as long as these are fibred categories associated to functors and the 1-morphism are induced by natural transformations of such functors.

We can summarize the previous results in the following theorem.

Theorem 2. *Given a functor $F : \mathcal{C}^{op} \longrightarrow$ **Cat** there is a canonical fibred category $p_F : \mathcal{S}_F \longrightarrow \mathcal{C}$. If $\alpha : F \longrightarrow G$ is a natural transformation of functors, there is a 1-morphism of fibred categories $\widetilde{\alpha} : \mathcal{S}_F \longrightarrow \mathcal{S}_G$ such that the construction is compatible with composition, identities and fibre product.*

With the notation as before, another fibre product can be constructed as follows:

Define the category $\mathcal{S}_{F,G,H}$ with

$$\mathrm{Ob}(\mathcal{S}_{F,G,H}) = \{(X,Y) \mid X \in \mathrm{Ob}(\mathcal{S}_F), Y \in \mathrm{Ob}(\mathcal{S}_G) \text{ and } \widetilde{\alpha}(X) = \widetilde{\beta}(Y)\}$$

Hence, if $X = (U,x)$ and $Y = (V,y)$ then $\widetilde{\alpha}(X) = \widetilde{\alpha}(U,x) = (U, \alpha_U(x))$ and $\widetilde{\beta}(Y) = \widetilde{\beta}(V,y) = (V, \beta_V y)$ and therefore $U = V$ and $\alpha_U(x) = \beta_U(y)$. Then an object of $\mathcal{S}_{F,G,H}$ has the form $((U,x),(U,y))$. A morphism from $((U,x_1),(U,y_1))$ to $((V,x_2),(V,y_2))$ in $\mathcal{S}_{F,G,H}$ is a pair (r,s) with $r : (U,x_1) \longrightarrow (V,x_2)$ and $s : (U,y_1) \longrightarrow (V,y_2)$ morphisms in $\mathcal{S}_F$ and $\mathcal{S}_G$ respectively such that $\widetilde{\alpha}(r) = \widetilde{\beta}(s)$. Hence $r = (f,a)$ where $f : U \longrightarrow V$ a morphism in $\mathcal{C}$ and $a : x_1 \longrightarrow F(f)(x_2)$ a morphism in $F(U)$. Similarly $s = (g,b)$ with $g : U \longrightarrow V$ and $b : y_1 \longrightarrow G(f)(y_2)$. Now, $\widetilde{\alpha}(r) = \widetilde{\beta}(s)$ if and only if $f = g$ and and $\alpha_U(a) = \alpha_U(b)$. Therefore a morphism has the form $((f,a),(f,b))$. Given a morphism (r',s') from $((V,x_2),(V,y_2))$ into a third object $((W,x_3),(W,y_3))$ of $\mathcal{S}_{F,G,H}$ we define $(r',s') \circ (r,s) := (r' \circ r, s' \circ s)$, making the compositions in $\mathcal{S}_F$ and $\mathcal{S}_G$ according to the case. Then $\mathcal{S}_{F,G,H}$ is a category and is easy to see that $\mathcal{S}_{F,G,H} \cong \mathcal{S}_{F \times_H G}$. Let $\varphi : \mathcal{S}_{F \times_H G} \longrightarrow \mathcal{S}_{F,G,H}$ be the functor defined by $\varphi(U,(x,y)) = ((U,x),(U,y))$ in the objects and $\varphi(f,(a,b)) = ((f,a),(f,b))$ in the morphisms. Then φ is an isomorphism. Therefore $\mathcal{S}_{F,G,H}$ is a fibre product of $\widetilde{\alpha} : \mathcal{S}_F \longrightarrow \mathcal{S}_H$ and $\widetilde{\beta} : \mathcal{S}_G \longrightarrow \mathcal{S}_H$.

In particular, if $\mathcal{X} = \mathcal{S}_F$, $\mathcal{Y} = \mathcal{S}_G$ and $\mathcal{Z} = \mathcal{S}_H$, the construction $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ we have done before in categories over $\mathcal{C}$ is also valid in fibred categories which are induced by functors. Here $(f^*x \longrightarrow x, f^*y \longrightarrow y) : (f^*x, f^*y) \longrightarrow (x,y)$ is the strongly cartesian lift of f.

Then if the categories are associated to functors $F : \mathcal{C} \longrightarrow \mathbf{Cat}$ and the 1-morphisms comes from natural transformations, the fibre product always exists and is given like any of the both equivalent constructions. In particular we can do this when the fibred categories are quotients $[U/R]_p$ of groupoids (U,R,s,t,c) and the 1-morphisms are induced by morphisms of groupoids. From now on we will concentrate the efforts in fibred categories and 1-morphisms of such type looking for a quite general result.

Definition 1.4.1 (Split fibred category). *(02XW) Let $\mathcal{C}$ be a category. A split fibred category over $\mathcal{C}$ is a fibred category over $\mathcal{C}$ 1-isomorphic to one of these categories $\mathcal{S}_F$, where $F : \mathcal{C}^{op} \longrightarrow \mathbf{Cat}$ is a functor.*

Lemma 1.4.1. *(02XX) Let $\mathcal{C}$ be a category. A fibred category $p : \mathcal{S} \longrightarrow \mathcal{C}$ is split if and only if for some choice of pullbacks, the functors $(f \circ g)^*$ and $g^* \circ f^*$ are equal.*

Lemma 1.4.2. *(004A) If $p : \mathcal{S} \longrightarrow \mathcal{C}$ is a fibred category, then there exists a functor $F : \mathcal{C}^{op} \longrightarrow \mathbf{Cat}$ such that $\mathcal{S}$ is equivalent to $\mathcal{S}_F$ as fibred category over $\mathcal{C}$. In other words, every fibred category is equivalent to a split one.*

The following sections introduce some classes of fibred categories depending in what additional structure the fibre categories have. The most important are groupoids, sets and setoids. There are some results that we have for any fibred category, but also each of these clases has particular properties. For the subsequent results in this work is enough with fibred categories in general or in some cases fibred categories in groupoids. Fibred categories in sets and setoids are important in order to define when a fibred category is representable, so we expose here the principal results.

1.5 Fibred categories in groupoids

Definition 1.5.1 (Fibred category in groupoids). *A category $p : \mathcal{S} \longrightarrow \mathcal{C}$ over $\mathcal{C}$ is a category fibred in groupoids if this is a fibred category over $\mathcal{C}$ and for any $U \in \mathrm{Ob}(\mathcal{C})$, the fibre category $\mathcal{S}_U$ is a groupoid.*

Remark. This definition is different than the presented in Stacks Project and some books, although they are equivalent. We prefer the presented here because is easier to recall and is more like the definitions for fibred categories in sets and setoids.

Lemma 1.5.1. *(03WQ) Given a fibred category $p : \mathcal{S} \longrightarrow \mathcal{C}$, let $\mathcal{S}'$ be the sub-category of $\mathcal{S}$ defined as follows:*

1. $\mathrm{Ob}(\mathcal{S}') = \mathrm{Ob}(\mathcal{S})$
2. *For $x, y \in \mathrm{Ob}(\mathcal{S}')$, the set of morphisms from x to y is the set of the strongly cartesian morphisms from x to y.*

Then $p' : \mathcal{S}' \longrightarrow \mathcal{C}$, defined as the restriction of p to $\mathcal{S}'$, is a fibred categories in groupoids over $\mathcal{C}$.

Lemma 1.5.2. *(003V) If $p : \mathcal{S} \longrightarrow \mathcal{C}$ is a fibred category in groupoids, then every morphisms of $\mathcal{S}$ is strongly cartesian.*

Proof. Let $x \xrightarrow{\varphi} y$ a morphism in $\mathcal{S}$ and $f = p(\varphi)$. Since $p : \mathcal{S} \longrightarrow \mathcal{C}$ is fibred, there is a strongly cartesian morphism $f^*y \xrightarrow{\psi} y$ which is a lift of f. Then, there exist $\alpha : x \longrightarrow f^*x$ in $\mathcal{S}_{p(x)}$ unique such that $x \xrightarrow{\alpha} f^*y \xrightarrow{\psi} y = x \xrightarrow{\varphi} y$. But $\mathcal{S}_{p(x)}$ is a groupoid and therefore α is an isomorphism and therefore is strongly cartesian. Then $\varphi = \psi \circ \alpha$ is strongly cartesian. $\square$

Definition 1.5.2. *(02XS) Let $\mathcal{C}$ be a category. The 2-category of fibred categories in groupoids over $\mathcal{C}$ is 2-category defined as follows:*

1. *Its objects are fibred categories in groupoids over $\mathcal{C}$.*
2. *The 1-morphisms $(\mathcal{S}, p) \longrightarrow (\mathcal{S}', p')$ are functors $G : \mathcal{S} \longrightarrow \mathcal{S}'$ such that $p' \circ G = p$*

3. *Its 2-morphisms* $t : G \longrightarrow H$ *for* $G, H : (\mathcal{S}, p) \longrightarrow (\mathcal{S}', p')$ *are natural transformations such that* $p'(t_x) = id_{p(x)}$ *for all* $x \in \mathrm{Ob}(\mathcal{S})$.

Remark. The 2-category of fibred categories in groupoids is a sub 2-category of the 2-category of fibred categories. Note that since every morphism of $\mathcal{S}$ is strongly cartesian, then any functor $G : \mathcal{S} \longrightarrow \mathcal{S}'$ preserve them. Moreover, every 2-morphism is isomorphism, and therefore this 2-category is actually a (2,1)-category.

Lemma 1.5.3. *(0041) Let* $\mathcal{C}$ *be a category. The 2-category of fibred categories in groupoids over* $\mathcal{C}$ *has 2-fibre products and they are described as in categories over* $\mathcal{C}$.

Proof. Given $U \in \mathcal{C}$ we have that $\mathcal{X}_U$, $\mathcal{Y}_U$ and $\mathcal{Z}_U$ are groupoids. Then let $(a, b) : (U, x, y, \alpha) \longrightarrow (U, x', y', \alpha')$ be a morphism in $(\mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y})_U$. Since $s(a, b) = id_U$, then $p(a) = q(b) = id_U$ and therefore $a : x \longrightarrow x'$ and $b : y \longrightarrow y'$ are morphisms in $\mathcal{X}_U$ and $\mathcal{Y}_U$ respectively. Hence a and b are isomorphisms. We shall see that (a^{-1}, b^{-1}) is a morphism from (U', x', y', α') to (U, x, y, α) in $(\mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y})_U$. Since $F(b) \circ \alpha = \alpha' \circ F(a)$ and $F(a^{-1}) = F(a)^{-1}$, $F(b^{-1}) = F(b)^{-1}$, then $\alpha \circ F(a^{-1}) = F(b^{-1}) \circ \alpha'$ and is clear that $s(a^{-1}, b^{-1}) = id_U$. Moreover, $(a^{-1}, b^{-1}) \circ (a, b) = (id_x, id_y) = id_{(U,x,y,\alpha)}$ and $(a, b) \circ (a^{-1}, b^{-1}) = (id_{x'}, id_{y'}) = id_{(U',x',y',\alpha')}$ and so (a, b) is an isomorphism and therefore $(\mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y})_U$ is a groupoid. □

Lemma 1.5.4. *(003Z) Let* $p : \mathcal{S} \longrightarrow \mathcal{C}$ *and* $p' : \mathcal{S}' \longrightarrow \mathcal{C}$ *be fibred categories in groupoids and let* $G : \mathcal{S} \longrightarrow \mathcal{S}'$ *be a functor over* $\mathcal{C}$.

1. *G is faithful (resp. fully faithful, resp. an equivalence) if and only if for any* $U \in \mathrm{Ob}(\mathcal{C})$ *the induced functor* $G_U : \mathcal{S}_U \longrightarrow \mathcal{S}'_U$ *is faithful (resp. fully faithful, resp. an equivalence).*

2. *If G is an equivalence, then G is an equivalence in the 2-category of fibred categories in groupoids over* $\mathcal{C}$.

Lemma 1.5.5. *(06N6) Let $p : \mathcal{S} \longrightarrow \mathcal{C}$ be a fibred category in groupoids, $x \longrightarrow z$ and $y \longrightarrow z$ morphisms of $\mathcal{S}$. If exists $p(x) \times_{p(z)} p(y)$, then exists $x \times_z y$ and $p(x \times_z y) = p(x) \times_{p(z)} p(y)$.*

Proof. It follows from the analog statement for categories over $\mathcal{C}$ taking in account that in a fibred categories in groupoids all the morphisms are strongly cartesian and therefore the conditions are satisfied automatically. □

Remark. If $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is a 1-morphism of fibred categories in groupoids, $\mathcal{X}$ is not necessarily a fibred category over $\mathcal{Y}$. However, the next lemma states that we are really close to this.

Lemma 1.5.6. *(06N7) If $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is a 1-morphism of fibred categories in groupoids over $\mathcal{C}$, Then there exists a factorization $\mathcal{X} \longrightarrow \mathcal{X}' \longrightarrow \mathcal{Y}$ of 1-morphisms of fibred categories in groupoids over $\mathcal{C}$ such that $\mathcal{X} \longrightarrow \mathcal{X}'$ is an equivalence over $\mathcal{C}$ and $\mathcal{X}'$ is a fibred category in groupoids over $\mathcal{Y}$.*

If $F : \mathcal{C}^{op} \longrightarrow$ **Gpds** is a functor, then it induces a fibred category $p_F : \mathcal{S}_F \longrightarrow \mathcal{C}$ which is a fibred category in groupoids. The same constructions made in fibred categories are possible here. In particular, a split fibred category in groupoids is a fibred category in groupoids equivalent to a category $\mathcal{S}_F$, where $F : \mathcal{C}^{op} \longrightarrow$ **Gpds** is a functor. The same result as before allow us to conclude that every fibred category in groupoids is equivalent over $\mathcal{C}$ to a split one.

1.6 Fibred categories in sets

Definition 1.6.1 (Discrete category). *(02Y0) A category is discrete if its only morphisms are the identities of the objects.*

Definition 1.6.2 (Fibred category in sets). *(0043) Let $\mathcal{C}$ be a category. A fibred category in sets is a fibred category whose fibre categories are discrete.*

Remark. Is easy to see that every category fibred in sets is a fibred category fibred in groupoids.

Example 8. If $F : \mathcal{C}^{op} \longrightarrow$ **Sets** is a functor, then $\mathcal{S}_F$ is the category fibred in sets with:

$$\mathrm{Ob}(\mathcal{S}_F) := \{(U, x) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in F(U)\}$$

where we have used that $F(U)$ is a set, that is to say, a discrete category. If (V, y) and (U, x) are in $\mathrm{Ob}(\mathcal{S}_F)$ we have

$$\mathrm{Hom}_{F(V)}((V, y), (U, x)) := \{f \in \mathrm{Hom}_{\mathcal{C}}(V, U) \mid F(f)(x) = y\}$$

More precisely, this is the set of pairs (f, a) where $f \in \mathrm{Hom}_{\mathcal{C}}(V, U)$ and $a \in \mathrm{Hom}_{F(V)}(y, F(f)(x))$, but in this case $F(V)$ is a discrete category and this means $y = F(f)(x)$ and a is the identity map. The functor $p_F : \mathcal{S}_F \longrightarrow \mathcal{C}$ is given by $(U, x) \longmapsto U$ and $f \longmapsto f$.

The composition in $\mathcal{S}_F$ is inherited from the composition in $\mathcal{C}$ and so $g^* \circ f^* = (f \circ g)^*$ for any pair of composable morphisms in $\mathcal{C}$. The functor $p_F : \mathcal{S}_F \longrightarrow \mathcal{C}$ is given by $p_F(U, x) = U$ and $p_F(f) = f$. Since every fibre category $\mathcal{S}_{F,U} = \{(U, x) \mid x \in F(U)\}$ is isomorphic to the set $F(U)$, then $\mathcal{S}_F$ is a fibred category in sets.

Definition 1.6.3 (2-category of fibred categories on sets). *(04S8) Let $\mathcal{C}$ be a category. The 2-category of fibred categories in sets over $\mathcal{C}$ is the sub 2-category of the 2-category of fibred categories in groupoids over $\mathcal{C}$ defined as follows:*

1. *The objects are categories $p : \mathcal{S} \longrightarrow \mathcal{C}$ fibred in sets.*
2. *The 1-morphisms $(\mathcal{S}, p) \longrightarrow (\mathcal{S}', p')$ are functors $G : \mathcal{S} \longrightarrow \mathcal{S}'$ such that $p' \circ G = p$.*

3. *The 2-morphisms $t : G \longrightarrow H$ for $G, H : (\mathcal{S}, p) \longrightarrow (\mathcal{S}', p)$ are natural transformations such that $p'(t_x) = id_{p(x)}$, for all $x \in \mathrm{Ob}(\mathcal{S})$.*

Remark. Since this is a full sub 2-category of the 2-category of fibred categories in groupoids, all the 2-morphisms are isomorphisms and therefore this is in fact a (2,1)-category.

Lemma 1.6.1. *(0047) Let $\mathcal{C}$ be a category. The 2-category of fibred categories in sets over $\mathcal{C}$ has 2-fibre product and is the same as fibred categories in groupoids.*

Here is a result that relate fibre and 2-fibre products imposing a strong condition over the fibred category $\mathcal{Z}$.

Proposition 1.6.1. *Let $p : \mathcal{X} \longrightarrow \mathcal{C}$, $q : \mathcal{Y} \longrightarrow \mathcal{C}$ and $r : \mathcal{Z} \longrightarrow \mathcal{C}$ be fibred categories over $\mathcal{C}$ and $F : \mathcal{X} \longrightarrow \mathcal{Z}$ and $G : \mathcal{Y} \longrightarrow \mathcal{Z}$ 1-morphisms of fibred categories over $\mathcal{C}$. If $\mathcal{Z}$ is fibred in sets, then the 2-fibre product 2-fibre product $\mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y}$ is also a fibre product, and therefore $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is a fibred category.*

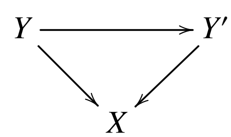
Proof. Recall that the objects of $\mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y}$ are cuadruples (U, x, y, α), where $U \in \mathrm{Ob}(\mathcal{C})$, $x \in \mathrm{Ob}(\mathcal{X}_U)$, $y \in \mathrm{Ob}(\mathcal{Y}_U)$ and $\alpha : F(x) \longrightarrow G(y)$ is an isomorphism in $\mathcal{Z}_U$. Since $\mathcal{Z}$ is fibred in sets, then the fibre category $\mathcal{Z}_U$ is discrete and therefore $F(x) = G(y)$ and α is the identity morphism. Hence $(x, y) \in \mathrm{Ob}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$, where $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is the fibre product which exists in categories over $\mathcal{C}$. Moreover, the morphisms are the same and so $\mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y} = \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$. □

Lemma 1.6.2. *(02Y2) Let $\mathcal{C}$ be a category. The only morphisms between fibred categories in sets over $\mathcal{C}$ are identities, that is to say, the 2-category of fibred categories in sets is a category. Moreover, there is an equivalence*

$$\left\{\begin{matrix}\text{category of presheaves}\\ \text{of sets over } \mathcal{C}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{category of fibred cate-}\\ \text{gories in sets over } \mathcal{C}\end{matrix}\right\}$$

The functor from left to right is $F \longmapsto \mathcal{S}_F$ and the functor in the other direction assign to $p : \mathcal{S} \longrightarrow \mathcal{C}$, the presheaf $U \longmapsto \mathrm{Ob}(\mathcal{S}_U)$.

Example 9. (0044) Let $\mathcal{C}$ be a category and $x \in \mathrm{Ob}(\mathcal{C})$. Recall that $\mathcal{C}/X$ is the category whose objects are morphisms $Y \longrightarrow X$ in $\mathcal{C}$ and the morphisms from $Y \longrightarrow X$ to $Y' \longrightarrow Y$ are the morphisms $Y \longrightarrow Y'$ in $\mathcal{C}$ such that the next triangle commutes



Consider the representable presheaf $h_X = \mathrm{Hom}_{\mathcal{C}}(\_, X)$, and the forgetful functor $p : \mathcal{C}/X \longrightarrow \mathcal{C}$. Since the fibre category $(\mathcal{C}/X)_U$ has as objects the morphisms $h : U \longrightarrow X$ and the morphisms are only the identities, the correspondence from the previous lemma implies

$$h_X \longleftrightarrow \mathcal{C}/X$$

Therefore the category $\mathcal{C}/X$ is canonically isomorphic to the category $\mathcal{S}_{h_X}$ associated to h_X.

Hence we could define a category to be representable if it is a fibred category in sets whose correspondent presheaf is representable in the usual sense. However it is preferable to have a notion which is invariant under equivalences. In order to make this precise, we are going to show which are the fibred categories in groupoids that are equivalent to fibred categories in sets.

1.7 Fibred categories in setoids

Definition 1.7.1 (Setoid). *(02XZ) A setoid is a groupoid where every object has exactly one automorphism: the identity.*

If C is a set together with an equivalence relation $\sim$, then we can construct a setoid $\mathcal{C}$ as follows: $\mathrm{Ob}(\mathcal{C}) = C$ and for $x, y \in C$ we define $\mathrm{Hom}_{\mathcal{C}}(x, y) = \emptyset$ unless $x \sim y$ in which case $\mathrm{Hom}_{\mathcal{C}}(x, y) = \{1\}$. Transitivity of $\sim$ means that morphisms can be composed. Reciprocally, every setoid defines a equivalence relation in its objets (isomorphism) such that the category can be recovered up to unique isomorphism with the previous construction.

Discrete categories are setoids. For any setoid $\mathcal{C}$ there is a canonical way to construct an equivalent discrete category, namely we replace $\mathrm{Ob}(\mathcal{C})$ by the set of isomorphism classes and identities morphisms are added. In terms of sets this corresponds to take the quotient by the equivalence relation.

Definition 1.7.2 (Fibred category in setoids). *(04SA) Let $\mathcal{C}$ be a category. A fibred category in setoids is a fibred category whose fibre categories are setoids.*

Definition 1.7.3 (2-category of fibred categories in setoids). *(02Y1) Let $\mathcal{C}$ be a category. The 2-category of fibred categories in setoids over $\mathcal{C}$ is the sub 2-category of fibred categories in groupoids over $\mathcal{C}$ defined as follows:*

1. *The Objects are categories $p : \mathcal{S} \longrightarrow \mathcal{C}$ fibred in setoids.*
2. *The 1-morphisms $(\mathcal{S}, p) \longrightarrow (\mathcal{S}', p')$ are functors $G : \mathcal{S} \longrightarrow \mathcal{S}'$ such that $p' \circ G = p$.*
3. *The 2-morphisms $t : G \longrightarrow H$ for $G, H : (\mathcal{S}, p) \longrightarrow (\mathcal{S}', p')$ are natural transformations such that $p'(t_x) = id_{p(x)}, \forall x \in \mathrm{Ob}(\mathcal{S})$.*

Once again this is a $(2, 1)$-category.

Lemma 1.7.1. *(04SB) Let $\mathcal{C}$ be a category. The 2-category of fibred categories in sets over $\mathcal{C}$ has 2-fibre product and is the same as fibred categories in groupoids.*

Lemma 1.7.2. *(0045) Let $p : \mathcal{S} \longrightarrow \mathcal{C}$ a fibred category.*

1. *If $S \longrightarrow \mathcal{S}'$ is an equivalence over $\mathcal{C}$ and $\mathcal{S}'$ is fibred in sets, then:*

   a) *$\mathcal{S}$ is a fibred category in setoids over $\mathcal{C}$,*

   b) *For each $U \in \mathrm{Ob}(\mathcal{C})$ the map $\mathrm{Ob}(\mathcal{S}_U) \longrightarrow \mathrm{Ob}(\mathcal{S}'_U)$ identify $\mathrm{Ob}(\mathcal{S}'_U)$ with the set of isomorphism classes of $\mathrm{Ob}(\mathcal{S}_U)$.*

2. *If $p : \mathcal{S} \longrightarrow \mathcal{C}$ is a fibred category in setoids, there is a fibred category in sets $p' : \mathcal{S}' \longrightarrow \mathcal{C}$ and a canonical equivalence $\mathcal{S} \longrightarrow \mathcal{S}'$ over $\mathcal{C}$.*

Lemma 1.7.3. *(04SC) The construction from the previous lemma gives a functor*

$$F : \left\{ \begin{array}{l} \text{2-category of fibred cate-} \\ \text{gories in setoids over } \mathcal{C} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{Category of fibred cate-} \\ \text{gories in sets over } \mathcal{C} \end{array} \right\}$$

This functor is an equivalence in the following sense:

1. *If $f, g : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ are 1-morphisms satisfying $F(f) = G(g)$, then there is a unique 2-isomorphism $f \longrightarrow g$.*

2. *For any morphism $h : F(\mathcal{S}_1) \longrightarrow F(\mathcal{S}_2)$ there is a 1-morphism $f : S_1 \longrightarrow \mathcal{S}_2$ such that $F(f) = h$.*

3. *Every fibred category in sets $\mathcal{S}$ is equal to $F(S)$.*

Definition 1.7.4 (Representable fibred category). *(0046) A fibred category $p : \mathcal{S} \longrightarrow \mathcal{C}$ is representable if there is $X \in \mathrm{Ob}(\mathcal{C})$ and an equivalence $j : \mathcal{S} \longrightarrow \mathcal{C}/X$. As usual we say that X represents $\mathcal{S}$.*

Lemma 1.7.4. *(02Y3) Let $p : \mathcal{S} \longrightarrow \mathcal{C}$ a fibred category.*

1. *$\mathcal{S}$ is representable if and only if the following conditions are satisfied:*

a) $\mathcal{S}$ is fibred in setoids.

b) The presheaf $U \longmapsto \mathrm{Ob}(\mathcal{S}_U)/\cong$ is representable.

2. If $\mathcal{S}$ is representable, the pair (X, j) is determined in a unique form up to isomorphism.

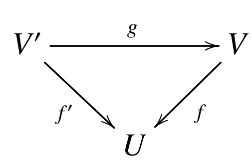
Chapter 2 Stacks

2.1 The Mor(x, y) presheaves

Assume $\mathcal{C}$ is a category and $p : \mathcal{S} \longrightarrow \mathcal{C}$ a fibred category. Given $U \in \text{Ob}(\mathcal{C})$ and $x, y \in \text{Ob}(\mathcal{S}_U)$ we will define a functor

$$\text{Mor}(x, y) : (\mathcal{C}/U)^{op} \longrightarrow \textbf{Sets}$$

Recall that $\mathcal{C}/U$ is the category whose objects are morphisms $f : V \longrightarrow U$ in en $\mathcal{C}$ and for $f : V \longrightarrow U$ and $f' : V' \longrightarrow U$, $\text{Hom}_{\mathcal{C}/U}(f', f)$ is the set of morphisms $g : V' \longrightarrow V$ such that the next diagram commutes



For $f : V \longrightarrow U$ we define $\text{Mor}(x, y)(f) := \text{Hom}_{\mathcal{S}_V}(f^*x, f^*y)$, wich make sense, since f^*x y f^*y are objects in $\mathcal{S}_V$ and we can consider the set $\text{Mor}_{\mathcal{S}_V}(f^*x, f^*y)$.

If $f' : V' \longrightarrow U$ is a second object in $\mathcal{C}/U$, then $\mathrm{Mor}(x,y)(f' : V' \longrightarrow U) = \mathrm{Hom}_{\mathcal{S}_{V'}}(f'^*x, f'^*y)$. Let $g : V' \longrightarrow V \in \mathrm{Mor}_{\mathcal{C}/U}(f', f)$, that is to say, $f \circ g = f'$. We want to define a function $\mathrm{Hom}_{\mathcal{S}_V}(f^*x, f^*y) \longrightarrow \mathrm{Hom}_{\mathcal{S}_{V'}}(f'^*x, f'^*y)$. For this, remember that exists a unique invertible natural transformation $\alpha_{g,f} : (f \circ g)^* \longrightarrow g^*f^*$ such that for each $x \in \mathcal{S}_U$ the following diagram is commutative

$$\begin{array}{ccccc} (f\circ g)^*x & \xrightarrow{(\alpha_{g,f})_x} & g^*f^*x & \longrightarrow & f^*x \\ & \searrow & & \swarrow & \\ & & x & & \end{array}$$

Therefore, we can do the next composition

$$\begin{array}{ccc} f'^*x & \xrightarrow{(\alpha_{g,f})_x} & g^*f^*x \\ {\scriptstyle \varphi|_{V'}} \Big\downarrow \vdots & & \Big\downarrow {\scriptstyle g^*\varphi} \\ f'^*y & \xleftarrow[(\alpha_{g,f})_y^{-1}]{} & g^*f^*y \end{array}$$

and then we can define

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{S}_V}(f^*x, f^*y) & \longrightarrow & \mathrm{Hom}_{\mathcal{S}_{V'}}(f'^*x, f'^*y) \\ \varphi & \longmapsto & \varphi|_{V'} \end{array}$$

That is to say $(\mathrm{Mor}(x,y)(g))(\varphi) = (\alpha_{g,f})_y^{-1} \circ g^*\varphi \circ (\alpha_{g,f})_x$.

We shall show that $\mathrm{Mor}(x,y)$ thus define is a functor. Let $f'' : V'' \longrightarrow V'$ another object in $\mathcal{C}/U$ and $g' : V'' \longrightarrow V'$ in $\mathrm{Hom}_{\mathcal{C}/U}(f'', f')$. Hence $f' \circ g' = f''$ and therefore $f'' = f' \circ g' = f \circ g \circ g'$. If $\varphi \in \mathrm{Hom}_{\mathcal{S}_V}(f^*x, f^*y)$, then $\mathrm{Mor}(x,y)(g \circ g') = (\alpha_{g\circ g',f})_y^{-1} \circ (g \circ g')^*\varphi \circ (\alpha_{g\circ g',f})_x$. Now, by definition of $\alpha_{g',g}$ the next diagram is commutative

$$\begin{array}{ccc} (g\circ g')^*f^*x & \xrightarrow{(\alpha_{g',g})_{f^*x}} & (g'^* \circ g^*)f^*x \\ {\scriptstyle (g\circ g')^*\varphi} \Big\downarrow & & \Big\downarrow {\scriptstyle g'^*g^*\varphi} \\ (g\circ g')^*f^*y & \overset{(\alpha_{g',g})_{f^*y}^{-1}}{\underset{(\alpha_{g',g})_{f^*y}}{\rightleftarrows}} & (g'^* \circ g^*)f^*y \end{array}$$

and we have $(g \circ g')^* \varphi = (\alpha_{g',g})^{-1}_{f^* y} \circ g'^* g^* \varphi \circ (\alpha_{g',g})_{f^* x}$. Thus $\mathrm{Mor}(x, y)(g \circ g')(\varphi) = (\alpha_{g \circ g', f})^{-1}_y \circ (\alpha_{g',g})^{-1}_{f^* y} \circ g'^* g^* \varphi \circ (\alpha_{g,g'})_{f^* x} \circ (\alpha_{g \circ g', f})_x$. Like the quadruple $(U \longmapsto \mathcal{S}_U, f \longmapsto f^*, \alpha_{g,f}, \alpha_U)$ is a pseudo functor from $\mathcal{C}^{op}$ to the (2,1)-category of categories, then $(\alpha_{g',g})_{f^*} \circ \alpha_{g' \circ g, f} = {}_{g'^*}(\alpha_{g,f}) \circ \alpha_{g', g \circ f}$ and in particular, for $x, y \in \mathrm{Ob}(\mathcal{S}_U)$ it follows $(\alpha_{g \circ g', f})^{-1}_y \circ (\alpha_{g',g})^{-1}_{f^* y} = (\alpha_{g', f})^{-1}_y \circ g'^*(\alpha_{g,f})^{-1}_y$ and $(\alpha_{g',g})_{f^* x} \circ (\alpha_{g \circ g', f})_x = g'^*(\alpha_{g,f})_x \circ (\alpha_{g', f'})_x$. Therefore

$$\begin{aligned} \mathrm{Mor}(x, y)(g \circ g')(\varphi) &= (\alpha_{g', f'})^{-1}_y \circ g'^*(\alpha_{g,f})^{-1}_y \circ g'^* g^* \varphi \circ g'^*(\alpha_{g,f})_x \circ (\alpha_{g', f'})_x \\ &= (\alpha_{g', f'})^{-1}_y \circ g'^*((\alpha_{g,f})^{-1}_y \circ g^* \varphi \circ (\alpha_{g,f})_x) \circ (\alpha_{g', f'})_x \end{aligned}$$

Consequently we have

$$\begin{aligned} (\mathrm{Mor}(x, y)(g') \circ \mathrm{Mor}(x, y)(g))(\varphi) &= \mathrm{Mor}(x, y)(g')(\mathrm{Mor}(x, y)(g)(\varphi)) \\ &= \mathrm{Mor}(x, y)(g')((\alpha_{g,f})^{-1}_y \circ g^* \varphi \circ (\alpha_{g,f})_x) \\ &= (\alpha_{g', f'})^{-1}_y \circ g'^*((\alpha_{g,f})^{-1}_y \circ g^* \varphi \circ (\alpha_{g,f})_x) \circ (\alpha_{g', f'})_x \\ &= \mathrm{Mor}(x, y)(g \circ g')(\varphi) \end{aligned}$$

Since φ is arbitrary $\mathrm{Mor}(x, y)(g \circ g') = \mathrm{Mor}(x, y)(g') \circ \mathrm{Mor}(x, y)(g)$, and we have stability by composition. Moreover, as $(\alpha_{id_V, f}) = id_{f^*}$ and $id_f = id_V$, then $\mathrm{Mor}(x, y)(id_V)(\varphi) = (\alpha_{id_V, f})^{-1}_y \circ id_V^* \varphi \circ (\alpha_{id_V, f})_x = \varphi$ and thus $\mathrm{Mor}(x, y)(id_f) = id_{\mathrm{Mor}_{\mathcal{S}_V}(f^* x, f^* y)} = id_{\mathrm{Mor}(x,y)(f)}$. We have stability by identity and we are done.

Definition 2.1.1 (Presheaf of morphisms). *(02ZB) Let $p : \mathcal{C} \longrightarrow \mathcal{C}$ be a fibred category. Given an object U of $\mathcal{C}$ and a pair of objects $x, y \in \mathcal{S}_U$, the functor* $\mathrm{Mor}(x, y)$ *defined as before is called presheaf of morphisms from x to y.*

Lemma 2.1.1. *(042V) If $F : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ is a 1-morphism of fibred categories over $\mathcal{C}$, $U \in$* $\mathrm{Ob}(\mathcal{C})$ *and $x, y \in$* $\mathrm{Ob}(\mathcal{S}_U)$*, then exists a canonical natural transformation of presheaves in $\mathcal{C}/U$*

$$\alpha : \mathrm{Mor}_{\mathcal{S}_1}(x, y) \Longrightarrow \mathrm{Mor}_{\mathcal{S}_2}(F(x), F(y))$$

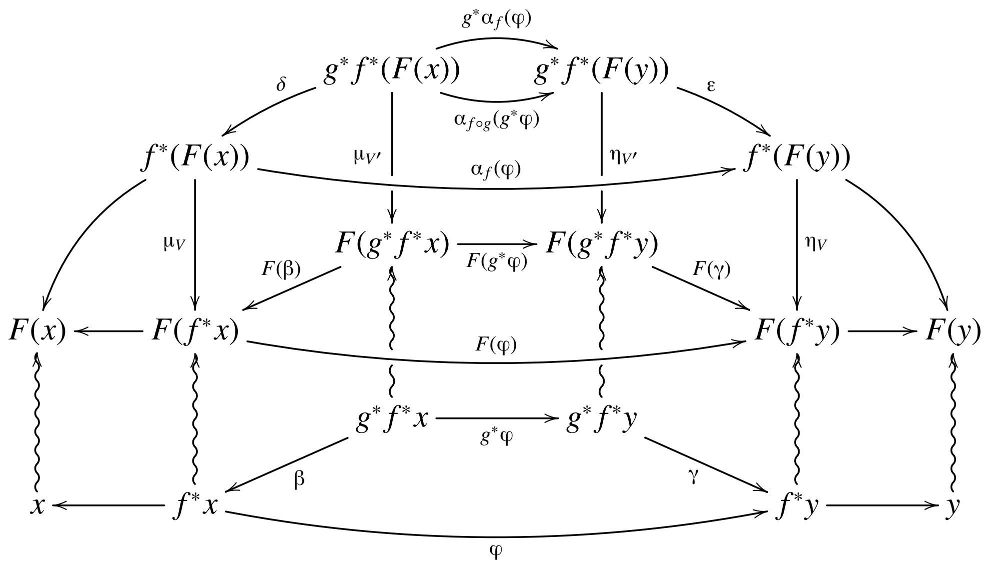
Moreover, if $F : S_1 \longrightarrow S_2$ is an equivalence of categories, then α is invertible.

Proof. We want to define for any $f \in \mathrm{Ob}(\mathcal{C}/U)$ a function $\alpha_f : \mathrm{Mor}_{\mathcal{S}_1}(x,y)(f) \longrightarrow \mathrm{Mor}_{\mathcal{S}_2}(F(x),F(y))(f)$, that is to say, $\alpha_f : \mathrm{Hom}_{S_{1,V}}(f^*x, f^*y) \longrightarrow \mathrm{Hom}_{\mathcal{S}_{2,V}}(f^*F(x), f^*F(y))$ so that $\alpha = \{\alpha_f\}_{f \in \mathcal{C}/U}$ be a natural transformation. Thus, given $\varphi \in \mathrm{Mor}_{\mathcal{S}_{1,V}}(f^*x, f^*y)$, we want $\alpha_f(\varphi) \in \mathrm{Mor}_{\mathcal{S}_{2,V}}(f^*F(x), f^*F(y))$.

The morphism $f^*x \longrightarrow x$ is strongly cartesian and then $F(f^*x) \longrightarrow F(x)$ is strongly cartesian and like $p' \circ F = p$, then $p'(F(f^*x) \longrightarrow F(x)) = p' \circ F(f^*x \longrightarrow x) = p(f^*x \longrightarrow x)$ and therefore $F(f^*x \longrightarrow F(x))$ is over f. But, as by definition $f^*F(x) \longrightarrow F(x)$ is also strongly cartesian over f, then there exists an unique isomorphism $\mu_V : f^*(F(x)) \longrightarrow F(f^*x)$ such that $f^*(F(x)) \xrightarrow{\mu_V} F(f^*) \longrightarrow F(x) = f^*(F(x)) \longrightarrow F(x)$. Moreover, $p'(F(f^*x)) = p(f^*x) = V$ and therefore $F(f^*) \in \mathcal{S}_V$. Similarly, given another $y \in \mathrm{Ob}(\mathcal{S}_U)$ there is an unique isomorphism $\eta_V : F^*(F(y)) \longrightarrow F(y)$ such that $f^*(F(y)) \xrightarrow{\eta_V} F(f^*) \longrightarrow F(y) = f^*(F(y)) \longrightarrow F(y)$. Therefore, given $\varphi \in \mathrm{Mor}_{S_{1,V}}(f^{*x}, f^*y)$ we define $\alpha_f(\varphi) = f^*(F(x)) \xrightarrow{\mu_V} F(f^*x) \xrightarrow{F(\varphi)} F(f^*y) \xrightarrow{\eta_V^{-1}} f^*(F(y))$, that is to say, $\alpha_f(\varphi) = \eta_V^{-1} \circ F(\varphi) \circ \mu_V$ and like μ_V and ν_V are isomorphisms, $\alpha_f(\varphi)$ is the unique morphisms in $\mathcal{S}_V$ such that $\eta_V \circ \alpha_f(\varphi) = F(\varphi) \circ \mu_V$. We shall see that $\alpha = \{\alpha_f\}$ is a natural transformation from $\mathrm{Mor}_{\mathcal{S}_1}(x,y)$ to $\mathrm{Mor}_{\mathcal{S}_2}(F(x),F(y))$. For this, we will prove that if $g : V' \longrightarrow V \in \mathrm{Mor}_{\mathcal{C}/U}(f', f)$, then the following diagram commutes

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{S}_1}(x,y)(f) & \xrightarrow{\alpha_f} & \mathrm{Mor}_{\mathcal{S}_2}(F(x),F(y))(f) \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g^*} \\ \mathrm{Mor}_{\mathcal{S}_1}(x,y)(f \circ g) & \xrightarrow[\alpha_{f \circ g}]{} & \mathrm{Mor}_{\mathcal{S}_2}(F(x),F(y))(f \circ g) \end{array}$$

This is the same as for $\varphi \in \mathrm{Mor}_{\mathcal{S}_1}(x,y)(f)$, we have $g^*(\alpha_f(\varphi)) = \alpha_{f\circ g}(g^*(\varphi))$. To do this consider the next diagram:



We want to prove that the two morphisms at the top are equal. By construction we have:

- $\alpha_f(\varphi)$ is the only morphism in $\mathcal{S}_V$ such that $\eta_V \circ \alpha_f(\varphi) = F(\varphi) \circ \mu_V$.
- $g^*\alpha_f(\varphi)$ is the only morphism in $\mathcal{S}_{V'}$ such that $\varepsilon \circ g^*\alpha_f(\varphi) = \alpha_f(\varphi) \circ \delta$.
- $\alpha_{f\circ g}(g^*\varphi)$ is the only morphism in $\mathcal{S}_V$ such that $\eta_{V'} \circ \alpha_{f\circ g}(g^*\varphi) = F(g^*\varphi) \circ \mu_{V'}$.

In order to see $\alpha_{f\circ g}(g^*\varphi) = g^*\alpha_f(\varphi)$, is enough to show $\varepsilon \circ \alpha_{f\circ g}(g^*\varphi) = \alpha_f(\varphi) \circ \delta$ ans since η_V is an isomorphism, this is equivalent to show $\eta_V(\varepsilon \circ \alpha_{f\circ g}(g^*\varphi)) = \eta_V(\alpha_f(\varphi) \circ \delta)$. Now, $\eta_V \circ \alpha_f(\varphi) \circ \delta = F(\varphi) \circ \mu_V \circ \delta$ and in the other hand $F(f^*x) \longrightarrow F(x)$ and $F(f^*y) \longrightarrow F(y)$ are strongly cartesian, and thus $F(\beta) \circ \mu_{V'} = \mu_V \circ \delta$ and $F(\gamma) \circ \eta_{V'} = \eta_V \circ \varepsilon$. Therefore we have

$$\begin{aligned} \eta_V \circ \varepsilon \circ \alpha_{f\circ g}(g^*\varphi) &= F(\gamma) \circ \eta_{V'} \circ \alpha_{f\circ g}(g^*\varphi) \\ &= F(\gamma) \circ F(g^*\varphi) \circ \mu_{V'} \end{aligned}$$

$$
\begin{aligned}
&= F(\gamma \circ g^*\varphi) \circ \mu_{V'} \\
&= F(\varphi \circ \beta) \circ \mu_{V'} \\
&= F(\varphi) \circ F(\beta) \circ \mu_{V'} \\
&= F(\varphi) \circ \mu_V \circ \delta
\end{aligned}
$$

So, $g^*\alpha_f(\varphi) = \alpha_{f \circ g}(g^*\varphi)$ and then α is a natural transformation. If $F : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ is fully faithful, then $\mathrm{Hom}_{\mathcal{S}_{1,V}}(f^*x, f^*y) \longleftrightarrow \mathrm{Hom}_{\mathcal{S}_{2,V}}(F(f^*x), F(f^*y))$. Moreover, by construction $\mathrm{Hom}_{\mathcal{S}_{2,V}}(F(f^*x), F(f^*y)) \longleftrightarrow \mathrm{Hom}_{\mathcal{S}_{2},V}(f^*F(x), f^*F(y))$ and then α_f is bijective. Furthermore, following the previous proof, we may see α_f^{-1} defines a natural transformation which is the inverse of α. Therefore, if $F : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ is fully faithful, α is an invertible natural transformation. □

2.2 Descent data in fibred categories

Before we treat about descent datums, we shall define the category of families of morphisms with fixed target in $\mathcal{C}$, which will be useful in the definition of the category of descent data.

Definition 2.2.1 (Category of families of fixed target). *Let $\mathcal{C}$ be a category $\mathcal{U} = \{f_i : U_i \longrightarrow U\}_{i \in I}$ and $\mathcal{V} = \{g_i : V_l \longrightarrow V\}$ families of morphisms in $\mathcal{C}$ with fixed target.*

1. *A morphism from $\mathcal{U}$ to $\mathcal{V}$ is a system (h, α, h_i), where $h : U \longrightarrow V$ is a morphisms in $\mathcal{C}$, $\alpha : I \longrightarrow J$ a function and $h_i : U_i \longrightarrow V_{\alpha(i)}$ a morphism for each $i \in I$, such that the following diagram commutes*

$$
\begin{array}{ccc}
U_i & \xrightarrow{h_i} & V_{\alpha(i)} \\
{\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle g_{\alpha(i)}} \\
U & \xrightarrow[h]{} & V
\end{array}
$$

2. *Given another family* $\mathcal{W} = \{W_m \longrightarrow W\}_{m\in M}$, *and* (k, β, k_l) *a morphism from* $\mathcal{V}$ *to* $\mathcal{W}$ *we define the composition of such morphisms as the system* $(k \circ h, \beta \circ \alpha, k_{\alpha_i} \circ h_i)$. *Then, commutativity of the next diagram is clear*

$$\begin{array}{ccccc} U_i & \xrightarrow{h_i} & V_{\alpha(i)} & \xrightarrow{k_{\alpha(i)}} & W_{\beta(\alpha(i))} \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle g_{\alpha(i)}} & & \downarrow{\scriptstyle k_{\beta(\alpha(i))}} \\ U & \xrightarrow[h]{} & V & \xrightarrow[g]{} & W \end{array}$$

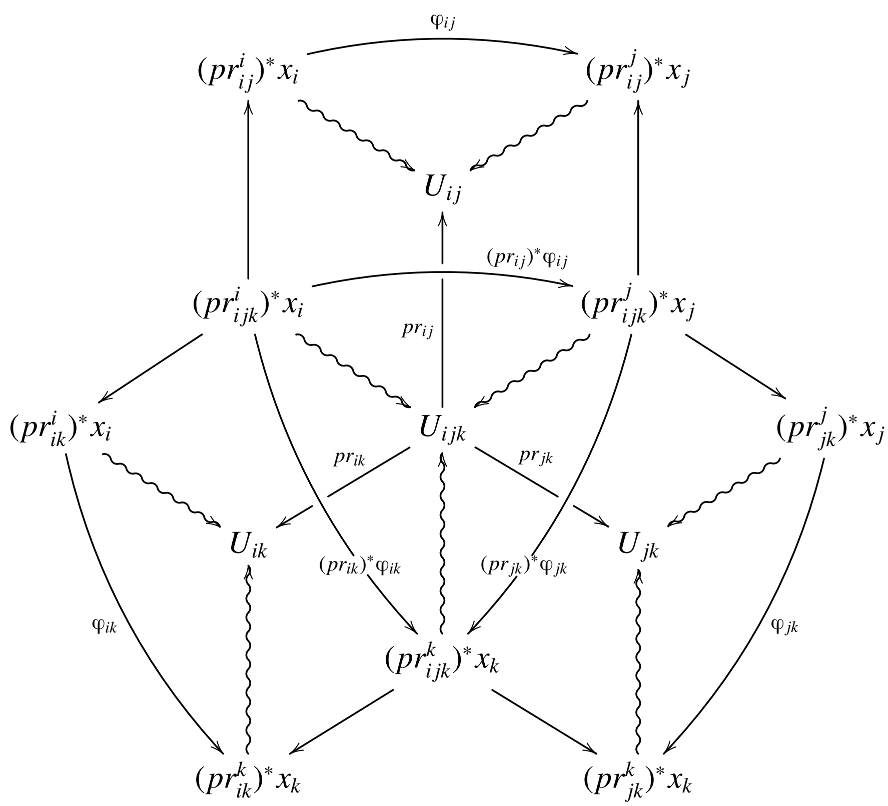
Moreover, there is a canonical morphisms from $\mathcal{U}$ *to* $\mathcal{U}$ *given by* (id_U, id_I, id_{U_i}), *which is the identity of the composition defined before.*

3. *In the case* $V = U$ *and* $U \longrightarrow V$ *the identity, we say* $\mathcal{U}$ *is a refinement of* $\mathcal{V}$.

Assume $\mathcal{C}$ is a category with fibre product and $p : \mathcal{S} \longrightarrow \mathcal{C}$ a fibred category. Let $\mathcal{U} = \{f_i : U_i \longrightarrow U\}_{i\in I}$ be a family of morphisms of $\mathcal{C}$. We will denote $U_{ij} = U_i \times_U U_j$, $U_{ijk} = U_i \times_U U_j \times_U U_k$, $pr^{ij}_{ijk} : U_{ijk} \longrightarrow U_{ij}$ as pr_{ij} and $pr^i_{ij} : U_{ij} \longrightarrow U_i$ as pr_i. Suppose this family is such that for all $i \in I$ exists $x_i \in \mathrm{Ob}(\mathcal{S}_{U_i})$ and for all $(i, j) \in I^2$ exists an isomorphism $\varphi_{ij} : pr_i^* x_i \longrightarrow pr_j^* x_j$ in $\mathcal{S}_{U_{ij}}$. Hence, considering pr_{ij} there exists a unique morphism $pr_{ij}^*\varphi_{ij} : (pr^i_{ijk})^* x_i \longrightarrow (pr^j_{ijk})^* x_j$ such that the following diagram commutes

$$\begin{array}{ccc} (pr^i_{ijk})^* x_i & \xrightarrow{pr_{ij}^*\varphi_{ij}} & (pr^j_{ijk})^* x_j \\ \downarrow & & \downarrow \\ pr_i^* x_i & \xrightarrow[\varphi_{ij}]{} & pr_j^* x_j \end{array}$$

Here we use $pr^i_{ijk} = pr_i \circ pr_{ij}$ and then $(pr^i_{ijk})^* = pr_{ij}^* \circ pr_i^*$, because we are omitting the 2-isomorphisms α_{pr_{ij}, pr_i}, and compatibility of this 2-isomorphism with the commutativity of the diagram above. Therefore consider the next diagram:



If the central triangle is commutative in $\mathcal{S}_{U_{ijk}}$, that is to say, if $pr^*_{jk}\varphi_{jk} \circ pr^*_{ij}\varphi_{ij} = pr^*_{jk}\varphi_{jk}$, we say cocycle condition is satisfied for (f_i, x_i, φ_{ij}).

Definition 2.2.2 (Category of descent data). *(026B) Let $\mathcal{C}$ be a category with fibre product, $p : \mathcal{S} \longrightarrow \mathcal{C}$ a fibred category and $\mathcal{U} = \{f_i : U_i \longrightarrow U\}_{i \in I}$ a family of morphisms in $\mathcal{C}$.*

1. *A descent datum in $\mathcal{S}$ relative a the family $\mathcal{U}$ is a system (x_i, φ_i), where x_i is an object of $\mathcal{S}_{U_i}$ for each $i \in I$, for each pair $(i, j) \in I^2$ a morphisms $\varphi_{ij} : pr^*_i x_i \longrightarrow pr^*_j x_j$ in $\mathcal{S}_{U_{ij}}$,*

such that for every triple $(i, j, k) \in I^3$ *the triangle*

$$\begin{array}{ccc} (pr^i_{ijk})^* x_i & \xrightarrow{pr^*_{ij}\varphi_{ij}} & (pr^j_{ijk})^* x_j \\ & {\scriptstyle pr^*_{ij}\varphi_{ik}} \searrow \quad \swarrow {\scriptstyle pr^*_{jk}\varphi_{jk}} & \\ & (pr^k_{ijk})^* x_k & \end{array}$$

is commutative, i.e, cocycle condition is satisfied for (f_i, x_i, φ_{ij})*.*

2. *A morphism* $\psi : (x_i, \varphi_{ij}) \longrightarrow (x'_i, \varphi'_{ij})$ *of descent data is a family of morphisms* $\psi_i : x_i \longrightarrow x_j$ *in* $\mathcal{S}_{U_i}$ *such that the diagrams:*

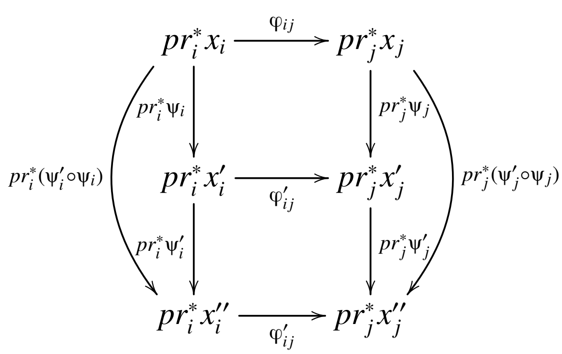
$$\begin{array}{ccc} pr^*_i x_i & \xrightarrow{\varphi_{ij}} & pr^*_j x_j \\ {\scriptstyle pr^*_i\psi_i} \downarrow & & \downarrow {\scriptstyle pr^*_j\psi_j} \\ pr^*_i x'_i & \xrightarrow[\varphi'_{ij}]{} & pr^*_j x'_j \end{array}$$

are commutative in $\mathcal{S}_{U_{ij}}$*.*

Lemma 2.2.1. *Let* $\mathcal{C}$ *be a category with fibre product,* $p : \mathcal{S} \longrightarrow \mathcal{C}$ *a fibred category and* $\mathcal{U} : \{f_i : U_i \longrightarrow U\}_{i \in I}$ *a family of morphisms. Then, descent data and morphisms of descent data defines are a category called category of descent data relative to* $\mathcal{U}$*, which is denoted* $DD(\mathcal{U})$*.*

Proof. If $\psi = (\psi_i) : (x_i, \varphi_{ij}) \longrightarrow (x'_i, \varphi'_{ij})$ and $\psi' = (\psi'_i) : (x'_i . \varphi'_{ij}) \longrightarrow (x''_i, \varphi''_{ij})$ are morphisms of descent data, we define $\psi' \circ \psi = (\psi'_i \circ \psi_i)$. Since pr^*_i is a functor from $\mathcal{S}_{U_i}$ to $S_{U_{ij}}$ then

$pr_i^*(\psi_i' \circ \psi_i) = pr_i^*\psi_i' \circ pr_i^*\psi_i$ and therefore the following diagram commutes



This prove that $\psi' \circ \psi$ is a morphism of descent data. Moreover $(id_{x_i}) : (x_i, \varphi_{ij}) \longrightarrow (x_i, \varphi_{ij})$ is the identity of the descent datum (x_i, φ_{ij}), because of pr_i^* is a functor, then $pr_i^*(id_{x_i}) = id_{pr_i^* x_i}$ thus commutativity of the respective triangle is trivial. □

Remark. Although the main interest in the definition of descent data is when $\mathcal{C}$ is a site and $\{U_i \longrightarrow U\}$ is a covering in the respective Grothendieck topology, the concept makes sense in general situations. For example we can consider the collection of all the families for which every descent datum is effective as we will see later in order to define stacks.

Pullback of descent data

Lemma 2.2.2. *(02XZD) Let $\mathcal{C}$ be a category with fibre product, $p : \mathcal{S} \longrightarrow \mathcal{C}$ a fibred category, $\mathcal{U} = \{f_i : U_i \longrightarrow U\}$ and $\mathcal{V} = \{g_l : V_l \longrightarrow V\}$ families of morphisms in $\mathcal{C}$ and (h, α, h_i) a morphism from $\mathcal{U}$ to $\mathcal{V}$.*

1. *If (Y_l, φ_{lm}) is a descent datum relative to $\mathcal{V}$, then the system*

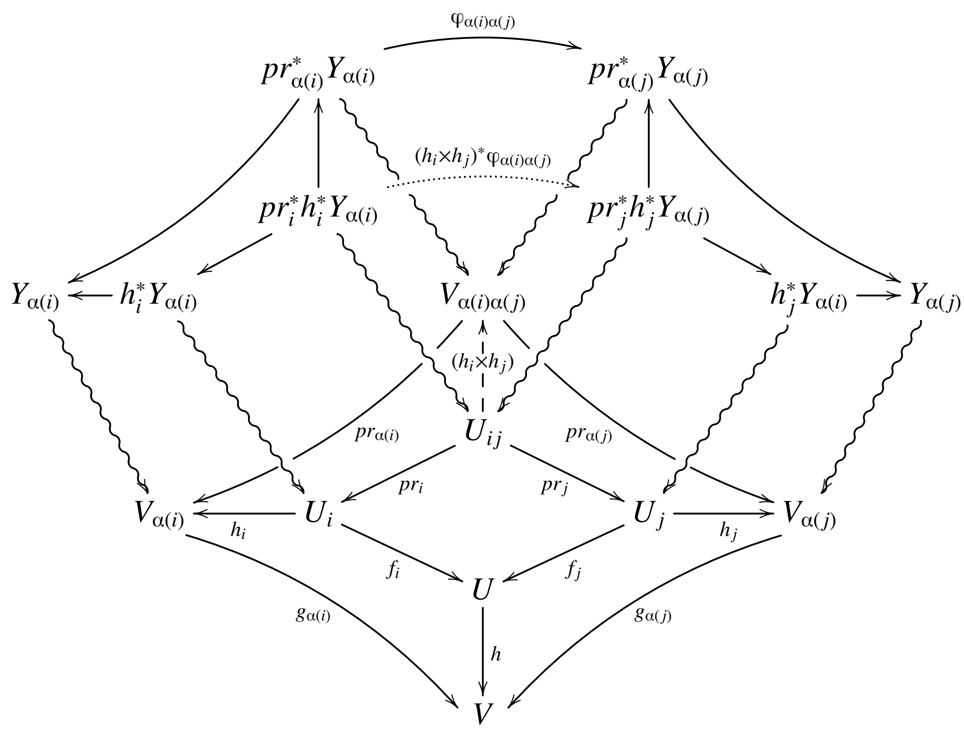
$$(h_i^* Y_{\alpha(i)}, (h_i \times h_j)^* \varphi_{\alpha(i)\alpha(j)})$$

is a descent datum relative to $\mathcal{U}$.

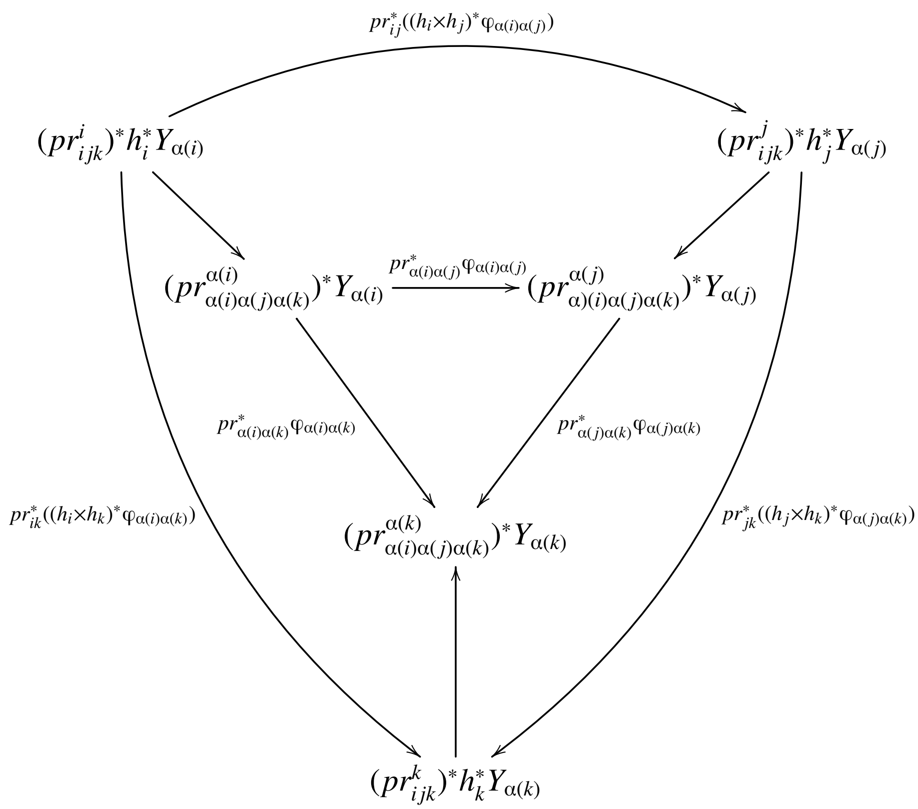
2. *This construction defines a functor* $DD(\mathcal{V}) \longrightarrow DD(\mathcal{U})$.

Proof.

1. Consider the following diagram



The lower rectangles are commutative by definition of morphism of families. Hence $g_{\alpha(i)} \circ h_i \circ pr_i = g_{\alpha(j)} \circ h_j \circ pr_j$ and the universal property of the fibre product $V_{\alpha(i)\alpha(j)}$ implies the existence of a morphism $h_i \times h_j : U_{ij} \longrightarrow V_{\alpha(i)\alpha(j)}$ unique such that $pr_{\alpha(i)} \circ (h_i \times h_j) = h_i \circ pr_i$ and $pr_{\alpha(j)} \circ (h_i \times h_j) = h_j \circ pr_j$. Then $(h_i \times h_j)^* \circ pr^*_{\alpha(i)} Y_{\alpha(i)} = pr^*_i h^*_i Y_{\alpha(i)}$ and $(h_i \times h_j)^* \circ pr^*_{\alpha(j)} Y_{\alpha(j)} = pr^*_j h^*_j Y_{\alpha(j)}$ and therefore $(h_i \times h_j)^* \varphi_{\alpha(i)\alpha(j)}$ is the only morphism such that te upper square commutes. Hence, considering a triple $(i, j, k) \in I^3$, in the next diagram the central triangle is commutative



This is because of the cocycle condition on (Y_l, φ_{lm}). Moreover, in the same way we construct $h_i \times h_j$, there is a morphism $h_i \times h_j \times h_k : U_{ijk} \longrightarrow V_{\alpha(i)\alpha(j)\alpha(k)}$ unique such that $pr_{\alpha(i)\alpha(j)} \circ (h_i \times h_j \times h_k) = (h_i \times h_j) \circ pr_{ij}$. Therefore $(h_i \times h_j \times h_k)^* \circ pr^*_{\alpha(i)\alpha(j)} = pr^*_{ij} \circ (h_i \times h_j)^*$ and so $pr^*_{ij}((h_i \times h_j)^* \varphi_{\alpha(i)\alpha(j)})$ is the only morphism making the upper rectangle commutes. Similarly, the other two rectangles are commutative and for unicity of $pr^*_{ik}((h_i \times h_k)^* \varphi_{\alpha(i)\alpha(k)})$, the external is commutative. Then for each triple $(i, j, k) \in I^3$ cocycle condition is satisfied and so $(h_i^* Y_{\alpha(i)}, (h_i \times h_j)^* \varphi_{\alpha(i)\alpha(j)})$ is a descent datum in $\mathcal{S}$ relative to $\mathcal{U}$.

2. Let (Y'_l, φ'_{lm}) be another descent datum in $\mathcal{S}$ relative to $\mathcal{V}$ and $\sigma = (\sigma_l)_{l \in L} : (Y_l, \varphi_{lm}) \longrightarrow (Y'_l, \varphi'_{lm})$ a morphism of descent data. Then $\sigma_l : Y_l \longrightarrow Y'_l$ is a morphism in $\mathcal{S}_{V_l}$ such

that the next diagram commutes

$$\begin{array}{ccc} pr_l^* Y_l & \xrightarrow{\varphi_{lm}} & pr_m^* Y_m \\ {\scriptstyle pr_l^*\sigma_l}\downarrow & & \downarrow{\scriptstyle pr_m^*\sigma_m} \\ pr_l^* Y_l' & \xrightarrow[\varphi_{lm}]{} & pr_m^* Y_m' \end{array}$$

We shall show that $(h_i^*\sigma_{\alpha(i)})_{i\in I}$ is a morphism of descent data from $(h_i^* Y_{\alpha(i)}, (h_i \times h_j)^*\varphi_{\alpha(i)\alpha(j)})$ to $(h_i^* Y'_{\alpha(i)}, (h_i \times h_j)^*\varphi'_{\alpha(i)\alpha(j)})$. In fact, recall that $pr_i^* h_i^* = (h_i \times h_i)^*(pr_{\alpha(i)}^*$ and so in the following diagram

$$\begin{array}{ccccccc} pr_i^* h_i^* Y_{\alpha(i)} & & & \xrightarrow{(h_i\times h_j)^*\varphi_{\alpha(i)\alpha(j)}} & & & pr_j^* h_j^* Y_{\alpha(j)} \\ & \searrow & & & & \swarrow & \\ & & pr_{\alpha(i)}^* Y_{\alpha(i)} & \xrightarrow{\varphi_{\alpha(i)\alpha(j)}} & pr_{\alpha(j)}^* Y_{\alpha(j)} & & \\ {\scriptstyle pr_i^*(h_i^*\sigma_{\alpha(i)})}\downarrow & & {\scriptstyle pr_{\alpha(i)}^*\sigma_{\alpha(j)}}\downarrow & & \downarrow{\scriptstyle pr_{\alpha(j)}^*\sigma_{\alpha(j)}} & & \downarrow{\scriptstyle pr_j^*(h_j^*\sigma_{\alpha(j)})} \\ & & pr_{\alpha(i)}^* Y'_{\alpha(i)} & \xrightarrow[\varphi'_{\alpha(i)\alpha(j)}]{} & pr_{\alpha(j)}^* Y'_{\alpha(j)} & & \\ & \nearrow & & & & \nwarrow & \\ pr_i^* h_i^* Y_{\alpha(i)} & & & \xrightarrow[(h_i\times h_j)^*\varphi'_{\alpha(i)\alpha(j)}]{} & & & pr_j^* h_j^* Y'_{\alpha(j)} \end{array}$$

the central square commutes because σ is a morphism of descent data in $\mathcal{S}$ relative to $\mathcal{V}$ and as $(h_i \times h_j)^*$ is a functor we have

$$\begin{aligned} pr_j^*(h_j^*\sigma_{\alpha(j)}) \circ (h_i \times h_j)^*(\varphi_{\alpha(i)\alpha(j)}) &= (h_i \times h_j)^*(pr_{\alpha(j)}^*\sigma_{\alpha(j)}) \circ (h_i \times h_j)^*(\varphi_{\alpha(i)\alpha(j)}) \\ &= (h_i \times h_j)^*(pr_{\alpha(j)}^*\sigma_{\alpha(j)} \circ \varphi_{\alpha(i)\alpha(j)}) \\ &= (h_i \times h_j)^*(\varphi'_{\alpha(i)\alpha(j)} \circ pr_{\alpha(j)}^*\sigma_{\alpha(i)}) \\ &= (h_i \times h_j)^*(\varphi'_{\alpha(i)\alpha(j)}) \circ (h_i \times h_j)^*(pr_{\alpha(i)}^*\sigma_{\alpha(i)}) \\ &= (h_i \times h_j)^*(\varphi'_{\alpha(i)\alpha(j)}) \circ pr_i^*(h_i^*\sigma_{\alpha(i)}) \end{aligned}$$

and so we have commutativity of external square, which is what we want. In particular, for $(id_{Y_l})_{l\in L} : (Y_l, \varphi_{lm}) \longrightarrow (Y_l, \varphi_{lm})$ we have $h_i^*(id_{Y_{\alpha(i)}}) = id_{h_i^* Y_{\alpha(i)}}$.

This shows that the asignation $(Y_l, \varphi_{lm}) \longmapsto (h_i^* Y_{\alpha(i)}, (h_i \times h_j)^* \varphi_{\alpha(i)\alpha(j)})$ and $(\sigma_l)_{l\in L} \longmapsto (h_i^* \sigma_{\alpha(i)})_{i\in I}$ is a functor from $DD\mathcal{V}$ to $DD\mathcal{U}$. □

Effective descent data

Lemma 2.2.3. (026E) *Let $\mathcal{C}$ be a category with fibre product, $p : \mathcal{S} \longrightarrow \mathcal{C}$ a fibred category and $\mathcal{U} = \{f_i : U_i \longrightarrow U\}$ a family of morphisms in $\mathcal{C}$*

1. *If x is an object of $\mathcal{S}_U$, then (x, id_x) is a descent datum relative to the family $\{id_U : U \longrightarrow U\}$ called trivial descent datum asociated to x.*

2. *Given an object x of $\mathcal{S}_U$, we have a canonical descent datum relative to the family of objects $f_i^* x$, obtained by changing of base the trivial descent datum (x, id_x) via the obvious morphism of families $\{f_i : U_i \longrightarrow U\} \longrightarrow \{id_U : U \longrightarrow U\}$. We denote this descent datum $(f_i^* x, can_{ij})$.*

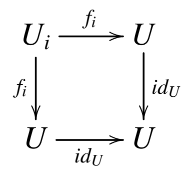
3. *The morphisms $can_{ij} : pr_i^* f_i^* x \longrightarrow pr_j^* f_j^* x$ are equal to $(\alpha_{pr_j, f_j})_x \circ (\alpha_{pr_i, f_i})_x^{-1}$.*

Proof.

1. Clearly (x, id_x) is a descent datum relative to the family $\{id_U : U \longrightarrow U\}$, because as $id_U^* x \cong x$ and $U \times_U U = U$, we can take a choice of pullbacks with $id_U^* x = x$ and so the conditions of descent are trivially satisfied.

2. The morphism of families $\{f_i : U_i \longrightarrow U\} \longrightarrow \{id_U : U \longrightarrow U\}$ is given by (id_U, α, f_i), where $\alpha : I \longrightarrow \{1\}$ is the constant function, being $\{1\}$ the set of indexes of the family

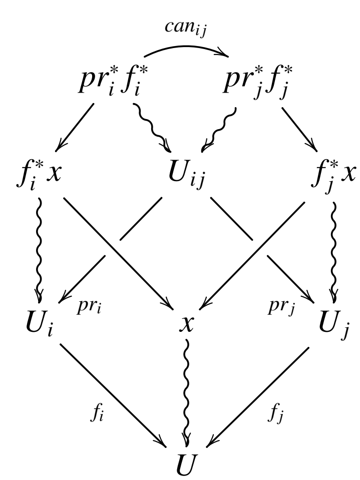
$\{id_U : U \longrightarrow U\}$. In this case the diagram



is obviously commutative. Since (x, id_x) is a descent datum in $\mathcal{S}$ relative to $\{id_U : U \longrightarrow U\}$, then changing the case with the morphism above we have a descent datum in $\mathcal{S}$ relative to $\mathcal{U}$ given by the system

$$(f_i^* x, (f_i \times f_j)^* id_x)$$

This can be summarized in the following diagram



where $can_{ij,x} = (f_i \times f_j)^* id_x$ is the canonical isomorphism given by the equality $f_i \circ pr_i = f_j \circ pr_j$, that is to say, is the only morphism such that the upper face of the cube commutes. Hence $(f_i^* x, can_{ij,x})$ is a descent datum in $\mathcal{S}$ relative to the family $f_i : U_i \longrightarrow U$.

3. We use $f_i \circ pr_i = pr_j \circ f_j$ as morphisms $U_i \times_U U_j \longrightarrow U$. Then in the next diagram

$$\begin{array}{ccccc} & & \overset{can_{ij,x}}{\frown} & & \\ pr_i^* f_i^* x & \xrightarrow[(\alpha_{pr_i,f_i})_x^{-1}]{} & (f_i \circ pr_i)^* x & \xrightarrow[(\alpha_{pr_j,f_j})_x]{} & pr_j^* f_j^* x \\ \downarrow & & \downarrow & & \downarrow \\ f_i^* x & \longrightarrow & x & \longleftarrow & f_j^* x \end{array}$$

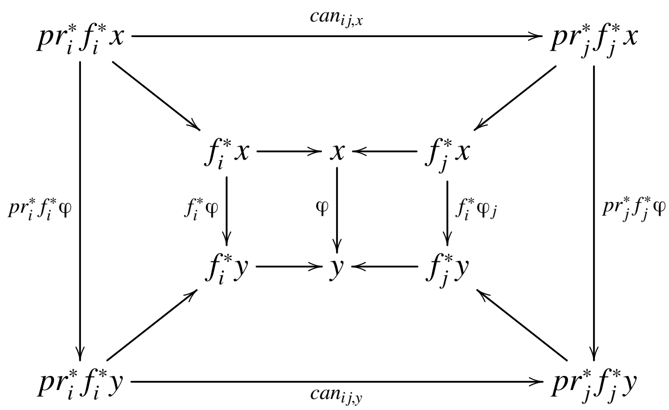
the isomorphisms $(\alpha_{pr_i,f_i})_x$ and $(\alpha_{pr_j,f_j})_x$ are unique such that the lower rectangles are commutatives an so $(\alpha_{pr_j,f_j})_x \circ (\alpha_{pr_i,f_i})_x^{-1}$ is unique such that the compose rectangle is commutative. But for previous numeral $can_{ij,x}$ is the unique with such property and therefore $can_{ij,x} = (\alpha_{pr_j,f_j})_x \circ (\alpha_{pr_i,f_i})_x^{-1}$. □

Definition 2.2.3 (Effective descent datum). (026E) *A descent datum (x_i, φ_{ij}) relative to $\{f_i : U_i \longrightarrow U\}$ is called effective is there exists an object x of $\mathcal{S}_U$ such that (x_i, φ_{ij}) is isomorphic to $(f_i^* x, can_{ij})$.*

Therefore a descent datum (x_i, φ_{ij}) relative to $\{f_i : U_i \longrightarrow U\}$ is effective if there is $x \in \mathrm{Ob}(\mathcal{S}_U)$ and morphisms $\psi_i : x_i \longrightarrow f_i^* x$ such that for each $i \in I$ the following diagram is commutative

$$\begin{array}{ccc} pr_i^* x_i & \xrightarrow{\varphi_{ij}} & pr_j^* x_j \\ {\scriptstyle pr_i^*\psi_i}\downarrow & & \downarrow{\scriptstyle pr_j^*\psi_j} \\ pr_i^* f_i^* x & \xrightarrow[can_{ij}]{} & pr_j^* f_j^* x \end{array}$$

Now, given a morphism $x \xrightarrow{\varphi} y$ en $\mathcal{S}_U$, we shall see that $(f_i^* \varphi)_{i \in I}$ is a morphism of descent data $(f_i^* x, can_{ij,x}) \longrightarrow (f_i^* y, can_{ij,y})$. For this we must prove that for every $(i, j) \in I^2$ the external rectangle in the next diagram is commutative



$can_{ij,x}$ is the only morphism such that $pr_i^* f_i^* x \overset{can_{ij,x}}{\longrightarrow} pr_j^* f_j^* x \longrightarrow f_j^* x \longrightarrow x = pr_i^* f_i^* x \longrightarrow f_i^* x \longrightarrow x$ an so in the previous diagram all the sub rectangles are commutative. Consecuently we have the following equality of diagrams

$$\begin{array}{ccccccccccccc}
pr_i^* f_i^* x & \xrightarrow{can_{ij,x}} & pr_j^* f_j^* x & \longrightarrow & f_j^* x & \longrightarrow & x & = & pr_i^* f_i^* x & \longrightarrow & f_i^* x & \longrightarrow & x \\
{\scriptstyle pr_i^* f_i^* \varphi}\downarrow & & {\scriptstyle pr_j^* f_j^* \varphi}\downarrow & & {\scriptstyle f_j^* \varphi}\downarrow & & \downarrow & & {\scriptstyle pr_i^* f_i^* \varphi}\downarrow & & {\scriptstyle f_i^* \varphi}\downarrow & & \downarrow \\
pr_i^* f_i^* y & \xrightarrow[can_{ij,y}]{} & pr_j^* f_j^* y & \longrightarrow & f_j^* y & \longrightarrow & y & = & pr_i^* f_i^* y & \longrightarrow & f_i^* y & \longrightarrow & y
\end{array}$$

On the right side both rectangles commutes and so $pr_i^* f_i^* x \overset{can_{ij,x}}{\longrightarrow} pr_j^* f_j^* x \overset{pr_j^* f_j^* \varphi}{\longrightarrow} pr_j^* f_j^* y$ and $pr_i^* f_i^* x \overset{pr_i^* f_i^* \varphi}{\longrightarrow} pr_i^* f_i^* y \overset{can_{ij,y}}{\longrightarrow} pr_j^* f_j^* y$ are morphisms which composed with $pr_j^* f_j^* y \longrightarrow f_j^* y \longrightarrow y$ are equal, and since the last one is strongly cartesian, being the composition of strongly cartesian morphisms, thus are equals y we have the required commutativity.

Now, if $y \overset{\psi}{\longrightarrow} z$ is another morphism in $\mathcal{S}_U$ and $(f_i^* \psi)_{i \in I}$ is the canonical morphism of descent data obtained as before, then $(f_i^*(\psi \circ \varphi))_{i \in I} = (f_i^* \psi \circ f_i^* \varphi)_{i \in I} = (f_i^* \psi)_{i \in I} \circ (f_i^* \varphi)_{i \in I}$ turns to be a morphism of canonical descent data. Also, considering $x \overset{id_x}{\longrightarrow} x$, by construction $(id_{f_i^* x})_{i \in I}$ is an endomorphism of the canonical descent datum $(f_i^* x, can_{ij})$ and is indeed the identity morphism. Thereby, given a family $\mathcal{U} = \{f_i : U_i \longrightarrow U\}$ of morphisms in $\mathcal{C}$ we have a functor

$$\mathcal{S}_U \longrightarrow DD(\mathcal{U})$$

Therefore, a descent datum is effective if this is in the essential image of that functor.

2.3 Stacks

Definition 2.3.1 (Stack). *(026F) Let $\mathcal{C}$ be a site. A stack over $\mathcal{C}$ is a fibred category $p : \mathcal{S} \longrightarrow \mathcal{C}$ satisfying the following conditions:*

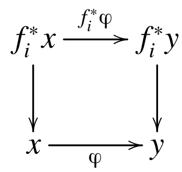
1. *For any $U \in \mathrm{Ob}(\mathcal{C})$ and every $x, y \in \mathcal{S}_U$, the functor $\mathrm{Mor}(x, y)$ is a sheaf in the site $\mathcal{C}/U$.*
2. *For each covering $\mathcal{U} = \{f_i : U_i \longrightarrow U\}_{i \in I}$ of the site $\mathcal{C}$, every descent datum in $\mathcal{S}$ relative to $\mathcal{U}$ is effective.*

Lemma 2.3.1. *(02ZF) Assume $\mathcal{C}$ is a site and $p : \mathcal{S} \longrightarrow \mathcal{C}$ a fibred category. The follow conditions are equivalent:*

1. *$\mathcal{S}$ is a stack over $\mathcal{C}$.*
2. *For any covering $\mathcal{U} = \{f_i : U_i \longrightarrow U\}_{i \in I}$ of $\mathcal{C}$, the functor $\mathcal{S}_U \longrightarrow DD(\mathcal{U})$ which associate to each object its canonical descent datum relative to $\mathcal{U}$ is an equivalence of categories.*

Proof.

Let $\mathcal{U} = \{f_i : U_i \longrightarrow U\}_{i \in I} \in \mathbf{Cov}(\tau)$. The functor $\mathcal{S}_U \longrightarrow DD(\mathcal{U})$ is defined for $x \in \mathrm{Ob}(\mathcal{S}_U)$ as the descent datum $(f_i^* x, can_{ij,x})$ and for $x \xrightarrow{\varphi} y$ a morphism in $\mathcal{S}_U$ as the morphism of descent data $(f_i^*\varphi)_{i\in I}$, where $f_i^*\varphi : f_i^*x \longrightarrow f_i^*y$ is the only morphism such that the following diagram commutes



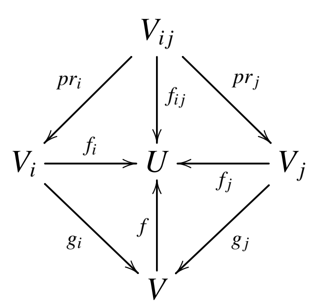
Therefore $\mathcal{S}_U \longrightarrow DD(\mathcal{U})$ is essentially surjective if and only if fiven (X_i, φ_{ij}) in $DD(\mathcal{U})$ there exists $x \in \mathrm{Ob}(\mathcal{S}_U)$ such that $(f_i^*x, can_{ij,x}) \cong (X_i, \varphi_{ij})$, that is to say every descent datum in $\mathcal{S}$ relative to $\mathcal{U}$ is effective.

Now, $\mathcal{S}_U \longrightarrow DD(\mathcal{U})$ is fully faithful if given $x, y \in \mathrm{Ob}(S_U)$, the canonical function

$$\xi : \mathrm{Mor}_{\mathcal{S}_V}(x, y) \longrightarrow \mathrm{Mor}_{DD(\mathcal{U})}((f_i^*x, can_{ij,x}), (f_i^*y, can_{ij,y}))$$

is bijective. This function is defined as follows: given $x \overset{\varphi}{\longrightarrow} y$, there is a morphism $\psi = (f_i^*\varphi)_{i\in I}$ from $(f_i^*x, can_{ij,x})$ to $(f_i^*y, can_{ij,y})$ and we define $\xi(\varphi) = \psi$.

If $\mathcal{C}$ is a category with fibre product and τ is a Grothendieck topology on $\mathcal{C}$, Then for $U \in \mathrm{Ob}(\mathcal{C})$, the category $\mathcal{C}/U$ has fibre product and there is a induced topology on $\mathcal{C}/U$ denoted τ/U. The fibre product in $\mathcal{C}/U$ is defined according to the next diagram:



More precisely $f_i \times_f f_j = f_{ij}$, where $f_{ij} := f_i \circ pr_i = f_j \circ pr_j$, being the last two equal because of $V_{ij} = V_i \times_V V_j$ and so the external square commutes and since g_i, g_j are morphisms of $\mathcal{C}/U$ the internal triangles are commutatives. We define $\mathbf{Cov}(\tau/U)$ as the families $\{g_i : V_i \longrightarrow V\}$

in $\mathbf{Cov}(\tau)$ wich$V_i, V \in \mathrm{Ob}(\mathcal{S}_U)$ and $g_i \in \mathrm{Mor}_{C/U}(V_i, V)$. Then it makes sense that $\mathrm{Mor}(x, y)$ to be a sheaf when $\mathcal{C}_\tau$ is a site.

In general, given a functor $F : \mathcal{C}^{op} \longrightarrow \mathcal{D}$ in a site $\mathcal{C}_\tau$ and $\mathcal{D}$ a category with arbitrary products, F is a sheaf provided that for all $\{V_i \xrightarrow{g_i} V\} \in \mathbf{Cov}(\tau)$ the following diagram is exact:

$$F(V) \xrightarrow{\pi} \prod_{i\in I} F(V_i) \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} \prod_{(i,j)\in I^2} F(V_i \times_V V_j)$$

In order to construct the morphisms π, π_1, π_2, consider the following: for each $i \in I$, $F(V_i) \in \mathrm{Ob}(\mathcal{D})$ and as $\mathcal{D}$ has arbitrary products, there is $\prod_{i\in I} F(V_i)$. Now $g_i : V_i \longrightarrow U$ and so $F(g_i) : F(V) \longrightarrow F(V_i)$ is a morphism in $\mathcal{D}$ for any $i \in I$. By the universal property of fibre product there is $\pi = \prod_{i\in I} F(g_i) : F(V) \longrightarrow \prod_{i\in I} F(V_i)$ unique such that the diagram

$$\begin{array}{ccc} F(V) & & \\ {\scriptstyle \prod F(g_i)} \downarrow & \searrow^{F(g_i)} & \\ \prod_{i\in I} F(V_i) & \xrightarrow[\pi_i]{} & F(V_i) \end{array}$$

is commutative. Given $(i, j) \in I^2$ we have a fibre product

$$\begin{array}{ccc} V_i \times_V V_j & \xrightarrow{pr^j_{ij}} & V_j \\ {\scriptstyle pr^i_{ij}} \downarrow & & \downarrow {\scriptstyle g_j} \\ V_i & \xrightarrow[g_i]{} & U \end{array}$$

Hence there is a morphism $F(V_i) \xrightarrow{F(pr^i_{ij})} F(V_i \times_V V_j)$ an so $\prod_{i\in I} F(V_i) \xrightarrow{\pi_i} F(V_i) \xrightarrow{F(pr^i_{ij})} F(V_i \times_V V_j)$. As before the universal property of the fibre product leads the existence of a unique morphism

π_1 such that the diagram

$$\begin{array}{ccc}
\prod_{i\in I} F(V_i) & \xrightarrow{\pi_i} & F(V_i) \\
{\scriptstyle \pi_1}\big\downarrow & & \big\downarrow {\scriptstyle F(pr^i_{ij})} \\
\prod_{(i,j)\in I^2} F(V_i \times_V V_j) & \xrightarrow[\pi_{ij}]{} & F(V_i \times_V V_j)
\end{array}$$

is commutative. Hence $\pi_{ij} \circ \pi_1 = F(pr^i_{ij}) \circ \pi_i$. In the same way there is π_2 unique such that $\pi_{ij} \circ \pi_2 = F(pr^j_{ij}) \circ \pi_j$. Therefore we have $\pi_1 = \prod_{(i,j)} F(pr^i_{ij}) \circ \pi_i$ and $\pi_2 = \prod_{(i,j)} F(pr^i_{ij}) \circ \pi_j$.

In the case of interest $F = \mathrm{Mor}(x, y)$, where where given $V \xrightarrow{f} U \in \mathrm{Ob}(\mathcal{C}/U)$, $\mathrm{Mor}(x, y)(f) := \mathrm{Hom}_{\mathcal{S}_V}(f^*x, f^*y)$ and $V' \xrightarrow{g} V \in \mathrm{Hom}_{\mathcal{C}/U}(f, f')$, $\mathrm{Mor}(x, y)(g)$ is the function defined as:

$$\begin{array}{rcl}
\mathrm{Hom}_{\mathcal{S}_V}(f^*x, f^*y) & \longrightarrow & \mathrm{Hom}_{\mathcal{S}_{V'}}(f'^*x, f'^*y) \\
\varphi & \longmapsto & (\alpha_{g,f})^{-1}_y \circ g^*\varphi \circ (\alpha_{g,f})_x
\end{array}$$

Then $\mathrm{Mor}(x, y)$ is a sheaf if and only if given $\{g_i : V_i \longrightarrow V\} \in \mathbf{Cov}(\tau/U)$ the following diagram is exact

$$\mathrm{Mor}(x, y)(f) \xrightarrow{\pi} \prod_{i\in I} \mathrm{Mor}(x, y)(f_i) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \prod_{(i,j)\in I^2} \mathrm{Mor}(x, y)(f_{ij})$$

where $\pi = \prod_{i\in I} \mathrm{Mor}(x, y)(g_i)$, $\pi_1 = \prod_{(i,j)\in I^2} \mathrm{Mor}(x, y)(pr^i_{ij}) \circ \pi_i$ and $\pi_2 = \prod_{(i,j)\in I^2} \mathrm{Mor}(x, y)(pr^j_{ij}) \circ \pi_j$.
Therefore $\mathrm{Mor}(x, y)$ is a sheaf if the next diagram is exact

$$\mathrm{Hom}_{\mathcal{S}_V}(f^*x, f^*y) \xrightarrow{\pi} \prod_{i\in I} \mathrm{Hom}_{\mathcal{S}_{V_i}}(f_i^*x, f_i^*y) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \prod_{(i,j)\in I^2} \mathrm{Hom}_{\mathcal{S}_{V_{ij}}}(pr_i^* f_i^* x, pr_i^* f_i^* y)$$

If $\psi = (\psi_i)_{i\in I} \in \prod_{i\in I} \mathrm{Hom}_{\mathcal{S}_{V_i}}(f_i^*x, f_i^*y)$ is such that $\pi_1(\psi) = \pi_2(\psi)$, there exists a unique $x \xrightarrow{\varphi} y \in \mathrm{Hom}_{\mathcal{S}_V}(x, y)$ such that $\pi(\varphi) = \psi$. Now, $\pi_1(\psi) = \prod_{(i,j)\in I^2} (\mathrm{Mor}(x, y)(pr_i) \circ \pi_i)(\psi) =$

$\prod_{(i,j)\in I^2} \mathrm{Mor}(x,y)(pr_i)(\psi_i) = \prod_{(i,j)\in I^2} (\alpha_{pr_i,f_i})_y^{-1} \circ pr_i^*\psi_i \circ (\alpha_{pr_i,f_i})_x$. Then $\pi_1(\psi) = \pi_2(\psi)$ if and only if por each $(i,j) \in I^2$ we have

$$(\alpha_{pr_i,f_i})_y^{-1} \circ pr_i^*\psi_i \circ (\alpha_{pr_i,f_i})_x = (\alpha_{pr_j,f_j})_y^{-1} \circ pr_j^*\psi_j \circ (\alpha_{pr_j,f_j})_x$$

On the other hand, an element of $\mathrm{Hom}_{DD(\mathcal{U})}((f_i^*x, can_{ij,x}), (f_i^*y, can_{ij,y}))$ is a family $\psi = (\psi_i)_{i\in I}$, where $\psi_i : f_i^*x \longrightarrow f_i^*y$ are morphisms in $\mathcal{S}_{V_i}$ such that the diagram

$$\begin{array}{ccc} pr_i^*f_i^*x & \xrightarrow{can_{ij,x}} & pr_j^*f_j^*x \\ {\scriptstyle pr_i^*\psi_i}\big\downarrow & & \big\downarrow{\scriptstyle pr_j^*\psi_j} \\ pr_i^*f_i^*y & \xrightarrow[can_{ij,y}]{} & pr_j^*f_j^*y \end{array}$$

is commutative in $\mathcal{S}_{V_{ij}}$, so $can_{ij,y} \circ pr_i^*\psi_i = pr_j^*\psi_j \circ can_{ij,x}$. Since $can_{ij,x} = (\alpha_{pr_j,f_j})_x \circ (\alpha_{pr_i,f_i})_x^{-1}$, it follows $(\alpha_{pr_j,f_j})_y \circ (\alpha_{pr_i,f_i})_y^{-1} \circ pr_i^*\psi_i = pr_j^*\psi_j \circ (\alpha_{pr_j,f_j})_x \circ (\alpha_{pr_i,f_i})_x^{-1}$. Hence $\mathrm{Hom}_{DD(\mathcal{U})}((f_i^*x, can_{ij,x}), (f_i^*y, can_{ij,y}))$ is equal to the set of $\psi = (\psi_i)_{i\in I} \in \prod_{i\in I} \mathrm{Hom}_{\mathcal{S}_{V_i}}(f_i^*x, f_i^*y)$ such that $\pi_1(\psi) = \pi_2(\psi)$.

Given $\psi = (\psi_i)_{i\in I}$, if exists $\varphi \in \mathrm{Hom}_{\mathcal{S}_U}(x,y)$ satisfying $\xi(\varphi) = \psi$, then $\pi(f^*\varphi) := \left(\prod_{i\in I} g_i^*\right)(f^*\varphi) = \prod_{i\in I}(g_i^*f^*\varphi) \approx (f_i^*\varphi)_{i\in I} =: \alpha(\varphi)$ and so $f^*\varphi$ is the unique such that $\pi(f^*\varphi) = \psi$. Reciprocally, given $\psi = (\psi_i)_{i\in I}$ such that $\pi_1(\psi) = \pi_2(\psi)$, there exists $f^*x \xrightarrow{\rho} f^*y$ such that $\pi(\rho) = \psi$, that is to say $g_i^*\rho = \psi_i$, for all $i \in I$. Then there exists $\varphi : x \longrightarrow y$ with $\rho = f^*\varphi$.

This shows that π is the equalizer of π_1 and π_2 if and only if α is bijective. In other words, $\mathrm{Mor}(x,y)$ is a sheaf if and only if $\mathcal{S}_U \longrightarrow DD(\mathcal{U})$ is fully faithful.

Summarizing $p : \mathcal{S} \longrightarrow \mathcal{C}$ is a stack if and only if for every covering $\mathcal{U} = \{f_i : V_i \longrightarrow V\}$ of $\mathcal{C}$, the functor $\mathcal{S}_U \longrightarrow DD(\mathcal{U})$ is fully faithful and essentially surjective, in other words $\mathcal{S}_U \longrightarrow DD(\mathcal{U})$ is an equivalence of categories. □

Now we are going to give a result which implies that any scheme or algebraic space can be seen as a stack.

Proposition 2.3.1. *Let $\mathcal{C}$ be a site. If $F : \mathcal{C}^{op} \longrightarrow$ **Sets** is a sheaf, then the fibred category in sets $p_F : \mathcal{S}_F \longrightarrow \mathcal{C}$ is a stack over $\mathcal{C}$.*

Proof. If $F : \mathcal{C}^{op} \longrightarrow$ **Sets** is a functor, then $\mathcal{S}_F$ is the category fibred in sets with:

$$\mathrm{Ob}(\mathcal{S}_F) := \{(U, x) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in F(U)\}$$

where we have used that $F(U)$ is a set, that is to say, a discrete category. If (V, y) and (U, x) are in $\mathrm{Ob}(\mathcal{S}_F)$ we have

$$\mathrm{Hom}_{F(V)}((V, y), (U, x)) := \{f \in \mathrm{Hom}_{\mathcal{C}}(V, U) \mid F(f)(x) = y\}$$

More precisely, this is the set of pairs (f, a) where $f \in \mathrm{Hom}_{\mathcal{C}}(V, U)$ and $a \in \mathrm{Hom}_{F(V)}(y, F(f)(x))$, but in this case $F(V)$ is a discrete category and this means $y = F(f)(x)$ and a is the identity map. The functor $p_F : \mathcal{S}_F \longrightarrow \mathcal{C}$ is given by $(U, x) \longmapsto U$ and $f \longmapsto f$.

Let $\mathcal{U} = \{f_i : U_i \longrightarrow U\}_{i \in I}$ be a covering in $\mathcal{C}$. We want to show the functor $\mathcal{S}_{F,U} \longrightarrow DD(\mathcal{U})$ is an equivalence of categories. If $(x_i, \varphi_{ij}) \in DD(\mathcal{U})$, then $x_i \in \mathcal{S}_{F,U_i} = F(U_i)$ and $\varphi_{ij} : (pr^i_{ij})^* x_i \longrightarrow (pr^j_{ij})^* x_j$ is an isomorphism for every $i, j \in I$. Since $F(U_i \times_U U_j)$ is discrete φ_{ij} must be the identity. The cocycle conditions are always satisfied and therefore are not required. Hence a descent datum is a collection $(x_i)_{i \in I}$ such that $x_i \in F(U_i)$ and $(pr^i_{ij})^* x_i = (pr^j_{ij})^* x_j$ for all $i, j \in I$. On the other hand if $x \in \mathcal{S}_{F,U} = F(U)$, the canonical descent datum relative to $\mathcal{U}$ is $(f_i^* x, can_{ij,x})$, where $can_{ij,x} = (\alpha_{pr_j, f_j})_x \circ (\alpha_{pr_i, f_i})_x^{-1} = id_{(f_i \times f_j)^* x}$, because we are in the case the pullback is compatible with composition. Now, since F is a sheaf the following diagram is exact

$$F(U) \xrightarrow{\pi} \prod_{i \in I} F(U_i) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \prod_{(i,j) \in I^2} F(U_i \times_U U_j)$$

This says that if $(x_i)_{i\in I}$ is such that $(pr^i_{ij})^* x_i = (pr^j_{ij})^* x_j$ for all $i, j \in I$, then there exists a unique $x \in F(U)$ such that $F(f_i)(x) = x_i$. Therefore, if F is a sheaf, then $\mathcal{S}_{F,U} \longrightarrow DD(\mathcal{U})$ is an equivalence of categories and so $p_f : \mathcal{S}_F \longrightarrow \mathcal{C}$ is a stack. □

Corollary 2.1. *If $\mathcal{C}$ is a site where every representable functor is a sheaf, then for any $X \in \mathrm{Ob}(\mathcal{C})$ the category fibred in sets $p_X : \mathcal{S}_X \longrightarrow \mathcal{C}$ is a stack over $\mathcal{C}$.*

Lemma 2.3.2. *(042W) Let $\mathcal{C}$ be a site and $\mathcal{S}_1$ and $\mathcal{S}_2$ categories over $\mathcal{C}$ which are equivalents as categories over $\mathcal{C}$. Then $\mathcal{S}_1$ is a stack over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is a stack over $\mathcal{C}$.*

2.4 Stackification of fibred categories

The statement of the following proposition is different from the correspondent in Stacks Project. The principal ideas of the proof are inspired in the ones exposed there, but some definitions have been changed in order to make more convenient the subsequent constructions. Also, although they are not presented here, we wrote all the details, since they are necessary in the functoriality properties that we have found.

Lemma 2.4.1. *(02ZN) Let $\mathcal{C}$ be a site and $p : \mathcal{S} \longrightarrow \mathcal{C}$ a fibred category over $\mathcal{C}$. There exists a stack $p' : \mathcal{S}' \longrightarrow \mathcal{C}$ and a 1-morphism $G : \mathcal{S} \longrightarrow \mathcal{S}'$ of fibred categories over $\mathcal{C}$ satisfying the following universal property: For any stack $q : \mathcal{X} \longrightarrow \mathcal{C}$ and every 1-morphism $H : \mathcal{S} \longrightarrow \mathcal{X}$ of fibred categories over $\mathcal{C}$, there exists a 1-morphism $H' : \mathcal{S}' \longrightarrow \mathcal{X}$ of stacks over $\mathcal{C}$ such that the following diagram is 2-commutative*

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\quad G \quad} & \mathcal{S}' \\ & {\scriptstyle H}\searrow \quad \swarrow{\scriptstyle H'} & \\ & \mathcal{X} & \end{array}$$

Proof.

We will construct the stack $\mathcal{S}'$ in three stages.

1. Locally equal morphisms: Given $x \in \mathrm{Ob}(S)$ over U and $y \in \mathrm{Ob}(\mathcal{S})$, we say that two morphisms $a, b : x \longrightarrow y$ of $\mathcal{S}$ such that $p(a) = p(b)$ are locally equal if there is a covering $\{f_i : U_i \longrightarrow U\}$ of $\mathcal{C}$ such that $f_i^*x \longrightarrow x \xrightarrow{a} y = f_i^*x \longrightarrow x \xrightarrow{b} y$. We shall see this define an equivalence relation. Clearly $a \sim a$ and $a \sim b$ implies $b \sim a$. Now, if $a \sim b$ and $b \sim c$, there are coverings $\{f_i : U_i \longrightarrow U\}_I$ and $\{g_j : V_j \longrightarrow U\}_J$ such that $f_i^*x \longrightarrow x \xrightarrow{a} y = f_i^*x \longrightarrow x \xrightarrow{b} y$ and $g_j^*x \longrightarrow x \xrightarrow{b} y = g_j^*x \longrightarrow x \xrightarrow{c} y$. Consider the covering $\{g_j \circ f_i' = f_i \circ g_j' : U_i \times_U V_j \longrightarrow U\}$ pictured in the diagram

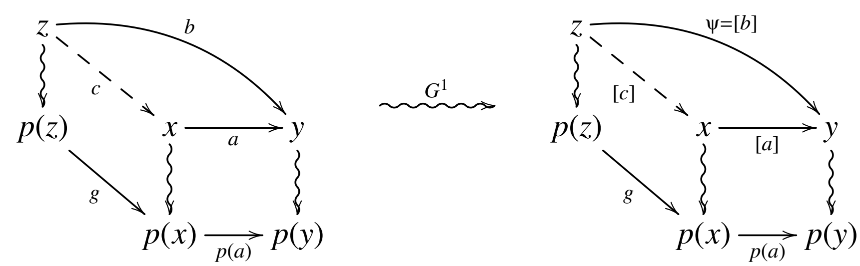
$$\begin{array}{ccc} U_i \times_U V_j & \xrightarrow{f_i'} & V_j \\ {\scriptstyle g_j'}\downarrow & & \downarrow{\scriptstyle g_j} \\ U_i & \xrightarrow[f_i]{} & U \end{array}$$

Then we have

$$\begin{aligned} (f_i \circ g_j')^*x \longrightarrow f_i^*x \longrightarrow x \xrightarrow{a} y &= (f_i \circ g_j')^*x \longrightarrow f_i^*x \longrightarrow x \xrightarrow{b} y \\ &= (g_j \circ f_i')^*x \longrightarrow g_j^*x \longrightarrow x \xrightarrow{b} y \\ &= (g_j \circ f_i')^*x \longrightarrow g_j^*x \longrightarrow x \xrightarrow{c} y \end{aligned}$$

That is to say, $(f_i \circ g_j')^*x \longrightarrow x \xrightarrow{a} y = (f_i \circ g_j')^*x \longrightarrow x \xrightarrow{c} y$ and so $a \sim c$. Hence $\sim$ is an equivalence relation and we can consider $\mathcal{S}/\sim$ in order to obtain a new category $\mathcal{S}^1$ identifying locally equal morphisms. We use the following fact: if $a, b : x \longrightarrow y$ and $c, d : y \longrightarrow z$ are locally equal, then $c \circ a, d \circ b : x \longrightarrow z$ are locally equal and therefore the composition of equivalence classes is well defined. Since $p(a) = p(b)$ when $a \sim b$, we also have a functor $p^1 : \mathcal{S}^1 \longrightarrow \mathcal{C}$ which is a fibred category over $\mathcal{C}$. Moreover we can define a 1-morphism $G^1 : \mathcal{S} \longrightarrow \mathcal{S}^1$ of fibred categories over $\mathcal{C}$ defined as the identity in the objects and for a morphism $a : x \longrightarrow y$, we denote $[a]$ the equivalence class of a in the previous relation and we make $G^1(a) := [a]$. Let $p^1 : \mathcal{S}^1 \longrightarrow \mathcal{C}$ defined by $p^1(x) = p(x)$ in the objects and

$p^1([a]) = p(a)$ in the morphisms. Note that the definition in the morphisms does not depends on the representative, because $a \sim b$ implies $p(a) = p(b)$. Then is clear that $p^1 : \mathcal{S}^1 \longrightarrow \mathcal{C}$ is a category over $\mathcal{C}$ and $G^1 : \mathcal{S} \longrightarrow \mathcal{S}^1$ is a morphisms of categories over $\mathcal{C}$. We shall see that G^1 preserves strongly cartesian morphisms and therefore S^1 is a fibred category. Let $a : x \longrightarrow y$ a strongly cartesian morphism in $\mathcal{S}$, we want to show that $[a] : x \longrightarrow y$ is strongly cartesian in $\mathcal{S}^1$.



Given $z \in \mathrm{Ob}(\mathcal{S})$ and a pair (ψ, g) where $\psi : z \longrightarrow y$ a morphism in $\mathcal{S}^1$ and $g : p(z) \longrightarrow p(x)$ such that $p^1(\psi) = p(a) \circ g$, we have $\psi = [b]$ for some morphism $b : z \longrightarrow y$ in $\mathcal{S}$. Hence $p^1(\psi) = p(b)$ and so $p(b) = p(a) \circ g$ in $\mathcal{S}$. Since $a : x \longrightarrow y$ is strongly cartesian, there exists a unique $c : z \longrightarrow x$ in $\mathcal{S}$ such that $p(c) = g$ and $a \circ c = b$. Then $[c] : z \longrightarrow x$ is a morphism in $\mathcal{S}^1$ such that $p^1([c]) = p(c) = g$ and $[a] \circ [c] = [a \circ c] = [b] = \psi$. Moreover, $[c]$ is unique with that property, because if $\gamma : z \longrightarrow x$ is a morphism in $\mathcal{S}^1$ such that $[a] \circ \gamma = \psi$, then for any representative $d : z \longrightarrow x$ of γ in $\mathcal{S}$ we have $[a \circ c] = [b] = [a] \circ [d] = [a \circ d]$, that is to say, there is a covering $\{f_i : U_i \longrightarrow p(z)\}$ satisfying $f_i^* z \longrightarrow z \xrightarrow{c} x \xrightarrow{a} y = f_i^* z \longrightarrow z \xrightarrow{d} x \xrightarrow{a} y$. But $a : x \longrightarrow y$ is strongly cartesian and therefore $f_i^* z \longrightarrow z \xrightarrow{c} x = f_i^* z \longrightarrow z \xrightarrow{d} x$ and this means that $[c] = [d]$. Hence $[a]$ is strongly cartesian. Since the objects are the same, this prove both $\mathcal{S}^1$ is a fibred category over $\mathcal{C}$ and the functor G^1 is a 1-morphism of fibred categories over $\mathcal{C}$.

Then without lost of generality we can assume $\mathcal{S} = \mathcal{S}^1$.

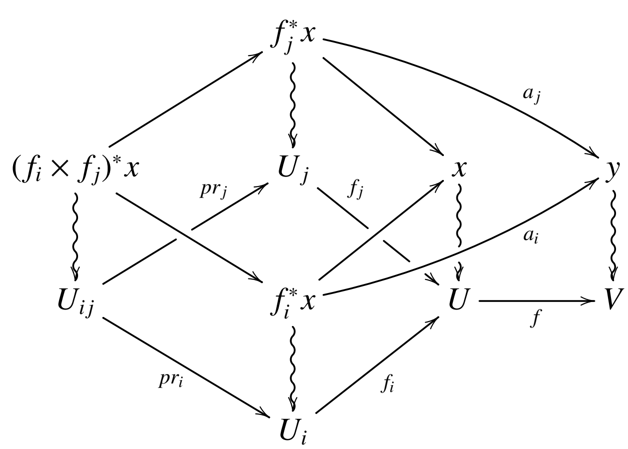
2. Locally defined morphisms: We will construct the category of locally defined morphisms associated to $\mathcal{S}$ by adding, if necessary, more morphisms as follows: Given x over U and y over V a locally defined morphism from x to y is a system $(f, \{f_i\}, a_i)$ where:

1. $f : U \longrightarrow V$ is a morphism of $\mathcal{C}$.

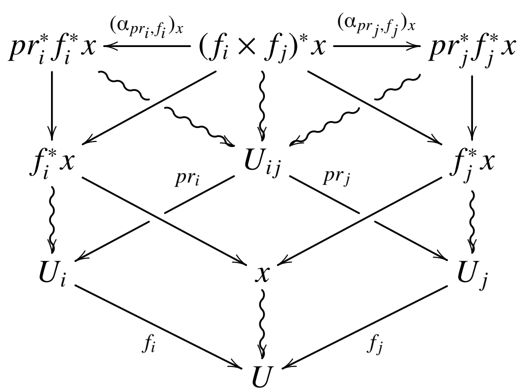
2. $\{f_i : U_i \longrightarrow U\}_{i \in I}$ is a covering of the topology on $\mathcal{C}$.

3. For every $i \in I$, $a_i : f_i^* x \longrightarrow y$ is a morphism such that $p(a_i) = f \circ f_i$. The compositions $(f_i \times f_j)^* x \longrightarrow f_i^* x \xrightarrow{a_i} y$ and $(f_i \times f_j)^* x \longrightarrow f_j^* x \xrightarrow{a_j} y$ are equal. Here $f_i \times f_j$ denotes both $f_i \circ pr_i$ and $f_j \circ pr_j$ which are equal.

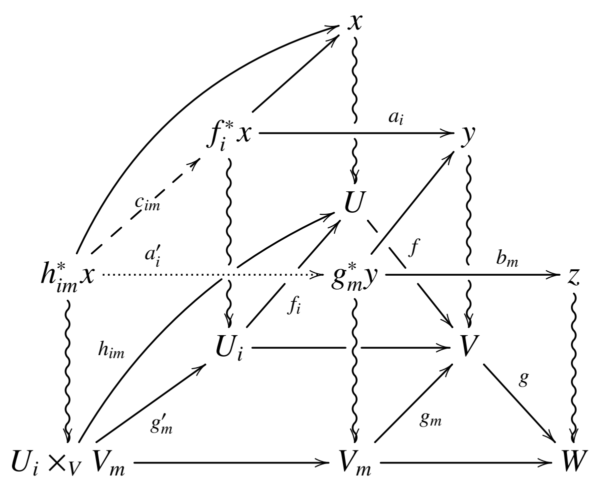
We can see it better in the next diagram:



Here we are using the 2-isomorphisms α_{pr_i, f_i} and α_{pr_j, f_j}. In rder to be more clear, consider the diagram



Then both $(f_i \times f_j)^* x \longrightarrow f_i^* x$ and $(f_i \times f_j)^* x \longrightarrow f_j^* x$ are strongly cartesian lifts of pr_i and pr_j respectively, and so the upper square is cartesian. A morphism $a : x \longrightarrow y$ in $\mathcal{S}$ determines a locally defined morphism $(p(a), \{id_{p(x)}\}, a)$. We say the locally defined morphisms $(f, \{f_i : U_i \longrightarrow U\}, a_i)$ and $(f, \{g_j : U_j \longrightarrow U\}, b_j)$ are equal if $f = g$ and the compositions $(f_i \times g_j)^* x \longrightarrow f_i^* x \xrightarrow{a_i} x$ and $(f_i \times g_j)^* x \longrightarrow g_j^* x \xrightarrow{b_j} x$ are equal (they coincide in the intersections). This is the right condition because we are assuming that locally equal morphisms are equal. We are going to define the composition of locally defined morphisms. Let $(f, \{f_i : U_i \longrightarrow U\}, a_i)$ be a locally defined morphism from x to y and $(g, \{g_m : V_m \longrightarrow V\}, b_m)$ a locally defined morphism from y to z and denote $p(z) = W$. Take $g \circ f : U \longrightarrow W$, the covering $\{h_{im} : U_i \times_V V_m \longrightarrow U\}$ and the morphisms $c_{im} : h_{im}^* x \xrightarrow{a_i'} g_m^* y \xrightarrow{b_m} z$ as is showed in the following diagram:



Since $f_i^*x \longrightarrow x$ is strongly cartesian, there is a morphism $h_{im}^*x \longrightarrow f_i^*x$, unique such that the triangle is commutative. The morphism $a_i' : h_{im}^*x \longrightarrow g_m^*y$ is the only one over pr_m making the square commutative in $\mathcal{S}$, which exists because $g_m^*y \longrightarrow y$ is strongly cartesian over g_m and using that $h_{im}^*x \longrightarrow f_i^*x \xrightarrow{a_i} y$ is a morphism over $(f \circ f_i) \circ pr_i$ and the commutativity of the lower square. We are going to show that $(g \circ f, \{h_{im}\}, c_{im})$ is a locally defined morphism from x to z and we will call this the composition of $(f, \{f_i : U_i \longrightarrow U\}, a_i)$ and $(g, \{g_m : V_m \longrightarrow V\}, b_m)$.

Then we can consider the category $\mathcal{S}^2$ with $\mathrm{Ob}(\mathcal{S}^2) = \mathrm{Ob}(\mathcal{S})$ and for $x, y \in \mathrm{Ob}(\mathcal{S}^2)$, $\mathrm{Hom}_{\mathcal{S}^2}(x, y)$ is the set of locally defined morphisms from x to y in $\mathcal{S}$. Then, $\mathcal{S}^2$ is a fibred category over $\mathcal{C}$ and for any $U \in \mathcal{C}$ and every $x, y \in \mathrm{Ob}(S_U^2)$, the functor $\mathrm{Mor}(x, y)$ is a sheaf. There is also a 1-morphism of fibred categories $G^2 : \mathcal{S} \longrightarrow \mathcal{S}^2$ given by the identity in the objects and for a morphism $a : x \longrightarrow y$, $G(a) = (p(a), id_{p(x)}, a)$. Then we can assume $\mathcal{S} = \mathcal{S}^2$.

Finally, let $p : \mathcal{S} \longrightarrow \mathcal{C}$ a fibred category such that for all $U \in \mathrm{Ob}(\mathcal{C})$ and $x, y \in \mathrm{Ob}(\mathcal{S}_U)$ the functor $\mathrm{Mor}(x, y)$ is a sheaf in the topology τ_U. We will define a stack $\mathcal{S}'$ and a canonical 1-morphism $G' : \mathcal{S} \longrightarrow \mathcal{S}'$.

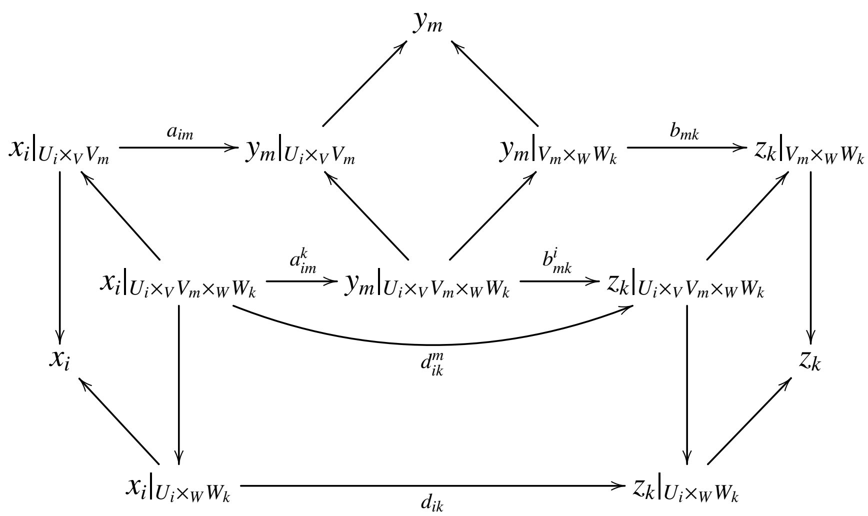
3. Effective descent data: Define $\mathrm{Ob}(\mathcal{S}')$ the class of pairs $(\mathcal{U}, \xi)$, where $\mathcal{U} = \{f_i : U_i \longrightarrow U\}_{i \in I}$ is a covering of τ and $\xi = (x_i, \varphi_{ij})$ is a descent datum in $\mathcal{S}$ relative to $\mathcal{U}$. Given $(\mathcal{U}, \xi) = (\{f_i : U_i \longrightarrow U\}, (x_i, \varphi_{ij}))$ and $(\mathcal{V}, \eta) = (\{g_m : V_m \longrightarrow V\}, (y_m, \psi_{mn}))$ in $\mathrm{Ob}(\mathcal{S}')$ we define $\mathrm{Hom}_{\mathcal{S}'}((\mathcal{U}, \xi), (\mathcal{V}, \eta))$ as the set of pairs (f, a_{im}) where $f : U \longrightarrow V$ is a morphism in $\mathcal{C}$ and $a_{im} : x_i|_{U_i \times_V V_m} \longrightarrow y_m|_{U_i \times_V V_m}$ are morphisms in $\mathcal{S}_{U_i \times_V V_m}$ satisfying the following condition:

Given $i, j \in I$ and $m, n \in M$ and denoting $A = (U_i \times_U U_j) \times_V (V_m \times_V V_n)$, the following rectangle is commutative:

$$\begin{array}{ccc} x_i|_A & \xrightarrow{a_{im}|_A} & y_m|_A \\ {\scriptstyle \varphi_{ij}|_A} \downarrow & & \downarrow {\scriptstyle \psi_{mn}|_A} \\ x_j|_A & \xrightarrow[a_{jn}|_A]{} & y_n|_A \end{array}$$

Given another $(\mathcal{W}, \theta) = (\{h_k : W_k \longrightarrow W\}, (z_k, \sigma_{kl}))$ in $\mathcal{S}'$ and $(g, b_{mk}) : (\mathcal{V}, \eta) \longrightarrow (\mathcal{W}, \theta)$ a morphism in $\mathcal{S}'$ we want to define a morphism $(g, b_{mk}) \circ (f, a_{im})$ from $(\mathcal{U}, \xi)$ to $(\mathcal{W}, \theta)$ in $\mathcal{S}'$.

Consider the following diagram:



Here a^k_{im} denotes the restriction of a_{im} to $U_i \times_V V_m \times_W W_k$ and the same is for b^i_{mk} and d^m_{ik}. Then we can make the composition $b^i_{mk} \circ a^k_{im}$ and write d^m_{ik}. It can be proved that $\{d^m_{ik}\}_{m\in M}$ determines a locally defined morphism $d_{ik} : x_i|_{U_i\times_W W_k} \longrightarrow z_k|_{U_i\times_W W_k}$ and this is such that $(g \circ f, d_{ik})$ is the wanted morphism. We this on mind, we can prove $\mathcal{S}'$ is a fibred category over $\mathcal{C}$ in which the functors $\mathrm{Mor}(x, y)$ are sheaves and any descent datum relative to a covering is effective, that is to say, $\mathcal{S}'$ is a stack over $\mathcal{C}$. There is also a canonical 1-morphism $G' : \mathcal{S} \longrightarrow \mathcal{S}'$ defined for $x \in \mathrm{Ob}(\mathcal{S})$ by $G'(x) := (\{id_{p(x)}\}, (x, id_x))$ and for $a : x \longrightarrow y$ a morphism in $\mathcal{S}$ we define $G'(a) = (p(a), a_x)$, where $a_x : x \longrightarrow y|_{p(x)}$ is the only morphism such that $x \xrightarrow{a_x} y|_{p(x)} \longrightarrow y = x \xrightarrow{a} y$. Here we are using the canonical identification $id^*_{p(x)}x = x$ in order to keep the notation simple. □

Definition 2.4.1 (Stackification). *Given fibred category $p : \mathcal{S} \longrightarrow \mathcal{C}$, the stack $p' : \mathcal{S}' \longrightarrow \mathcal{C}$ constructed in the previous lemma is called stackification of the fibred category $\mathcal{S}$.*

Next we are going to give some conditions that allow us to conclude when a fibred category is actually a stack. This is used in the example that follow the result.

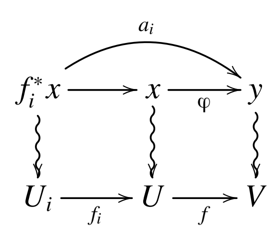
Proposition 2.4.1. *Let $\mathcal{S}$ be a category over $\mathcal{C}$ satisfying the following conditions:*

- *Every object of $\mathcal{C}$ has a lift.*
- *For any morphism $f : U \longrightarrow V$ in $\mathcal{C}$ and lifts x and y of U and V respectively, there is exactly one lift of f from x to y.*

Then S is a stack over $\mathcal{C}$.

Proof. We will show that $\mathcal{S}$ is fibred category. Given y over V and $f : U \longrightarrow V$, there exists x over U and a unique morphism $\varphi : x \longrightarrow y$ over f. If $z \in \mathrm{Ob}(\mathcal{S})$ over W, $g : W \longrightarrow U$ is a morphism in $\mathcal{C}$ and $\psi : z \longrightarrow y$ is a morphism in $\mathcal{S}$ over $f \circ g$, which is unique with such property. There is a unique morphism $\gamma : z \longrightarrow x$ and so $\varphi \circ \gamma = \psi$.

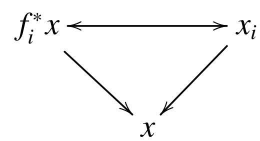
1. Let $x \in \mathcal{S}_U$, $y \in \mathcal{S}_V$ and $(f, \{f_i\}, a_i)$ a locally defined morphism in $\mathcal{S}$ from x to y, this is pictured in the following diagram



By hypothesis, there is a morphism $\varphi : x \longrightarrow y$ and $\varphi|_{U_i} = a_i$. Therefore, φ is the global extension of $(f, \{f_i\}, a_i)$. Hence $\mathrm{Mor}(x, y)$ is a sheaf.

2. Let $\mathcal{U} = \{f_i : U_i \longrightarrow U\}$ a covering in $\mathcal{C}$ and (x_i, φ_{ij}) a descent datum relative to $\mathcal{U}$. Given x over $\mathcal{U}$, there is a morphism $x_i \longrightarrow x$ and therefore we have that $f_i^* x \longrightarrow x$ and $x_i \longrightarrow x$ are strongly cartesian lifts of f_i and so there is a unique isomorphism

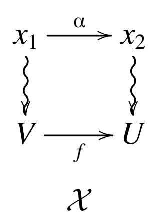
$\psi_i : f_i^* x \longrightarrow x_i$ such that the following triangle commutes

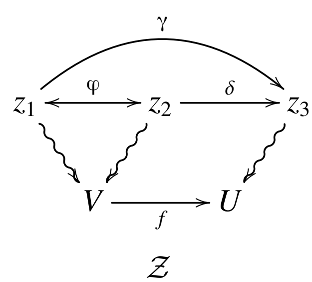


This defines an isomorphism $\psi = \{\psi_i\}$ from (x_i, φ_{ij}) to $(f_i^* x, can_{ij})$. Hence any descent datum is effective.

Numerals (1) and (2) shows that $\mathcal{S}$ is a stack over $\mathcal{C}$. □

Example 10. Let $p : \mathcal{X} \longrightarrow \mathcal{C}$ and $r : \mathcal{Z} \longrightarrow \mathcal{C}$ be the fibred categories pictured in the following diagrams:





The previous result shows that $\mathcal{X}$ and $\mathcal{Z}$ are stacks over $\mathcal{C}$. In the three cases the topology is the same.

2.5 Functoriality in stackification

Stackification has functorial properties. In this section we are going to show some of them. As in the stackification lemma, we will proceed in three stages.

Theorem 3. *Let $\mathcal{X}$ and $\mathcal{Y}$ fibred categories over $\mathcal{C}$. If $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is a 1-morphism of fibred categories over $\mathcal{C}$, then there exists a canonical 1-morphism $F' : \mathcal{X}' \longrightarrow \mathcal{Y}'$ such such that* *the following diagram commutes*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{F} & \mathcal{Y} \\ {\scriptstyle G'}\downarrow & & \downarrow{\scriptstyle H} \\ \mathcal{X}' & \xrightarrow[F']{} & \mathcal{Y}' \end{array}$$

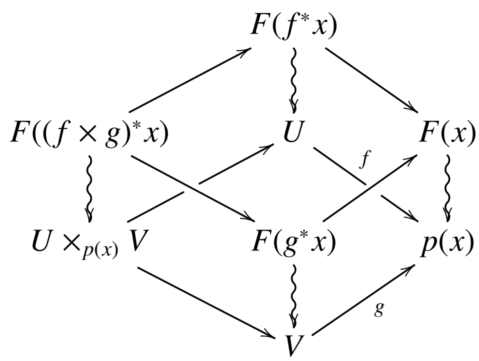
Furthermore, if $F : \mathcal{X} \longrightarrow \mathcal{Y}$ *is fully faithful (respectively essentially surjective) also is* $F' : \mathcal{X}' \longrightarrow \mathcal{Y}'$*. If* $R : \mathcal{Y} \longrightarrow \mathcal{Z}$ *is another morphism, then* $(R \circ F)' = R' \circ F'$*.*

Proof.

The proof will be done in three stages, and in many parts of this we will use the following fact: If $x \in \mathcal{X}$ is over U, $f : U \longrightarrow V$ is a morphism in $\mathcal{C}$ and F is a 1-morphism of fibred categories, then if $f^*x \longrightarrow x$ is the pullback of x over f, so is $F(f_i^*x) \longrightarrow F(x)$ and therefore there exists an isomorphism $\alpha : f_i^*F(x) \longrightarrow F(f_i^*x)$ over id_U which is unique such that the triangle

$$\begin{array}{ccc} f_i^*F(x) & \xrightarrow{\alpha_i} & F(f_i^*x) \\ & \searrow \quad \swarrow & \\ & F(x) & \end{array}$$

is commutative. So, pullback of $F(x)$ over f is up to a unique isomorphism equal to $F(f^*x \longrightarrow x)$, that is to say, the image of the pullback is almost the pullback of the image. Another fact that we will use extensively is the following: If $x \in \mathrm{Ob}(\mathcal{X})$ and $f : U \longrightarrow p(x)$ and $g : V \longrightarrow p(x)$ are morphism in $\mathcal{C}$, then in the following diagram the upper square is cartesian:



This is because $f^*x \longrightarrow x$ and $(f \times g)^*x \longrightarrow g^*x)$ are strongly cartesian, and since F is a 1-morphisms also are $F(f^*x) \longrightarrow F(x)$ and $F((f \times g)^*x) \longrightarrow F(g^*x))$, and the lower square is cartesian.

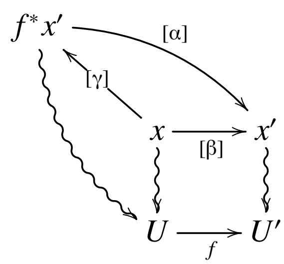
1. Locally equal morphisms: There is a canonical 1-morphism $F^1 : \mathcal{X}^1 \longrightarrow \mathcal{Y}^1$ of fibred categories over $\mathcal{C}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{F} & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{X}^1 & \xrightarrow[F^1]{} & \mathcal{Y}^1 \end{array}$$

If $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is fully faithful (resp. essentially surjective) also is $F^1 : \mathcal{X}^1 \longrightarrow \mathcal{Y}^1$.

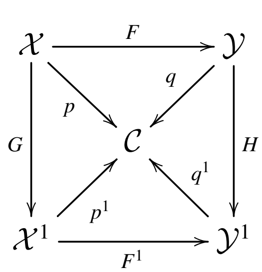
Let $F^1(x) = x$ in the objects and for a morphism $a : x \longrightarrow x'$ a morphism in $\mathcal{X}$ define $F^1[a] = [F(a)]$. Lets see that if $[a] = [b]$, then $[F(a)] = [F(b)]$ and therefore $F^1 : \mathcal{X}' \longrightarrow \mathcal{Y}'$ is well defined. The equality $[a] = [b]$ means that $p(a) = p(b)$ and this is the same that $q \circ F(a) = q \circ F(b)$. Moreover, there is a covering $\{f_i : U_i \longrightarrow U\}$ such that $f_i^*x \longrightarrow x \xrightarrow{a} y = f_i^*x \longrightarrow x \xrightarrow{b} y$ and therefore $F(f_i^*x) \longrightarrow F(x) \xrightarrow{F(a)} F(y) = F(f_i^*x) \longrightarrow F(x) \xrightarrow{F(b)} F(y)$. Hence $f_i^*F(x) \xrightarrow{\alpha_i} F(f_i^*x) \longrightarrow F(x) \xrightarrow{a} F(y) = f_i^*F(x) \xrightarrow{\alpha_i} F(f_i^*x) \longrightarrow F(x) \xrightarrow{a} F(y)$ and this means $[f(a)] = [f(b)]$. Given $[a'] : x' \longrightarrow x''$, the equality $[a'] \circ [a] = [a' \circ a]$ and the fact F is a functor implies $F^1([a'] \circ [a]) = F^1([a']) \circ F^1([a])$ and since id_x in $\mathcal{X}^1$ is $[id_x]$, then $F^1[id_x] = [id_{F(x)}]$ proving that $F^1 : \mathcal{X}^1 \longrightarrow \mathcal{Y}^1$ is a functor.

Now we will see that F^1 is a 1-morphisms of fibred categories over $\mathcal{C}$. Clearly $q^1 \circ F^1 = p^1$ in the objects. If $[a] : x \longrightarrow x'$ is a morphism in $\mathcal{X}^1$, then $q^1 \circ F^1([a]) = q^1([F(a)]) = q(F(a)) = p(a) = p^1([a])$, and so F^1 is a 1-morphism of categories over $\mathcal{C}$. We need to show that if $[\beta] : x \longrightarrow x'$ is strongly cartesian in $\mathcal{X}^1$, then $F^1([\beta])$ is strongly cartesian in $\mathcal{Y}^1$. Let $f = p(b)$ and $\alpha : f^*x' \longrightarrow x'$ the strongly cartesian lift of f in $\mathcal{X}$. Hence $[\alpha]$ is the strongly cartesian lift of f in $\mathcal{X}^1$ and so there exists an isomorphism $[\gamma] : x \longrightarrow f^*x'$ unique such that in the following diagram the upper tirangle is commutative



Hence $F^1([\beta]) = F^1([\alpha]) \circ F^1([\gamma])$ and since F^1 is a 1-morphism of fibred categories, $F^1([\alpha])$ is strongly cartesian. Moreover $F^1([\gamma])$ is isomorphism and therefore is strongly cartesian. Hence $F^1([\beta])$ is strongly cartesian being the composition of strongly cartesian morphisms. Then $F^1 : \mathcal{X}^1 \longrightarrow \mathcal{Y}^1$ is a 1-morphism of fibred categories over $\mathcal{C}$.

In particular, in the following diagram all the triangles and therefore the rectangle are commutative



We are going to prove now the second part of the statement.

- If $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is fully faithful, then given $x, x' \in \mathrm{Ob}(\mathcal{X}^1) = \mathrm{Ob}(\mathcal{X})$ we have $F : \mathrm{Hom}_{\mathcal{X}}(x, x') \longrightarrow \mathrm{Hom}_Y(F(x), F(x'))$ is a bijection. We want to prove that the same is true for $F^1 : \mathrm{Hom}_{\mathcal{X}^1}(x, x') \longrightarrow \mathrm{Hom}_{\mathcal{Y}^1}(F(x), F(x'))$.

  Surjectivity: If $[c] \in \mathrm{Hom}_{\mathcal{Y}^1}(F(x), F(y))$, then $c \in \mathrm{Hom}_{\mathcal{Y}}(F(x), F(x'))$ and so there exists $a \in \mathrm{Hom}_{\mathcal{X}}(x, x')$ such that $F(a) = c$ and therefore $[F(a)] = [c]$.

  Injectivity: If $[a], [b] \in \mathrm{Hom}_{\mathcal{X}^1}(x, x')$ are morphisms such that $F^1([a]) = F^1([b])$, then $[F(a)] = [F(b)]$ and so $F(a)$ and $F(b)$ are locally equal morphisms in $\mathcal{Y}$. Hence $q(F(a)) = q(F(b))$, that is to say $p(a) = p(b)$, and there exists a covering $\{f_i : U_i \longrightarrow p(x)\}$ such that $f_i^* F(x) \longrightarrow F(x) \xrightarrow{F(a)} F(x') = f_i^* F(x) \longrightarrow F(x) \xrightarrow{F(b)} F(x')$ and therefore $F(f_i^* x) \longrightarrow F(x) \xrightarrow{F(a)} F(x') = F(f_i^* x) \longrightarrow F(x) \xrightarrow{F(b)} F(x')$, where $F(f_i^* x) \xrightarrow{\alpha_i^{-1}} f_i^* F(x) \longrightarrow F(x) = F(f_i^* x \longrightarrow x)$. Since F is bijective $f_i^* x \longrightarrow x \xrightarrow{a} x' = f_i^* x \longrightarrow x \xrightarrow{b} x'$ and this means that $[a] = [b]$.

  Then F^1 is fully faithful.

- If $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is essentially surjective, given $y \in \mathrm{Ob}(\mathcal{Y}') = \mathrm{Ob}(\mathcal{Y})$, there is $x \in \mathrm{Ob}(\mathcal{X}) = \mathrm{Ob}(\mathcal{X}')$ and an isomorphism $\varphi : F(x) \longrightarrow y$ in $\mathcal{Y}$. Hence $[\varphi] : F(x) \longrightarrow y$ is an isomorphism in $\mathcal{Y}^1$ and therefore $F^1 : \mathcal{X} \longrightarrow \mathcal{Y}$ is essentially surjective.

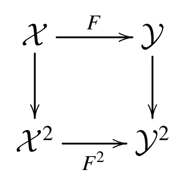
Now, given $R : \mathcal{Y} \longrightarrow \mathcal{Z}$ we have $(R \circ F)^1 = R^1 \circ F^1$ in the objects, and for a morphism $[a]$ in $\mathcal{X}$,

$$\begin{aligned}(R \circ F)^1([a]) &= [(R \circ F)(a)] \\ &= [R(a) \circ F(a)] \\ &= [R(a)] \circ [F(a)]\end{aligned}$$

$$= R^1([a]) \circ F^1([a])$$
$$= (R^1 \circ F^1)([a])$$

Therefore $(R \circ F)^1 = R^1 \circ F^1$ in the morphisms.

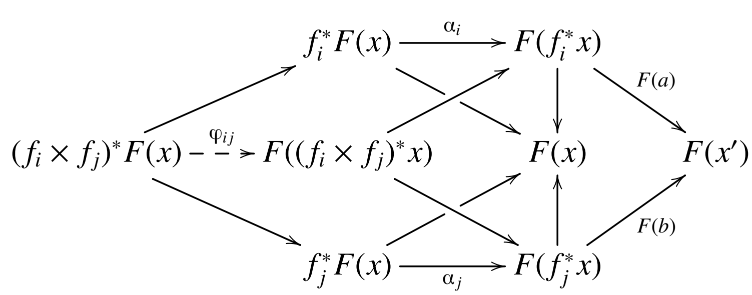
2. Locally defined morphisms: If $\mathcal{X}^1 = \mathcal{X}$ and $\mathcal{Y}^1 = \mathcal{Y}$, then there is a canonical 1-morphism $F^1 : \mathcal{X}^2 \longrightarrow \mathcal{Y}^2$ of fibred categories over $\mathcal{C}$ such that the following diagram commutes



If $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is fully faithful (resp. essentially surjective) also is $F^2 : \mathcal{X}^2 \longrightarrow \mathcal{Y}^2$.

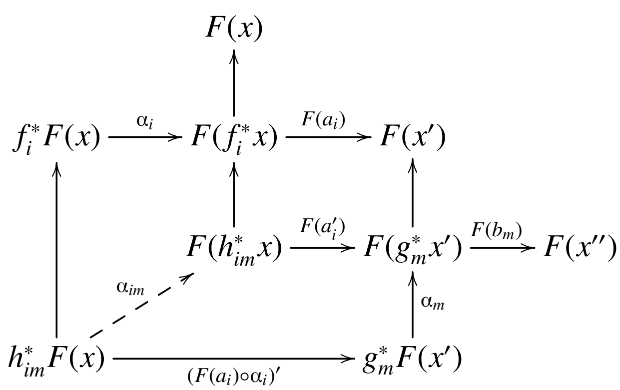
The previous result says that we can assume that $\mathcal{X}^1 = \mathcal{X}$ and $\mathcal{Y}^1 = \mathcal{Y}$, that is to say, locally equal morphisms are equal. Let $F^2(x) = F(x)$ in the objects. If $a \in \mathrm{Hom}_{\mathcal{X}^2}(x, x')$, then $a = (f, \{f_i\}, a_i)$ is a locally defined morphism in $\mathcal{X}$. Let $F^2(a) = (f, \{f_i\}, F(a_i) \circ \alpha_i)$ and see that $F^2(a) \in \mathrm{Hom}_{\mathcal{Y}^2}(F(x), F(x'))$.

We have $q(F(a_i) \circ \alpha_i) = q(F(a_i)) \circ q(\alpha_i) = f \circ f_i$. In order to prove that $F^2(a)$ is a locally defined morphism, consider the following diagram



For the remarks at beginning of this proof that in the front face the left square is cartesian. The external one is commutative because F is a functor. Since the triangles with α_i and α_j

are commutative, by the universal property of the fibre product there is a unique morphism $\varphi_{ij} : (f_i \times f_j)^* F(x) \longrightarrow F((f_i \times f_j)^* x)$ such that the two rectangles determined by this are commutative and therefore $(f_i \times f_j)^* F(x) \longrightarrow f_i^* F(x) \overset{F(a)\circ\alpha_i}{\longrightarrow} F(x') = (f_i \times f_j)^* F(x) \longrightarrow f_j^* F(x) \overset{F(a)\circ\alpha_j}{\longrightarrow} F(x')$ and we are done. We want to show that $F^2 : \mathcal{X}^2 \longrightarrow \mathcal{Y}^2$ is a functor. For this, let $b = (g, \{g_m\}, b_m)$ be a morphism in $\mathcal{X}^2$ from x' to x''. Then $b \circ a = (g \circ f, \{h_{im}\}, b_m \circ a_i')$, and therefore $F^2(b \circ a) = (g \circ f, \{U_i \times_V V_m, F(b_m \circ a_i') \circ \alpha_{im}\})$, where $\alpha_{im} : h_{im}^* F(x) \longrightarrow F(h_{im} * x)$ is the only morphism such that the rectangles in the following diagram commutes



On the other hand,

$$
\begin{aligned}
F^2(b) \circ F^2(a) &= (f, \{f_i\}, F(a_i) \circ \alpha_i) \circ (g, \{g_m\}, F(b_m) \circ \alpha_m) \\
&= (g \circ f, \{h_{im}\}, (F(b_m) \circ \alpha_m) \circ (F(a_i) \circ \alpha_i)') \\
&= (g \circ f, \{h_{im}\}, F(b_m) \circ F(a_i') \circ \alpha_{im}) \\
&= F^2(b \circ a)
\end{aligned}
$$

Then F^2 is compatible with composition. Given $x \in \mathrm{Ob}(\mathcal{X}^2)$, the identity id_x in $\mathcal{X}^2$ is $(id_{p(x)}, \{id_{p(x)}\}, id_{p(x)}^* x \longrightarrow x)$ and therefore

$$
\begin{aligned}
F^2(id_x) &= (id_{p(x)}, \{id_{p(x)}\}, F(id_{p(x)}^* x \longrightarrow x) \circ \delta) \\
&= (id_{p(x)}, \{id_{p(x)}\}, id_{p(x)}^* F(x) \longrightarrow F(x))
\end{aligned}
$$

$$= id_{F(x)}$$

where $\delta : id^*_{p(x)}F(x) \longrightarrow F(id^*_{p(x)}x)$ is the only morphism such that $id^*_{p(x)}F(x) \xrightarrow{\delta} F(id^*_{p(x)}x) \longrightarrow F(x) = id^*_{p(x)}F(x) \longrightarrow F(x)$. Hence F^2 preserves identities and so $F^2\mathcal{X}^2 \longrightarrow \mathcal{Y}^2$ is a functor. Moreover, the following diagram is commutative

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{F} & \mathcal{Y} \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle H} \\ \mathcal{X}^2 & \xrightarrow[F^2]{} & \mathcal{Y}^2 \end{array}$$

Given $x \in \mathrm{Ob}(\mathcal{X})$, then $F^2 \circ G(x) = F^2(x) = F(x) = H(F(x)) = H \circ F(x)$, since G and H are identities in the objects. If $a \in \mathrm{Hom}_{\mathcal{X}}(x, x')$, then

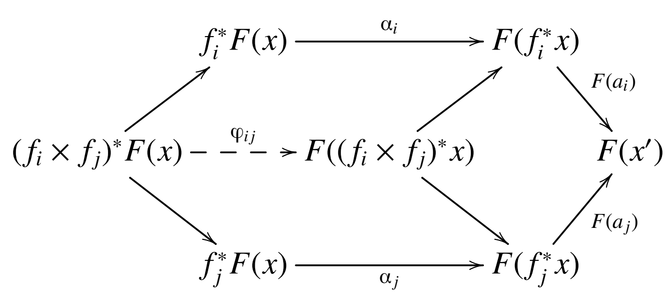
$$\begin{aligned} F^2 \circ G(a) &= F(p(a), \{id_{p(x)}\}, id^*_{p(x)} \longrightarrow x \xrightarrow{a} x') \\ &= (p(a), \{id_{p(x)}\}, id^*_{p(x)}F(x) \xrightarrow{\delta} F(id^*_{p(x)}x) \longrightarrow F(x) \xrightarrow{F(a)} F(x')) \\ &= H(F(a)) \end{aligned}$$

Now, given a strongly cartesian morphism $b = (f, \{f_i\}, b_i)$ from x to x' in $\mathcal{X}^2$, let $a = (f, \{f_i\}, f^*x' \xrightarrow{a} x')$ the pullback of x' over f in $\mathcal{X}^2$. Then there exists a unique isomorphism $c : x \longrightarrow f^*x'$ in $\mathcal{X}^2_U$ such that $a \circ c = b$. Therefore $F^2(b) = F^2(a \circ c) = F^2(a) \circ F^2(c)$. Since $F^2(c)$ is an isomorphism, it is enough to show that $F^2(a)$ is a strongly cartesian morphism. By construction, $a = G(\alpha)$, where $\alpha : f^*x' \longrightarrow x$ is the pullback of x' over f in $\mathcal{X}$, and therefore $F^2(a) = F^2 \circ G(\alpha) = H \circ F(\alpha)$. Since F and H are 1-morphism of fibred categories, then $H \circ F$ also is and so $H \circ F(\alpha)$ is a strongly cartesian morphism in $\mathcal{Y}^2$. Hence $F^2 : \mathcal{X}^2 \longrightarrow \mathcal{Y}^2$ is a 1-morphism of fibred categories over $\mathcal{C}$.

- If $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is fully faithful, then given $x, x' \in \mathrm{Ob}(\mathcal{X}^2) = \mathrm{Ob}(\mathcal{X})$ we have $\mathrm{Hom}_{\mathcal{X}}(x, x') \longleftrightarrow \mathrm{Hom}_{\mathcal{Y}}(F(x), F(x'))$.

Injectivity: If $a = (f, \{f_i\}, a_i)$ and $b = (g, \{g_m\}, b_m)$ are morphisms in $\mathrm{Hom}_{\mathcal{X}^2}(x, x')$ such that $F^2(a) = F^2(b)$, then $(f, \{f_i\}, F(a_i) \circ \alpha_i) = (g, \{g_m\}, F(b_m) \circ \beta_m)$. Hence $f = g$, $\{f_i\} = \{g_m\}$ and we can take $M = I$ and $f_i = g_i$ for all $i \in I$. Then $\alpha_i = \beta_i$ and therefore $F(a_i) \circ \alpha_i = F(b_i) \circ \alpha_i$. Since α_i is isomorphism, we have $F(a_i) = F(b_i)$ and since F is injective, $a_i = b_i$, for all $i \in I$. Hence $a = b$.

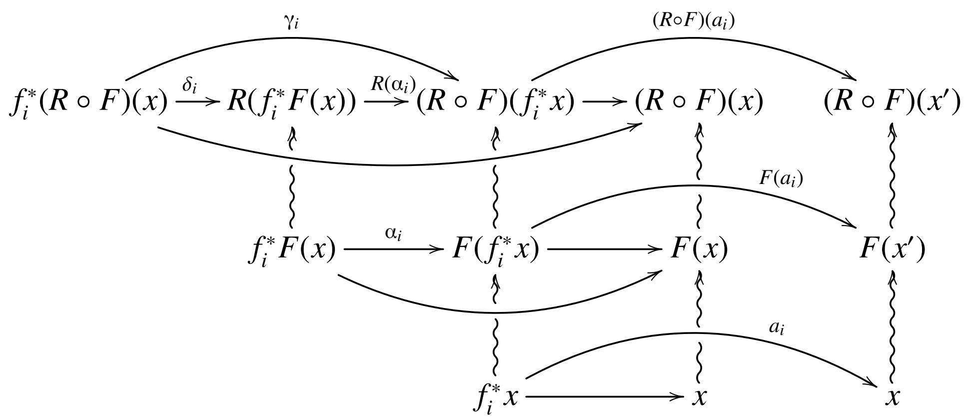
Surjectivity: If $c \in \mathrm{Hom}_{\mathcal{Y}^2}(F(x), F(x'))$, then $x = (f, \{f_i\}, c_i)$, where $c_i : F_i^* F(x) \longrightarrow F(x')$ is a morphism in $\mathcal{Y}$. Composing with α_i^{-1} we have $c \circ \alpha_i^{-1} : F(f_i^* x) \longrightarrow F(x')$ and therefore there exists $a_i \in \mathrm{Hom}_{\mathcal{X}}(f_i^* x, x')$ which is unique such that $F(a_i) = c_i \circ \alpha_i^{-1}$ or $F(a_i) \circ \alpha_i = c_i$. We need to see that $a = (f, \{f_i\}, a_i)$ is a locally defined morphism in $\mathcal{X}$. For this, consider the following diagram



Since $c = (f, \{f_i\}, c_i)$ is a locally defined morphism and $c_i = F(a_i) \circ \alpha_i$, then the external diagram and the two rectangles at the left are commutative. But φ_{ij} is an isomorphism and so the rectangle at the right is also commutative. The conclusion follows from F is fully faithful. Hence $a = (f, \{f_i\}, a_i)$ is a morphism in $\mathcal{X}^2$ and $F^2(a) = c$ which means that F^2 is is surjective.

- The same proof done before for F^1 aloow us to conclude that if $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is essentially surjective, also is $F^2 : \mathcal{X} \longrightarrow \mathcal{Y}$.

Given $R : \mathcal{Y} \longrightarrow \mathcal{Z}$ we have again $(R \circ F)^2 = R^2 \circ F^2$ in the objects. Let $(f, \{f_i\}, a_i)$ be a morphism in $\mathcal{X}$, and consider the following diagram



Then $(R \circ F)^2(f, \{f_i\}, a_i) = (f, \{f_i\}, (R \circ F)(a) \circ \gamma_i)$, where $\gamma_i : f_i^*(R \circ F)x \longrightarrow (R \circ F) f_i^* x$ is the only morphism such that the triangle with $(R \circ F)x$ is commutative. On the other hand

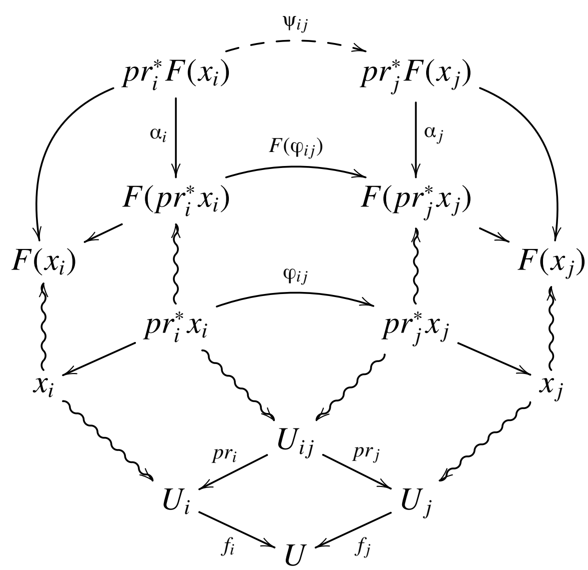
$$\begin{aligned}(R^2 \circ F^2)(f, \{f_i\}, a_i) &= R^2(f, \{f_i\}, F(a_i) \circ \alpha_i) \\ &= (f, \{f_i\}, R(F(a_i) \circ \alpha_i) \circ \delta_i) \\ &= (f, \{f_i\}, (R \circ F)(a_i) \circ R(\alpha_i) \circ \delta_i)\end{aligned}$$

where $\delta_i : f_i^* R(F(x)) \longrightarrow R(f_i^* F(x))$ is the only morphism such that the triangle with $(R \circ F)(x)$ commutes. Since $R \circ F$ is a 1-morphism of fibred categories, the morphism $(R \circ F)(f_i^* x) \longrightarrow (R \circ F)(x)$ is strongly cartesian and by construction the composition of this morphism with $R(\alpha_i) \circ \delta_i$ and γ_i are equal and so $R(\alpha_i) \circ \delta_i = \gamma_i$. Then $(R \circ F)^2(f, \{f_i\}, a_i) = (R^2 \circ F^2)(f, \{f_i\}, a_i)$ and so $(R \circ F)^2 = R^2 \circ F^2$.

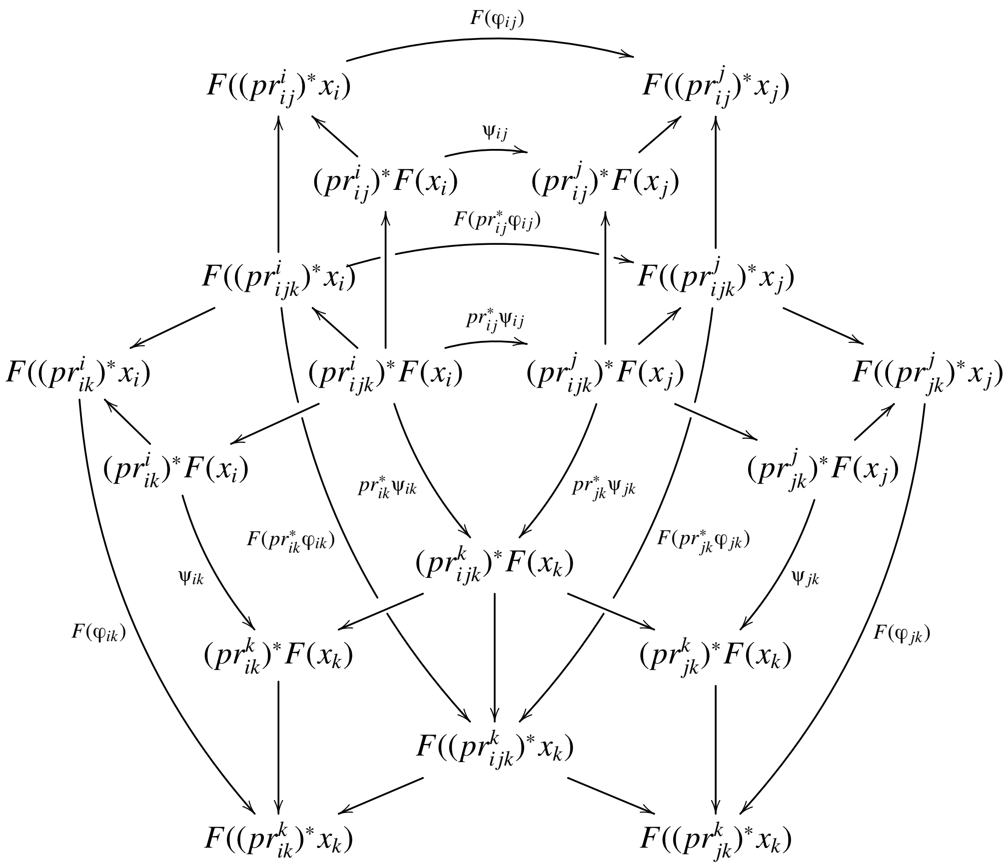
3. Effective descent data: Suppose that $X^2 = \mathcal{X}^1 = \mathcal{X}$ and $\mathcal{Y}^2 = \mathcal{Y}^1 = \mathcal{Y}$

Given $(\mathcal{U}, \xi) = (\{f_i : U_i \longrightarrow U\}, (x_i, \varphi_{ij}))$ in $\mathrm{Ob}(\mathcal{X}')$, then $F(x_i)$ is an object of $\mathcal{Y}$ such that $q(y_i) = q \circ F(x_i) = p(x_i) = U_i$ and since $\varphi_{ij} : pr_i^* x_i \longrightarrow pr_j^* x_j$ is an isomorphism in $\mathcal{X}_{U_{ij}}$, then

$F(\varphi_{ij}) : F(pr_i^* x_i) \longrightarrow F(pr_j^* x_j)$ is an isomorphism in $\mathcal{Y}_{U_{ij}}$. Consider the following diagram



where $\alpha_i : pr_i^* F(x_i) \longrightarrow F(pr_i^* x_i)$ is the unique morphism such that the respective triangle is commutative. Hence $\psi_{ij} = \alpha_j^{-1} \circ F(\varphi_{ij}) \circ \alpha_i$ is an isomorphism from $pr_i^* F(x_i)$ to $pr_j^* F(x_i)$. We will see that $(F(x_i), \psi_{ij})$ is a descent datum in $\mathcal{Y}$ relative to $\mathcal{U}$. For this, consider the diagram



We want to prove that the central triangle commutes. Because (x_i, φ_{ij}) is a descent datum relative to $\mathcal{U}$, and F is a functor, the rectangles and the triangle in the back of the diagram are commutative. By definition, ψ_{ij} is the only morphism such that the rectangle with $F(\varphi_{ij})$ is commutative. Also, $pr^*_{ij}\psi_{ij}$ is the only morphism such that the rectangle with ψ_{ij} is commutative. The squares with none of its arrows labeled are commutative by construction and all the morphisms in they are strongly cartesian. Therefore, the rectangle with $pr^*_{ij}\psi_{ij}$ and $F(pr^*_{ij}\varphi_{ij})$ is commutative. In the same way, the other two rectangles connecting the two triangles are commutative. Hence $pr_{jk}^*\psi_{jk} \circ pr_{ij}^*\psi_{ij}$ and $pr_{ik}^*\psi_{ik}$ are morphisms such that the rectangle

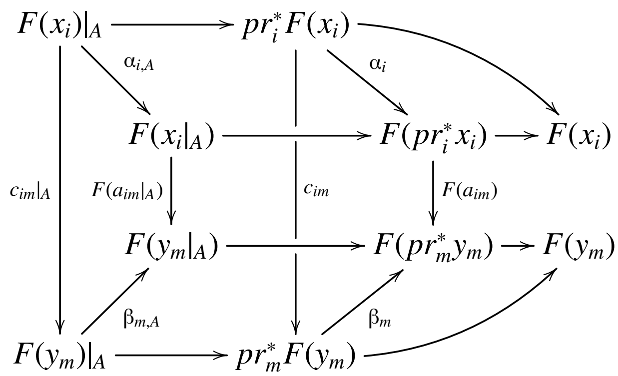
$$\begin{array}{ccc} (pr_{ijk}^i)^*F(x_i) & \longrightarrow & (pr_{ijk}^k)^*F(x_k) \\ \downarrow & & \downarrow \\ F((pr_{ijk}^i)^*x_i) & \longrightarrow & F((pr_{ijk}^k)^*x_k) \end{array}$$

is commutative, but there is only one such morphism with this property and therefore $pr_{jk}^*\psi_{jk} \circ pr_{ij}^*\psi_{ij} = pr_{ik}^*\psi_{ik}$. Then $(F(x_i), \psi_{ij})$ is a descent datum in $\mathcal{Y}$ relative to $\mathcal{U}$ and so $F(\mathcal{U}, \xi) = (\mathcal{U}, (F(x_i), \psi_{ij}))$ is an object in $\mathcal{Y}'$.

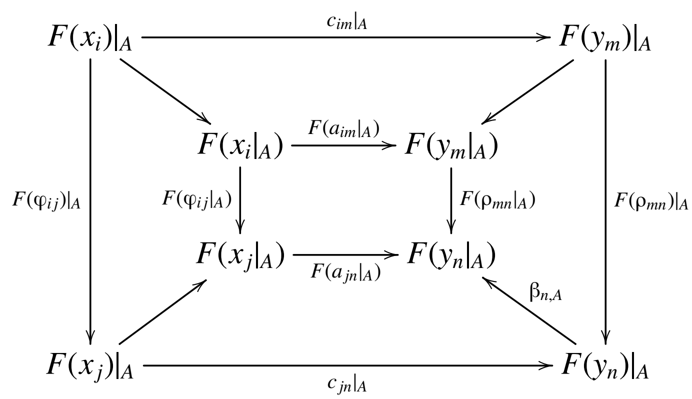
If $(\mathcal{V}, \eta) = (\{g_m : V_m \longrightarrow V\}, (y_m, \rho_{mn}))$ is another object in $\mathcal{X}'$ and $(f, a_{ij}) : (\mathcal{U}, \xi) \longrightarrow (\mathcal{V}, \eta)$ is a morphism in $\mathcal{X}'$, we have $f : U \longrightarrow V$ is a morphism in $\mathcal{C}$ and $a_{im} : x_i|_{U_i \times_V V_m} \longrightarrow y_m|_{U_i \times_V V_m}$ is a morphism over $id_{U_i \times_V V_m}$ such that for $i, j \in I$ and $m, n \in M$, the following rectangle commutes:

$$\begin{array}{ccc} x_i|_A & \xrightarrow{a_{im}|_A} & y_m|_A \\ {\scriptstyle \varphi_{ij}|_A}\downarrow & & \downarrow{\scriptstyle \psi_{mn}|_A} \\ x_j|_A & \xrightarrow[a_{jn}|_A]{} & y_n|_A \end{array}$$

By construction $F(\mathcal{U}, \xi) = (\mathcal{U}, (F(x_i), \psi_{ij}))$ and $F(\mathcal{V}, \eta) = (\mathcal{V}, (F(y_m), \sigma_{mn}))$, where $\psi_{ij} = \alpha_j^{-1} \circ F(\varphi_{ij}) \circ \alpha_i$ and $\sigma_{mn} = \beta_n^{-1} \circ F(\rho_{mn}) \circ \beta_m$, with $\alpha_i : pr_i^* F(x_i) \longrightarrow F(pr_i^* x_i)$ is the canonical isomorphism making commutative the right triangle in the next diagram

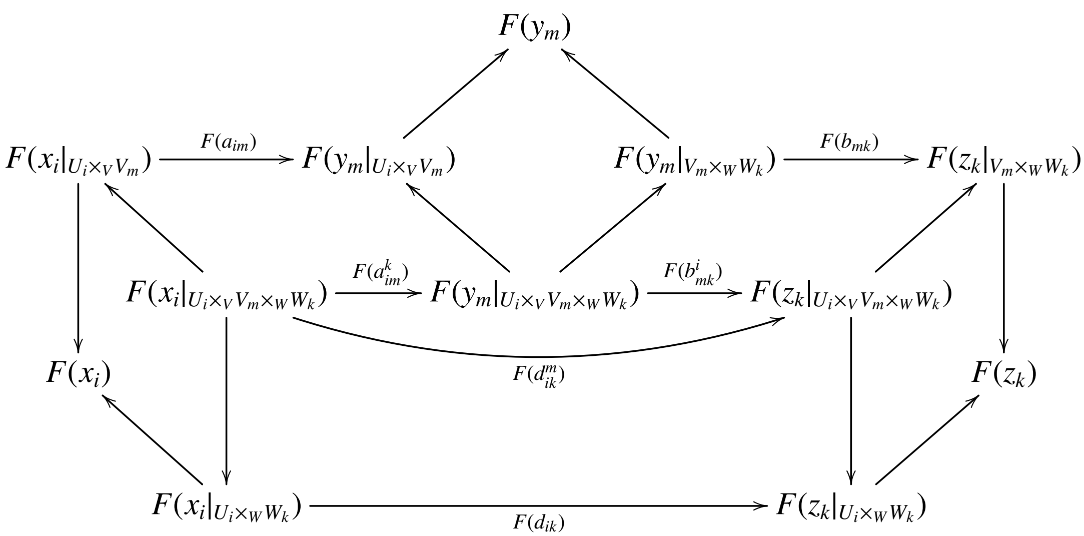


In the same way, we have an isomorphism $\alpha_{i,A}$ which is unique such that the triangle with $F(x_i)$ is commutative. Since F is a 1-morphism of fibred categories, the morphism $F(pr_i^* x_i) \longrightarrow F(x_i)$ is strongly cartesian and therefore $\alpha_{i,A}$ is the only morphism such that the square α_i is commutative. We have analog properties for β_m and $\beta_{m,A}$. Also, the central square is commutative since F is a functor. Letting $c_{im} = \beta_m^{-1} \circ F(a_{im}) \circ \alpha_i$, we will show that (f, c_{im}) is a morphism from $F(\mathcal{U}, \xi)$ to $F(\mathcal{V}, \eta)$. By definition, $c_{im}|_A$ is the only morphism such that the back rectangle commutes. Note that all the horizontal arrows are strongly cartesian morphisms and so $c_{im}|_A$ is the only morphism making the left rectangle commutative. We can make the same with the descent data, that is to say, changing $F(a_{im})$ for $F(\varphi_{ij})$ (resp. $F(\rho_{mn})$) and c_{im} for ψ_{ij} (resp. σ_{mn}). Hence, in the following diagram c_{im}, c_{jn}, $F(\varphi_{ij})|_A$ and $F(\sigma_{mn})|_A$ are unique making its respective rectangle commutative

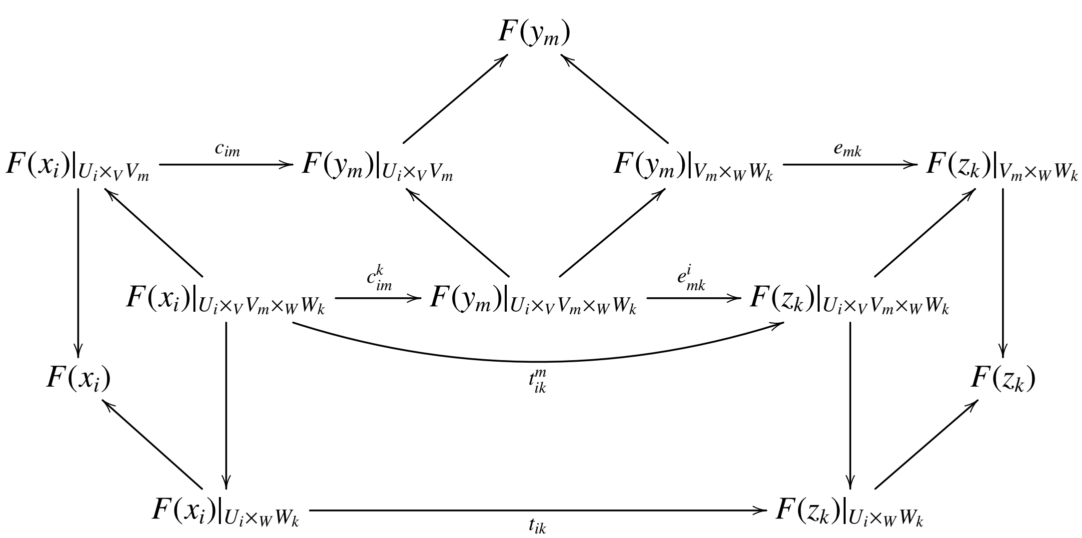


Since the centrar rectangle is clearly commutative, and $\beta_{n,A}$ is isomorphism, then the external rectangle is also commutative and so (f, c_{im}) is a morphism in $\mathcal{Y}'$ and therefore $F' : \mathcal{X} \longrightarrow \mathcal{Y}'$ is well defined in the morphisms.

Now, if $(\mathcal{W}, \theta) = (\{h_k : W_k \longrightarrow W\}, (z_k, \mu_{kl}))$ is an object of $\mathcal{X}'$ and $(g, b_{mk}) : (\mathcal{V}, \eta) \longrightarrow (\mathcal{W}, \theta)$ is another morphism, then $(g, b_{mk}) \circ (f, a_{im}) = (g \circ f, d_{ik})$, where $d_{ik} : x_i|_{U_i \times_W W_k} \longrightarrow z_k|_{U_i \times_W W_k}$ is the only morphism such that $d^m_{ik} = b^i_{mk} \circ a^k_{im}$. Applying the funtor F to the composition diagram we have the next diagram in $\mathcal{Y}$



Now we make the identification $F(a^k_{im}) \cong F(a_{im})|_{U_i\times_V V_m\times_W W_k}$ and the same for $F(b^i_{mk})$ and $F(d^m_{ik})$. Then we use $F(d^m_{ik}) = F(b^i_{mk}) \circ F(a^k_{im})$. Letting $F'(f, a_{im}) = (f, c_{im})$, $F'(g, b_{mk}) = (g, e_{mk})$ and $F'(g\circ f, d_{ik}) = (g\circ f, t_{ik})$, we have the following diagram



For each vertex there is a canonical arrow towards the corresponding vertex in the previous diagram, such that for any arrow in one of the diagram, the square with the corresponding arrow is commutative and this arrow is unique with such property as was proved before. Then $t^m_{ik} = e^i_{mk} \circ c^k_{im}$ and therefore $(g \circ f, t_{ik}) = (g, e_{mk}) \circ (f, c_{im})$, that is to say $F' : \mathcal{X}' \longrightarrow \mathcal{Y}'$ preserves composition.

Moreover, given $(\mathcal{U}, \xi)$ as before, $id_{(\mathcal{U},\xi)} = (id_U, \varphi_{ij})$ and $F'(id_U, \varphi_{ij}) = (id_U, \alpha_j^{-1} \circ F(\varphi_{ij}) \circ \alpha_i) = (id_U, \psi_{ij}) = id_{F(\mathcal{U},\xi)}$. Hence $F' : \mathcal{X}' \longrightarrow \mathcal{Y}'$ is a functor. We shall prove that the following diagram commutes

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{F} & \mathcal{Y} \\ {\scriptstyle G'}\downarrow & & \downarrow{\scriptstyle H'} \\ \mathcal{X}' & \xrightarrow[F']{} & \mathcal{Y}' \end{array}$$

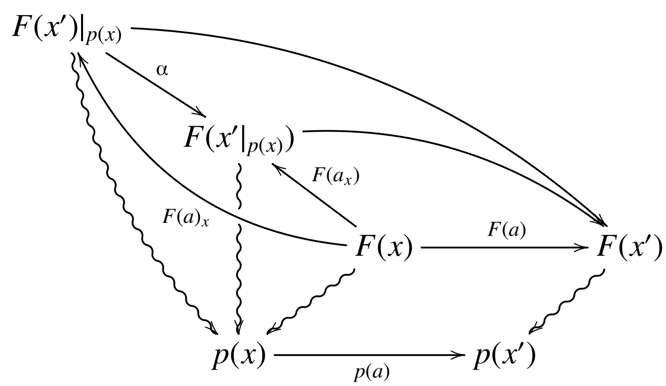
Given $x \in \mathrm{Ob}(\mathcal{X})$, we have $G'(x) = (\{id_{p(x)}\}, (x, id_x))$ and if $x \xrightarrow{a} x'$ is a morphism in $\mathcal{X}$, $G'(a) = (p(a), a_x)$, where $a_x : x \longrightarrow x'|_{p(x)}$ is the only morphism such that $x \xrightarrow{a_x} x'|_{p(x)} \longrightarrow x' = a$. Hence $H'(F(x)) = (\{id_{q(F(x))}\}, (F(x), id_{F(x)}))$ and also

$$\begin{aligned} F'(G'(x)) &= F'(\{id_{p(x)}\}, (x, id_x)) \\ &= (\{id_{p(x)}\}, (F(x), \alpha^{-1} \circ F(id_x) \circ \alpha)) \\ &= (\{id_{p(x)}\}, (F(x), id_{F(x)})) \end{aligned}$$

Then we have the equality in the objects. On the other hand $H'(F(a)) = (q(F(a)), F(a)_x) = (p(a), F(a)_x)$ and also

$$\begin{aligned} F'(G'(a)) &= F'(p(a), a_x) \\ &= (p(a), \alpha^{-1} \circ F(a_x) \circ id_{F(x)}) \\ &= (p(a), F(a)_x) \end{aligned}$$

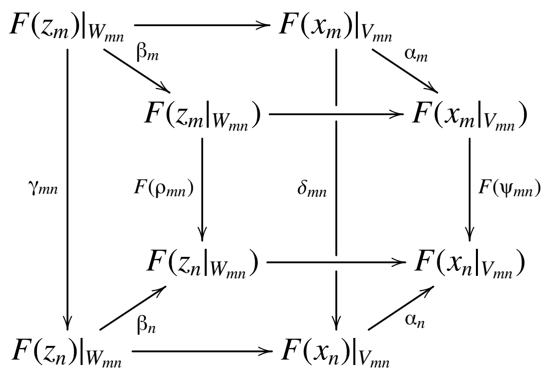
as is showed in the next diagram



Here, $F(a)_x$ is the only morphism such that the triangle with $F(x')$ commutes. The two triangles at the right commute because F is a functor and by definition of α. Hence $F(x) \xrightarrow{F(a)_x} F(x')|_{p(x)} \xrightarrow{\alpha} F(x'|_{p(x)}) \longrightarrow F(x') = F(a_x) \xrightarrow{F(a_x)} F(x'|_{p(x)}) \longrightarrow F(x')$ and since $F(x'|_{p(x)}) \longrightarrow F(x')$ is strongly cartesian, it follows that $\alpha \circ F(a)_x = F(a_x)$, and we have commutativity in the morphisms.

In order to see that $F' : \mathcal{X}' \longrightarrow \mathcal{Y}'$ is a 1-morphism of fibred categories, we must to show that $q' \circ F' = p'$ which is easy and that F' preserves strongly cartesian morphisms. The difficulty here is that the objects in $\mathcal{X}'$ are not necessarily the same as the objects in $\mathcal{X}'$ and therefore, the argument used with F^1 and F^2 cannot be applied. However, we can do it partially. Given an object $(\mathcal{V}, \eta) = (\{g_m : V_m \longrightarrow V\}, (x_m, \psi_{mn}))$ of $\mathcal{X}'$, let $(f, \rho_{nm}) : f^*(\mathcal{V}, \eta) \longrightarrow (\mathcal{V}, \eta)$ be the pullback of $(\mathcal{V}, \eta)$ in $\mathcal{X}'$ over f. Then $f^*(\mathcal{V}, \eta) = (\{g'_m : W_m \longrightarrow U\}, (z_m, \rho_{mn}))$, where $W_m = V_m \times_V U$, $z_m = x_m|_{W_m}$ and $\rho_{mn} = \psi_{mn}|_{W_{mn}}$. Hence $F'(f, \rho_{mn}) = (f, \gamma_{mn})$ where

$\gamma_{mn} = \beta_n^{-1} \circ F(\rho_{mn}) \circ \beta_m$, as showed in the next diagram



As before, it remains to show that $F'(f, \rho_{mn})$ is a strongly cartesian morphism in $\mathcal{Y}'$. We have

$$\begin{aligned} F'(\mathcal{V}, \eta) &= F'(\{g_m\}, (x_m, \psi_{mn})) \\ &= (\{g_m\}, (F(x_m), \alpha_n^{-1} \circ F(\psi_{mn}) \circ \alpha_m)) \\ &= (\{g'_m\}, (F(x_m), \delta_{mn})) \end{aligned}$$

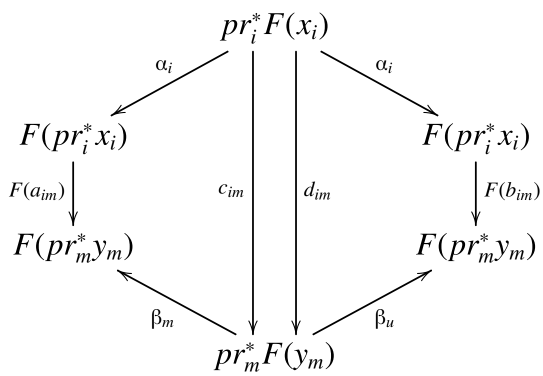
and therefore $f^* F'(\mathcal{V}, \eta) = (\{g'_m\}, (F(x)|_{W_{mn}}, \gamma_{mn}))$ and in the other hand

$$\begin{aligned} F'(f^*(\mathcal{V}, \eta)) &= F'(\{g'_m\}, (z_m, \rho_{mn})) \\ &= (\{g'_m\}, (F(z_m), \beta_n^{-1} \circ F(\rho_{mn}) \circ \beta_m)) \\ &= (\{g'_m\}, (F(x_m|_{W_{mn}}), \gamma_{mn})) \end{aligned}$$

and so $F'(f^*(\mathcal{V}, \eta)) \cong f^* F'(\mathcal{V}, \eta)$ via the canonical isomorphism induced by the identification between $F(x)|_{W_{mn}}$ and $F(x_m|_{W_{mn}})$. Then $F'(f, \rho_{mn})$ is the composition of the pullback of $F(\mathcal{V}, \eta)$ over f in $\mathcal{Y}'$ and one isomorphism and so this is a strongly cartesian morphism and therefore $F' \mathcal{X}' \longrightarrow \mathcal{Y}'$ is a 1-morphism of fibred categories.

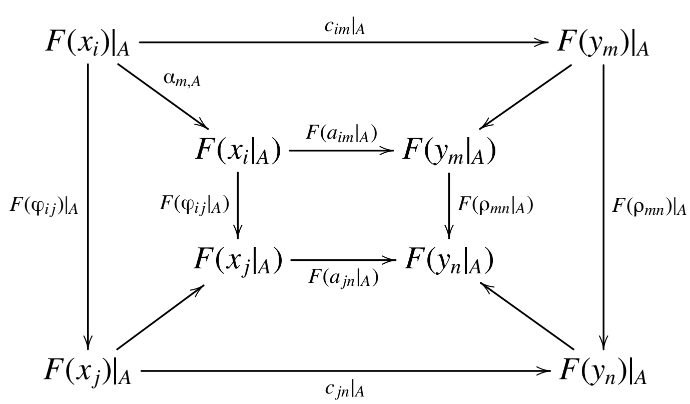
Suppose now that $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is fully faithful. Given $(\mathcal{U}, \xi)$ and $(\mathcal{V}, \eta)$ in $\mathrm{Ob}(\mathcal{X}')$ we want to see that $\mathrm{Hom}_{\mathcal{X}'}((\mathcal{U}, \xi), (\mathcal{V}, \eta)) \longleftrightarrow \mathrm{Hom}_{\mathcal{Y}'}(F'(\mathcal{U}, \xi), F'(\mathcal{V}, \eta))$.

Injectivity: If (f, a_{im}) and (g, b_{im}) are morphisms from $(\mathcal{U}, \xi)$ to $(\mathcal{V}, \eta)$ such that $F'(f, a_{im}) = F'(g, b_{im})$, then $(f, c_{im}) = (g, d_{im})$, where c_{im} and d_{im} are defined as in the following diagram



Hence $f = g$ and $c_{im} = d_{im}$. Since α_i and β are isomorphims we have $F(a_{im}) = F(b_{im})$ and so $a_{im} = c_{im}$, because F is fully faitfull.

Surjectivity: Given $(f, c_{im}) : F'(\mathcal{U}, \xi) \longrightarrow (\mathcal{V}, \eta)$, we want to construct a morphism $(f, a_{im}) : (\mathcal{U}, \xi) \longrightarrow (\mathcal{V}, \eta)$ such that $F'(f, a_{im}) = (f, c_{im})$. Let $\gamma_{im} : F(pr_i^* x_i) \longrightarrow F(pr_m^* y_m)$ the composition $\beta_m \circ c_{im} \circ \alpha_i^{-1}$. Since F is fully faithful there is a unique morphism $a_{im} : pr_i^* x_i \longrightarrow pr_m^* y_m$ such that $\gamma_{im} = F(a_{im})$. We will see that (f, a_{im}) is a morphism, which by construction satisfies $F'(f, a_{im}) = (f, c_{im})$. Letting $A = (U_i \times_U U_j) \times_V (V_m \times_V V_n)$ we have, in a reciprocal way to the part when we proved that $F' : \mathcal{X}' \longrightarrow \mathcal{Y}'$ is well defined in the objects, that in the following diagram the external rectangle is commutative



Then, because $\alpha_{m,A}$ is isomorphism, the internal rectangle is also commutative and since F is a functor $F(\rho_{mn}|_A \circ a_{im}|_A) = F(a_{jn}|_A \circ \varphi_{ij}|_A)$ and therefore $\rho_{mn}|_A \circ a_{im}|_A = a_{jn}|_A \circ \varphi_{ij}|_A$ and we are done.

Given $F : \mathcal{Y} \longrightarrow \mathcal{Z}$, if $(\mathcal{U}, \xi)$ is an object of $\mathcal{X}$, where $\mathcal{U} = \{f_i\}$ is a covering and $\xi = (x_i, \varphi_{ij})$ is a descent datum in $\mathcal{X}$ relative to $\mathcal{U}$, then

$$\begin{aligned}
(R' \circ F')(\mathcal{U}, \xi) &= R'(\mathcal{U}, (F(x_i), \alpha_j^{-1} \circ F(\varphi_{ij}) \circ \alpha_i)) \\
&= (\mathcal{U}, \delta_j^{-1} \circ R(\alpha_j^{-1} \circ F(\varphi_{ij}) \circ \alpha_i) \circ \delta_i) \\
&= (\mathcal{U}, \delta_j^{-1} \circ R(\alpha_j)^{-1} \circ R(F(\varphi_{ij})) \circ \mathcal{R}(\alpha_i) \circ \delta_i) \\
&= (\mathcal{U}, (R(\alpha_j) \circ \delta_j)^{-1} \circ (R \circ F)(\varphi_{ij}) \circ (R(\alpha_i) \circ \delta_i)) \\
&= (\mathcal{U}, \gamma_j^{-1} \circ (R \circ F)(\varphi_{ij}) \circ \gamma_i) \\
&= (R \circ F)'(\mathcal{U}, \xi)
\end{aligned}$$

and therefore, $(R \circ F)' = R' \circ F'$ in the objects.

If $(\mathcal{V}, \eta)$, with $\eta = (y_m, \rho_{mn})$, is another object of $\mathcal{X}$ and (f, a_{im}) is a morphism from $(\mathcal{U}, \xi)$ to $(\mathcal{V}, \eta)$, then letting λ_m and κ_m the analogs of γ_m and δ_m respectively, we have

$$\begin{aligned}
(R' \circ F')(f, a_{im}) &= R'(f, \beta_m^{-1} \circ F(a_{im}) \circ \alpha_i) \\
&= (f, \kappa_m^{-1} \circ R(\beta_m^{-1} \circ F(a_{im}) \circ \alpha_i) \circ \delta_i) \\
&= (f, \kappa_m^{-1} \circ R(\beta_m)^{-1} \circ R(F(a_{im})) \circ R(\alpha_i) \circ \delta_i) \\
&= (f, (R(\beta_m) \circ \kappa_m)^{-1} \circ (R \circ F)(a_{im}) \circ (R(\alpha_i) \circ \delta_i)) \\
&= (f, \lambda_m^{-1} \circ (R \circ F)(a_{im}) \circ \gamma_i) \\
&= (R \circ F)'(f, a_{im})
\end{aligned}$$

and therefore $(R \circ F)' = R' \circ F'$ in the morphisms. In particular stackification preserves commutative diagrams. □

Corollary 3.1. *Let $p : \mathcal{X} \longrightarrow \mathcal{C}$, $q : \mathcal{Y} \longrightarrow \mathcal{C}$ and $r : \mathcal{Z} \longrightarrow \mathcal{C}$ be fibred categories over $\mathcal{C}$ and $F : \mathcal{X} \longrightarrow \mathcal{Z}$ and $G : \mathcal{Y} \longrightarrow \mathcal{Z}$ 1-morphisms of fibred categories over $\mathcal{C}$ such that the category $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ over $\mathcal{C}$ has componentwise pullbacks. Then*

$$H' : (\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})' \longrightarrow (\mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y})'$$

is a fully faithful functor. If the functor H is essentially surjective, then H' is an equivalence of categories.

Stackification and fibre product

It is a known fact that stakification preserves 2-fibre products, that is to say, the stackification of a 2-fibre product of fibred categories is the same as the 2-fibre product of the stackifications of the fibred categories. When we begin this research, we though it can be proved that stackification of a fibre product will be isomorphic to the stackification of the 2-fibre product, since in Stacks Project we find that stackification sends commutative squares to 2-commutative squares. However a similar result about fibre product, when this exist is not available in the literature and doesn't follows from the result about 2-fibre products. We are going to prove that this functorial properties are also compatible with fibre products, provided that the fibre product is the constructed in categories over $\mathcal{C}$.

Theorem 4. *Let $p : \mathcal{X} \longrightarrow \mathcal{C}$, $q : \mathcal{Y} \longrightarrow \mathcal{C}$ and $r : \mathcal{Z} \longrightarrow \mathcal{C}$ be fibred categories over $\mathcal{C}$ such that the category $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ over $\mathcal{C}$ has componentwise pullbacks, then*

$$(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})' \cong \mathcal{X}' \times_{\mathcal{Z}'} \mathcal{Y}'$$

Proof.

The proof is again divided in three stages:

1. Locally equal morphisms: $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})^1 \cong \mathcal{X}^1 \times_{\mathcal{Z}^1} \mathcal{Y}^1$.

The objects in $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$, $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})^1$ and $\mathcal{X}^1 \times_{\mathcal{Z}_1} \mathcal{Y}^1$ are the same. Given $(x, y), (x', y') \in \mathrm{Ob}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})^1$ we have $x, x' \in \mathrm{Ob}(\mathcal{X})$, $y, y' \in \mathrm{Ob}(\mathcal{Y})$ and $F(x) = G(y)$, $F(x') = G(y')$. Let $[(a, b)] : (x, y) \longrightarrow (x', y')$ be a morphism in $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})^1$. Hence $(a, b) : (x, y) \longrightarrow (x', y')$ is morphism in $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ and so $a : x \longrightarrow x'$ and $b : y \longrightarrow y'$ are morphisms in $\mathcal{X}$ and $\mathcal{Y}$ respectively such that $F(a) = G(b)$. Then $[a] : x \longrightarrow x'$ and $[b] : y \longrightarrow y'$ are morphisms in $\mathcal{X}'$ and $\mathcal{Y}'$ respectively such that $F^1[a] = G^1[b]$, which means $([a], [b])$ is a morphism in $\mathcal{X}' \times_{\mathcal{Z}'} \mathcal{Y}'$ from (x, y) to (x', y'). If $[(a, b)] = [(a', b')]$, then letting $U = p(x) = q(y)$ and $V = p(x') = q(y')$, there is a covering $\{f_i : U_i \longrightarrow U\}$ such that $(a, b)|_{U_i} = (a', b')|_{U_i}$, that is to say, $(a|_{U_i}, b|_{U_i}) = (a'|_{U_i}, b'|_{U_i})$ and so $a|_{U_i} = a'|_{U_i}$ and $b|_{U_i} = b'|_{U_i}$. Thus $a, a' : x \longrightarrow x'$ and $b, b' : y \longrightarrow y'$ are two pairs of locally equal morphisms in $\mathcal{X}$ and $\mathcal{Y}$ respectively.

Defining $K^1 : (\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})^1 \longrightarrow \mathcal{X}^1 \times_{\mathcal{Z}^1} \mathcal{Y}^1$ as the identity in the objects and by $K^1[(a, b)] = ([a], [b])$ in the morphisms is easy to see that if $[(c, d)] : (x', y') \longrightarrow (x'', y'')$ is another morphism in $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})^1$, then $K^1([(c, d)] \circ [(a, b)]) = K^1[(c, d)] \circ K^1[(a, b)]$ and $K^1[id_{(x,y)}] = id_{(x,y)}$. Therefore is a well defined functor. Reciprocally, if $([a], [b])$ is a morphism in $\mathcal{X}^1 \times_{\mathcal{Z}^1} \mathcal{Y}^1$, then $[a] : x \longrightarrow x'$ and $[b] : y \longrightarrow y'$ are morphisms in $\mathcal{X}^1$ and $\mathcal{Y}^1$ respectively such that $F^1[a] = G^1[b]$, that is to say $F(a) = G(b)$ and so (a, b) is a morphism in $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ and therefore $[(a, b)]$ is a morphism in $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})^1$. This defines a functor $L^1 : \mathcal{X}^1 \times_{\mathcal{Z}^1} \mathcal{Y}^1 \longrightarrow (\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})^1$ which is the inverse of K^1.

In particular K is an equivalence of categories and therefore $\mathcal{X}^1 \times_{\mathcal{Z}^1} \mathcal{Y}^1$ is a fibred category. Moreover, $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})^1$ is a fibre product.

2. Locally defined morphisms: $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})^2 \cong \mathcal{X}^2 \times_{\mathcal{Z}^2} \mathcal{Y}^2$.

As before, $\mathrm{Ob}((\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})^2) = \mathrm{Ob}(\mathcal{X}^2 \times_{\mathcal{Z}^2} \mathcal{Y}^2) = \mathrm{Ob}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$, an without lost of generality we can assume $\mathcal{X}^1 = \mathcal{X}$, $\mathcal{Y}^1 = \mathcal{Y}$ and $\mathcal{Z}^1 = \mathcal{Z}$. Given $(x, y), (x', y') \in \mathrm{Ob}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$ a morphism from (x, y) to (x', y') in $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})^2$ is a locally defined morphism in $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$, that is to say, a triple $(f, \{f_i\}, c_i)$ where $f : U \longrightarrow V$ is a morphism in $\mathcal{C}$, $\{f_i : U_i \longrightarrow U\}$ is a covering and $c_i : f_i^*(x, y) \longrightarrow (x', y')$ is a morphism in $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ such that $c_i|_{U_{ij}} = c_j|_{U_{ij}}$ for all $i, j \in I$. Now, we are in the case $f_i^*(x, y) = (f_i^* x, f_i^* y)$ and therefore $c_i = (a_i, b_i)$, where $a_i : f_i^* x \longrightarrow x'$, $b_i : f_i^* y \longrightarrow y'$ and $F(a_i) = G(b_i)$. Now, $c_i|_{U_{ij}} = c_j|_{U_{ij}}$ if and only if $a_i|_{U_{ij}} = a_j|_{U_{ij}}$ and $b_i|_{U_{ij}} = b_j|_{U_{ij}}$, and therefore $(f, \{f_i\}, a_i)$ and $(f, \{f_i\}, b_i)$ are locally defined morphisms in $\mathcal{X}$ and $\mathcal{Y}$ respectively such that $F^2(f, \{f_i\}, a_i) = G^2(f, \{f_i\}, b_i)$, and so $((f, \{f_i\}, a_i), (f, \{f_i\}, b_i))$ is a morphism in $\mathcal{X}^2 \times_{\mathcal{Z}^2} \mathcal{Y}^2$ from (x, y) to (x', y'). This defines a functor $K^2 : (\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})^2 \longrightarrow \mathcal{X}^2 \times_{\mathcal{Z}^2} \mathcal{Y}^2$. If $(g, \{g_m\}, (d_m, e_m)) : (x', y') \longrightarrow (x'', y'')$ is another morphism, then

$$\begin{aligned}(g, \{g_m\}, (d_m, e_m)) \circ (f, \{f_i\}, (a_i, b_i)) &= (g \circ f, \{h_{im}\}, (a_i, b_i)' \circ (d_m, e_m)) \\ &= (g \circ f, \{h_{im}\}, (a_i', b_i') \circ (d_m, e_m)) \\ &= (g \circ f, \{h_{im}\}, (d_m \circ a_i', e_m \circ b_i'))\end{aligned}$$

and therefore

$$\begin{aligned}K^2((g, \{g_m\}, (d_m, e_m)) \circ (f, \{f_i\}, (a_i, b_i))) &= ((g \circ f, \{h_{im}\}, d_m \circ a_m'), (g \circ f, \{h_{im}\}, e_m \circ b_m')) \\ &= ((g, \{g_m\}, d_m) \circ (f, \{f_i\}, a_i), (g, \{g_m\}, e_m) \circ (f, \{f_i\}, b_i)) \\ &= ((g, \{g_m\}, d_m), (g, \{g_m\}, e_m)) \circ ((f, \{f_i\}, a_i), (f, \{f_m\}, b_i)) \\ &= K^2(g, \{g_m\}, (d_m, e_m)) \circ K^2(f, \{f_i\}, (a_i, b_i))\end{aligned}$$

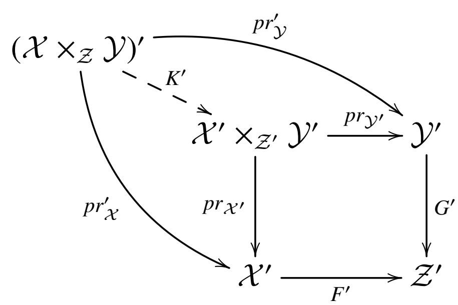
Then K^2 is compatible with composition and is easy to see that is also compatible with identities. Reciprocally, if $((f, \{f_i\}, a_i), (g, \{g_m\}, b_m))$ is a morphism in $\mathcal{X}^2 \times_{\mathcal{Z}^2} \mathcal{Y}^2$, then $F^2(f, \{f_i\}, a_i) = G^2(g, \{g_m\}, b_m)$ and so $(f, \{f_i\}, F(a_i) \circ \alpha_i) = (g, \{g_m\}, G(b_m) \circ \beta_m)$. It follows that $f = g$, $\{f_i\} = \{g_m\}$ and therefore by a change of index we have $F(a) \circ \alpha_i = G(b_m) \circ \alpha_i$. Since α_i is an isomorphism, $F(a_i) = G(b_i)$ and so (a_i, b_i) is a morphism in $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ from $(f_i^* x, f_i^* y)$

to (x', y'). Moreover, since $a_i|_{U_{ij}} = a_j|_{U_{ij}}$ and $b_i|_{U_{ij}} = b_j|_{U_{ij}}$, then $(a_i, b_i)|_{U_{ij}} = (a_j, b_j)|_{U_{ij}}$ and therefore $(f, \{f_i\}, (a_i, b_i))$ is a locally defined morphism in $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ from (x, y) to (x', y'). This defines a functor $L^2 : \mathcal{X}^2 \times_{\mathcal{Z}^2} \mathcal{Y}^2 \longrightarrow (\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})^2$ which is clearly the inverse of K^2.

In particular K^2 is an equivalence of categories and therefore $\mathcal{X}^2 \times_{\mathcal{Z}^2} \mathcal{Y}^2$ is a fibred category. Moreover, $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})^2$ is a fibre product.

3. Effective descent data: $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})' \cong \mathcal{X}' \times_{\mathcal{Z}'} \mathcal{Y}'$.

Without lost of generality we can assume $\mathcal{X}^2 = \mathcal{X}$, $\mathcal{Y}^2 = \mathcal{Y}$ and $\mathcal{Z}^2 = \mathcal{Z}$. Stackification preserves commutative diagrams, and therefore in the following diagram the external square is commutative



Since the internal square is cartesian, there exists a 1-morphism $K' : (\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})' \longrightarrow \mathcal{X}' \times_{\mathcal{Z}'} \mathcal{Y}'$ of categories over $\mathcal{C}$, unique such that the triangles are commutative.

Given an object $(\{f_i\}, ((x_i, y_i), (\varphi_{ij}, \rho_{ij})))$ of $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$ we have (x_i, y_i) and $(\varphi_{ij}, \rho_{ij})$ are an object and a morphism in $\in \mathrm{Ob}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$. Then $F(x_i) = G(y_i)$ and $F(\varphi_{ij}) = G(\rho_{ij})$ and so $pr'_{\mathcal{X}}(\{f_i\}, ((x_i, y_i), (\varphi_{ij}, \rho_{ij}))) = (\{f_i\}, (pr_{\mathcal{X}}(x_i, y_i), \gamma_j^{-1} \circ pr_{\mathcal{X}}(\varphi_{ij}) \circ \gamma_i)) = (\{f_i, (x_i, \varphi_{ij})\})$. We are used that since $pr_i^*(x_i, y_i) = (pr_i^* x_i, pr_i^* y_i)$ and $pr_{\mathcal{X}}$, $pr_{\mathcal{Y}}$ are defined pointwise, then the morphisms $\gamma_i : pr_i^* pr_{\mathcal{X}}(x_i, y_i) \longrightarrow pr_{\mathcal{X}}(pr_i^*(x_i, y_i))$ is the identity. In the same way $pr'_{\mathcal{Y}}(\{f_i\}, ((x_i, y_i), (\varphi_{ij}, \rho_{ij}))) = (\{f_i, (x_i, \varphi_{ij})\})$. Since $pr_{\mathcal{X}'}$ and $pr_{\mathcal{Y}'}$ are also defined pointwise this implies that $K'(\{f_i\}, ((x_i, y_i), (\varphi_{ij}, \rho_{ij})))$ must be equal to $((\{f_i\}, (x_i, \varphi_{ij})), (\{f_i\}, (y_i, \rho_{ij})))$. If $(f, (a_{im}, b_{im}))$ is a morphism in $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})'$, following a similar argument we have

$pr'_{\mathcal{X}}(f,(a_{im},b_{im})) = (f,a_{im})$ and $pr'_{\mathcal{Y}}(f,(a_{im},b_{im})) = (f,b_{im})$ and $K'(f,(a_{im},b_{im}))$ must be $((f,a_{im}),(f,b_{im}))$.

Reciprocally, if $((\{f_i\},(x_i,\varphi_{ij})),(\{g_m\},(y_m,\rho_{mn})))$ is an object of $\mathcal{X}' \times_{\mathcal{Z}'} \mathcal{Y}'$, we have $(\{f_i\},(x_i,\varphi_{ij})) \in \mathrm{Ob}(\mathcal{X}')$, $(\{g_m\},(y_m,\rho_{mn})) \in \mathrm{Ob}(\mathcal{Y}')$ and $F'(\{f_i\},(x_i,\varphi_{ij})) = G'(\{g_m\},(y_m\rho_{mn}))$, that is to say $(\{f_i\},(F(x_i),\alpha_j^{-1} \circ F(\varphi_{ij}) \circ \alpha_i)) = (\{g_m\},(F(y_m),\beta_n^{-1} \circ G(\rho_{mn}) \circ \beta_m))$ and so $\{f_i\} = \{g_m\}$ and we can change the index m to have $f_i = g_i$ for all $i \in I$. Also $F(x_i) = G(y_i)$ and therefore $(x_i,y_i) \in \mathrm{Ob}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$. Moreover $pr_i^*(x_i,y_i) = (pr_i^*x_i, pr_i^*y_i)$ and the pull-back $pr_i^*(x_i,y_i) \longrightarrow (x_i,y_i)$ is the pair $(pr_i^*x_i \longrightarrow x_i, pr_i^*y_i \longrightarrow y_i)$. This means that $F(pr_i^*x_i \longrightarrow x_i) = G(pr_i^*y_i \longrightarrow y_i)$ and this implies that $\alpha_i = \beta_i$ for all $i \in I$. Since $\alpha_j^{-1} \circ F(\varphi_{ij}) \circ \alpha_i = \beta_n^{-1} \circ G(\rho_{ij}) \circ \beta_m$ and α_i is isomorphism, then $F(\varphi_{ij}) = G(\rho_{ij})$ and therefore (φ_{ij},ρ_{ij}) is a morphism in $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$. Hence $((x_i,y_i),(\varphi_{ij},\rho_{ij}))$ is a descent datum in $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ relative to $\{f_i\}$ and so $(\{f_i\},((x_i,y_i),(\varphi_{ij},\rho_{ij})))$ is an object in $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})'$. Moreover, if $((f,a_{im}),(g,b_{im}))$ is a morphism in $\mathcal{X}' \times_{\mathcal{Z}'} \mathcal{Y}'$, then (f,a_{im}) and (g,b_{im}) are morphisms in $\mathcal{X}'$ and $\mathcal{Y}'$ respectively and $F'(f,a_{im}) = G(g,b_{im})$, that is to say $(f,\beta_m^{-1} \circ F(a_{im}) \circ \alpha_i) = (g,\beta_m^{-1} \circ G(b_{im}) \circ \alpha_i)$ and so $f = g$ and $F(a_{im}) = G(b_{im})$, which means (a_{im},b_{im}) is a morphism in $\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y}$ and since restrictions are made pointwise, $(f,(a_{im},b_{im}))$ is a morphism in $(\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y})'$. We shall prove that this defines a functor $L' : \mathcal{X}' \times_{\mathcal{Z}'} \mathcal{Y}' \longrightarrow (\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})'$. If $((g,c_{mk}),(g,e_{mk}))$ is another morphism wich is composable with $((f,a_{im}),(f,b_{im}))$ then we have

$$\begin{aligned}
L'(((g,c_{mk}),(g,e_{mk})) \circ ((f,a_{im}),(f,b_{im}))) &= L'((g,c_{mk}) \circ (f,a_{im}),(g,e_{mk}) \circ (f,b_{im})) \\
&= L'((g \circ f, c_{mk} \circ a'_{im}),(g \circ f, e_{mk} \circ b'_{im})) \\
&= L'(g \circ f,(c_{mk} \circ a_{im}, e_{mk} \circ b_{im})) \\
&= (g \circ f,(c_{mk},b_{mk}) \circ (a'_{im},b'_{im})) \\
&= (g \circ f,(c_{mk},b_{mk}) \circ (a_{im},b_{im})') \\
&= (g,(c_{mk},e_{mk})) \circ (f,(a_{im},b_{im}))
\end{aligned}$$

$$= L'((g, c_{mk}), (g, e_{mk})) \circ L'((f, a_{im}), (f, b_{im}))$$

and therefore L' is compatible with composition. In the other hand, the identity of $((\{f_i\}, (x_i, \varphi_{ij}), (\{f_i\}, (y_i, \rho_{ij})))$ is $((id_U, \varphi_{ij}), (id_U, \rho_{ij}))$ and $L'((id_U, \varphi_{ij}), (id_U), \rho_{ij}) = (id_U, (\varphi_{ij}, \psi_{ij}))$ which is the identity of $L'(\{f_i\}, (x_i, \varphi_{ij}), (\{f_i\}, (y_i, \psi_{ij})) = (\{f_i\}, (x_i, y_i), (\varphi_{ij}, \psi_{ij}))$. Then L' is a functor and is easy to see that this is an inverse functor for K'. Hence K' is an isomorphism and so $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})'$ is a fibre product. In particular K' is an equivalence of categories over $\mathcal{C}$ and therefore $\mathcal{X}' \times_{\mathcal{Z}'} \mathcal{Y}'$ is a fibred category over $\mathcal{C}$. □

Remark. Previously we give an example, the heart shaped diagram, of fibred categories $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ such that $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ and $\mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y}$ where not equivalent categories. Later we proved that these categories are actually stacks and therefore their stackifications are isomorphic to them. Hence $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is a stack, not equivalent to the stack $\mathcal{X} \times^2_{\mathcal{Z}} \mathcal{Y}$, which is indeed the 2-fibre product of the stackifications. This shows that the fibre product of the stackifications is not in general isomorphic nor categorically equivalent to de 2-fibre product of the stackifications.

Chapter

3 Groupoids in Algebraic Spaces

3.1 Groupoid Categories

We are going to define groupoid categories since they are the basic concept in order to consider quotient stacks. The theory presented here is more general than the found in Stacks Project, since we are not restricting $\mathcal{C}$ to be the category of schemes of algebraic spaces and therefore the objects U, R and the morphisms s, t, c, e, i in the next definition has not to be functors and natural transformations respectively. Instead our approach is based on the relations between the objects and morphisms without having to go down to the category of sets. The most important construction here is the fibre product, which is going to be really useful in the next chapters. Originally, this was the first chapter in this work, since the existence of this fibre product was the first important step in the solution of the wanted results in quotient stacks. However, it was move down in order to have a more comprehensive structure. The reader can notice the difference in the tag (0231).

GROUPOID CATEGORIES

Definition 3.1.1 (Groupoid category). *Let $\mathcal{C}$ be a category with fibre products. A groupoid category is a quintuple (U, R, s, t, c), where U and R are objects of $\mathcal{C}$ and $s, t : R \longrightarrow U$ and $c : R \times_{s,U,t} R \longrightarrow R$ are morphisms satisfying the following properties:*

1. ***(Associativity)*** *$c \circ (c, 1) = c \circ (1, c)$ as morphisms $R \times_{s,U,t} R \times_{s,U,t} R \longrightarrow R$, i.e. the following diagram commutes:*

$$\begin{array}{ccc} R \times_U R \times_U R & \xrightarrow{(1,c)} & R \times_U R \\ {\scriptstyle (c,1)}\downarrow & & \downarrow{\scriptstyle c} \\ R \times_U R & \xrightarrow[c]{} & R \end{array}$$

2. ***(Identity)*** *There exists a morphism $e : U \longrightarrow R$ such that:*

   a) $s \circ e = t \circ e = id_U$

   b) $c \circ (1, e \circ s) = c \circ (e \circ t, 1) = id_R$

3. ***(Inverse)*** *There exists a morphism $i : R \longrightarrow R$ such that:*

   a) $s \circ i = t$ *and* $t \circ i = s$

   b) $c \circ (1, i) = e \circ t$ *and* $c \circ (i, 1) = e \circ s$

Example 11. Recall that a groupoid is a category $\mathcal{C}$ in which every morphism is an isomorphism. In that case, we can consider $U = \mathrm{Ob}(\mathcal{C})$ the class of objects and $R = \mathrm{Ar}(\mathcal{C})$ the class of all the morphisms on $\mathcal{C}$.

The morphisms $s, t : \mathrm{Ar}(\mathcal{C}) \longrightarrow \mathrm{Ob}(\mathcal{C})$, are the source and target maps, $c : \mathrm{Ar} \times_{t,\mathrm{Ob},s} \mathrm{Ar} \longrightarrow \mathrm{Ar}$ is the usual composition and $e : \mathrm{Ob} \longrightarrow \mathrm{Ar}$ and $i : \mathrm{Ar} \longrightarrow \mathrm{Ar}$ are the identity and inverse maps.

The morphisms $s, t : \mathrm{Ar}(\mathcal{C}) \longrightarrow \mathrm{Ob}(\mathcal{C})$, are the source and target maps which are defined as follows: for any $f : X \longrightarrow Y$ in $\mathrm{Ar}(\mathcal{C})$, $s(f) = X$ and $t(f) = Y$. The fibre product $\mathrm{Ar} \times_{s,\mathrm{Ob},t} \mathrm{Ar}$

is the class of pairs (f, g) such that $s(f) = t(g)$ that is to say the source of f is the target of g. The morphism $c : \mathrm{Ar}\times_{s,\mathrm{Ob},t}\mathrm{Ar} \longrightarrow \mathrm{Ar}$ is the usual composition $c(f, g) := f \circ g$. Associativity says that given f, g, h with $s(f) = t(g)$ and $s(g) = t(h)$ we have

$$\begin{aligned} f \circ (g \circ h) &= c(f, c(g, h)) = (c \circ (1, c))(f, g, h) \\ &= (c \circ (c, 1))(f, g, h) \\ &= c(c(f, g), h) \\ &= (f \circ g) \circ h \end{aligned}$$

which is the usual associativity property on $\mathcal{C}$. The identity of an object X is given by $e(x) = id_X$, which satisfy $s \circ e(X) = X = t \circ e(X)$, $c \circ (1, e \circ s)(f) = c(f, id_{s(f)}) = f \circ id_{s(f)} = f$ and $c \circ (e \circ t, 1)(f) = c(id_{t(f)}, f) = id_{t(f)} \circ f = f$. Since every morphism is actually isomorphism then each morphism has an inverse f^{-1}. The morphism $i : \mathrm{Ar} \longrightarrow \mathrm{Ar}$ is given by $i(f) = f^{-1}$. Hence, if $f : X \longrightarrow Y$, then $f^{-1} : Y \longrightarrow X$ is such that $f \circ f^{-1} = id_Y$ and $f^{-1} \circ f = id_X$. Therefore $s \circ i(f) = s(f^{-1}) = t(f)$, $t \circ i(f) = t(f^{-1}) = s(f)$, $c \circ (1, i)(f) = c \circ (f, i(f)) = f \circ f^{-1} = id_{t(f)} = e \circ t(f)$ and $c \circ (i, 1)(f) = c(i(f), f) = f^{-1} \circ f = id_{s(f)} = e \circ s(f)$.

Consequently every groupoid defines canonically a groupoid category.

Lemma 3.1.1. *The morphisms $e : U \longrightarrow R$ and $i : R \longrightarrow R$ are uniquely determined by the properties* 2) *and* 3) *in the definition. Furthermore $i \circ i = 1_R$.*

Proof. Let $e_1, e_2 : U \longrightarrow R$ be morphisms satisfying the conditions above. Then $e_2 = c \circ (e_1 \circ t, 1) \circ e_2 = c \circ (e_1 \circ t \circ e_2, e_2) = c \circ (e_1, e_2)$ and $e_1 = c \circ (1, e_2 \circ s) \circ e_1 = c \circ (e_1, e_2 \circ s \circ e_1) = c \circ (e_1, e_2)$ and so $e_1 = e_2$.

Suppose now that $i_1, i_2 : R \longrightarrow R$ are morphisms which $s \circ i_1 = t = s \circ i_2$, $t \circ i_1 = s = t \circ i_2$, $c \circ (1, i_1) = e \circ t = c \circ (1, i_2)$ and $c \circ (i_1, 1) = e \circ s = c \circ (i_2, 1)$. Using associativity, $c \circ (c, 1) \circ (i_1, 1, i_2) = c \circ (1, c) \circ (i_1, 1, i_2)$. But $c \circ (c, 1)(i_1, 1, i_2) = c \circ (c \circ (i_1, 1), i_2) = c \circ (e \circ s, i_2)$

and $c \circ (1, c)(i_1, 1, i_2) = c \circ (i_1, c \circ (1, i_2)) = c \circ (i_1, e \circ t)$. Then $c \circ (e \circ s, i_2) = c \circ (i_1, e \circ t)$. Also $s = t \circ i_2$ and $t = s \circ i_1$, hence $c \circ (e \circ t \circ i_2, i_2) = c \circ (i_1, e \circ s \circ i_1)$ and we have $c \circ (e \circ t, 1) \circ i_2 = c \circ (1, e \circ s) \circ i_1$, but from definition of $e : U \longrightarrow R$ we know that $c \circ (e \circ t, 1) = c \circ (1, e \circ s) = id_R$ and we conclude $i_1 = i_2$.

We shall show that i is idempotent. Again by associativity $c \circ (c, 1)(i \circ i, i, 1) = c \circ (1, c)(i \circ i, i, 1)$ and also $s \circ i \circ i = s$, as $s \circ i = t$ and $t \circ i = s$. But

$$\begin{aligned} c \circ (c, 1)(i \circ i, i, 1) &= c \circ (c \circ (i \circ i, i), 1) = c \circ (c \circ (i, 1) \circ i, 1) \\ &= c \circ (e \circ s \circ i, 1) \\ &= c \circ (e \circ t, 1) \\ &= id_R \end{aligned}$$

and on the other hand

$$\begin{aligned} c \circ (1, c)(i \circ i, i, 1) &= c \circ (i \circ i, c \circ (i, 1)) \\ &= c \circ (i \circ i, e \circ s) \\ &= c \circ (i \circ i, e \circ s \circ i \circ i) \\ &= c \circ (1, e \circ s) \circ i \circ i \\ &= i \circ i \end{aligned}$$

because $c \circ (1, e \circ s) = id_R$. Therefore $i \circ i = id_R$. □

Definition 3.1.2 (Morphism of groupoid categories). *Let $\mathcal{G} = (U, R, s, t, c)$ and $\mathcal{G}' = (U', R', s', t', c')$ be groupoid categories. A morphism $F : \mathcal{G} \longrightarrow \mathcal{G}'$ is a pair of morphisms $\varphi : U \longrightarrow U'$ and $\widetilde{\varphi} : R \longrightarrow R'$ such that $s' \circ \widetilde{\varphi} = \varphi \circ s$, $t' \circ \widetilde{\varphi} = \varphi \circ t$ and $c' \circ (\widetilde{\varphi}, \widetilde{\varphi}) = \widetilde{\varphi} \circ c$.*

Lemma 3.1.2. *Let $(\varphi, \widetilde{\varphi}) : (U, R, s, t, c) \longrightarrow (U', R', s', t', c')$ and $(\psi, \widetilde{\psi}) : (U', R', s', t', c') \longrightarrow (U'', R'', s'', t'', c'')$ be morphisms of groupoid categories.*

i) The pair $(\psi\circ\varphi, \widetilde{\psi}\circ\widetilde{\varphi})$ *is a morphism of groupoid categories, called composition of* $(\varphi, \widetilde{\varphi})$ *and* $(\psi, \widetilde{\psi})$*, which is denoted* $(\varphi, \widetilde{\varphi}) \circ (\psi, \widetilde{\psi})$.

ii) The pair $(id_{U'}, id_{R'})$ *is a morphism* $(U', R', s', t', c') \longrightarrow (U', R', s', t', c')$ *and for all* $(\varphi, \widetilde{\varphi})$ *and* $(\psi, \widetilde{\psi})$ *as before we have* $(id_{U'}, id_{R'})\circ(\varphi, \widetilde{\varphi}) = (\varphi, \widetilde{\varphi})$ *and* $(\psi, \widetilde{\psi})\circ(id_{U'}, id_{R'}) = (\psi, \widetilde{\psi})$. *Then that pair is de identity with the composition rule defined as in part i).*

Proof.

i) In fact, that pair makes sense, because there exist the compositions $U \xrightarrow{\varphi} U' \xrightarrow{\psi} U''$ and $R \xrightarrow{\widetilde{\varphi}} R' \xrightarrow{\widetilde{\psi}} R''$. We must to show that pair is a functor of groupoid categories.

- We have

$$\begin{aligned} s'' \circ (\widetilde{\psi} \circ \widetilde{\varphi}) &= (s'' \circ \widetilde{\psi}) \circ \widetilde{\varphi} \\ &= (\psi \circ s') \circ \widetilde{\varphi} = \psi \circ (s' \circ \widetilde{\varphi}) \\ &= \psi \circ (\varphi \circ s) \\ &= (\psi \circ \varphi) \circ s \end{aligned}$$

- In the same way $t'' \circ (\widetilde{\psi} \circ \widetilde{\varphi}) = (\psi \circ \varphi) \circ t$.
- Also

$$\begin{aligned} c'' \circ (\widetilde{\psi} \circ \widetilde{\varphi}, \widetilde{\psi} \circ \widetilde{\varphi}) &= c'' \circ ((\widetilde{\psi}, \widetilde{\psi}) \circ (\widetilde{\varphi}, \widetilde{\varphi})) \\ &= (c'' \circ (\widetilde{\psi}, \widetilde{\psi})) \circ (\widetilde{\varphi}, \widetilde{\varphi}) \\ &= (\widetilde{\psi} \circ c') \circ (\widetilde{\varphi}, \widetilde{\varphi}) \\ &= \widetilde{\psi} \circ (c' \circ (\widetilde{\varphi}, \widetilde{\varphi})) \\ &= \widetilde{\psi} \circ (\widetilde{\varphi} \circ c) \\ &= (\widetilde{\psi} \circ \widetilde{\varphi}) \circ c \end{aligned}$$

This shows that $(\psi, \circ\varphi, \widetilde{\psi}\circ\widetilde{\varphi})$ is a functor of groupoid categories. In particular $(\widetilde{\psi}\circ\widetilde{\varphi})\circ e = e'' \circ (\psi \circ \varphi)$ and $(\widetilde{\psi} \circ \widetilde{\varphi}) \circ i = i'' \circ (\widetilde{\psi} \circ \widetilde{\varphi})$.

ii) Once again, the pair $(id_{U'}, id_{R'})$ makes sense. Moreover, $s' \circ id_{R'} = s' = id_{U'} \circ s'$, $t' \circ id_{R'} = t' = id_{U'} \circ t'$ and $c' \circ (id_{R'}, id_{R'}) = c' = id_{R'} \circ c'$. Then $(id_{U'}, id_{R'})$ is a functor from (U', R', s', t', c') to (U', R', s', t', c'). Furthermore, composing we have $(\psi, \widetilde{\psi}) \circ (id_{U'}, id_{R'}) = (\psi \circ id_{U'}, \widetilde{\psi} \circ id_{R'}) = (\psi, \widetilde{\psi})$ and $(id_{U'}, id_{R'}) \circ (\varphi, \widetilde{\varphi}) = (id_{U'} \circ \varphi, id_{R'} \circ \widetilde{\varphi}) = (\varphi, \widetilde{\varphi})$. □

Definition 3.1.3 (Category of groupoids categories). *The category of groupoid categories is the category whose objects are groupoids categories and the morphisms are morphisms of groupoid categories, with the composition and identities given by the previous lemma. Consequently* $(\psi, \widetilde{\psi}) \circ (\varphi, \widetilde{\varphi}) = (\psi \circ \varphi, \widetilde{\psi} \circ \widetilde{\varphi})$ *and* $id_{(U,R,s,t,c)} = (id_U, id_R)$.

Lemma 3.1.3. *With the notation as before, if* $(\varphi, \widetilde{\varphi}) : \mathcal{G} \longrightarrow \mathcal{G}'$ *is a functor of groupoid categories, then* $\widetilde{\varphi} \circ e = e' \circ \varphi$ *and* $\widetilde{\varphi} \circ i = i' \circ \widetilde{\varphi}$.

Proof.

In order to show that $\widetilde{\varphi} \circ e = e' \circ \varphi$, consider the following:

i) $c \circ (e, e) = c \circ (e, e \circ s \circ e) = c \circ (1, e \circ s) \circ e = e$. Therefore $c' \circ (\widetilde{\varphi}, \widetilde{\varphi}) \circ (e, e) = \widetilde{\varphi} \circ c \circ (e, e) = \widetilde{\varphi} \circ e$.

ii)

$$\begin{aligned} c' \circ (i' \circ \widetilde{\varphi}, \widetilde{\varphi}) \circ e &= c' \circ (i', 1) \circ \widetilde{\varphi} \circ e \\ &= e' \circ s' \circ \widetilde{\varphi} \circ e \\ &= e' \circ \varphi \circ s \circ e \\ &= e' \circ \varphi \end{aligned}$$

iii)

$$\begin{aligned} c' \circ (i' \circ \widetilde{\varphi}, \widetilde{\varphi}) \circ e &= c' \circ (i' \circ \widetilde{\varphi} \circ e, \widetilde{\varphi} \circ e) \\ &= c' \circ (i' \circ e, c' \circ (\widetilde{\varphi}, \widetilde{\varphi}) \circ (e, e)) \\ &= c' \circ (1, c') \circ (i' \circ \widetilde{\varphi}, \widetilde{\varphi}, \widetilde{\varphi}) \circ e \\ &= c' \circ (c', 1) \circ (i' \circ \widetilde{\varphi}, \widetilde{\varphi}, \widetilde{\varphi}) \circ e \\ &= c' \circ (c' \circ (i' \circ \widetilde{\varphi}, \widetilde{\varphi}) \circ e, \widetilde{\varphi} \circ e) \\ &= c' \circ (e' \circ \varphi, \widetilde{\varphi} \circ e) \\ &= c' \circ (e' \circ t' \circ \widetilde{\varphi} \circ e, \widetilde{\varphi} \circ e) \\ &= c' \circ (e' \circ t', 1) \circ \widetilde{\varphi} \circ e \\ &= \widetilde{\varphi} \circ e \end{aligned}$$

The last two equations show $\widetilde{\varphi} \circ e = e' \circ \varphi$ that is to say, the compatibility with identity. On the other hand, to show $\widetilde{\varphi} \circ i = i' \circ \widetilde{\varphi}$, consider the following equalities:

i)

$$\begin{aligned} c' \circ (i' \circ \widetilde{\varphi}, \widetilde{\varphi}) &= c' \circ (i', 1) \circ \widetilde{\varphi} \\ &= e' \circ s' \circ \widetilde{\varphi} \\ &= e' \circ \varphi \circ s \\ &= \widetilde{\varphi} \circ e \circ s \end{aligned}$$

ii)

$$\begin{aligned} c' \circ (i' \circ \widetilde{\varphi}, c' \circ (\widetilde{\varphi}, \widetilde{\varphi} \circ i)) &= c' \circ (i' \circ \widetilde{\varphi}, c' \circ (\widetilde{\varphi}, \widetilde{\varphi}) \circ (1, i)) \\ &= c' \circ (i' \circ \widetilde{\varphi}, \widetilde{\varphi} \circ c \circ (1, i)) \end{aligned}$$

$$
\begin{aligned}
&= c' \circ (i' \circ \widetilde{\varphi}, \widetilde{\varphi} \circ e \circ t) \\
&= c' \circ (i' \circ \widetilde{\varphi}, e' \circ \varphi \circ t) \\
&= c' \circ (i' \circ \widetilde{\varphi}, e' \circ t' \circ \widetilde{\varphi}) \\
&= c' \circ (i', e' \circ t') \circ \widetilde{\varphi} \\
&= c' \circ (i', e' \circ s' \circ i') \circ \widetilde{\varphi} \\
&= c' \circ (1, e' \circ s') \circ i' \circ \widetilde{\varphi} \\
&= i' \circ \widetilde{\varphi}
\end{aligned}
$$

iii)

$$
\begin{aligned}
c' \circ (i' \circ \widetilde{\varphi}, c' \circ (\widetilde{\varphi}, \widetilde{\varphi} \circ i)) &= c' \circ (1, c') \circ (i' \circ \widetilde{\varphi}, \widetilde{\varphi}, \widetilde{\varphi} \circ i) \\
&= c' \circ (c', 1) \circ (i' \circ \widetilde{\varphi}, \widetilde{\varphi}, \widetilde{\varphi} \circ i) \\
&= c' \circ (c' \circ (i' \circ \widetilde{\varphi}, \widetilde{\varphi}), \widetilde{\varphi} \circ i) \\
&= c' \circ (\widetilde{\varphi} \circ e \circ s, \widetilde{\varphi} \circ i) \\
&= c' \circ (\widetilde{\varphi}, \widetilde{\varphi}) \circ (e \circ s, i) \\
&= \widetilde{\varphi} \circ c \circ (e \circ s, i) \\
&= \widetilde{\varphi} \circ c \circ (e \circ t \circ i, i) \\
&= \widetilde{\varphi} \circ i
\end{aligned}
$$

Once again, the last two equations show that $\widetilde{\varphi} \circ i = i' \circ \widetilde{\varphi}$ and we have compatibility with inverse. □

Remark. In categories in general is not automatic that $\widetilde{\varphi} \circ e = e' \circ \varphi$. Note that here we have used the existence of inverse.

3.2 Fibre product of groupoid categories

In this section we will see that in groupoid categories the fibre product always exists provided that the base category $\mathcal{C}$ admits fibre products.

Theorem 5. *If $\mathcal{C}$ is a category with fibre products, then the category of groupoid categories on $\mathcal{C}$ have fibre products.*

Proof.

Let (U, R, s, t, c), (U', R', s', t', c') and $(U'', R'', s'', t'', c'')$ be groupoids on $\mathcal{C}$ and $(\varphi, \widetilde{\varphi}) : (U, R, s, t, c) \longrightarrow (U'', R'', s'', t'', c')$ and $(\psi, \widetilde{\psi}) : (U', R', s', t', c') \longrightarrow (U'', R'', s'', t'', c'')$ morphisms of groupoids. Hence $\varphi : U \longrightarrow U'$ and $\widetilde{\varphi} : R \longrightarrow R'$ are morphisms on $\mathcal{C}$ such that $s' \circ \widetilde{\varphi} = \varphi \circ s$, $t' \circ \widetilde{\varphi} = \varphi \circ t$ and $c' \circ (\widetilde{\varphi}, \widetilde{\varphi}) = \widetilde{\varphi} \circ c$. In the same way $\psi : U' \longrightarrow U''$ and $\widetilde{\psi} : R' \longrightarrow R''$ are morphisms on $\mathcal{C}$ satisfying $s'' \circ \widetilde{\psi} = \psi \circ s'$, $t'' \circ \widetilde{\psi} = \psi \circ t$ and $c'' \circ (\widetilde{\psi}, \widetilde{\psi}) = \widetilde{\psi} \circ c'$.

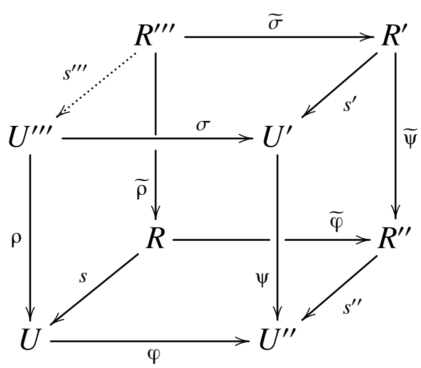
We will construct a groupoid category $(U''', R''', s''', t''', c''')$ and morphisms of $(\rho, \widetilde{\rho}) : (U''', R''', s''', t''', c''') \longrightarrow (U, R, s, t, c)$ and $(\sigma, \widetilde{\sigma}) : (U''', R''', s''', t''', c''') \longrightarrow (U', R', s', t', c')$ such that the diagram

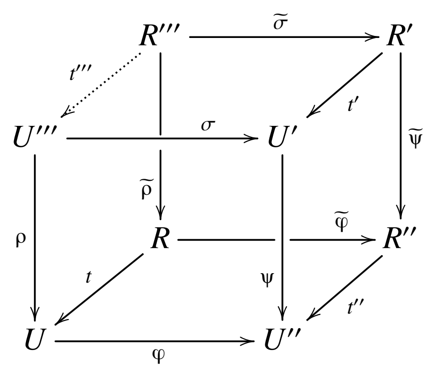
$$\begin{array}{ccc} (U''', R''', s''', t''', c''') & \xrightarrow{(\sigma, \widetilde{\sigma})} & (U', R', s', t', c') \\ \downarrow{\scriptstyle (\rho, \widetilde{\rho})} & & \downarrow{\scriptstyle (\psi, \widetilde{\psi})} \\ (U, R, s, t, c) & \xrightarrow[(\varphi, \widetilde{\varphi})]{} & (U'', R'', s'', t'', c'') \end{array}$$

is cartesian in the category of groupoids on $\mathcal{C}$. For this, using the fact on $\mathcal{C}$ there are fibre product, define $U''', R''', \varphi, \psi, \widetilde{\varphi}, \widetilde{\psi}$ so that the following diagrams are cartesian squares:

$$\begin{array}{ccc} U''' & \xrightarrow{\sigma} & U' \\ \rho\downarrow & & \downarrow\psi \\ U & \xrightarrow[\varphi]{} & U'' \end{array} \qquad\qquad \begin{array}{ccc} R''' & \xrightarrow{\widetilde{\sigma}} & R' \\ \widetilde{\rho}\downarrow & & \downarrow\widetilde{\psi} \\ R & \xrightarrow[\widetilde{\varphi}]{} & R'' \end{array}$$

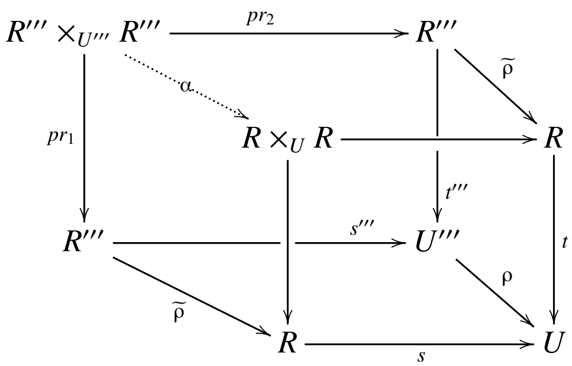
to define the morphisms $s''', t''' : R''' \longrightarrow U'''$ look at the next diagrams:

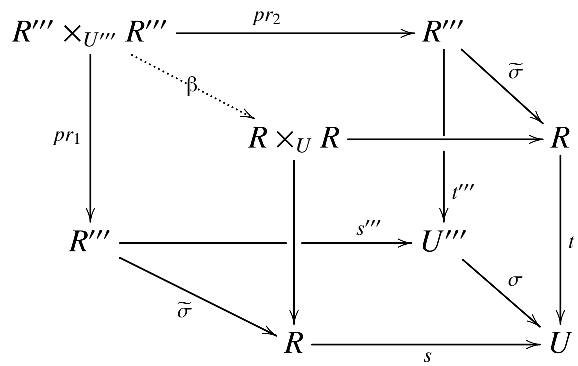




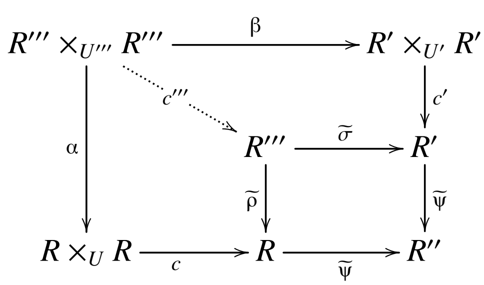
where in each case the front and back faces of the cube are cartesian squares and by construction of U''' and R''', the lower and right faces are commutative because of definition of morphism of groupoids. The universal property of fibre product ensure the existence of the morphisms $s''', t''' : R''' \longrightarrow U'''$ as the only ones such that $\rho \circ s''' = s \circ \widetilde{\rho}$, $\sigma \circ s''' = s' \circ \widetilde{\sigma}$, $\rho \circ t''' = t \circ \rho$ and $\sigma \circ t''' = t' \circ \widetilde{\sigma}$.

Now we will define $c''' : R''' \times_{s''', U''', t'''} R''' \longrightarrow R'''$. Consider the following diagrams:





Both are commutatives and again for the universal property of fibre product $R \times_{s,U,t} R$ there exist α and β unique such that the respective cubes commute. Then $\alpha = (\widetilde{\rho} \circ pr_1, \widetilde{\rho} \circ pr_2) = c \circ (\widetilde{\rho}, \widetilde{\rho}) \circ (pr_1, pr_2) = c \circ (\widetilde{\rho}, \widetilde{\rho})$ and $\beta = (\widetilde{\sigma} \circ pr_1, \widetilde{\sigma} \circ pr_2) = c' \circ (\widetilde{\sigma}, \widetilde{\sigma}) \circ (pr_1, pr_2) = c' \circ (\widetilde{\sigma}, \widetilde{\sigma})$, since $(pr_1, pr_2) = id_{R'''}$. To define c''' regard the following diagram:



We claim the external rectangle commutes. In fact,

$$
\begin{aligned}
\widetilde{\varphi} \circ c \circ \alpha &= c'' \circ (\widetilde{\varphi}\widetilde{\varphi}) \circ \alpha \\
&= c'' \circ (\widetilde{\varphi}, \widetilde{\varphi}) \circ (\widetilde{\rho} \circ pr_1, \widetilde{\rho} \circ pr_2) \\
&= c'' \circ (\widetilde{\varphi} \circ \widetilde{\rho} \circ pr_1, \widetilde{\varphi} \circ \widetilde{\rho} \circ pr_2) \\
&= c'' \circ (\widetilde{\psi} \circ \widetilde{\sigma} \circ pr_1, \widetilde{\psi} \circ \widetilde{\sigma} \circ pr_2) \\
&= c'' \circ (\widetilde{\psi}, \widetilde{\psi}) \circ (\widetilde{\sigma} \circ pr_1, \widetilde{\sigma} \circ pr_2) \\
&= \widetilde{\psi} \circ c' \circ \beta
\end{aligned}
$$

Thus, for the universal property of fibre product, there exists $c''' : R''' \times_{U'''} R''' \longrightarrow R'''$ unique such that $\widetilde{\rho} \circ c''' = c \circ \alpha$ and $\widetilde{\sigma} \circ c''' = c' \circ \beta$. Moreover $c''' = (c \circ (\widetilde{\rho}, \widetilde{\rho}), c' \circ (\widetilde{\sigma}, \widetilde{\sigma}))$.

Let us see that this is a groupoid category. By construction we have $s''' = (s \circ \widetilde{\rho}, s' \circ \widetilde{\sigma})$, $t''' = (t \circ \widetilde{\rho}, t' \circ \widetilde{\sigma})$ and $c''' = (c \circ (\widetilde{\rho}, \widetilde{\rho}), c' \circ (\widetilde{\sigma}, \widetilde{\sigma}))$. To see that $(U''', R''', s''', t''', c''')$ is a groupoid category we need to show that the morphisms satisfy the properties:

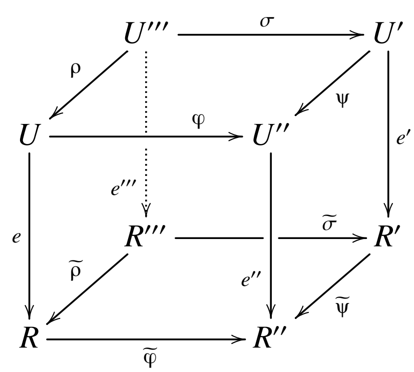
1. Associativity.

$$\begin{aligned}
\widetilde{\rho} \circ (c''' \circ (c''', 1)) &= (\widetilde{\rho} \circ c''') \circ (c''', 1) \\
&= c \circ (\widetilde{\rho}, \widetilde{\rho}) \circ (c''', 1) \\
&= c \circ (\widetilde{\rho} \circ c''', \widetilde{\rho}) \\
&= c \circ (c \circ (\widetilde{\rho}, \widetilde{\rho}), \widetilde{\rho}) \\
&= c \circ (c, 1) \circ (\widetilde{\rho}, \widetilde{\rho}, \widetilde{\rho})
\end{aligned}$$

In the same way $\widetilde{\rho} \circ (c''' \circ (1, c''')) = c \circ (1, c) \circ (\widetilde{\rho}, \widetilde{\rho}, \widetilde{\rho})$ since $c \circ (c, 1) = c \circ (1, c)$, it follows that $\widetilde{\rho} \circ (c''' \circ (c''', 1)) = \widetilde{\rho} \circ (c''', (1, c'''))$. In the same way $\widetilde{\sigma} \circ (c''' \circ (c''', 1)) = \widetilde{\sigma} \circ (1, c''')$. But the universal property of the fibre product $R''' = R \times_{R''} R'$ ensure the unicity of such a morphism. Then we have $c''' \circ (c''', 1) = c''' \circ (1, c''')$.

We will define now the morphisms $e''' : U''' \longrightarrow R'''$ and $i''' : R''' \longrightarrow R'''$.

2. Identity. In order to define $e''' : U''' \longrightarrow R'''$ consider the diagram



The upper and lower faces are cartesian squares, the front and back ones are commutative, since $(\varphi, \widetilde{\varphi})$ and $(\psi, \widetilde{\psi})$ are morphisms of groupoids. Then by the universal property of fibre product, exists a unique morphism $e''' : U''' \longrightarrow R'''$ such that $\widetilde{\sigma} \circ e''' = e' \circ \sigma$ and $\widetilde{\rho} \circ e''' = e \circ \rho$.

a) We have the equality

$$\begin{aligned} s''' \circ e''' &= (s \circ \widetilde{\rho}, s' \circ \widetilde{\sigma}) \circ (e \circ \rho, e' \circ \sigma) \\ &= (s \circ \widetilde{\rho} \circ e \circ \rho, s' \circ \widetilde{\sigma} \circ e' \circ \sigma) \\ &= (s \circ e \circ \rho, s' \circ e' \circ \sigma) \\ &= (\rho, \sigma) \\ &= id_{U'''} \end{aligned}$$

Similarly $t''' \circ e''' = id_{U'''}$.

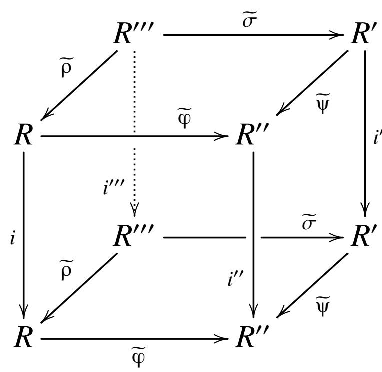
b) Also we have

$$\begin{aligned} c''' \circ (1, e'' \circ s''') &= (c \circ (\widetilde{\rho}, \widetilde{\rho}), c' \circ (\widetilde{\sigma}, \widetilde{\sigma})) \circ (1, e''' \circ s''') \\ &= (c \circ (\widetilde{\rho}, \widetilde{\rho}) \circ (1, e''' \circ s'''), c' \circ (\widetilde{\sigma}, \widetilde{\sigma}) \circ (1, e''' \circ s''')) \\ &= (c \circ (\widetilde{\rho}, \widetilde{\rho} \circ e''' \circ s'''), c' \circ (\widetilde{\sigma}, \widetilde{\sigma} \circ e''' \circ s''')) \end{aligned}$$

$$
\begin{aligned}
&= (c \circ (\widetilde{\rho}, e \circ \rho \circ s'''), c' \circ (\widetilde{\sigma}, e' \circ \sigma \circ s'''))\\
&= (c \circ (\widetilde{\rho}, e \circ s \circ \widetilde{\rho}), c' \circ (\widetilde{\sigma}, e' \circ s' \circ \widetilde{\sigma}))\\
&= (c \circ (1, e \circ s) \circ \widetilde{\rho}, c' \circ (1, e' \circ s') \circ \widetilde{\sigma})\\
&= (\widetilde{\rho}, \widetilde{\sigma})\\
&= id_{R'''}
\end{aligned}
$$

where we have used $c \circ (1, e \circ s) = id_R$ and $c' \circ (1, e' \circ s') = id_{R'}$. In the same way $c''' \circ (e''' \circ t''', 1) = id_{R'''}$.

3. Inverse. Likewise, to define $i''' : R''' \longrightarrow R'''$ consider the diagram



Just as before there is a morphism $i''' : R''' \longrightarrow R'''$ such that $\widetilde{\rho} \circ i''' = i \circ \widetilde{\rho}$ and $\widetilde{\sigma} \circ i''' = i' \circ \widetilde{\sigma}$.

a)

$$
\begin{aligned}
s''' \circ i''' &= (s \circ \widetilde{\rho}, s' \circ \widetilde{\sigma}) \circ (i \circ \widetilde{\rho}, i' \circ \widetilde{\sigma})\\
&= (s \circ i \circ \widetilde{\rho}, s' \circ i' \circ \widetilde{\sigma})\\
&= (t \circ \widetilde{\rho}, t' \circ \widetilde{\sigma})\\
&= t'''
\end{aligned}
$$

Thus, by using the properties $t''' \circ i''' = s'''$.

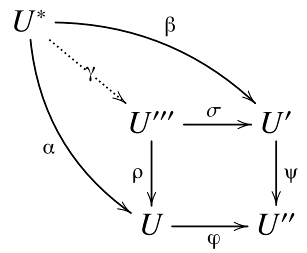
b)

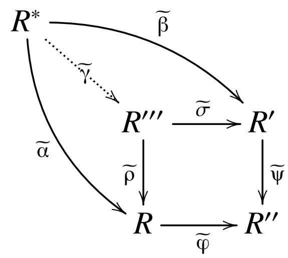
$$\begin{aligned}
c''' \circ (1, i''') &= (c \circ (\widetilde{\rho}, \widetilde{\rho})) \circ (1, i''') \\
&= (c \circ (\widetilde{\rho}, \widetilde{\rho}) \circ (1, i'''), c' \circ (\widetilde{\sigma}, \widetilde{\sigma} \circ i''')) \\
&= (c \circ (\widetilde{\rho}, i\widetilde{\rho}), c' \circ (\widetilde{\sigma}, i' \circ \widetilde{\sigma})) \\
&= (c \circ (1, i) \circ \widetilde{\rho}, c' \circ (1, i') \circ \widetilde{\sigma}) \\
&= (e \circ t \circ \widetilde{\rho}, e' \circ t' \circ \widetilde{\sigma}) \\
&= (e \circ \rho \circ t''', e' \circ \sigma \circ t''') \\
&= (\widetilde{\rho} \circ e''' \circ t''', \widetilde{\sigma} \circ e''' \circ t''') \\
&= (\widetilde{\rho}, \widetilde{\sigma}) \circ e''' \circ t''' \\
&= e''' \circ t'''
\end{aligned}$$

and also $c''' \circ (i''', 1) = e''' \circ s'''$.

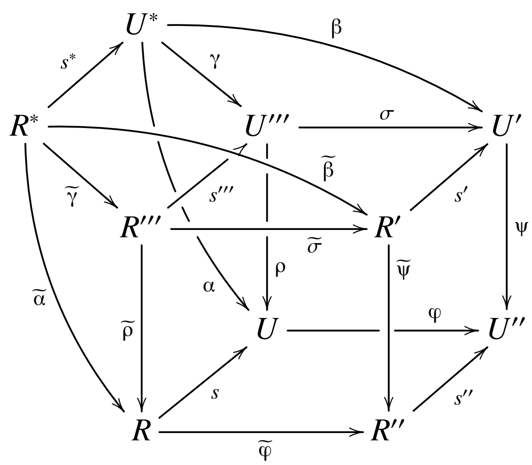
Consecuently $(U''', R''', s''', t''', c''')$ is a groupoid category. Furhermore we have shown that $s \circ \widetilde{\rho} = \rho \circ s'''$, $t \circ \widetilde{\rho} = \rho \circ t'''$ and $c \circ (\widetilde{\rho}, \widetilde{\rho}) = \widetilde{\rho} \circ c'''$, so that $(\rho, \widetilde{\rho})$ is a morphism of groupoid categories from $(U''', R''', s''', t''', c''')$ to (U, R, s, t, c). In the same way $(\sigma, \widetilde{\sigma})$ is a morphism from $(U''', R''', s''', t''', c''')$ to (U', R', s', t', c').

We now can show that $(U''', R''', s''', t''', c''')$ and the morphisms $(\rho, \widetilde{\rho})$ and $(\sigma, \widetilde{\sigma})$ are a fibre product in the category of groupoids on $\mathcal{C}$. At this point we only need to prove de universal property. Let $(U^*, R^*, s^*, t^*, c^*)$ be a groupoid and $(\alpha, \widetilde{\alpha}) : (U^*, R^*) \longrightarrow (U, R)$ and $(\beta, \widetilde{\beta}) : (U^*, R^*) \longrightarrow (U', R')$ morphisims such that $(\varphi, \widetilde{\varphi}) \circ (\alpha, \widetilde{\alpha}) = (\psi, \widetilde{\psi}) \circ (\beta, \widetilde{\beta})$. Hence $(\varphi \circ \alpha, \widetilde{\varphi} \circ \widetilde{\alpha}) = (\psi \circ \beta, \widetilde{\psi} \circ \widetilde{\beta})$ and we get $\varphi \circ \alpha = \psi \circ \beta$ and $\widetilde{\varphi} \circ \widetilde{\alpha} = \widetilde{\psi} \circ \widetilde{\beta}$. Then, the following diagrams are commutative:





Moreover the inner squares are cartesian and therefore there exist $\gamma : U^* \longrightarrow U'''$ and $\widetilde{\gamma} : R^* \longrightarrow R'''$ unique making the respective diagrams commutative. More precisely $\gamma = (\alpha, \beta)$ and $\widetilde{\gamma} = (\widetilde{\alpha}, \widetilde{\beta})$. We claim the pair $(\gamma, \widetilde{\gamma})$ is a morphism of groupoids. For this look at the next diagram:



Here $s'' \circ \widetilde{\gamma}$ and $\gamma \circ s^*$ are morphisms from R^* to U''' satisfying $\rho \circ (s''' \circ \widetilde{\gamma}) = s \circ \widetilde{\rho} \circ \widetilde{\gamma} = s \circ \widetilde{\alpha} = \alpha \circ s^*$, $\sigma \circ (s''' \circ \widetilde{\gamma}) = s \circ \widetilde{\sigma} \circ \widetilde{\gamma} = s \circ \widetilde{\beta} = \beta \circ s^*$, $\rho \circ (\gamma \circ s^*) = \alpha \circ s^*$ and $\sigma \circ (\gamma \circ s^*) = \beta \circ s^*$. But there is a unique morphism with that property an thus $s''' \circ \widetilde{\gamma} = \gamma \circ s^*$. Similarly $t''' \circ \widetilde{\gamma} = \gamma \circ t^*$.

We know that $c \circ (\widetilde{\alpha}, \widetilde{\alpha}) = \widetilde{\alpha} \circ c^*$ and $c' \circ (\widetilde{\beta}, \widetilde{\beta}) = \widetilde{\beta} \circ c^*$. We see that $c''' \circ (\widetilde{\gamma}, \widetilde{\gamma}) = \widetilde{\gamma} \circ c^*$. In fact

$$
\begin{aligned}
c''' \circ (\widetilde{\gamma}, \widetilde{\gamma}) &= (c \circ (\widetilde{\rho}, \widetilde{\rho}), c' \circ (\widetilde{\sigma}, \widetilde{\sigma})) \circ (\widetilde{\gamma}, \widetilde{\gamma}) \\
&= (c \circ (\widetilde{\rho}, \widetilde{\rho}) \circ (\widetilde{\gamma}, \widetilde{\gamma}), c' \circ (\widetilde{\sigma}, \widetilde{\sigma}) \circ (\widetilde{\gamma}, \widetilde{\gamma}))
\end{aligned}
$$

$$= (c \circ (\widetilde{\rho} \circ \widetilde{\gamma}, \widetilde{\rho} \circ \widetilde{\gamma}), c' \circ (\widetilde{\sigma} \circ \widetilde{\gamma}, \widetilde{\sigma} \circ \widetilde{\gamma}))$$

$$= (c \circ (\widetilde{\alpha}, \widetilde{\alpha}), c' \circ (\widetilde{\beta}, \widetilde{\beta}))$$

$$= (\widetilde{\alpha} \circ c^*, \widetilde{\beta} \circ c^*)$$

$$= (\widetilde{\alpha}, \widetilde{\beta}) \circ c^*$$

$$= \widetilde{\gamma} \circ c^*$$

Then $(\gamma, \widetilde{\gamma})$ is a morphism of groupoids and by contruction $(\rho, \widetilde{\rho}) \circ (\gamma, \widetilde{\gamma}) = (\rho \circ \gamma, \widetilde{\rho} \circ \widetilde{\gamma}) = (\alpha, \widetilde{\alpha})$ and also $(\sigma, \widetilde{\sigma}) \circ (\gamma, \widetilde{\gamma}) = (\sigma \circ \gamma, \widetilde{\sigma} \circ \widetilde{\gamma}) = (\beta, \widetilde{\beta})$. If $(\delta, \widetilde{\delta})$ is a morphism from (U^*, R^*) to (U''', R''') such that $(\rho, \widetilde{\rho}) \circ (\delta, \widetilde{\delta}) = (\alpha, \widetilde{\alpha})$ and $(\sigma, \widetilde{\sigma}) \circ (\delta, \widetilde{\delta}) = (\beta, \widetilde{\beta})$, then $(\rho \circ \delta, \widetilde{\rho} \circ \widetilde{\delta}) = (\alpha, \widetilde{\alpha})$ and $(\sigma \circ \delta, \widetilde{\sigma} \circ \widetilde{\delta}) = (\beta, \widetilde{\beta})$. Therefore $\delta : U^* \longrightarrow U'''$ and $\widetilde{\delta} : R^* \longrightarrow R'''$ are morphisms such that $\rho \circ \delta = \alpha$, $\sigma \circ \delta = \beta$, $\widetilde{\rho} \circ \widetilde{\delta} = \widetilde{\alpha}$ y $\widetilde{\sigma} \circ \widetilde{\delta} = \widetilde{\beta}$. But γ and $\widetilde{\gamma}$ are the unique with such property $(\delta, \widetilde{\delta}) = (\gamma, \widetilde{\gamma})$. It follows that $(\gamma, \widetilde{\gamma})$ is the only morphism satisfying $(\rho, \widetilde{\rho}) \circ (\gamma, \widetilde{\gamma}) = (\alpha, \widetilde{\alpha})$ and $(\sigma, \widetilde{\sigma}) \circ (\gamma, \widetilde{\gamma}) = (\beta, \widetilde{\beta})$.

All this shows $(U''', R''', s''', t''', c''')$ togehter with the morphisms $(\rho, \widetilde{\rho})$ and $(\sigma, \widetilde{\sigma})$ are a fibre product of $(\varphi, \widetilde{\varphi})$ and $(\psi, \widetilde{\psi})$, so we are done. □

3.3 Groupoids in algebraic spaces

The category of algebraic spaces has fibre products, so the following definition makes sense.

Definition 3.3.1 (Groupoid in algebraic spaces). *A groupoid in algebraic spaces is a groupoid category in the category of algebraic spaces.*

Therefore, a groupoid in algebraic spaces is a septuple (U, R, s, t, c, e, i), where U and R are algebraic spaces and s, t, c, e, i are morphisms of algebraic spaces. Since an algebraic space is a functor $F : \mathbf{Sch}/S \longrightarrow \mathbf{Sets}$ and a morphism of algebraic spaces is a natural transformation, then por each scheme T over S it can be considered the septuple

$(U(T), R(T), s_T, t_T, c_T, e_T, i_T)$. It is easy to see that this septuple is a groupoid category in the category of sets and this determines a groupoid in the usual sense.

If $T' \stackrel{\alpha}{\longrightarrow} T$ is a morphisms of schemes, there are functions $U(\alpha) : U(T) \longrightarrow U(T')$ and $R(\alpha) : R(T) \longrightarrow R(T')$. Also, as $s, t : R \longrightarrow U$ and $c : R \times_{s,U,t} R \longrightarrow R$ are natural transformations, the following diagrams are commutative

$$\begin{array}{ccc} R(T) & \xrightarrow{R(\alpha)} & R(T') \\ {\scriptstyle t_T}\downarrow\downarrow{\scriptstyle s_T} & & {\scriptstyle s_{T'}}\downarrow\downarrow{\scriptstyle t_{T'}} \\ U(T) & \xrightarrow[U(\alpha)]{} & U(T') \end{array} \qquad \begin{array}{ccc} (R \times_{s,U,t} R)(T) & \xrightarrow{(R\times_{s,U,t}R)(\alpha)} & (R \times_{s,U,t} R)(T) \\ \| & \| & \| \\ R(T) \times_{s_T,U(T),t_T} R(T) & \xrightarrow[R(\alpha),R(\alpha)]{} & R(T') \times_{s_{T'},U(T'),t_{T'}} R(T') \\ {\scriptstyle c_T}\downarrow & & \downarrow{\scriptstyle c_{T'}} \\ R(T) & \xrightarrow[R(\alpha)]{} & R(T') \end{array}$$

Hence $U(\alpha) \circ t_T = t_{T'} \circ R(\alpha)$, $U(\alpha) \circ s_T = s_{T'} \circ R(\alpha)$ and $C_{T'} \circ (R(\alpha), R(\alpha)) = R(\alpha) \circ c_T$. Thus $(U(\alpha), R(\alpha))$ is a morphism of groupoids in the category of sets.

3.4 Group algebraic spaces

Definition 3.4.1 (Group algebraic space). *(043H) Let $B \longrightarrow S$ an algebraic space over S.*

1. *A group algebraic space over B is a pair (G, m), where G is an algebraic space over B and $m : G \times G \longrightarrow G$ is a morphism of algebraic spaces satisfying the following property: For every scheme T over B, the pair $(G(T), m)$ is a group.*

2. *A morphism $\psi : (G, m) \longrightarrow (G', m')$ of group algebraic spaces over B is a morphism $\psi : G \longrightarrow G'$ of algebraic spaces over B such that for every scheme T over B the induced map $\psi_T : G(T) \longrightarrow G'(T)$ is a homomorphism of groups.*

If (G, m) is a group algebraic space over B, then we have morphisms of algebraic spaces $e : B \longrightarrow G$ and $i : G \longrightarrow G$ such that for every T the quadruple $(G(T), m_T, e_T, i_T)$ satisfies the axioms of a group.

3.5 Actions of group algebraic spaces

Definition 3.5.1 (Action of a group category). *(043Q) Let (G, m) be a group algebraic space over B and X an algebraic space.*

1. *An action of G on X is a morphism $a : G \times_B X \longrightarrow X$ over B such that for every scheme T over B the map $a_T : G(T) \times X(T) \longrightarrow X(T)$ defines an action of $G(T)$ on $X(T)$, that is to say, the following diagrams commute.*

$$\begin{array}{ccc} G \times_B G \times_B X & \xrightarrow{(m, id_X)} & G \times_B X \\ \downarrow{\scriptstyle (id_G, a)} & & \downarrow{\scriptstyle a} \\ G \times_B X & \xrightarrow[a]{} & X \end{array} \qquad \begin{array}{ccc} X = B \times_B X & \xrightarrow{(e_G, id_X)} & G \times_B X \\ & \searrow{\scriptstyle id_X} & \downarrow{\scriptstyle a} \\ & & X \end{array}$$

2. *Suppose that X and Y are algebraic spaces over B each endowed with an action of the group algebraic space (G, m). A G-equivariant morphism $\psi : X \longrightarrow Y$ is a morphism of algebraic spaces over B such that for every T over B the map $\psi_T : X(T) \longrightarrow Y(T)$ is a morphism of $G(T)$-sets, which means the following diagram is commutative.*

$$\begin{array}{ccc} G \times_B X & \xrightarrow{(id_G, \psi)} & G \times_B Y \\ \downarrow{\scriptstyle a_X} & & \downarrow{\scriptstyle a_Y} \\ X & \xrightarrow[\psi]{} & Y \end{array}$$

3.6 Groupoid category associated to a group action

Lemma 3.6.1. *(0444) Let $B \longrightarrow S$ an algebraic space over $\mathcal{S}$, let (G, m) a group algebraic space over B with identity e_G and inverse i_G. Given an algebraic space X over B and $a : G \times_B X \longrightarrow X$ a group action of G into X over B, then we have a groupoid in algebraic spaces (U, R, s, t, c, e, i) over B in the following manner*

1. *$U = X$ and $R = G \times_B X$.*
2. *$s : R \longrightarrow U$ is defined by $(g, x) \longmapsto x$.*
3. *$t : R \longrightarrow U$ is defined by $(g, x) \longmapsto a(g, x)$.*
4. *$c : R \times_{s,U,t} R \longrightarrow R$ given by $((g, x), (g', x')) \longmapsto (m(g, g'), x')$.*
5. *$e : U \longrightarrow R$ is defined by $x \longmapsto (e_G(x), x)$.*
6. *$i : R \longrightarrow R$ is given by $(g, x) \longmapsto (i_G(g), a(g, x))$.*

Chapter

4 Quotient Stacks

4.1 Fibred category associated to a groupoid in algebraic spaces

Let (U, R, s, t, c) be a groupoid in algebraic spaces. Consider the functor

$$F : \mathbf{Sch}\,/S \longrightarrow \mathbf{Gpds}$$

defined for a scheme T by $(U(T), R(T), s_T, t_T, c_T)$ and for a morphism $T' \stackrel{\alpha}{\longrightarrow} T$ of schemes as the morphism of groupoids $(U(\alpha), R(\alpha))$. We can consider the fibred category associated $p_F : \mathcal{S}_F \longrightarrow \mathbf{Sch}\,/S$. Then $\mathcal{S}_F$ is the category with

$$\mathrm{Ob}(\mathcal{S}_F) = \{(T, x) \mid T \in \mathrm{Ob}(\mathbf{Sch}\,/S), x \in U(T)\}$$

and for (T, x) and (T', y) in $\mathrm{Ob}(\mathcal{S}_F)$

$$\mathrm{Hom}_{\mathcal{S}_F}((T', y), (T, x)) = \left\{(\alpha, a) \mid \alpha \in \mathrm{Hom}_{\mathbf{Sch}\,/S}(T', T), a \in R(T'), s_{T'}(a) = y, t_{T'}(a) = U(\alpha)x\right\}$$

Composition of $(T', y) \stackrel{(\alpha,a)}{\longrightarrow} (T, x)$ and $(T'', z) \stackrel{(\beta,b)}{\longrightarrow} (T', y)$ is thus defined by $(\alpha \circ \beta, c_{T''}(R(\beta)(a), b))$. This composition rule makes sense because $y \stackrel{\alpha}{\longrightarrow} U(\alpha)x \in R(T')$, $z \stackrel{b}{\longrightarrow}$

$U(\beta)y \in R(T'')$ and $R(\beta)(a)$ is such that $s_{T''}(R(\beta)(a)) = (s_{T''} \circ R(\beta))(a) = U(\beta) \circ s_T(a) = U(\beta)y$, so that there exists $c_{T''}(R(\beta)(a), b)$.

Now, $(T, x) \overset{(id_T, e_T(x))}{\longrightarrow} (T, x)$ is an endomorphism of (T, x), inasmuch as $id_T : T \longrightarrow T$ is a morphism of schemes, and $e_T(x) \in R(T)$ is such that $s_T(e_T(x)) = (s_T \circ e_T)(x) = x$ and $t_T(e_T(x)) = (t_T \circ e_T)(x) = x = U(id_T)x$. Moreover, $(\alpha, a) \circ (id_T, e_T x) = (\alpha \circ id_T, c_T(R(id_T)(a), e_T(x))) = (\alpha, c_T(id_{R(T)}(a), e_T(x))) = (\alpha, c_T(a, e_T x)) = (\alpha, a)$. In the same way $(id_T, e_T(x)) \circ (\beta, b) = (\beta, b)$ and then $(id_T, e_T(x)) = id_{(T,x)}$. By construction $p_F(T, x) = T$ and $p_F(\alpha, a) = \alpha$.

In this case, the fibred category $\mathcal{S}_F$ is fibred in groupoids. Indeed, given $U \in \mathrm{Ob}(\mathcal{C})$, the fibre category $(\mathcal{S}_F)_U$ is equivalent to the category $F(U)$:

$$\begin{aligned} \mathrm{Ob}((\mathcal{S}_F)_U) &= \{(U, x) \mid x \in \mathrm{Ob}(F(U))\} \\ &\simeq F(U) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}_{(\mathcal{S}_F)_U}((U, x), (U, x')) &= \{(f, a) \mid p_F(f, a) = id_U\} \\ &= \{(id_U, a) \mid a \in \mathrm{Hom}_{F(U)}(x, x')\} \\ &\simeq \mathrm{Hom}_{F(U)}(x, x') \end{aligned}$$

The result follows because $F(U)$ is a groupoid by hypotesis.

Definition 4.1.1 (Fibred category associated to a groupoid in algebraic spaces). *The fibred category in groupoids $\mathcal{S}_F$ constructed as before is denoted as $[U/R]_p$ and is called quotient prestack associated to the groupoid in algebraic spaces (U, R, s, t, c).*

Functoriality of quotient prestacks

In this subsection we are going to show functorial properties about the fibred category associated to a functor when this comes from a groupoid in algebraic spaces. We found that this construction is compatible with fibre products, but in general this not a 2-fibre product. This is the reason why we decide to investigate the relation between fibre product of fibred categories and stackification.

Consider now a morphism of groupoids in algebraic spaces

$$f = (\varphi, \widetilde{\varphi}) : (U, R, s, t, c) \longrightarrow (U', R', s', t', c')$$

We will see that this induces a canonical morphism

$$[f]_p : [U/R]_p \longrightarrow [U'/R']_p$$

of fibred categories in groupoids. In fact, given $(T, x) \in \mathrm{Ob}([U/R]_p)$, we define $[f]_p(T, x) := (T, \varphi_T(x))$, which is in $\mathrm{Ob}([U'/R']_p)$, because $T \in \mathrm{Ob}(\mathbf{Sch}/S)$ and, since $\varphi : U \longrightarrow U'$ is a natural transformation we have that $\varphi_T : U(T) \longrightarrow U'(T)$ is a function, and because $x \in U(T)$ we have $\varphi_T(x) \in U'(T)$. Also, given $(\alpha, a) \in \mathrm{Hom}_{[U/R]_p}((T', y), (T, x))$, we define $[f]_p(\alpha, a) := (\alpha, \widetilde{\varphi}_{T'}(a))$, which is in $\mathrm{Hom}_{[U'/R']_p}([f]_p(T', y), [f]_p(T, x)) = \mathrm{Hom}_{[U'/R']_p}((T', \varphi_{T'}(y)), (T, \varphi_T(x)))$, insomuch as $\alpha \in \mathrm{Hom}_{\mathbf{Sch}/S}(T', T)$, $a \in R(T')$, $s_{T'}(a) = y$, $t_{T'}(a) = U(\alpha)x$ and since $\widetilde{\varphi} : R \longrightarrow R'$ is a natural transformation, then $\widetilde{\varphi}_{T'} : R(T') \longrightarrow R'(T')$ is a function and therefore $\widetilde{\varphi}_{T'}(a) \in R'(T')$. Moreover $s_{T'}(\widetilde{\varphi}_{T'}(a)) = (s'_{T'} \circ \widetilde{\varphi}_{T'})(a) = (\varphi_{T'} \circ s_{T'})(a) = \varphi_{T'}(s_{T'}(a)) = \varphi_{T'}(y)$ and $t'_{T'}(\widetilde{\varphi}_{T'}(a)) = (t'_{T'} \circ \widetilde{\varphi}_{T'})(a) = (\varphi_{T'} \circ t_{T'})(a) = \varphi_{T'}(t_{T'}(a)) = \varphi_{T'}(U(\alpha)x) = (\varphi_{T'} \circ U(\alpha))(x) = (U'(\alpha) \circ \varphi_T)(x) = U'(\alpha)(\varphi_T(x))$, where we used that $\varphi : U \longrightarrow U'$ is a natural transformation and then the following diagram commutes

$$\begin{array}{ccc} U(T) & \xrightarrow{\varphi_T} & U'(T) \\ {\scriptstyle U(\alpha)}\downarrow & & \downarrow{\scriptstyle U'(\alpha)} \\ U(T') & \xrightarrow[\varphi_T]{} & U'(T') \end{array}$$

Which means $\varphi_{T'} \circ U(\alpha) = U'(\alpha) \circ \varphi_T$. In order to prove the compatibility with composition, let $(T'', z) \xrightarrow{(\beta, b)} (T', y)$ be another morphism in $[U/R]_p$. We can consider the composition

$(\alpha, a) \circ (\beta, b) = (\alpha \circ \beta, c_{T''}(R(\beta)(a), b))$. Then we have

$$\begin{aligned}
[f]_p((\alpha, a) \circ (\beta, b)) &= [f]_p(\alpha \circ \beta, c_{T''}(R(\beta)(a), b)) \\
&= (\alpha \circ \beta, \widetilde{\varphi}_{T''}(c_{T''}(R(\beta)(a), b))) \\
&= (\alpha \circ \beta, (\widetilde{\varphi}_{T''} \circ c_{T''})(R(\beta)(a), b)) \\
&= (\alpha \circ \beta, (c'_{T''} \circ (\widetilde{\varphi}_{T''}, \widetilde{\varphi}_{T''}))(R(\beta)(a), b)) \\
&= (\alpha \circ \beta, c'_{T''}(\widetilde{\varphi}_{T''}(R(\beta)(a), b)))
\end{aligned}$$

On the other hand

$$\begin{aligned}
[f]_p(\alpha, a) \circ [f]_p(\beta, b) &= (\alpha, \widetilde{\varphi}_{T'}(a)) \circ (\beta, \widetilde{\varphi}_{T''}(b)) \\
&= (\alpha \circ \beta, c'_{T''}(R'(\beta)(\widetilde{\varphi}_{T'}(a)), \widetilde{\varphi}_{T''}(b))
\end{aligned}$$

Is enough to show $R'(\beta) \circ \widetilde{\varphi}_{T'} = \widetilde{\varphi}_{T''} \circ R(\beta)$, which follows the fact $\widetilde{\varphi} : R \longrightarrow R'$ is a natural transformation and so for $T'' \xrightarrow{\beta} T'$, the following diagram commutes

$$\begin{array}{ccc}
R(T') & \xrightarrow{\widetilde{\varphi}_{T'}} & R'(T') \\
{\scriptstyle R(\beta)}\downarrow & & \downarrow{\scriptstyle R'(\beta)} \\
R(T'') & \xrightarrow[\widetilde{\varphi}_{T'}]{} & R'(T'')
\end{array}$$

Then $[f]_p((\alpha, a) \circ (\beta, b)) = [f]_p(\alpha, a) \circ [f]_p(\beta, b)$. Also $[f]_p(id_T, e_T(x)) = (id_T, \widetilde{\varphi}_T(e_T(x))) = (id_T, e'_T(\varphi(x)))$ and on the other hand $id_{[f]_p(T,x)} = id_{(T, \widetilde{\varphi}_T(x))} = (id_T, e'_T(\widetilde{\varphi}(x)))$ and so $[f]_p(id_{(T,x)}) = id_{[\varphi_p](T,x)}$. Therefore $[f]_p : [U/R]_p \xrightarrow{U'/R'}_p$ is a functor and the following triangle is clearly commutative

$$\begin{array}{ccc}
[U/R]_p & \xrightarrow{[\varphi]_p} & [U'/R']_p \\
& \searrow \quad \swarrow & \\
& \mathbf{Sch}/S &
\end{array}$$

Hence $[f]_p$ is a 1-morphism of fibred categories in groupoids over $\mathbf{Sch}/S$.

We will see that if $g = (\psi, \widetilde{\psi}) : (U', R') \longrightarrow (U'', R'')$ is another morphism of groupoids in algebraic spaces, then $[g \circ f]_p = [g]_p \circ [f]_p$ as 1-morphisms of fibred categories from $[U/R]_p$ to $[U''/R'']_p$. Given $(T, x) \in \mathrm{Ob}([U/R]_p)$ we have $[g \circ f]_p(T, x) = (T, (\psi \circ \varphi)_T(x)) = (T, \psi_T \circ \varphi_T(x))$ and on the other hand $([g]_p \circ [f]_p)(T, x) = [g]_p([f]_p(T, x)) = [g]_p(T, \varphi_T(x)) = (T, \psi_T(\varphi_T(x)))$ and we have the equality in the objects. Let $(\alpha, a) \in \mathrm{Hom}_{[U/R]_p}((T', y), (T, x))$. Hence $[g \circ f](\alpha, a) = (\alpha, (\widetilde{\psi} \circ \widetilde{\varphi})_{T'}(a)) = (\alpha, \widetilde{\psi}_{T'} \circ \widetilde{\varphi}_{T'}(a))$ and also $([g]_p \circ [f]_p)(\alpha, a) = [g]_p([f]_p(\alpha, a)) = [g]_p(\alpha, \widetilde{\varphi}_{T'}(a)) = (\alpha, \widetilde{\psi}_{T'} \circ \widetilde{\varphi}_{T'}(a))$ and we have the equality on morphisms.

In this way we have a functor $[\ ]_p$: **GpdsAlgSp** $\longrightarrow$ **FibCatGpds** from the category of groupoids in algebraic spaces to the 2-category of fibred categories in groupoids. We shall see that this functor preserves fibre products. Consider a cartesian square in the category of groupoids in algebraic spaces

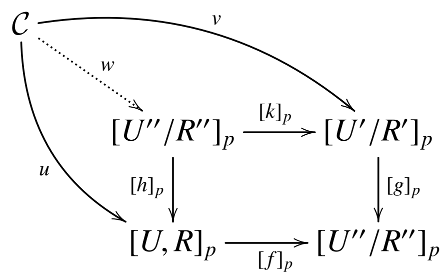
$$\begin{array}{ccc} (U''', R''') & \xrightarrow{k} & (U', R') \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ (U, R) & \xrightarrow[f]{} & (U'', R'') \end{array}$$

where $f = (\varphi, \widetilde{\varphi})$, $g = (\psi, \widetilde{\psi})$, $h = (\rho, \widetilde{\rho})$ and $k = (\sigma, \widetilde{\sigma})$ are like in the construction of the fibre product of groupoids as before. We are going to show that the following square is also cartesian in the category of fibred categories in groupoids over **Sch** $/S$

$$\begin{array}{ccc} [U''/R'']_p & \xrightarrow{[k]_p} & [U'/R']_p \\ \downarrow{\scriptstyle [h]_p} & & \downarrow{\scriptstyle [g]_p} \\ [U, R]_p & \xrightarrow[{[f]_p}]{} & [U''/R'']_p \end{array}$$

This diagram is commutative, inasmuch as $[f]_p \circ [h]_p = [f \circ h]_p = [g \circ k]_p = [g]_p \circ [k]_p$. We shall see this satisfy the universal property of fibre product.

Let $p : \mathcal{C} \longrightarrow$ **Sch** $/S$ be a fibred category in groupoids and let $u : \mathcal{C} \xrightarrow{U/P}_p$ and $v : \mathcal{C} \xrightarrow{U'/R'}_p$ be 1-morphisms of fibred categories in groupoids such that $[f]_p \circ u = [g]_p \circ v$. Look at the next diagram



Given $A \in \mathrm{Ob}(\mathcal{C})$, we have $u(A) \in \mathrm{Ob}([U/R]_p)$ and so $u(A) = (T, x)$, with $T \in \mathbf{Sch}/S$ and $x \in U(T)$. In the same way $v(A) \in \mathrm{Ob}([U'/R']_p)$ and so $v(a) = (T', x')$. Hence it follows

$$[f]_p \circ u(A) = [f]_p(T, x) = (T, \varphi_T(x))$$

$$[g]_p \circ v(A) = [g]_p(T', x') = (T', \psi_T(x))$$

and by hypotesis $(T, \varphi_T(x)) = (T', \psi_{T'}(x'))$ and therefore $T = T'$ and $\varphi_T(x) = \psi_{T'}(x')$. Thus $u(A) = (T, x)$ and $v(A) = (T, x')$, where $x \in U(T)$, $x' \in U'(T)$ and $\varphi_T(x) = \psi_T(x')$. Therefore $(x, x') \in U(T)\times_{\varphi_T(x),U(T),\psi_T} = U'''(T)$ and so $(T, (x, x')) \in \mathrm{Ob}([U'''/R''']_p)$. We define

$$w(A) := (T, (x, x'))$$

Hence $[h]_p(w(A)) = [h]_p(T, (x, x')) = (T, \rho_T(x, x')) = (T, x) = u(A)$ and $[k]_p(w(A)) = [h]_p(T, (x, x')) = (T, \sigma_T(x, x')) = (T, x') = v(A)$. Let $B \in \mathrm{Ob}(\mathcal{C})$ and $B \xrightarrow{m} A$ morphism in $\mathcal{C}$. Thereby $u(B) = (T', y)$, $v(B) = (T', y')$ and $w(B) = (T', (y, y'))$ as before. Moreover, $u(m) \in \mathrm{Mor}_{[U/R]_p}((T', y), (T, x))$ and therefore $u(m) = (\alpha, a)$, where $\alpha \in \mathrm{Hom}_{\mathbf{Sch}/S}(T', T)$, $a \in R(T')$, $s_{T'}(a) = y$ and $t_{T'}(a) = U(\alpha)(x)$. Consecuently $[f]_p \circ u(m) = (\alpha, \widetilde{\varphi}_{T'}(a))$. Similarly $v(m) = (\beta, b)$ and $[g]_p \circ v(m) = (\beta, \widetilde{\psi}_{T'}(b))$ y by hypotesis $(\alpha, \widetilde{\varphi}_{T'}(a)) = (\beta, \widetilde{\psi}_{T'}(b))$, and so $\alpha = \beta$ and $\widetilde{\varphi}_{T'}(a) = \widetilde{\psi}_{T'}(b)$. Then $(a, b) \in R(T')\times_{\widetilde{\varphi}_{T'},R''(T'),\widetilde{\psi}_{T'}} = R'''(T')$ and satisfy satisface $s'''_{T'}(a, b) = (s_{T'} \circ \widetilde{\rho}_{T'}, s'_{T'} \circ \widetilde{\sigma}_{T'})(a, b) = (s_{T'}(a), s'_{T'}(b)) = (y, y')$, $t'''_{T'}(a, b) = (t_{T'}(a), t'_{T'}(b)) = (U(\alpha)(x), U'(\beta)(x')) = U'''(\alpha, \beta)(x, x')$. Therefore $(\alpha, (a, b)) \in$

$\mathrm{Hom}_{[U'''/R''']_p}((T', (y, y')), (T, (x, x')))$. We define

$$w(m) := (\alpha, (a, b))$$

Then $[h]_p \circ w(m) = (\alpha, \widetilde{\rho}(a, b)) = (\alpha, a)$ y $[k]_p \circ w(m) = (\alpha, \widetilde{\sigma}(a, b)) = (\alpha, b) = v(m)$.

Let us show that $w : \mathcal{C} \longrightarrow [U'''/R''']_p$ defined in this way is a 1-morphism of fibred categories in groupoids. For this it is suffices to show compatibility with composition and identity. Consider the identity morphism $id_A : A \longrightarrow A$. Then $u(id_A) = id_{u(A)} = id_{(T,x)} = (id_T, e_T(x))$. Similarly $v(id_A) = (id_T, e'_T(x'))$ and so $w(id_A) = (id_T, (e_T(x), e'_T(x'))) = (id_T, e'''_T(x, x')) = id_{(T,(x,x'))} = id_{w(A)}$. Let $C \stackrel{n}{\longrightarrow} B$ be another morphism in $\mathcal{C}$. Hence $u(C) = (T'', z)$, $v(C) = (T'', z')$ and $w(C) = (T'', (z, z'))$. Also $u(n) = (\gamma, c)$ and $v(n) = (\gamma, d)$ where $\gamma : T'' \longrightarrow T' \in \mathrm{Hom}_{\mathbf{Sch}/S}(T'', T')$, $c \in R()T''$, $d \in R'(T'')$ and $w(n) = (\gamma, (c, d))$. Therefore $u(m \circ n) = u(m) \circ u(n) = (\alpha, a) \circ (\gamma, c) = (\alpha \circ \gamma, c_{T''}(R(\gamma)(a), c))$ and $v(m \circ n) = v(m) \circ v(n) = (\alpha, b) \circ (\gamma, d) = (\alpha \circ \gamma, c'_{T''}(R'(\gamma)(b), d))$ and then we have

$$w(m \circ n) = (\alpha \circ \gamma, (c_{T''}(R(\gamma)(a), c), c'_{T''}(R'(\gamma)(b), d))$$

On the other hand

$$\begin{aligned}
w(m) \circ w(n) &= (\alpha, (a, b)) \circ (\gamma, (c, d)) \\
&= (\alpha \circ \gamma, c'''_{T''}(R'''(\gamma)(a, b), (c, d))) \\
&= (\alpha \circ \gamma, c'''_{T''}((R(\gamma)(a), R'(\gamma)(b)), (c, d))) \\
&= (\alpha \circ \gamma, (c_{T''} \circ (\widetilde{\rho}_{T''}, \widetilde{\rho}_{T''}), c'_{T''} \circ (\widetilde{\sigma}_{T''}, \widetilde{\sigma}_{T''}))((R(\gamma)(a), R'(\gamma)(b)), (c, d))) \\
&= (\alpha \circ \gamma, (c_{T''}(\widetilde{\rho}_{T''}, \widetilde{\rho}_{T''})((R(\gamma)(a), R'(\gamma)(b)), (c, d))), c'_{T''}(\widetilde{\sigma}_{T''}, \\
&\quad \widetilde{\sigma}_{T''})((R(\gamma)(a), R'(\gamma)(b)), (c, d))) \\
&= (\alpha \circ \gamma, c_{T''}(\widetilde{\rho}_{T''}(R(\gamma)(a), R'(\gamma)(b)), (c, d)), c'_{T''}(\widetilde{\sigma}_{T''}(R(\gamma)(a), \\
&\quad R'(\gamma)(b)), (c, d)))
\end{aligned}$$

$$= (\alpha \circ \gamma, (c_{T''}(R(\gamma)(a), c), c'_{T''}(R'(\gamma)(b), d)))$$

and therefore $w(m \circ n) = w(m) \circ w(n)$.

So $w : \mathcal{C} \longrightarrow [U'''/R''']_p$ is a functor and by definition $[h]_p \circ w = u$ and $[k]_p \circ w = v$. Let's see this is unique with such a property. Let $\eta : \mathcal{C} \longrightarrow [U'''/R''']_p$ be a functor such that $[h]_p \circ \eta = u$ and $[k]_p \circ \eta = v$. Since η, u and $[h]_p$ are morphisms of categories over $\mathbf{Sch}/S$, given $A \in \mathrm{Ob}(\mathcal{C})$ if $u(A) = (T, x)$ and $v(A) = (T, x')$ it follows that $\eta(A) = (T, (y, y'))$ where $y \in U(T)$ and $y' \in U'(T)$ and therefore $[h]_p \circ \eta(A) = (T, y)$ and $[k]_p \circ \eta(A) = (T, y')$. But, for hypotesis $(T, y) = (T, x)$ and $(T, y') = (T, x')$, so that $(y, y') = (x, x')$ and then $\eta(A) = (T, (x, x')) = w(A)$. In the same way, if $B \xrightarrow{m} A$ is a morphism in $\mathcal{C}$ and if $u(m) = (\alpha, a)$ and $v(m) = (\alpha, b)$ it follows $\eta(m) = (\alpha, (c, d))$ where $c \in R(T')$ and $d \in R'(T')$ and so $[h]_p \circ \eta(m) = (\alpha, c)$ and $[k]_p \circ \eta(m) = (\alpha, d)$. Again by hypotesis $(\alpha, c) = (\alpha, a)$ and $(\alpha, d) = (\alpha, b)$ so that $(c, d) = (a, b)$ and then $\eta(m) = (\alpha, (a, b)) = w(m)$. Therefore w is the unique morphism with those properties and this proves that $[U''', R''']_p$ is a fibre product.

we can summarize the previous results in the following statement

Theorem 6. *Let $f = (\varphi, \widetilde{\varphi}) : (U, R, s, t, c) \longrightarrow (U', R', s', t', c')$ be a morphism of groupoids in algebraic spaces. This induces a canonical morphism of quotient prestacks*

$$[f]_p : [U/R]_p \longrightarrow [U'/R']_p.$$

*In this way we have a functor $[\ ]_p$: **GpdsAlgSp** $\longrightarrow$ **FibCatGpds** from the category of groupoids in algebraic spaces to the 2-category of fibred categories in groupoids. This functor is compatible with fibred products.*

4.2 Quotient stacks

Let (U, R, s, t, c) a groupoid in the category of algebraic spaces. Consider the functor

$$\begin{aligned}(\mathbf{Sch}\,/S\,)^{op} &\longrightarrow \mathbf{Gpds}\\ T &\longmapsto (U(T), R(T), s_T, t_T, c_T)\end{aligned}$$

This functor determines a category fibred in groupoids over $\mathbf{Sch}\,/S$ which is denoted $[U/R]_p \longrightarrow \mathbf{Sch}\,/S$.

Definition 4.2.1 (Quotient stack). *(044Q) Let B an algebraic space over S.*

1. *The quotient stack $p : [U/R] \longrightarrow \mathbf{Sch}\,/S$ associated to a groupoid in algebraic spaces (U, R, s, t, c) over B is the stackification of the fibred category in groupoids $[U/R]_p \longrightarrow \mathbf{Sch}\,/S$.*

2. *If (G, m) be a group algebraic space over B and $a : G \times_B X \longrightarrow X$ is an action of G in the algebraic space X, the quotient stack $p : [X/G] \longrightarrow \mathbf{Sch}\,/S$ is the stack associated to the groupoid in algebraic spaces $(X, G \times_B X, s, t, c)$.*

Lemma 4.2.1. *Let (U, R, s, t, c) a groupoid in algebraic spaces.*

1. *(044R) There are 1-morphisms of stacks $\pi : \mathcal{S}_U \longrightarrow [U/R]$ and $[U/R] \longrightarrow \mathcal{S}_B$ such that the composition $\mathcal{S}_U \longrightarrow [U/R] \longrightarrow \mathcal{S}_B$ is the 1-morphism associated to the structural morphism $U \longrightarrow B$.*

2. *(044S) There is a canonical 2-isomorphism making 2-commutative the following diagram:*

$$\begin{array}{ccc}\mathcal{S}_R & \xrightarrow{s} & \mathcal{S}_U\\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \pi}\\ \mathcal{S}_U & \xrightarrow[\pi]{} & [U/R]\end{array}$$

From now on we won't distinguish an algebraic space from its associated stack $\mathcal{S}_X$. Therefore we can consider $\pi : U \longrightarrow [U/R]$ and $[U/R] \longrightarrow B$. After we are going to show this square is actually 2-cartesian.

4.3 Presentations of algebraic stacks

Given an algebraic stack over $\mathcal{S}$ we are going to build a groupoid in algebraic spaces over $\mathcal{S}$ whose associated quotient stack is the initial algebraic stack.

If (U, R, s, t, c) is a groupoid in algebraic spaces over $\mathcal{S}$, then $[U/R]$ Is not in general an algebraic stack.

Lemma 4.3.1. *(04T4) Let $\mathcal{X}$ be an algebraic stack over S, $\mathcal{U}$ an algebraic stack over S which is representable by an algebraic space U, and $f : \mathcal{U} \longrightarrow \mathcal{X}$ a 1-morphism. Then:*

1. *The 2-fibre product $\mathcal{R} = \mathcal{U} \times_{f,\mathcal{X},f} \mathcal{U}$ is representable by an algebraic space R.*
2. *There is an equivalence*

$$\mathcal{U} \times_{f,\mathcal{X},f} \mathcal{U} \times_{f,\mathcal{X},f} \mathcal{U} = \mathcal{R} \times_{pr_1,\mathcal{U},pr_2} \mathcal{R}$$

3. *There is a 1-morphism induced via 2.*

$$pr_3 : \mathcal{R} \times_{pr_1,\mathcal{U},pr_2} \mathcal{R} \longrightarrow \mathcal{R}$$

4. *If $t, s : R \longrightarrow U$ and $c : R \times_{s,U,t} R \longrightarrow R$ are the morphisms $pr_1, pr_2 : \mathcal{R} \longrightarrow \mathcal{U}$ and $pr_3 : \mathcal{R} \times_{pr_1,\mathcal{U},pr_2} \mathcal{R} \longrightarrow \mathcal{R}$, then (U, R, s, t, c) is a groupoid in algebraic spaces over S.*
5. *The morphism f induces a 1-morphism $f_{can} : [U/R] \longrightarrow \mathcal{X}$ of stacks in groupoids over S, which is fully faithful.*

6. *If $f : \mathcal{S}_U \longrightarrow \mathcal{X}$ is surjective and smooth, then the 1-morphisms s, t are smooth and the 1-morphism $f_{can} : [U/R] \longrightarrow \mathcal{X}$ is an equivalence.*

Remark. If the morphism $f : \mathcal{S}_U \longrightarrow \mathcal{X}$ is only assumed surjective, flat and locally of finite presentation, then $f_{can} : [U/R] \longrightarrow \mathcal{X}$ is also an equivalence, and the morphisms s, t are flat and locally of finite presentation, but not smooth in general.

Definition 4.3.1 (Smooth groupoid). *(04TH) A groupoid in algebraic spaces (U, R, s, t, c) over a scheme S is said to be a smooth groupoid if $s, t : R \longrightarrow \mathcal{S}$ are smooth morphisms of algebraic spaces.*

Definition 4.3.2 (Presentation of an algebraic stack). *(04TI) Let $\mathcal{X}$ be an algebraic stack over S. A presentation of $\mathcal{X}$ is an equivalence $f : [U/R] \longrightarrow \mathcal{X}$, where $[U/R]$ is the quotient stack associated to a smooth groupoid in algebraic spaces (U, R, s, t, c).*

The previous lemma states that every algebraic stack has a presentation. Reciprocally, we will see every smooth groupoid determines an algebraic stack.

Lemma 4.3.2. *Let (U, R, s, t, c) a groupoid in algebraic spaces over S. Then*

1. *(04WZ) The diagonal 1-morphism of $[U/R]$ is representable by algebraic spaces.*

2. *(04X0) If (U, R, s, t, c) is a smooth groupoid, the 1-morphism $\pi : \mathcal{S}_U \longrightarrow [U/R]$ is smooth surjective.*

Theorem 7. *(04TK) Let (U, R, s, t, c) a smooth groupoid in algebraic spaces over S. Then the quotient stack $[U/R]$ is an algebraic stack over S.*

4.4 Quotient stacks and fibre product

Here is a proposition which result from applying which results from the union of the lemmas about fibre product of groupoids, the functorial properties of the fibred categories associated to a functor and functorial properties of stackification.

Theorem 8. *Let (U,R), (U',R') and (U'',R'') be groupoids on $\mathcal{C}$ and $(\varphi,\widetilde{\varphi}) : (U,R) \longrightarrow (U'',R'')$ and $(\psi,\widetilde{\psi}) : (U',R') \longrightarrow (U'',R'')$ morphisms of groupoids. Then*

1. *The following diagram is cartesian in the 2-category of stacks over $\mathcal{C}$.*

$$\begin{array}{ccc} [U \times_{U''} U/R \times_{R''} R'] & \longrightarrow & [U'/R'] \\ \downarrow & & \downarrow \\ [U/R] & \longrightarrow & [U''/R''] \end{array}$$

2. *If (U,R) and (U',R') are smooth groupoids in algebraic spaces and $s'',t'' : R'' \longrightarrow U''$ are monomorphisms, then $(U \times_{U''} U', R \times_{R''} R')$ is a smooth groupoid, and therefore $[U \times_{U''} U/R \times_{R''} R']$ is an algebraic stack.*

Proof.

1. The construction of the fibre product of groupoids categories shows that $(U,R) \times_{(U'',R'')} (U',R') = (U \times_{U''} U', R \times_{R''} R')$ and then we construct the associated fibred categories and the functorial properties which means $[U,R]_p \times_{[U'',R'']_p} [U',R']_p = [U \times_{U''} U', R \times_{R''} R']_p$. Finally, since those fibred categories comes from functors, then the fibre product has componentwise pullbacks and so, when we take the stackification of each fibred category, the functorial properties show that $[U,R] \times_{[U'',R'']} [U',R'] \cong [U \times_{U''} U', R \times_{R''} R']$.

2. This is because smooth morphisms are a stable class which is local in the domain.

□

4.5 Open immersions

Open immersions have interesting properties. In this chapter we will see some of them. In particular we are going to study the case when an algebraic stack has a coarse moduli space and we will give some results that are helpful when we want to know what is the change of base via an open subscheme of the moduli. The initial question that leads to this work was: if we have an algebraic stack $\mathcal{X}$ which has a coarse moduli space X and $Y \longrightarrow X$ is an open immersion, what is the form of the 2-fibre product $U \times_X \mathcal{X}$? Since many of the most interesting algebraic stacks arise from the action of an algebraic group on an algebraic space, and on many cases the coarse moduli space is the action of a subgroup, then we fall in the case of quotient stacks of the form $\mathcal{X} = [U/R]$ and $X = [U/R']$. In those cases $U \times_X \mathcal{X}$ is easy to compute. We generalize this idea and give some considerations. Also, at the end some more questions arise and we explain possible ways to solve them.

Invariant subspaces

Definition 4.5.1 (Invariant open subspace). *(044F)*

Let (U, R, s, t, c) a groupoid in algebraic spaces over B.

1. *We say an open subspace $W \subset U$ is R-invariant if $t(s^{-1}(W)) \subset W$.*
2. *A locally closed subspace $Z \subset U$ is called R-invariant if $t^{-1}(Z) = s^{-1}(Z)$ as locally closed subspaces of R.*

3. A monomorphism of algebraic spaces $T \longrightarrow U$ is R-invariant if $T \times_{U,t} R = R \times_{s,U} T$ as algebraic spaces over B.

Remark. For an open subspace $W \subset U$, the R-invariance is equivalent to require $s^{-1}(W) = t^{-1}(W)$. If $W \subset U$ is R-invariant, the restriction of R to W is $R_W = s^{-1}(W) = t^{-1}(w)$.

Immersions

Definition 4.5.2 (Open immersion). *(04YL) A 1-morphism $\mathcal{Y} \longrightarrow \mathcal{Z}$ of algebraic stacks is called open immersion if it is representable and for every scheme X and any 1-morphism $X \longrightarrow \mathcal{Z}$, the morphism of schemes $X \times^2_{\mathcal{Z}} \mathcal{Y} \longrightarrow X$ is an open immersion.*

Lemma 4.5.1. *(0501)(0502)(0504) Open immersions are monomorphisms and they are stable by change of base and composition.*

In the same way, closed immersions or general immersions are defined and the same lemma is true mutatis mutandis. Note that the morphism $X \times^2_{\mathcal{Z}} \mathcal{Y} \longrightarrow X$ is the change of base of $\mathcal{Y} \longrightarrow \mathcal{Z}$ via X and it is a 2-fibre product. This consideration is important in the results we are going to give in the last part.

Lemma 4.5.2. *(0505) Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces and $i : \mathcal{Z} \longrightarrow [U/R]$ is an immersion, then there is a locally close subspace $Z \subset U$, R-invariant and a presentation $[Z/R_Z] \longrightarrow \mathcal{Z}$ such that*

$$\begin{array}{ccc} [Z/R_Z] & \longrightarrow & \mathcal{Z} \\ & \searrow & \swarrow_{i} \\ & [U/R] & \end{array}$$

is 2-commutative. If the morphism i is an open (resp. closed) immersion, then Z is an open (resp. closed) subspace of U.

Lemma 4.5.3. *(04YN) Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces, $\mathcal{X} = [U/R]$ the associated quotient stack, and $Z \subset U$ a locally closed R-invariant subspace. Then*

$$[Z/R_Z] \longrightarrow [U/R]$$

is an immersion of algebraic stacks. If $Z \subset U$ is open (resp. closed), then the morphism is an open (resp. closed) immersion of algebraic stacks.

Definition 4.5.3 (Open substack). *(04YM) Let $\mathcal{X}, \mathcal{X}'$ be algebraic stacks with $\mathcal{X}'$ a strictly full subcategory of $\mathcal{X}$. We say $\mathcal{X}'$ is an open (resp. closed, locally closed) substack if $\mathcal{X}' \longrightarrow \mathcal{X}$ is a open (resp. closed, locally closed) immersion of stacks.*

Remark. If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is an equivalence of algebraic stacks and $\mathcal{X}'$ is an open substack of $\mathcal{X}$, then it is not necessarily the case the subcategory $f(\mathcal{X}')$ is an open substack of $\mathcal{Y}$. The problem is that it may not be a strictly full subcategory, but this is also the only problem.

Lemma 4.5.4. *(0506) For any immersion $i : \mathcal{Z} \longrightarrow \mathcal{X}$ of stacks, there exists a unique locally closed substack $\mathcal{X}'$ of $\mathcal{X}$ such that i factors as the composition of an equivalence $i' : \mathcal{Z} \longrightarrow \mathcal{X}'$ followed by the inclusion $\mathcal{X}' \longrightarrow \mathcal{X}$. If i is an open (resp. closed) immersion, then $\mathcal{X}'$ is an open (resp. closed) substack of $\mathcal{X}$.*

Lemma 4.5.5. *(0507) Let $[U/R] \longrightarrow \mathcal{X}$ a presentation of an algebraic stack. There is a canonical bijection between the locally closed R-invariant subspaces of U and the locally closed substacks of $\mathcal{X}$, where if Z corresponds to $\mathcal{Z}$, then $[Z/R_Z] \longrightarrow \mathcal{Z}$ is a presentation such that the diagram*

$$\begin{array}{ccc} [Z/R_Z] & \longrightarrow & [U/R] \\ \downarrow & & \downarrow \\ \mathcal{Z} & \longrightarrow & \mathcal{X} \end{array}$$

is 2-commutative. Similarly for open and closed substacks.

With previous definitions and lemmas in mind, and using the results given in the other chapters we can prove the following result and its corollary, which led to the realization of this work.

Theorem 9. *Let* $[X/R] \longrightarrow [X/R']$ *and* $[U, R'] \longrightarrow [X/R'_U]$ *be 1-morphisms of stacks induced by the morphisms of algebraic spaces* $\widetilde{\varphi} : R \longrightarrow R'$ *and* $i : U \longrightarrow X$*, where* U *is an open* R'*-invariant algebraic subspace of* X*, then* U *is a* R*-invariant subspace of* X *and the following diagram is a cartesian square*

$$\begin{array}{ccc} [U/R_U] & \longrightarrow & [X/R] \\ \downarrow & & \downarrow \\ [U/R'_U] & \longrightarrow & [X/R'] \end{array}$$

Moreover, $[U/R_U] \longrightarrow [X/R]$ *is an open immersion of stacks and therefore if* $[X/R]$ *is algebraic, so is* $[U/R_U]$.

Proof. Let (X, R, s, t, c) and (X, R', s', t', c') be groupoids in algebraic spaces, $(id_X, \widetilde{\varphi}) : (X, R) \longrightarrow (X, R')$ a morphism of groupoids and U and open R'-invariant subspace of X. We shall see that U is also R-invariant. Indeed, we have $s = s' \circ \widetilde{\varphi}$ and $t = t' \circ \widetilde{\varphi}$ and therefore:

$$\begin{aligned} t(s^{-1}(U)) &= (t' \circ \widetilde{\varphi})((s' \circ \widetilde{\varphi})^{-1}(U)) \\ &\subseteq (t' \circ \widetilde{\varphi})(\widetilde{\varphi}^{-1}(s'^{-1}(U))) \\ &\subseteq t'(s'^{-1}(U)) \\ &\subset U \end{aligned}$$

Then by a previous lemma in this section $[U/R_U] \longrightarrow [X/R]$ is an open immersion of stacks and when we have open immersions and the target is algebraic, so is the source. The diagram is cartesian, because at the level of groupoids, the fibred product is taken component-wise and we already proved that this induces the same at the level of stacks. □

Corollary 9.1. *Let G be a group space acting over an algebraic space X and consider G' to be a quotient of G. If U is an open subspace of X which is G'-invariant, then $[U/G'] \longrightarrow [X/G']$ and the following diagram is cartesian*

$$\begin{array}{ccc} [U/G] & \longrightarrow & [X/G] \\ \downarrow & & \downarrow \\ [U/G'] & \longrightarrow & [X/G'] \end{array}$$

Moreover U is also G-invariant.

Proof. This is only a restatement of the theorem considering the groupoids in algebraic spaces induced by the actions. □

Remark. This corollary was the initial question we have to give an answer. The problem was to ensure U is G-invariant.

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