CIMAT

Centro de Investigación en Matemáticas, A.C.

MATRIX LIE GROUPS AND LIE CORRESPONDENCES

T H E S I S

Summited in partial fulfillment of the requirement for the master degree of science in basic mathematics

A thesis by:

Monyrattanak Seng

Under direction of advisor:

Professor. Raúl Quiroga Barranco

Guanajuato, Gto., August, 02nd 2017

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Abstract

Lie group is a differentiable manifold equipped with a group structure in which the group multiplication and inversion are smooth. The tangent space at the identity of a Lie group is called Lie algebra. Most Lie groups are in (or isomorphic to) the matrix forms that is topologically closed in the complex general linear group. We call them matrix Lie groups. The Lie correspondences between Lie group and its Lie algebra allow us to study Lie group which is an algebraic object in term of Lie algebra which is a linear object.

In this work, we concern about the two correspondences in the case of matrix Lie groups; namely,

1. The one-one correspondence between Lie group and it Lie algebra and
2. The one-one correspondence between Lie group homomorphism and Lie algebra homomorphism.

However, the correspondences in the general case is not much different of those in the matrix case. To achieve these goals, we will present some matrix Lie groups and study their topological and algebraic properties. Then, we will construct their Lie algebras and develop some important properties that lead to the main result of the thesis.

Contents

CHAPTER 1

Differentiable manifolds

The concept of differentiable manifolds is useful because it allows us to locally describe and understand more complicated structure on those manifolds in term of relatively properties on Euclidean space. The goal of this chapter is to give a basic understanding of differentiable manifolds.

1. Differentiable Manifolds

Definition 1.1. By a *neighborhood* of a point p in a topological space M, one means that an open set containing p. A topological space M is *a n-dimensional locally Euclidean* if every point p in M has an neighborhood U such that there is a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. We call the pair $(U, \phi : U \to \mathbb{R}^n)$ a *chart*, U a *coordinate neighborhood* or a *coordinate open set*, and ϕ a *coordinate map* or a *coordinate system* on U. We say that a chart (U, ϕ) is *center* at $p \in U$ if $\phi(p) = 0$.

Definition 1.2. A *Hausdorff* space is a topological space M such that whenever p and q are distinct points of M, there are disjoint open sets U and V in M with $p \in U$ and $q \in V$. A *differentiable structure* or *smooth structure* on a Hausdorff, second countable (that is, its topology has a countable base), Locally Euclidean space M is a collection of chat $\mathscr{F} = \{(U_\alpha, \phi_\alpha) | \alpha \in I\}$ satisfying the following three properties:

P1: $M = \bigcup\limits_{\alpha \in I} U_\alpha$.

P2: $\phi_\alpha \circ \phi_\beta^{-1}$ is C^∞ for all $\alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \varnothing$.

P3: The collection $\mathscr{F}$ is maximal with respct to P2; that is, if (U, ϕ) is a chart such that ϕ and ϕ_α are *compatible* for all $\alpha \in I$, that is, $\phi \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ \phi^{-1}$ are C^∞ for all $\alpha \in I$, then $(U, \phi) \in \mathscr{F}$.

A pair $(M, \mathscr{F})$ is a *differentiable manifold* or *smooth manifold* and is said to have *dimension* n if M is n-dimensional locally Euclidean.

Remark 1.1.

(1). A Hausdorff, second countable, locally Euclidean space is called a *topological manifold.*

(2). We think of the map ϕ as defining local coordinate functions $x_1, ..., x_n$ where x_k is the continuous function from U into $\mathbb{R}$ given by $x_k(p) = \phi(p)_k$ (the k^{th} component of $\phi(p)$). We call $x_1, ..., x_n$ a *local coordinate system.*

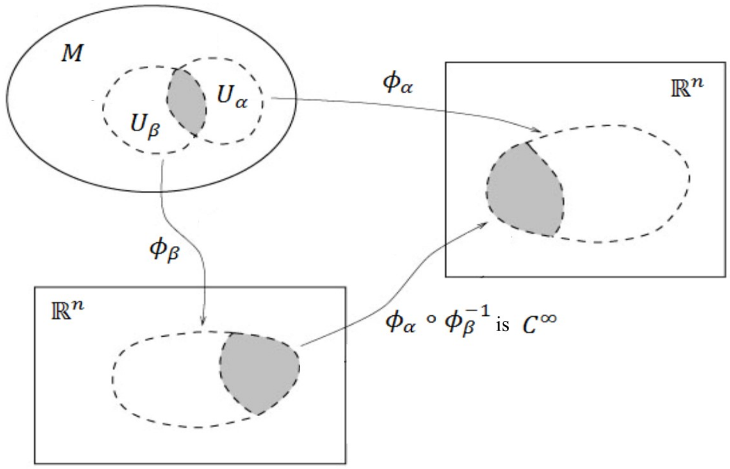


FIGURE 1. Differentiable Manifold

(3). In P2, the map $\phi_\alpha \circ \phi_\beta^{-1}$ define from $\phi_\beta(U_\alpha \cap U_\beta)$ to $\phi_\alpha(U_\alpha \cap U_\beta)$ is called the *change of coordinate*.

(4). To prove P3, it is suffice to prove that $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are smooth for a fixed coordinate map ψ since $\phi \circ \phi_\alpha^{-1} = (\phi \circ \psi^{-1}) \circ (\psi \circ \phi_\alpha^{-1})$ and $\phi_\alpha \circ \phi^{-1} = (\phi_\alpha \circ \psi^{-1}) \circ (\psi \circ \phi^{-1})$.

(5). If $\mathscr{F}_o$ is a collection of chat (U_α, ϕ_α) that satisfies the properties P1 and P2, then we can extend $\mathscr{F}_o$ uniquely to $\mathscr{F}$ that in addition satisfies the condition P3. Namely,

$$\mathscr{F} = \{(U, \phi) | \phi \circ \phi_\alpha^{-1} \text{ and } \phi_\alpha \circ \phi^{-1} \text{ are } C^\infty \text{ for all } \phi_\alpha \in \mathscr{F}_o\}$$

This remark tells us that to show that a Hausdorff, second countable space M is a differential manifold, it is suffice to construct a collection of chat that satisfies the properties P1 and P2. Thus; without any doubt, we will also call $\mathscr{F}$ that satisfies the properties P1 and P2 a differentiable structure.

Example 1.1.
(1). The cross $C_p \in \mathbb{R}^2$ such that $C_p = \{(x, y) \in \mathbb{R}^2 | x = p_1\} \cup \{(x, y) \in \mathbb{R}^2 | y = p_2\}$ where $p = (p_1, p_2)$ is not a differentiable manifold since it is not locally Eculidean at p. To see this, suppose that C_p is n-dimensional locally Euclidean at p and let ϕ be a homeomorphism from a neighborhood U of p to an open ball $B_r(0) \in \mathbb{R}^n$ that maps p to 0. then ϕ induces homeomorphism from $U \setminus \{p\} \to B_r(0) \setminus \{0\}$. This lead to a contradiction since $B_r(0) \setminus \{0\}$ is connected if $n \geq 2$ or has 2 connected components if $n = 1$ but $U \setminus \{p\}$ has 4 connected components.

(2). The pendulum P that is a union of a sphere $S^2 \subset \mathbb{R}^3$ with a semi vertical line $L = \{(x_N, y_N, z) | z \geq z_N\}$, where the north pole of sphere $N = (x_N, y_N, z_N)$, is not a differentiable manifold since it is not locally Euclidean at N. Suppose this is the case; as in example (1), $U \setminus \{N\}$ and $B_r(0) \setminus \{0\}$ are homeomorphic. Now $U \setminus \{N\}$ has 2 connected components then the only case is when $n = 1$ that is $B_r(0) \setminus \{0\}$ has

2 connected component which are open interval. However one connected component of $U\setminus\{N\}$ is homeomorphic to the deleted open disk of 0,(that is, $D_r(0)\setminus\{0\}\subset\mathbb{R}^2$). This is a contradiction.

FIGURE 2. The cross and the pendulum

(3). The Euclidean space $\mathbb{R}^n$ with the standard differentiable structure $\mathscr{F}$ that is the maximal containing the single chat $(\mathbb{R}^n, id)$, where $id : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map satisfies the properties P1 and P2. Also, $\mathbb{R}^n\setminus\{0\}$ is a differentialble manifold.

(4). An open set A of a differentiable manifold $(M, \mathscr{F}_M)$ is itself a differentiable manifold. Indeed, if (U_α, ϕ_α) are charts of differentiable manifold M, we define

$$\mathscr{F}_A = \{(A\cap U_\alpha, \phi_\alpha|_{A\cap U_\alpha})|(U_\alpha,\phi_\alpha)\in\mathscr{F}_M\}$$

where $\phi_\alpha|_{A\cap U_\alpha}$ is a restriction of ϕ_α in $A\cap U_\alpha$

then $\mathscr{F}_\alpha$ is a differentiable structure on A.

(5). The set $M(n,\mathbb{R})$ which is isomorphic to $\mathbb{R}^{n\times n}$ is a vector space of all $n\times n$ real matrices. Since $\mathbb{R}^{n\times n}$ isomophic to $\mathbb{R}^{n^2}$, we give it a topology of $\mathbb{R}^{n^2}$. Then $M(n,\mathbb{R})$ is a differentiable manifold. The *real general linear group* is a collection of invertible $n\times n$ real matrices that we can define by

$$GL(n,\mathbb{R}) = \{M\in M(n,\mathbb{R})|\det(M)\neq 0\}$$

since the matrix M (real or complex) is invertible if and only if $\det(M)\neq 0$.

Now, the determinant map $\det : M(n,\mathbb{R})\to\mathbb{R}$ is continuous. Since $GL(n,\mathbb{R}) = \det^{-1}(\mathbb{R}\setminus\{0\})$ where $\mathbb{R}\setminus\{0\}$ is an open set in $\mathbb{R}$, then $GL(n,\mathbb{R})$ is an open set in $M(n,\mathbb{R})$ which itself is a differentiable manifold.

(6). The *complex general linear group* $GL(n,\mathbb{C})$ is a subset of a complex vector space $\mathbb{C}^{n\times n}\cong\mathbb{R}^{2n^2}$ is defined to be a collection of all invertible $n\times n$ complex matrices. The similar argument in the real case tell us that $GL(n,\mathbb{C})$ is a differentiable manifold.

(7). The $n-$*sphere* $S^n\subset\mathbb{R}^{n+1}$ is defined by

$$S^n = \left\{(x_1,x_2,...,x_{n+1})\in\mathbb{R}^{n+1}\middle|\sum_{i=1}^{n+1}x_i^2 = 1\right\}$$

is a differentiable manifold. To see this, Let $N = (0, ..., 0, 1)$ be the north pole and $S = (0, ..., 0, -1)$ be the south pole of S^n and consider the *stereographic projection* from the north pole and south pole

$$\phi_N : S^n \setminus \{N\} \to \mathbb{R}^n, (x_1, x_2, ..., x_{n+1}) \mapsto (\frac{x_1}{1 - x_{n+1}}, ..., \frac{x_n}{1 - x_{n+1}})$$

$$\phi_S : S^n \setminus \{S\} \to \mathbb{R}^n, (x_1, x_2, ..., x_{n+1}) \mapsto (\frac{x_1}{1 + x_{n+1}}, ..., \frac{x_n}{1 + x_{n+1}})$$

that take $p = (x_1, ..., x_{n+1}) \in S^n \setminus \{N\}$ (or $S^n \setminus \{S\}$) into the intersection of the hyperplane $x_{n+1} = 0$ with the line that pass through p and N (or S)

These maps ϕ_N and ϕ_S are differentiable, injective and map onto the hyperplane $x_{n+1} = 0$

It is easy to check that the inverse maps ϕ_N^{-1} and ϕ_S^{-1} are also differentiable. This implies that S^n is locally Euclidean. Moreover, $S^n = (S^n \setminus \{N\}) \cup (S^n \setminus \{S\})$ and the change of coordinates (by a direct calculation) $\phi_N \circ \phi_S^{-1}$ on $\mathbb{R}^n$ is given by

$$y'_j = \frac{y_j}{\sum\limits_{i=1}^{n} y_i^2} \qquad , j = 1, ..., n$$

is smooth and also $\phi_S \circ \phi_N^{-1} = (\phi_N \circ \phi_S^{-1})^{-1}$ is smooth.
Therefore, $\mathscr{F} = \{(S^n \setminus \{N\}, \phi_N), (S^n \setminus \{S\}, \phi_S)\}$ is differentiable structure on S^n.

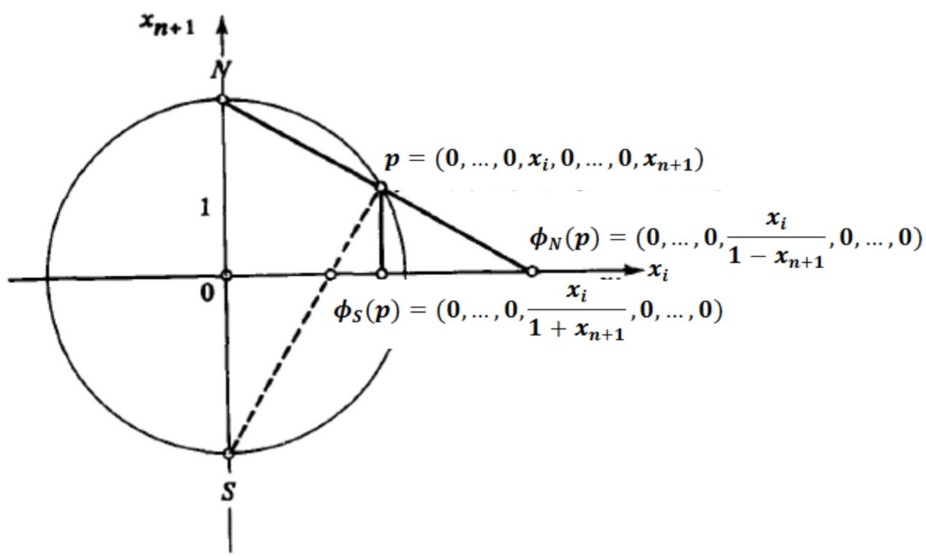


FIGURE 3. The stereotype projections from north and south poles of S^n

(8). For a subset of $A \subset \mathbb{R}^n$ and a function $f : A \to \mathbb{R}^m$, the *graph* of f is defined to be the subset

$$\Gamma(f) = \{(x, f(x)) \in A \times \mathbb{R}^m\}$$

If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ is C^∞, then the two maps

$$\phi : \Gamma(f) \to U, (x, f(x)) \mapsto x$$

and

$$\psi : U \to \Gamma(f), x \mapsto (x, f(x))$$

are continuous and inverse to each other, and so are homeomorphisms. The set $\Gamma(f)$ of a C^∞ function $f : U \to \mathbb{R}^m$ has a single chat $(\Gamma(f), U)$ that satisfies the property P1 and P2.

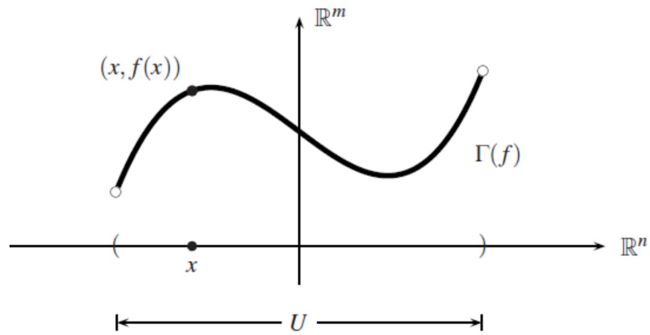


FIGURE 4. The graph of a smooth function $f : \mathbb{R}^n \supset U \to \mathbb{R}^m$

(9)(*Product manifolds*). Let $(M^k, \mathscr{F})$ and $(N^l, \mathscr{G})$ be differentiable manifolds of dimensions k and l, respectively. Then $M^k \times N^l$ becomes a differentiable manifold of dimension $k + l$, with differentiable structure $\mathscr{H}$ the maximal collection containing:

$$\{(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^k \times \mathbb{R}^l) | (U_\alpha, \phi_\alpha) \in \mathscr{F}, (V_\beta, \psi_\beta) \in \mathscr{G}\}$$

(10). The *n-torus* T^n can be consider as a n-product of circle $S^1 \subset \mathbb{R}^2$ that is, $T^n = \underbrace{S^1 \times S^1 \times ... \times S^1}_{n\text{-times}}$ is a differentiable manifold since S^1 is a differentiable manifolds.

Remark 1.2. One may ask if there exists a topological manifold that is not a differentiable manifold. The answer is yes. A triangulable closed manifold M_0 of dimension 10 is a topological manifold (in fact, it is a *piecewise linear manifold*) that does not have any smooth structures. For the construction of this manifold M_0, consult **[16]**, A manifold which does not admit any Differentiable Structure by Michel A. Kerviare.

Note that in examples above, we do not concern about Hausdorff and second countibility since the subspace with relative topology of a Hausdorff and second countable space is Hausdorff and second countable space.

From now on, "manifold" is refered to "differentiable manifold".

2. Tangent Spaces and Differential Forms

Definition 1.3. Let M^k be k-dimensional manifold and N^l be l-dimensional manifold. A map $f : M^k \to N^l$ is said to be C^∞ or *smooth* at $p \in M^k$ if given a chart (V, ψ) about $f(p) \in N^l$, there exists a chat (U, ϕ) about $p \in M^k$ such that

$f(U) \subset V$ and the composition map $\psi \circ f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^l$ is C^∞ at $\phi(p)$. The map f is said to be smooth if it is smooth at every point of M^k.

Remark 1.3.
(1). It follows from P2 of definition 1.2 that the above definition of the smoothness of a map $f : M^k \to N^l$ at a point $p \in M^k$ is independent of the choice of charts. Indeed, if (U^*, φ) is any chart about p then $\psi \circ f \circ \varphi^{-1} = (\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \varphi^{-1})$ is C^∞ since it is the composition of C^∞ maps.

(2). If (U, ϕ) is a chart at $p \in M^n$ with $\phi = (x_1, \dots, x_n)$. It follows from the above definition and the condition P2 that ϕ and ϕ^{-1} are smooth and then the coordinate functions $x_i \in \mathscr{D}(M)$ where $\mathscr{D}(M)$ denotes the set of real value smooth function on M.

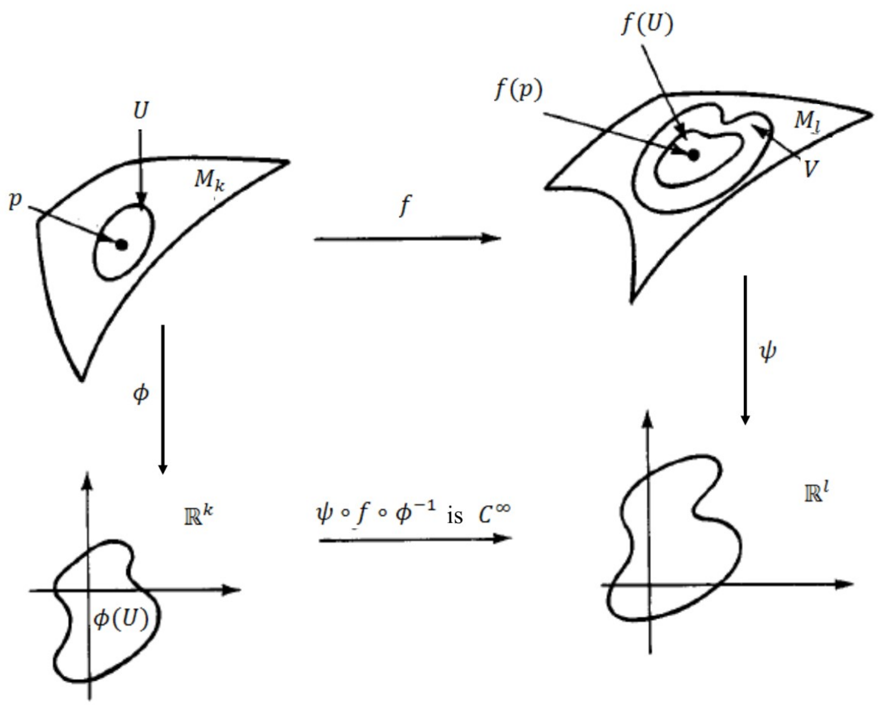


FIGURE 5. A smooth map between two manifolds

Definition 1.4. The *tangent vector* at $p \in M$ is the linear maps $u : \mathscr{D}(M) \to \mathbb{R}$ from the set of real value smooth function on a manifold M ($\mathscr{D}(M)$ or simply $\mathscr{D}$) to a set of real number that satisfies the production rule of derivation. That is, for all$f, g \in \mathscr{D}$ and $\lambda \in \mathbb{R}$,

1. $u(f + \lambda g) = u(f) + \lambda u(g)$ (linearity),

2. $u(fg) = u(f)g(p) + f(p)u(g)$ (product rule of derivation).

The collection of all tangent vectors at p is said to be a *tangent space* at p and denoted by T_pM.

Remark 1.4. The linear map $0 \in T_pM$ and if we define $(u+v)(f) := u(f)+v(f)$ and $(\lambda u)(f) := \lambda u(f)$ for all $u, v \in T_pM$ and $\lambda \in \mathbb{R}$, then it is easy to see that $(u+v)(f)$ and $(\lambda u(f))$ again are tangent vectors at p. Thus, T_pM is a real vector space.

Definition 1.5. Let $\phi = (x_1, \ldots, x_n)$ be a coordinate system in a manifold M^n at p. The *partial differential* of a smooth function f on M at p is defined by:

$$\frac{\partial f}{\partial x_i}(p) = \frac{\partial (f \circ \phi^{-1})}{\partial u_i}(\phi(p)) \qquad\qquad 1 \leq i \leq n,$$

where $u_1, \ldots, u_n$ are the natural coordinate functions of $\mathbb{R}^n$.

A straightforward computation then shows that the function

$$\partial_i|_p = \left.\frac{\partial}{\partial x_i}\right|_p : \mathscr{D}(M) \to \mathbb{R}$$

sending each $f \in \mathscr{D}(M)$ to $(\partial f/\partial x_i)(p)$ is a tangent vector to M at p. We can picture $\partial_i|_p$ as an arrow at p tangent to the x_i-coordinate curve through p.

Lemma 1.1. Let $v \in T_pM$.
(1). If $f, g \in \mathscr{D}(M)$ are equal on a neighborhood U of p, then $v(f) = v(g)$.
(2). If $h \in \mathscr{D}(M)$ is constant on a neighborhood of p, then $v(h) = 0$.

Proof.
To prove (1), we make use of the result that for any neighborhood U of $p \in M$, there exists a function $f \in \mathscr{D}$, called a *bump function* at p, such that
a. $0 \leq f \leq 1$ on M.
b. $f = 1$ on some neighborhood of p.
c. $\mathrm{supp} f = \{x | f(x) \neq 0\} \subset U$.
Let h be a bump function at p such that $\mathrm{supp} h \subset U$ then $(f-g)h = 0$ on M. But $v(0) = v(0+0) = v(0) + v(0)$ implies $v(0) = 0$. Thus,

$$0 = v((f-g)h) = v(f-g)h(p) + (f-g)(p)v(h) = v(f-g) = v(f) - v(g)$$

Therefore, $v(f) = v(g)$

For (2), observe that $v(1) = v(1.1) = v(1)1 + 1v(1) = 2v(1)$ which implies $v(1) = 0$
Thus, if $h = c$ on M then $v(h) = v(c.1) = cv(1) = 0$. This completes the proof.
□

Proposition 1.1. Let M be an n-dimensional manifold and $p \in M$. Let (U, ϕ) be a chart about p with a coordinate system $x_1, ..., x_n$. Then T_pM has basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ and any tangent vector $u \in T_pM$ can be expressed (uniquely) as

$$u(f) = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}$$

where $a_i = u(x_i) \in \mathbb{R}$ and $f \in \mathscr{D}$.

In this way T_pM is a real vector space of dimension n (as the same as dimension of M).

Remark 1.5. If we taking a chart (U, ϕ) of M at p such that for $u_i : \phi(U) \to \mathbb{R}$, the natural coordinate function on $\mathbb{R}^n$, the partial derivative operator $\frac{\partial}{\partial u_i}(\phi(p))$ in $\mathbb{R}^n$ for $i = 1, \cdots, n$ eventually yield a base $\frac{\partial}{\partial x_i}(p)$ for $i = 1, \cdots, n$ of T_pM. Thus, let $f \in \mathscr{D}$ where f is defined on a neighborhood V of p then f is smooth on $U \cap V \subset U$ so that we can write f in term of $\phi = (x_1, \cdots, x_n)$ where $x_i = u_i \circ \phi$. Therefore, for $\phi(U)$ which is an open set in $\mathbb{R}^n$, the function $g = f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth on $\phi(U)$ and $f = g \circ \phi = g(x_1, \cdots, x_n)$.

Proof. Without loss of generality, we can assume that $\phi(p) = 0$ since a translation $x_i = y_i + t$ yields $\partial/\partial x_i = \partial/\partial y_i$. Shrinking U if necessary give $\phi(U) = \{q \in \mathbb{R}^n | \|q\| < \epsilon\}$ for some ϵ.

Let g be a smooth function on $\phi(U)$ and for each $1 \leq i \leq n$, we define

$$g_i(q) = \int_0^1 \frac{\partial g}{\partial u_i}(tq)dt \qquad \text{for all} \qquad q \in \phi(U)$$

It follows from the fundamental theorem of calculus that:

$$g(q) - g(0) = \int_0^1 g'(tq)dt = \int_0^1 \frac{\partial g}{\partial (tq)} q dt$$

since $\frac{\partial g}{\partial (tq)} = (\frac{\partial g}{\partial u_1}(tq), \cdots, \frac{\partial g}{\partial u_n}(tq))$ and $q = (q_1, \cdots, q_n)$ then:

$$g(q) = g(0) + \int_0^1 \sum_{i=1}^{n} \frac{\partial g}{\partial u_i}(tq) q_i = g(0) + \sum_{i=1}^{n} \int_0^1 \frac{\partial g}{\partial u_i}(tq) q_i = g(0) + \sum_{i=1}^{n} g_i(q) u_i(q)$$

since $u_i(q) = q_i$. Thus, $g = g(0) + \sum\limits_{i=1}^{n} g_i u_i$

Now, let $f \in \mathscr{D}$ and set $g = f \circ \phi^{-1}$ then:

$$f \circ \phi^{-1} = f \circ \phi^{-1}(0) + \sum_{i=1}^{n} g_i u_i = f(p) + \sum_{i=1}^{n} g_i u_i$$

So we obtain:

$$f = f(p) + \sum_{i=0}^{n} (g_i u_i) \circ \phi = f(p) + \sum_{i=0}^{n} f_i (u_i \circ \phi) = f(p) + \sum_{i=0}^{n} f_i x_i$$

since $f(p)$ is constant, $f_i = g_i \circ \phi$ and $x_i = u_i \circ \phi$

Thus, from lemma 1.1, apply $u \in T_pM$ we obtain:

$$u(f) = 0 + \sum_{i=1}^{n} u(f_i x_i) = \sum_{i=1}^{n} u(f_i) x_i(p) + \sum_{i=1}^{n} f_i(p) u(x_i) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) u(x_i)$$

since $x_i(p) = 0$ and $f_i(p) = g_i \circ \phi(p) = g_i(0) = \frac{\partial g}{\partial u_i}(0) = \frac{\partial (f \circ \phi^{-1})}{\partial u_i}(\phi(p)) = \frac{\partial f}{\partial x_i}(p)$
Since f is arbitrary, let $a_i = u(x_i) \in \mathbb{R}$ then $u(f) = \sum\limits_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) u(x_i)$

It remains to prove that the coordinate vector ∂_i are linearly independent. Suppose that $\sum\limits_{i=0}^{n} \alpha_i \partial_i = 0$ then apply to x_j yields

$$0 = \sum_{i=1}^{n} \alpha_i \frac{\partial x_j}{x_i} = \sum_{i=0}^{n} \alpha_i \sigma_{ij} = \alpha_j \qquad \forall j$$

□

Definition 1.6. Let $f : M^m \to \mathbb{R}$ be a smooth function. We define the differential of f at $p \in M$ to be the map $df_p : T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$ by $(df_p)(v) = v(f)$.

More general, if $f : M^m \to N^n$ be a smooth function and let $p \in M$. The *differential* of f at p is the map $df_p : T_pM \to T_{f(p)}M$ such that for any $u \in T_pM$, $df_p(u)$ is to be a tangent vector at $f(p)$. On the other hand, if g is a smooth function on neighborhood of $f(p)$, we define $df_p(u)(g) = u(g \circ f)$.

Proposition 1.2. df_p is a linear map. In addition, if (U, ϕ) is a chart at p and (V, ψ) is a chart at $f(p)$, then df_p has a matrix which is the Jacobian matrix of f represented in these coordinates.

Proof. Let $u, v \in T_pM$ then for $\lambda \in \mathbb{R}$ and $g \in \mathscr{D}(N)$, we have:

$$\begin{aligned} df_p(u + \lambda v)(g) &= (u + \lambda v)(g \circ f) \\ &= u(g \circ f) + \lambda v(g \circ f) \\ &= df_p(u)(g) + \lambda df_p(v)(g) \end{aligned}$$

This proves the linearity of df_p.

Now, let $\phi = (x_1, \ldots, x_m)$ and $\psi = (y_1, \ldots, y_n)$ be the given coordinate functions so that f can be expressed in term of coordinates in the neighborhood V as:

$$f_i = y_i \circ f \qquad\qquad i = 1, \ldots, n$$

Let $\partial/\partial x_j$ and $\partial/\partial y_i$ be a basis for T_pM and $T_{f(p)}N$, respectively and let $[a_{ij}]$ be a matrix of df_p. We will prove that $a_{ij} = \partial f_i/\partial x_j$

We have $df_p(\partial/\partial x_j) = \sum\limits_i a_{ij} \partial/\partial y_i \in T_{f(p)}N$. Using the fact that $y_k \in \mathscr{D}(N)$, we obtain:

$$\frac{\partial f_k}{\partial x_j} = \frac{\partial}{\partial x_j}(y_k \circ f) = df_p(\frac{\partial}{\partial x_j})(y_k) = \sum_i a_{ij} \frac{\partial y_k}{\partial y_i} = a_{kj}$$

The last equality is followed by using $\partial y_k/\partial y_i = \delta_{ki}$
Thus, $a_{ij} = \partial f_i/\partial x_j$ is the desired Jacobian matrix.

$\square$

Proposition 1.3. Let $f : M^m \to N^n$ and $g : N^n \to K^k$ be smooth functions. Then for each $p \in M$,

$$d(g \circ f)_p = dg_{f(p)} \circ df_p$$

PROOF. First, note that if $u \in T_pM$ then the map $u_f : \mathscr{D}(N) \to \mathbb{R}$ sending each h to $u(h \circ f)$ is a tangent vector to N at $f(p)$. To see this, let $h_1, h_2 \in \mathscr{D}(N)$ and $\lambda \in \mathbb{R}$ then:

$$\begin{aligned}
u_f(h_1 + \lambda h_2) &= u((h_1 + \lambda h_2) \circ f) = u((h_1 \circ f) + (\lambda h_2 \circ f)) \\
&= u(h_1 \circ f) + \lambda u(h_2 \circ f) = u_f(h_1) + \lambda u_f(h_2) \\
u_f(h_1 h_2) &= u(h_1 h_2 \circ f) = u((h_1 \circ f)(h_2 \circ f)) \\
&= u(h_1 \circ f)(h_2(f(p)) + h_1(f(p))u(h_2 \circ f) \\
&= u_f(h_1)(h_2(f(p)) + h_1(f(p))u_f(h_2)
\end{aligned}$$

Now, let $u \in T_pM$ and $h \in \mathscr{D}(P)$, then:

$$d(g \circ f)_p(u)(h) = u(h \circ g \circ f) = df_p(u)(h \circ g) = [dg_{f(p)} \circ df_p(u)](h)$$

Thus,

$$d(g \circ f)_p = dg_{f(p)} \circ df_p$$

$\square$

Definition 1.7.
(1). The *curve* on a manifold M is a smooth map $\alpha : I \to M$ where I is an open interval in the real line $\mathbb{R}$(I can be half infinity or all of $\mathbb{R}$). As an open submanifold of $\mathbb{R}$, I has a coordinate system consisting of the identity map u of I then the coordinate vector $(d/du)(t) \in T_t(\mathbb{R})$.
(2). Let $\alpha : I \to M$ be a curve. The *velocity vector* of α at $t \in I$ is

$$\alpha'(t) = d\alpha_t \left(\frac{d}{du} \right) \in T_{\alpha(t)}M$$

Remark 1.6.
(1). By the definition of $d\alpha$, the tangent vector $\alpha'(t)$ applied to a function $f \in \mathscr{D}(M)$ gives:

$$\alpha'(t)f = d\alpha_t \left(\frac{d}{du} \right) f = \frac{d(f \circ \alpha)}{du}(t)$$

Thus, if α is any curve with say $\alpha'(0) = v$, then:

$$v(f) = \frac{d(f \circ \alpha)}{dt}(0)$$

.

(2). If $f : M \to N$ is smooth and α is a curve on M then $f \circ \alpha$ is a curve on N and from the chain rule, we have $(f \circ \alpha)'(t) = df_{\alpha(t)}(\alpha'(t)) \in T_{(f\circ\alpha)(t)}N$.

Lemma 1.2. consider the product manifold $M^m \times N^n$ and the canonical projections $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$.

(1). The map $\alpha : u \to (d\pi_1(u), d\pi_2(u))$ is an isomorphism of $T_{(m,n)}(M \times N) \to T_m M \oplus T_n N$.

(2). If $(m_0, n_0) \in M \times N$, and define injections $i_{n_0} : M \to M \times N$ and $i_{m_0} : N \to M \times N$ by:

$$i_{n_0}(m) = (m, n_0)$$
$$i_{m_0}(n) = (m_0, n)$$

Let $u \in T_{(m_0,n_0)}(M \times N), u_1 = d\pi_1(u), u_2 = d\pi_2(u)$ and $f \in \mathscr{D}(M \times N)$. Then

$$u(f) = u_1(f \circ i_{n_0}) + u_2(f \circ i_{m_0})$$

Proof.

(1). First, it is easy to see that the projections π_i $(i = 1, 2)$ are smooth. The map α is clearly a linear map since $d\pi_i$ $(i = 1, 2)$ are linear map. Now, observe that the Jacobian matrices of $d\pi_1$ and $d\pi_2$ are $[I_m\ 0]_{m,m+n}$ and $[0\ I_n]_{n,m+n}$ respectively. Thus, $d\pi_i$ are surjective and so is α. To see that α is injective, let $\alpha(u) = \alpha(u')$ for some $u = (x_1, \dots, x_m, y_1, \dots, y_n)$ and $u' = (x'_1, \dots, x'_m, y'_1, \dots, y'_n)$ in $T_{(m,n)}(M \times N)$ then $d\pi_1(u) = d\pi_1(u')$ and $d\pi_2(u) = d\pi_2(u')$ so that $(x_1, \dots, x_m) = (x'_1, \dots, x'_m)$ and $(y_1, \dots, y_n) = (y'_1, \dots, y'_n)$ which imply $u = u'$.

(2). It is not difficult to see that injections i_{m_0} and i_{n_0} are smooth. We have from definition of u, linearity of $df_{(m_0,n_0)}$ and (1) that:

$$\begin{aligned} u(f) &= df_{(m_0,n_0)}(u) = df_{(m_0,n_0)}(u_1, u_2) \\ &= df_{(m_0,n_0)}[(u_1, 0) + (0, u_2)] \\ &= df_{(m_0,n_0)}(u_1, 0) + df_{(m_0,n_0)}(0, u_2) \end{aligned}$$

Now, choose the curve $\alpha(t) = (\alpha_1(t), n_0) = i_{n_o}(\alpha_1(t))$ with $\alpha'_1(0) = u_1$ then:

$$\begin{aligned} df_{(m_0,n_0)}(u_1, 0) &= df_{(m_0,n_0)}(\alpha'(0)) = \alpha'(0)(f) \\ &= \frac{d(f \circ \alpha)}{dt}(0) = \frac{d((f \circ i_{n_0}) \circ \alpha_1)}{dt}(0) \\ &= \alpha'_1(0)(f \circ i_{n_0}) = u_1(f \circ i_{n_0}) \end{aligned}$$

Similarly, if we chose the curve $\beta(t) = (m_0, \beta_2(t)) = i_{m_0}(\beta_2(t))$ with $\beta'_2(0) = u_2$, then:

$$df_{(m_0,n_0)}(0, u_2) = u_2(f \circ i_{m_0})$$

$\square$

Definition 1.8. Let M^k and N^l be manifolds of dimension k and l, respectively. A map $f : M^k \to N^l$ is said to be *diffeomorphism* if it is bijective and bi-smooth i.e. f and its inverse map f^{-1} are smooth. On the other hand, f is said to be *locally diffeomorphism* at $p \in M^k$ if there exist a neighborhood U of p and V of $f(p)$ such that $f : U \to V$ is diffeomorphism.

Remark 1.7. It is immediate that if $f : M^k \to N^l$ is a diffeomorphism, then $df_p : T_pM^k \to T_{f(p)}N^l$ is an isomorphism for all $p \in M^k$;in particular, the dimensions of M^k and N^l are equal. The converse is not true; however, the local converse is true i.e. if $f : M^k \to N^k$ be smooth map and let $p \in M^k$ such that $df_p : T_pM^k \to T_{f(p)}N^k$ is an isomorphism, then f is locally diffeomorphism at p. This result is followed immediately from the Inverse Function Theorem.

Proposition 1.4. Let $(M^n, \mathscr{F})$ be a manifold and a function $f : \mathbb{R}^n \to \mathbb{R}^n$ be diffeomorphsim. If (U, ϕ) is a chart in $\mathscr{F}$, then $(U, f \circ \phi) \in \mathscr{F}$.

Proof. Let (V, ψ) be a chart in $\mathscr{F}$ such that $U \cap V \neq \varnothing$ then:
$\psi \circ (f \circ \phi)^{-1} = (\psi \circ \phi^{-1}) \circ f^{-1}$ and $(f \circ \phi) \circ \psi^{-1} = f \circ (\phi \circ \psi^{-1})$ are smooth since $f, f^{-1}, \psi \circ \phi^{-1}, \phi \circ \psi^{-1}$ are smooth. From the maximality of $\mathscr{F}$, we obtain $(U, f \circ \phi) \in \mathscr{F}$.

□

3. Submanifolds

Definition 1.9. Let $(M, \mathscr{F})$ be a $n+k$-dimensional manifolds. An *n-dimensional embedded submanifold* in M is a subset $N \subset M$ such that for each $p \in N$, there is a chart $(U, \phi : U \to V)$ of $\mathscr{F}$ with $p \in U$ such that $\phi(U \cap N) = V \cap (\mathbb{R}^n \times \{0\}) \subset \mathbb{R}^n \times \mathbb{R}^k$.

On the other hand, An *n-dimensional immersed submanifold* in M is a topological space $N \subset M$ such that for each $p \in N$, there is a chart $(U, \phi : U \to V)$ of $\mathscr{F}$ with $p \in U$ such that for a neiborhood W of p in N with $W \subset U$, we have $\phi(W) = V \cap (\mathbb{R}^n \times \{0\}) \subset \mathbb{R}^n \times \mathbb{R}^k$.

In this definition, we identify $\mathbb{R}^{n+k}$ with $\mathbb{R}^n \times \mathbb{R}^k$ and often write $\mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^k$ instead of $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^k$ to signify the subset of all points with the k last coordinates equal to zero.

Remark 1.8.
(1). If $(M, \mathscr{F})$ is a manifold and $N \subset M$ is a embedded submanifold, then we can give N a differentiable structure

$$\mathscr{F}_N = \{\ (N \cap U_\alpha, \phi_\alpha|_{N \cap U_\alpha}) | (U_\alpha, \phi_\alpha) \in \mathscr{F}\}$$

Note that the inclusion map $i : N \hookrightarrow M$ is smooth. Also note that from this smooth structure, the topology in N is a induced topology from M.

(2). The topological of a immersed submanifold $N \subset M$ need not be the induced topology of the containing manifold. Also note that the dimension of a submanifold (embedded or immersed) is less than or equal to the dimension of its containing manifold and in the case of equality we just obtain open submanifolds.

Example 1.2.
(1). Let n be a natural number. Then $K_n = \{(x, x^n) \in \mathbb{R}^2 | x \in \mathbb{R}\} \subset \mathbb{R}^2$ is a submanifold. Indeed, if we give $\mathbb{R}^2$ a differentiable structure with a single chart

$(\mathbb{R}^2, \phi)$ where $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(x, y) \mapsto (x, y - x^n)$ then $\phi(K_n) = \mathbb{R} \times \{0\}$.

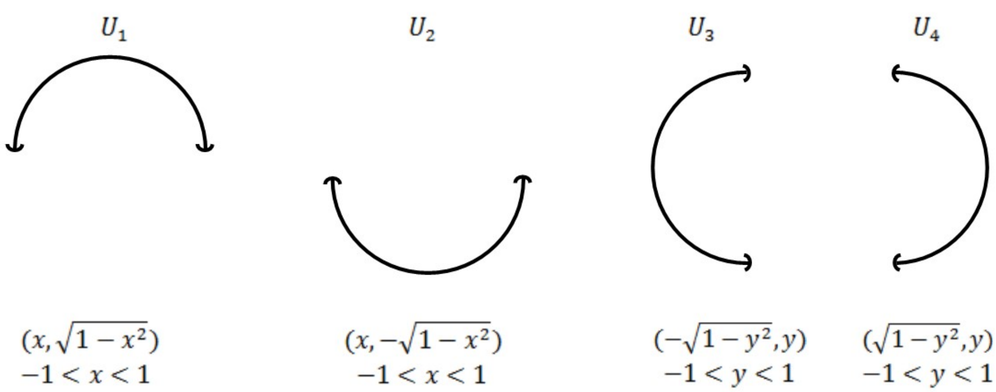
(2). Consider the unit sphere $S^1 \subset \mathbb{R}^2$. In $\mathbb{R}^2$, we can construct a smooth structure $\mathscr{F}$ to be the maximal collection containing

$$F = \{(\mathbb{R}^2, id)\}$$

Now consider the maps $\phi_i : U_i \to \mathbb{R}^2$ as below:

$$\phi_1(x,y) = (x, y - \sqrt{1-x^2}), \qquad \phi_2(x,y) = (x, y + \sqrt{1-x^2}),$$
$$\phi_3(x,y) = (x + \sqrt{1-y^2}, y), \qquad \phi_4(x,y) = (x - \sqrt{1-y^2}, y)$$

and U_i are open sets indicating in the following picture:
It is easy to see that $\phi_1^{-1} = \phi_2, \phi_3^{-1} = \phi_4$ and all ϕ_i are smooth on U_i. Then



$(U_i, \phi_i) \in \mathscr{F}$ since ϕ_i and id are compatible.

Let $p \in S^1$, p is contained in one of U_i.
The case p is contained in the upper or the lower half of the circle, we have:

$$\phi_1(U_1 \cap S^1) = \phi_1(U_1) = \{(x,0) | -1 < x < 1\} = \phi_1(U_1) \cap (\mathbb{R} \times \{0\})$$

$$\phi_2(U_2 \cap S^1) = \phi_2(U_2) = \{(x,0) | -1 < x < 1\} = \phi_2(U_2) \cap (\mathbb{R} \times \{0\})$$

For the case p is in the left or the right half of the circle, we have:
$(U_3, f \circ \phi_3), (U_4, f \circ \phi_4) \in \mathscr{F}$, where $f : (x, y) \to (y, x)$ is the interchange of coordinate function (Proposition 1.4) and:

$$f \circ \phi_3(U_3 \cap S^1) = f \circ \phi_3(U_3) = \{(y,0) | 1 < y < 1\} = f \circ \phi_3(U_3) \cap (\mathbb{R} \times \{0\})$$

$$f \circ \phi_4(U_4 \cap S^1) = f \circ \phi_4(U_4) = \{(y,0) | 1 < y < 1\} = f \circ \phi_4(U_4) \cap (\mathbb{R} \times \{0\})$$

This proves that S^1 is a submanifold of $\mathbb{R}^2$. Also, S^1 is a submanifold of $\mathbb{R}^2 \setminus \{0\}$.

4. Vector Fields, Brackets

Definition 1.10. A *vector filed* X on a manifold M is a correspondence that associates to each point $p \in M$ a vector $X(p) \in T_pM$. In term of a coordinate map $\phi : M \supset U \to \mathbb{R}^n$, we can write

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i}$$

where each $a_i : U \to \mathbb{R}$ is a function on U and $\{\frac{\partial}{\partial x_i}\}$ is the basis associated to ϕ, $i = 1, ..., n$. It is said to be smooth if a_i are smooth.

Occasionally, it is convenient to use this idea and think of a vector field as a map $X : \mathscr{D} \to \mathscr{F}$ from the set $\mathscr{D}$ of smooth function on M to the set $\mathscr{F}$ of function on M, defined in the following way

$$X_p(f) = (Xf)(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial f}{\partial x_i}(p)$$

It is easy to check that the function Xf does not depend on the choice of coordinate map ϕ. In this context, it is immediate that X is smooth if and only if $Xf \in \mathscr{D}$ for all $f \in \mathscr{D}$.

Proposition 1.5. Let X and Y be smooth vector fields on a manifold M^n. Then there exists a unique vector field Z such that for all $f \in \mathscr{D}$,

$$Zf = (XY - YX)f = X(Yf) - Y(Xf)$$

.

Proof. Let $p \in M$ and $x_1, ..., x_n$ be a local coordinate system about p. Then, we can express the vector field X and Y uniquely as

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i}, \qquad Y(p) = \sum_{j=1}^{n} b_j(p) \frac{\partial}{\partial x_j}$$

Then for all $f \in \mathscr{D}$,

$$X(Yf)(p) = X\left(\sum_j b_j(p) \frac{\partial f}{\partial x_j}\right)(p) = \left(\sum_{i,j} a_i(p) \frac{\partial b_j}{\partial x_i} \frac{\partial f}{\partial x_j} + \sum_{i,j} a_i(p) b_j(p) \frac{\partial^2 f}{\partial x_i \partial x_j}\right)(p)$$

$$Y(Xf)(p) = Y\left(\sum_i a_i(p) \frac{\partial f}{\partial x_i}\right)(p) = \left(\sum_{i,j} b_j(p) \frac{\partial a_i}{\partial x_j} \frac{\partial f}{\partial x_i} + \sum_{i,j} a_i(p) b_j(p) \frac{\partial^2 f}{\partial x_i \partial x_j}\right)(p)$$

Thus,

$$(Zf)(p) = X(Yf)(p) - Y(Xf)(p) = \sum_{i,j} \left(a_i \frac{\partial b_j}{\partial x_i} - b_j \frac{\partial a_i}{\partial x_j}\right) \frac{\partial f}{\partial x_j}(p)$$

is a vector field in coordinate neighborhood of p and is unique. Since p is arbitrary, we can define Z_p in each coordinate neighborhood U_p of differentiable structure $\{(U_\alpha, \phi_\alpha)\}$ on M by the above expression. The uniqueness implies $Z_p = Z_q$ on

$U_p \cap U_q \neq \varnothing$, which allows us to define Z over the entire manifold M and also this Z is unique.

□

Definition 1.11. The vector field Z defined in proposition 1.5 is called *bracket* and is defined by $[X, Y] = XY - YX$ of X and Y. $[X, Y]$ is obviously smooth.

Proposition 1.6. If X, Y and Z are smooth vector fields on M, a, b are real numbers and f, g are smooth functions, then:

(1). $[X, Y] = -[Y, X]$ (anticommutativity),
(2). $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ (linearity),
(3). $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$ (Jacobi identity),
(4). $[fX, gX] = fg[X, Y] + f(Xg)Y - g(Yf)X$.

Proof.

(1) is obvious and (2) is immediate from the linearity of derivation. For (3), we have:

$$\begin{aligned} [[X, Y], Z] &= [X, Y]Z - Z[X, Y] \\ &= (XY - YX)Z - Z(XY - YX) = XYZ - YXZ - ZXY + ZYX \end{aligned}$$

and by interchanging X, Y and Z, we obtain:

$$[[Z, X], Y] = ZXY - XZY - YZX + YXZ$$
$$[[Y, Z], X] = YZX - ZYX - XYZ + XZY$$

Thus,

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$$

(4). Let $h \in \mathscr{D}$, then:

$$\begin{aligned} [fX, gY]h &= [(fX)(gY) - (gY)(fX)]h \\ &= fX(gYh) - gY(fXh) \\ &= f(Xg)(Yh) + fgX(Yh) - g(Yf)(Xh) - fgY(Xh) \\ &= f(Xg)(Yh) + fg(X(Yh) - Y(Xh)) - g(Yf)(Xh) \\ &= (f(Xg)Y + fg[X, Y] - g(Yf)X)h \end{aligned}$$

Thus,

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$$

□

5. Connectedness of Manifolds

Definition 1.12. Let M be a manifold.

(1). M is said to be *connected* if it cannot separate by any two non empty and distinct open or closed subsets in M. More precisely, if there is no non empty open sets or closed sets A and B in M such that $A \cap B = \varnothing$, and $A \cup B = G$. A manifold which is not connected can be decomposed (uniquely) as a union of several pieces, called *components*, such that two elements in the same component can be joined by a continuous path and two elements of different component cannot.

(2). M is said to be *path-connected* if given any two point x and y in M, there exists a continuous path $\alpha(t), a \leq t \leq b$, lying in M with $\alpha(a) = x$ and $\alpha(b) = y$. On the other hand, it said to be *locally path connected* if every point is contained in a path-connected neighborhood.

Proposition 1.7. Let M be a path-connected manifold. Then M is connected.

PROOF. Suppose that M is not connected. Then there exist non empty open sent $A, B \in M$ such that $M = A \cup B$ and $A \cap B = \varnothing$. Let $a \in A$ and $b \in B$. Since M is path-connected then there is a continuous path $\alpha : [0,1] \to M$ such that $\alpha(0) = a$ and $\alpha(1) = b$. Thus, $[0,1] = \alpha^{-1}(A) \cup \alpha^{-1}(B)$ with $\alpha^{-1}(A) \cap \alpha^{-1}(B) = \varnothing$ and they are non empty open sets in $[0,1]$. This contradicts the connectivity of $[0,1]$. Therefore, M is connected.

□

Proposition 1.8. Let M^n be a manifold. Then M is connected if and only if M is path-connected.

PROOF. The second part is a result from the previous proposition. To prove the first part, observe that M is locally path connected. Indeed, let $x \in M$, there exists a homeomorphism between the neighborhood of x and a open subset of $\mathbb{R}^n$. So, we can take an open ball (with is a path-connected) containing the image of x and it is clearly that the preimage of this ball is path-connected neighborhood of x.

Now, Let M be connected and $x \in M$ and consider the set M_x defined by:

$$M_x = \{y \in M | \exists \alpha : [0,1] \to M, \alpha(0) = x, \alpha(1) = y\}$$

Then $M_x \neq \varnothing$ since $x \in M_x$. Let $y \in M_x$ there is path-connected neighborhood U_y of y. Let $z \in U_y$, then z can connect to x by a path from x to y and then from y to z. Therefore, $U_y \subset M_x$ which implies $M_x = \bigcup\limits_{y \in M_x} U_y \subset M$ is an open set in M and is path-connected by definition.

Let $p \in M \setminus M_x$ then $M_p \subset M \setminus M_x$. Let $q \in M_p$, then q can connect to p but not to x. Thus, $M_p \subset M \setminus M_x$ which implies $M \setminus M_x = \bigcup\limits_{p \in M \setminus M_x} M_p$ is an open subset in M. So, $M = M_x \cup (M \setminus M_x)$. Since M is connected and $M_x \neq \varnothing$, then $M \setminus M_x = \varnothing$. Therefore, $M = M_x$ is path-connected.

□

Example 1.3.
(1). The n-sphere S^n for any $n \geq 2$ is connected then it is path-connected.

(2). The n-torus T^n is path-connected since it is connected.

(3). The Euclidean space $\mathbb{R}^n$ is connected and then is path-connected. The set of $n \times n$ square matrices over real or complex number which are isomorphic to $\mathbb{R}^{n^2}$ or $\mathbb{R}^{2n^2}$ respectively, are path-connected.

CHAPTER 2

Lie groups and matrix Lie groups

Lie group is one of the most important type of differentiable manifolds. It is a differentiable manifold, which is also a group in which the group operations are smooth. This allows us to use algebraic properties to study this smooth manifold. Most Lie groups are appear in the matrix forms which are (topologically) closed in the general linear group $GL(n, \mathbb{C})$. We call them matrix Lie groups. Our goal is to study some topological and algebraic properties of matrix Lie groups.

1. Lie Groups and Matrix Lie groups

Definition 2.1. A *Lie group* G is a differentiable manifold which is also endowed with a group structure such that the map $G \times G \to G, (a, b) \mapsto ab$ and the map $G \to G, a \mapsto a^{-1}$ are C^∞.

Example 2.1.

(1). The Euclidean space $\mathbb{R}^n$ under vector addition and the non-zero complex number $\mathbb{C}^*$ under multiplication are Lie groups.

(2). The unit circle $S^1 \subset \mathbb{C}^*$ is a Lie group with the multiplication induced from $\mathbb{C}^*$.

(3). The n-torus T^n which is a manifold can be view as a set consists of all $n \times n$ diagonal matrices with complex entries of modulus 1, that is for any $M \in T^n$

$$M = \begin{bmatrix} e^{2\pi i\theta_1} & 0 & \cdots & 0 \\ 0 & e^{2\pi i\theta_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2\pi i\theta_n} \end{bmatrix} \qquad \text{where } \theta_i \text{ are real} \tag{2.1}$$

Then T^n is a group and the group operations, matrix multiplication and inversion are clearly smooth. Therefore, n-torus is Lie group.

(4). The real and complex general linear groups $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ which are differentiable manifolds ((5) and (6) in example 1.1) are also groups under matrix multiplication. It is easy to see that the maps $(a, b) \mapsto ab$ and $a \mapsto a^{-1}$ are C^∞. Thus, they are Lie groups.

(5). Let $G = \mathbb{R} \times \mathbb{R} \times S^1 = \{(x, y, z) | x, y, \in \mathbb{R}, z \in S^1\}$ and define the group product $G \times G \to G$ by:

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, e^{ix_1y_2} z_1z_2)$$

Now, we will check that G is a group. For associativity, we have

$$\begin{aligned}
&[(x_1, y_1, z_1) \cdot (x_2, y_2, z_2)] \cdot (x_3, y_3, z_3) \\
&\qquad = (x_1 + x_2, y_1 + y_2, e^{ix_1y_2} z_1 z_2) \cdot (x_3, y_3, z_3) \\
&\qquad = ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3, e^{i(x_1+x_2)y_3} e^{ix_1y_2} z_1 z_2 z_3) \\
&\qquad = (x_1 + x_2 + x_3, y_1 + y_2 + y_3, e^{i(x_1y_2+x_1y_3+x_2y_3)} z_1 z_2 z_3) \\
&(x_1, y_1, z_1) \cdot [(x_2, y_2, z_2) \cdot (x_3, y_3, z_3)] \\
&\qquad = (x_1, y_1, z_1) \cdot (x_2 + x_3, y_2 + y_3, e^{ix_2y_3} z_2 z_3) \\
&\qquad = (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3), e^{ix_1(y_2+y_3)} z_1 e^{ix_2y_3} z_2 z_3) \\
&\qquad = (x_1 + x_2 + x_3, y_1 + y_2 + y_3, e^{i(x_1y_2+x_1y_3+x_2y_3)} z_1 z_2 z_3)
\end{aligned}$$

This proves associativity. In addition, G has identity element; namely, $(0, 0, 1)$ and the inverse $(-x, -y, e^{ixy} z^{-1})$ for all $(x, y, z) \in G$. It is clear that the product map and the inverse map are smooth. Thus, G is Lie group.

(6). Let $G = \mathbb{R}^* \times \mathbb{R}$ be a product manifold and we define the product operator on G by

$$(a_1, x_1) \cdot (a_2, x_2) = (a_1a_2, a_1x_2 + x_1)$$

Under this operation, $(1, 0)$ is an identity element of G and $(a^{-1}, -a^{-1}x)$ is an inverse element for each $(a, x) \in G$. The associativity is easy to check and it is clear that the product and inverse maps are smooth. Therefore, G is Lie group and is called the *group of affine motions of* $\mathbb{R}$. If we identify the element (a, x) of G with the affine motion $t \mapsto at + x$, then the multiplication in G is composition of affine motions.

(7). Let $G = GL(n, \mathbb{R}) \times \mathbb{R}^n$ be a product manifold and we define the product operator on G by

$$(A, u) \cdot (B, v) = (AB, Av + u)$$

Again, G is a Lie group with identity $(I, 0)$ and the inverse element $(A^{-1}, -A^{-1}u)$ for each $(A, u) \in G$. G is called the *group of affine motion of* $\mathbb{R}^n$. If we identify the element (A, u) of G with the affine motion $x \mapsto Ax + u$, then the multiplication in G is composition of affine motions.

Definition 2.2.
(1). Let $M(n, \mathbb{C})$ denote the space of all $n \times n$ matrices with complex entries and A_m be a sequence of complex matrices in $M(n, \mathbb{C})$. We say that A_m is *converges* to a matrix A if each entry of A_m converges (as $m \to \infty$) to the corresponding entry of A (that is, if $(A_m)_{kl}$ converges to A_{kl} for all $1 \leq k, l \leq n$).

(2). A *matrix Lie group* is any subgroup G of $GL(n, \mathbb{C})$ with the property that, if A_m is any sequence of matrices in G, and A_m converges to some matrix A then either $A \in G$, or A is not invertible. It is equivalent to say that a matrix Lie group is a closed subgroup of $GL(n, \mathbb{C})$ (This does not necessary closed in $M(n, \mathbb{C})$).

Remark 2.1. From the definition above, for any arbitrary collection of matrix Lie groups $\{G_i\}_{i\in I}$ then the intersection $\bigcap\limits_{i\in I} G_i$ is again a matrix Lie group.

The following examples of matrix Lie groups, except $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$, have a strong properties that they are closed in $M(n,\mathbb{C})$.

Example 2.2.

(1). $GL(n,\mathbb{C})$ is itself a subgroup of $GL(n,\mathbb{C})$ and if A_m any sequence in $GL(n,\mathbb{C})$ converges to a matrix A then either A is in $GL(n,\mathbb{C})$, or A is not invertible. $GL(n,\mathbb{R})$ is a subgroup of $GL(n,\mathbb{C})$ and if any sequence $A_m \subset GL(n,\mathbb{R})$ converges to a matrix A then the entries of A are real and thus either $A \in GL(n,\mathbb{R})$, or A is not invertible.

Thus, the Lie groups $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$ are also the matrix Lie groups.

(2). The *special linear group* over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is defined by $SL(n,\mathbb{K})$ is the group of $n \times n$ invertible matrices (with entries over $\mathbb{K}$) having determinant 1. It is clearly a subgroup of $GL(n,\mathbb{C})$. Indeed, for any matrices $A, B \in SL(n,\mathbb{K})$, we have $\det(AB^{-1}) = \det A \det(B^{-1}) = \det A(\det B)^{-1} = 1$ which implies that $AB^{-1} \in SL(n,\mathbb{K})$. In addition, if $\{A_m\}$ is any sequence in $SL(n,K)$ that converges to a matrix A then all A_m have determinant 1 and so does A since determinant is a continuous function. Thus, $SL(n,\mathbb{R})$ and $SL(n,\mathbb{C})$ are matrix Lie groups.

(3). We define the *orthogonal group* $O(n)$ by

$$O(n) = \{A \in M(n,\mathbb{R}) | A^T A = AA^T = I_n\}$$

where A^T denote the transpose matrix of A and I_n is an identity matrix of size n.

It is clearly that $O(n)$ is a subgroup of $GL(n,\mathbb{C})$ since for any matrix $A \in O(n)$, A has inverse $A^T \in O(n)$ and for any matrices $A, B \in O(n)$, we have $AB \in O(n)$ since:

$$(AB)^T(AB) = B^T A^T AB = B^T I_n B = B^T B = I_n$$
$$(AB)(AB)^T = ABB^T A^T = AI_n A^T = AA^T = I_n$$

To see that $O(n)$ is closed in $GL(n,\mathbb{C})$, notice that the set of the identity matrix $\{I_n\}$ is closed in $GL(n,\mathbb{C})$ and whenever we have $A^T A = I_n$ in $GL(n,\mathbb{C})$ then $AA^T = I_n$ and viceversa. So that $O(n)$ can be express as

$$\{A \in GL(n,\mathbb{C}) | A^T A = I_n\} \text{ or } \{A \in GL(n,\mathbb{C}) | AA^T = I_n\}$$

Consider the map $T : M(n,\mathbb{C}) \to M(n,\mathbb{C})$ defined by $A \mapsto A^T A$. T is continuous since the entries of $A^T A$ are polynomials of entries of A; namely, $\sum\limits_{k=1}^{n} a_{ki}a_{kj}$ where $A = [a_{ij}]$. Then $O(n) = T^{-1}(\{I_n\})$ is closed in $M(n,\mathbb{C})$ and thus closed in $GL(n,\mathbb{C})$. This makes $O(n)$ becomes a matrix Lie group.

Now consider the determinant map restricted on $O(n)$, $\det_{O(n)} : O(n) \to \mathbb{R}$ and observe that for any matrix $A \in O(n)$,

$$[\det(A)]^2 = \det A \det A = \det(A^T) \det A = \det(A^T A) = \det(I_n) = 1$$

This implies $\det A = \pm 1$ so we obtain that

$$O(n) = O^+(n) \cup O^-(n)$$

where $O^+(n) = \{A \in O(n) | \det A = 1\}$, $O^-(n) = \{A \in O(n) | \det A = -1\}$ with $O^+(n) \cap O^-(n) = \varnothing$.

We define the *special orthogonal group* by:

$$SO(n) = \{A \in GL(n, \mathbb{C}) | A^T A = I_n \text{ and } \det A = 1\} = O^+(n)$$

$SO(n)$ is clearly a subgroup of $GL(n, \mathbb{C})$ and is closed since $SO(n) = O(n) \cap SL(n, \mathbb{R})$ is the intersection of two closed subgroup of $GL(n, \mathbb{C})$ (also of $M(n, \mathbb{C})$). Therefore, $SO(n)$ is matrix Lie group.

Remark 2.2. Geometrically, element of $O(n)$ are either rotations or combinations of rotations and reflections. The elements of $SO(n)$ are just the rotations. Thus, occasionally, we call $SO(n)$ the *rotation group*.

(4). We define the *unitary group* $U(n)$ and the *special unitary group* $SU(n)$ as below:

$$\begin{aligned} U(n) &= \{A \in GL(n, \mathbb{C}) | A^* A = AA^* = I_n\} \\ &= \{A \in GL(n, \mathbb{C}) | A^* A = I_n\} \\ &= \{A \in GL(n, \mathbb{C}) | AA^* = I_n\} \\ SU(n) &= \{A \in GL(n, \mathbb{C}) | A^* A = I_n \text{ and } \det A = 1\} \\ &= U(n) \cap SL(n, \mathbb{C}) \end{aligned}$$

where A^* denotes the *adjoint* of A ($(A^*)_{ji} = \overline{A_{ij}}$). $U(n)$ is a subgroup of $GL(n, \mathbb{C})$ since for any $A, B \in U(n)$,

$$(AB^{-1})^*(AB^{-1}) = (AB^*)^*(AB^*) = BA^*AB^* = BI_nB^* = BB^* = I_n$$

And also $SU(n)$ is clearly a subgroup of $U(n)$. Similar to the case of $O(n)$, $U(n)$ is closed in $M(n, \mathbb{C})$ and thus closed in $GL(n, \mathbb{C})$ since it is a inverses image of continuous function $A \mapsto A^*A$ of a closed set $\{I_n\}$. $SU(n)$, which is the intersection of two closed sets, is closed.
Therefore, $U(n)$ and $SU(n)$ are matrix Lie groups.

Remark 2.3. If $A \in U(n)$, then $\det(A^*A) = \det A^* \det A = \overline{\det A} \det A = |\det A|^2 = \det I_n = 1$. This implies $|\det A| = 1$ so that $\det A = e^{i\theta}$ for any real θ. So, $SU(n)$ is a smaller subset of $U(n)$ then $SO(n)$ is of $O(n)$. Particularly, $SO(n)$ has the same dimension as $O(n)$, whereas $SU(n)$ has dimension one less than that of $U(n)$.

(5). The *skew-symmetric bi-linear* form S on $\mathbb{R}^{2n}$ defined as follows:

$$S[x, y] = \sum_{k=1}^{n} (x_k y_{n+k} - x_{n+k} y_k) \text{ where } x, y \in \mathbb{R}^{2n} \tag{2.2}$$

The set of all real $2n \times 2n$ matrices which preserve S is the *real symplectic group* denoted by:

$$Sp(n, \mathbb{R}) = \{A \in M(n, \mathbb{R}) | S[Ax, Ay] = S[x, y] \text{ for all } x, y \in \mathbb{R}^{2n}\}$$

Define a $2n \times 2n$ matrix $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ where I_n is the $n \times n$ identity matrix. Then for all $x, y \in \mathbb{R}^{2n}$ where $x^T = (x_1, x_2, \cdots, x_{2n})$ and $y^T = (y_1, y_2, \cdots, y_{2n})$ we have:

$$\begin{aligned}
\langle x, Jy \rangle &= \langle (x_1 \cdots x_n, x_{n+1}, \cdots, x_{2n})^T, (y_{n+1}, \cdots, y_{2n}, -y_1, \cdots, -y_n)^T \rangle \\
&= (x_1 y_{n+1} - x_{n+1} y_1) + (x_2 y_{n+2} - x_{n+2} y_2) + \cdot + (x_n y_{2n} - x_{2n} y_n) \\
&= \sum_{k=1}^{n} (x_k y_{n+k} - x_{n+k} y_k) = S[x, y]
\end{aligned}$$

Therefore,

$$\begin{aligned}
A \in Sp(n, \mathbb{R}) \quad &\Longleftrightarrow \quad \forall x, y \in \mathbb{R}^{2n}, S[Ax, Ay] = S[x, y] \\
&\Longleftrightarrow \quad \forall x, y \in \mathbb{R}^{2n}, \langle Ax, JAy \rangle = \langle x, Jy \rangle \\
&\Longleftrightarrow \quad \forall x, y \in \mathbb{R}^{2n}, (Ax)^T (JAy) = x^T (Jy) \\
&\Longleftrightarrow \quad \forall x, y \in \mathbb{R}^{2n}, x^T (A^T JA) y = x^T Jy \\
&\Longleftrightarrow \quad \forall x, y \in \mathbb{R}^{2n}, x^T (A^T JA - J) y = 0 \\
&\Longleftrightarrow \quad A^T JA - J = 0 \\
&\Longleftrightarrow \quad A^T JA = J \\
&\Longrightarrow \quad \det(A^T JA) = \det J \\
&\Longrightarrow \quad [\det(A)]^2 \det J = \det J \\
&\Longrightarrow \quad [\det(A)]^2 = 1 \\
&\Longrightarrow \quad \det A = \pm 1
\end{aligned}$$

So, we can view $Sp(n, \mathbb{R})$ as:

$$Sp(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) | A^T JA = J\}$$

where J is defined as above. We claim that $Sp(n, \mathbb{R})$ is a matrix Lie group. Indeed, for any matrices $A, B \in Sp(n, \mathbb{R})$ we have:

$$\begin{aligned}
&(AB)^T J(AB) = B^T (A^T JA) B = B^T JB = J \Rightarrow AB \in Sp(n, \mathbb{R}) \\
&A^T JA = J \Rightarrow J = (A^T)^{-1} JA^{-1} \Rightarrow (A^{-1})^T JA^{-1} = J \Rightarrow A^{-1} \in Sp(n, \mathbb{R})
\end{aligned}$$

This implies that $Sp(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$. To see that this group is closed, observe that the map $A \mapsto A^T JA$ is continuous on $M(n, \mathbb{C})$. Thus, any sequences $\{A_m\}$ in $Sp(n, \mathbb{R})$ such that $A_m \to A$ imply that $A \in Sp(n, \mathbb{R})$.

Similarly, we define the skew-symmetric bilinear form S on $\mathbb{C}^{2n}$ by the same formula (2.2) and then the *complex symplectic group* $Sp(n,\mathbb{C})$ is defined by:

$$\begin{aligned} Sp(n,\mathbb{C}) &= \{A \in M(n,\mathbb{C}) | S[Ax, Ay] = S[x, y], \forall x, y \in \mathbb{C}^{2n}\} \\ &= \{A \in GL(n,\mathbb{C}) | A^T JA = J\} \end{aligned}$$

Thus, $Sp(n,\mathbb{C})$ is a matrix Lie group.

Lastly, we define the *compact symplectic group* $Sp(n)$ by:

$$Sp(n) = Sp(n,\mathbb{C}) \cap U(2n)$$

Thus, $Sp(n)$ is a matrix Lie group.

Remark 2.4. The determinant of symplectic groups is always equal to 1. For more information, consult [**15**], On the Determinant of Symplectic matrices by D. Steven Mackey and Niloufer Mackey.

(6). The *Heisenberg group* H is the set of all 3×3 real matrices of the form:

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \tag{2.3}$$

It is easy to see that H is a subset of $GL(n,\mathbb{R})$ and is closed under multiplication of matrices. The identity matrix I_3 is clearly in H and the inverse of any matrices of the form (2.3) is :

$$\begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

Moreover, the limit of sequence of matrices of the form (2.3) is again of that form. These make H becomes a matrix Lie group.

(7). The groups $\mathbb{R}^*, \mathbb{C}^*, S^1, T^n, \mathbb{R}$ and $\mathbb{R}^n$ are all matrix Lie groups since:
The group $\mathbb{R}^* = \mathbb{R} \setminus \{0\}, \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ under multiplication are isomorphic to $GL(1,\mathbb{R}), GL(1,\mathbb{C})$, respectively.
The group S^1 of complex numbers with absolute value one is isomorphic to $U(1)$.
The torus T^n in the matrix from as (2.1) is clearly a matrix Lie group.
The group $\mathbb{R}$ under addition is isomorphic to $GL(1,\mathbb{R})^+$ (1×1 real matrix with positive determinant) via the map $x \mapsto [e^x]$.
The group $\mathbb{R}^n$ with vector addition is isomorphic to the group of diagonal matrices with positive diagonal entries, via the map:

$$(x_1, x_2, ..., x_n) \mapsto \begin{bmatrix} e^{x_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{x_n} \end{bmatrix}$$

(8). This is a counter example of a subgroup of $GL(n,\mathbb{C})$ which is not closed and hence is not a matrix Lie group. The set $GL(n,\mathbb{Q})$ of all invertible $n \times n$ matrices with rational entries is clearly a subgroup of $GL(n,\mathbb{C})$ but the density of rational and irrational number in $\mathbb{R}$ implies that there exist a sequence A_n of matrices with rational entries that converges to a invertible matrix A with irrational entries. Thus, $GL(n,\mathbb{Q})$ is not a matrix Lie group.

Remark 2.5. As we can see from examples 2.1 and 2.2 above, $\mathbb{R}^n$ with vector addition, $\mathbb{C}^*$ under complex multiplication, the circle S^1, the torus T^n and the general linear groups $GL(n,\mathbb{R}), GL(n,\mathbb{C})$ are Lie groups and also matrix Lie groups. Thus, there are two natural questions arise:

1. Are all Lie groups matrix Lie groups?
2. Are all matrix Lie groups Lie groups?

The answer for the first question is no. That is, most but not all Lie groups are matrix Lie groups and even not isomorphic to any matrix Lie groups. We will provide a counter example of this fact in the last section of the next chapter. The second question has the positive answer by the closed subgroups theorem.

Closed Subgroup Theorem: Every closed subgroups of a Lie group is an embedded submanifold of that Lie group and thus a Lie group.

We will provide the proof of this theorem in the next chapter in the case of matrices. However, keep in mind that matrix Lie groups are Lie groups so that all properties that apply to Lie groups are also true in case of matrix Lie groups.

2. Compactness

We concern about compactness of matrix Lie groups. Since $M(n,\mathbb{C})$ is isomorphic to $\mathbb{C}^{n^2}$ and the subset in the Euclidean space is compact if and only if it is closed and bounded, so we can give the definition of compact Matrix Lie groups as below:

Definition 2.3. A matrix Lie group G is said to be *compact* if the following two conditions are satisfied:
1. G is closed in $M(n,\mathbb{C})$.
2. There exists a constant C such that for all $A \in G, |A_{ij}| \leq C, \forall 1 \leq i,j \leq n$.

Remark 2.6. All of our examples, except $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$, are closed in $M(n,\mathbb{C})$. Thus, we are only concern about the boundedness condition. We will study the compactness of all examples of matrix Lie groups in the previous section.

Example 2.3.
(1). The general linear groups $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$ are not compact since they are open in $M(n,\mathbb{R})$ and $M(n,\mathbb{C})$, respectively.

(2). The special linear groups $SL(n,\mathbb{R})$ and $SL(n,\mathbb{C})$ $(n \geq 2)$ are not compact since they are not bounded. To see this, for any constants C, there is a positive integer k such that $C < k$. Consider the matrix $A_k \in SL(n,\mathbb{R}) \subset SL(n,\mathbb{C})$ such that:

$$A_k = \begin{bmatrix} k & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{k} & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} = diag_n(k, \frac{1}{k}, 1, \dots, 1)$$

A_k is clearly has determinant 1 and is not bounded by C.

(3). The symplectic groups $Sp(n, \mathbb{R})$ and $Sp(n, \mathbb{C})$ are noncompact since they are unbounded. Indeed, for any constants C, choose a positive integer k such that $C < k$ and consider the matrix A_k defined by:

$$A_k = \begin{bmatrix} \frac{1}{k} I_n & 0_n \\ 0_n & k I_n \end{bmatrix}$$

where 0_n represents the zero matrix of size n. A_k has determinant 1 and is the element of $Sp(n, \mathbb{R})$ and $Sp(n, \mathbb{C})$ Since:

$$\begin{aligned} A_k^T J A_k &= \begin{bmatrix} \frac{1}{k} I_n & 0_n \\ 0_n & k I_n \end{bmatrix} \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{k} I_n & 0_n \\ 0_n & k I_n \end{bmatrix} \\ &= \begin{bmatrix} 0_n & \frac{1}{k} I_n \\ -k I_n & 0_n \end{bmatrix} \begin{bmatrix} \frac{1}{k} I_n & 0_n \\ 0_n & k I_n \end{bmatrix} \\ &= \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} = J \end{aligned}$$

However, A_k is not bounded by C.

(4). The Heisenberg group H is not bounded and then is not compact. Choose a positive integer k such that $C < k$ for any constants C then $A_k = \begin{bmatrix} 1 & k & k \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \in H$ is not bounded by C.

(5). The group $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^*$ and C^* are unbounded and then are noncompact.

Example 2.4.

(1). The orthogonal group $O(n)$ and special orthogonal group $SO(n)$ are compact since they are bounded by 1. To see this, observe that if A is a element of $O(n)$ or $SO(n)$ then $A^T A = I_n$ which implies that all the column vectors of A has norm 1 and hence $|A_{ij}| \leq 1$ for all $1 \leq i, j \leq n$.

(2). Similar argument implies that the unitary group $U(n)$ and special unitary group $SU(n)$ are bounded by 1 and hence are compact.

(3). Clearly, the compact symplectic group $Sp(n) = Sp(n, \mathbb{C}) \cap U(2n)$ is bounded and then is compact.

(4). The unit circle S^1 and the $n-$torus T^n that are bounded by 1, are compact.

3. Connectedness

As before, we are interested in matrix Lie groups. However, the following definitions are for Lie groups in general.

Definition 2.4.
(1). A Lie group G is said to be *connected* if it cannot be separated by any two non empty and distinct open or closed subsets in G. More precisely, if there is no non empty open sets or closed sets A and B in G such that $A \cap B = \varnothing$, and $A \cup B = G$. A Lie group which is not connected can be decomposed (uniquely) as a union of several pieces, called *components*, such that two elements in the same component can be joined by a continuous path and two elements of different component cannot.

(2). A Lie group G is said to be *path-connected* if given any two points x and y in G, there exists a continuous path (or simply a path) $A(t), a \leq t \leq b$, lying in G with $A(a) = x$ and $A(b) = y$.

Proposition 2.1. The component containing identity of a matrix Lie group G is a subgroup of G. We call it the *identity component*.

Proof. Let A and B be in the identity component of G, then there exist continuous paths $A(t)$ and $B(t)$ connected I to A and I to B, respectively such that $A(0) = B(0) = I$ and $A(1) = A, B(1) = B$. Since matrix multiplication and matrix inversion are continuous, it is clearly that $A(t)B(t)$ is a continuous path that stating at I and ending at AB and $(A(t))^{-1}$ is a continuous path that starting at I and ending at A^{-1}. These show that AB and A^{-1} are again in the identity component of G. This completes the proof.

□

Now, we will study the connectedness of some Matrix Lie groups we have already known. From proposition 1.8, we see that any Lie groups are connected if and only if they are path-connected. So, study the connectedness of the following matrix Lie groups is as the same as studying their path-connectedness.

Example 2.5.
(1). The group $GL(n, \mathbb{C})$ is connected for all $n \geq 1$. It is clearly for the case $n = 1$ since $GL(1, \mathbb{C}) \cong \mathbb{C}^*$ with is path-connected. For the case $n \geq 2$, given any matrix $A \in GL(n, \mathbb{C})$, we will show that there is a continuous path that connects A to the identity I and so any two matrices $A, B \in GL(n, \mathbb{C})$ can be connected by a continuous path form A to I and then from I to B.

To see this, we make use of the fact from linear algebra that every matrices is similar to an upper triangular matrix. That is, there is an invertible matrix C such that $A = CTC^{-1}$ where T is an upper triangular matrix:

$$T = \begin{bmatrix} \lambda_1 & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \text{ with } \lambda_i \neq 0, \forall i \text{ are eigenvalues of } A \text{ (since } A \text{ is invertible).}$$

We set $T(t)$ by multiplying each entries of T above the diagonal by $(1-t)$ for $0 \leq t \leq 1$ and define $A(t) = CT(t)C^{-1}$. Thus $A(t)$ is continuous and connects A (when $t = 0$) to the matrix $A(1) = CT(1)C^{-1} = T(1)$ where $T(1) = diag_n(\lambda_1, \dots, \lambda_n)$. This path $A(t)$ lies in $GL(n, \mathbb{C})$ since $\det A(t) = \det T(t) = \prod_{i=1}^{n} \lambda_i = \det A \neq 0$.

Now, we define a continuous path $B(t)$ from $A(1)$ to I for $1 \leq t \leq 2$ by defining a continuous path $\lambda_i(t)$ from each λ_i to 1 such that $\lambda_i(t) \neq 0, \forall 1 \leq t \leq 2$. $\lambda_i(t)$ exist for the fact that $\lambda_i, 1 \in \mathbb{C}^*$ and $\mathbb{C}^*$ is path-connected.
Define $B(t) = diag_n(\lambda_1(t), \dots, \lambda_n(t))$. It is clearly that $B(t)$ is continuous and connects $A(1)$ to I for $1 \leq t \leq 2$. This $B(t)$ is again lies in $GL(n, \mathbb{C})$ since $\det B(t) = \prod_{i=1}^{n} \lambda_i(t) \neq 0$. Therefore, A can be connected to I by a continuous path $A(t)$ for $0 \leq t \leq 1$ and $B(t)$ for $1 \leq t \leq 2$ in $GL(n, \mathbb{C})$.

Similarly, The group $SL(n, \mathbb{C})$ for all $n \geq 1$. The case $n = 1$ is trivial. For $n \geq 2$, the proof is almost the same as for $GL(n, \mathbb{C})$. Let $A \in SL(n, \mathbb{C})$, we repeat the proof for the continuous path $A(t)$. The additional condition $\det A = 1$ implies that $\det A(t) = 1$ so that $A(t)$ lies in $SL(n, \mathbb{C})$. For the construction of $B(t)$, we define $\lambda_i(t)$ for $1 \leq i \leq n-1$ as before and $\lambda_n(t) = [\lambda_1(t) \dots \lambda_{n-1}(t)]^{-1}$. It is clear that λ_n is continuous and $\lambda_n(1) = (\lambda_1 \dots \lambda_{n-1})^{-1} = \lambda_n$ and $\lambda_n(2) = (1 \dots 1)^{-1} = 1$ (here, we use $\prod_{i=1}^{n} \lambda_i = 1$). Therefore, if we define $B(t)$ as before, then it is a continuous path from $A(1)$ to I and lies in $SL(n, \mathbb{C})$ since $\det B(t) = 1$ and thus we are done.

(2). The group $GL(n, \mathbb{R})$ is not connected since for two matrices $A, B \in GL(n, \mathbb{R})$ with $\det A > 0$ and $\det B < 0$, the image via determinant of any continuous paths connecting A and B are closed intervals containing 0. This means that there is a matrix lies in the path with determinant 0 so that there is no continuous path connecting A and B and lies in $GL(n, \mathbb{R})$. In fact, $GL(n, \mathbb{R})$ has 2 components. Those are $GL(n, \mathbb{R})^+$, the set of invertible $n \times n$ real matrices with positive determinant and $GL(n, \mathbb{R})^-$ of those with negative determinant.

(3). The groups $U(n)$ and $SU(n)$ are connected, for all $n \geq 1$. Recall the result from linear algebra that any unitary matrix U can be written as:

$$U = V \begin{bmatrix} e^{i\theta_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{i\theta_n} \end{bmatrix} V^{-1} \text{ where } V \text{ is unitary and } \theta_i \in \mathbb{R} \tag{2.4}$$

We define:

$$U(t) = V \begin{bmatrix} e^{i(1-t)\theta_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{i(1-t)\theta_n} \end{bmatrix} V^{-1} \text{ for } 0 \leq t \leq 1 \tag{2.5}$$

Thus, $U(t)$ is continuous path from U to I and it lies in $U(n)$ since:

$$\begin{aligned}
U^*(t)U(t) &= V \begin{bmatrix} e^{-i(1-t)\theta_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{-i(1-t)\theta_n} \end{bmatrix} V^*V \begin{bmatrix} e^{i(1-t)\theta_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{i(1-t)\theta_n} \end{bmatrix} V^* \\
&= V diag_n(e^{-i(1-t)\theta_1}, \dots, e^{-i(1-t)\theta_n}) diag_n(e^{i(1-t)\theta_1}, \dots, e^{i(1-t)\theta_n}) V^* \\
&= V I_n V^* = I_n
\end{aligned}$$

Here, we use the fact that $VV^* = V^*V = I_n$ and $V^{-1} = V^*, (V^*)^* = V$

For the case of $U \in SU(n)$, U can express in form (2.4) with additional condition that $\sum_{i=1}^{n} \theta_i = 0$ (since $\det U = 1$) and again we define $U(t)$ as in the form (2.5). Thus, we only need to check the determinant of $U(t)$. Since $\det U(t) = e^{i(1-t)\sum_{i=1}^{n}\theta_i} = e^0 = 1$, we are done.

(4). The group $O(n) = SO(n) \cup O^-(n)$ is non connected since it is a disjoint union of two non empty closed subgroups $SO(n)$ and $O^-(n)$.

(5). The Heisanberg group H in the form (2.3) is clearly connected since all its elements are connected to I_3 via a continuous path obtained by multiplying all the entries above the diagonal by $(1-t)$ for $0 \le t \le 1$.

Definition 2.5. A Lie group G is said to be *simply connected* if it is path-connected and in addition, for any two path $\alpha(t)$ and $\beta(t)$ with $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$, α can be "continuously deformed" to β, that is there exists a continuous map $H : [0,1] \times [0,1] \to G$ that satisfies the following properties:

(1). $H(s,0) = \alpha(0) = \beta(0)$ and $H(s,1) = \alpha(1) = \beta(1)$ for all s

(2). $H(0,t) = \alpha(t), H(1,t) = \beta(t)$ for all t

Example 2.6.

The group $SU(2)$ is simply connected. To see this, we will prove that:

$$SU(2) = \left\{ \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix} \Big| |\alpha|^2 + |\beta|^2 = 1, \alpha, \beta \in \mathbb{C} \right\} \tag{2.6}$$

Let $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$ and let $A = \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix}$ then:

$$\begin{aligned}
AA^* &= \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix} \begin{bmatrix} \overline{\alpha} & \overline{\beta} \\ -\beta & \alpha \end{bmatrix} = \begin{bmatrix} |\alpha|^2 + |\beta|^2 & 0 \\ 0 & |\beta|^2 + |\alpha|^2 \end{bmatrix} = I_2 \\
\det A &= |\alpha|^2 + |\beta|^2 = 1
\end{aligned}$$

This proves that $A \in SU(2)$.

On the other hand, Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SU(2)$, we want to prove that $A = \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix}$

with $|\alpha|^2 + |\beta|^2 = 1$. We have:

$$AA^* = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{bmatrix} = \begin{bmatrix} |a|^2 + |b|^2 & a\overline{c} + b\overline{d} \\ c\overline{a} + d\overline{b} & |c|^2 + |d|^2 \end{bmatrix} \text{ and } \det A = ad - bc$$

Then, we obtain:

$$(i).|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$$
$$(ii).a\overline{c} + b\overline{d} = 0$$
$$(iii).ad - bc = 1$$

Multiply (ii) by d and substitute ad from (iii) we obtain from (i) that $b + \overline{c} = 0$
Multiply (ii) by c and substitute bc from (iii) we obtain from (i) that $a - \overline{d} = 0$
Let $a = \alpha$ and $c = \beta$ then A is in the form we desire and thus (2.6) holds.
Therefore, $SU(2)$ can be thought of as the 3-dimensional sphere S^3 in $\mathbb{C}^2 \cong \mathbb{R}^4$. Since S^3 is simply connected, so is $SU(2)$.

The following table lists the compactness and connectedness of some matrix Lie groups we have known.

Group	Compactness	Simply connectedness	connectedness	Components
$GL(n,\mathbb{C})$	No	No	Yes	1
$GL(n,\mathbb{R})$	No	No	No	2
$SL(n,\mathbb{C})$	No$(n \geq 2)$	Yes	Yes	1
$SL(n,\mathbb{R})$	No$(n \geq 2)$	No$(n \geq 2)$	Yes	1
$O(n)$	Yes	No	No	2
$SO(n)$	Yes	No$(n \geq 2)$	Yes	1
$U(n)$	Yes	No	Yes	1
$SU(n)$	Yes	Yes	Yes	1
$Sp(n,\mathbb{C})$	No	Yes	Yes	1
$Sp(n,\mathbb{R})$	No	No	Yes	1
$Sp(n)$	Yes	Yes	Yes	1
Heisenberg	No	Yes	Yes	1

TABLE 1. Compactness and Connectedness of some matrix Lie groups

4. Subgroups and Homomorphism

Definition 2.6. Let G and H be Lie groups. H is said to be a *Lie subgroup* of G if H is a submanifold of G and also a subgroup of G.

Example 2.7.
(1). The unit circle S^1 which is a Lie group is a submanifold of $\mathbb{R}^2 \setminus \{0\} \cong \mathbb{C}^*$ and is also a subgroup of $\mathbb{C}^*$. Thus, S^1 is a Lie subgroup of $\mathbb{C}^*$.

(2). All matrix Lie groups that we have studied are embedded submanifolds of $GL(n, \mathbb{C})$ by the closed subgroup theorem and thus they are Lie subgroups of $GL(n, \mathbb{C})$.

Definition 2.7. Let G and H be Lie groups. The map $f : G \to H$ is called a *Lie group homomorphism* if f is smooth and is also a group homomorphism. In addition, f is said to be a *Lie group isomorphism* if f is bijective and f^{-1} is a Lie group homomorphism i.e. f is both a group isomorphism and diffeomorphism. If $G = H$, f is called a *Lie group automorphism.* If f is a Lie group isomorphism, then G and H are said to be *isomorphic* and we write $G \cong H$. Two Lie groups which are isomorphic should be thought of as being essentially the same group.

Proposition 2.2. Let G and H be matrix Lie groups and $f : G \to H$ is a group homomorphism. If f is continuous, then f is smooth.

Remark 2.7. In fact, this result is valid for general Lie groups. However, the proof is more difficult in the general case. As before, the proof of this proposition will be provided in the next chapter.

Example 2.8.
(1). The determinant map $\det : GL(n, \mathbb{C}) \to \mathbb{C}^*$ is a Lie group homomorphism since det is continuous and $\det(AB) = \det(A)\det(B)$ for any matrices $A, B \in GL(n, \mathbb{C})$.

(2). The map $f : \mathbb{R} \to SO(2)$ given by $f(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$ is a Lie group homomorphism since f is clearly continuous and for any $t, s \in \mathbb{R}$, we have:

$$\begin{aligned} f(t+s) &= \begin{bmatrix} \cos(t+s) & -\sin(t+s) \\ \sin(t+s) & \cos(t+s) \end{bmatrix} \\ &= \begin{bmatrix} \cos(t)\cos(s) - \sin(t)\sin(s) & -\sin(t)\cos(s) - \sin(s)\cos(t) \\ \sin(t)\cos(s) + \sin(s)\cos(t) & \cos(t)\cos(s) - \sin(t)\sin(s) \end{bmatrix} \\ &= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{bmatrix} = f(t)f(s) \end{aligned}$$

CHAPTER 3

Lie algebra

Lie algebras play an important role in the theory of Lie groups. Each Lie group gives rise to Lie algebra. In this chapter, we will study some properties of Lie algebra and the relations between Lie group and its Lie algebra.

1. The Exponential Map

The construction of exponential map is more difficult in the general case. In this section, we only study the exponential map in the case of matrix groups.

Definition 3.1. Let X be $n \times n$ complex matrix. We define the *exponential* of X to be the usual power series:

$$\exp X = \sum_{k=0}^{\infty} \frac{X^k}{k!} \tag{3.1}$$

Remark 3.1. We recall that the Hilbert-Schmidt norm of any matrices $A \in M(n, \mathbb{C})$ is defined by:

$$\|X\| = \left(\sum_{i,j=1}^{n} |x_{ij}|^2 \right)^{\frac{1}{2}} \tag{3.2}$$

This norm satisfies the inequalities:

$$\|A + B\| \leq \|A\| + \|B\|$$
$$\|AB\| \leq \|A\|\|B\|$$

The series (3.1) converges uniformly and the exponential map $\exp : M(n, \mathbb{C}) \to M(n, \mathbb{C})$ is continuous. To see this, given $R > 0$ then for any X such that $\|X\| \leq R$, we have:

$$\left\| \sum_{k=0}^{\infty} \frac{X^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \left\| \frac{X^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \frac{\|X\|^k}{k!} \leq \sum_{k=0}^{\infty} \frac{R^k}{k!} = \exp R < \infty$$

The Weierstrass M-test implies that the series converges absolutely and uniformly on the set $\{\|X\| \leq R\}$. Since R is arbitrary, the series converges uniformly. For the continuity, observe that X^k is a continuous function of X then the partial sums of the series are continuous. Since the series converges uniformly, then exp is continuous.

Now we state the basic properties of the exponential.

Proposition 3.1. Let X and Y be arbitrary matrices. Then,
(1). $e^0 = I$.
(2). $(e^X)^* = e^{X^*}$ where X^* denotes the conjugate transpose of X.
(3). If $XY = YX$, then $e^{X+Y} = e^X e^Y = e^Y e^X$.
(4). e^X is invertible, and $(e^X)^{-1} = e^{-X}$.
(5). $e^{(a+b)X} = e^{aX} e^{bX}$ for all $a, b \in \mathbb{C}$.
(6). If C is invertible, then $e^{CXC^{-1}} = Ce^X C^{-1}$.
(7). $\|e^X\| \leq e^{\|X\|}$.

Proof. (1) is obvious from the definition of exponential and (2) is follows from the fact that $(X^k)^* = (X^*)^k$. To prove (3), we multiply the series $\exp X$ and $\exp Y$ term by term and collection term where the power of X plus the power of Y equal m, we get:

$$\begin{aligned} e^X e^Y &= \left(I + X + \frac{X^2}{2!} + \cdots\right)\left(I + Y + \frac{Y^2}{2!} + \cdots\right) \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{X^k}{k!} \frac{Y^{m-k}}{(m-k)!} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} X^k Y^{m-k} \\ &= \sum_{m=0}^{\infty} \frac{(X+Y)^m}{m!} \\ &= e^{X+Y} = e^Y e^X \end{aligned}$$

Here, the fourth equality follows from the commute of X and Y. To prove (4), from (3) we get:

$$e^{-X} e^X = e^X e^{-X} = e^0 = I$$

and (5) follows by substituting $X = aX$ and $Y = bY$ in (3). To prove (6), notice that $(CXC^{-1})^k = CX^kC^{-1}$ and (7) is the result from the proof of the previous remark. □

Proposition 3.2. In a neighborhood of 0, $\exp : M(n, \mathbb{C}) \to M(n, \mathbb{C})$ has a local inverse $\log : M(n, \mathbb{C}) \to M(n, \mathbb{C})$, defined in a neighborhood of I by the series:

$$\log Y = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (Y - I)^k \tag{3.3}$$

This series converges in norm and $\log Y$ is continuous for $\|Y - I\| < 1$. Also,

$$\log(\exp X) = X \text{ for } \|X\| < \log 2 \tag{3.4}$$

$$\exp(\log Y) = Y \text{ for } \|Y - I\| < 1 \tag{3.5}$$

Proof. For $\|Y - I\| < 1$, we have:

$$\|\log Y\| = \left\| \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (Y - I)^k \right\| \leq \sum_{k=1}^{\infty} \frac{1}{k} \|Y - I\|^k < \sum_{k=1}^{\infty} \|Y - I\|^k < \infty$$

The Weierstrass M-test implies that the series converges absolutely and uniformly. For continuity, observe that the partial sums of the series are continuous and since the series converges uniformly then $\log Y$ is continuous. To prove (3.4), we will consider 2 cases:

Case 1: If X is diagonal matrix then $X = Cdiag_n(\lambda_1, \cdots, \lambda_n)C^{-1}$ for some invertible matrix C where λ_i for $i = 1, \cdots, n$ are eigenvalues of X. Then $(\exp X - I) = Cdiag_n(e^{\lambda_1} - 1, \cdots, e^{\lambda_n} - 1)C^{-1}$ so that we obtain:

$$(\exp X - I)^k = C \begin{bmatrix} (e^{\lambda_1} - 1)^k & & 0 \\ & \ddots & \\ 0 & & (e^{\lambda_n} - 1) \end{bmatrix} C^{-1}$$

Thus,

$$\begin{aligned} \log(\exp X) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (\exp X - I)^k \\ &= C \begin{bmatrix} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(e_1^{\lambda} - 1)^k & & 0 \\ & \ddots & \\ 0 & & \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(e_n^{\lambda} - 1)^k \end{bmatrix} C^{-1} \\ &= Cdiag_n(\log(e^{\lambda_1}), \cdots, \log(e^{\lambda_n}))C^{-1} \\ &= Cdiag_n(\lambda_1, \cdots, \lambda_n)C^{-1} \\ &= X \end{aligned}$$

The third and fourth equalities follow from the properties of usual exp and log in real number.

Case 2: If $X \in M(n, \mathbb{C})$ is not diagonal then $X = CUC^{-1}$ for some invertible matrix C and U is an upper triangle matrix with diagonal entries $\lambda_1, \cdots, \lambda_l$,which are eigenvalues of X, with multiplicity $n_1, \cdots, n_l$ respectively where $n_1 + \cdots + n_l = n$ $(l < n)$ i.e.

$$U = \begin{bmatrix} \lambda_1 & & & & & & * \\ & \ddots & & & & & \\ & & \lambda_1 & & & & \\ & & & \ddots & & & \\ & & & & \lambda_l & & \\ & & & & & \ddots & \\ & 0 & & & & & \lambda_l \end{bmatrix}$$

Let $\lambda = \min_{1 \leq i \neq j \leq l} |\lambda_i - \lambda_j|$ and for $m \in \mathbb{N}$, we define U_m from U by changing the diagonal entries of U by μ_p such that

$$\mu_p = \begin{cases} \lambda_i - \frac{1}{m+R_p} & \text{if } \lambda_i > 0 \\ \lambda_i + \frac{1}{m+R_p} & \text{if } \lambda_i \leq 0 \end{cases}$$

where $1 \leq p \leq n_1$ if $i = 1$ and $n_1 + \ldots + n_{i-1} + 1 \leq p \leq n_1 + \ldots + n_i$ if $i = 2, \ldots, l$ and R_p $(1 \leq p \leq n)$ are chosen distinct and sufficiently large such that:

$$\frac{1}{m + R_p} < \frac{\lambda}{2}$$

Then for $1 \leq p \neq q \leq n$, we have $\mu_p \neq \mu_q$. Indeed, if $\mu_p = \lambda_i \pm \frac{1}{m+R_p}$, $\mu_q = \lambda_j \pm \frac{1}{m+R_q}$ and suppose that $\lambda_i \pm \frac{1}{m+R_p} = \lambda_j \pm \frac{1}{m+R_q}$ then:

$$\lambda \leq |\lambda_i - \lambda_j| = \left|\frac{1}{m + R_p} \pm \frac{1}{m + R_q}\right| \leq \left|\frac{1}{m + R_p}\right| + \left|\frac{1}{m + R_q}\right| < \frac{\lambda}{2} + \frac{\lambda}{2} = \lambda$$

This leads to the contradiction.

Let $\{X_m\}$ be a sequence such that $X_m = CU_mC^{-1}$. It is clear that X_m are diagonal matrices for all m and $X_m \to X$ which implies that $\|X_m\| \to \|X\| < \log 2$. Then for m sufficiently large, we have $\|X_m\| < \log 2$ and Since "exp" and "log" are continuous, we obtain:

$$\log(\exp X) = \lim_{m \to \infty} \log(\exp X_m) = \lim_{m \to \infty} X_m = X$$

This proves (3.4). Similar argument shows that $\exp(\log Y) = Y$ for $\|Y - I\| < 1$.

$\square$

Lemma 3.1. There exists a constant c such that for all $n \times n$ matrices Y with $\|Y\| < 1/2$, then $\|\log(I + Y) - Y\| \leq c\|Y\|^2$.

Proof. First, note that

$$\log(I + Y) - Y = \sum_{k=2}^{\infty} (-1)^{k+1} \frac{Y^k}{k} = Y^2 \sum_{k=2}^{\infty} (-1)^{k+1} \frac{Y^{k-2}}{k}$$

Then:

$$\|\log(I + Y) - Y\| \leq \|Y\|^2 \sum_{k=2}^{\infty} \frac{\left(\frac{1}{2}\right)^{k-2}}{k}$$

But

$$\sum_{k=2}^{\infty} \frac{\left(\frac{1}{2}\right)^{k-2}}{k} < \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

so that $\sum_{k=2}^{n} \frac{\left(\frac{1}{2}\right)^{k-2}}{k}$ converges and has limit c. Thus, we are done.

$\square$

Remark 3.2. We may restate the lemma in more concise way by saying that

$$\log(I + Y) = Y + O(\|Y\|^2)$$

where $O(\|Y\|^2)$ denotes the quantity of order $\|Y\|^2$ i.e. a quantity that is bounded by a constant times $\|Y\|^2$ for all sufficiently small values of $\|Y\|$.

Proposition 3.3. (*Lie Product Formula*) Let X and Y be $n \times n$ complex matrices. Then, $e^{X+Y} = \lim_{m \to \infty} \left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right)^m$.

PROOF. First, observe that

$$e^{X/m}e^{Y/m} = (I + \frac{X}{m} + \frac{X^2}{2m^2} + \cdots)(I + \frac{Y}{m} + \frac{Y^2}{2m^2} + \cdots) = I + \frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)$$

Then $e^{X/m}e^{Y/m} \to I$ as $m \to \infty$ so that $e^{X/m}e^{Y/m}$ is in the domain of the logarithm for all sufficiently large m. By lemma 3.1, we obtain:

$$\begin{aligned}
\log\left(e^{X/m}e^{Y/m}\right) &= \log\left(I + \frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)\right) \\
&= \frac{X}{m} + \frac{Y}{m} + O\left(\left\|\frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)\right\|^2\right) \\
&= \frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)
\end{aligned}$$

Then:

$$e^{X/m}e^{Y/m} = \exp\left(\frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)\right)$$

implies that,

$$\left(e^{X/m}e^{Y/m}\right)^m = \exp\left(X + Y + O\left(\frac{1}{m}\right)\right)$$

By the continuity of exponential, we conclude that:

$$\lim_{m\to\infty}\left(e^{\frac{X}{m}}e^{\frac{Y}{m}}\right)^m = e^{X+Y}$$

□

Definition 3.2. A function $A : \mathbb{R} \to GL(n, \mathbb{C})$ is called a *one-parameter subgroup* of $GL(n, \mathbb{C})$ if:
(1). A is continuous,
(2). $A(0) = I$,
(3). $A(t + s) = A(t)A(s)$ for all $t, s \in \mathbb{R}$.

Remark 3.3. From the above definition, we obtain that the inverse of $A(t)$ is $A(-t)$ since $A(-t)A(t) = A(-t + t) = A(0) = I$ and also $A(t)A(-t) = I$.

Proposition 3.4. (*One-Parameter Subgroups*) If A is a one-parameter subgroup of $GL(n, \mathbb{C})$, then there exists a unique $n \times n$ complex matrix X such that $A(t) = e^{tX}$.

PROOF. The uniqueness is immediate, since if there is an X such that $A(t) = e^{tX}$, then $X = \frac{d}{dt}\Big|_{t=0} A(t)$. For the existence, let $B_\epsilon(0)$ be the open ball of radius ϵ with center zero in $M(n, \mathbb{C})$. Assume that $\epsilon < \log 2$ then "exp" maps $B_\epsilon(0)$ injectively into $M(n, \mathbb{C})$ with continuous inverse "log". Let $U = \exp(B_{\epsilon/2}(0))$, which is an open set in $GL(n, \mathbb{C})$, We need the following lemma.

Lemma 3.2. Every $g \in U$ has a unique square root h in U, given by $h = \exp\left(\frac{1}{2}\log g\right)$.

PROOF. Since $g \in U$ then $\frac{1}{2}\log g \in B_{\epsilon/2}(0)$ and so $h = \exp\left(\frac{1}{2}\log g\right) \in U$ and is a square root of g, since $h^2 = \left[\exp\left(\frac{1}{2}\log g\right)\right]^2 = \exp(\log g) = g$. For uniqueness, let $k \in U$ such that $k^2 = g$ and let $X = \log g, Y = \log k$ then $\exp Y = k$ and $\exp(2Y) = (\exp Y)^2 = k^2 = g = \exp X$.

Since $Y \in B_{\epsilon/2}(0)$ then $2Y \in B_\epsilon(0)$ and $X \in B_{\epsilon/2}(0) \subset B_\epsilon(0)$ then $\exp(2Y) = \exp(X)$ implies that $2Y = X$ since exp is injective on $B_\epsilon(0)$. Thus, $k = \exp Y = \exp(X/2) = \exp\left(\frac{1}{2}\log g\right) = h$. This proved uniqueness. $\square$

Now, returning to the proof of proposition. Choose $\epsilon' > 0$ such that $B_{\epsilon'}(I) \subset U = \exp(B_{\epsilon/2}(0))$ since U is open and "exp" maps 0 to I. The continuity of A at 0 implies that exists $t_0 > 0$ such that for all t with $|t| \leq t_0$ then $A(t) \in B_{\epsilon'}(I) \subset U$ since $A(0) = I$. Let $X = \frac{1}{t_0}\log(A(t_0))$. We have $A(t_0)$ and $A(t_0/2)$ are in U and $A(t_0/2)^2 = A(t_0/2)A(t_0/2) = A(t_0)$. By the lemma, $A(t_0)$ has a unique square root in U that equals $\exp\left(\frac{1}{2}\log A(t_0)\right) = \exp(t_0X/2)$. Thus $A(t_0/2) = \exp(t_0X/2)$. Repeat this procedure, we obtain for all positive integer k that

$$A(t_0/2^k) = \exp(t_0X/2^k)$$

Then for any integer m, we have

$$A\left(\frac{mt_0}{2^k}\right) = A\left(\frac{t_0}{2^k}\right)^m = \exp\left(\frac{t_0X}{2^k}\right)^m = \exp\left(\frac{mt_0X}{2^k}\right)$$

Thus, $A(t) = \exp(tX)$ for all $t = \frac{mt_0}{2^k}$ where $m \in \mathbb{Z}, k \in \mathbb{N}$. To prove that $A(t) = \exp(tX)$ for all real number t, we make use the fact from analysis that if f, g be continuous functions on $\mathbb{R}$ such that $f = g$ on a dense subset of $\mathbb{R}$ then $f = g$ on $\mathbb{R}$. Since "A" and "exp" are continuous, we only need to prove that the set in the form $\frac{mt_0}{2^k}$ is dense in $\mathbb{R}$. Indeed, for $x, y \in \mathbb{R}$ such that $x < y$, we can choose k sufficiently large so that there is m such that $2^k x < mt_0 < 2^k y$ then $x < \frac{mt_0}{2^k} < y$. This proves the density and thus we are done. $\square$

2. Lie Algebra of Lie group

Definition 3.3. A *finite-dimensional real or complex Lie algebra* is a finite-dimensional real or complex vector space $\mathfrak{g}$, together with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which is called a *bracket*, with the following properties:

(1). $[\cdot,\cdot]$ is bilinear.

(2). $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$.

(3). $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$ for all $X, Y, Z \in \mathfrak{g}$.

Example 3.1.

(1). Any vector spaces become Lie algebras if all brackets are set equal to 0. Such a Lie algebra is called *abelian*.

(2). The vector space of all smooth vector fields on the manifold M form a Lie algebra under the bracket operation on vector fields.

(3). Let $\mathfrak{g} = \mathbb{R}^3$ and define $[x, y]$ to be the cross product $x \times y$. It is easy to check that the cross product satisfies all three properties in the definition above.

(4). The vector spaces $M(n,\mathbb{R})$ and $M(n,\mathbb{C})$ are real Lie algebra and complex Lie algebra, respectively with respect to the bracket operation $[A,B] = AB - BA$. Similarly, Let V be a finite-dimensional real or complex vector space and $gl(V)$ denote the space of linear map on V, then $gl(V)$ is a real or complex Lie algebra with the bracket operation $[A,B] = AB - BA$.

Now, we are going to define a Lie algebra of a Lie group G.

Definition 3.4. Let G be a Lie group and $a \in G$. *Left translation* by a and *right translation* by a are respectively the diffeomorphisms l_a and r_a of G defined by:

$$l_a(x) = ax$$
$$r_a(x) = xa$$

for all $x \in G$.

A vector field X (not necessary smooth) on G is called *left invariant* if for each $a \in G$, X is l_a-related to itself; that is, $dl_a \circ X = X \circ l_a$.

$$\begin{tikzcd} G \arrow[r, "l_a"] \arrow[d, "X"'] \arrow[dr, phantom, "\circlearrowleft"] & G \arrow[d, "X"] \\ T_x G \arrow[r, "(dl_a)_x"'] & T_{ax} G \end{tikzcd}$$

The set of all left invariant vector fields on G will be denoted by $\mathfrak{g}$.

Proposition 3.5. Let G be a Lie group and $\mathfrak{g}$ its set of left invariant vector fields.

(1). $\mathfrak{g}$ is a real vector space, and the map $\alpha : \mathfrak{g} \to T_eG$ defined by $\alpha(X) = X(e)$ is an isomorphism of $\mathfrak{g}$ with the tangent space T_eG at the identity. Consequently, $\dim \mathfrak{g} = \dim T_eG = \dim G$.

(2). Left invariant vector fields are smooth.

(3). The bracket of two left invariant vector fields is itself a left invariant vector field.

(4). $\mathfrak{g}$ form a (real) Lie algebra under the bracket operation on vector fields.

PROOF.

(1). Let $X, Y \in \mathfrak{g}$ and $k \in \mathbb{R}$ then:

$$\begin{aligned} dl_a \circ (X+Y) &= dl_a \circ X + dl_a \circ Y && (dl_a \text{ is linear}) \\ &= X \circ l_a + Y \circ l_a && (X, Y \in \mathfrak{g}) \\ &= (X+Y) \circ l_a \\ dl_a(kX) &= k dl_a(X) = k[X(l_a)] = (kX)(l_a) \end{aligned}$$

This proves that $\mathfrak{g}$ is a real vector space. On the other hand, α is clearly linear by its definition.

α is injective: Let $X, Y \in \mathfrak{g}$ with $\alpha(X) = \alpha(Y)$ then $X(e) = Y(e)$. Thus, for $a \in G$, we have:

$$X(a) = X(ae) = X(l_a(e)) = dl_a(X(e)) = dl_a(Y(e)) = Y(l_a(e)) = Y(ae) = Y(a)$$

This implies that $X = Y$.

α is surjective: Let $u \in T_eG$ and for $a \in G$, define $X(a) = dl_a(u)$ then $\alpha(X) = X(e) = dl_e(u) = u$. X is left invariant since for $b \in G$:

$$X(l_b(a)) = X(ba) = dl_{ba}(u) = dl_b(dl_a(u)) = dl_b(X(a))$$

The third equality follows from $l_{ba} = l_b l_a$ and proposition 1.3. This proves surjectivity and part (1) is done.

(2). To see that $X \in \mathfrak{g}$ is smooth, we only need to prove that $X(f)$ is smooth for any $f \in \mathscr{D}(G)$. Let $a \in G$ then:

$$Xf(a) = X_a(f) = X_{l_a(e)}f = dl_a(X_e)f = X_e(f \circ l_a)$$

Now, let $\phi : G \times G \to G$ denotes the group multiplication, $\phi(a, b) = ab$ which is smooth. Also, let $i_e^1, i_a^2 : G \to G \times G$ be injection maps such that $i_e^1(b) = (b, e)$ and $i_a^2(b) = (a, b)$.

Let Y be any smooth vector field on G such that $Y(e) = X(e)$ then $(0, Y)$ is a smooth vector field on G and $[(0, Y)(f \circ \phi)] \circ i_e^1$ is a smooth function on G since f, ϕ and i_e^1 are smooth. Using the result from lemma 1.2, we obtain:

$$\begin{aligned}
[(0, Y)(f \circ \phi)] \circ i_e^1(a) &= [(0, Y)(f \circ \phi)](a, e) \\
&= (0, Y)_{(a,e)}(f \circ \phi) \\
&= 0_a(f \circ \phi \circ i_e^1) + Y_e(f \circ \phi \circ i_a^2) \\
&= X_e(f \circ \phi \circ i_a^2) \\
&= X_e(f \circ l_a)
\end{aligned}$$

The last equality follows since:

$$f \circ \phi \circ i_a^2(b) = f \circ \phi(a, b) = f(ab) = f \circ l_a(b)$$

Thus, $X_e(f \circ l_a)$ is smooth so that X is smooth.

(3). Since X and Y in $\mathfrak{g}$ are smooth from (2), the bracket is defined. Let $a \in G$ and f be a smooth function on a neighborhood of $l_a(x)$ where $x \in G$ then:

$$\begin{aligned}
dl_a[X, Y](f) &= [X, Y](f \circ l_a) \\
&= X[Y(f \circ l_a)] - Y[X(f \circ l_a)] \\
&= X[dl_a(Y)(f)] - Y[dl_a(X)(f)] \\
&= X[Y(l_a)(f)] - Y[X(l_a)(f)] \\
&= (XY)(l_a)(f) - (YX)(l_a)(f) \\
&= [X, Y](l_a)(f)
\end{aligned}$$

Thus, $[X, Y]$ is invariant.

(4). It is immediate from the properties of the bracket of vector fields.

$\square$

Definition 3.5. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two (both are complex or real) Lie algebras. The map $f : \mathfrak{g} \to \mathfrak{h}$ is called *Lie algebra homomorphism* or *homomorphism of Lie algebra* if it is linear and preserves the bracket i.e. $f([x, y]) = [f(x), f(y)]$ for all $x, y \in \mathfrak{g}$. In addition, if f is bijective then f is called *Lie algebra isomorphism.* A Lie algebra isomorphism of a Lie algebra with itself is called *Lie algebra automorphism.*

proposition 3.5 tell us that any Lie groups give rise to their Lie algebra as below:

Definition 3.6. Let G be a Lie group. The *Lie algebra* $\mathfrak{g}$ of G is the set of left invariant vector fields on G. Alternatively, the *Lie algebra* $\mathfrak{g}$ of G is the tangent space T_eG of G at the identity with Lie algebra structure induced by requiring the vector space isomorphism $X \mapsto X(e)$ of the previous proposition from $\mathfrak{g}$ to T_eG to be an isomorphism of Lie algebra.

Example 3.2. The real line $\mathbb{R}$ is a Lie group under addition and the left invariant vector fields are simply the constant vector fields $\left\{\lambda \frac{d}{dr} | \lambda \in \mathbb{R}\right\}$. The bracket of any two such vector fields is 0.

Proposition 3.6. The Lie algebra of the real general linear group $GL(n, \mathbb{R})$ is $M(n, \mathbb{R})$ that is denoted by $\mathfrak{gl}(n, \mathbb{R})$. Similarly, the Lie algebra of the complex general linear group $GL(n, \mathbb{C})$ is $M(n, \mathbb{C})$ that is denoted by $\mathfrak{gl}(n, \mathbb{C})$.

PROOF. We will proof the first case since the second case can be considered analogously from the first case. Let $\mathfrak{g}$ be the Lie algebra of $GL(n, \mathbb{R})$. It is sufficient to prove that there is Lie algebra isomorphism between $\mathfrak{g}$ and $\mathfrak{gl}(n, \mathbb{R})$. To see this, let x_{ij} be the natural coordinate functions on $\mathfrak{gl}(n, \mathbb{R})$ which assign to each matrix its ij^{th} entry and let $\alpha : T_e(\mathfrak{gl}(n, \mathbb{R})) \to \mathfrak{gl}(n, \mathbb{R})$ be the canonical identification i.e. if $u \in T_e(\mathfrak{gl}(n, \mathbb{R}))$,

$$\alpha\left(u = \sum_{i,j=1}^{n} u(x_{ij}) \frac{\partial}{\partial x_{ij}}\right) = \sum_{i,j=1}^{n} u(x_{ij}) E_{ij}$$

where E_{ij} is a standard basis for the matrix space.
Then

$$\alpha(u)_{ij} = u(x_{ij})$$

But $T_e(GL(n, \mathbb{R})) = T_e(\mathfrak{gl}(n, \mathbb{R}))$ since $GL(n, \mathbb{R})$ is a subset of $\mathfrak{gl}(n, \mathbb{R})$. Thus, we can define a map $\beta : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{R})$ by:

$$\beta(X) = \alpha(X(e))$$

β is clearly a vector space isomorphism since the map $X \to X(e)$ and α are isomorphisms. So, we only need to prove that for any $X, Y \in \mathfrak{g}$,

$$\beta([X, Y]) = [\beta(X), \beta(Y)]$$

We have $(x_{ij} \circ l_A)(B) = x_{ij}(AB) = \sum_k x_{ik}(A) x_{kj}(B)$ where
$A = (x_{ij}(A))_{ij}, B = (x_{ij}(B))_{ij} \in G$ and since Y is a left invariant vector field, then:

$$\begin{aligned}
(Y(x_{ij}))(A) &= Y_A(x_{ij}) \\
&= dl_A(Y_e)(x_{ij}) = Y_e(x_{ij} \circ l_A) \\
&= Y_e\Big(\sum_k x_{ik}(A) x_{kj}\Big) = \sum_k x_{ik}(A) Y_e(x_{kj}) \\
&= \sum_k x_{ik}(A) \alpha(Y_e)_{kj} = \sum_k x_{ik}(A) \beta(Y)_{kj}
\end{aligned}$$

From this result,we can compute the ij^{th} component of $\beta([X,Y])$:

$$\begin{aligned}
\beta([X,Y])_{ij} &= \alpha([X,Y]_e)_{ij} \\
&= [X,Y]_e(x_{ij}) \\
&= X_e(Y(x_{ij})) - Y_e(X(x_{ij})) \\
&= X_e\left(\sum_k x_{ik}\beta(Y)_{kj}\right) - Y_e\left(\sum_k x_{ik}\beta(X)_{kj}\right) \\
&= \sum_k X_e(x_{ik})\beta(Y)_{kj} - \sum_k Y_e(x_{ik})\beta(X)_{kj} \\
&= \sum_k \alpha(X_e)_{ik}\beta(Y)_{kj} - \sum_k \alpha(Y_e)_{ik}\beta(X)_{kj} \\
&= \sum_k \beta(X)_{ik}\beta(Y)_{kj} - \sum_k \beta(Y)_{ik}\beta(X)_{kj} \\
&= \beta(X)\beta(Y)_{ij} - \beta(Y)\beta(X)_{ij} \\
&= [\beta(X),\beta(Y)]_{ij}
\end{aligned}$$

Thus β is Lie algebra isomorphism and we are done.

□

3. Properties of Lie algebra

In this section we will study some properties of Lie algebra of matrix Lie groups. Also, we will compute Lie algebras of some matrix Lie groups.

Lemma 3.3. Let G be a Lie subgroup of $GL(n,\mathbb{C})$ (not necessarily closed) with its Lie algebra $\mathfrak{g}$. Then "exp" maps $\mathfrak{g}$ to G.

Proof. In this proof, we view $\mathfrak{g}$ as the tangent space at the identity I of G. Let $X_1,\dots,X_m$ be a basis of $\mathfrak{g}$ and choose curves $\alpha_k(t)$ for $1 \le k \le m$ such that $\alpha_k(0) = I$ and $\alpha_k'(0) = X_k$. Define:

$$g(t_1X_1 + \ldots + t_mX_m) = \alpha_1(t_1)\alpha_2(t_2)\ldots\alpha_m(t_m)$$

Then $g : \mathfrak{g} \to G \subset \mathfrak{gl}(n,\mathbb{C})$ and $dg_0X = X$ for all $X \in \mathfrak{g}$ since:

$$\begin{aligned}
dg_0X_k &= \frac{d}{dt}\Big|_{t=0}(g(tX_k)) \\
&= \frac{d}{dt}\Big|_{t=0}(\alpha_1(0)\ldots\alpha_k(t)\ldots\alpha_m(0)) \\
&= \frac{d}{dt}\Big|_{t=0}(\alpha_k(t)) = \alpha_k'(0) = X_k
\end{aligned}$$

Choose a subspace $\mathfrak{s}$ of $\mathfrak{gl}(n,\mathbb{C})$ complement to $\mathfrak{g}$ so that $\mathfrak{gl}(n,\mathbb{C}) = \mathfrak{g} \oplus \mathfrak{s}$ and define a smooth map $h : \mathfrak{s} \to \mathfrak{gl}(n,\mathbb{C})$ in a neighborhood of 0 in $\mathfrak{s}$ such that $h(0) = I$ and $dh_0Y = Y$ for all $Y \in \mathfrak{s}$; for example, $h(Y) = 1 + Y$.
Now, define $f : \mathfrak{g} \times \mathfrak{s} \to \mathfrak{gl}(n,\mathbb{C})$ by $f(X,Y) = g(X)h(Y)$. Then f is defined and smooth in a neighborhood of 0 in $\mathfrak{g} \times \mathfrak{s}$ and

$$df_{(0,0)}(X,Y) = dg_0Xh(0) + g(0)dh_0Y = X + Y$$

This differential is invertible since if $df_{(0,0)}(X,Y) = X+Y = 0$ then $X=0, Y=0$ so that $Ker(df_{(0,0)}) = \{0\}$. The Inverse Function Theorem implies that f has a local inverse $\overline{f}$ from the neighborhood of I in $\mathfrak{gl}(n,\mathbb{C})$ to a neighborhood of 0 in $\mathfrak{g} \times \mathfrak{s}$ which we defined by $\overline{f}(A) = (f_1(A), f_2(A))$. Explicitly, for any A in a neighborhood of I in $\mathfrak{gl}(n,\mathbb{C})$, A is in the form $A = g(X)h(Y)$ for unique $(X,Y) \in \mathfrak{g} \times \mathfrak{s}$ with $X = f_1(A), Y = f_2(A)$.
If $f_2(A) = 0$ then $A = g(X)h(0) = g(X) \in G$. This mean that

$$f_2(A) = 0 \text{ implies } A \in G \tag{3.6}$$

Let (X,Y) close to 0 so that $\overline{f}$ is defined near $A = f(X,Y) = g(X)h(Y)$. Given $Z \in \mathfrak{g}$, then for real t close to 0, $(X+tZ, Y)$ close to 0 and $g(X+tZ)h(Y)$ close to I so we have:

$$f_2(g(X+tZ)h(Y)) = Y$$

Differentiate with respect to t at $t=0$ gives:

$$(df_2)_A(dg_X Z h(Y)) = 0$$

Since $h(Y) = g(X)^{-1}A$ then we obtain:

$$(df_2)_A(dg_X Z g(X)^{-1}A) = 0 \tag{3.7}$$

Observe that the matrix $dg_X Z g(X)^{-1}$ is in $\mathfrak{g}$ since:

$$dg_X Z g(X)^{-1} = \frac{d}{dt}\Big|_{t=0}(g(X+tZ)g(X)^{-1})$$

where $g(X+tZ)g(X)^{-1}$ is a curve on G that is equal I when $t=0$.
So, for any $X \in \mathfrak{g}$, we can define a map $F_X : \mathfrak{g} \to \mathfrak{g}$ by $F_X(Z) = dg_X Z g(X)^{-1}$. F_X is a linear transformation of $\mathfrak{g}$ depending continuously on X. Then (3.7) becomes:

$$(df_2)_A(F_X(Z)A) = 0 \tag{3.8}$$

If $X=0$ then $F_0(Z) = dg_0 Z g(0)^{-1} = Z$. That is F_0 is the identity transformation of $\mathfrak{g}$. This implies that $\det F_0 \neq 0$. By continuity, $\det F_X \neq 0$ for X in a neighborhood of 0 in $\mathfrak{g}$. This means that F_X is invertible for X close to 0 in $\mathfrak{g}$. Explicitly, F_X is injective for X near 0 in $\mathfrak{g}$ so that F_X is surjective since F_X is linear. Thus, every element $Z' \in \mathfrak{g}$ can be written as $F_X(Z)$ for some $Z \in \mathfrak{g}$. Then (3.8) becomes:

$$(df_2)_A(Z'A) = 0 \tag{3.9}$$

for all $Z' \in \mathfrak{g}$ and all A in a neighborhood of I in $\mathfrak{gl}(n,\mathbb{C})$
Returning to the assertion of proposition, let $X \in \mathfrak{g}$ and set $A(t) = e^{tX}$. By (3.9), for t in an interval about 0 in $\mathbb{R}$, we have:

$$0 = (df_2)_{A(t)}(XA(t)) = \frac{d}{dt}(f_2(A(t)))$$

since $A'(t) = \frac{d}{dt}(e^{tX}) = Xe^{tX} = XA(t)$. Thus, $f_2(A(t))$ is constant. But for $t=0$, $f_2(A(0)) = f_2(I) = 0$ then $f_2(A(t)) \equiv 0$ for all t in an interval about 0 in $\mathbb{R}$. From (3.6), we obtain $A(t) = e^{tX} \in G$. Thus for $N \in \mathbb{N}$ sufficiently large, $e^{X/N} \in G$ so that $e^X = \left(e^{X/N}\right)^N \in G$.

□

Corollary 3.1. If G is a Lie subgroup of $GL(n, \mathbb{C})$ with its Lie algebra $\mathfrak{g}$, then there exists a neighborhood U of 0 in $\mathfrak{g}$ and a neighborhood V of I in G such that the exponential map takes U diffeomorphically onto V.

PROOF. Since $\exp : \mathfrak{g} \to G$ is smooth then the differential $d\exp_0$ maps $\mathfrak{g}$ to $\mathfrak{g} = T_I G$. Now, if $d\exp_0(X) = 0$ then $0 = d\exp_0(X) = \frac{d}{dt}\Big|_{t=0}(\exp(tX)) = X$. Thus $d\exp_0$ is injective so that invertible. The Inverse Function Theorem implies that the exponential map is locall diffeomorphism at 0. This is what we desire.

□

Corollary 3.2. Let G be a Lie subgroup of $GL(n, \mathbb{C})$ with its Lie algebra $\mathfrak{g}$. Then

$$\mathfrak{g} = \{X \in \mathfrak{gl}(n, \mathbb{C}) | e^{tX} \in G, \forall t \in \mathbb{R}\}$$

PROOF. Let $X \in \mathfrak{gl}(n, \mathbb{C})$ such that $e^{tX} \in G$ for all real t then $X \in T_I G = \mathfrak{g}$ since $\alpha(t) = e^{tX}$ is a curve on G and $\alpha(0) = I, \alpha'(0) = X$. Conversely, if $X \in \mathfrak{g}$, then from lemma 3.3, $e^{tX} \in G$ for all real t.

□

Corollary 3.3. Let G be a Lie subgroup of $GL(n, \mathbb{C})$ and X an element of its Lie algebra. Then, e^X is an element of the identity component of G.

PROOF. We have that $e^{tX} \in G$ for all real t. However, as t varies from 0 to 1, e^{tX} is a curve on G connecting the identity to e^X. Thus, e^X lies in the identity component of G.

□

Corollary 3.4. Let G be a Lie subgroup of $GL(n, \mathbb{C})$ with Lie algebra $\mathfrak{g}$. Let $X \in \mathfrak{g}$ and $A \in G$ then $AXA^{-1} \in \mathfrak{g}$.

PROOF. We have $e^{t(AXA^{-1})} = Ae^{tX}A^{-1} \in G$ for all real t, then $AXA^{-1} \in \mathfrak{g}$.

□

The corollary 3.2 is useful for computing Lie algebras of Lie subgroups of $GL(n, \mathbb{C})$ (including matrix Lie groups). We can give the definition of Lie algebras of Lie subgroups of $GL(n, \mathbb{C})$ as below:

Definition 3.7. Let G be a Lie subgroup of $GL(n, \mathbb{C})$. *Lie algebra* $\mathfrak{g}$ of G is defined by:

$$\mathfrak{g} = \{X \in M(n, \mathbb{C}) | e^{tX} \in G, \forall t \in \mathbb{R}\}$$

Example 3.3.

(1). Consider the real special linear group $SL(n, \mathbb{R})$. We denote its Lie algebra by $\mathfrak{sl}(n, \mathbb{R})$. Then $X \in \mathfrak{sl}(n, \mathbb{R})$ implies that $e^{tX} \in SL(n, \mathbb{R})$ for all real t. Thus,

$$\det\left(e^{tX}\right) = 1$$

It is easy to check that $\det\left(e^{tX}\right) = e^{Tr(tX)} = e^{tTr(X)}$ where $Tr(X)$ denote the *trace* of the matrix X. This implies:

$$e^{tTr(X)} = 1 \text{ for all real } t$$

So, $Tr(X) = 0$. Conversely, if $X \in M(n, \mathbb{C})$ with $Tr(X) = 0$ then $X \in \mathfrak{sl}(n, \mathbb{R})$. Hence, $\mathfrak{sl}(n, \mathbb{R}) = \{X \in \mathfrak{gl}(n, \mathbb{R}) | Tr(X) = 0\}$.

Similarly, the Lie algebra of complex special linear group $SL(n, \mathbb{C})$ is:

$$\mathfrak{sl}(n, \mathbb{C}) = \{X \in \mathfrak{gl}(n, \mathbb{C}) | Tr(X) = 0\}.$$

(2). Recall that the unitary group $U(n)$ consists of all invertible complex matrices A such that $A^{-1} = A^*$. Thus, for all real t, $e^{tX} \in U(n)$ implies that:

$$e^{-tX} = \left(e^{tX}\right)^{-1} = \left(e^{tX}\right)^* = e^{tX^*} \text{ for all real } t \tag{3.10}$$

The equation (3.10) holds if:

$$-tX = tX^* \text{ for all } t \text{ which implies that } -X = X^*$$

Conversely, if (3.10) holds, by differentiating with respect to t at $t = 0$ give $-X = X^*$ Therefore, the Lie algebra $\mathfrak{u}(n)$ of $U(n)$ is:

$$\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) | -X = X^*\}$$

From this and example (1), it is easy to see that the Lie algebra $\mathfrak{su}(n)$ of the special unitary group $SU(n)$ is:

$$\mathfrak{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) | Tr(X) = 0, -X = X^*\}$$

(3). The special orthogonal group $SO(n)$ is the identity component of the orthogonal group $O(n)$. From corollary of the proposition 3.2, the exponential of a matrix in the Lie algebra $\mathfrak{o}(n)$ of the group $O(n)$ is in $SO(n)$. Thus, the Lie algebra $\mathfrak{o}(n)$ is as the same as the Lie algebra $\mathfrak{so}(n)$ of the group $SO(n)$. Recall the real matrix A is orthogonal if $A^{-1} = A^T$. Thus, for all real t, e^{tX} is orthogonal if

$$e^{tX^T} = \left(e^{tX}\right)^T = \left(e^{tX}\right)^{-1} = e^{-tX} \tag{3.11}$$

If (3.11) holds, by differentiating with respect to t at $t = 0$, we obtain $X^T = -X$. Conversely, if $X^T = -X$ then (3.11) holds. Thus,

$$\mathfrak{o}(n) = \mathfrak{so}(n) = \{X \in \mathfrak{gl}(n, \mathbb{R}) | X^T = -X\}$$

Note that the condition $X^T = -X$ implies that the diagonal entries of X are 0 so that the trace $Tr(X) = 0$.

(4). Recall that the Heisenberg group H is the set of 3×3 matrices of the form

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

We claim that the Lie algebra $\mathfrak{h}$ of H is the set of 3×3 matrices of the form

$$X = \begin{bmatrix} 0 & m & n \\ 0 & 0 & p \\ 0 & 0 & 0 \end{bmatrix}$$

where $m, n, p \in \mathbb{R}$. Note that

$$X^2 = \begin{bmatrix} 0 & 0 & mp \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } X^n = 0, \forall n \geq 3$$

Using the series formula of the exponential, we obtain:

$$e^{tX} = \begin{bmatrix} 1 & tm & tn + \frac{t^2}{2}mp \\ 0 & 1 & pt \\ 0 & 0 & 1 \end{bmatrix} \in H$$

Conversely, if X is any matrices such that $e^{tX} \in H$ then the entries of X which on and below the diagonal are zero since $X = \frac{d}{dt}\Big|_{t=0} e^{tX}$. Thus

$$\mathfrak{h} = \left\{ \begin{bmatrix} 0 & m & n \\ 0 & 0 & p \\ 0 & 0 & 0 \end{bmatrix} \Big| m, n, p \in \mathbb{R} \right\}$$

Theorem 3.1. Let G and H be matrix Lie groups with their Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Suppose that $\Phi : G \to H$ is a Lie group homomorphism. Then there exists a unique (real) Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ such that $\Phi \circ \exp = \exp \circ \phi$. This ϕ is also satisfies $\phi(AXA^{-1}) = \Phi(A)\phi(X)\Phi(A)^{-1}$ for all $A \in G, X \in \mathfrak{g}$.

Proof. Since Φ maps the identity element in G to the identity elements in H then its differential at the identity $d\Phi_I$ maps Lie algebra $\mathfrak{g}$ of G to Lie algebra $\mathfrak{h}$ of H. We define $\phi = d\Phi_I$ and claim that ϕ satisfies the conditions we desire. First note that ϕ is (real) linear. Now, let $X \in \mathfrak{g}$, then for all real t, we have:

$$\phi(X) = d\Phi_I X = \frac{d}{dt}\Big|_{t=0} (\Phi(e^{tX}))$$

$\Phi(e^{tX})$ is a one-parameter subgroup since Φ is continuous homomorphism and $\Phi(e^{tX})$ equal I when $t = 0$. From proposition 3.4, there exists a unique complex matrix Z such that $\Phi(e^{tX}) = e^{tZ}$. This Z is in $\mathfrak{h}$ since $X \in \mathfrak{g}$ then $e^{tX} \in G$ so that $e^{tZ} = \Phi(e^{tX}) \in H$ for all real t. Thus,

$$\phi(X) = \frac{d}{dt}\Big|_{t=0} (e^{tZ}) = Z \text{ where } \Phi(e^{tX}) = e^{tZ}$$

If $t = 1$ then $\Phi(e^X) = e^Z = e^{\phi(X)}$. Thus, $\Phi \circ \exp = \exp \circ \phi$.

To prove that $\phi(AXA^{-1}) = \Phi(A)\phi(X)\Phi(A)^{-1}$ for all $X \in \mathfrak{g}, A \in G$, observe that for all real t,

$$\begin{aligned} e^{t\phi(AXA^{-1})} &= e^{\phi(tAXA^{-1})} = \Phi(e^{tAXA^{-1}}) \\ &= \Phi(Ae^{tX}A^{-1}) = \Phi(A)\Phi(e^{tX})\Phi(A)^{-1} \\ &= \Phi(A)e^{\phi(tX)}\Phi(A)^{-1} \\ &= \Phi(A)e^{t\phi(X)}\Phi(A)^{-1} \end{aligned}$$

By differentiating with respect to t at $t = 0$ gives:

$$\phi(AXA^{-1}) = \Phi(A)\phi(X)\Phi(A)^{-1}$$

Now, the only thing we have to prove is that ϕ preserves the bracket; that is,

$$\phi([X, Y]) = [\phi(X), \phi(Y)] \text{ for all } X, Y \in \mathfrak{g}$$

Since $X = \frac{d}{dt}\Big|_{t=0}(e^{tX})$, it follows that $XY = \frac{d}{dt}\Big|_{t=0}(e^{tX}Y)$. So, by the product rule, we have:

$$\frac{d}{dt}\Big|_{t=0}(e^{tX}Ye^{-tX}) = (XY)e^0 + (e^0Y)(-X) = XY - YX = [X,Y]$$

Thus,

$$\begin{aligned}\phi([X,Y]) &= \phi(\frac{d}{dt}\Big|_{t=0}(e^{tX}Ye^{-tX})) \\ &= \frac{d}{dt}\Big|_{t=0}(\phi(e^{tX}Ye^{-tX})) \\ &= \frac{d}{dt}\Big|_{t=0}(\Phi(e^{tX})\phi(Y)\Phi(e^{tX})^{-1}) \\ &= \frac{d}{dt}\Big|_{t=0}(e^{t\phi(X)}\phi(Y)e^{-t\phi(X)}) \\ &= [\phi(X), \phi(Y)]\end{aligned}$$

The second equality follows from the fact that a derivative commutes with the linear map ϕ. For the uniqueness of ϕ, let $\tilde{\phi}$ be a Lie algebra associated of Φ then for any $X \in \mathfrak{g}$,

$$\begin{aligned}\phi(X) = d\Phi_I X &= \frac{d}{dt}\Big|_{t=0}(\Phi(e^{tX})) \\ &= \frac{d}{dt}\Big|_{t=0}(e^{t\tilde{\phi}(X)})) \\ &= \tilde{\phi}(X)\end{aligned}$$

This completes the proof.

□

Corollary 3.5. Let $G, H,$ and K be matrix Lie groups and let $\Phi : G \to H$ and $\Psi : H \to K$ be Lie group homomorphisms with associated Lie algebra homomorphisms ϕ and ψ respectively. If λ is a associated Lie algebra homomorphism of $\Psi \circ \Phi$ then $\lambda = \psi \circ \phi$.

Proof. Let $X \in \mathfrak{g}$ then for all real t,

$$e^{t\lambda(X)} = \Psi(\Phi(e^{tX})) = \Psi(e^{t\phi(X)}) = e^{t(\psi\circ\phi(X))}$$

By differentiating respect to t at $t = 0$, gives what we desire.

□

Corollary 3.6. Let G and H be matrix Lie groups and $\Phi : G \to H$ be a group homomorphism. If Φ is continuous than Φ is smooth. (This is exactly the proposition 2.2)

Proof. Let U be a neighborhood of 0 of Lie algebra $\mathfrak{g}$ and V be a neighborhood of I in G such that $\exp : U \to V$ is diffeomorphism. Let $A \in G$ then $A \in AV$. Thus for any $B \in AV$, $B = Ae^X$ for some $X \in U$ then

$$\Phi(B) = \Phi(A)\Phi(e^X) = \Phi(A)e^{\phi(X)}$$

where ϕ is the correspondence Lie algebra homomorphism.

This means that in exponential coordinates near A, Φ is a composition of the linear map ϕ, the exponential map, and multiplication on the left by $\Phi(A)$, all of which are smooth. This shows that Φ is smooth near any point A in G. Since A is arbitrary, Φ is smooth.

□

Proposition 3.7. Let G be a Lie subgroup of $GL(n,\mathbb{C})$ with Lie algebra $\mathfrak{g}$. If G is connected then

$$G = \{e^{X_1}e^{X_2}\dots e^{X_m} | X_1,\dots,X_m \in \mathfrak{g}\}$$

Proof. Let $A \in G$ then there is a curve $\alpha : [0,1] \to G$ such that $\alpha(0) = I$ and $\alpha(1) = A$. Let $V = e^U$ as in the previous corollary and let $W \subset V$ such that $W^{-1} = W$ and $W^2 \subset V$; for example, $W = W_o \cap W_o^{-1}$ for some neighborhood W_o of I in V. Since $[0,1]$ is compact then $\alpha([0,1])$ is compact. Observe that $\alpha([0,1]) \subset \bigcup\limits_{t\in[0,1]} \alpha(t)W$ then exist $0 = t_o, t_1, \dots, t_n = 1 \in [0,1]$ such that:

$$\alpha([0,1]) \subset \bigcup_{i=0}^{n} \alpha(t_i)W$$

Note that since $\alpha([0,1])$ is connected, then we can choose and arrange $t_0,\dots,t_n$ so that $\alpha(t_{i-1})W \cap \alpha(t_i)W \neq \varnothing$. Thus, there exist $W_1, W_2 \in W$ such that $\alpha(t_i)W_1 = \alpha(t_{i+1})W_2$ then $\alpha(t_{i-1})^{-1}\alpha(t_i) = W_2W_1^{-1} \in W^2 \subset V$ for $i = 1,\dots,n$. Thus,

$$A = \alpha(0)\alpha(1) = (\alpha(t_o)^{-1}\alpha(t_1))(\alpha(t_1)^{-1}\alpha(t_2))\dots(\alpha(t_{n-1})^{-1}\alpha(t_n))$$

If we choose $X_i \in U \subset \mathfrak{g}$ such that $e^{X_i} = \alpha(t_{i-1})^{-1}\alpha(t_i)$ for $i = 1,\dots,n$ then $A = e^{X_1}\dots e^{X_n}$. This completes the proof.

□

Corollary 3.7. Let G and H be Lie subgroups of $GL(n,\mathbb{C})$ such that G is connected. Let Φ_1, Φ_2 are Lie groups homomorphisms of G into H and ϕ_1, ϕ_2 be the associated Lie algebra homomorphisms. If $\phi_1 = \phi_2$ then $\Phi_1 = \Phi_2$.

Proof. Let $A \in G$ then from proposition, $A = e^{X_1}e^{X_2}\dots e^{X_m}$ for some $X_1,\dots,X_m \in \mathfrak{g}$. Thus,

$$\begin{aligned}
\Phi_1(A) &= \Phi_1(e^{X_1}e^{X_2}\dots e^{X_m}) \\
&= \Phi_1(e^{X_1})\Phi_1(e^{X_2})\dots\Phi_1(e^{X_m}) \\
&= e^{\phi_1(X_1)}e^{\phi_1(X_2)}\dots e^{\phi_1(X_m)} \\
&= e^{\phi_2(X_1)}e^{\phi_2(X_2)}\dots e^{\phi_2(X_m)} \\
&= \Phi_2(A)
\end{aligned}$$

□

4. The Closed Subgroup Theorem

As we have already known that all matrix Lie groups are Lie groups by the closed subgroup theorem. In this section, we are going to prove this theorem in the case of matrices.

Theorem 3.2. (**Closed Subgroup Theorem**) Every closed subgroup of the Lie group $GL(n, \mathbb{C})$ is an embedded submanifold of $GL(n, \mathbb{C})$ and thus a Lie group.

PROOF. For simplicity, we define $G := GL(n, \mathbb{C})$ with Lie algebra $\mathfrak{g} := \mathfrak{gl}(n, \mathbb{C})$ and $dimG = 2n^2 := N$. Let H be a closed subgroup of G with Lie algebra $\mathfrak{h}$ and $dimH = k < N$. First, note that

$$\mathfrak{h} = \{X \in \mathfrak{g} | e^{tX} \in H, \forall t \in \mathbb{R}\}$$

is a (real) subspace of $\mathfrak{g}$ since for any $X, Y \in \mathfrak{h}, \lambda \in \mathbb{R}$ and for all real t, the Lie product formula 3.3 implies that:

$$e^{t(X+\lambda Y)} = e^{tX+t\lambda Y} = \lim_{m\to\infty} \left(e^{\frac{tX}{m}} e^{\frac{t\lambda Y}{m}}\right)^m \in H$$

The last equation follows from the fact that H is closed. Now, let $\mathfrak{s}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ and consider the smooth map $f : \mathfrak{h} \times \mathfrak{s} \to G$ defined by:

$$f(X, Y) = e^X e^Y$$

The differential at $(0, 0)$, $df_{(0,0)} : \mathfrak{h} \times \mathfrak{s} \to \mathfrak{g}$ is:

$$\begin{aligned} df_{(0,0)}(X, Y) &= \frac{d}{dt}\Big|_{t=0} f(tX, tY) \\ &= \frac{d}{dt}\Big|_{t=0} e^{tX} e^{tY} = X + Y \end{aligned}$$

If $X + Y = 0$ then $X = 0, Y = 0$ since X, Y are orthogonal complement of each other. This proves that $Ker\left(df_{(0,0)}\right)$ is trivial and thus $df_{(0,0)}$ is invertible. The Inverse Function Theorem implies that there exist neighborhoods $U_{\mathfrak{h}}$ and $U_{\mathfrak{s}}$ of 0 in $\mathfrak{h}$ and $\mathfrak{s}$ respectively and a neighborhood V of the identity I in G such that $f : U_{\mathfrak{h}} \times U_{\mathfrak{s}} \to V$ is diffeomorphism.

If we identity $\mathfrak{h} \times \mathfrak{s}$ with $\mathbb{R}^N$ and define $\phi = f^{-1} : V \to U_{\mathfrak{h}} \times U_{\mathfrak{s}}$ then (V, ϕ) is a chart of G. We need the following lemma:

Lemma 3.4. There exists a neighborhood $U_{\mathfrak{s}}$ of 0 in $\mathfrak{s}$ such that

$$H \cap \exp(U_{\mathfrak{s}} \setminus \{0\}) = \varnothing$$

PROOF. Assume the contrary that there exists a non-zero sequence $\{Y_i\} \in \mathfrak{s}$ with $Y_i \to 0$ such that for any integer M, there is $i \geq M$ such that $e^{Y_i} \in H$ (that is, $\lim_{i\to 0} e^{Y_i} \in H$). Let $\tilde{Y}_i = \frac{Y_i}{\|Y_i\|}$ be a sequence in a unit sphere which is a compact set. Then there exists a subsequence $\tilde{Y}_{i_j}$ converges to $Y \neq 0$ in $\mathfrak{s}$.

Let $t \in \mathbb{R}$. For each j, choose n_j such that $\frac{t}{\|Y_{i_j}\|} - 1 < n_j \leq \frac{t}{\|Y_{i_j}\|}$. Since $\|Y_{i_j}\| \to 0$, then $n_j \|Y_{i_j}\| \to t$. Therefore,

$$e^{tY} = \lim_{j \to \infty} \exp(n_j \|Y_{i_j}\| \tilde{Y}_{i_j}) = \lim_{j \to \infty} \exp(n_j Y_{i_j}) = \lim_{j \to \infty} \exp(Y_{i_j})^{n_j} \in H$$

The limit is in H since H is closed. This proves that $Y \in \mathfrak{h}$ and thus $Y \in \mathfrak{h} \cap \mathfrak{s}$. Then $Y = 0$ which leads to a contradiction.

□

Returning to the theorem, we can choose $U_{\mathfrak{s}}$ satisfying the lemma. Therefore,

$$\begin{aligned} Z \in V \cap H &\iff Z = e^X e^Y \in H \text{ for some } X \in U_{\mathfrak{h}}, Y \in U_{\mathfrak{s}} \\ &\iff e^Y \in H \text{ for some } Y \in U_{\mathfrak{s}} \\ &\iff Y = 0 \text{ for some } Y \in U_{\mathfrak{s}} \end{aligned}$$

Thus,

$$\phi(V \cap H) = U_{\mathfrak{h}} \cap \{0\} = (U_{\mathfrak{h}} \times U_{\mathfrak{s}}) \cap (\mathbb{R}^k \times \{0\})$$

Therefore, for any point $A \in H$, we have $A \in AV$ and $\phi \circ l_{A^{-1}} : AV \to U_{\mathfrak{h}} \times U_{\mathfrak{s}}$ is diffeomorphism and thus $(AV, \phi \circ l_{A^{-1}})$ is a chart of G such that

$$\begin{aligned} \phi \circ l_{A^{-1}}(AV \cap H) &= \phi[A^{-1}(AV \cap H)] \\ &= \phi(V \cap H) \text{ since } A^{-1}(AV \cap H) = V \cap H \\ &= (U_{\mathfrak{h}} \times U_{\mathfrak{s}}) \cap (\mathbb{R}^k \times \{0\}) \end{aligned}$$

Thus, H is an embedded submanifold of G and it is Lie group since its multiplication map and inverse map that induce from $G = GL(n, \mathbb{C})$ are smooth. To see this, observe the inclusion map $i : H \hookrightarrow G$ is smooth then the map $\psi = f \circ i$, where $f : G \to G, g \mapsto g^{-1}$ is the inverse operation on G, is smooth. The induced map $\tilde{f}$ on H satisfies $i \circ \tilde{f} = \psi$ and in addition $\psi(H) = i(H) = H$. From theorem 1.32 in [**3**]: Foundations of differentiable manifolds and Lie groups by Frank W. Warner, implies that $\tilde{f}$ is smooth. Similarly, for the induced map $\tilde{g}$ of the multiplication map $f : G \times G \to G, (a, b) \mapsto ab$ to H. See the figure below. □

$$\begin{tikzcd} G \arrow[r, "f"] & G \\ H \arrow[u, hook, "i"] \arrow[r, "\tilde{f}"] \arrow[ur, "\psi"] & H \arrow[u, hook, "i"'] \end{tikzcd} \qquad \begin{tikzcd} G \times G \arrow[r, "g"] & G \\ H \times H \arrow[u, hook, "i"] \arrow[r, "\tilde{g}"] \arrow[ur, "\varphi"] & H \arrow[u, hook, "i"'] \end{tikzcd}$$

5. Lie Group that is not a Matrix Lie Group

We shall give a counterexample of a Lie group that is not a matrix Lie group and even not isomorphic to any matrix Lie groups. Recall from example 2.1 the group $G = \mathbb{R} \times \mathbb{R} \times S^1$ with the group multiplication:

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, e^{ix_1 y_2} z_1 z_2)$$

is a Lie group. We claim that G is not isomorphic to any matrix group (and thus not a matrix Lie group). Let H be a Heisenberg group (point (6) of example 2.2) and define a map $F : H \to G$ by:

$$F\left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\right) = (a, c, e^{ib})$$

F is continuous of Lie groups and thus is smooth. In addition,

$$\begin{aligned} F\left(\begin{bmatrix} 1 & a_1 & b_1 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a_2 & b_2 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{bmatrix}\right) &= F\left(\begin{bmatrix} 1 & a_1 + a_2 & b_1 + b_2 + a_1 c_2 \\ 0 & 1 & c_1 + c_2 \\ 0 & 0 & 1 \end{bmatrix}\right) \\ &= \left(a_1 + a_2, c_1 + c_2, e^{i(b_1 + b_2 + a_1 c_2)}\right) \\ &= \left(a_1 + a_2, c_1 + c_2, e^{ia_1 c_2} e^{ib_1} e^{ib_2}\right) \\ &= \left(a_1, c_1, e^{ib_1}\right)\left(a_2, c_2, e^{ib_2}\right) \\ &= F\left(\begin{bmatrix} 1 & a_1 & b_1 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{bmatrix}\right) F\left(\begin{bmatrix} 1 & a_2 & b_2 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{bmatrix}\right) \end{aligned}$$

Thus, F is a Lie group homomorphism. Let $N = KerF$ and recall that the center $Z(H)$ of H is the set of any elements in H that commute with all elements in H. The direct computation show that:

$$N = \left\{ \begin{bmatrix} 1 & 0 & 2\pi n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| n \in \mathbb{Z} \right\}$$

$$Z(H) = \left\{ \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| t \in \mathbb{R} \right\}$$

We need the following lemmas:

Lemma 3.5. Recall that a matrix X is *nilpotent* if there exists a positive integer k such that $X^k = 0$ or equivalently, if all eigenvalues of X are zero. Let $X \neq 0$ and is a nilpotent matirx. Then for all non-zero real number t, $e^{tX} \neq I$.

PROOF. Since X is nilpotent, for $t \in \mathbb{R} \setminus \{0\}$, e^{tX} is a finite degree polynomial of matrix tX and thus all the entries of e^{tX} is a finite degree polynomial of t, that is:

$$(e^{tX})_{ij} = (P_{ij}(t)),$$

where $P_{ij}(t)$ is a finite degree polynomial of t of the entry ij. Suppose that there exists $t_o \neq 0$ such that $e^{t_o X} = I$ then for all $n \in \mathbb{N}$, we have:

$$e^{nt_o X} = (e^{t_o X})^n = I^n = I$$

This proves that $P_{ij}(nt_o) = \delta_{ij}$ for all $n \in \mathbb{N}$. Since each P_{ij} is a finite degree polynomial and equal to the same constant number δ_{ij} for infinitely many $t = nt_0, n \in \mathbb{N}$ then

$$P_{ij}(t) = \delta_{ij}, \forall t \neq 0 \text{ so that } e^{tX} = I, \forall t \in \mathbb{R}$$

Then

$$X = \frac{d}{dt}\Big|_{t=0} e^{tX} = \frac{d}{dt}\Big|_{t=0} I = 0 : \text{ contradiction}$$

□

Lemma 3.6. Let $\Phi : H \to GL(n, \mathbb{C})$ be a Lie group homomorphism. If $N \subset Ker\Phi$ then $Z(H) \subset Ker\Phi$.

PROOF. From (4) of example 3.3, the Lie algebra $\mathfrak{h}$ of the Heisenberg group H is:

$$\mathfrak{h} = \left\{ \begin{bmatrix} 0 & m & n \\ 0 & 0 & p \\ 0 & 0 & 0 \end{bmatrix} \middle| m, n, p \in \mathbb{R} \right\}$$

$\mathfrak{h}$ has a basis:

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

that satisfies:

$$[E_1, E_3] = E_2, [E_1, E_2] = [E_3, E_2] = 0$$

Let $\phi : \mathfrak{h} \to \mathfrak{gl}(n, \mathbb{C})$ be associated Lie algebra of Φ.Then:

$$[\phi(E_1), \phi(E_3)] = \phi(E_2), [\phi(E_1), \phi(E_2)] = [\phi(E_3), \phi(E_2)] = 0$$

We claim that $\phi(E_2)$ is nilpotent. Indeed, let λ be an eigenvalue of $\phi(E_2)$ and V_λ be an eigenspace corresponding to λ. V_λ is invariant under $\phi(E_2)$ since for any $u \in V_\lambda$,

$$\phi(E_2)(\lambda u) = \lambda(\lambda u) \Longrightarrow \phi(E_2)u = \lambda u \in V_\lambda$$

In addition, $[\phi(E_1), \phi(E_2)] = [\phi(E_3), \phi(E_2)] = 0$ implies that $\phi(E_1)$ and $\phi(E_3)$ commute with $\phi(E_2)$. Thus, for any $u \in V_\lambda$, $i \in \{1, 3\}$

$$\phi(E_2)[\phi(E_i)u] = \phi(E_2)\phi(E_i)u = \phi(E_i)\phi(E_2)u = \phi(E_i)[\phi(E_2)u] = \lambda[\phi(E_i)u]$$

So, V_λ is invariant under $\phi(E_1)$ and $\phi(E_3)$.

Now, on V_λ, $\phi(E_2) = \lambda I$, then

$$\lambda I = [\phi(E_1)|_{V_\lambda}, \phi(E_3)|_{V_\lambda}]$$

Thus,

$$\lambda \dim V_\lambda = tr(\lambda I) = tr[\phi(E_1)|_{V_\lambda}, \phi(E_3)|_{V_\lambda}] = 0$$

Since $\dim V_\lambda \neq 0$ then $\lambda = 0$ and then $\phi(E_2)$ is nilpotent.

Observe that $(E_2)^2 = 0$ and thus $e^{tE_2} = \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ so that:

$$Z(H) = \{e^{tE_2} | t \in \mathbb{R}\}$$
$$N = \{e^{2\pi n E_2} | n \in \mathbb{Z}\}$$

Since $N \in Ker\Phi$, we have

$$I = \Phi(e^{2\pi n E_2}) = e^{\phi(2\pi n E_2)} = e^{2\pi n \phi(E_2)}$$

Since $2\pi n \phi(E_2)$ is nilpotent then from lemma 3.5, we obtain $\phi(E_2) = 0$. Thus,

$$\Phi(e^{tE_2}) = e^{t\phi(E_2)} = I$$

Therefore, $Z(H) \subset Ker\Phi$.

$\square$

Now, let $\Psi : G \to GL(n, \mathbb{C})$ be a Lie group homomorphism. We want to prove that Ψ cannot be injective. We have $\Psi \circ F : H \to GL(n, \mathbb{C})$ is a Lie group homomorphism and $N \subset Ker(\Psi \circ F)$. From lemma 3.6, then $Z(H) \subset Ker(\Psi \circ F)$. Thus,

$$I = \Psi \circ F \left(\begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \Psi(0, 0, e^{it}), \forall t \in \mathbb{R}$$

This proves that $Ker\Psi$ is not trivial and thus Ψ is not injective.

CHAPTER 4

Lie correspondences

From the previous chapter, we obtain the following results:

1. Every Lie group gives rise to a (real) Lie algebra which is the set of left invariant vector fields on the group or the tangent space of the group at the identity. In the case of matrix Lie group G, Lie algebra $\mathfrak{g}$ is given in the form:

$$\mathfrak{g} = \{X \in \mathfrak{gl}(n, \mathbb{C}) | e^{tX} \in G, \forall t \in \mathbb{R}\}$$

2. For every Lie homomorphism $\Phi : G \to H$ between two matrix Lie groups G and H with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively, there is a unique correspondence (real) Lie algebra $\phi : \mathfrak{g} \to \mathfrak{h}$ such that $\Phi \circ \exp = \exp \circ \phi$.

In this chapter, we shall answer the following questions.

1. Given a matirx Lie group G with Lie algebra $\mathfrak{g}$. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, are there any matrix Lie subgroups H of G with Lie algebra $\mathfrak{h}$? Conversely, if H is a matrix Lie group and is a subgroup of G with Lie algebra $\mathfrak{h}$. Is $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$?

2. Given matirx Lie groups G and H with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. If $\phi : \mathfrak{g} \to \mathfrak{h}$ is a (real) Lie algebra homomorphism, are there any Lie group homomorphisms $\Phi : G \to H$ such that $\Phi \circ \exp = \exp \circ \phi$?

To answer these questions, we need some new concepts and we will recall all of them here.

Definition 4.1. Let $\mathfrak{g}$ be a Lie algebra of a Lie group G. A *Lie subalgebra* of $\mathfrak{g}$ is a subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $[h_1, h_2] \in \mathfrak{h}$ for any $h_1, h_2 \in \mathfrak{h}$. If $\mathfrak{g}$ is a complex Lie algebra and $\mathfrak{h}$ is a real subspace of $\mathfrak{g}$ which is closed under brackets, then $\mathfrak{h}$ is called *real Lie subalgebra* of $\mathfrak{g}$.

Definition 4.2. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. For each $A \in G$, define a linear map $Ad_A : \mathfrak{g} \to \mathfrak{g}$ by

$$Ad_A(X) = AXA^{-1}$$

This map is called the *adjoint representation* or the *adjoint mapping.*

Remark 4.1.

(1). Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$ and let $GL(\mathfrak{g})$ denotes the group of all invertible linear transformations of $\mathfrak{g}$. Then for each $A \in G$, Ad_A is an invertible linear transformation of $\mathfrak{g}$ with inverse $Ad_{A^{-1}}$, and the map $A \mapsto Ad_A$ is a Lie group homomorphism of G into $GL(\mathfrak{g})$. Furthermore, for each $A \in G$, Ad_A

satisfies:

$$Ad_A([X,Y]) = [Ad_A(X), Ad_A(Y)] \text{ for all } X, Y \in \mathfrak{g}$$

(2). If $ad : \mathfrak{g} \to gl(\mathfrak{g})$ is the associated Lie algebra homomorphism of Lie group homomorphism $Ad : G \to GL(\mathfrak{g})$ as defined above then for all $X, Y \in \mathfrak{g}$, we have:

$$ad_X(Y) = [X,Y]$$

and for all $Y \in \mathfrak{g}$,

$$e^{ad_X}Y = Ad_{e^X}Y = e^X Y e^{-X}$$

(3). For any $X, Y \in \mathfrak{g}$, $ad_{[X,Y]} = [ad_X, ad_Y]$

Now, consider the function:

$$g(z) = \frac{\log z}{1 - \frac{1}{z}}$$

This function is defined and analytic in the open disk $D_1(1)$ and thus for z in this disk, $g(z)$ can be written as:

$$g(z) = \sum_{k=0}^{\infty} a_k (z-1)^k$$

for some set of constant $\{a_k\}$. This series has radius of convergence one.

Suppose V is a finite-dimensional vector space. Choose an arbitrary basis for V so that V can be identified with $\mathbb{C}^n$ and thus, the norm of a linear operator on V can be defined. Then for any operator A on V with $\|A - I\| < 1$, we can define:

$$g(A) = \sum_{k=0}^{\infty} a_k (A-I)^k$$

The Baker-Campbell-Hasudorff's formula (BCH formula). For all $n \times n$ complex matrix X and Y with $\|X\|$ and $\|Y\|$ sufficiently small,

$$\log(e^X e^Y) = X + \int_0^1 g(e^{ad_X} e^{tad_Y})(Y)dt \tag{4.1}$$

Note that $e^{ad_X} e^{tad_Y}$ and $g(e^{ad_X} e^{tad_Y})$ are linear operators on the space $\mathfrak{gl}(n, \mathbb{C})$. In (4.1), this operator is being applied to the matrix Y. The condition that X and Y are assume to be small guarantees that $e^{ad_X} e^{tad_Y}$ is closed to identity so that $g(e^{ad_X} e^{tad_Y})$ is well defined.

Remark 4.2. If we define $C(X,Y) = X + \int_0^1 g(e^{ad_X} e^{tad_Y})(Y)dt$ then $C(\cdot, \cdot)$ is continuous. To see this, Let X_n and Y_n be sequences such that when $n \to \infty$, $X_n \to X$ and $Y_n \to Y$. We shall prove that $C(X_n, Y_n) \to C(X,Y)$ as $n \to \infty$. Let $F(X,Y,t) = g(e^{ad_X} e^{tad_Y})(Y)$ and $F(X,Y) = \int_0^1 g(e^{ad_X} e^{tad_Y})(Y)dt$, then we only need to prove that $F(X_n, Y_n) \to F(X,Y)$ as $n \to \infty$. Since $e^{ad_X} e^{tad_Y}$ and $g(e^{ad_X} e^{tad_Y})$ are linear operators then:

$$F(X_n, Y_n, t) \to F(X, Y, t) \text{ as } n \to \infty$$

$F(X, Y, t)$ is continuous in (X, Y). Given $\epsilon_o > 0$, there exist $\delta_o > 0$ such that

$$\|X' - X\| < \delta_0, \|tY' - tY\| \leq \|Y' - Y\| < \delta_0, t \in [0, 1]$$

Then:

$$\|F(X', Y', t) - F(X, Y, t)\| < \epsilon_0$$

Choose $N \in \mathbb{N}$ such that $\|X_n - X\| < \delta_0, \|Y_n - Y\| < \delta_0$ for all $n \geq N$ then

$$\begin{aligned}\|F(X_n, Y_n, t)\| &\leq \|F(X_n, Y_n, t) - F(X, Y, t)\| + \|F(X, Y, t)\| \\ &< \epsilon_o + \sup_{t\in[0,1]} F(X, Y, t) = M_{(X,Y)} < \infty\end{aligned}$$

By Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty} \int_0^1 g(e^{ad_{X_n}} e^{tad_{Y_n}})(Y_n)dt = \int_0^1 g(e^{ad_X} e^{tad_Y})(Y)dt$$

This proves that $F(X_n, Y_n) \to F(X, Y)$ as $n \to \infty$.

The BCH formula has the following consequence.

Corollary 4.1. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Suppose that $\phi : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$ is a Lie algebra homomorphism. Then, for all sufficiently small X and Y in $\mathfrak{g}$, $\log(e^X e^Y)$ is in $\mathfrak{g}$ and

$$\phi[\log(e^X e^Y)] = \log\left(e^{\phi(X)} e^{\phi(Y)}\right)$$

1. Lie Group-Lie Algebra Correspondence

The answer to the second part of the first question, in general, is true for any Lie subgroup H of a matrix Lie group G. For the first part, the answer is no in general. However, if we restrict the question to find any connected Lie subgroup H of a matrix Lie group G, which is not necessary a matrix Lie subgroup, then the answer is yes.

Theorem 4.1. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Then there exists a unique connected Lie subgroup H of G with Lie algebra $\mathfrak{h}$ (H is not necessary a matrix Lie group).
One the other hand, if G is a matrix Lie group with Lie algebra $\mathfrak{g}$ and H is a Lie subgroup of G with Lie algebra $\mathfrak{h}$. Then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.

The proof of the theorem is followed from [**1**], Brian C.Hall: Lie Groups, Lie Algebras, and Representations: An Elementary Introduction.

PROOF. We start with the second part of the theorem. Since H is a Lie subgroup of G then $\mathfrak{h} = T_I H$ is a subspace of $\mathfrak{g} = T_I G$. Let $X, Y \in \mathfrak{h}$. The corollary 3.4 implies that for all real t,

$$e^{tX} Y e^{-tX} \in \mathfrak{h}$$

Therefore, since $\mathfrak{h}$ is a vector space and thus a topologically closed subset of $M(n, \mathbb{C})$ then:

$$[X, Y] = XY - YX = \frac{d}{dt}\Big|_{t=0} e^{tX} Y e^{-tX}$$
$$= \lim_{h \to 0} \frac{e^{hX} Y e^{-hX} - Y}{h} \in \mathfrak{h}$$

Thus, $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.

To prove the first part, let

$$H = \{e^{X_1} e^{X_2} \dots e^{X_m} | X_1, X_2, \dots, X_m \in \mathfrak{h}\}$$

It is easy to see that this H is unique since if there is a connected Lie subgroup H' of G with Lie algebra $\mathfrak{h}$ then $H \subset H'$. Also, for any $A \in H'$, A can be written as $A = e^{X_1} e^{X_2} \dots e^{X_m} \in H$ for some $X_1, X_2, \dots, X_m \in \mathfrak{h}$. Thus, $H' \subset H$ and we conclude that $H' = H$

Now, H is a subgroup of G and it is connected since for all $A \in H$ such that $A = e^{X_1} e^{X_2} \dots e^{X_m}$, A can be connected to the identity by the curve:

$$t \mapsto e^{tX_1} e^{tX_2} \dots e^{tX_m}, \qquad 0 \leq t \leq 1$$

Thus we only need to prove that the Lie algebra of H is $\mathfrak{h}$ and H is a submanifold of G.

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ where $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{s}$ is a orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. As shown in the proof of the closed subgroup theorem, the map $\mathfrak{h} \times \mathfrak{s} \to \mathfrak{g}, (X, Y) \mapsto e^X e^Y$ is local diffeomorphism then there exist neighborhood U and V of the origin in $\mathfrak{h}$ and $\mathfrak{s}$ respectively, and a neighborhood W of I in G such that each $A \in W$ can be written uniquely as:

$$A = e^X e^Y, \qquad X \in U, Y \in V$$

in such a way that X, Y depend continuously on A. We need the following lemma:

Lemma 4.1. The set

$$E = \{Y \in V | e^Y \in H\} \text{ is at most countable}$$

We assume the lemma and go on our proof. Let $\mathfrak{h}'$ be a Lie algebra of H. It is clear that $\mathfrak{h} \subset \mathfrak{h}'$ since for every $X \in \mathfrak{h}$, by the definition of H, $e^{tX} \in H$ for all real t then $X \in \mathfrak{h}'$. For $Z \in \mathfrak{h}'$, we can write for all sufficiently small t (t being sufficiently small to guarantee that $e^{tZ} \in W$),

$$e^{tZ} = e^{X(t)} e^{Y(t)}$$

where $X(t) \in U \subset \mathfrak{h}$ and $Y(t) \in V \subset \mathfrak{s}$ are continuously functions of t with $X(0) = Y(0) = 0$. Since e^{tZ} and $e^{X(t)}$ are in H then $e^{Y(t)} \in H$.

If $Y(t)$ is not constant then the set E in lemma 4.1 is uncountable which leads to a contradiction. Thus $Y(t)$ is constant and it is identically zero since $Y(0) = 0$. Therefore,

$$e^{tZ} = e^{X(t)}$$

we can make t small enough if necessary so that exp is injective and then $tZ = X(t) \in \mathfrak{h}$ which implies that $Z \in \mathfrak{h}$. So, $\mathfrak{h}' \subset \mathfrak{h}$ and we conclude that $\mathfrak{h}' = \mathfrak{h}$.

To prove that H is a submanifold of G, we assume that dim$G$=dim$\mathfrak{g}$=n and dim$H$=dim$\mathfrak{h}$ $= k$. Let $X_1, X_2, \ldots, X_k$ be a basis in $\mathfrak{h}$ and $X_1, X_2, \ldots, X_n$ be a extended basis in $\mathfrak{g}$.

Consider the map:

$$\mathbb{R}^n \xrightarrow{F} G$$
$$(t_1, t_2, \ldots, t_n) \mapsto e^{t_1 X_1} e^{t_2 X_2} \ldots e^{t_n X_n}$$

If we identify $\mathfrak{g}$ with $\mathbb{R}^n$, then the differential dF_0 at 0 is an identity. From the Inverse Function's theorem, there exists an open set U of I in G and V of 0 in $\mathbb{R}^n$ such that:

$$U \xrightarrow{\phi := F^{-1}} V$$
$$e^{t_1 X_1} e^{t_2 X_2} \ldots e^{t_n X_n} \mapsto (t_1, t_2, \ldots, t_n)$$

is a diffeomorphism of a neighborhood of I in G. This (U, ϕ) is a chart of G. Thus if we define:

$$W = \phi^{-1}(V \cap (\mathbb{R}^k \times \{0\}))$$

Then $W \in H$ since $e^{tX_1} e^{t_2 X_2} \ldots e^{t_k X_k} \in H$ and $\phi\Big|_W : W \to V \cap (\mathbb{R}^k \times \{0\})$ is diffeomorphism. Thus, if we define W as an open set in H then

$$\phi(W) = V \cap (\mathbb{R}^k \times \{0\})$$

This is for any point near I in H. For $A \in H$, we have $AW \subset H$ is an open subset in H. Then $(AU, \phi \circ l_{A^{-1}})$ is a chart of G and $\phi \circ l_{A^{-1}}(AW) = \phi \circ l_{A^{-1}} \circ l_A(W) = \phi(W) = V \cap (\mathbb{R}^k \times \{0\})$. This proves that H is a (immersed) submanifold of G. Note that if we define a new topology on H generating from the subset $\{AW | A \in H, I \in W\}$ and construct a smooth structure $\mathscr{H}$ to be a maximal collection of

$$\{(AW, \phi \circ l_{A^{-1}}|_{AW}) | (AU, \phi \circ l_{A^{-1}}) \text{ is a chart in } G\}$$

then H is a manifold. Now, to prove that H is a Lie subgroup of G is to prove that the group multiplication and inversion induced from G are smooth. To achieve this goal, we need some new concepts of involutive distribution and integral manifold.

Definition 4.3. Let M^m be a manifold and $1 \leq k \leq m$. A *k-dimensional distribution* $\mathcal{D}$ on M is a choice of a k-dimensional subspace $\mathcal{D}(p)$ of T_pM for each p in M. $\mathcal{D}$ is said to be smooth if for each $p \in M$, there is a neighborhood U of p and there are k smooth vector fields $X_1, \cdots, X_k$ on U which span $\mathcal{D}$ at each point of U. A vector field X on M is lie in $\mathcal{D}(X \in \mathcal{D})$ if $X(p) \in \mathcal{D}(p)$ for each $p \in M$. A smooth distribution $\mathcal{D}$ is called *involutive* if $[X, Y] \in \mathcal{D}$ whenever X and Y are smooth vector fields lie in $\mathcal{D}$. A submanifold N of M is an *integral manifold* of a distribution $\mathcal{D}$ on M if $T_qN = \mathcal{D}(q)$ for each $q \in N$.

Lemma 4.2. Let $f : M^m \to N^n$ be smooth map and P is a integral manifold of distribution $\mathcal{D}$ on N and $f(M) \subset P$. Let $\tilde{f} : M \to P$ be the unique map such that $i \circ \tilde{f} = f$ where $i : P \hookrightarrow N$ is an inclusion map. Then $\tilde{f}$ is smooth.

This result follows from theorem 1.32(a) and theorem 1.62 in [**3**]: Foundation of differentiable manifold and Lie groups by Frank W. Warner.

Let f and g be the inversion map and group multiplication on G respectively and $\tilde{f}$ and $\tilde{g}$ be the induced inversion map and multiplication map on H respectively. Then $\psi = f \circ i$ and $\phi = i \circ g$ are smooth. From 4.2, we only need to prove that H is

$$\begin{tikzcd} G \arrow[r, "f"] & G & & G \times G \arrow[r, "g"] & G \\ H \arrow[u, hook, "i"] \arrow[r, "\tilde{f}"'] \arrow[ur, "\psi"] & H \arrow[u, hook, "i"'] & & H \times H \arrow[u, hook, "i"] \arrow[r, "\tilde{g}"'] \arrow[ur, "\varphi"] & H \arrow[u, hook, "i"'] \end{tikzcd}$$

an integral manifold of an involutive distribution $\mathcal{D}$ on G. Let $\mathcal{D}$ be a distribution on G such that $\mathcal{D}(a) = (dl_a)_I(\mathfrak{h})$ for each $a \in G$ where $(dl_a)_I : T_I G \cong \mathfrak{g} \to T_a G$. $\mathcal{D}$ is clearly a distribution on G since $(dl_a)_I(\mathfrak{h}) \in T_a G$ is a subspace of $T_a G$ for each point $a \in G$. Observe that if $v \in T_I G$ then

$$(dl_a)_I(v) = \frac{d}{dt}\Big|_{t=0} l_a(e^{tv}) = \frac{d}{dt}\Big|_{t=0} ae^{tv} = av$$

Then $(dl_a)_I(v) = 0$ implies $v = 0$ since $a \neq 0$. Note that if G is a group under addition then $(dl_a)_0(v) = \frac{d}{dt}\Big|_{t=0} l_a(e^{tv}) = \frac{d}{dt}\Big|_{t=0}(a + e^{tv}) = v$. Thus, $(dl_a)_I$ is injective and then we obtain

$$\mathcal{D}(a) = (dl_a)_I(\mathfrak{h}) \cong \mathfrak{h} \text{ for each } a \in G$$

Then $dim\mathcal{D} = dim(\mathfrak{h}) = k$. Let $X_1, \cdots, X_k$ be a basis of $\mathfrak{h}$ then this basis span $\mathcal{D}$ at each point of G. This proves that $\mathcal{D}$ is smooth. Since $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, it is closed under bracket. Identifying $\mathcal{D}(a)$ with $\mathfrak{h}$ for each $a \in G$, we obtain that $\mathcal{D}$ is an involutive distribution on G. Let $h \in H$, we have that $dim(T_h H) = dim(\mathcal{D}(h))$. Let $p \in \mathcal{D}(a)$, then there is $v \in \mathfrak{h}$ such that

$$p = \frac{d}{dt}\Big|_{t=0}(he^{tv}) = \frac{d}{dt}\Big|_{t=0}(e^{X_1} \cdots e^{X_m} e^{tv})$$

where $X_1, \cdots, X_m \in \mathfrak{h}$. Let α be the curve such that $\alpha(t) = e^{X_1} \cdots e^{X_m} e^{tv}$ then α lies in H and is smooth. In addition, $\alpha(0) = h$ and $\alpha'(0) = p$. Thus $\mathcal{D} \subset T_h H$ so that $T_h H = \mathcal{D}(h)$. This proves that H is an integral manifold of the involutive distribution $\mathcal{D}$ on G. This proves the theorem. $\square$

Now, we only need to prove the lemma 4.1. We need the following lemma.

Lemma 4.3. Pick a basis for $\mathfrak{h}$ and call an element of $\mathfrak{h}$ *rational* if its coefficients with respect to this basis are rational. Then for every $\delta > 0$ and every $A \in H$, there exist rational elements $R_1, \ldots, R_k$ of $\mathfrak{h}$ such that

$$A = e^{R_1} e^{R_2} \ldots e^{R_k} e^X$$

where $X \in \mathfrak{h}$ with $\|X\| < \delta$.

PROOF. Choose $\epsilon > 0$ so that for all $X, Y \in \mathfrak{h}$ with $\|X\|, \|Y\| < \epsilon$, the Baker-Campbell-Hausdorff holds for X and Y; that is,

$$\log(e^X e^Y) = X + \int_0^1 g(e^{ad_X} e^{tad_Y})(Y)dt$$

Let $C(X,Y) = X + \int_0^1 g(e^{ad_X} e^{tad_Y})(Y)dt$ then we have:

$$e^X e^Y = e^{C(X,Y)}$$

We have $C(\cdot,\cdot)$ is continuous and if the lemma hold for some $\delta > 0$, it also holds for any $\delta' > \delta$. Thus we can assume that $\delta < \epsilon$ and sufficiently small so that if $\|X\|, \|Y\| < \delta$, we have $\|C(X,Y)\| < \epsilon$.

Since $e^X = \left(e^{X/l}\right)^l$ for sufficiently large non zero positive integer l with $\left\|\frac{X}{l}\right\| < \delta$. Then every element A in H can be written as:

$$A = e^{X_1} e^{X_2} \dots e^{X_m} \tag{4.2}$$

with $X_i \in \mathfrak{h}$ and $\|X_i\| < \delta$. By induction, suppose that $m = 0$ then $A = I = e^0$ and there is nothing to prove. Assume the lemma until $m-1$; that is,

$$e^{X_1} e^{X_2} \dots e^{X_{m-1}} = e^{R_1} e^{R_2} \dots e^{R_{k-1}} e^X$$

for some rational elements $R_j, X \in \mathfrak{h}$ with $\|X\| < \delta$. Now, for A as in (4.2), A can be written as:

$$\begin{aligned} A &= e^{R_1} e^{R_2} \dots e^{R_{k-1}} e^X e^{X_m} \\ &= e^{R_1} e^{R_2} \dots e^{R_{k-1}} e^{C(X,X_m)} \end{aligned}$$

where $\|C(X,X_m)\| < \epsilon$. Since $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ then $[X, X_m] \in \mathfrak{h}$ so that $C(X,X_m) \in \mathfrak{h}$.

Let R_k be a rational element in $\mathfrak{h}$ that is close enough to $C(X,X_m)$ and such that $\|R_k\| < \epsilon$. Then:

$$\begin{aligned} A &= e^{R_1} e^{R_2} \dots e^{R_{k-1}} e^{R_k} e^{-R_k} e^{C(X,X_m)} \\ &= e^{R_1} e^{R_2} \dots e^{R_{k-1}} e^{R_k} e^{C(-R_k, C(X,X_m))} \end{aligned}$$

where R_j are rational elements in $\mathfrak{h}$ and $C(-R_k, C(X,X_m)) \in \mathfrak{h}$. Observe that $C(-Z,Z) = log(e^{-Z} e^Z) = 0$ for all small Z. Thus the condition R_k is close enough to $C(X,X_m)$ implies that $\|C(-R_k, C(X,X_m))\| < \delta$. This completes the proof of the lemma.

□

Returning to the lemma 4.1. Fix $\delta > 0$ small enough so that for all X, Y with $\|X\|, \|Y\| < \delta$, the quantity $C(X,Y)$ is defined and contain in U. We claim that for each sequence $R_1, \dots, R_k$ of rational elements in $\mathfrak{h}$, there is at most one $X \in \mathfrak{h}$ with $\|X\| < \delta$ such that:

$$e^{R_1} e^{R_2} \dots e^{R_k} e^X \in e^V$$

To see this, suppose that,

$$e^{R_1}e^{R_2}\dots e^{R_k}e^{X_1} = e^{Y_1}$$
$$e^{R_1}e^{R_2}\dots e^{R_k}e^{X_2} = e^{Y_2}$$

where $X_1, X_2 \in \mathfrak{h}$ and $Y_1, Y_2 \in V$. Then,

$$e^{-Y_1}e^{Y_2} = (e^{-X_1}e^{-R_m}\dots e^{-R_1})(e^{R_1}e^{R_2}\dots e^{R_k}e^{X_2}) = e^{-X_1}e^{X_2}$$

and so,

$$e^{-Y_1} = e^{-X_1}e^{X_2}e^{-Y_2} = e^{C(-X_1,X_2)}e^{-Y_2} \in e^U e^V$$

But, each element of $e^U e^V$ has a unique representation as $e^X e^Y$ with $X \in U, Y \in V$. Therefore, we must have $Y_1 = Y_2$ and then $e^{X_1} = e^{X_2}$. we can choose δ sufficiently small if necessary such that exp is injective. Thus, $X_1 = X_2$.

By lemma 4.3, every element $A \in H$ can be expressed as:

$$A = e^{R_1}e^{R_2}\dots e^{R_k}e^X$$

with rational elements $R_j \in \mathfrak{h}$ and $X \in \mathfrak{h}$ with $\|X\| < \delta$.

Now, there are countably many rational elements in $\mathfrak{h}$ and thus only countably many expressions of the form $e^{R_1}e^{R_2}\dots e^{R_k}$, each of which produces at most one element $e^{R_1}e^{R_2}\dots e^{R_k}e^X \in e^V$. Thus the set:

$$E = \{Y \in V | e^Y \in H\} \text{ is at most countable.}$$

□

The following example illustrates the fact that H need not be a matirx Lie group.

Example 4.1. Consider $G = GL(2,\mathbb{C})$ and fix irrational number a, we define:

$$\mathfrak{h} = \left\{ \begin{bmatrix} it & 0 \\ 0 & ita \end{bmatrix} \middle| t \in \mathbb{R} \right\}$$

It is easy to see that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g} = \mathfrak{gl}(2,\mathbb{C})$. Suppose that $\mathfrak{h}$ is a Lie algebra of a matrix Lie subgroup $H \subset G$ then since H is closed, H would be contain the closure of the group:

$$H_1 = \left\{ \begin{bmatrix} e^{it} & 0 \\ 0 & e^{ita} \end{bmatrix} \middle| t \in \mathbb{R} \right\}$$

However, the closure of H_1 is:

$$\overline{H_1} = \left\{ \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{it\beta} \end{bmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\}$$

Thus, the Lie algebra $\mathfrak{h}$ of H must contains Lie algebra of $\overline{H_1}$ which is two dimensional. This proves the contradiction. However, the Lie subgroup of G with Lie algebra $\mathfrak{h}$ is simply the group H_1 which is connected but is not closed in G.

2. Lie Group-Lie Algebra Homomorphism Correspondence

The answer for the second question is valid if G is given to be a simply connected Lie group.

Theorem 4.2. Let G and H be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively and $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. If G is simply connected then there exists a unique Lie group homomorphism $\Phi : G \to H$ such that $\Phi(\exp X) = \exp(\phi(X))$ for all $X \in \mathfrak{g}$.

We need the following lemma.

Lemma 4.4. Suppose that $f : K \to GL(n, \mathbb{C})$ is continuous where $K \subset \mathbb{R}^m$ is compact. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that if $s, t \in K$ satisfy $\|s-t\| < \delta$, then $\|f(s)f(t)^{-1} - I\| < \epsilon$.

Proof. First, observe that

$$\begin{aligned}\|f(s)f(t)^{-1} - I\| &= \|(f(s) - f(t))f(t)^{-1}\| \\ &\leq \|f(s) - f(t)\|\|f(t)^{-1}\|\end{aligned}$$

Since the map $t \to \|f(t)^{-1}\|$ is continuous and K is compact, then there exists a constant $C > 0$ such that $\|f(t)^{-1}\| < C$. On the other hand, f is uniformly continuous. Thus, given $\epsilon > 0$ there is $\delta > 0$ such that for $s, t \in K$ with $\|s - t\| < \delta$, we have

$$\|f(s) - f(t)\| < \frac{\epsilon}{C}$$

This implies

$$\|f(s)f(t)^{-1} - I\| < \epsilon$$

□

Now, we are ready for providing the proof of the theorem. The proof of the theorem is followed from [**1**], Brian C.Hall: Lie Groups, Lie Algebras, and Representations: An Elementary Introduction.

Proof. First, note that the uniqueness of Φ is followed from the corollary 3.7 . So, we only need to prove the existence. Let U be a neighborhood of 0 in $\mathfrak{g}$ and V be a neighborhood of I in G such that $\exp : U \to V$ is diffeomorphism with the inverse $\log : V \to U$ and V is small enough such that for all $e^X, e^Y \in V$ where $X, Y \in U$ the BCH formula applies for $\log(e^X e^Y)$.

Define $\Phi_o : V \to H$ by $\Phi_0(A) = e^{\phi(\log(A))}$ that is,

$$\Phi_o = \exp \circ \phi \circ \log$$

Then Φ_o is continuous and by the corollary 4.1, Φ_o is homomorphism since for any $A = e^X, B = e^Y \in V$ with $X, Y \in U$,

$$\begin{aligned}\Phi_o(AB) &= \exp \circ \phi \circ \log(e^X e^Y) \\ &= \exp \circ \log(e^{\phi(X)} e^{\phi(Y)}) \\ &= \exp \circ \log(e^{\phi(\log A)} e^{\phi(\log B)}) \\ &= e^{\phi(\log A)} e^{\phi(\log B)} \\ &= \Phi_o(A)\Phi_o(B)\end{aligned}$$

Since G is simply connected, it is connected and also path-connected. Let $A \in G$, then there exists a path $\alpha : [0,1] \to G$ such that $\alpha(0) = I$ and $\alpha(1) = A$. From lemma 4.4, choose $0 = t_0 < t_1 < \ldots < t_n = 1$ such that for $|t_{i+1} - t_i| < \delta$, we have

$$\alpha(t_{i+1})\alpha(t_i)^{-1} \in B_\epsilon(I) \subset V \text{ for some } \epsilon > 0 \tag{4.3}$$

Thus for all s, t such that $t_i \leq s \leq t \leq t_{i+1}$, we have

$$\alpha(t)\alpha(s)^{-1} \in V \tag{4.4}$$

Since $\alpha(0) = I$ we have $\alpha(t_1) = [\alpha(t_1)\alpha(t_o)^{-1}] \in V$. Thus, $A = \alpha(1)$ can be written as:

$$A = [\alpha(1)\alpha(t_{n-1})^{-1}][\alpha(t_{n-1})\alpha(t_{n-2})^{-1}] \ldots [\alpha(t_2)\alpha(t_1)^{-1}]\alpha(t_1)$$

Therefore, we can define $\Phi : G \to H$ by:

$$\Phi(A) = \Phi_o[\alpha(1)\alpha(t_{n-1})^{-1}]\Phi_o[\alpha(t_{n-1})\alpha(t_{n-2})] \ldots \Phi_o[\alpha(t_2)\alpha(t_1)^{-1}]\Phi_o[\alpha(t_1)]$$

To prove that Φ is well-defined, it is sufficiently to prove that Φ is independence of the partition and independence of the path.

Φ is independent of the partition.

We will prove that the value of Φ does not change in the particular partition $(t_0, \ldots, t_n)$ and its refinement, that is a partition which contains all the point t_i. Now if we add a point s in $[t_i, t_{i+1}]$ then from (4.4), we have $\alpha(t_{i+1})\alpha(s)^{-1}$ and $\alpha(s)\alpha(t_i)^{-1}$ are in V and then $\Phi(A)$ in this path $(t_0, \ldots, t_i, s, t_{i+1}, \ldots, t_n)$ is equal:

$$\begin{aligned} &\Phi_o[\alpha(1)\alpha(t_{n-1})^{-1}] \ldots \Phi_o[\alpha(t_{i+1})\alpha(s)^{-1}]\Phi_o[\alpha(s)\alpha(t_i)^{-1}] \ldots \Phi_o[\alpha(t_2)\alpha(t_1)^{-1}]\Phi_o[\alpha(t_1)] \\ &= \Phi_o[\alpha(1)\alpha(t_{n-1})^{-1}] \ldots \Phi_o[\alpha(t_{i+1})\alpha(t_i)^{-1}] \ldots \Phi_o[\alpha(t_2)\alpha(t_1)^{-1}]\Phi_o[\alpha(t_1)] \end{aligned}$$

The equality follows from the fact that Φ_o is homomorphism. By repeating this argument, the value of $\Phi(A)$ is not change by adding finitely many point to a partition $(t_0, \ldots, t_n)$. Thus for any two partitions, they have common refinement; that is their union. Thus, the value of Φ in the first partition is as the same as in their common refinement and so is the same as in the second partition.

Φ is independent of the path.

Let $A \in G$ and $\alpha, \beta : [0,1] \to G$ be paths such that $\alpha(0) = \beta(0) = I$ and $\alpha(1) = \beta(1) = A$. Since G is simply connected, there exists a continuous map $H : [0,1] \times [0,1] \to G$ with

$$\begin{aligned} H(t,0) &= \alpha(t), \qquad H(t,1) = \beta(t), \qquad \forall t \in [0,1] \\ H(0,s) &= I, \qquad H(1,s) = A, \qquad \forall s \in [0,1] \end{aligned}$$

Lemma 4.4 guarantees that there exists an integer N such that for all $(t_1, s_1), (t_2, s_2) \in [0,1] \times [0,1]$ with $|t_2 - t_1| < \frac{3}{N}$ and $|s_2 - s_1| < \frac{3}{N}$, we have

$$H(t_1, s_1)H(t_2, s_2)^{-1} \in V \tag{4.5}$$

We now define the sequence of paths $B_{l,k}$ with $k = 0, \ldots, N-1$ and $l = 0, \ldots, N$ such that:

$$B_{l,k}(t) = \begin{cases} H(t, \frac{k+1}{N}), & \text{for } 0 \leq t \leq \frac{l-1}{N} \\ H(t, \frac{k}{N}), & \text{for } \frac{l}{N} \leq t \leq 1 \\ \text{H(t,s)}, & \text{for the values of } (t,s) \text{ that goes diagonally in (t,s)-plane from } (\frac{l-1}{N}, \frac{k+1}{N}) \text{ to } (\frac{l}{N}, \frac{k}{N}) \end{cases}$$

Observe that when $l = 0$, there is no t between 0 and $\frac{l-1}{N}$ then $B_{0,k} = H(t, \frac{k}{N})$ for all $t \in [0, 1]$ and thus $B_{0,0} = H(t, 0) = \alpha(t)$. See figure 6.

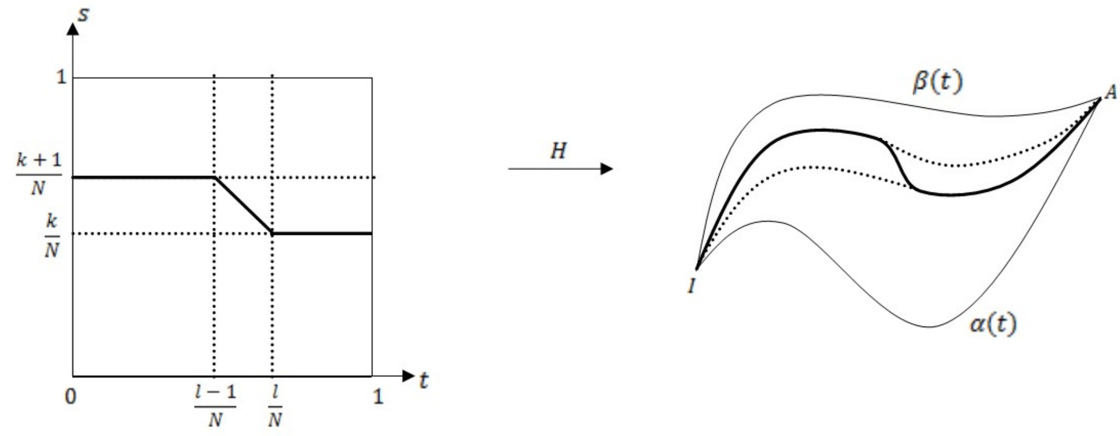


FIGURE 6. The sequence of path $B_{l,k}$

We will deform the path α to the path β in steps. The first step, we will deform $B_{l,k}$ to $B_{l+1,k}$ and the second step, we will deform $B_{N,k}$ to $B_{0,k+1}$. Thus we can deform $\alpha = B_{0,0}$ into $B_{1,0}$ until $B_{N,0}$ (in step 1) and then from $B_{N,0}$ to $B_{0,1}$ (in step 2) and then deform to $B_{1,1}$ until $B_{N,1}$. We repeat this procedure until we reach $B_{N,N-1}$ and finally deform $B_{N,N-1}$ to β. The following figure illustrates our process.

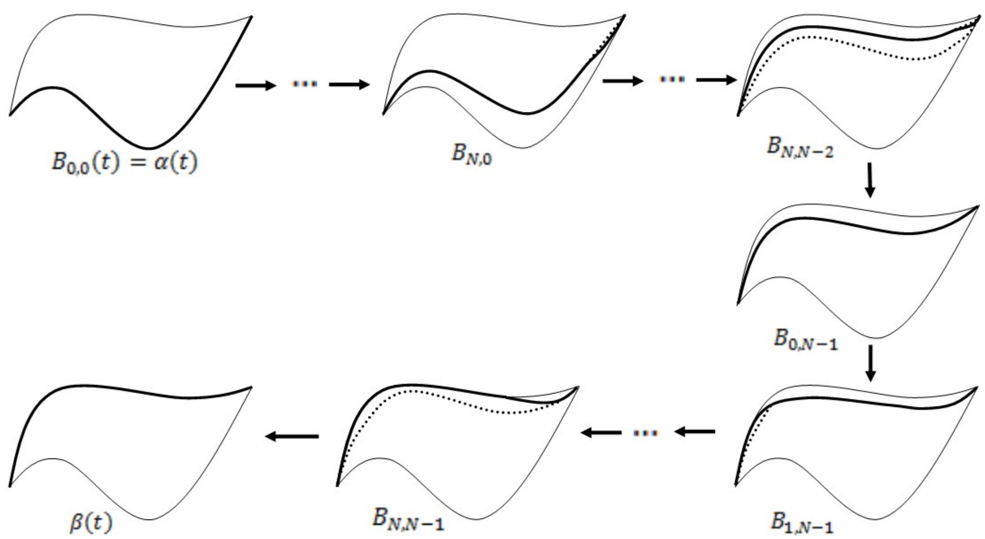
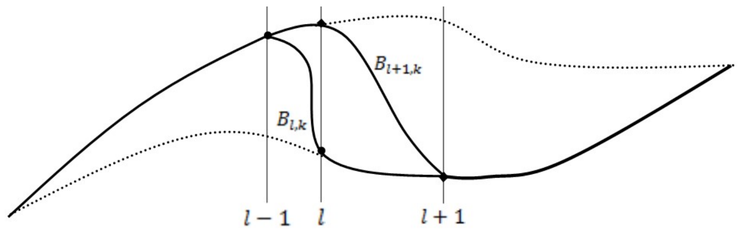


FIGURE 7. The deformation from $B_{0,0}(t) = \alpha(t)$ to $\beta(t)$

We only need to prove that in each step the value of $\Phi(A)$ does not change.

Step 1: Observe that the paths $B_{l,k}$ and $B_{l+1,k}$ are coincide except for $l - 1 < t < l + 1$



Since Φ is independent of partition, we choose the partition:

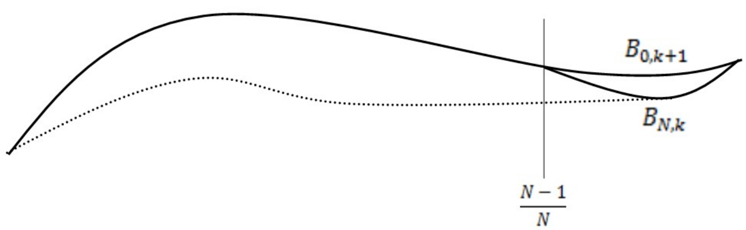
$$t_0 = 0, t_1 = \frac{1}{N}, \ldots, t_{l-1} = \frac{l-1}{N}, t_l = \frac{l+1}{N}, \ldots, t_{N-1} = 1$$

for both $B_{l,k}$ and $B_{l+1,k}$. Note that the distances between two consecutive points in this partition less than $\frac{3}{N}$. From (4.5), when $s_1 = s_2 = 0$ and when $s_1 = s_2 = 1$, for all $t_i \leq t \leq t' \leq t_{i+1}$, we have:

$$\alpha(t')\alpha(t)^{-1} \in V \text{ and } \beta(t')\beta(t)^{-1} \in V$$

Thus, the value of $\Phi(A)$ with respects to the path $B_{l,k}$ and the path $B_{l+1,k}$ are the same since $\alpha(t_i) = \beta(t_i)$ for all $i = 0, \ldots, N-1$.

Step 2: The paths $B_{N,k}$ and $B_{0,k+1}$ are coincide except for $\frac{N-1}{N} < t < N$.



Using the same partition for both $B_{N,k}$ and $B_{0,k+1}$, the value of $\Phi(A)$ is unchange from the path $B_{N,k}$ to the path $B_{0,k+1}$ and so is unchange from the path $B_{N,N-1}$ to the path $\beta(t)$.

Φ is a Lie homomorphism.
Let $A, B \in G$ and $\alpha, \beta : [0,1] \to G$ be paths in G such that $\alpha(0) = \beta(0) = I$ and $\alpha(1) = A, \beta(1) = B$. Define $\gamma : [0,1] \to G$ by:

$$\gamma(t) = \begin{cases} \beta(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \alpha(2t-1)B & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

γ is a path connecting I to AB.

Let $(0 = s_0, s_1, \ldots, s_m = 1)$ and $(0 = t_0, t_1, \ldots, t_n = 1)$ be partitions of α and β respectively that satisfy (4.3). we claim that:

$$\frac{t_0}{2}, \ldots, \frac{t_n}{2}, \frac{1+s_0}{2}, \ldots, \frac{1+s_m}{2}$$

is a partition of γ that also satisfies (4.3). To see this, note that for $i = 0, \ldots, n-1$ and $j = 0, \ldots, m-1$, we have:

$$\gamma\left(\frac{t_{i+1}}{2}\right)\gamma\left(\frac{t_i}{2}\right)^{-1} = \beta(t_{i+1})\beta(t_i)^{-1} \in V$$

$$\gamma\left(\frac{1+s_{i+1}}{2}\right)\gamma\left(\frac{1+s_i}{2}\right)^{-1} = \alpha(s_{i+1})BB^{-1}\alpha(s_i)^{-1} = \alpha(s_{i+1})\alpha(s_i)^{-1} \in V$$

$$\gamma\left(\frac{1+s_0}{2}\right)\gamma\left(\frac{t_n}{2}\right)^{-1} = \alpha(0)B\beta(1)^{-1} = I \in V$$

Thus,

$$\begin{aligned}\Phi(AB) &= \Phi_o\left[\gamma\left(\frac{1+s_m}{2}\right)\gamma\left(\frac{1+s_{m-1}}{2}\right)^{-1}\right]\ldots\Phi_o\left[\gamma\left(\frac{t_1}{2}\right)\right] \\ &= \left(\Phi_o\left[\alpha\left(s_m\right)\alpha\left(s_{m-1}\right)^{-1}\right]\ldots\Phi_o\left[\alpha\left(s_1\right)\right]\right)\left(\Phi_o\left[\beta\left(t_n\right)\beta\left(t_{n-1}\right)^{-1}\right]\ldots\Phi_o\left[\beta\left(t_1\right)\right]\right) \\ &= \Phi(A)\Phi(B)\end{aligned}$$

Since Φ is smooth by its definition then Φ is a Lie group homomorphism.

Φ **satisfies** $\Phi \circ \exp = \exp \circ \phi$**.**
Since Φ is Lie group homomorphism, theorem 3.1 implies that $d\Phi_I$ is the associated Lie algebra homomorphism that satifies:

$$\Phi \circ \exp = \exp \circ d\Phi_I$$

However, $\Phi(A) = \Phi_o(A) = \exp \circ \phi \circ \log(A)$ for $A \in V$ (A near the identity) then

$$\begin{aligned} d\Phi_I(X) &= \left.\frac{d}{dt}\right|_{t=0}\Phi_o(e^{tX}) \\ &= \left.\frac{d}{dt}\right|_{t=0} e^{\phi(tX)} \\ &= \phi(X)\end{aligned}$$

Thus,

$$\Phi \circ \exp = \exp \circ \phi$$

This completes the proof.

□

Corollary 4.2. Let G and H be simply connected matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. If $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic then so are G and H.

Proof. Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism with the inverse Lie algebra homomorphism $\phi^{-1} : \mathfrak{h} \to \mathfrak{g}$. From theorem, there exist Lie group homomorphisms $\Phi : G \to H$ and $\Psi : H \to G$ such that:

$$\Phi \circ \exp = \exp \circ \phi \qquad \text{and} \qquad \Psi \circ \exp = \exp \circ \phi^{-1}$$

From corollary 3.5, Lie algebra homomorphism associated to $\Phi \circ \Psi$ is $\phi \circ \phi^{-1} = Id_{\mathfrak{g}}$ and Lie algebra homomorphism associated to $\Psi \circ \Phi$ is $\phi^{-1} \circ \phi = Id_{\mathfrak{h}}$. From corollary 3.7, we obtain $\Phi \circ \Psi = Id_G$ and $\Psi \circ \Phi = Id_H$ and thus G is isomorphic to H.

□

The following example shows that the condition that G is simply connected in the theorem cannot be omitted.

Example 4.2. Consider the matrix Lie groups $SO(3)$ and $SU(2)$. The Lie algebra $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are both 3-dimensional real vector spaces with the following bases:

$$\mathfrak{so}(3): \quad E_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathfrak{su}(2): \quad F_1 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad F_2 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad F_3 = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

The direct calculations show that the non-trivial Lie brackets among these are:

$$[E_1, E_2] = E_3 \qquad [E_2, E_3] = E_1 \qquad [E_3, E_1] = E_2$$
$$[F_1, F_2] = F_3 \qquad [F_2, F_3] = F_1 \qquad [F_3, F_1] = F_2$$

This implies that the real linear isomorphism

$$\phi : \mathfrak{su}(2) \to \mathfrak{so}(3), \phi(a_1E_1 + a_2E_2 + a_3E_3) = a_1F_1 + a_2F_2 + a_3F_3$$

satisfies:

$$\phi[X, Y] = [\phi(X), \phi(Y)]$$

This proves that $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are (Lie algebra) isomorphic. Now, suppose that the theorem is still valid although G is not simply connected. From the corollary, we obtain that $SO(3)$ and $SU(2)$ are (Lie group) isomorphic. However, this is a contradiction since $SU(2)$ is simply connected but $SO(3)$ is not (from the table in chapter 2).

Conclusion:

Let G be a matrix Lie group. If G is connected then there is a one-one correspondence between G and its Lie algebra $\mathfrak{g}$. Thus, we can study some properties on G (such as abelian, nilpotent and solvable) by studying those properties on $\mathfrak{g}$ and viceversa.

Similarly, if G and H are matrix Lie groups such that G is simply connected and $\Phi : G \to H$ is a Lie group homomorphism, then there is a one-one correspondence between Φ and its Lie algebra homomorphism associated $\phi : \mathfrak{g} \to \mathfrak{h}$. Thus, some properties on Φ (such as representation) can be done by studying those on ϕ and viceversa.

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