CIMAT

Centro de Investigación en Matemáticas, A.C.

# CHAOS AND PERIOD FORCING OF A FAMILY OF PIECEWISE MONOTONIC REAL-VALUED FUNCTIONS

**T H E S I S**

As a requirement to obtain the degree of

**Master of Science in Mathematics**

With speciality in

**Pure Mathematics**

**Thesis presented by**

Sophea Sin

**Under the advise of**

Dra. Mónica Moreno Rocha

Guanajuato, Gto., October 11th, 2017

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**Authorization of the final version**

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# Contents

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# Introduction

In this work we consider a family of one-real parameter of piecewise monotone real-valued functions given by $f_b(x) = \wp_\Lambda(x) + b$, where $b$ is a real parameter and $\wp_\Lambda$ denotes the Weierstrass $\wp$ function defined over a real square lattice $\Lambda$, and restricted over the real line.

Each element in the family $f_b$ defines a periodic function over the real line with singularities at the integer multiples of its real period. When restricted to a fundamental interval, the family $f_b$ exhibits some dynamical similarities to the quadratic family $Q_c(x) = x^2 + c$. One of the main problems addressed in this thesis is to show that under certain conditions on the parameter $b$ and the lattice $\Lambda$, $f_b$ acts over the real line as a **chaotic dynamical system**.

The second problem considered in this work is related to Sharkovskii's Theorem, one of the most celebrated theorems in real dynamics. This theorem states a **period forcing** result: if $f$ is a continuous function over the real line that has a periodic point of period $n$, then it must also have a periodic point of period $k$, with $k$ smaller than $n$ in the **Sharkovskii ordering**. Taking into account that each $f_b$ is no longer continuous in the whole real line, we provide a partial period forcing result for the family $f_b$ following Sharkovskii's ordering.

Chapter 1 presents fundamental concepts in real discrete dynamical systems. We start by considering the family of quadratic functions defined by

$$Q_c(x) = x^2 + c$$

with $c$ is a real parameter. This family serves as a model example that will allow us to understand the dynamical behavior of one-parameter family $f_b$ in Chapter 3. As the parameter $c$ changes, the dynamics of $Q_c$ also changes remarkably for certain values of $c$. The noticeable dynamics happens when $c < -2$. But for simplicity, we are interested in studying the family only with the case $c < -\frac{5 + 2\sqrt{5}}{4}$. Our aim is to define and understand the properties of a chaotic dynamical system. According to R. L. Devaney in [3] and [4], a dynamical system $(X, f)$ is said to be chaotic if the following three conditions are satisfied: **density of periodic points**, **transitivity** and **sensitivity to initial conditions**. To achieve our purpose, the concept of **symbolic dynamics** is introduced, which plays a very important role in order to prove a system is chaotic. For this task, we construct an **itinerary map** that defines a conjugacy between the action of $Q_c$ restricted to the invariant set $\Lambda$ and the symbolic system $(\Sigma_2, \sigma)$. The main result in this section is that each element of the quadratic family is chaotic whenever

$c < -\dfrac{5+2\sqrt{5}}{4}$.

Chapter 1 ends with an introduction to elliptic functions and properties of the Weierstrass $\wp$ function, defined by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda-\{0\}} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right],$$

where $\Lambda = \{m\lambda_1 + n\lambda_2 \mid m, n \in \mathbb{Z}\}$ is a lattice over the complex plane $\mathbb{C}$. Two straightforward properties of the Weierstrass $\wp$ elliptic function are that $\lambda_1$ and $\lambda_2$ are its periods and has double poles at its lattice points.

In Chapter 2, we provide a complete proof of Sharkovskii's theorem which plays an important role in real dynamics. The Sharkovskii's ordering is an ordering of all natural numbers in a little bit strange way as follows:

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \rhd \cdots 2^p \rhd 2^{p-1} \rhd \cdots 2^2 \rhd 2 \rhd 1.$$

The proof is based on the idea of P. Stefan given in [11] and also described in [9]. This technique is based on the action of the function $f$ restricted to a given interval whose image may contain another interval. For an $n$-cycle with $n$ odd, the **transition graph** of a function $f$ over a partition of intervals determined by the $n$-cycle contains a special subgraph called **Stefan's transition graph**. This allows us to prove the period forcing for a real-valued continuous function $f : \mathbb{R} \to \mathbb{R}$.

In Chapter 3, we study the dynamics of the family $f_b(x) = \wp_\Lambda(x) + b$, where $\Lambda$ is the **central real square lattice**. This lattice has been introduced in [7] and has the property that the real critical point $\lambda/2$ is fixed under $\wp_\Lambda$. Here we provide the main results of our thesis. First, we study the period forcing properties of each element of the family $f_b$ for any $b \in \mathbb{R}$. Under this assumption and the restriction of $\wp_\Lambda$ to the real line, we obtain a complete directed graph, whose vertices are fundamental intervals and the arrows represent when the image of one interval under $f_b$ contains another interval. Ultimately, we end up that regardless of parameter $b$, each member $f_b$ has the property that for any periodic point of odd period $n$, then there exists a point of period $k$ with $k$ smaller that $n$ in the Sharkovskii's ordering.

On the other hand, for the second result, we just consider the case $b = 0$. It is shown that on each fundamental interval $I_j = (j\lambda, (j+1)\lambda)$ for any integer $j$, the function $\wp|_{I_j}$ is piecewise monotone and unicritical, thus, dynamically similar to the quadratic family studied in Chapter 1. The results presented in Chapter 1 inspire the proof that $f_b$ exhibits a chaotic behavior when restricted to an invariant subset of the real line. Unlike the quadratic case, each element $f_b$ family is piecewise continuous with discontinuities at $\lambda$ and its integer multiples. However, as Theorem 3.21 shows, on each fundamental interval $I_j$ there exists an invariant set $\Gamma_j$ on which $f_b : \Gamma_j \to \Gamma_j$ is chaotic, for each $j \geqslant 1$.

# Chapter 1

# Preliminaries

In this section we gather the fundamental concepts of dynamical systems and several necessary results of a particular complex-valued function called Weierstrass $\wp$ function which is an elliptic function with poles on the associated lattice. Here we also mainly discuss the meaning of chaotic dynamical system. The basis of the material that we present in this chapter come principally from [12] and [9].

## 1.1 Discrete dynamical systems

Through out this work we are only interested in a real-valued function of one variable as encountered in elementary calculus. We will start our work by discussing the **quadratic** family $Q_c(x) = x^2 + c$ where $c \in \mathbb{R}$ is constant. One of the important questions we will address in this chapter is how the dynamics of the function changes as the parameter changes.

The **n-iteration** of a function $f$ is a composition of the function $f$ with itself $n$ times.

**Example 1.1.** For $Q_c(x) = x^2 + c$, the second iteration of $Q_c$ is $Q_c^2(x) = Q_c(Q_c(x)) = (x^2 + c)^2 + c = x^4 + 2cx^2 + c^2 + c$ and so forth.

Let $f : \mathrm{D}(f) \to \mathbb{R}$ be a real-valued function with domain $\mathrm{D}(f) \subset \mathbb{R}$ an open and non-empty set. Given $x_0 \in \mathrm{D}(f)$, the **orbit** of $x_0$ under $f$, denoted by $\mathcal{O}_f(x_0)$, is the set

$$\mathcal{O}_f(x_0) = \{x_0, f(x_0), f^2(x_0), \cdots\},$$

whenever the iterate $f^n(x_0)$ is defined for all $n > 0$. A point $x_0$ is called a **fixed point** if $f(x_0) = x_0$. Since the fixed points are roots of the algebraic equation $f(x) = x$, thus geometrically we could examine the fixed points of a function by finding the intersection points between the graph of function and the diagonal line $y = x$.

Another important type of orbit is a **periodic orbit** or a **cycle**. A point $x_0$ is called **periodic point** of period $n \in \mathbb{N}$ if $f^n(x_0) = x_0$ for some $n$. The minimum $n$ that satisfied this equation is called **least** period of $x_0$.

A point $x_0$ is called **eventually fixed** and **eventually peroidic** point if it is not fixed nor periodic but for some point in the orbit of $x_0$ it becomes fixed or periodic. In addition, let us look at the following example to see other types of orbit.

**Example 1.2.** Let $f$ and $g$ be real-valued functions on the real line defined by

$$f(x) = 3x \text{ and } g(x) = x/3.$$

$f$ and $g$ has a single fixed point at $x = 0$. But for any point $x \neq 0$, we have

$$|f^n(x)| = |3^n x| \to \infty \text{ as } n \to \infty;$$

$$|g^n(x)| = |x/3^n| \to 0 \text{ as } n \to \infty.$$

We have discussed several types of orbit such as fixed, periodic, eventually fixed, eventually periodic, tending to infinity and tending to a specific value. Still, there are some more complicated orbits as we will see in Section 1.2.2.

**Theorem 1.3** (**Fixed Point Theorem**)**.** Let $f : [a, b] \to [a, b]$ be continuous. Then there is a fixed point for $f$ in $[a, b]$.

*Proof.* Let $g(x) = f(x) - x$. Since $f$ is continuous, then $g$ is continuous. Also we have $g(a) = f(a) - a \geqslant 0$ and $g(b) = f(b) - b \leqslant 0$. Then it follows by the Intermediate Value Theorem that there exists a point $c$ with $a < c < b$ such that $g(c) = 0$, that is, $f(c) = c$. Hence there is a fixed point for $f$ in $[a, b]$. □

**Definition 1.4.** Let $x_0$ be a fixed point for $f$. There are three remarkable types of fixed points. The point $x_0$ is an **attracting**, **repelling** or **neutral** fixed point if $|f'(x_0)| < 1$, $|f'(x_0)| > 1$ or $|f'(x_0)| = 1$, respectively.

We say that the point $x_0$ is **super-attracting** fixed point if $|f'(x_0)| = 0$, that is, $x_0$ is attracting and also a critical point of $f$.

**Theorem 1.5** (**Mean Value Theorem**). Suppose that $f$ is real-valued function which a continuous function on the closed interval $[a, b]$ and differentiable at any point on the open interval $(a, b)$. Then there exists $c$ such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This theorem simply says that the slope of line connecting the endpoints of the closed interval is the same as the derivative of the function at some interior point of the interval.

**Theorem 1.6** (**Attracting Fixed Point Theorem**). Let $x_0$ be an attracting fixed point for a function $f$. Then there exists a neighborhood $I$ of $x_0$ such that for every $x \in I$ then $f^n(x) \in I$ for all $n \in \mathbb{N}$ and moreover $f^n(x) \to x_0$ as $n \to \infty$.

**Theorem 1.7** (**Repelling Fixed Point Theorem**). Let $x_0$ be a repelling fixed point for a function $f$. Then there exists a neighborhood $I$ of $x_0$ such that for every $x \in I - \{x_0\}$, there exists $N = N(x) \in \mathbb{N}$ such that $f^N(x) \notin I$.

This theorem tells us that for any point in the deleted neighborhood of $x_0$, the iteration will leave that interval for some large enough iteration.

**Theorem 1.8** (**Chain Rule**). Let $x_0, x_1, \cdots, x_{n-1}$ be a cycle of period $n$ for a function $f$ with $x_j := f^j(x_0)$. Then

$$(f^n)'(x_0) = \prod_{j=0}^{n-1} f'(x_j).$$

This theorem allow us to classify a periodic point $x_0$ of period $n$ as either attracting, repelling, or neutral by considering absolute value of $(f^n)'$ evaluated at any point in the cycle.

**Corollary 1.9.** Let $x_0, x_1, \cdots x_{n-1}$ lie on a cycle of period $n$ for a function $f$. Then

$$(f^n)'(x_0) = (f^n)'(x_1) = \cdots = (f^n)'(x_{n-1}).$$

**Example 1.10.** For the quadratic family $Q_c$, we have two fixed points: the solutions of the equation $Q_c(x) = x$ are given by $p_- = \frac{1}{2}(1 - \sqrt{1-4c})$ and $p_+ = \frac{1}{2}(1 + \sqrt{1-4c})$. In order to classify these fixed points, observed that $Q_c'(x) = 2x$. Then

$$Q_c'(p_-) = 1 - \sqrt{1-4c}\ ,\ Q_c'(p_+) = 1 + \sqrt{1-4c}.$$

Analyzing the expression of these derivatives, we can conclude the following:

- If $c > 1/4$, there is no fixed point and all orbits tend to infinity.
- If $c < 1/4$, $Q_c$ has two fixed points at $p_-$ and $p_+$.
- If $c = 1/4$, $Q_c$ has a single fixed point at $p_+ = p_- = \frac{1}{2}$ that is neutral.

The fixed point $p_+$ is always repelling. If $c < -1/4$ we can classify $p_-$ in terms of $c$ as:

- If $-3/4 < c < 1/4$, $p_-$ is attracting.
- If $c < -3/4$, $p_-$ is repelling.
- If $c = -3/4$, $p_-$ is neutral.

For further discussion see [4].

### 1.1.1 Bifurcations

**Definition 1.11.** A one-parameter family of functions $F_c$ goes through a **saddle-node** bifurcation at the parameter value $c_0$ if there is an open interval $I$ and an $\epsilon > 0$ such that:

1. For $c_0 < c < c_0 + \epsilon$, $F_c$ has no fixed points in the interval $I$.
2. For $c = c_0$, $F_c$ has one fixed point in $I$ and this fixed point is neutral.
3. For $c_0 - \epsilon < c < c_0$, $F_c$ has two fixed points in $I$, one attracting and one repelling.

From the previous analysis of the quadratic family $Q_c(x) = x^2 + c$, we see that $Q_c$ goes through a saddle-node bifurcation at $c = 1/4$ choosing $\epsilon = 1/2$.

Periodic points of period 2 are the solutions to the equation $Q_c^2(x) = x$. A quick computation shows the existence of two periodic points of least period 2 given by

$$q_- = \frac{1}{2}(-1 - \sqrt{-4c-3}) \text{ and } q_+ = \frac{1}{2}(-1 + \sqrt{-4c-3}).$$

For the family $Q_c(x) = x^2 + c$ :

- For $-3/4 < c < 1/4$, $Q_c$ has an attracting fixed point at $p_-$ and no 2-cycle.
- For $c = -3/4$, $Q_c$ has a neutral fixed point at $p_- = q_\pm$ and no 2-cycle.
- For $-5/4 < c < -3/4$, $Q_c$ has repelling fixed point at $p_-$ and attracting 2-cycle at $q_\pm$.

**Definition 1.12.** A one-parameter family of functions $F_c$ goes through a **period-doubling** bifurcation at the parameter value $c = c_0$ if there is an open interval $I$ and an $\epsilon > 0$ such that:

1. For each $c$ in the interval $[c_0 - \epsilon,\ c_0 + \epsilon]$, there is a unique fixed point $p_c$ for $F_c$ in $I$.
2. For $c_0 < c < c_0 + \epsilon$, $p_c$ is attracting and $F_c$ has no cycles of period 2 in $I$.
3. For $c_0 - \epsilon < c < c_0$, there is a unique 2-cycle $q_c^1$ , $q_c^2$ in $I$ with $F_c(q_c^1) = q_c^2$. This 2-cycle is attracting. Meanwhile, the fixed point $p_c$ becomes repelling.
4. As $c \searrow c_0$, we have $q_c^i \to p_{c_0}$.

Thus the quadratic family $Q_c(x) = x^2 + c$ goes through period-doubling bifurcation at $c = -3/4$ choosing $\epsilon = 1/2$.

### 1.1.2 Invariant set $\Lambda_c$

In this section we consider only the case when $c < -2$. Form Example 1.10 $Q_c$ has two fixed points, $p_-$ and $p_+$. Let $I = [-p_+, p_+]$ and let us consider the square formed by vertices $(p_+, p_+)$, $(-p_+, p_+)$, $(p_+, -p_+)$ and $(-p_+, -p_+)$. It follows that $p_-$ is in the square since $|p_-| < p_+$. The intersection points of the graph of quadratic function $Q_c$ and the bottom edge of the square are roots of the equation

$$x^2 + c = -p_+.$$

Solving this equation we obtain

$$x_-^1 = -\sqrt{\frac{-2c - 1 + \sqrt{1-4c}}{2}} \text{ and } x_+^1 = \sqrt{\frac{-2c - 1 + \sqrt{1-4c}}{2}}. \tag{1.1}$$

Since the quadratic map is even, then $Q_c(x_-^1) = Q_c(x_+^1) = \frac{-1+\sqrt{1-4c}}{2}$, that is, $Q_c(x_-^1) = Q_c(x_+^1) = -p_+$. Observe that $-p_+ = -\frac{1}{2}(1 + \sqrt{1-4c}) > c$ for $c < -2$. This inequality is equivalent to $4c(c+2) < 0$ whenever $c < -2$. Thus, the points $x_-^1$ and $x_+^1$ lie outside the interval $I = [-p_+, p_+]$.

Let us denote $A_1 = (x_-^1, x_+^1)$ the set of points that escape from interval $I$ just after one iteration of $Q_c$. For each $n \geqslant 1$ let

$$A_n = \{x \in I \mid Q_c^k(x) \in I, k = 1, 2, \cdots, n-1, Q_c^n(x) \notin I\}.$$

Denote by $\Lambda_c$ the set of points in $I$ that never leave the interval $I$, that is,

$$\Lambda_c = \{x \in I \mid Q_c^n(x) \in I, \forall n \in \mathbb{N}\}.$$

Notice that $p_-$ and $p_+$ lie in $\Lambda_c$.

**Definition 1.13.** A set $C$ is a **Cantor Set** if it is **nonempty**, **closed**, **totally disconnected** and **perfect**. A set is totally disconnected if it contains no interval. A set is perfect if every point in it is an accumulation point or a limit point of other points in the set. Equivalently, a set is perfect if it is closed and contains no isolated points.

**Example 1.14.** The Cantor Middle-Thirds set is a Cantor set. And any other Cantor set is homeomorphic to the Cantor Middle-Thirds set as the one below.

**Theorem 1.15.** For $c < -2$, $\Lambda_c$ is nonempty, closed, totally disconnected and perfect.

*Proof.* First, we have $p_-$ and $p_+$ are fixed points inside $I_0$. So $p_-$ and $p_+$ lie in $\Lambda_c$. Now, notice that $\Lambda_c = I - \bigcup_{n=1}^{\infty} A_n$ and since $A_n$ is open for every $n \in \mathbb{N}$, it follows that $\Lambda_c$ is closed.

Suppose that $\Lambda_c$ contains an interval $J$ with $l = \text{length}(J) > 0$ (when $l = 0$, the interval $J$ reduces to a point). By Mean Value Theorem, for any two points $x, y \in J$ such that $x \neq \pm y$, then there exists a point $z$ in an open interval with endpoints $x$ and $y$ such that

$$\left|Q_c'(z)\right| = \frac{\left|Q_c(x) - Q_c(y)\right|}{|x-y|}.$$

For simplicity, let us assume that $c < -\frac{5+2\sqrt{5}}{4}$. Then for any $x$ lies in $I_0$ or $I_1$, we have $|Q_c'(x)| > 1$. Now we can choose $\lambda > 1$ such that $|Q_c'(x)| > \lambda$. Then it follows that $\left|Q_c(x) - Q_c(y)\right| > \lambda|x-y|$. By definition of $\Lambda_c$ and $J \subset \Lambda_c$, we have $Q_c(J) \subset \Lambda_c$. Then $Q_c(x)$ and $Q_c(y)$ lie in $\Lambda_c$ with $Q_c(x) \neq \pm Q_c(y)$ since $Q_c$ is even. Applying Mean Value Theorem again, we have

$$\frac{\left|Q_c^2(x) - Q_c^2(y)\right|}{\left|Q_c(x) - Q_c(y)\right|} > \lambda.$$

Then $\left|Q_C^2(x) - Q_C^2(y)\right| > \lambda\left|Q_c(x) - Q_c(y)\right| > \lambda^2|x-y|$. For $n$ iteration, we have

$$\left|Q_c^n(x) - Q_c^n(y)\right| > \lambda^n|x-y|.$$

Then

$$\lim_{n\to\infty}\left|Q_c^n(x) - Q_c^n(y)\right| > \lim_{n\to\infty} \lambda^n|x-y|.$$

Since $|x-y| > 0$ and $\lambda > 1$, then $\lambda^n|x-y| \to \infty$. But $\lim_{n\to\infty}\left|Q_c^n(x) - Q_c^n(y)\right| < l$ and $l$ is finite. This gives us a contradiction and thus $\Lambda_c$ contains no interval.

Finally, we will prove that $\Lambda_c$ is perfect. We have seen that it is closed, so it remains to see that it does not contain an isolated point. First notice that $Q_c$ has a single critical point at $x = 0$ and the critical value $Q_c(0) = c$ lies outside the interval $I = [-p_+, p_+]$. Also, the endpoints of the intervals $A_n$ belong to $\Lambda_c$ for all $n \geqslant 1$. Moreover, if $x$ is an endpoint of $A_n$, then $Q_c^n(x) = -p_+$ and $Q_c^{n+1}(x) = p_+$. Indeed, if $n = 1$ we have $A_1 = (x_-^1, x_+^1)$ where $x_-^1$ and $x_+^1$ given in Equation 1.1. We then have $Q_c(x_-^1) = Q_c(x_-^1) = -p_+$ so that $Q_c^2(x_-^1) = Q_c^2(x_-^1) = p_+$. Assume that it is true for case $n$. To prove the case $n+1$, assume that $x$ is the endpoint of $A_{n+1}$. Then $x = Q_c^{-1}(y)$ for some $y$ which is the endpoint of $A_n$. Now $Q_c^{n+1}(x) = Q_c^{n+1}(Q_c^{-1}(y)) = Q_c^n(y) = -p_+$ and so $Q_c^{n+2}(x) = p_+$.
Now, assume $p \in \Lambda_c$ is an isolated point. We have two cases as follows. First, if $p$ is not an endpoint of any interval $A_n$, then there exists a sequence of endpoints $a_j$ that converges to $p$. This implies that $p$ is a limit point of elements in $\Lambda_c$, which is a contradiction. Second, assume $p$ is an endpoint of $A_k$ for some $k \geqslant 1$. Then, there exists a neighborhood $N(p)$ of $p$ such that $N(p) \cap A_k \neq \emptyset$. Then for every $x \in N(p)$, $Q_c^k(x) < Q_c^k(p) = -p_+$. This says that $p$ is a local maximum of $Q_c^k|N(p)$. That is, $(Q_c^k)'(p) = 0$ and by the chain rule,

$$(Q_c^k)'(p) = \prod_{j=0}^{k-1} Q_c'(Q_c^j(p)) = 0$$

so there must exist at least one $j \in \{0, \ldots, k-1\}$ such that $Q_c'(Q_c^j(p)) = 0$. In other words, $Q_c^j(p) = 0$. But then, $Q_c^{j+1}(p) = c \notin I$, a contradiction as $p \in \Lambda_c$ and hence all its iterates remain in $\Lambda_c$. We completed the second case and thus the theorem is proved.

□

**Theorem 1.16.** Let $c < -2$. The invariant set $\Lambda_c$ for a quadratic map $Q_c$ is a Cantor set.

## 1.2 Symbolic and chaotic dynamics

### 1.2.1 Symbolic dynamics

Let $I_0$ and $I_1$ be two closed intervals lying respectively the left and the right of $A_1$, so that $I_0 \cup A_1 \cup I_1 = I$.

**Definition 1.17.** Let $x \in \Lambda_c$. The **itinerary** of $x$ is the infinite sequence given by

$$S(x) = (s_0 s_1 s_2 \cdots)$$

where $s_j = 0$ if $Q_c^j(x) \in I_0$ and $s_j = 1$ if $Q_c^j(x) \in I_1$.

**Example 1.18.** $S(p_+) = (111\cdots)$, $S(-p_+) = (011\cdots)$.

**Definition 1.19.** The **sequence space** on two symbols is the set

$$\Sigma_2 = \{(s_0 s_1 s_2 \cdots) | s_j = 0 \text{ or } 1\}.$$

Let $s = (s_0 s_1 s_2 \cdots)$ and $t = (t_0 t_1 t_2 \cdots)$ be two points in $\Sigma_2$. The **distance** between $s$ and $t$ is given by

$$d(s,t) = \sum_{j=0}^{\infty} \frac{|s_j - t_j|}{2^j}.$$

**Proposition 1.20.** $(\Sigma_2, d)$ is a metric space, and it is homeomorphic to the Cantor set.

**Lemma 1.21.** Let $s = (s_j)$ and $t = (t_j)$ be two points in $\Sigma_2$. If $s_j = t_j$ for $j = 0, 1, \cdots, n$, then $d(s,t) \leqslant \frac{1}{2^n}$. Conversely, if $d(s,t) < \frac{1}{2^n}$, then $s_j = t_j$ for $j = 0, 1, \cdots, n$.

**Definition 1.22.** The **shift map** $\sigma : \Sigma_2 \to \Sigma_2$ is defined by

$$\sigma(s_0 s_1 s_2 \cdots) = (s_1 s_2 s_3 \cdots).$$

**Proposition 1.23.** The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is continuous.

*Proof.* Let $\epsilon > 0$ and $s \in \Sigma_2$. There is a natural number $n$ such that $\frac{1}{2^n} < \epsilon$. We take $\delta = \frac{1}{2^{n+1}}$. Then for any point $t$ in $\Sigma_2$ such that $d(s,t) < \delta$, by Lemma 1.21, we have $s_j = t_j$ for $j = 0, 1, \cdots, n+1$. In other words, $t = (s_0 \cdots s_{n+1} t_{n+2} \cdots)$. Applying the shift $\sigma$ on $t$, we get $\sigma(t) = (s_1 \cdots s_{n+1} t_{n+2} t_{n+3} \cdots)$. But $\sigma(s) = (s_1 \cdots s_{n+1} s_{n+2} s_{n+3} \cdots)$. Since $\sigma(s)$ and $\sigma(t)$ coincide in the first $n+1$ terms, by Lemma 1.21, we obtain

$$d(\sigma(s), \sigma(t)) \leqslant \frac{1}{2^n} < \epsilon.$$

Hence, this completes the proof. □

### 1.2.2 Chaotic dynamics

**Definition 1.24.** Let $X$ and $Y$ be topological spaces. Suppose that $F : X \to X$ and $G : Y \to Y$ are two continuous functions. We say that $F$ and $G$ are **conjugate** if there is a homeomorphism $h : X \to Y$ such that $h \circ F = G \circ h$. The map $h$ is called a **conjugacy**.

**Theorem 1.25.** Considering a family of nonempty, closed and bounded intervals $I_n = [a_n, b_n]$, where $n \in \mathbb{N}$. If for every $n$, we have $a_n < a_{n+1} < b_{n+1} < b_n$, then

$$\bigcap_{n=0}^{\infty} I_n \neq \varnothing.$$

The main goal of this section is to prove the following results.

**Theorem 1.26.** Suppose $c < -\dfrac{5+2\sqrt{5}}{4}$. Then the itinerary map $S : \Lambda_c \to \Sigma_2$ is a homeomorphism.

*Proof.* We will prove that $S$ is one-to-one, onto and continuous together with a continuous inverse $S^{-1}$.

**One-to-one**: Let $x, y \in \Lambda_c$ with $x \neq y$. Suppose that $S(x) = S(y)$. This means that $Q_c^n(x)$ and $Q_c^n(y)$ always lie in the same subinterval $I_0$ or $I_1$ for every $n \in \mathbb{N}$. We know that $Q_c$ is one-to-one on each of these intervals and condition $c < -\frac{5+2\sqrt{5}}{4}$ implies that $|Q_c'(x)| > \lambda > 1$ for all $x \in I_0 \cup I_1$ and some $\lambda$. Let us assume that the interval $[x, y] \subset I_0$. For each $n \in \mathbb{N}$, $Q_c^n$ maps interval $[x, y]$ onto interval $[Q_c^n(x), Q_c^n(y)]$. Mean Value Theorem implies that

$$|Q_c^n(x) - Q_c^n(y)| \geqslant \lambda^n |x - y|.$$

Since $\lambda > 1$ and thus $\lambda^n \to \infty$, we have a contradiction unless $x = y$. Similarly, we prove the case where $[x, y] \subset I_1$. Thus $Q_c$ is one-to-one.

**Onto**: We first introduce the following notation. Let $J \subset I$ be a closed interval. Let

$$Q_c^{-n}(J) = \{x \in I \mid Q_c^n(x) \in J\}.$$

In particular, $Q_c^{-1}(J)$ denotes the preimage of $J$ inside $I$. Notice that if $J \subset I$ is a closed interval, then $Q_c^{-1}(J)$ consists of two closed subintervals, one in $I_0$ and one in $I_1$. Now let $s = (s_0 s_1 s_2 \cdots) \in \Sigma_2$ be arbitrary. To find $x \in \Lambda_c$ with $S(x) = s = (s_0 s_1 s_2 \cdots)$, we define

$$I_{s_0 s_1 \cdots s_n} = \{x \in I \mid x \in I_{s_0}, Q_c(x) \in I_{s_1}, \cdots, Q_c^n(x) \in I_{s_n}\}.$$

Since $s_j = 0$ or $1$ for each $j$, the set $I_{s_j}$ is equal to either one of $I_0$ or $I_1$ depending on the digit $s_j$. We may rewrite $I_{s_0 s_1 \cdots s_n}$ as follows

$$\begin{aligned}
I_{s_0 s_1 \cdots s_n} &= I_{s_0} \cap Q_c^{-1}(I_{s_1}) \cap \cdots \cap Q_c^{-n}(I_{s_n}) \\
&= I_{s_0} \cap Q_c^{-1}(I_{s_1} \cap \cdots \cap Q_c^{-(n-1)}(I_{s_n})) \\
&= I_{s_0} \cap Q_c^{-1}(I_{s_1 \cdots s_n}).
\end{aligned}$$

We claim that the $I_{s_0 s_1 \cdots s_n}$ are closed intervals that are nested. Clearly $I_{s_0}$ is a closed interval. By induction we assume that $I_{s_1 \cdots s_n}$ is a closed interval. Then $Q_c^{-1}(I_{s_1 \cdots s_n})$ consists of a pair of intervals, one in $I_0$ and one in $I_1$. In either event, $I_{s_0} \cap Q_c^{-1}(I_{s_1 \cdots s_n}) = I_{s_0 s_1 \cdots s_n}$ is a single closed interval.
These intervals are nested because

$$I_{s_0 \cdots s_n} = I_{s_0 \cdots s_{n-1}} \cap Q_c^{-n}(I_{s_n}) \subset I_{s_0 \cdots s_{n-1}}.$$

Therefore we conclude that

$$\bigcap_{n=0}^{\infty} I_{s_0 s_1 \cdots s_n}$$

is nonempty by Theorem 1.25. Let us choose $x \in \bigcap_{n=0}^{\infty} I_{s_0 s_1 \cdots s_n}$, then $x \in I_{s_0}, Q_c(x) \in I_{s_1}$, and so forth. Hence $S(x) = (s_0 s_1 \cdots)$. This prove that $S$ is onto.

Note that $\bigcap_{n=0}^{\infty} I_{s_0 s_1 \cdots s_n}$ consists of a unique point. This follows immediately from the fact that $S$ is one-to-one. In particular, from the hypothesis $c < -\frac{5+2\sqrt{5}}{4}$, it follows that $Q_c^{-1}$ is a strict contraction in $I$. Then diam $I_{s_0 s_1 \cdots s_n} \to 0$ as $n \to \infty$.

**Continuity**: Let $x \in \Lambda_c$ and suppose that $S(x) = (s_0 s_1 s_2 \cdots)$. Let $\epsilon > 0$ and then pick $n \in \mathbb{N}$ so that $\frac{1}{2^n} < \epsilon$. Let $J_n$ be closed subinterval $J_n \subset I_{s_n}$ such that $Q_c^n(x) \in J_n$. Then $Q_c^{-1}(x)$ consists of two closed intervals, one in $I_0$ and one in $I_1$. Let $J_{n-1}$ be closed subinterval of $I_{s_{n-1}}$ such that $Q_c^{n-1}(x) \in J_{n-1}$. We apply this for $0 \leqslant j \leqslant n$ so that we obtain

$$J_j \subset I_{s_j} \quad \text{with} \quad Q_c^j(x) \in J_j.$$

Then for $x, y \in J_0$ we have $Q_c^j(x), Q_c^j(x) \in J_j$ for $j = 1, 2, \cdots, n$. It follows that $S(y)$ agrees with $S(x)$ in the first $n+1$ terms. By Lemma 1.21, we have

$$d(S(x), S(y)) \leqslant \frac{1}{2^n} < \epsilon.$$

This proves the continuity of $S$.

**Continuity of** $S^{-1}$: Since $S$ is a bijection, then there exists $S^{-1}$, inverse function of $S$, defined from $\Sigma_2$ to $\Lambda_c$. Let $s \in \Sigma_2$ and $\epsilon > 0$. Then there is $n \in \mathbb{N}$ such that $\dfrac{1}{2^n} < \epsilon$. Since $S$ is a bijection, so there is $x \in \Lambda$ such that $s = S(x)$. Take $\delta = \dfrac{1}{2^n} > 0$. Suppose $t = S(y) \in \Sigma_2$ for some $y \in \Lambda$ such that $d(s,t) < \dfrac{1}{2^n}$. This says that for

$0 \leqslant j \leqslant n$, $Q_c^j(x)$ and $Q_c^j(y)$ always lie on the same interval $I_0$ or $I_1$. Let us suppose it is $I_0$. For $c < -\dfrac{5+2\sqrt{5}}{4}$, there is $\lambda > 1$ such that $\left|Q_c^k(x) - Q_c^k(y)\right| > \lambda^k |x-y|$ for any $k$. Equivalently,

$$|x-y| < \frac{\left|Q_c^k(x) - Q_c^k(y)\right|}{\lambda^k}.$$

But $\left|Q_c^k(x) - Q_c^k(y)\right| < l = \text{length}(I_0) < 1$. Then $|x-y| < \dfrac{l}{\lambda^k} < \dfrac{1}{\lambda^k}$. Now choose $k \in \mathbb{N}$ such that

$$\frac{1}{\lambda^k} < \frac{1}{2^n}.$$

Then $|x-y| < \frac{1}{2^n} < \epsilon$. Equivalently, $|S^{-1}(S(x)) - S^{-1}(S(y))| < \epsilon$. Then,

$$\left|S^{-1}(s) - S^{-1}(t)\right| < \epsilon.$$

Thus, $S^{-1} : \Sigma_2 \to \Lambda_c$ is continuous. Hence, $S$ is a homeomorphism. □

**Theorem 1.27.** For any $x \in \Lambda_c$, we have $S \circ Q_c(x) = \sigma \circ S(x)$.

*Proof.* Let $x \in \Lambda_c$ and let the itinerary of $x$ be given by $S(x) = (s_0 s_1 s_2 \cdots)$. This means that $Q_c^j(x) \in I_{s_j}$ for $j \geqslant 0$. Then, $\sigma(S(x)) = (s_1 s_2 s_3 \cdots)$. Now, $S(Q_c(x)) = (s_1 s_2 s_3 \cdots)$. □

**Corollary 1.28.** For any $x \in \Lambda_c$ and $n \in \mathbb{N}$, we have $S \circ Q_c^n(x) = \sigma^n \circ S(x)$.

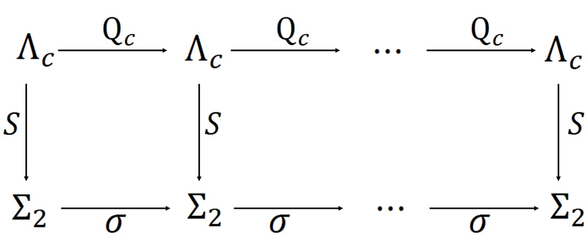


Figure 1.1

From Theorem 1.26 and Theorem 1.27, we can conclude that the itinerary map $S : \Lambda_c \to \Sigma_2$ is the conjugacy for $\sigma$ and $Q_c$.

**Theorem 1.29.** The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is conjugate to the quadratic map $Q_c : \Lambda_c \to \Lambda_c$ whenever $c < -\dfrac{5+2\sqrt{5}}{4}$.

**Definition 1.30.** A dynamical system is **transitive** if for any pair of points $x$ and $y$ and any $\epsilon > 0$ there is a third point $z$ within $\epsilon$ of $x$ whose orbit comes within $\epsilon$ of $y$.

**Definition 1.31.** A dynamical system $(X, F)$ **depends sensitively on initial conditions** if there is a $\beta > 0$ such that for any $x \in X$ and any $\epsilon > 0$ there is a $y \in X$ within $\epsilon$ of $x$ and an integer $k$ such that the distance between $F^k(x)$ and $F^k(y)$ is at least $\beta$.

**Definition 1.32.** Let $X$ be a topological space and $F : X \to X$ be a continuous map. We say that a dynamical system $(X, F)$ is **chaotic** if:

1. Periodic points for $F$ are dense in $X$. [**Density Property**]

2. $F$ is transitive. [**Transitivity Property**]

3. $F$ depends sensitively on initial conditions. [**Sensitivity Property**]

The following theorems show the equivalence between being chaotic and the existence of a dense $F$-orbit when some properties of the space $X$ is added.

**Theorem 1.33.** If $X$ has **no isolated points** and it has a **dense $F$-orbit**, then the dynamical system $(X, F)$ is **transitive**.

**Theorem 1.34.** If $X$ is a **separable** space, of **second category** and $(X, F)$ is **transitive**, then $X$ has a **dense $F$-orbit**.

**Proposition 1.35.** Periodic points under the shift map $\sigma$ form a dense subset of $\Sigma_2$.

*Proof.* Let $s = (s_j) \in \Sigma_2$ and $\epsilon > 0$. Then there is $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \epsilon$. Let us consider the sequence of periodic points $(t_j)_{j \geqslant 0}$ with $t_j = (\overline{s_0 s_1 \cdots s_j})$. Then for any $j \geqslant n$, we have $d(t_j, s) \leqslant \frac{1}{2^n} < \epsilon$. Hence, the set of periodic points of the shift map $\sigma$ forms a dense subset of $\Sigma_2$. □

**Proposition 1.36.** The shift map $\sigma$ is transitive.

*Proof.* We observe that $\Sigma_2$ has no isolated points. By Theorem 1.33, we need to find a point whose orbit form a dense subset of $\Sigma_2$. Let us consider

$$s^* = (\underbrace{01}_{\text{1 blocks}}\ \underbrace{00\ 01\ 10\ 11}_{\text{2 blocks}}\ \underbrace{000\ 001\ 010\ 011\ 100\ 101\ 110\ 111}_{\text{3 blocks}} \cdots) \in \Sigma_2.$$

In words, $s^*$ is constructed successively listing all block of 0's and 1's of length $n$ and then $n+1$ and so forth. We claim that the orbit of this point forms a dense subset for the sequence space $\Sigma_2$. Indeed, let $s = (s_j) \in \Sigma_2$ be arbitrary and $\epsilon > 0$. Then there exists $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \epsilon$. By the expression of $s^*$, we have that there is a block of length $n+1$ consisting $s_0 s_1 \cdots s_n$. Then there exists some integer $k$ so that

$$\sigma^k(s^*) = (s_0 s_1 \cdots s_n \cdots).$$

Now we see that the first $n+1$ terms of $s$ and $\sigma^k(s^*)$ coincide, it follows from Lemma 1.21 that

$$d(\sigma^k(s^*), s) \leqslant \frac{1}{2^n} < \epsilon.$$

Thus, we proved that there is a dense orbit for $\sigma$ on $\Sigma_2$. □

**Proposition 1.37.** The shift map $\sigma$ depends sensitively on initial conditions.

*Proof.* Let us choose $\beta = 1$. Let $s \in \Sigma_2$ and $\epsilon$ be any positive number. Then again we can find a natural number $n$ such that $\frac{1}{2^n} < \epsilon$. Let $t \in \Sigma_2$ such that $d(s,t) < \frac{1}{2^n}$ with $s \neq t$. Again, by Lemma 1.21, we have $s_j = t_j$ for $j = 0, 1, \cdots, n$. Since $s \neq t$, then there exists a natural number $k$ with $k > n$ such that $s_k \neq t_k$ so that $|s_k - t_k| = 1$. Now let us find the distance between $\sigma^k(x)$ and $\sigma^k(y)$:

$$d(\sigma^k(x), \sigma^k(y)) = \sum_{j=0}^{\infty} \frac{|s_{j+k} - t_{j+k}|}{2^j} = 1 + \sum_{j=1}^{\infty} \frac{|s_{j+k} - t_{j+k}|}{2^j} \geqslant 1.$$

This proves that the shift map $\sigma$ depends sensitively on initial conditions. □

From Propositions 1.35, 1.36 and 1.37 we can conclude the following result.

**Theorem 1.38.** The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is a chaotic dynamical system.

**Lemma 1.39.** Let $X$ and $Y$ be two topological spaces. Assume that $f : X \to Y$ is continuous and onto. If the subset $D \subset X$ is dense, the image of $D$ under $f$ is also dense in $Y$.

Now we arrive to the main result of this section.

**Theorem 1.40.** Suppose $c < -\dfrac{5+2\sqrt{5}}{4}$, then the quadratic map $Q_c$ is chaotic on the invariant set $\Lambda_c$.

*Proof.* **Density**: $\Sigma_2$ has a dense periodic subset $D$ by Proposition 1.35. Since $S$ is a homeomorphism, then in particular we have $S^{-1}$ is onto and continuous from $\Sigma_2$ to $\Lambda$. It follows by Lemma 1.39 that $S^{-1}(D) \subset \Lambda_c$ is a dense subset. By Corollary 1.28, we have $S^{-1}(D)$ is the set of periodic points of $Q_c$ and it is a dense subset of $\Lambda_c$. Thus $Q_c$ satisfies the density property.

**Transitivity**: Since $\Sigma_2$ has a point $s^*$ whose orbit $\mathcal{O}_\sigma(s^*)$ forms a dense subset of $\Sigma_2$. By Lemma 1.39, we have that $S^{-1}(\mathcal{O}_\sigma(s^*)) = \mathcal{O}_{Q_c}(S^{-1}(s^*))$ which is dense in $\Lambda_c$ and $S^{-1}(s^*) \in \Lambda_c$. Thus, $\Lambda_c$ contains the point $S^{-1}(s^*)$ whose orbit is dense in $\Lambda_c$. This proves $Q_c$ is transitive.

**Sensitivity**: Let us choose $\beta = (x_+^1 - x_-^1)/2 > 0$ where $x_-^1$ and $x_+^1$ are the end points of $A_1$ and are given in Equation 1.1. Now given a point $x \in \Lambda_c$ and $\epsilon > 0$. Then there is $y \in \Lambda_c$ different from $x$ and is within $\epsilon$ of $x$, since $\Lambda_c$ has no isolated point. Since the itinerary map $S$ is injective, then

$$s = (s_0 s_1 s_2 ...) = S(x) \neq S(y) = (t_0 t_1 t_2 ...) = t.$$

Then there is a natural number $k$ such that $s_k \neq t_k$. This says that $Q_c^k(x)$ and $Q_c^k(y)$ lie in different intervals. Thus the distance between $Q_c^k(x)$ and $Q_c^k(y)$ is greater than $\beta$, that is, $|Q_c^k(x) - Q_c^k(y)| > \beta$. This shows that the quadratic map $Q_c$ depends sensitively on initial conditions. □

## 1.3 The Weierstrass $\wp$ function

### 1.3.1 Elliptic functions

**Definition 1.41.** Let $\Omega$ be an open set in $\mathbb{C}$ and $f$ be a complex-valued function on $\Omega$. The function $f$ is **holomorphic at the point** $z_0 \in \mathbb{C}$ if the quotient

$$\frac{f(z_0 + h) - f(z_0)}{h}$$

converges to a limit as $h \to 0$. Notice that $h \in \mathbb{C}$ with $h \neq 0$ and $z_0 + h \in \Omega$. When the limit exists we denote it by $f'(z_0)$ and call it the **derivative of $f$ at** $z_0$:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

The function $f$ is said to be **holomorphic on** $\Omega$ if $f$ is holomorphic at every point of $\Omega$. If $C$ is a closed subset of $\mathbb{C}$, we say that $f$ is holomorphic on $C$ if $f$ is holomorphic on some open set containing $C$. Last, if $f$ is holomorphic in the whole complex plane $\mathbb{C}$, we say that $f$ is an **entire** function.

A **singularity** of a complex function $f$ is a complex number $z_0 \in \mathbb{C}$ for which $f$ is defined in a deleted neighborhood of $z_0$ but not at the point $z_0$ itself. We say that $f$ defined on a deleted neighborhood of $z_0$ has a **pole** at $z_0$ if the function $1/f$ is well defined and is holomorphic in the whole neighborhood.

**Definition 1.42.** A function $f$ on an open set $\Omega$ is **meromorphic** if there exists a sequence of points (not necessarily distinct) $\{z_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ that has no limit points in $\Omega$, and such that
(i) the function $f$ is holomorphic on $\Omega - \{z_1, z_2, \cdots\}$ and
(ii) $f$ has poles at the $\{z_1, z_2, \cdots\}$.

A nonzero number $\omega \in \mathbb{C}$ is called a **period** of $f$ if $f(z+\omega) = f(z)$ for all $z \in \mathbb{C}$. A function $f$ is called **doubly periodic** if $f$ has two distinct periods, $\omega_1$ and $\omega_2$. If $\omega_1$ and $\omega_2$ are linearly dependent over $\mathbb{R}$, that is $\omega_2/\omega_1 \in \mathbb{R}$, is uninteresting. On one hand, if $\omega_2/\omega_1 \in \mathbb{Q}$, $f$ has a simple period. On the other hand, $\omega_2/\omega_1 \notin \mathbb{Q}$, $f$ is constant. Now we only consider the case when $\omega_1$ and $\omega_2$ are linearly independent over $\mathbb{R}$. We will do a normalization. Let $\tau = \omega_2/\omega_1 \in \mathbb{C}$. Since $\mathrm{Im}(\omega_1)$ and $\mathrm{Im}(\omega_2)$ have opposite sign and $\tau \in \mathbb{C}$, we could assume that $\mathrm{Im}(\tau) > 0$. Now let $F(z) = f(\omega_1 z)$. Observe that $f$ has two periods $\omega_1$ and $\omega_2$ if and only if $F$ has two periods 1 and $\tau$. $f$ is a meromorphic function if and only if $F$ is a meromorphic function. Thus we may assume without loss of generality that $f$ is a meromorphic function on $\mathbb{C}$ with periods 1 and $\tau$ with $\mathrm{Im}(\tau)$ $> 0$.

**Definition 1.43.** Let $\lambda_1$ and $\lambda_2$ be $\mathbb{R}$-linearly independent. A **lattice** $\Lambda$ in the complex plane $\mathbb{C}$ is a set of points of the form

$$\Lambda = \{m\lambda_1 + n\lambda_2 \mid m, n \in \mathbb{Z}\}.$$

We say that $\lambda_1$ and $\lambda_2$ are **generators** of the lattice $\Lambda$ and let us denote $\Lambda^* = \Lambda - \{0\}$.

**Example 1.44.** The lattice generated by $\lambda_1 = 1$ and $\lambda_2 = i$ coincides with $\mathbb{Z}[i]$, the **Gaussian integers**.

**Definition 1.45.** Let $\Lambda$ and $\Gamma$ be two lattices. We say that lattices $\Lambda$ and $\Gamma$ are **similar** if there is a nonzero complex number $a$ such that $\Gamma = a\Lambda$ where $a\Lambda = \{a\lambda \mid \lambda \in \Lambda\}$.

Notice that similarity of lattices is an equivalence relation.

**Proposition 1.46.** Let $\lambda_1, \lambda_2 \in \mathbb{C} - \{0\}$ so that $\mathrm{Im}(\lambda_2/\lambda_1) > 0$. Consider the lattice $\Lambda = \{m\lambda_1 + n\lambda_2 \mid m, n \in \mathbb{Z}\}$. If $\tau = \lambda_2/\lambda_1$, then $\Lambda$ is similar to the lattice $\{m + n\tau \mid m, n \in \mathbb{Z}\}$.

*Proof.* Let us choose $\lambda_1 \in \Lambda$ so that $\lambda_1 \neq 0$. Now,

$$\lambda_1\Gamma = \{\lambda_1\gamma \mid \gamma \in \Gamma\} = \{\lambda_1(m + n\tau) \mid m, n \in \mathbb{Z}\} = \{m\lambda_1 + n\lambda_2 \mid m, n \in \mathbb{Z}\} = \Lambda.$$

□

**Remark 1.47.** By Proposition 1.46, it will be indistinguishable to consider $\{\lambda_1, \lambda_2\}$ or $\{1, \tau\}$ as the set of generators of $\Lambda$.

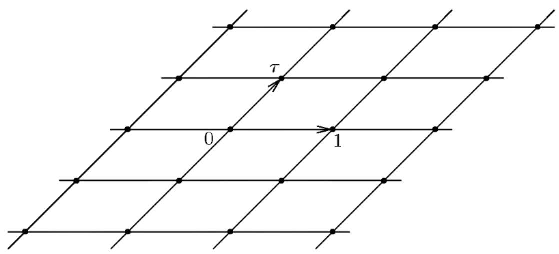


Figure 1.2: The lattice generated by 1 and $\tau$

**Definition 1.48.** We say $\Lambda$ is a **real** lattice if $\bar{\Lambda} = \Lambda$, that is, the complex conjugate of every element in $\Lambda$ belongs again to $\Lambda$. The lattice $\Lambda$ is **square** if $i\Lambda = \Lambda$. Equivalently, there exist generators $\lambda_1, \lambda_2$ of $\Lambda$ so that $\lambda_1 > 0$ and $\lambda_2 = i\lambda_1$.

**Definition 1.49.** A closed, connected subset $Q$ of the complex plane is said a **fundamental region** for $\Lambda$ if

1. for each $z \in \mathbb{C}$, $\Lambda$ contains at least one point in the same $\Lambda$-orbit as $z$;
2. no two points in the interior of $\Lambda$ are in the same orbit.

If $Q$ is any fundamental region for $\Lambda$, then for any $s \in \mathbb{C}$, the set

$$Q + s = \{z + s \mid z \in Q\}$$

is also a fundamental region. Usually, we choose $Q$ to be a polygon with finite number of parallel sides. In this case we call $Q$ a **period parallelogram** for $\Lambda$.

**Example 1.50.** Given the lattice $\Lambda = \{m + n\tau \mid m, n \in \mathbb{Z}\}$ with $\text{Im}(\tau) > 0$, then the period parallelogram is given by $P_0 = \{z \in \mathbb{C} \mid z = a + b\tau,\ 0 \leqslant a < 1,\ 0 \leqslant b < 1\}$.

Let us denote the **upper-half plane** $\mathbb{H}$ in the complex plane as below:

$$\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}.$$

**Lemma 1.51.** The two series

$$(a) \sum_{(m,n)\neq(0,0)} \frac{1}{(|m|+|n|)^r} \text{ and } (b) \sum_{m+n\tau\in\Lambda^*} \frac{1}{|m+n\tau|^r}$$

converge whenever $r > 2$.

*Proof.* (a) We will sum in $m$ first and then in $n$. For any $n \neq 0$, we have

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \frac{1}{(|m|+|n|)^r} &= \frac{1}{|n|^r} + 2\sum_{m\geqslant 1} \frac{1}{(|m|+|n|)^r}\\ &= \frac{1}{|n|^r} + 2\sum_{k\geqslant |n|+1} \frac{1}{k^r}\\ &\leqslant \frac{1}{|n|^r} + 2\int_{|n|}^{\infty} \frac{dx}{x^r}\\ &\leqslant \frac{1}{|n|^r} + C\frac{1}{|n|^{r-1}}.\end{aligned}$$

Thus, for any $r > 2$, we have

$$\begin{aligned}\sum_{(m,n)\neq(0,0)} \frac{1}{(|m|+|n|)^r} &= \sum_{m\neq 0} \frac{1}{|m|^r} + \sum_{n\neq 0}\sum_{m\in\mathbb{Z}} \frac{1}{(|m|+|n|)^r}\\ &\leqslant \sum_{m\neq 0} \frac{1}{|m|^r} + \sum_{n\neq 0}\left(\frac{1}{|n|^r} + C\frac{1}{|n|^{r-1}}\right)\\ &< \infty.\end{aligned}$$

We used the fact that each series converges since the $p$-series converges whenever $p > 1$.

(b) To prove this part, it is enough to show that there exists a positive constant $c$ for which

$$|n| + |m| \leqslant c|n + m\tau|$$

for all $m, n \in \mathbb{Z}$. For any two real numbers $x$ and $y$, we write that $x \preceq y$ if there is a positive constant $\alpha$ such that $x \leqslant \alpha y$. We also write $x \approx y$ if $x \preceq y$ and $y \preceq x$. Notice that for any two positive numbers $A$ and $B$, we have

$$(A^2 + B^2)^{1/2} \approx A + B.$$

Indeed, we have $A \leqslant (A^2 + B^2)^{1/2}$ and $B \leqslant (A^2 + B^2)^{1/2}$, so $A + B \leqslant 2(A^2 + B^2)^{1/2}$. On the other hand, we have $A^2 + B^2 \leqslant (A + B)^2$. Then $(A^2 + B^2)^{1/2} \leqslant A + B$.

We get the following observation:

$$|n| + |m| \approx |n + m\tau| \text{ whenever } \tau \in \mathbb{H}.$$

Indeed, if $\tau = s + it$ with $s, t \in \mathbb{R}$ and $t > 0$, then

$$|n + m\tau| = [(n + ms)^2 + (mt)^2]^{1/2} \approx |n + ms| + |mt| \approx |n + ms| + |m| \approx |n| + |m|.$$

□

**Theorem 1.52.** An entire doubly periodic function is constant.

By the result of the Theorem 1.52, we are only interested in doubly periodic meromorphic functions.

**Definition 1.53.** An **elliptic function** is a non-constant doubly-periodic meromorphic function.

Given an elliptic function $f$ with periods 1 and $\tau$, we could consider the associated lattice $\Lambda = \{m + n\tau \mid m, n \in \mathbb{Z}\}$. Then we obtain the following properties of an elliptic function.

**Theorem 1.54.** The total number of poles of an elliptic function on a period parallelogram $P_0$ is greater than or equal to 2, counting multiplicity.

Such a number is called **order** of the elliptic function.

**Theorem 1.55.** Every elliptic function of order $m$ has $m$ zeros in $P_0$.

For the proofs of Theorem 1.54 and Theorem 1.55 can be found in [12].

### 1.3.2 The Weierstrass $\wp$ function

**Definition 1.56.** Let $\Lambda = \{m\lambda_1 + n\lambda_2 \mid m, n \in \mathbb{Z}\}$ be a lattice where $\lambda_1$ and $\lambda_2$ are $\mathbb{R}$-linearly independent. The **Weierstrass $\wp$ function** over the lattice $\Lambda$ is defined by

$$\wp(z) = \wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda^*} \left[ \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right].$$

Notice that $\wp$ is an even function. Indeed,

$$\wp(-z) = \frac{1}{(-z)^2} + \sum_{\lambda \in \Lambda^*} \left[ \frac{1}{(-z - \lambda)^2} - \frac{1}{\lambda^2} \right] = \frac{1}{z^2} + \sum_{\lambda \in \Lambda^*} \left[ \frac{1}{(z + \lambda)^2} - \frac{1}{\lambda^2} \right] = \wp(z).$$

**Proposition 1.57.** $\wp$ is a meromorphic function with double poles at its lattice points.

*Proof.* To prove that $\wp$ is a meromorphic function on $\mathbb{C}$ with poles at its lattice points, let us suppose $|z| < R$ for $R$ any positive real number. Then we the expression of $\wp$ can be written as

$$\wp(z) = \frac{1}{z^2} + \sum_{|\lambda|\leqslant 2R}\left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right] + \sum_{|\lambda|>2R}\left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right].$$

First, notice that the first term exhibits poles at the lattice points in the disc $|z| < R$. For the second term, for any $|z| < R$ and for any $|\lambda| > 2R$, we have

$$\left|\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right| = \left|\frac{2\lambda z - z^2}{\lambda^2(\lambda - z)^2}\right| = \left|\frac{z\left(2 - \frac{z}{\lambda}\right)}{\lambda^3\left(1 - \frac{z}{\lambda}\right)^2}\right| \leqslant \frac{10R}{|\lambda|^3}.$$

By Lemma 1.51, taking $r = 3$, we obtain $\sum_{\lambda\in\Lambda*} \frac{10R}{|\lambda|^3}$ converges on $|z| < R$. So the second sum defines a holomorphic function on $|z| < R$. Since $R$ is arbitrary, we conclude that the Weierstrass $\wp$ function is meromorphic with poles at its lattice points. □

**Proposition 1.58.** The Weierstrass $\wp$ function is doubly periodic with periods determined by the generators of lattice $\Lambda$.

*Proof.* Without lost of generality, assume $\Lambda$ has generators 1 and $\tau$. By differentiating term by term the series defining $\wp$, we can obtain a series for $\wp'$:

$$\wp'(z) = -2\sum_{\lambda\in\Lambda} \frac{1}{(z-\lambda)^3}. \tag{1.2}$$

By Lemma 1.51, taking $r = 3$, the differentiated series converges absolutely on $\mathbb{C} - \Lambda$. It can be seen clearly that the derivative of $\wp$ has two periods, 1 and $\tau$. Then it follows that there exist real numbers $a$ and $b$ such that

$$\wp(z+1) = \wp(z) + a \text{ and } \wp(z+\tau) = \wp(z) + b.$$

Plugging $z = -1/2$ and $z = -\tau/2$ into above equations respectively, we obtain

$$\wp(1/2) = \wp(-1/2) + a \text{ and } \wp(\tau/2) = \wp(-\tau/2) + b.$$

Since $\wp$ is even, we have $\wp(1/2) = \wp(-1/2)$ and $\wp(\tau/2) = \wp(-\tau/2)$. Then it follows immediately that $a = b = 0$. Thus we obtain $\wp(z+1) = \wp(z)$ and $\wp(z+\tau) = \wp(z)$. Hence, $\wp$ is doubly periodic with periods 1 and $\tau$. □

By Propositions 1.57 and 1.58, we conclude that $\wp = \wp_\Lambda$ is an elliptic function with periods given by the generators of $\Lambda$ and has double poles at the lattice points.

Let $\mathrm{Crit}(f)$ denote the set of critical points of $f$, that is,

$$\mathrm{Crit}(f) = \{z \in D(f) \mid f'(z) = 0\}.$$

If $z_0$ is a critical point then $f(z_0)$ is called a critical value of $f$.

**Proposition 1.59.** In the period parallelogram $P_0$,

$$\mathrm{Crit}(\wp|_{P_0}) = \{1/2, \tau/2, (1+\tau)/2\}.$$

*Proof.* By expression given in Equation 1.2, it is clear that $\wp'$ is odd. Since $\wp'$ is doubly periodic with periods 1 and $\tau$, we have

$$\wp'(1/2) = \wp'(\tau/2) = \wp'((1+\tau)/2) = 0.$$

Indeed, for example, $\wp'(1/2) = -\wp'(-1/2) = -\wp'(-1/2+1) = -\wp(1/2)$. From the expression in Equation 1.2, we see that $\wp'$ has triple poles in the period parallelogram at zero. In other words, we say that $\wp'$ is of order 3. So the three points $1/2$, $\tau/2$ and $(1+\tau)/2$ are only three roots of $\wp'$ in the period parallelogram. Moreover, each of them has multiplicity 1. ∎

Let us denote

$$\omega_1 = 1/2, \quad \omega_2 = \tau/2, \quad \omega_3 = (1+\tau)/2.$$

$$\wp(1/2) = e_1, \quad \wp(\tau/2) = e_2, \quad \wp\big((1+\tau)/2\big) = e_3.$$

**Remark 1.60.** In the whole complex plane, the equation $\wp(z) = e_j$ has double roots $z = \omega_j + \Lambda$, for any $\lambda \in \Lambda$, since $\wp$ is $\Lambda$-periodic. In particular, the three numbers $e_1$, $e_2$ and $e_3$ are distinct. Indeed, if some two of these are equal, then $\wp$ has at least four roots in the fundamental parallelogram which is a contradiction since $\wp$ is of order 2.

The derivative $\wp'$ squared can be written as a polynomial in $\wp$ in the following theorem.

**Theorem 1.61.** The function $(\wp')^2$ can be written as a cubic polynomial in term of $\wp$, namely

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3).$$

*Proof.* Let $F(z) = (\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$. In the fundamental parallelogram $P_0$, $F(z) = 0$ if and only if $\wp(z) = e_1$, $\wp(z) = e_2$ or $\wp(z) = e_3$. By Remark 1.60, we have $z \in \{\omega_1, \omega_2, \omega_3\}$ and each root has multiplicity 2. By Proposition 1.59, $\wp'$ has roots at these half periods with multiplicity 1. It follows that $(\wp')^2$ has double roots at $\omega_1$, $\omega_2$ and $\omega_3$. We also observe that $F$ has poles of order 6 at the lattice points since $\wp$ has poles of order 2 there. And since $\wp'$ has poles of order 3 at half periods, then $(\wp')^2$ also has poles of order 6. We conclude that $\dfrac{(\wp')^2}{F}$ is holomorphic. Furthermore, we have

$$\begin{aligned}
\frac{(\wp')^2}{F}(z+1) &= \frac{(\wp')^2(z+1)}{F(z+1)} \\
&= \frac{(\wp')^2(z)}{(\wp(z+1) - e_1)(\wp(z+1) - e_2)(\wp(z+1) - e_3)} \\
&= \frac{(\wp')^2(z)}{F(z)} \\
&= \frac{(\wp')^2}{F}(z).
\end{aligned}$$

and we do similarly for period $\tau$. Then $\dfrac{(\wp')^2}{F}$ is constant by Theorem 1.52. We have $\wp(z) = \frac{1}{z^2} + O(z^2)$ and $\wp'(z) = -\frac{2}{z^3} + O(z)$, then near the origin, we have $\dfrac{\wp'(z)^2}{F(z)} = 4$. Therefore, $(\wp'(z))^2 = 4F = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$. □

The elliptic function $\wp_\Lambda$ satisfies the following **homogeneity condition**: for any $z \in \Lambda$ and any $k \in \mathbb{C} - \{0\}$,

$$\wp_{k\Lambda}(kz) = \frac{1}{k^2}\wp_\Lambda(z). \tag{1.3}$$

### 1.3.3 Modular character of elliptic functions and Eisenstein series

Let us consider the lattice $\Lambda = \{m + n\tau \mid m, n \in \mathbb{Z}\}$ with $\mathrm{Im}(\tau) > 0$.

**Proposition 1.62.** We have two properties as follow:

$$(a)\ \wp_\tau(z) = \wp_{\tau+1}(z) \ \text{ and } \ (b)\ \wp_{-1/\tau} = \tau^2\wp_\tau(\tau z).$$

*Proof.* (a) By definition of $\wp_\tau$, we have

$$\begin{aligned}\wp_\tau(z) &= \frac{1}{z^2} + \sum_{\lambda\in\Lambda^*}\left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right]\\ &= \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)}\left[\frac{1}{(z-(m+n\tau))^2} - \frac{1}{(m+n\tau)^2}\right].\end{aligned}$$

Then

$$\begin{aligned}\wp_{\tau+1} &= \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)}\left[\frac{1}{(z-m-n-n\tau)^2} - \frac{1}{(m+n+n\tau)^2}\right]\\ &= \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)}\left[\frac{1}{(z-m-n\tau)^2} - \frac{1}{(m++n\tau)^2}\right]\\ &= \wp_\tau(z).\end{aligned}$$

(b) We have

$$\begin{aligned}\wp_\tau(\tau z) &= \frac{1}{\tau^2 z^2} + \sum_{\lambda\in\Lambda^*}\left[\frac{1}{(\tau z-\lambda)^2} - \frac{1}{\lambda^2}\right].\\ \tau^2\wp_\tau(\tau z) &= \frac{1}{z^2} + \tau^2\sum_{\lambda\in\Lambda^*}\left[\frac{1}{(\tau z-\lambda)^2} - \frac{1}{\lambda^2}\right]\\ &= \frac{1}{z^2} + \sum_{\lambda\in\Lambda^*}\left[\frac{1}{(z-\frac{\lambda}{\tau})^2} - \left(\frac{\tau}{\lambda}\right)^2\right]\\ &= \frac{1}{z^2} + \sum_{\lambda\in\Lambda^*}\left[\frac{1}{(z-\frac{\lambda}{\tau})^2} - \left(\frac{1}{\frac{\lambda}{\tau}}\right)^2\right].\end{aligned}$$

Since $\lambda = m + n\tau$, then $\frac{\lambda}{\tau} = \frac{m}{\tau} + n$ and $\frac{\tau}{\lambda} = \frac{\tau}{m+n\tau} = \frac{1}{\frac{m}{\tau}+n}$. Then we have

$$\tau^2\wp_\tau(\tau z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)}\left[\frac{1}{(z-\frac{m}{\tau}-n)^2} - \frac{1}{(\frac{m}{\tau}+n)^2}\right].$$

On the other hand,

$$\wp_{-\frac{1}{\tau}}(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)}\left[\frac{1}{(z-m+\frac{n}{\tau})^2} - \frac{1}{(-m+\frac{n}{\tau})^2}\right].$$

Hence we have $\wp_{1/\tau}(z) = \tau^2\wp(\tau z)$. □

**Definition 1.63.** The **Eisenstein series** of order $k$ is defined by

$$E_k = E_k(\tau) = \sum_{\lambda \in \Lambda^*} \frac{1}{\lambda^k} = \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^k}$$

where $k \geqslant 3$ and $\tau \in \mathbb{C}$ with $\operatorname{Im}(\tau) > 0$.

Now we provide some properties of the Eisenstein series as follow.

**Proposition 1.64.** Each Eisenstein series satisfies the following:

1) $E_k(\tau)$ converge if $k \geqslant 3$ and is holomorphic in $\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}$.
2) $E_k(\tau) = 0$ if $k$ is odd.
3) $E_k(\tau + 1) = E_k(\tau)$ and $E_k(-\frac{1}{\tau}) = \tau^k E_k(\tau)$.

*Proof.* 1) This is the consequence of Lemma 1.51 taking $k \geqslant 3$.
2) For $k$ odd, we have

$$\begin{aligned} E_k(\tau) &= \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^k} \\ &= \sum_{(m,n)\neq(0,0)} \frac{1}{(-m-n\tau)^k} \\ &= -\sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^k} \\ &= -E_k(\tau). \end{aligned}$$

Hence, $E_k(\tau) = 0$ whenever $k$ is odd.
3)

$$\begin{aligned} E_k(\tau+1) &= \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau+n)^k} \\ &= \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n+n\tau)^k} \\ &= \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^k} \\ &= E_k(\tau). \end{aligned}$$

$$\begin{aligned}
E_k(-\frac{1}{\tau}) &= \sum_{(m,n)\neq(0,0)} \frac{1}{[m+n(-\frac{1}{\tau})]^k} \\
&= \tau^{-k} \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau-n)^k} \\
&= \tau^{-k} \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^k} \\
&= \tau^{-k} E_k(\tau).
\end{aligned}$$

□

**Proposition 1.65.** The numbers $E_k(\tau)$ for $k$ even appear as coefficients in the power series expansion of $\wp_\Lambda$ around the origin, namely

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)E_{2k+2}z^{2k}(\tau). \tag{1.4}$$

*Proof.* We have by definition, $\wp(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda*}\left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right]$. The geometric series of $\frac{1}{1-\lambda}$ is given by

$$\frac{1}{1-\lambda} = 1 + \lambda + \lambda^2 + \cdots$$

for $\mid \lambda \mid < 1$. Then $\frac{1}{(1-\lambda)^2} = 1 + 2\lambda + 3\lambda^2 + \cdots = \sum_{0\leqslant\ell\leqslant\infty}(\ell+1)\lambda^\ell$ for $\mid \lambda \mid < 1$. For $\mid z \mid < \mid \lambda \mid$, then $\frac{\mid z \mid}{\mid \lambda \mid} < 1$. It follows that

$$\frac{1}{(1-\frac{z}{\lambda})^2} = \sum_{0\leqslant\ell\leqslant\infty}(\ell+1)(\frac{z}{\lambda})^\ell.$$

Then $\frac{1}{(z-\lambda)^2} = \frac{1}{(\lambda-z)^2} = \frac{1}{\lambda^2}\sum_{0\leqslant\ell\leqslant\infty}(\ell+1)(\frac{z}{\lambda})^\ell = \frac{1}{\lambda^2} + \frac{1}{\lambda^2}\sum_{1\leqslant\ell\leqslant\infty}(\ell+1)(\frac{z}{\lambda})^\ell$. We can now write

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{\lambda\in\Lambda^*}\sum_{1\leqslant\ell\leqslant\infty}(\ell+1)\frac{z^\ell}{\lambda^{\ell+2}}\\ &= \frac{1}{z^2} + \sum_{1\leqslant\ell\leqslant\infty}(\ell+1)(\sum_{\omega\in\Lambda^*}\frac{1}{\omega^{\ell+2}})z^\ell\\ &= \frac{1}{z^2} + \sum_{1\leqslant\ell\leqslant\infty}(\ell+1)E_{\ell+2}z^\ell.\end{aligned}$$

Hence, $\wp(z) = \frac{1}{z^2} + \sum\limits_{1\leqslant k\leqslant\infty}(2k+1)E_{2k+2}z^{2k} = \frac{1}{z^2} + 3E_4z^2 + 5E_6z^4 + \cdots$ near 0.

□

From the power expansion in (1.4) we can infer the following result.

**Proposition 1.66.** There exist values $g_2 = 60E_4$ and $g_3 = 140E_6$ that satisfy the equation

$$(\wp'_\Lambda(z))^2 = 4(\wp_\Lambda(z))^3 - g_2\wp_\Lambda(z) - g_3, \tag{1.5}$$

for all $z \in \mathbb{C}$.

*Proof.* By Equation (1.3) we have

$$\wp'(z) = -\frac{2}{z^3} + 6E_4z + 20E_6z^3 + \cdots .$$

Then

$$\begin{aligned}(\wp'(z))^2 &= \frac{4}{z^6} - \frac{24E_4}{z^2} - 80E_6 + \cdots, \text{ and}\\ (\wp'(z))^3 &= \frac{1}{z^6} + \frac{9E_4}{z^2} + 15E_6 + \cdots .\end{aligned}$$

We now have

$$\begin{aligned}&(\wp'(z))^2 - 4(\wp(z))^3 + 60E_4\wp(z) + 140E_6\\ &= \frac{4}{z^6} - \frac{24E_4}{z^2} - 80E_6 - \frac{4}{z^6} - \frac{36E_4}{z^2} - 60E_6 + \frac{60E_4}{z^2} + 180E_4^2z^2 + 300E_4E_6z^4 + 140E_6\\ &= 300E_4E_6z^4 + 180E_4^2z^2,\end{aligned}$$

which is a polynomial, so it is holomorphic on a neighborhood of the origin. In fact, it is entire. Notice that the difference is also doubly periodic. By Theorem 1.52 we conclude that it is constant. Also at the origin, $(\wp'(0))^2 - 4(\wp(0))^3 + 60E_4\wp(0) + 140E_6 = 0$. Thus it follows that it is identically zero. Denote

$$g_2 = 60E_4 = 60 \sum_{\lambda \in \Lambda^*} \frac{1}{\lambda^4} \text{ and } g_3 = 140E_6 = 140 \sum_{\lambda \in \Lambda^*} \frac{1}{\lambda^6}.$$

Hence,

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3.$$

□

# Chapter 2

# Sharkovskii's theorem

The aim of this chapter is to provide a complete proof of the Sharkovskii's Theorem which was published in [10]. We will consider a real continuous function acting over the real line and examine the enforcement of its periods. The proof presented here is based on P. Stefan's work in [11] where he introduced the study of graphs that encode admissible transitions. All intervals in this chapter are supposed to be non-trivial.

## 2.1 Preparatory Lemmas

**Definition 2.1.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous real-valued function on the real line. Let $I, J \subset \mathbb{R}$ be intervals. The interval $I$ **f-covers** interval $J$ if $f(I) \supset J$. We write $I \to J$.

**Lemma 2.2.** Let $I$ and $J$ be closed intervals. If $I$ f-covers $J$, then there exists a closed subinterval $K \subset I$ such that $f(K) = J$, $f(\text{int}(K)) = \text{int}(J)$ and $f(\partial K) = \partial J$.

*Proof.* Let $J = [b_1, b_2]$ with $b_1 < b_2$. Then $b_1 = f(a_1)$ and $b_2 = f(a_2)$ for some $a_1$ and $a_2$ in $I$ with $a_1 \neq a_2$. Assume first that $a_1 < a_2$.
Let $x_1 = \sup\{x \mid a_1 \leqslant x \leqslant a_2, f(x) = b_1\}$. Since $x_1$ is the supremum, there is a sequence (possibly constant) of $x_n$ with $a_1 \leqslant x_n \leqslant a_2$ and $f(x_n) = b_1$ such that

$$\lim_{n\to\infty} x_n = x_1.$$

By continuity of $f$, $f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} b_1 = b_1$. Thus $f(x_1) = b_1$. It follows that $x_1 < a_2$ for if $x_1 = a_2$ then $f(a_2) = b_1$ contradicting $f(a_2) = b_2$ and $b_2 \neq b_1$. Let $x_2 = \inf\{x \mid x_1 \leqslant x \leqslant a_2, f(x) = b_2\}$. Similarly, it follows that $f(x_2) = b_2$. Now assume that $a_2 < a_1$. Let $x_1 = \inf\{a_2 \leqslant x \leqslant a_1, f(x) = b_1\}$ and $x_2 = \sup\{a_2 \leqslant x \leqslant$

$x_1, f(x) = b_2\}$. Thus $f(\{x_1, x_2\}) = \{b_1, b_2\}$. Let $K = [x_1, x_2]$. We get $f(\partial K) = \partial J$. And $f(\text{int}(K)) \cap \partial(J) = f((x_1, x_2)) \cap \{b_1, b_2\} = \varnothing$. It follows that $f(\text{int}(K)) = \text{int}(J)$ and therefore $f(K) = J$.

□

**Lemma 2.3.** (i) Let $a, b \in \mathbb{R}$ such that $a \neq b$, $f(a) > a$, $f(b) < b$ and $[a, b] \subset D(f)$, domain of $f$. Then there exists a fixed point between $a$ and $b$.
(ii) If $I, J$ are closed intervals with $I \subset J$ and $I \to J$, then $f$ has a fixed point in $I$.

*Proof.* (i) Let $g(x) = f(x) - x$. Then $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$. By Intermediate Value Theorem, there exists a point $c$ between $a$ and $b$ such that $g(c) = 0$, so $f(c) = c$. Therefore there is a fixed point between $a$ and $b$.
(ii) Since $I \to J$, by Lemma 2.2, there is $K = [x_1, x_2] \subset I$ (with $x_1 < x_2$) such that $f(K) = J = [a, b]$. We have now two possibilities.
a. $f(x_1) = a \leqslant x_1$ and $f(x_2) = b \geqslant x_2$, or b. $f(x_1) = b > x_1$ and $f(x_2) = a < x_2$.

If $f(x_1) = x_1$ or $f(x_2) = x_2$ we are done. The remaining cases $f(x_1) < x_1$, $f(x_2) > x_2$ and $f(x_1) > x_1$, $f(x_2) < x_2$ follow from part (i). Therefore, $f$ has a fixed point in $I$. □

**Corollary 2.4.** If $I \to I$, then $f$ has a fixed point in $I$.

**Lemma 2.5.** Let $I_0 \to I_1 \to \cdots \to I_n = I_0$ be a loop with $I_j \to I_{j+1}$ for all $j = 0, 1, \cdots, n-1$.

(i) Then there exists a fixed point $x_0$ of $f^n$ with $f^j(x_0) \in I_j$ for all $j = 0, 1, \cdots, n$.

(ii) Suppose further that: (1) this loop is not a product loop formed by going $p$ times around a shorter loop of length $m$ where $mp = n$, and (2) $\text{int}(I_j) \cap \text{int}(I_k) = \varnothing$ unless $I_j = I_k$.

If the periodic point $x_0 \in \text{int}(I_0)$, then it has a least period $n$.

*Proof.* First we will prove the following statement:
There exists a subinterval $K_l \subset I_0$ such that for $i = 1, 2, \cdots, l$,

$$f^i(K_l) \subset I_i,\ f^i(\text{int}(K_l)) \subset \text{int}(I_i) \text{ and } f^l(K_l) = I_l.$$

We will give a proof by induction on $l$. Now for the case $l = 1$, it is true by Lemma 2.2, that is, there exists $K_1 \subset I_0$ such that $f(\text{int}(K_1)) \subset \text{int}(I_1)$ and $f(K_1) = I_1$. Suppose that the statement is true for $l-1$. That is, there exists such a $K_{l-1} \subset I_0$. Then $f^l(K_{l-1}) = f(f^{l-1}(K_{l-1})) = f(I_{l-1}) \supset I_l$. By Lemma 2.2, there exists a subinterval $K_l \subset K_{l-1}$ such that $f^l(K_l) = I_l$ and $f^l(\text{int}(K_l)) = \text{int}(I_l)$. Adding the induction assumption, there exists $K_l \subset K_{l-1} \subset I_0$ such that for all $i = 1, 2, \cdots, l-1, l$ we get

$$f^i(K_l) \subset f^i(K_{l-1}) \subset I_i,\ f^i(\text{int}(K_l)) \subset f^i(\text{int}(K_{l-1})) \subset \text{int}(I_i) \text{ and } f^l(K_l) = I_l.$$

Therefore, the statement is proved.

Now for the case $l = n$ we obtain $f^n(K_n) = I_n = I_0$. By Corollary 2.4, there is a fixed point $x_0 \in K_n$ for $f^n$. Since $x_0 \in K_n \subset I_0$, it follows that for each $j = 0, 1, ..., n$, $f^j(x_0) \in I_j$. This proves part (i).

For part (ii), if $x_0 \in \text{int}(I_0)$ and since $f^n(\text{int}(K_n)) = \text{int}(I_0)$, then $x_0 \in \text{int}(K_n)$ and $f^j(x_0) \in \text{int}(I_j)$ for $j = 1, 2, \cdots, n$. Since the loop is not a product one, it implies that $x_0$ must have period $n$. □

**Definition 2.6.** The Sharkovskii's ordering is the ordering of the natural numbers in the following form:

$$\begin{array}{ccccccccccccc}
3 & \rhd & 5 & \rhd & 7 & \rhd & \cdots & & & & & & \\
2\cdot 3 & \rhd & 2\cdot 5 & \rhd & 2\cdot 7 & \rhd & \cdots & & & & & & \\
2^2\cdot 3 & \rhd & 2^2\cdot 5 & \rhd & 2^2\cdot 7 & \rhd & \cdots & & & & & & \\
\vdots & \rhd & \vdots & \rhd & \vdots & \rhd & \vdots & & & & & & \\
2^m\cdot 3 & \rhd & 2^m\cdot 5 & \rhd & 2^m\cdot 7 & \rhd & \cdots & & & & & & \\
2^{m+1}\cdot 3 & \rhd & 2^{m+1}\cdot 5 & \rhd & 2^{m+1}\cdot 7 & \rhd & \cdots & & & & & & \\
\vdots & \rhd & \vdots & \rhd & \vdots & \rhd & \vdots & & & & & & \\
2^{m+1} & \rhd & 2^m & \rhd & \cdots & \rhd & 2^2 & \rhd & 2 & \rhd & 1 & &
\end{array}$$

**Remark 2.7.** In the Sharkovskii's ordering, all odd integers greater than 1 are positioned in backward order by the symbol $\rhd$. Then, all integers which are two times an odd integer follow. Next, all odd integers times increasing powers of two follow. Finally, all the integers that are the power of two are added. It turns out that this ordering seem strange but it expresses the forcing of period in the Sharkovskii's Theorem.

**Definition 2.8.** If $n$ and $k$ are two positive integers such that $n \rhd k$ in Sharkovskii's ordering, we say that **period** $n$ **forces period** $k$.

## 2.2 Proof

**Theorem 2.9.** (**Sharkovskii's Theorem**) Let $I \subset \mathbb{R}$ be an interval. Let $f : I \to \mathbb{R}$ be a continuous function. If $f$ has a periodic point of least period $n$ and $n \rhd k$ in the above ordering, then $f$ has a periodic point of least period $k$.

**Remark 2.10.** The requirement for period in the statement of the theorem is necessarily least. For example, if $f$ has a fixed point at $x_0$, then $x_0$ is trivially a periodic point of period 5, so $f$ should also have a point of period 7 which is not necessarily true.

**Definition 2.11.** For a given closed subinterval $I \subset \mathbb{R}$, let $\mathcal{A} = \{I_1, I_2, \cdots, I_r\}$ be a finite partition of $I$ into closed intervals $I_j$ with disjoint interiors, $j = 1, 2, \cdots, r$. A **transition graph** of $f$ for the partition $\mathcal{A}$ is a directed graph with vertices representing $I_j$ and directed edges defined from $I_j$ to $I_k$ if $I_j \rightarrow I_k$.

We shall prove the theorem by using Stefan's technique found in [11]. He had the idea to prove the existence of an orbit on the real line with a special pattern. Let $x \in I$ be a periodic point of $f$ of least period $n > 1$ such that

$$f^{n-1}(x) < f^{n-3}(x) < \cdots < f^4(x) < f^2(x) < x < f(x) < f^3(x) < \cdots < f^{n-4}(x) < f^{n-2}(x).$$

A periodic point with such an ordering of its orbit on the real line is called **Stefan cycle**. Lemma 2.12 will prove its existence. Now let us denote $I_1 = [x, f(x)]$, $I_2 = [f^2(x), x]$ and

$$I_{2j-1} = [f^{2j-3}(x), f^{2j-1}(x)],$$
$$I_{2j} = [f^{2j}(x), f^{2j-2}(x)]$$

for $j = 2, 3, \cdots, \frac{n-1}{2}$. By continuity of the function $f$, we will have (i) $I_1$ covers $I_1$ and $I_2$, (ii) $I_j$ covers $I_{j+1}$ for $j = 2, \cdots, n-2$ and (iii) $I_{n-1}$ covers $I_j$ for $j$ odd. The existence of Stefan cycle proves that the transition graph of $f$ for the partition $\mathcal{A}$ contains a special subgraph called **Stefan transition graph** as given in Figure 2.2. Lemma 2.5 and Stefan transition graph will prove the period forcing in the Sharkovskii's ordering.

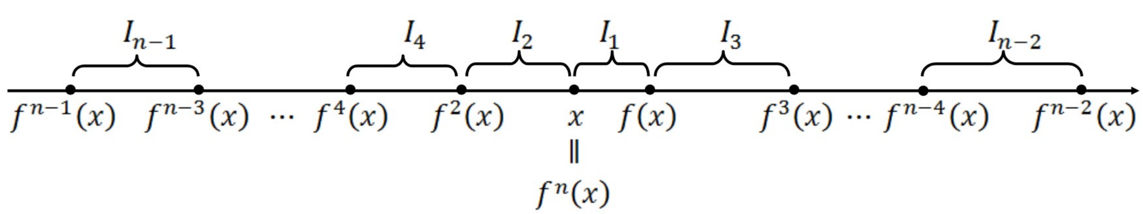


Figure 2.1: Stefan Cycle

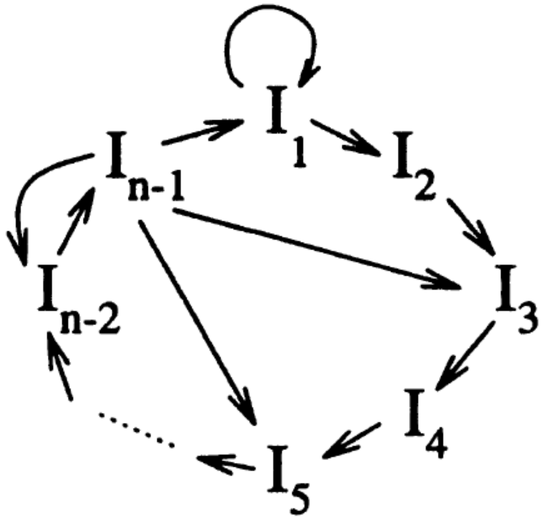


Figure 2.2: Stefan Transition Graph

**Lemma 2.12.** Let $n$ be an odd integer and $n > 1$. Suppose that $f$ has a point $x$ of least period $n$ and suppose that for any odd value $1 < k < n$, $f$ has no periodic points of period $k$. Let $J = [\min(\mathcal{O}(x)), \max(\mathcal{O}(x))]$. Let $\mathcal{A}$ be the partition of $J$ by the elements of $\mathcal{O}(x)$.

Then the transition graph of $f$ for $\mathcal{A}$ contains a transition subgraph of the following form:

(i) $I_1$ covers $I_1$ and $I_2$

(ii) $I_j$ covers $I_{j+1}$ for $2 \leqslant j \leqslant n-2$, and

(iii) $I_{n-1}$ covers $I_j$ for $j$ odd.

*Proof.* First, we label the elements in the orbit of $x$ in increasing order, so that $\mathcal{O}(x) = \{x_1, x_2, \cdots, x_n\}$ with $x_1 < x_2 < \cdots < x_n$. Since $f(x_n) = x_j$ for some $j = 1, 2, \cdots, n-1$, then $f(x_n) < x_n$. Similarly, we have $f(x_1) > x_1$ since $f(x_1) = x_j$ for some $j = 2, 3, \cdots, n$.
Let $a = \max\{y \in \mathcal{O}(x) \mid f(y) > y\}$. Clearly, $a \neq x_n$. Let $b$ be the next point in the orbit larger than $a$. Let $I_1 = [a, b]$. Notice that $I_1 \in \mathcal{A}$ and is the candidate for the Lemma 2.12.

We divide the proof into several claims as follow.

**Claim 2.12.1.** The image of $I_1$ covers itself, that is, $I_1 \to I_1$.

*Proof.* We have $f(a) > a$, so $f(a) \geqslant b$. Also, $f(b) < b$, so $f(b) \leqslant a$. Then $f(I_1) \supset I_1$. This means that $I_1 \to I_1$. □

**Claim 2.12.2.** The $(n-2)$-image of $I_1$ covers $J$, that is, $f^{n-2}(I_1) \supset J$.

*Proof.* Since $f(I_1) \supset I_1$, then $f^{k+1}(I_1) \supset f^k(I_1)$ for $k \in \mathbb{N}$, so the iterates are nested. Since the number of points in $\mathcal{O}(x) - \{a, b\}$ is $n-2$, then $x_n \in f^k(I_1)$ for some $0 \leqslant k \leqslant n-2$. By the nested property, $x_n \in f^{n-2}(I_1)$. Similarly, we get $x_1 \in f^{n-2}(I_1)$. Since $I_1 = [a, b]$ and it is connected, then $f^{n-2}(I_1) \supset J$. □

**Claim 2.12.3.** There exists $K_0 \in \mathcal{A}$ such that $K_0 \neq I_1$ and $K_0 \to I_1$.

*Proof.* Since $n$ is odd, there are more elements of $\mathcal{O}(x)$ on one side of $\mathrm{int}(I_1)$ than the other. Let

$$\mathcal{P} = \{x_i \in \mathcal{O}(x) \mid x_i \text{ is on the side of } \mathrm{Int}(I_1) \text{ with more elements}\}.$$

There exist $y_1$ and $y_2 \in \mathcal{P}$ such that $f(y_1) \in \mathcal{P}$ and $f(y_2) \in \mathcal{O}(x) - \mathcal{P}$. Let $K_0 = [y_1, y_2]$ by assuming that $y_1 < y_2$. Then $f(K_0) \supset I_1$ and $K_0 \neq I_1$. □

**Claim 2.12.4.** There exists a loop

$$I_1 \to I_2 \to \cdots \to I_k \to I_1$$

with $I_2 \neq I_1$. The shortest such loop with $k \geqslant 2$ has $k = n-1$.

*Proof.* Let $K_0$ be as in Claim 2.12.3. So we get $f(K_0) \supset I_1$ by Claim 2.12.3 and $f^{n-2}(I_1) \supset K_0$ by Claim 2.12.2. There are only $n-1$ distinct intervals in $\mathcal{A}$, so there exists such a loop with $2 \leqslant k \leqslant n-1$. Now suppose that $2 \leqslant k < n-1$. Since the loop is the shortest, none of intervals can be repeated and it cannot be shortened. Then $k$ or $k+1$ is odd. Let $m$ be this odd integer so $3 \leqslant m \leqslant n-3$.
Consider the loop with $m$ intervals

$$I_1 \to I_2 \to \cdots \to I_k \to I_1$$

or

$$I_1 \to I_2 \to \cdots \to I_k \to I_1 \to I_1$$

depending on $m = k$ or $m = k+1$. Then $f^m(z) = z$. By Lemma 2.5 (i), there is a point $z$ such that $f^m(z) = z$. Suppose $z \in \partial I_1$. Then this point has period $n > m$ which is a contradiction. Thus $z \in \mathrm{int}(I_1)$. By Lemma 2.5 (ii), so $z$ has least period $m$. Since $m$ is odd, this contradicts the fact that $n$ is maximal. Therefore, $k = n-1$. □

For the remaining proof, we fix $I_1$, $I_2$, $\cdots$, $I_{n-1}$ as in Claim 2.12.4.

**Claim 2.12.5.** (i) If $f(I_j) \supset I_1$, then $j = 1$ or $n-1$.
(ii) For $j > i+1$, there is no directed edge from $I_i$ to $I_j$ in a transition graph.
(iii) The interval $I_1$ covers only $I_1$ and $I_2$.

*Proof.* For part (i), if $j = 1$, then $f(I_1) \supset I_1$ by Claim 2.12.1.
If $j \neq 1$, then $j = n-1$ by Claim 2.12.4. (ii) and (iii) follows from the fact that the loop is the shortest one. □

**Claim 2.12.6.** Either (i) the ordering (in the sense of the real line) of the intervals $I_j$ in the loop of Claim 2.12.4 is

$$I_{n-1} \leqslant I_{n-3} \leqslant \cdots \leqslant I_2 \leqslant I_1 \leqslant I_3 \leqslant \cdots \leqslant I_{n-2}$$

and the order of the orbit is

$$f^{n-1}(a) < f^{n-3}(a) < \cdots < f^2(a) < a < f(a) < f^3(a) < \cdots < f^{n-4}(a) < f^{n-2}(a) \text{ or}$$

(ii) Both of these orderings are exactly reversed.

*Proof.* $I_1$ covers only $I_1$ and $I_2$ by Claim 2.12.5 (iii). So $I_1$ and $I_2$ are next to each other. Otherwise $I_1$ covers other intervals.
Assume $I_2 \leqslant I_1$ (the other possibility gives the reverse order). From here it follows that $f(a) = b$ and $f(b)$ is the left endpoint of $I_2$. Next, $f(\partial I_2) = \partial I_3$. Since one of the endpoints is $f(a) = b$ which is above the $\text{int}(I_1)$, then the other endpoint is above the $\text{int}(I_1)$. By claim 2.12.5 (i), we have $I_2 \nrightarrow I_1$ and (ii), we have $I_2 \nrightarrow I_j$ for $j > 3$. Then $I_3$ must be adjacent to $I_1$. Continuing the argument by induction. For $k < n-1$ , $I_k \nrightarrow I_1$ and $I_k \nrightarrow I_j$ for $j > k+1$. Then $I_{k+1}$ must be adjacent to $I_{k-1}$. And therefore the claim is proved □

**Claim 2.12.7.** $I_{n-1}$ covers all $I_j$ for all $j$ odd.

*Proof.* Let $I_{n-1} = [f^{n-1}(a), f^{n-3}(a)]$. Then $f(f^{n-1}(a)) = f^n(a) = a$ and $f^{n-3}(a) \in I_{n-3}$. So $f(f^{n-3}(a)) = f^{n-2}(a) \in I_{n-2}$. Therefore $f(I_{n-1}) \supset [a, f^{n-2}(a)] = I_1 \cup I_3 \cup \cdots \cup I_{n-2}$. Therefore, $I_{n-1} \to I_j$ for $j$ odd. □

Now we begin the proof of Sharkovskii's Theorem.

*Proof.* Case 1 : Assume $n > 1$ is odd and maximal in the Sharkovskii's ordering. If $n \rhd k$, then $f$ has a periodic point of period $k$.

Case 1.1 : $k < n$ and $k$ is even.
Consider $I_{n-1} \to I_{n-k} \to I_{n-k+1} \to \cdots \to I_{n-1}$ of length $k$. By Lemma 2.5 (i), there exists $x_0 \in I_{n-1}$ such that $f^k(x_0) = x_0$.

If $x_0 \in \partial I_{n-1}$, then $x_0$ has period $n$ contradicting that $k < n$. Then $x_0 \in \text{int}(I_{n-1})$. Therefore $x_0$ has period $k$ by Lemma 2.5 (ii).

Case 1.2 : $k > n$ and $k$ is either even or odd.
Consider $I_1 \to I_2 \to \cdots \to I_{n-1} \to I_1 \to I_1 \to \cdots \to I_1$ of length $k$. There exists $x_0 \in I_1$ such that $f^k(x_0) = x_0$. If $x_0 \in \partial I_1$, then $x_0$ has period $n$. By Lemma 2.5, $n|k$ and this implies $k \geqslant 2n \geqslant n + 3$. Since $f^n(x_0) \in I_1$ then $f^{n+1}(x_0) \notin I_1$ which contradicts Lemma 2.5 (i). So $x_0 \notin \partial I_1$. Therefore, $x_0$ has period $k$ by Lemma 2.5 (ii).

Case 2 : $n = 2^m$ and $n \rhd k$ with $k = 2^s$, $0 \leqslant s < m$.

Case 2.1 : $s = 0$, that is, $f$ has a fixed point.
We have $f(a) \geqslant b$ and $f(b) < b$. Then $f(b) \leqslant a$. Thus $f(I_1) \supset I_1$ where $I_1 = [a, b]$. By Corollary 2.4, it follows that $f$ has a fixed point in $I_1$. Since endpoints of $I_1$ have period $n > 1$, the fixed point lies in the interior of $I_1$. □

Case 2.2 : $s = 1$, that is, $f$ has a point of period 2. First we shall prove the following lemma.

**Lemma 2.13.** If $f$ has a point of even period, then it has a point of period 2.

*Proof.* Let $J = [\min \mathcal{O}(x), \max \mathcal{O}(x)] = [x_1, x_n]$ and let $I_1 = [a, b]$. First suppose there exists $K_0 \in \mathcal{A}$ such that $K_0 \neq I_1$ and $K_0 \to I_1$. Then the loop in Claim 2.12.4 contains a shorter loop with $2 \leqslant k \leqslant n - 1$. $I_k$ covers all $I_j$ on the other sides. Thus $I_{n-1} \to I_{n-2} \to I_{n-1}$ is a loop of length 2 and $f$ has a point of period 2.
Now suppose that there does not exist $K_0 \in \mathcal{A}$ with $K_0 \neq I_1$ such that $K_0 \to I_1$. Notice first that $a \neq x_1$. For otherwise suppose that $a = x_1$. Recall by definition that $a = \max\{y \in \mathcal{O}(x) \mid f(y) > y\}$. So $f(x_1) > x_1$. Then $f(x_1) \geqslant b$ and $f(x_j) < x_j$ for all $j = 2, 3, \cdots, n$. Then $f(b) = f(x_2) = x_1 = a$. There exists $[b, x_3] \in \mathcal{A}$ such that $[b, x_3] \neq I_1$ and $[b, x_3] \to I_1$. It contradicts to the above assumption. But $b$ could be either $b = x_n$ or $b < x_n$.

Case (a) : $a \neq x_1$ and $b < x_n$. We first claim the following two facts.
(i) $\forall x_j \in \mathcal{O}(x), x_j \leqslant a$ we have $f(x_j) \geqslant b$. (ii) $\forall x_j \in \mathcal{O}(x), x_j \geqslant b$ we have $f(x_j) \leqslant b$.

For part (i), if $x_j = a$, then $f(x_j) > a$, that is, $f(x_j) \geqslant b$ as $b$ is the next point in the $n$-orbit larger than $a$. If $x_j < a$, suppose $f(x_j) \leqslant a$. By definition of $a$, $f(a) \geqslant b$. Then there exists an interval $K_0 = [x_j, a]$ distinct from $I_1$ and such that $K_0 \to I_1$, which contradicts the previous assumption. Thus part (i) is proved. For part (ii), if $x_j = b$, then $f(x_j) \leqslant b$. If $x_j > b$, suppose that $f(x_j) > b$. Thus there exists $K_0 = [b, x_j]$ such that $K_0 \neq I_1$ but $K_0 \to I_1$. This again contradicts the above assumption. Thus we proved the second part.

Now we claim that $[x_1, a] \to [b, x_n] \to [x_1, a]$. First suppose that $f([x_1, a]) \not\supset [b, x_n]$. Then there exists $x_j \in [b, x_n]$ such that $x_j \notin f([x_1, a])$. For $x_j$, there exists $\hat{x}_j \in \mathcal{O}(x)$ such that $f(\hat{x}_j) = x_j$. By item (i), $\hat{x}_j \in [x_1, a]$ and so $x_j = f(\hat{x}_j) \in f([x_1, a])$ which is a contradiction. Thus, $f([x_1, a]) \supset [b, x_n]$. Similarly, suppose that $f([b, x_n]) \not\supset [x_1, a]$. Then there exists $x_k \in [x_1, a]$ such that $x_k \notin f([b, x_n])$. For $x_k$, there exists $\tilde{x}_k \in \mathcal{O}(x)$ such that $f(\tilde{x}_k) = x_k$. By item (ii), $\tilde{x}_k \in [b, x_n]$ and so $x_k = f(\tilde{x}_k) \in f([b, x_n])$ which is a contradiction. Thus $f([b, x_n]) \supset [x_1, a]$. Therefore, $f$ has a point of period 2.

Case (b) : $x_1 < a < b = x_n$.
First let us notice that $f(a) = x_n$. Indeed, by definition of $a$, $f(a) > a$. So $f(a) \geqslant b$, that is, $f(a) \geqslant x_n$ but $f(a) \leqslant x_n$, then $f(a) = x_n$. Furthermore, since $b = x_n$, then $a = x_{n-1}$. If $f(x_{n-1}) = a$, then $a$ is a fixed point which impossible. If $f(x_n) = a$, that is, $f(b) = a$, then $f^2(b) = f(f(b)) = f(a) = b$. This shows that $b$ is a point of period 2 which again impossible. Thus $f(x_k) = a$ for some $k = 1, 2, \cdots, n-2$. Let $x_j$ be the image of $b$ under $f$, that is, $f(b) = x_j$. Under the action of $f$, we have $x_k \to a \to b \to x_j$.

Now we get three possibilities for $x_j$.
(a) For $x_j = x_k$, then $a$ lies in a 3-cycle. Thus we get 2-loop $[x_j, a] \to I_1 \to [x_j, a]$. Therefore, $f$ has a point of period two in $\text{int}(I_1)$ since these intervals has empty intersection of their interiors.
(b) For $x_j < x_k$, let us consider $I_2 = [x_k, a]$. It follows that $I_2 \to I_1 \to I_2$. Therefore, $f$ has a point of period two in $\text{int}(I_1)$.
(c) For $x_j > x_k$, let us consider $I_3 = [x_j, a]$. It follows that $I_3 \to I_1 \to I_3$. Thus, $f$ has a point of period two in $\text{int}(I_1)$. therefore the lemma is proved. □

The proof of the Case 2.2 just follows immediately from Lemma 2.13 as $2^m$ is even.

Case 2.3 : $n = 2^m, k = 2^s, 1 < s < m$.
Let $g = f^{k/2} = f^{2^{s-1}}$. Since $f$ has a point $\hat{x}$ of period $2^m$, so $f^{2^m}(\hat{x}) = \hat{x}$. Then we have $g^{2^{m-s+1}}(\hat{x}) = (f^{2^{s-1}})^{2^{m-s+1}}(\hat{x}) = f^{2^m}(\hat{x}) = \hat{x}$. Then $g$ has a point of period $2^{m-s+1}$ which is even with $m - s + 1 \geqslant 2$ . By Lemma 2.13, $g$ has a point $x_0$ of period 2. So $g^2(x_0) = (f^{2^{s-1}})^2(x_0) = f^{2^s}(x_0) = x_0$ and $g(x_0) = f^{k/2}(x_0) = f^{2^{s-1}}(x_0) \neq x_0$. Thus $f$ has a point $x_0$ of period $2^t$ for some $t \leqslant s$. For now suppose $t < s$. Then $x_0$ is fixed by $g$ which is impossible. So we obtain $t = s$. Therefore, $f$ has point $x_0$ of period $2^s = k$.

Case 3 : $n = 2^m \cdot p$ , $p > 1$, $p$ odd, $m \geqslant 1$, $n$ maximal in the Sharkovskii ordering and $n \rhd k$.

Case 3.1 : $k = 2^s \cdot q$, $s \geqslant m + 1$, $q \geqslant 1$, $q$ odd.

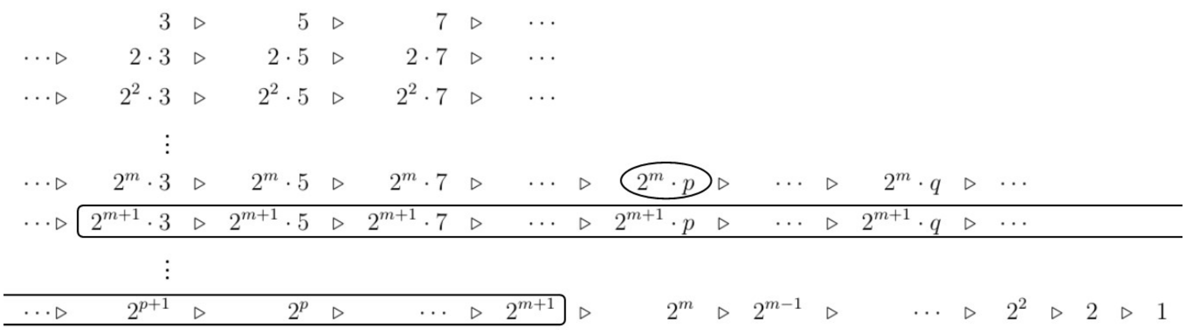


Figure 2.3: In Case 3.1: $k = 2^s \cdot q$, $s \geqslant m + 1$, $q \geqslant 1$, $q$ odd.

Let $g = f^{2^m}$. Since $f$ has a periodic point $x_0$ of period $2^m \cdot p$, then $f^{2^m \cdot p}(x_0) = x_0$. So $(f^{2^m})^p(x_0) = x_0$. Thus, $g$ has a periodic point of period $p$. Since $p$ is odd, by Case 1, $g$ has a point of period $2^{s-m} \cdot q$, where $s - m \geqslant 1$. Therefore $f$ has a point of period $2^s \cdot q$ with $s \geqslant m + 1$, $q \geqslant 1$, $q$ odd.

Case 3.2 : $k = 2^s, s \leqslant m$.

Let $g = f^p$. Since $f$ has a periodic point $x_0$ of period $2^m \cdot p$, then $f^{2^m \cdot p}(x_0) = x_0$. So $(f^p)^{2^m}(x_0) = x_0$. Then $g$ has a periodic point of period $2^m$. By Case 2, $g$ has a point of period $2^s$, $0 \leqslant s < m$. Thus $f$ has a point of period $2^s \cdot p$ with $p > 1$, $p$ odd. By Case 3.1, $f$ has a point of period $2^t \cdot q$ with $t \geqslant s + 1$ and $q \geqslant 1$, $q$ odd. Choosing $t_0 \geqslant m + 1$ and choosing $q = 1$, we obtain $f$ has a point of period $2^{t_0}$ with $s \leqslant m < t_0$. By Case 2, $f$ has a point of period $2^s$ with $s \leqslant m$.

Case 3.3 : $k = 2^m \cdot q$, $q$ odd, $q > p$.

Let $g = f^{2^m}$. Since $f$ has a periodic point $x_0$ of period $2^m \cdot p$, then $f^{2^m \cdot p}(x_0) = x_0$. So $(f^{2^m})^p(x_0) = x_0$. Thus, $g$ has a periodic point of period $p$. By Case 1.2, then $g$ has a periodic point of period $q$ odd since $q > p$. Therefore, $f$ has a periodic point of period $2^m \cdot q$ with $q$ odd, $q > p$.

$\square$

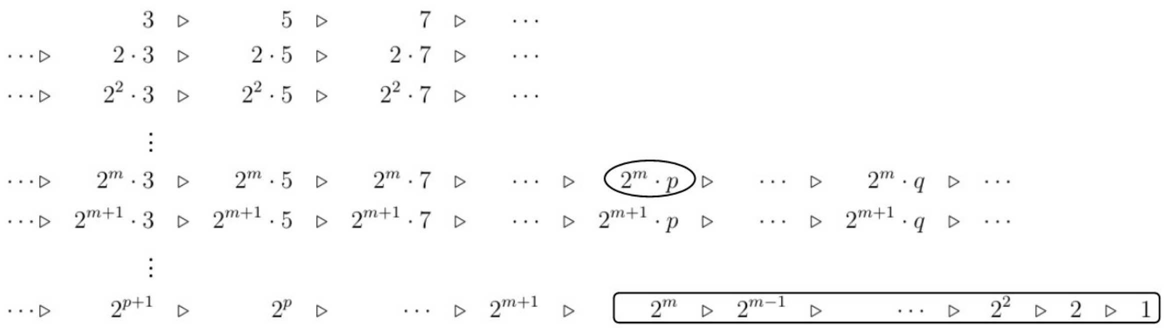


Figure 2.4: In Case 3.2: $k = 2^s, s \leqslant m$.

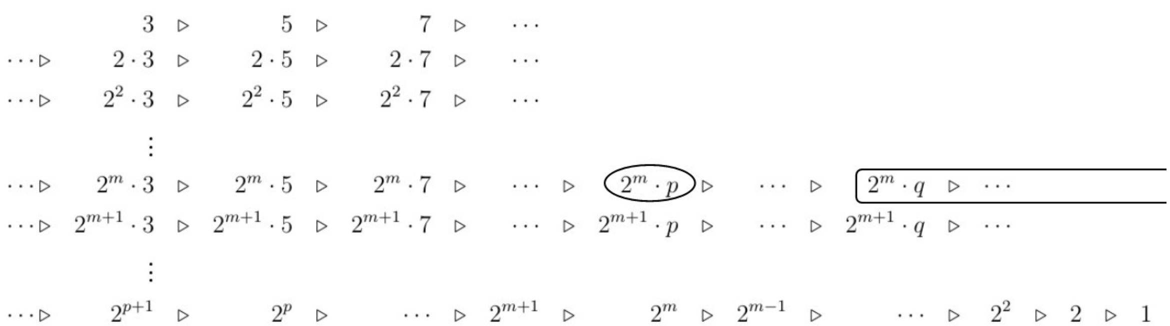


Figure 2.5: In Case 3.3: $k = 2^m \cdot q$, $q$ odd, $q > p$.

# Chapter 3

# Results

## 3.1 Introduction

My thesis project has its origin from the article [7] where the authors study parameter space and dynamics of the family of complex valued functions

$$F_b(z) = \wp_\Lambda(z) + b,\ z, b \in \mathbb{C},$$

where $\wp_\Lambda$ represents the Weierstrass $\wp$ function defined over a lattice $\Lambda$. When $\Lambda$ is fixed and the parameter $b$ is restricted to real values, the authors showed that the orbits of the critical values of $F_b$ lie in the real line, thus in order to understand the dynamics of $F_b$, it is enough to study the dynamics of the family of real-valued functions

$$f_b(t) = \wp_\Lambda(t) + b,\ t, b \in \mathbb{R}.$$

The global dynamics of $F_b|_\mathbb{R}$ for real parameters $b$ (and hence of $f_b : \mathbb{R} \to \mathbb{R}$) can be determined.

Here we will study the dynamics of the Weierstrass $\wp_\Lambda$ function restricted to the real line under the assumption that $\Lambda$ is the **central lattice**.

Recall that Weierstrass $\wp$ function, $\wp_\Lambda : \mathbb{C} \to \mathbb{C}$, is given by the expression

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda^*} \left[ \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right]$$

where the sum ranges over all nonzero lattice points $\lambda \in \Lambda$, where

$$\Lambda = \{m\lambda_1 + n\lambda_2 \mid \mathrm{Im}(\lambda_2/\lambda_1) > 0,\ m, n \in \mathbb{Z}\}.$$

We denote by $g_2$ and $g_3$ the two **invariants** of the lattice $\Lambda$. If $g_2 > 0$ and $g_3 = 0$, then the lattice $\Lambda$ is called a **real square lattice**. We recall some of the main properties of the Weierstrass $\wp$ function that have been previously discussed in Chapter 1, Section 2. First, $\wp_\Lambda$ is a meromorphic function with double poles at the points of $\Lambda$. Second, $\wp_\Lambda$ is doubly periodic with periods $\lambda_1$ and $\lambda_2$, the generators of the lattice $\Lambda$. Finally, $\wp'_\Lambda(z)$ is odd, double periodic and has zeros at $\dfrac{\lambda_1}{2}, \dfrac{\lambda_2}{2}, \dfrac{\lambda_1+\lambda_2}{2}$ and all its translation by $\Lambda$.

The last property describes the critical points of $\wp_\Lambda$ which are typically denoted as

$$\omega_1 = \frac{\lambda_1}{2}, \quad \omega_2 = \frac{\lambda_2}{2}, \quad \omega_3 = \frac{\lambda_1+\lambda_2}{2} = \omega_1 + \omega_2.$$

The constant $\kappa = \frac{\Gamma(1/4)^2}{4\sqrt{\pi}} \approx 1.85407467$ appears in the solution of the elliptic integral

$$\omega_1 = \int_{\frac{\sqrt{g_2}}{2}}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t}} = \frac{\kappa}{\sqrt{g_2}}.$$

and implies that $\wp_\Lambda(\omega_1) = \dfrac{\sqrt{g_2}}{2}$ [1]. If $g_2 = (2\kappa)^{\frac{4}{3}}$, we will see in Proposition 3.2 that $\wp_\Lambda(\omega_1) = \omega_1$, that is, the real critical point $\omega_1$ is fixed under $\wp_\Lambda$ where $\Lambda$ is the central lattice.

Consider the real square central lattice $\Lambda$ with invariants $g_2 = (2\kappa)^{\frac{4}{3}} \approx 5.73953$ and $g_3 = 0$. The aim of the thesis is see whether or not and how far the Sharkovskii's theorem applies to the family of $f_b(x) = \wp_\Lambda(x) + b$, with $b \in \mathbb{R}$. Also, in the last part, we prove that for the case $f(x) = \wp(x)$, this function is chaotic on a certain invariant set on each fundamental interval of definition.

## 3.2 The central lattice

We begin by establishing some basic properties of real square lattices.

**Proposition 3.1.** If $\Lambda$ is a real square lattice, then the critical values of $\wp_\Lambda$ are all real numbers and are given by $e_1 = \dfrac{\sqrt{g_2}}{2}$, $e_2 = -e_1$ and $e_3 = 0$.

*Proof.* Since any real square lattice is similar to the lattice generated by $\{1, i\}$, we assume without lost in generality that $\Lambda = \{m + ni \mid m, n \in \mathbb{Z}\}$. First, we note that $g_2 > 0$ and $g_3 = 0$. Indeed, by Proposition 1.63, Eisenstein series of order $k$ satisfy $E_k(\tau) = \tau^{-k} E_k(-1/\tau)$. If $\tau = i$, then

$$E_6(i) = i^{-6} E_6(i) = -E_6(i).$$

Then $E_6(i) = 0$ and so $g_3 = 140E_6(i) = 0$. We have

$$g_2 = E_4(i) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m+ni)^4}$$
$$= \sum_{(m,n)\neq(0,0)} \frac{m^4+n^4-6m^2n^2}{|m+in|^8} + i \sum_{(m,n)\neq(0,0)} \frac{mn(m^2-n^2)}{|m+in|^8}.$$

Using symmetry argument for the imaginary part of $E_4(i)$ we get $\mathrm{Im}(E_4) = 0$ and a rearrangement of terms in its real part we obtain $\mathrm{Re}(E_4) > 0$. Thus $g_2$ is positive.

The critical values of $\wp_\Lambda$ are given by the expressions

$$\wp_\Lambda\left(\frac{1}{2}\right) = e_1, \quad \wp_\Lambda\left(\frac{i}{2}\right) = e_2 \quad \text{and} \quad \wp_\Lambda\left(\frac{1+i}{2}\right) = e_3.$$

The homogeneity condition in (1.3) for $k = i$, combined with the fact that $\Lambda$ is square (so $i\Lambda = \Lambda$), we obtain that, for any $z \in \mathbb{C}$,

$$\wp_\Lambda(iz) = \wp_{i\Lambda}(iz) = -\wp_\Lambda(z).$$

This implies that $\wp_\Lambda(i/2) = -\wp_\Lambda(1/2)$, in other words, $e_2 = -e_1$.

By Theorem 1.61, the differential equation that appears in Proposition 1.66 can be written in terms of the (distinct) critical values $e_1, e_2$ and $e_3$ as

$$(\wp'_\Lambda(z))^2 = (\wp_\Lambda(z) - e_1)(\wp_\Lambda(z) - e_2)(\wp_\Lambda(z) - e_3). \tag{3.1}$$

Expanding the right-hand side of (3.1) and comparing coefficients with the right-hand side of (1.5), combined with the fact $g_3 = 0$, $g_2 > 0$, we obtain the relations

$$e_1 + e_2 + e_3 = 0, \quad e_1e_2 + e_1e_3 + e_2e_3 = -\frac{g_2}{4}, \quad e_1e_2e_3 = 0.$$

The last relation implies that one (and only one) of the critical values is zero. Since $e_2 = -e_1$, we conclude $e_3 = 0$. Finally, from the second relation, we obtain

$$-e_1^2 = -\frac{g_2}{4}, \quad \text{hence} \quad e_1 = \frac{\sqrt{g_2}}{2} > 0,$$

and all three critical values are real numbers. □

If $\Lambda$ is a real square lattice with generators $\{\lambda, i\lambda\}$ for some $\lambda > 0$, then $\wp_\Lambda$ has a real critical point at $\lambda/2$ so that $\wp_\Lambda(\lambda/2) = e_1$. Moreover, $\lambda/2$ can be expressed in terms of the integral

$$\frac{\lambda}{2} := \int_{e_1}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2t}} = \frac{\Gamma(1/4)^2}{4\sqrt{\pi}(g_2)^{1/4}}.$$

**Proposition 3.2.** Given $\Lambda$ real square lattice generated by $\{\lambda, i\lambda\}$, $\lambda > 0$, there exists a unique value $g_2 > 0$ for which the equation $\wp_\Lambda(\lambda/2) = \lambda/2$ is satisfied.

*Proof.* Let $\kappa = \Gamma(1/4)^2/(4\sqrt{\pi})$ so $\lambda/2 = \kappa/(g_2)^{1/4}$. Since $e_1 = \wp_\Lambda(\lambda/2)$ and $\Lambda$ is real square, then solving $\wp_\Lambda(\lambda/2) = \lambda/2$ is equivalent to solve

$$e_1 = \frac{\sqrt{g_2}}{2} = \frac{\kappa}{(g_2)^{1/4}} = \frac{\lambda}{2}$$

in terms of $g_2$. A quick computation shows that $g_2 = (2\kappa)^{4/3} \approx 5.739529$.

□

**Definition 3.3.** The **center lattice** is the real square lattice $\Lambda$ with generators $\{\lambda, i\lambda\}$ and $(g_2, g_3)$ as given in Proposition 3.2. That is $\lambda = 2\kappa/(g_2)^{1/4}$ for $g_2 \approx 5.739529$.

## 3.3 Weierstrass $\wp$ function restricted on $\mathbb{R}$

In this section, we gather properties of the Weierstrass $\wp$ function restricted on the real line. In fact, we will study its behavior on each fundamental interval $I_j = (j\lambda, (j+1)\lambda)$ for each $j \in \mathbb{Z}$.

**Theorem 3.4.** The following are equivalent:

1) $\wp_\Lambda$ is a real function, that is, $\wp_\Lambda(\overline{z}) = \overline{\wp_\Lambda(z)}$ for all $z \in \mathbb{C}$.
2) $\Lambda$ is a real lattice.
3) $g_2$, $g_3$ are real numbers.

The proof of this theorem can be found in [8]. If $\Lambda$ is a real lattice (for example, if $\Lambda$ is the central lattice) then it follows from Theorem 3.4 that $\wp_\Lambda|_\mathbb{R}$ is a real-valued function.

The following lemma is based on Lemma 4.7 in [6]. This lemma describes the crucial properties of $\wp$ when restricted over the real line. We provide its proof for completeness.

**Lemma 3.5.** If $\wp_\Lambda$ is real, then it is periodic as a map on $\mathbb{R}$ and has infinitely many critical points and at least one real critical value. There is at least one nonnegative critical value that is the minimum of $\wp$ on $\mathbb{R}$. In particular, if $e_r$ denotes the real critical value, then $\wp|_\mathbb{R} : \mathbb{R} \to [e_r, \infty]$ is piecewise monotonic and onto.

*Proof.* Since $\wp_\Lambda$ is real, then by Theorem 3.4, the lattice $\Lambda$ is real rectangular. It follows then that $\Lambda = \{m\lambda_1 + n\lambda_2 \mid m, n \in \mathbb{Z}\}$ with $\lambda_1 \in \mathbb{R}$ and $\lambda_2 \in i\mathbb{R}$. Then it is clear that the real period of $\wp_\Lambda$ is $\lambda_1$.

Since the critical points are half lattice points, and the sums of the half lattice points, then we have $\lambda/2$ is the only half period in the fundamental interval. Thus, all points of the form $\lambda/2 + m\lambda$ are critical points on the real line where $m$ is an integer.

Since there are infinitely many critical points, one in each fundamental interval, then their common value is a real critical value. Since $\wp'$ is strictly increasing, and is negative to the left of a critical point and positive to the right (in each fundamental interval), then $e_r$ is the minimum on $\mathbb{R}$. In particular, if $\lambda$ is the real period of $\wp_\Lambda$ and $m$ is any integer, then $\wp_\Lambda : (m\lambda, m\lambda + \lambda/2] \to [e_r, \infty)$ is monotonic and onto, as is $\wp : [m\lambda + \lambda/2, (m+1)\lambda) \to [e_r, \infty)$. □

**Theorem 3.6.** Fix a lattice $\Lambda$. Then for any $z \in \mathbb{C} - \Lambda$, we have

$$\wp(z \pm \omega_i) = \frac{(e_i - e_j)(e_i - e_k)}{\wp(z) - e_i} + e_i.$$

**Remark 3.7.** It implies from Theorem 3.6 that for each $j \in \mathbb{Z}$, the Weierstrass $\wp$ function is even on each $I_j$ and symmetric with respect to the vertical line passing through critical point $(j+1)\lambda/2$.

From the result of Theorem 3.4 and Theorem 3.6, we can conclude the following theorem.

**Theorem 3.8.** For the central lattice $\Lambda$, the Weierstrass function $\wp_\Lambda : \mathbb{R} \to \mathbb{R}$ has following properties.

(i) $\wp_\Lambda$ is continuous, even, strictly monotone, $\lambda$-periodic function on each fundamental interval $I_j = (j\lambda, (j+1)\lambda)$ for $j \in \mathbb{Z}$.

(ii) On each fundamental interval $I_j$, $\wp_\Lambda|_{I_j}$ attains the minimum value $\lambda/2$ at $\lambda/2 + j\lambda$ for each $j \in \mathbb{Z}$.

## 3.4 Period forcing for Weierstrass $\wp$ function

In this section, we will consider the family of functions $f_b$ defined by

$$f_b(x) = \wp(x) + b$$

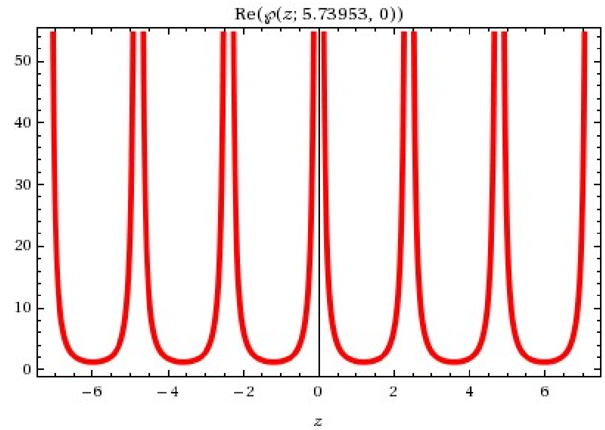


Figure 3.1: Graph of Weierstrass $\wp$ function restricted to the real line and defined over the central lattice.

with $x, b$ are real numbers. By Lemma 3.5, if $b$ is given, then we have $f_b(\mathbb{R}) = (\wp + b)|_{\mathbb{R}}(\mathbb{R}) = [\lambda/2 + b, \infty]$. The smallest integer $j$ that satisfies $j\lambda > \lambda/2 + b$ or equivalently $j > b/\lambda + 1/2$ specifies the index of the fundamental interval $I_{j-1}$ where $b$ lies in.

We aim to provide an extension of Sharkovskii's period forcing theorem even when the function $f_b$ is no longer continuous on its domain.

Notice that for any real number $x$, we denote the smallest integer greater that $x$ by $\lceil x \rceil$ and the greatest integer part of $x$ by $\lfloor x \rfloor$.

**Lemma 3.9.** The action of $f_b$, $b \in \mathbb{R}$, on each interval $I_j = (j\lambda, (j+1)\lambda)$, $j \in \mathbb{Z}$ determines the following transitions.

(i) For all $j \geqslant \lceil b/\lambda + 1/2 \rceil$ , $I_j$ $f_b$-covers itself and $I_k$ for all $k \geqslant \lceil b/\lambda + 1/2 \rceil$.

(ii) For $j \leqslant \lfloor b/\lambda + 1/2 \rfloor$, $I_j$ does not $f_b$-covers itself, while for every $k \geqslant \lceil b/\lambda + 1/2 \rceil$, $I_j$ $f_b$-covers $I_k$.

For convenience, we just consider the case $b = 0$. For the other values of $b$, the graph of $f_b$ is merely a translation of the graph of $\wp$ up or down $b$ units.

**Lemma 3.10.** The transition graph of $\wp$ contains a subgraph with the following conditions.

(i) For all $j, k \geqslant 1$, $I_j$ $\wp$-covers $I_j$ itself and $I_k$; and $I_k$ $\wp$-covers $I_j$ and itself.

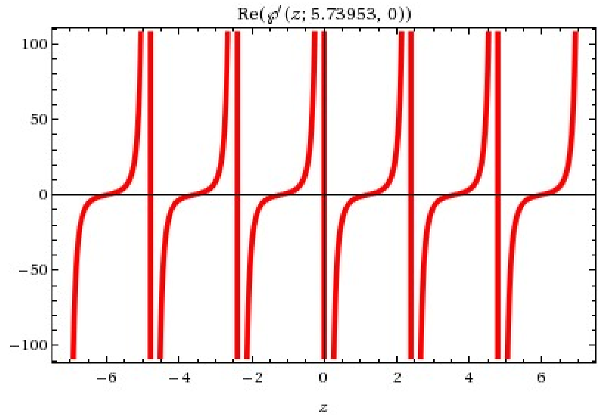


Figure 3.2: Graph of the derivative of Weierstrass $\wp$ function same as above.


(ii) For $j \leqslant 0$, $k \geqslant 1$, we have $I_j$ $\wp$-cover $I_k$.

*Proof.* (i) It follows form Lemma 3.5 that for each $j \geqslant 1$ we have $\wp(I_j) = [\lambda/2, \infty)$ and each $I_j \subset [\lambda/2, \infty)$ for $j \geqslant 1$.
(ii) Also, for $j \leqslant 0$, we have $\wp(I_j) = [\lambda/2, \infty)$ which covers any $I_k$ for $k \geqslant 1$. □

In Chapter 2, we proved the very important Lemma 2.5 which ensure the existence of $n$-cycle for a continuous function on its domain. Likewise, the following Lemma guarantees the existence of an $n$-cycle for the Weierstrass $\wp$ function, which is continuous over the real line except at $j\lambda$ for any $j \in \mathbb{Z}$. This Lemma will allow us to find a periodic point of least period $n$ for the Weierstrass $\wp$ function. We denote the fundamental interval $J_i$ for $i \geqslant 1$ so that $J_i = I_j$ for some $j \geqslant 1$.

**Lemma 3.11.** Let $J_i$ be a fundamental interval for $i = 0, 1, \cdots, n$ and let $J_0 \to J_1 \to \cdots \to J_n = J_0$ be a loop with $J_i \to J_{i+1}$ for all $i = 0, 1, \cdots, n-1$.

(i) Then there exists a fixed point $x_0$ in $J_0$ such that $\wp^m(x_0) = x_0$ with $m \mid n$ and $m$ is the least period.

(ii) Suppose further that this loop is not a product loop formed by going $p$ times around a shorter loop of length $m$ where $mp = n$.

If the periodic point $x_0 \in J_0$, then $\wp$ has a least period $n$.

*Proof.* Let us first prove the following statement by induction. There exists a closed interval $K_l \subset J_0$ such that for each $i = 1, 2, \cdots, l$,

$$\wp^i(K_l) \subset J_i,\ \wp^i(\mathrm{Int}(K_l)) \subset J_i \text{ and } \wp^l(K_l) = \bar{J}_l.$$

Here, we have $J_i = I_j = (j\lambda, (j+1)\lambda)$ for some $j$.

Case $l = 1$: There exists $K_1 = [a_1, b_1] \subset J_1$ with $a_1 < \frac{(j+1)\lambda}{2} < b_1$ and $\wp(a_1) = \wp(b_1) = (j+1)\lambda$. It follows that $\wp(K_1) \supset \bar{J}_1$ and $K_1 \subset \bar{J}_1$. By Lemma 2.5, it implies that there is a closed subset $L_1 \subset K_1 \subset J_1$ such that $\wp(L_1) = \bar{J}_1$, $\wp(\mathrm{Int}(L_1)) \subset \mathrm{Int}(\bar{J}_1) = J_1$ and $\wp(\partial L_1) = \partial \bar{J}_1 = \{j\lambda, (j+1)\lambda\}$.
Case $l-1$: There exists a closed subset $K_{l-1} \subset J_0$ such that for every $i = 1, 2, \cdots, l-1$, $\wp^i(K_{l-1}) \subset J_i$, $\wp^j(\mathrm{Int}(K_{l-1})) \subset J_i$ and $\wp^l(K_{l-1}) = \bar{J}_l$.

For Case $l$: we have $\wp^l(K_{l-1}) = \wp(\wp^{l-1}(K_{l-1})) = \wp(\bar{J}_{l-1}) \supset \bar{I}_l$ by Lemma 3.10. So by Lemma 2.5, there exists a closed subset $K_l \subset K_{l-1} \subset J_0$ such that $\wp^l(K_l) = \bar{J}_l, \wp^l(\mathrm{int}(K_l)) = J_l$ and $\wp^l(\partial K_l) = \{l\lambda, (l+1)\lambda\}$. Now combining this result with hypothesis from the Case $l-1$, we proved the statement.

Now in order to prove that $\wp^n$ has a fixed point in $J_0$, we just consider the case $l = n-1$. By the previous statement, there is a close subset $K_{n-1} \subset J_0$ such that $\wp^{n-1}(K_{n-1}) = \bar{J}_{n-1}$. Moreover, $\wp(\bar{J}_{n-1}) = [\frac{\lambda}{2}, \infty]$. Since $[\frac{\lambda}{2}, \infty] \supset \bar{I}_1$, it implies that $\wp(\bar{J}_{n-1}) \supset \bar{J}_0$. Again, by Lemma 2.2, there exists a closed subset $K \subset \bar{J}_{n-1}$ such that $\wp(K) = \bar{J}_0$. Then there exists a closed subset $K_0 \subset \mathrm{int}(K_{n-1}) \subset J_0$ such that $\wp^n(K_0) = \wp(\wp^{n-1}(K_0)) = \wp(K) = \bar{J}_0$. By Corollary 2.4, $\wp^n$ has a fixed point in $K_0 \subset J_0$.

(ii) Now we have $x_0 \in J_0$. Since $\wp^n(\mathrm{Int}(K_n)) = J_0$, then $x_0 \in \mathrm{Int}(K_n)$ and $\wp^i(\mathrm{Int}(K_n)) \in J_i$ for $i = 1, 2, \cdots n$. Also, since the loop is not a product, $x_0$ must have period $n$. This competes the proof of the lemma. □

Notice that by a complete directed graph $(V, E)$ where $V$ and $E$ denote the vertice and the edge, respectively we mean for any pair of vertices, there exists a pair of opposite directed edges adjoining the two vertices.

By Lemma 3.10 (i), it is a complete directed graph. It follows then that it induce a Stefan cycle and thus the Sharkovskii's theorem applies to $\wp$ by Lemma 3.5. Moreover, by the completeness of the transition graph, there are more properties on period forcing. More precisely, the forcing period holds for any period.

**Theorem 3.12.** Given $n$ and $k$ two positive integers such that $n \rhd k$ in the Sharkovskii's ordering, that is, $n$ forces $k$. If the Weierstrass $\wp$ function has a point of odd period $n$ with $n \geqslant 3$, then $\wp$ has a point of period $k$ with $n \rhd k$.

*Proof.* Case a: $k < n$ and $k$ even. Consider the loop $I_{n-1} \to I_{n-k} \to I_{n-k+1} \to \cdots I_{n-1}$ of length $(n-1)-(n-k)+1 = k$. By Lemma 3.11 (ii), there exists $x_0 \in I_{n-1}$, $\wp^k(x_0) = x_0$.

Case b: $k > n$, $k$ is even or odd. Consider $I_1 \to I_2 \to \cdots \to I_{n-1} \to I_1 \to I_1 \to \cdots \to I_1$ of length $k$. By Lemma 3.11 (ii), there exists $x_0 \in I_1$ such that $\wp^k(x_0) = x_0$. □

This theorem provides a complete existence of any $n$-cycle of $\wp$.

**Theorem 3.13.** The Weierstrass $\wp$-function has a periodic point of period $n$ for $n \geqslant 1$ on each $I_j$ for all $j \geqslant 1$.

*Proof.* Consider the loop $I_j \to I_{j+1} \to \cdots \to I_{n+1+j} = I_j$ of length $n$. By Lemma 3.11, for all $j \geqslant 1$, then there exists a point $x_0 \in I_j$ of period $n$. □

The last theorem generalizes the period forcing of $\wp$ in a broader sense than Sharkovskii's ordering.

**Theorem 3.14.** Any periodic point of least period $n > 1$ of the Weierstrass $\wp$-function forces periodic points of other least period $k \geqslant 1$.

*Proof.* Let $x_0$ be the periodic point of least period $n$. There are two possibilities for $k$.

- $k > n$ : Let us consider the loop $J_0 \to J_1 \to \cdots \to J_n = J_0 \to J_0 \cdots \to J_0$ of length $k$. Then by Lemma 3.11 (ii), there exists a periodic point $x_0 \in J_0$ of least period $k$.
- $1 \leqslant k < n$ : Let us consider the loop $J_0 \to J_1 \to \cdots J_{k-1} \to J_0$ of length $k$. Hence, there also exists a periodic point $x_0 \in J_0$ of least period $k$ by Lemma 3.11 (ii).

□

**Conclusion**: For a family $f_b(x) = \wp(x) + b$ where $x, b$ are real numbers, the period forcing provided by Theorem 3.12 holds in any given parameter $b$. In fact, by completeness of the transition graph provide the period forcing for any period. The noticeable fact is that the fundamental interval where periodic points lie changes depending on the parameter $b$.

## 3.5 Chaotic behavior of Weierstrass $\wp$ function

Let us recall that Theorem 1.26 in Chapter 1, we proved that the quadratic family $Q_c(x) = x^2 + c$ is chaotic on the invariant set $\Lambda$ under the assumption $c < -\dfrac{5+2\sqrt{5}}{4}$. To understand the dynamics of the quadratic family, we constructed a sequence space of two symbols and the shift map on that new space. Also, we produced the itinerary map from $\Lambda_c$ to $\Sigma_2$. This map was shown to be a homeomorphism. It is much simpler to study the dynamics on the sequence space and the itinerary serves as a conjugacy between those spaces. This leads us to understand the dynamics of the quadratic map more easily. Now we consider the Weierstrass $\wp$ function.

Now we will first study the chaotic behavior of $\wp$ restricted on $I_0$.

**Theorem 3.15.** The Weierstrass $\wp$-function restricted on $I_0$ is not chaotic.

*Proof.* We observe that $\bar{I}_0 = [0, \lambda]$ is separable and of second category. Thus, by Theorem 1.34, if the dynamical system $(I_0, \wp)$ is transitive then it has a dense $\wp$-orbit. We will prove that every point in $\bar{I}_0$ has no dense $\wp$-orbit which proves that $(\wp, I_0)$ is not transitive. Since $\wp$ has a minimum at $\lambda/2$, it follows that $\wp'(\lambda/2) = 0$. Also, we have $\wp(\lambda/2) = \lambda/2$ so that $\lambda/2$ is a fixed point of $\wp$. It follows that $\lambda/2$ is a superattracting fixed point. Then there is a neighborhood $V$ of $\lambda/2$ such that for every $x \in V$, we have $\wp^n(x) \to \lambda/2$ as $n \to \infty$. Now if there is $x_0 \in I_0$ such that $\mathcal{O}_\wp(x_0) = \{x_0, \wp(x_0), \wp^2(x_0), \cdots\}$ is dense in $I_0$, then we have $x_0 \notin V$. Accordingly, assume $x_0 \in I_0 - V$ and $\overline{\mathcal{O}_\wp(x_0)} = I_0$. Then there is $N \in \mathbb{N}$ such that $\wp^N(x_0) \in V \cap I_0$ (the immediate basin of $\lambda/2$) but then $\wp^{N+m}(x_0) \to \lambda/2$ as $m \to \infty$. This shows that any point in $I_0$ has no dense $\wp$-orbit. Thus the Weierstrass $\wp$ function restricted to $I_0$, $\wp|_{I_0} : [0, \lambda] \to [0, \lambda]$ fails the transitivity condition and therefore it is not chaotic on $I_0$. □

**Theorem 3.16.** For any $j \leqslant -1$, the Weierstrass $\wp$ function restricted on $I_j$ has no periodic cycle of any period.

*Proof.* Notice that for any $j \leqslant -1$, the $I_j$ is an interval of negative numbers. Also, we have $\wp(I_j) = [\lambda/2, \infty)$ for any $j \leqslant -1$. Thus, for each $I_j$, $\wp$ maps the negative numbers to positive numbers. Thus, $\wp$ has no fixed point on $I_j$ for $j \leqslant -1$. Now for any $n > 1$, we have $\wp^n([\lambda/2, \infty)) = [\lambda/2, \infty]$. Indeed, we have

$$\wp([\lambda/2, \infty)) = \wp([\lambda/2, \lambda]) \bigcup_{j=1}^{\infty} \wp([j\lambda, (j+1)\lambda]) = [\lambda/2, \infty].$$

Thus, the image under $\wp$ of a negative number becomes positive and subsequent iterates remains positive. Then, $\wp$ cannot have a positive cycle over the negative axis. □

For each $j \geqslant 1$, let us consider the Weierstrass $\wp$ function on each interval $I_j$. We have seen that the Weierstrass $\wp$ function is an even function on each fundamental interval $I_j$ for $j \in \mathbb{Z}$. Also, it attains a minimum at the single critical point $(j+1)/2$ on each interval $I_j$. Therefore, its graph on each interval is similar to the quadratic family $Q_c$ that we already discuss its dynamics by the help of symbolic dynamics with two symbols, 0 and 1. As before we are interested in finding the invariant subset of $I_j$.

By Lemma 3.5, it implies that $\wp_\Lambda$ has two fixed points for all $j \geqslant 1$. Let $\mathrm{Fix}(\wp_\Lambda|_{I_j}) = \{p_j, q_j\}$. Since $\wp$ attains the local minimum at $(j+1)\lambda/2$, then $j\lambda < p_j < (j+1)\lambda/2 < q_j < (j+1)\lambda$.

**Lemma 3.17.** For each $j \geqslant 1$, the fixed points $p_j, q_j \in I_j$ are repelling for $\wp_\Lambda$, $\Lambda$ the central lattice.

*Proof.* Since $\Lambda$ is the central lattice, then $\wp(\lambda/2) = \wp(\lambda + j\lambda) = \lambda/2$ for all $j \in \mathbb{Z}$. The critical points of $\wp|_\mathbb{R}$ are exactly the points $\lambda/2 + j\lambda$, $j \in \mathbb{Z}$. Then all critical points of $\wp|_\mathbb{R}$ are mapped after one iteration into $\lambda/2$. By Theorem 4.6 in [7], $\wp|_\mathbb{R}$ cannot have any other attracting or parabolic periodic point in $\mathbb{R}$. Hence $p_j, q_j \in I_j$ for $j \geqslant 1$ are repelling. □

Let us consider a closed subinterval $J_j = [(j+2)\lambda - q_j, q_j]$ of $I_j$. Let us defined the subset $A_j^1$ of $J_j$ that leave the interval $J_j$ just after one iteration, that is,

$$A_j^1 := \{x \in J_j \mid \wp_\Lambda(x) \notin J_j\}.$$

Notice that the endpoints of the interval $A_j^1$ are $p_j^1$ and $q_j^1$ which are satisfied the equation of intersection between the graph of $\wp_\Lambda$ on each interval $I_j$ and the horizontal line $y = (j+2)\lambda - q_j$ for $j \geqslant 1$:

$$\wp_\Lambda(x) = (j+2)\lambda - q_j. \tag{3.2}$$

Define $A_j^n$ to be the subset of $J_j$ that leaves the interval $J_j$ after $n$ iterations,

$$A_j^n = \{x \in J_j \mid \wp_\Lambda^i(x) \in J_j \text{ for } i = 1, 2, \cdots, n-1 \text{ but } \wp_\Lambda^n(x) \notin J_j\}.$$

Now let us define the subset of points that never leave $J_j$ for any iteration, namely

$$\Gamma_j = \{x \in J_j \mid \wp^n(x) \in J_j \text{ for any } n \in \mathbb{N}\} \subset I_j,\ j \geqslant 1.$$

In other words, we can express $\Gamma_j$ as

$$\Gamma_j = J_j - \bigcup_{n=1}^{\infty} A_j^n.$$

It should be noticed that by definition $\Gamma_j$ is closed and nonempty. Let us denote $K_j^0$ and $K_j^1$ two closed subintervals of $J_j$ on the the left and the right of $A_j^1$ so that $K_j^0 \cup A_j^1 \cup K_j^1 = J_j$.

**Lemma 3.18.** For any point $x \in K_j^0 \cup K_j^1$, we have $|\wp'_\Lambda(x)| > 1$ for all $j \geqslant 1$.

**Theorem 3.19.** For $j \geqslant 1$, we have $\Gamma_j$ is a Cantor set.

*Proof.* Notice that $q_j$ is a fixed point for $\wp$ so that $q_j \in \Gamma_j$. For any $j \geqslant 2$ and $n \geqslant 2$, we have $A_j^n = \wp^{1-n}(A_j^1)$. Recall $A_j^1 = (p_1^1, q_1^1)$ which is open as the subset of the real line $\mathbb{R}$. Also, $\wp$ is continuous on each fundamental interval $I_j$. It follows that $A_j^n = \wp^{1-n}(A_j^1)$ is open for all $j \geqslant 2,\ n \geqslant 2$. Since $J_j = [(j+2)\lambda - q_j, q_j]$ is closed, it implies that $\Gamma_j$ is closed.

Now suppose that $\Gamma_j$ contains an interval $J$ of positive length $l$. For any two distinct points $x$, $y$, the Mean Value Theorem implies that there exists a point $z$ on the open interval with endpoints $x$ and $y$ such that

$$\left|\wp'(z)\right| = \frac{\left|\wp(x) - \wp(y)\right|}{|x-y|}.$$

By Lemma 3.18, then it follows that $\left|\wp(x) - \wp(y)\right| > \lambda|x-y|$ for some $\lambda > 1$. By definition of $\Gamma_j$ and $J \subset \Gamma_j$, we have $\Gamma_j(J) \subset \Gamma_j$. Then $\wp(x)$ and $\wp(y) \in \Gamma_j$ with $\wp(x) \neq \pm\wp(y)$ since $\wp$ is even on each interval with respect to the vertical line passing through its critical point. Applying Mean Value Theorem again, we have

$$\frac{\left|\wp^2(x) - \wp^2(y)\right|}{\left|\wp(x) - \wp(y)\right|} > \lambda.$$

Then $\left|\wp^2(x) - \wp^2(y)\right| > \lambda\left|\wp(x) - \wp(y)\right| > \lambda^2|x-y|$. For the $n$ iteration, we have

$$\left|\wp^n(x) - \wp^n(y)\right| > \lambda^n|x-y|.$$

Then

$$\lim_{n\to\infty}\left|\wp^n(x) - \wp^n(y)\right| \geqslant \lim_{n\to\infty} \lambda^n|x-y|.$$

Since $|x - y| > 0$ and $\lambda > 1$, then $\lambda^n|x - y| \to \infty$. But $\lim_{n\to\infty}\left|\wp^n(x) - \wp^n(y)\right| < l$ and $l$ is finite. This gives us a contradiction and thus $\Gamma_j$ contains no interval. This shows that $\Gamma_j$ is totally disconnected

It remains to prove that $\Gamma_j$ is perfect. But we have seen that $\Gamma_j$ is closed, it is enough to prove that it has an isolated point. However, the proof of this part are the same as the one for the quadratic map $Q_c$ provided in Theorem 1.15. Hence, $\Gamma_j$ is the Cantor set for any $j \geqslant 1$. □

Let us define an **itinerary map** $S$ from the invariant set $\Gamma_j$ to the space of two symbols $\Sigma_2$. For any $x \in \Gamma_j$, let

$$S(x) = (s_0 s_1 s_2 \cdots)$$

where $s_i = 0$ if $\wp^i(x) \in K_j^0$ and $s_i = 1$ if $\wp^i(x) \in K_j^1$.

**Theorem 3.20.** The itinerary map $S$ is a homeomorphism with the property that for any natural number $n$,

$$S \circ \wp^n|_{\Gamma_j}(x) = \sigma^n \circ S(x).$$

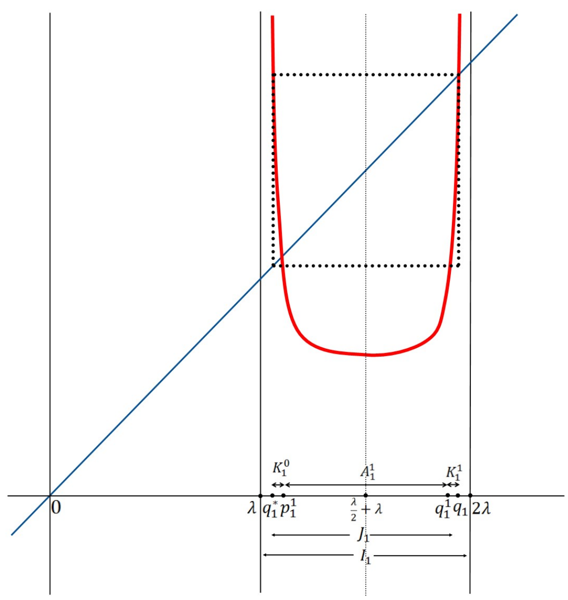


Figure 3.3: The Weierstras $\wp$-function on $I_1$

Recall that from Proposition 1.23 in Chapter 1, the shift map $\sigma$ is continuous on $\Sigma_2$. Moreover, for any $j \geqslant 1$, $\wp$ is continuous on each $J_j$. Together with the Theorem 3.20, we conclude that $\wp|_{J_j}$ for $j \geqslant 1$ and the shift map $\sigma$ are conjugate under the conjugacy $S$.

**Theorem 3.21.** Let $\Lambda$ be a central lattice. Then, for any $j \geqslant 1$, $\wp_\Lambda$ is chaotic in $\Gamma_j$.

The proof of the theorem is similar to the proof provided for the quadratic map $Q_c|_{\Lambda_c}$ whenever $c < -\dfrac{5+2\sqrt{5}}{4}$. For our case, the central lattice guarantees sufficient expansion of $\wp|_{\Gamma_j}$.

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