CIMAT

Centro de Investigación en Matemáticas A. C.

CR Structures and Partially Pure Spinors

T E S I S

Que para obtener el grado de

Doctor en ciencias
con orientación en

Matemáticas Básicas

P R E S E N T A:

Iván Téllez Téllez

Director de tesis:

Dr. Rafael Herrera Guzmán

GUANAJUATO, GTO. Octubre de 2017

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A Naomi

Agradecimientos

Quiero agradecer a todas las personas que me dieron la oportunidad de realizar el doctorado en el Centro de Investigación en Matemáticas. Pricipalmente al Dr. Rafael Herrera Guzmán quien me permitió estudiar geometría spin. Él siempre tuvo tiempo y paciencia para responder mis dudas y para dirigir el presente trabajo. Gracias Rafael.

Agradezco el tiempo y la dedicación de quienes revisaron el presente escrito y me ayudaron a obtener una mejor versión del mismo. Gracias a los Dres. Oscar Adolfo Sánchez Valenzuela, Luis Hernández Lamoneda, Andrés Pedroza, Rafael Herrera Guzmán y Gerardo Arizmendi Echegaray.

Agradezco a las personas que estuvieron siempre a tiempo y de forma eficiente en los diferentes trámites que tenía que realizar, en particular los correspondientes a las ayudantías que di, las becas que recibí y las estancias que realicé. Gracias a Lolita, Rosa Inés, Norma, Jannet, Claudia, Lalo y a Memo.

El CIMAT es un lugar excelente para hacer matemáticas. Agradezco las facilidades y los materiales que se me otorgaron para estudiar y para asistir a distintos seminarios y conferencias tanto nacionales como internacionales. Siempre tuve un lugar de primera, una buena computadora, impresiones y café. Gracias a los que hicieron todo esto posible. Me considero privilegiado.

Quiero agradecer a quienes me apoyaron con sus proyectos de investigación para que realizara estancias o asistiera a conferencias. Gracias a los señores Oscar Adolfo Sánchez Valenzuela, Rafael Herrera Guzmán y Francisco Sánchez Sánchez.

Gracias al Consejo Nacional de Ciencia y Tecnología por otorgarme una beca de doctorado y asignarme parte de los recursos para realizar una estancia en el ICTP, en Trieste, Italia.

Gracias a mi madre, la señora Gudelia Téllez Barrera, por ser mi ejemplo y mi fuente de inspiración. Gracias a mis hermanas, Selene y Diana, por su afecto y sus ánimos.

Agradezco también a mis amigos por su motivación y valiosa compañía, principalmente a Imelda, Rubí, Aurelio, Gerardo, Hugo, Jonatán y William.

Agradezco a Ale y a Naomi por cambiar todo a mi alrededor. Gracias por las palabras de aliento y gracias por los buenos momentos. Ustedes son todo para mí.

Contents

Introduction

Spinors have played an important role in both physics and mathematics ever since they were discovered by É. Cartan in 1913. To have a picture of their importance we refer the reader to Hitchin's seminal paper [13], the textbooks [9, 16], as well as to [21, 31] for the more recent development of Seiberg-Witten theory and its notorious results on 4-manifold geometry and topology.

Cartan defined *pure spinors* [2, 3, 5] in order to characterize (almost) complex structures and, one hundred years later, they are still central in tackling geometrical problems. Furthermore, spinor fields have also been related to the notion of calibrations on a Spin manifold by Harvey and Lawson [10, 6], since distinguished differential forms are naturally associated to a spinor field. Pure spinors are also present in the Penrose formalism in General Relativity as they are implicit in Penrose's notion of "flag planes" [26, 27, 28].

The main subject of this work is the study of CR manifolds via twisted spin geometry. Our main motivation is the aformentioned relationship between classical pure spinors and orthogonal almost complex structures. More precisely, a classical pure spinor $\phi \in \Delta_{2m}$ is a spinor such that the (isotropic) subspace of complexified vectors $X - iY \in \mathbb{R}^{2m} \otimes \mathbb{C}$, $X, Y \in \mathbb{R}^{2m}$, which annihilate ϕ under Clifford multiplication

$$(X - iY) \cdot \phi = 0$$

is of maximal dimension, where $m \in \mathbb{N}$ and Δ_{2m} is the standard complex representation of the Spin group $Spin(2m)$ (cf. [16]). This means that for every $X \in \mathbb{R}^{2m}$ there exists a $Y \in \mathbb{R}^{2m}$ satisfying

$$X \cdot \phi = iY \cdot \phi.$$

By setting $Y = J(X)$, one can see that a pure spinor determines a complex structure on $\mathbb{R}^{2m}$. Geometrically, the two subspaces $TM \cdot \phi$ and $iTM \cdot \phi$ of Δ_{2m} coincide, which means $TM \cdot \phi$ is a complex subspace of Δ_{2m}, and the effect of multiplication by the number $i = \sqrt{-1}$ is transferred to the tangent space TM in the form of J.

In [14] the space of spinors $\Delta_n, n \leq 12$, is decomposed into classes of orbits under the action of the Spin group. The authors of [4, 30] investigated (the classification of) non-pure classical spinors by means of their isotropic subspaces. In [30], the authors noted that there may be many spinors (in different orbits under the action of the Spin group) admitting isotropic subspaces of the same dimension, and that there is a gap in the possible dimensions of such isotropic subspaces.

Following these geometrical considerations, we investigated how to define twisted partially pure spinors in order to spinorially characterize subspaces of Euclidean space endowed with a complex structure. We characterize subspaces of Euclidean space $\mathbb{R}^n$, $n \geq 2m$, endowed with an orthogonal complex structure by means of twisted spinors. Thus, we define twisted partially pure spinors (cf. Definition 2.2.1) in order to establish a one-to-one correspondence between subspaces of Euclidean space (of fixed codimension and endowed with a orthogonal complex structures and oriented orthogonal complements), and orbits of such spinors under a particular subgroup of the twisted Spin group (cf. Theorem 2.2.1). By using spinorial twists we avoid having different orbits under the full twisted Spin group and also the aforementioned gap in the dimensions.

The need to establish such a correspondence arises from our interest in developing a spinorial setup to study the geometry of manifolds admitting (almost) CR structures (of arbitrary codimension). Since such manifolds are not necessarily Spin nor Spin^c, we are led to consider spinorially twisted Spin groups, representations, structures, etc. Thus, we develop a spinorial description of CR structures of arbitrary codimension. More precisely, we characterize almost CR structures of arbitrary codimension on (Riemannian) manifolds by the existence of a Spin^{c,r} structure carrying a partially pure spinor field.

The notion of abstract CR structures in odd dimensions generalizes that of complex structure in even dimensions. This notion aims to describe intrinsically the property of being a hypersurface of a complex space form. This is done by distinguishing a distribution whose sections play the role of the holomorphic vector fields tangent to the hypersurface. There exists also the notion of almost CR structure of arbitrary codimension, in which a fixed codimension subbundle of the tangent bundle carries a complex structure. It has been proved that every codimension one, strictly pseudoconvex CR manifold has a canonical Spin^c structure [29]. Naturally, this led us to ask if it is possible to characterize almost CR structures of arbitrary codimension (and a choice of compatible metric) by means of a twisted Spin structure carrying a special spinor field.

The spinorially twisted Spin group is defined as follows

$$Spin^{c,r}(n) = \frac{Spin(n) \times Spin^c(r)}{\{\pm(1,1)\}}.$$

It will be the structure group for the twisted Spin structures (cf. Definition 3.2.1). One of its representations will contain the partially pure spinors we need and the parameter r will eventually play the role of the codimension of an almost CR structure. Such twisted Spin structures involve not only the principal bundle of orthonormal frames, but also two auxiliary principal bundles.

The existence of a partially pure spinor field ϕ on a Riemannian Spin^{c,r} manifold M implies the splitting of the tangent bundle TM into two orthogonal distributions V^ϕ and $(V^\phi)^\perp$, where the former is endowed with an automorphism J^ϕ satisfying $(J^\phi)^2 = -\mathrm{Id}_{V^\phi}$, i.e. M has an almost CR hermitian structure. In fact, the converse is also true (cf. Theorem 4.1.1). Furthermore, we characterize the integrability condition of a CR structure (with metric) by an equation involving covariant derivatives of the partially pure spinor (cf. Theorem 4.1.2). We proceed to study other natural "integrability conditions" of the partially pure spinor field, such as being parallel in the V^ϕ directions (cf. Theorem 4.1.3), or being Killing in the $(V^\phi)^\perp$ directions (cf. Theorem 4.1.5), etc. We present a family of homogeneous spaces as examples for the different theorems. Finally, we study various integrability conditions of the almost CR structure in our spinorial setup, including the classical integrability of a CR structure as well as those implied by Killing-type conditions on the partially pure spinor field.

The thesis is organized as follows:

- In chapter one we recall some results related to Clifford algebras, Spin groups and almost complex structures. These results are introductory material for the subject and are used in the subsequent chapters.
- In chapter two we define the twisted Spin groups and representations that will be used. We present the space of anti-symmetric 2-forms and endomorphims associated to twisted spinors and show some results on subgroups and branching of representations. In Section 2.2, we define partially pure spinors, deduce their basic properties and prove Theorem 2.2.1, which establishes the aforementioned one-to-one correspondence.
- In chapter three we use the background material from the definition of partially pure spinors and describe the isotropy representation of a family of homogeneous spaces (partial flag manifolds) that will be used to produce examples of some results throughout the thesis. In Section 3.2, we define $\mathrm{Spin}^{c,r}$ structures, study their existence, define twisted Dirac and Laplace operators, prove some curvature identities and a Schrödinger-Lichnerowicz-type formula, and derive some Bochner-type results.
- In chapter four we develop a spinorial description of CR structures of arbitrary codimension. We characterize almost CR structures of arbitrary codimension on (Riemannian) manifolds by the existence of a $\mathrm{Spin}^{c,r}$ structure carrying a partially pure spinor field. We study various integrability conditions of the almost CR structure in our spinorial setup, including the classical integrability of a CR structure as well as those implied by Killing-type conditions on the partially pure spinor field.

Summarizing, this work lays the foundations for the geometrical study of CR structures of arbitrary codimension by means of spinorially twsited spin geometry, some of which has been published in [12]. Our future work will focus on generalizing codimension CR constructions, such as the Fefferman space and metric.

Chapter 1

Preliminaries

In this chapter we study some results needed for the next chapters. Most of the results of Clifford algebras, Spin groups and Spin structures were taken from [9] and [16]. These results will help clarify notation appearing in the next chapters and make the material more self contained.

1.1 Clifford Algebras

To introduce the concept of spinor we need to recall some results about Clifford algebras and the Spin group. Consider a finite dimensional vector space V together with a quadratic form q over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. The Clifford algebra of (V, q), that we denote by $Cl(V, q)$, is the quotient space $T(V)/I$ where $T(V)$ is the tensor algebra of V and I is the two sided ideal generated by all elements of the form

$$\{v \otimes v - q(v) \cdot 1 : v \in V\}.$$

The algebra $Cl(V, q)$ can be charaterized as follows. We say that $(C(q), j)$ is a Clifford algebra for (V, q) if

i) $C(q)$ is an associative $\mathbb{K}-$algebra with 1;

ii) $j : V \to C(q)$ is a linear map and $j(v)^2 = q(v) \cdot 1$ for all $v \in V$;

iii) if A is another algebra with 1 and $u : V \to A$ a linear map satisfying $u(v)^2 = q(v) \cdot 1$, then there exists one and only one algebra homomorphism $\tilde{U} : C(q) \to A$ such that $u = \tilde{U} \circ j$.

With these defining properties it can be proved that $(Cl(V, q), j)$, j being the inclusion of V in $T(V)$ followed by projection to $Cl(V, q)$, is a Clifford algebra for (V, q) and also if there exists another Clifford algebra $(C'(q), j')$ for (V, q) then there exists an isomorphism of algebras, $f : Cl(V, q) \to C'(q)$, satisfying $f \circ j = j'$, i.e. the diagram

$$\begin{tikzcd} V \arrow[d, "j"'] \arrow[dr, "j'"] & \\ Cl(V,q) \arrow[r, "f"'] & C'(q) \end{tikzcd}$$

commutes.

The map $j : V \to Cl(V,q)$ is injective and the set $j(V)$ generates $Cl(V,q)$ multiplicatively. If $\dim(V) = n$ then $\dim(Cl(V,q)) = 2^n$; so, an orthogonal basis, $v_1, \dots, v_n$, for V with respect to $B(u,v) = \frac{1}{2}(q(u+v) - q(u) - q(v))$ also generates the algebra $Cl(V,q)$ and since $v_i^2 = q(v)^2$ we have $(v_i + v_j)^2 = q(v_i + v_j) = q(v_i) + q(v_j) = v_i^2 + v_j^2$ which implies the relations

$$v_i v_j + v_j v_i = 0, \qquad i \neq j.$$

Consider Cl_n, the Clifford algebra of $\mathbb{R}^n$ with $q(v) = -|v|^2$. According to the previous discussion this Clifford algebra is generated by the usual orthonormal vectors $e_1, e_2, \dots, e_n \in \mathbb{R}^n$ subject to the relations

$$e_j e_k + e_k e_j \quad = \quad -2\langle e_j, e_k \rangle,$$

where $\langle , \rangle$ denotes the standard inner product in $\mathbb{R}^n$.

For $n = 1$, the Clifford algebra Cl_1 is generated by 1 and by the basic element e_1 with the single relation $e_1^2 = -1$. Thus $Cl_1 = \mathbb{C}$. If $n = 2$ then the set $\{e_1, e_2\}$ generates $\mathbb{R}^2$ and in consequence the elements 1, $i = e_1$, $j = e_2$, $k = e_1 e_2$ generate Cl_2. Since $e_i^2 = -1$ and $e_1 e_2 + e_2 e_1 = 0$, these elements satisfy the relations $i^2 = j^2 = k^2 = -1$ and $ij = k$, $jk = i$, $ki = j$. These are the relations of the algebra of quaternions. So, $Cl_2 = \mathbb{H}$.

Here is a table with the first eight Clifford algebras Cl_n

n	Cl_n
1	$\mathbb{C}$
2	$\mathbb{H}$
3	$\mathbb{H} \oplus \mathbb{H}$
4	$M_2(\mathbb{H})$
5	$M_4(\mathbb{C})$
6	$M_8(\mathbb{R})$
7	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$
8	$M_{16}(\mathbb{R})$

For $n \geq 9$, the algebras Cl_n are isomorphic to tensor products of these (see [16] p. 27).

We will concentrate on the algebras

$$\mathbb{C}l_n = Cl_n \otimes_{\mathbb{R}} \mathbb{C},$$

the complexification of Cl_n. These algebras satisfy

$$\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_2.$$

The algebra $\mathbb{C}l_2$ is generated by the elements e_1, e_2 satisfying the relations $e_i^2 = -1$ and $e_1 e_2 + e_2 e_1 = 0$. Now, in the basis

$$Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

of $M_2(\mathbb{C})$, the elements g_1 and g_2 satisfy the same relations: $g_i^2 = -1$ and $g_1g_2 + g_2g_1 = 0$. This implies $\mathbb{C}l_2 = M_2(\mathbb{C})$ and

$$\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes_{\mathbb{C}} M_2(\mathbb{C}). \tag{1.1}$$

We can describe now $\mathbb{C}l_n$ explicitly: we start with $\mathbb{C}l_1 = Cl_1 \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$ and $\mathbb{C}l_2 = M_2(\mathbb{C})$. For $n \geq 3$, we use (1.1) and obtain

$$\mathbb{C}l_n \cong \begin{cases} \mathrm{End}(\Delta_n), & \text{if } n = 2k, \\ \mathrm{End}(\Delta_n) \oplus \mathrm{End}(\Delta_n), & \text{if } n = 2k+1. \end{cases}$$

Where the complex space

$$\Delta_n := (\mathbb{C}^2)^{\otimes k} = \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{k \text{ times}},$$

is the tensor product of $k = [n/2]$ copies of $\mathbb{C}^2$. From now on we refer to Δ_n as the space of *spinors*.

The map

$$\kappa : \mathbb{C}l_n \longrightarrow \mathrm{End}(\Delta_n)$$

is defined to be either the above mentioned isomorphism if n is even, or the isomorphism followed by the projection onto the first summand if n is odd.

In terms of the generators $e_1, \ldots, e_n$, κ can be described explicitly as follows,

$$\begin{array}{rcl} e_1 & \mapsto & Id \otimes Id \otimes \ldots \otimes Id \otimes Id \otimes g_1, \\ e_2 & \mapsto & Id \otimes Id \otimes \ldots \otimes Id \otimes Id \otimes g_2, \\ e_3 & \mapsto & Id \otimes Id \otimes \ldots \otimes Id \otimes g_1 \otimes T, \\ e_4 & \mapsto & Id \otimes Id \otimes \ldots \otimes Id \otimes g_2 \otimes T, \\ \vdots & & \ldots \\ e_{2k-1} & \mapsto & g_1 \otimes T \otimes \ldots \otimes T \otimes T \otimes T, \\ e_{2k} & \mapsto & g_2 \otimes T \otimes \ldots \otimes T \otimes T \otimes T, \end{array}$$

and, if $n = 2k+1$,

$$e_{2k+1} \mapsto i\, T \otimes T \otimes \ldots \otimes T \otimes T \otimes T.$$

With this explicit isomorphism the Clifford algebras $\mathbb{C}l_n$ are

n	$\mathbb{C}l_n$
1	$\mathbb{C} \oplus \mathbb{C}$
2	$M_2(\mathbb{C})$
3	$M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$
4	$M_4(\mathbb{C})$
5	$M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$
6	$M_8(\mathbb{C})$
$\vdots$	$\vdots$

Now, the vectors

$$u_{+1} = \frac{1}{\sqrt{2}}(1, -i) \qquad \text{and} \qquad u_{-1} = \frac{1}{\sqrt{2}}(1, i),$$

form a unitary basis of $\mathbb{C}^2$ with respect to the standard Hermitian product $\langle z, w\rangle = z_1\bar{w}_1 + z_2\bar{w}_2$. Thus,

$$\{u_{\varepsilon_1,\ldots,\varepsilon_k} = u_{\varepsilon_1} \otimes \ldots \otimes u_{\varepsilon_k} \,|\, \varepsilon_j = \pm 1,\ j = 1, \ldots, k\},$$

is a unitary basis for the space of spinors Δ_n with respect to the naturally induced Hermitian product. From now on we will denote inner and Hermitian products by the same symbol $\langle\cdot,\cdot\rangle$ trusting that the context will make clear which product is being used.

The explicit description of κ and the basis for Δ_n allows us to compute the action of any element $a \in \mathbb{C}l_n$ on the space Δ_n. In particular the Clifford multiplication, of a vector with a spinor, is defined by

$$\begin{array}{rcl} \mu_n : \mathbb{R}^n \otimes \Delta_n & \longrightarrow & \Delta_n \\ x \otimes \psi & \mapsto & \mu_n(x \otimes \psi) := \kappa(x)(\psi). \end{array}$$

We use the notation $x \cdot \psi$ for Clifford multiplication.

Example 1.1.1. *Let $\psi = u_{+1} \otimes u_{+1} \in \Delta_4$. The elements of the standard basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}^4$ act on ψ as follows*

$$\begin{array}{rcl} e_1 \cdot \psi & = & u_{+1} \otimes g_1(u_{+1}) = iu_{+1} \otimes u_{-1}, \\ e_2 \cdot \psi & = & u_{+1} \otimes g_2(u_{+1}) = u_{+1} \otimes u_{-1}, \\ e_3 \cdot \psi & = & g_1(u_{+1}) \otimes T(u_{+1}) = -iu_{-1} \otimes u_{+1}, \\ e_4 \cdot \psi & = & g_2(u_{+1}) \otimes T(u_{+1}) = -u_{-1} \otimes u_{+1}. \end{array}$$

Observe that $e_1 \cdot \psi = ie_2 \cdot \psi$ and $e_3 \cdot \psi = ie_4 \cdot \psi$, these equations imply that ψ is an example of spinor, called pure spinor, which we define later.

1.2 The Spin Group and its Standard Representation

We will focus now on the Spin group, $Spin(n) \subset Cl_n$ is the subset

$$Spin(n) = \{x_1 x_2 \cdots x_{2l-1} x_{2l} \in Cl_n \,|\, x_j \in \mathbb{R}^n,\ |x_j| = 1,\ l \in \mathbb{N}\},$$

endowed with the product of the Clifford algebra. If $x \in \mathbb{R}^n \subset Cl_n$ then $x \cdot x = -|x|^2$, thus if $g = x_1 \cdot x_2 \cdots x_{2l} \in Spin(n)$ then its inverse is $g^{-1} = x_{2l} \cdots x_2 \cdot x_1$.

Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. For $n = 1$ there are just two elements for which $|\lambda e_1| = 1$, this implies $Spin(1) = \mathbb{Z}_2$. For $n = 2$ the product g of an even number of elements x_i is of the form $g = a + be_1e_2$, $a, b \in \mathbb{R}$, and its inverse is $g^{-1} = a - be_1e_2$ from where we have that $a^2 + b^2 = 1$, i.e. $Spin(2) = U(1)$. For $n = 3$, the product mentioned must be of the form $g = a + be_1e_2 + ce_1e_3 + de_2e_3$, $a, b, c, d \in \mathbb{R}$, and its inverse must be $g^{-1} = a - be_1e_2 - ce_1e_3 - de_2e_3$, multiplying out we obtain $gg^{-1} = 1$ if and only if $a^2 + b^2 + c^2 + d^2 = 1$. So, $Spin(3) = S^3 \cong SU(2)$.

The first Spin groups are

n	$Spin\ (n)$
1	$\mathbb{Z}_2$
2	$U(1)$
3	$SU(2)$
4	$SU(2)\times SU(2)$
5	$Sp(2)$
6	$SU(4)$

The group $Spin(n)$ is the double cover of $SO(n)$. The covering map

$$\lambda_n : Spin(n) \to SO(n)$$

acts on $v\in\mathbb{R}^n$ as follows

$$\lambda_n(g)(v) = g\cdot v\cdot g^{-1}.$$

This product is always an element of $\mathbb{R}^n$, the copy inside Cl_n, if $g\in Spin(n)$. It is clear that g and $-g$ give the same transformation under λ_n for all n.

For $x\in\mathbb{R}^n\subset Cl_n$ such that $|x|=1$ and $v\in\mathbb{R}^n$

$$R_x(v) = x\cdot v\cdot x = x\cdot(-x\cdot v - 2\langle x,v\rangle) = |x|^2 v - 2\langle x,v\rangle x = v - 2\langle x,v\rangle x$$

is the reflection of v with respect to the plane $x^\perp = \{y\in\mathbb{R}^n : \langle y,x\rangle = 0\}$. According to the Cartan-Dieudonné Theorem, $A\in O(n)$ is obtained composing a finite number of reflections R_x. For a finite composition of reflections $R_1\circ R_2\circ\cdots\circ R_k\in SO(n)$ if and only if k is even. So, $\lambda_n(g)\in SO(n)$ for all $g\in Spin(n)$ and λ_n is onto.

Now we prove $\ker(\lambda_n)=\mathbb{Z}_2$. The map λ_n is a group homomorphism. If there exists $g\in Spin(n)$ such that $\lambda(g)=1$ then $g\cdot v = v\cdot g$ for all $v\in\mathbb{R}^n$. This implies g commutes with every element of Cl_n and, in particular, with every element of the Clifford subalgebra generated by $\{e_1,\ldots,e_{n-1}\}\subset\mathbb{R}^n$, i.e. g is in the intersection of the centers of these algebras which can be proved (see [9], p. 9) to be just $\mathbb{R}$. So, $g\in\mathbb{R}$ and from $|g|=1$ it follows that $g=\pm1$.

The group $Spin(n)$ is connected for $n\geq 2$ and simply connected for $n\geq 3$. For the connectedness it is enough to connect the elements of the form $v_1\cdot v_2$, $|v_1|=|v_2|=1$, by a path $\gamma(t)$ to 1. If $v_2=\lambda v_1$ then $\lambda=\pm1$ and $v_1\cdot v_2=\mp1$. The path

$$\begin{aligned}\gamma(t) &= -\cos(\pi t) - \sin(\pi t)e_1e_2\\ &= (\cos(\pi t/2)e_1 + \sin(\pi t/2)e_2)\cdot(\cos(\pi t/2)e_1 - \sin(\pi t/2)e_2)\end{aligned}$$

is in $Spin(n)$ for all $0\leq t\leq 1$ and connects -1 with 1.

Asume v_1 and v_2 linearly independent. Let W be the plane spanned by v_1 and v_2 and $w\in v_1^\perp\cap W$ such that $|w|=1$. Rotating the plane W we get a path $\alpha(t)$ connecting v_2 and w, this path satisfies $|\alpha(t)|=1$. The path $\beta(t)=v_1\cdot\alpha(t)$ connects $v_1\cdot v_2$ with $v_1\cdot w$ and is in $Spin(n)$. Finally connect 1 to $v_1\cdot w$ with

$$\gamma(t) = \cos(\pi t/2) + \sin(\pi t/2)v_1 w$$

$$= \quad (\cos(\pi t/4)v_1 + \sin(\pi t/4)w) \cdot (\sin(\pi t/4)w - \cos(\pi t/4)v_1)$$

From the previous description of the first Spin groups we have $\pi_1(Spin(3)) = 0$, so, as λ_3 is a covering map, we have $\pi_1(SO(3)) = \mathbb{Z}_2$. Recalling that $\pi_2(S^n) = 0$ for $n \geq 3$ and considering the fibration $SO(n) \hookrightarrow SO(n+1) \to S^n$, for $n \geq 3$, we have $\pi_1(SO(n)) = \pi_1(SO(3)) = \mathbb{Z}_2$, $n \geq 4$. Finally, using the map λ_n, this implies that $\pi_1(Spin(n))$ is a subgroup of index two of $\mathbb{Z}_2$, which gives us $\pi_1(Spin(n)) = 0$ for $n \geq 3$.

Example 1.2.1. *To see explicitly the covering map λ_n we make a computation. For $n = 2$, let $v = xe_1 + ye_2$ and $g = a + be_1e_2$ as described earlier. Multiplying one obtains*

$$\begin{aligned} g \cdot v \cdot g^{-1} &= (a + be_1e_2) \cdot (xe_1 + ye_2) \cdot (a - be_1e_2) \\ &= ((a^2 - b^2)x - 2aby)e_1 + ((a^2 - b^2)y + 2abx)e_2. \end{aligned}$$

As $Spin(2) = U(1)$ we can put $a = \cos(\theta)$ and $b = \sin(\theta)$ to obtain

$$g \cdot v \cdot g^{-1} = (x\cos(2\theta) - y\sin(2\theta))e_1 + (y\cos(2\theta) + x\sin(2\theta))e_2$$

which is equal to Av where

$$A = \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix}.$$

Thus $\lambda_2 : Spin(2) = U(1) \to U(1)$ is just the map $z \mapsto z^2$.

The Lie algebra of $Spin(n)$ is

$$\mathfrak{spin}(n) = \mathrm{span}\{e_ie_j \,|\, 1 \leq i < j \leq n\},$$

with commutator $[x, y] = x \cdot y - y \cdot x$. The differential of λ_n is given by $\lambda_{n*}(e_ie_j) = 2E_{ij}$, where E_{ij}, $i < j$, is the standard basis of the skew-symmetric matrices.

For $i \neq j$, the path $\cos(t) + \sin(t)e_i \cdot e_j$ is in $Spin(n)$ and passes trough 1 at $t = 0$. As in example 1.2.1 we have

$$\lambda_n(\gamma(t))(xe_i + ye_j) = (x\cos(2t) - y\sin(2t))e_i + (y\cos(2t) + x\sin(2t))e_j$$

and

$$\lambda_n(\gamma(t))(e_k) = e_k, \qquad k \neq i,\, k \neq j.$$

On the other hand

$$\exp(te_i \cdot e_j) = \cos(t) + \sin(t)e_ie_j = \gamma(t).$$

Deriving and evaluating at $t = 0$ we get $\lambda_{n*}(e_i \cdot e_j)(e_i) = 2e_j$, $\lambda_{n*}(e_i \cdot e_j)(e_j) = -2e_i$ and $\lambda_{n*}(e_i \cdot e_j)(e_k) = 0$. Hence we have an isomorphism between $\mathfrak{spin}(n)$ and the linear span of $e_i \cdot e_j$, $i < j$, which we showed is isomorphic to $\mathfrak{so}(n)$.

To end this section we describe the usual representation of $Spin(n)$. There exists a positive Hermitian product on Δ_n such that the Clifford multiplication satisfies

$$\langle x \cdot \psi_1, \psi_2 \rangle = -\langle \psi_1, x \cdot \psi_2 \rangle. \tag{1.2}$$

The restriction of κ to $Spin(n)$ defines the Lie group representation

$$Spin(n) \longrightarrow GL(\Delta_n).$$

This is a faithful and a special unitary representation with respect to the product in (1.2) with corresponding Lie algebra representation

$$\mathfrak{spin}(n) \longrightarrow \mathfrak{gl}(\Delta_n).$$

Finally, the multiplication μ_n can be extended to an homomorphism

$$\begin{array}{rcl} \mu_n : \bigwedge^*(\mathbb{R}^n) \otimes \Delta_n & \longrightarrow & \Delta_n \\ \omega \otimes \psi & \mapsto & \omega \cdot \psi \end{array}$$

as follows. Each element $\omega^k \in \bigwedge^*(\mathbb{R}^n)$ can be written, with respect to the orthonormal basis $\{e_1, \ldots, e_n\}$, as

$$\omega^k = \sum_{i_1 < \cdots i_k} w_{i_1 \ldots i_k} \, e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*.$$

We define

$$\omega^k \cdot \psi = \sum_{i_1 < \cdots i_k} w_{i_1 \ldots i_k} \, e_{i_1} \cdot \ldots \cdot e_{i_k} \cdot \psi.$$

Furthermore, if we abuse notation and also denote by λ_n the induced representation of λ_n on $\bigwedge^* \mathbb{R}^n$, it can be proved that μ_n is a homomorphism of $Spin(n)$-representations. Which means that for every $g \in Spin(n)$

$$\kappa(g)(\omega^k \cdot \psi) = (\lambda_n(g)\omega^k) \cdot (\kappa(g)\psi).$$

1.3 Twisted Spin Groups

Consider the following groups:

1. By using the unit complex numbers $U(1)$, the Spin group can be twisted [9]

$$Spin^c(n) = (Spin(n) \times U(1))/\{\pm(1,1)\} = Spin(n) \times_{\mathbb{Z}_2} U(1),$$

   with Lie algebra

$$\mathfrak{spin}^c(n) = \mathfrak{spin}(n) \oplus i\mathbb{R}.$$

2. In [8] the twisted Spin groups $Spin^r(n)$, $r \in \mathbb{N}$, have been considered and are defined as follows

$$Spin^r(n) = (Spin(n) \times Spin(r))/\{\pm(1,1)\} = Spin(n) \times_{\mathbb{Z}_2} Spin(r).$$

   The Lie algebra of $Spin^r(n)$ is

$$\mathfrak{spin}^r(n) = \mathfrak{spin}(n) \oplus \mathfrak{spin}(r).$$

3. Here, we will also consider the group

$$\begin{aligned} Spin^{c,r}(n) &= (Spin(n) \times Spin^c(r))/\{\pm(1,1)\} \\ &= Spin(n) \times_{\mathbb{Z}_2} Spin^c(r), \end{aligned}$$

where $r \in \mathbb{N}$, whose Lie algebra is

$$\mathfrak{spin}^{c,r}(n) = \mathfrak{spin}(n) \oplus \mathfrak{spin}(r) \oplus i\mathbb{R}.$$

This group fits into the exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin^{c,r}(n) \xrightarrow{\lambda_n \times \lambda_r \times \lambda_2} SO(n) \times SO(r) \times U(1) \longrightarrow 1,$$

where

$$(\lambda_n \times \lambda_r \times \lambda_2)([g,[h,z]]) = (\lambda_n(g), \lambda_r(h), z^2).$$

Note that for $r = 0, 1$, $Spin^{c,r}(n) = Spin^c(n)$.

These groups have the following twisted representations

1. The Spin representation Δ_n extends to a representation of $Spin^c(n)$ by letting

$$\begin{aligned} Spin^c(n) &\longrightarrow GL(\Delta_n) \\ [g,z] &\mapsto z\kappa_n(g) =: zg. \end{aligned}$$

2. The twisted $Spin^{c,r}(n)$ representation

$$\begin{aligned} Spin^{c,r}(n) &\longrightarrow GL(\Delta_r \otimes \Delta_n) \\ [g,[h,z]] &\mapsto z\,\kappa_r(h) \otimes \kappa_n(g) =: zh \otimes g. \end{aligned}$$

which is also unitary with respect to the natural Hermitian metric. Sometimes we will refer to this representation as $\kappa_n^{c,r}$.

3. For $r = 0, 1$, the twisted spin representation is simply the $Spin^c(n)$ representation Δ_n.

For the sake of future notation we will set

$$SO(0) = \{1\}, \qquad SO(1) = \{1\},$$

$$Spin(0) = \{\pm 1\}, \qquad Spin(1) = \{\pm 1\},$$

and

$$\Delta_0 = \Delta_1 = \mathbb{C}$$

a trivial 1-dimensional representation.

1.4 Spin Structures

We introduce now the concept of Spin structure. Let M be a Riemannian connected oriented smooth manifold of dimension n and $(Q, \pi_Q, M; SO(n))$ the bundle of oriented orthonormal frames over M. A Spin structure on the principal bundle Q is a pair (P, Λ) where

i) P is a $Spin(n)$ principal bundle over M,

ii) $\Lambda : P \to Q$ is a two fold covering for which the diagram

$$\begin{tikzcd} P \times Spin(n) \arrow[r] \arrow[d, "\Lambda\times\lambda_n"] & P \arrow[d, "\Lambda"] \arrow[dr, "\pi_P"] & \\ & & M \\ Q \times SO(n) \arrow[r] & Q \arrow[ur, "\pi_Q"'] & \end{tikzcd}$$

commutes. Here an horizontal arrow represents the action of the corresponding group on the respective principal bundle.

For an oriented manifold M, with $n = \dim(M) \geq 3$, the bundle of orthonormal frames has fiber F diffeomorphic to $SO(n)$, this implies $\pi_1(F) = \mathbb{Z}_2$. The embedding $i : F \to Q$ induces a homomorphism of fundamental groups $i_\# : \pi_1(F) \to \pi_1(Q)$. Thus for α_F, the nontrivial element of $\pi_1(F)$, $i_\#(\alpha_F) \in \pi_1(Q)$.

Let (P, Λ) be a Spin structure on Q. From covering theory $H = \Lambda_\#(\pi_1 P) \subset \pi_1(Q)$ is a subgroup of index two that does not contain the element $i_\#(\alpha_F)$. For if $i_\#(\alpha_F) \in H$ implies that there exist a lift $I : F \to P$ such that the diagram

$$\begin{tikzcd} & P \arrow[d, "\Lambda"] \\ F \arrow[ur, "I"] \arrow[r, "i"] & Q \end{tikzcd}$$

commutes. This implies that $I(F) \subset F_1$, with F_1 the fibre of P diffeomorphic to $Spin(n)$, from where $\Lambda_\# \circ I_\# = Id_{\pi_1(F)}$. This gives a contradiction since $\pi_1(F) = \mathbb{Z}_2$ and $\pi_1(F_1) = 0$.

With the same kind of arguments one can show that *the equivalence classes of Spin structures on the orthonormal frame bundle Q of a connected manifold M are in bijective correspondence with those subgroups $H \subset \pi_1(Q)$ of index two which do not contain α_F.*

Another characterization of the Spin structures can be given using this result. The fibration $F \hookrightarrow Q \to M$ induce the long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_2(M) \xrightarrow{\partial} \pi_1(F) \xrightarrow{i_\#} \pi_1(Q) \xrightarrow{\pi_\#} \pi_1(M) \longrightarrow 0. \qquad (1.3)$$

Now, a subgroup $H \subset \pi_1(Q)$ of index two gives us a nontrivial homomorphism

$$f : \pi_1(Q) \longrightarrow \pi_1(Q)/H = \mathbb{Z}_2$$

and vice versa. This, together with $\alpha_F \notin H$, implies that the composition

$$f \circ i_\# : \pi_1(F) = \mathbb{Z}_2 \longrightarrow \pi_1(Q) \longrightarrow \pi_1(Q)/H = \mathbb{Z}_2$$

is the identity. Thus *the Spin structures on Q over a connected manifold M are in one-to-one correspondence with the homomorphisms splitting the sequence* (1.3), *i.e. the homomorphisms $f : \pi_1(Q) \to \pi_1(F)$ satisfying $f \circ i_\# = Id_{\pi_1(F)}$.*

We have the following consequences of the previous observations

Proposition 1.4.1. *Let Q be the frame of orthonormal bundles over M.*

i) If Q has a Spin structure then

a) $\pi_1(Q) = \pi_1(F) \oplus \pi_1(X)$

b) $\pi_2(Q) = \pi_2(M)$

ii) If M is simply connected then Q has a Spin structure if and only if $\pi_1(Q) = \mathbb{Z}_2$.

Example 1.4.1. *The bundle of orthonormal frames of the sphere S^n, $n \geq 2$, is $SO(n+1)$ which can be seen from the fibration*

$$SO(n) \hookrightarrow SO(n+1) \to S^n.$$

Since $\pi_1(SO(n+1)) = \mathbb{Z}_2$ and S^n is simply connected, the sphere S^n, $n \geq 2$, has a Spin structure.

We can reformulate the description of the Spin structures in terms of the second Stiefel-Whitney class. Since

$$\begin{aligned} H^1(Q;\mathbb{Z}_2) &= Hom(H_1(Q);\mathbb{Z}_2) \\ &= Hom(\pi_1(Q)/[\pi_1(Q),\pi_1(Q)];\mathbb{Z}_2) = Hom(\pi_1(Q);\mathbb{Z}_2), \end{aligned}$$

every map $f : \pi_1(Q) \to \pi_1(F) = \mathbb{Z}_2$ defines an element $f \in H^1(Q;\mathbb{Z}_2)$. Note that $H^1(F;\mathbb{Z}_2) = Hom(\mathbb{Z}_2;\mathbb{Z}_2) = \mathbb{Z}_2$, so the condition $f \circ i_\# = Id_{\pi_1(F)}$ is equivalent to say that $f \in H^1(Q;\mathbb{Z}_2)$ remains nontrivial when we apply the restriction map $i^* : H^1(Q;\mathbb{Z}_2) \to H^1(F;\mathbb{Z}_2) = \mathbb{Z}_2$. Thus, *the Spin structures over a connected manifold M are in one-to-one correspondence with those $f \in H^1(Q;\mathbb{Z}_2)$ for which $i^*(f) \in H^1(F;\mathbb{Z}_2) = \mathbb{Z}_2$ is nonzero.*

Now, the fibration $F \hookrightarrow Q \to M$ induces the long exact sequence of cohomology groups

$$0 \longrightarrow H^1(M;\mathbb{Z}_2) \longrightarrow H^1(Q;\mathbb{Z}_2) \xrightarrow{i^*} H^1(F;\mathbb{Z}_2) \xrightarrow{\partial} H^2(M;\mathbb{Z}_2) \longrightarrow \cdots \tag{1.4}$$

Let $\alpha \in H^1(F;\mathbb{Z}_2)$ be the nontrivial element. The image

$$w_2(Q) := \partial(\alpha) \in H^2(M;\mathbb{Z}_2)$$

is called the second Stiefel-Whitney class of Q. If $w_2(Q) = 0$ then, by the exactness of (1.4), there exists $f \in H^1(Q;\mathbb{Z}_2)$ such that $i^*(f) = \alpha \in H^1(F;\mathbb{Z}_2)$. Conversely, if there exists such an element in $H^1(Q;\mathbb{Z}_2)$ then, by exactness, $\partial(\alpha) = 0$. Aditionally if $\partial(\alpha) = 0$ then, from the sequence (1.4), we deduce that

$$\frac{H^1(Q;\mathbb{Z}_2)}{H^1(M;\mathbb{Z}_2)} \cong \mathbb{Z}_2.$$

Hence,

Proposition 1.4.2. *The bundle of orthonormal frames Q over M has a Spin structure if and only if the second Stiefel-Whitney class $w_2(Q)$ vanishes. If $w_2(Q) = 0$ then, using the sequence (1.4), the Spin structures are clasified by $H^1(M;\mathbb{Z}_2)$.*

Example 1.4.2. *It is well known that the tangent space of the real projective space satisfies $T(\mathbb{R}\mathrm{P}^n) \oplus \epsilon^1 \cong ({\gamma^1}^*)^{n+1}$ where γ^1 is the tautological line bundle on $\mathbb{R}\mathrm{P}^n$ and ϵ^1 is the trivial line bundle.*

Using properties of the Stiefel-Whitney classes one obtains the total Stiefel-Whitney class of $\mathbb{R}\mathrm{P}^n$:

$$w(\mathbb{R}\mathrm{P}^n) = (1+x)^{n+1}$$

where $x \in H^1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2)$ is the generator and the first Stiefel-Whitney class of ${\gamma^1}^*$.

Developing the last expression we obtain

$$w_1(\mathbb{R}\mathrm{P}^n) = (n+1)x, \qquad w_2(\mathbb{R}\mathrm{P}^n) = \frac{n(n+1)}{2}x^2.$$

So, $\mathbb{R}\mathrm{P}^n$ is orientable if and only if $n = 2k+1$ and, in this case, it has a Spin structure if and only if $(k+1)(2k+1) \equiv 0 \mod 2$ which means that $\mathbb{R}\mathrm{P}^n$ has a Spin structure if and only if $n \equiv 3 \mod 4$.

Let us analize the case $M = \mathbb{C}\mathrm{P}^n$. We have $T(\mathbb{C}\mathrm{P}^n) \oplus \epsilon^1 \cong ({\gamma^1}^*)^{n+1}$ where γ^1 is the complex tautological line bundle on $\mathbb{C}P^n$ and ϵ^1 is the complex trivial line bundle. So the total Chern class of $\mathbb{C}\mathrm{P}^n$ is

$$c(\mathbb{C}\mathrm{P}^n) = (1+x)^{1+n}$$

where $x \in H^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is the generator and the first Chern class of ${\gamma^1}^*$.

Developing we have

$$c_1(\mathbb{C}\mathrm{P}^1) = (n+1)x.$$

Using $w_2(\mathbb{C}\mathrm{P}^n) = c_1(\mathbb{C}\mathrm{P}^n) \mod 2$ (see [16] p. 82, and [20]), one has that $w_2(\mathbb{C}\mathrm{P}^n) = 0$ if and only if $n \equiv 1 \mod 2$, i.e. $\mathbb{C}\mathrm{P}^n$ has a Spin structure if and only if $n \equiv 1 \mod 2$.

It is now clear that not every manifold has a Spin structure. Now we introduce Spin^c structures.

Definition 1.4.1. [16, p. 391] *Let Q be the bundle of oriented orthonormal frames over a Riemannian oriented manifold M. A Spin^c structure on Q consist of a principal $U(1)$-bundle $P_{U(1)}$ over M and a principal Spin$^c(n)$-bundle $P_{Spin^c(n)}$ with a Spin$^c(n)$-equivariant bundle map*

$$\Lambda : P_{Spin^c(n)} \longrightarrow Q \tilde{\times} P_{U(1)},$$

which is a 2-fold covering.

A $U(1)$-bundle over M is often called a *circle bundle*. The set of all circle bundles has structure of abelian group, in fact

Proposition 1.4.3. [1, p. 15-19] *If $\mathcal{P}(M, S^1)$ is the set of classes of circle bundles over a smooth manifold M then*

$$\mathcal{P}(M, S^1) \cong H^2(M; \mathbb{Z}).$$

There exists a relation between the class of a circle bundle, $c \in H^2(M; \mathbb{Z})$, giving a Spin^c structure and the class $w_2(Q) \in H^2(M; \mathbb{Z}_2)$ giving the existence of a Spin structure on Q. This relation will be clear in the next section.

1.5 Spin Structures on G Principal Bundles

In the third chapter of this work we will introduce the concept of Spin^{c,r} structure. It is a concept related to principal fiber bundles with fiber the twisted Spin group $Spin^{c,r}$. The existence of such a structure can be characterized by the existence of a Spin structure.

Let $G \subset SO(n)$ be a connected compact subgroup for which the map

$$i_\# : \pi_1(G) \to \pi_1(SO(n))$$

is onto, so the homogeneous space $SO(n)/G$ is simply connected.

Example 1.5.1. *The inclusion $i : SO(n) \to SO(n+1)$ given by*

$$i(A) = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

induces a surjective map $i_\# : \pi_1(SO(n)) \to \pi_1(SO(n+1))$ which is the unique nontrivial homomorphism of groups $f : \mathbb{Z} \to \mathbb{Z}_2$ when $n = 2$ and $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ for $n \geq 3$.

The fibration $SO(n) \hookrightarrow SO(n+1) \to S^n$ induces the long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_2(S^n) \xrightarrow{\partial} \pi_1(SO(n)) \xrightarrow{i_\#} \pi_1(SO(n+1)) \xrightarrow{\pi_\#} \pi_1(S^n) \longrightarrow 0.$$

This sequence is

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{i_\#} \mathbb{Z}_2 \xrightarrow{\pi_\#} 0 \longrightarrow 0$$

for $n = 2$ and

$$\cdots \longrightarrow 0 \longrightarrow 0 \xrightarrow{\partial} \mathbb{Z}_2 \xrightarrow{i_\#} \mathbb{Z}_2 \xrightarrow{\pi_\#} 0 \longrightarrow 0$$

for $n \geq 3$. Thus, the generator

$$\alpha_t = \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix}, \qquad 0 \leq t \leq 1,$$

of $\pi_1(SO(2)) = \mathbb{Z}$, gives the generator

$$\begin{pmatrix} \alpha_t & 0 \\ 0 & Id_{n-2} \end{pmatrix}$$

of $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n \geq 3$.

Definition 1.5.1. *Suppose Q is a G-principal bundle over M, we say that Q has a Spin structure if the associated bundle $Q^* = Q \times_G SO(n)$ has a Spin structure.*

Proposition 1.5.1. [9, p. 47] *Let $G \subset SO(n)$ be a connected compact subgroup such that the group $\pi_1(SO(n)/G)$ is trivial. A G-principal bundle P over M has a Spin structure if and only if there exists a homomorphism $f : \pi_1(P) \to \pi_1(SO(n))$ for which the diagram*

$$\begin{tikzcd} \pi_1(G) \arrow[r, "i_\#"] \arrow[d] & \pi_1(SO(n)) \\ \pi_1(P) \arrow[ur, "f"'] & \end{tikzcd}$$

commutes.

In the preceding diagram the vertical arrow is part of the long exact sequence of homotopy groups induced by the fibration $G \hookrightarrow P \to M$. The homomorphism f defines an element $f \in H^1(P;\mathbb{Z}_2) = Hom(\pi_1(P);\mathbb{Z}_2)$ whose restriction to the fibre, $i^*(f) \in H^1(G;\mathbb{Z}_2)$, has to coincide with $i_\# : \pi_1(G) \to \pi_1(SO(n)) = \mathbb{Z}_2$.

Proposition 1.5.2. *For $n \geq 3$*

a) $\pi_1(Spin^c(n)) = \mathbb{Z}$. The map $l_\# : \pi_1(Spin^c(n)) \to \pi_1(SO(2))$, induced by

$$l([g, z]) = z^2,$$

is an isomorphism.

b) If

$$\alpha \in \pi_1(Spin^c(n)), \quad \beta \in \pi_1 SO(n), \quad \gamma \in \pi_1(SO(2))$$

are the generators of these groups with $l_\#(\alpha) = \gamma$ then

$$(\lambda_n \times \lambda_2)_\#(\alpha) = \beta + \gamma.$$

Corollary 1.5.1. *Let $n \geq 3$. For the inclusion $i : SO(n) \times SO(2) \to SO(n+2)$ the following holds*

$$i_\# \circ (\lambda_n \times \lambda_2)_\# = 0.$$

Proof. It follows from Proposition 1.5.2 and Example 1.5.1. □

Proposition 1.5.3. *Let Q be the bundle of orthonormal frames over an oriented smooth manifold M. Q admits a $Spin^c$ structure if and only if there exists an $U(1)$-principal bundle $P_{U(1)}$ over M such that the fibre product $Q \tilde{\times} P_{U(1)}$ has a Spin structure.*

Proof. If Q has a Spin^c structure then there exists a $U(1)$-bundle $P_{U(1)}$ such that $P = Q \tilde{\times} P_{U(1)}$ is an $SO(n) \times SO(2)$ bundle over M together with a 2 fold cover P' of P which is a $Spin^c(n)$-bundle over M.

There exists an injective homomorphism $\tilde{\imath}$ which makes the diagram

$$\begin{tikzcd} Spin^c(n) \arrow[r, "\tilde{\imath}"] \arrow[d, "\lambda_n \times \lambda_2"'] & Spin(n+2) \arrow[d, "\lambda_{n+2}"] \\ SO(n) \times SO(2) \arrow[r, "i"'] & SO(n+2) \end{tikzcd}$$

commute. We describe this homomorphism. $Spin(n)$ is a subgroup of $Spin(n+2)$ generated by elements $x \in \mathbb{R}^n \subset \mathbb{R}^{n+2}$ as described in section 1.1 and there is also a copy of S^1 inside $Spin(n+2)$ given by the elements $\cos(t) + \sin(t)e_{n+1} \cdot e_{n+2}$. The intersection of this groups inside $Spin(n+2)$ is $\{1, -1\} = \mathbb{Z}_2$. We define $\tilde{\imath}([g, z]) = g \cdot z$ where the right side is a product in $Spin(n+2)$ after the identification described earlier has been made.

This map is well defined since $\tilde{\imath}([g, z]) = g \cdot z = (-g) \cdot (-z)$. If $g \cdot z = 1$ then z is the inverse of $g \in Spin(n) \subset Spin(n+2)$ and g is the inverse of z in $S^1 \subset Spin(n+2)$ thus $g, z \in Spin(n) \cap S^1$ which implies $z = g = 1$ or $g = z = -1$. Hence $\ker \tilde{\imath}$ is trivial.

Back to the proof, let $\alpha \in \pi_1(SO(n) \times SO(2))$ be an element of the fundamental group of the fibre of P. There exists a lift $\alpha' \in Spin^c(n)$ via $\lambda_n \times \lambda_2$. Now

$$(\lambda_{n+2} \circ \tilde{\imath})(\alpha') \in \pi_1(SO(n+2)).$$

Using the previous commutative diagram, the composition described is just $i_\#$. This implies, according to proposition 1.5.1, that P has a Spin structure in the sense of definition 1.5.1.

Conversely, let $P = Q \tilde{\times} P_{U(1)}$, $\lambda = \lambda_n \times \lambda_2$ and $F = SO(n) \times U(1)$. According to proposition 1.5.1, due to the existence of f, $H = \ker(f) \subset \pi_1(P)$ is a subroup of index 2. Therefore, there exists a double covering space $\Lambda : P_{Spin^c(n)} \to P$ corresponding to H. Let $\mu : P \times F \to P$ be the action of F in P and consider the composition of induced maps on fundamental groups

$$\pi_1(P_{Spin^c(n)} \times Spin^c(n)) \xrightarrow{(\Lambda \times \lambda)_\#} \pi_1(P \times F) \xrightarrow{\mu_\#} \pi_1(P).$$

If $(\sigma, \tau) \in \pi_1(P_{Spin^c(n)}) \times \pi_1(Spin^c(n))$, by means of the inclusion $h : \pi_1(F) \to \pi_1(P)$, then

$$\mu_\# \circ (\Lambda \times \lambda)_\#(\sigma, \tau) \quad = \quad \Lambda_\#(\sigma) h(\lambda_\#(\tau)).$$

We know that

$$\Lambda_\#(\sigma) \in H \qquad \text{and} \qquad f(h(\lambda_\#(\tau))) = i_\#(\lambda_\#(\tau)) = 0$$

by Corollary 1.5.1 and Proposition 1.5.1. Thus, $h(\lambda_\#(\tau)) \in H$ and $\Lambda_\#(\sigma)h(\lambda_\#(\tau)) \in H$. Hence, there exists a lift $\tilde{\mu} : P_{Spin^{c,r}(n)} \times Spin^{c,r}(n) \to P_{Spin^{c,r}(n)}$ which gives the equivariance in definition 1.4.1. $\square$

Corollary 1.5.2. *Let Q be the bundle of orthonormal frames over M, the following are equivalent*

a) Q has a $Spin^c$ structure.

b) There exists a circle bundle $P_{U(1)}$ such that $w_2(Q \tilde{\times} P_{U(1)}) = 0$.

c) There exists a circle bundle $P_{U(1)}$ such that $w_2(Q) \equiv c_1(P_{U(1)}) \mod 2$.

d) There exists a cohomology class $z \in H^2(M;\mathbb{Z})$ such that $w_2(Q) \equiv z \mod 2$

Proof. $a) \Leftrightarrow b)$ follows from propositions 1.4.2 and 1.5.3. $b) \Leftrightarrow c)$ follows from the properties of Stiefel-Whitney classes and the fact that $w_2(P_{U(1)}) = c_1(P_{U(1)}) \mod 2$ where c_1 is the first Chern class of $P_{U(1)}$. $c) \Leftrightarrow d)$ is due to Proposition 1.4.3. □

Note. We have used $w_2(Q)$ to denote an element in $H^2(M;\mathbb{Z}_2)$, we would replace this notation by $w_2(M)$ when Q is the bundle of oriented orthonormal frames of M and in this case we won't distinguish between saying that there exists a Spin (Spin^c) structure "on Q" or "on M" but we will keep the notation $w_2(Q)$ since Q is not necessarily this bundle.

Example 1.5.2. *If M has a Spin structure then it has a Spin^c structure.*

Proof. Take $P_{U(1)}$ as the trivial bundle and use the Corollary 1.5.2. □

Now we know what is the role of the $U(1)$-bundles over M on the existence of Spin^c structures. In terms of characteristic classes, we know from Corollary 1.5.2 which manifolds have a Spin^c structure that does not come from a Spin structure on the bundle of orthonormal frames of M. Next, we give a characterization of the Spin^c structures which do not come from a Spin structure on a simply connected manifold in terms of homotopy groups.

Proposition 1.5.4. *Let M be simply connected and Q the bundle of orthonormal frames over M. The following are equivalent*

1. *There exists a $U(1)$ bundle P_1 over M such that in the long exact sequence*

$$\cdots \longrightarrow \pi_2(X) \xrightarrow{\partial} \pi_1(SO(n)\times SO(2)) \xrightarrow{h} \pi_1(Q\tilde{\times}P_1) \longrightarrow \pi_1(X) = 0,$$

   $Im(\partial) \cong \langle(1,p)\rangle$ for $p \in \mathbb{N}$ odd.

2. *Q has a Spin^c but not a Spin structure.*

Proof. If P is a Spin^c structure on Q then there exists a $P_{U(1)}$ bundle over M such that, by Proposition 1.5.3, the fibre product $Q\tilde{\times}P_1$ has a Spin structure. According to Proposition 1.5.1 this means that there exists a map $f : \pi_1(Q\tilde{\times}P_1) \to \pi_1(SO(n+2))$, such that the diagram

$$\begin{tikzcd} \pi_1(SO(n)\times SO(2)) = \mathbb{Z}_2\oplus\mathbb{Z} \arrow[r, "i_\#"] \arrow[d, "h"'] & \pi_1(SO(n+2)) = \mathbb{Z}_2 \\ \pi_1(Q\tilde{\times}P_1) \arrow[ur, "f"'] & \end{tikzcd}$$

commutes.

The existence of f implies than $h : \mathbb{Z}_2\oplus\mathbb{Z} \to \pi_1(Q\tilde{\times}P_1)$ maps the generators, $(\alpha,0)$ and $(0,\beta)$, of each summand nontrivially and satisfies $2h(\alpha,0) = 0$.

Since $\pi_1(M)$ is trivial h is onto. In consequence $\pi_1(Q\tilde{\times}P_1) = h(\mathbb{Z}_2\oplus\mathbb{Z})$, with $h(\alpha,0)$ a non trivial element satisfying $2h(\alpha,0) = 0$. This happens if $\pi_1(Q\tilde{\times}P_1)$ contains $\mathbb{Z}_{2p}$, $p \in \mathbb{N}$, as a subgroup. These observations imply that $\pi_1(Q\tilde{\times}P_1)$ is isomorphic to $\mathbb{Z}_2\oplus\mathbb{Z}$ or $\mathbb{Z}_{2p}$.

If Q does not have a Spin structure then, by Proposition 1.4.1, $\pi_1(Q) = 0$. This implies that the map k is onto in the commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \\
 & \pi_2(M) \arrow[d, "\partial"] & & \\
 & \mathbb{Z}_2 \oplus \mathbb{Z} \arrow[d, "h"'] & & \\
\mathbb{Z} = \pi_1(U(1)) \arrow[ur, "j_\#"] \arrow[r, "k"] & \pi_1(Q \tilde{\times} P_1) \arrow[r] \arrow[d] & \pi_1(Q) \arrow[r] & 0 \\
 & \pi_1(M) = 0 & &
\end{tikzcd}$$

In consequnce $\pi_1(Q \tilde{\times} P_1)$ must be cyclic.

So $\pi_1(Q \tilde{\times} P_1) = \mathbb{Z}_{2p}$ and $h(\alpha, 0) = [p]$ with p odd and $h(0, \beta) = [c]$ with $\gcd(c, 2p) = 1$. Observe that $h(a, b) \equiv 0 \mod 2p$ if and only if $(ap + bc) \equiv 0 \mod 2p$ with $a \in \mathbb{Z}_2$ and $b \in \mathbb{Z}$. This equation is satisfied if and only if the equations

$$bc \equiv 0 \mod 2p, \qquad p + bc \equiv 0 \mod 2p,$$

are satisfied.

The condition $\gcd(c, 2p) = 1$ implies (a, b) satisfies these equations if and only if $(a, b) \in \langle(1, p)\rangle$, the cyclic group generated by the element $(1, p) \in \mathbb{Z}_2 \oplus \mathbb{Z}$. So $\mathrm{Im}(\partial) = \langle(1, p)\rangle = \ker(h)$.

If $\mathrm{Im}(\partial) = \langle(1, p)\rangle$ for some $p \in \mathbb{N}$ odd then, since h is onto, $\pi_1(Q \tilde{\times} P_1) \cong (\mathbb{Z}_2 \oplus \mathbb{Z})/\langle(1, p)\rangle \cong \mathbb{Z}_{2p}$. If H is the cyclic group $\langle(1, p)\rangle$ then the quotient group is given by the classes

$$(0, r) + H, \qquad r = 1, \ldots, 2p,$$

so h is defined by $h(\alpha, 0) = (0, p) + H$ since h is a homomorphism and $h(0, \beta) = (0, c) + H$ with $\gcd(c, 2p) = 1$ since h is onto. This shows that $h(0, \beta)$ also generates $\pi_1(Q \times P_{U(1)})$ so the map $k = h \circ j_\#$ is onto and this implies that $\pi_1(Q) = 0$, i.e. Q does not have a Spin structure according to Proposition 1.4.1.

Let $f : (\mathbb{Z}_2 \oplus \mathbb{Z})/H \to \mathbb{Z}_2$ be given by $(0, r) + H \mapsto r \mod 2$. For the generators of $\mathbb{Z}_2 \oplus \mathbb{Z}$ one has

$$f \circ h(\alpha, 0) = p \mod 2 = 1 = i_\#(\alpha, 0), \qquad f \circ h(0, \beta) = c \mod 2 = 1 = i_\#(0, \beta)$$

so $f \circ h = i_\#$ and Q has a spin^c structure. □

Similar results can be proved for other twisted structures defined using twisted Spin groups. For example in [8] the Spin^r structures are defined using the group $Spin(r)$ instead of S^1 used to define Spin^c structures. In this work we are interested in Spin^{c,r} structures but we introduce them in chapter 3.

1.6 Almost Complex Structures

Definition 1.6.1. *Let M be a smooth manifold. If there exists $J \in \mathrm{End}(TM)$, an endomorphism of the tangent bundle, such that $J^2 = -Id_{TM}$ we say that J is an almost complex structure and M is an almost complex manifold.*

Example 1.6.1. *For $M = \mathbb{R}^{2n}$ we have $TM \cong \mathbb{R}^{2n} \times \mathbb{R}^{2n}$, i.e. for every $p \in \mathbb{R}^n$ we have that $T_p(M) \cong \mathbb{R}^{2n}$. This latter space has its standard basis $\{(p, e_1), \dots, (p, e_{2n})\}$, let $J_p((p, e_{2i-1})) = (p, e_{2i})$ and $J_p((p, e_{2i})) = (p, -e_{2i-1})$. We will refer to this particular J as the standard almost complex structure.*

If M is a complex manifold (a manifold having a cover by charts to $\mathbb{C}^n$ such that the transition functions are holomorphic), with local complex coordinates $z_j = x_j + iy_j$ then the map $J(\partial/\partial x_i) = \partial/\partial y_i$ and $J(\partial/\partial y_i) = -\partial/\partial x_i$ can be glued to define a global almost complex structure.

On an overlapping chart V, let $w_j = u_j + iv_j$ be local complex coordinates and define $\tilde{J}(\partial/\partial u_i) = \partial/\partial v_i$ and $\tilde{J}(\partial/\partial v_i) = -\partial/\partial u_i$. On a point of the intersection we have

$$\begin{aligned} \frac{\partial}{\partial x_j} &= \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_j}\frac{\partial}{\partial u_k} + \frac{\partial v_k}{\partial x_j}\frac{\partial}{\partial v_k}, \\ \frac{\partial}{\partial y_j} &= \sum_{k=1}^{n} \frac{\partial u_k}{\partial y_j}\frac{\partial}{\partial u_k} + \frac{\partial v_k}{\partial y_j}\frac{\partial}{\partial v_k}. \end{aligned}$$

Since the transition functions are holomorphic, the change of coordinates satisfies

$$\begin{aligned} \frac{\partial u_k}{\partial x_j} &= \frac{\partial v_k}{\partial y_j}, \\ \frac{\partial v_k}{\partial x_j} &= -\frac{\partial u_k}{\partial y_j}. \end{aligned}$$

Thus

$$\begin{aligned} \tilde{J}\frac{\partial}{\partial x_j} &= \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_j}\frac{\partial}{\partial v_k} - \frac{\partial v_k}{\partial x_j}\frac{\partial}{\partial u_k} = \sum_{k=1}^{n} \frac{\partial v_k}{\partial y_j}\frac{\partial}{\partial v_k} + \frac{\partial u_k}{\partial y_j}\frac{\partial}{\partial u_k} = \frac{\partial}{\partial y_j}, \\ \tilde{J}\frac{\partial}{\partial y_j} &= \sum_{k=1}^{n} \frac{\partial u_k}{\partial y_j}\frac{\partial}{\partial v_k} - \frac{\partial v_k}{\partial y_j}\frac{\partial}{\partial u_k} = -\sum_{k=1}^{n} \frac{\partial v_k}{\partial x_j}\frac{\partial}{\partial v_k} + \frac{\partial u_k}{\partial x_j}\frac{\partial}{\partial u_k} = -\frac{\partial}{\partial x_j}. \end{aligned}$$

Hence J and $\tilde{J}$ coincide.

The existence of an almost complex structure implies that M is even dimensional and naturally orientable as follows. On every fiber T_xM of TM, J_x is an isomorphism such that $J_x^2 = -Id_{T_xM}$. If $\dim(M) = n$ then

$$(-1)^n = \det(J_x)^2 > 0.$$

For the orientability take a local basis $\{e_1, \dots, e_{2n}\}$ on $U \subset M$. Let $V \subset M$ such that $U \cap V \neq \varnothing$ with local basis $e_i' = Ae_i$. If $p \in U \cap V$ then, assuming J_p is the standard almost complex structure,

$$J_pAe_{2i-1} = J_pe_{2i-1}' = e_{2i}' = Ae_{2i} = AJ_pe_{2i-1}.$$

Similarly $J_pAe_{2i} = AJ_pe_{2i}$, which means that $AJ_p = J_pA$. Writing the product explicitly, the components of the matrix A satisfy

$$a_{2i+1,2j+1} = a_{2i-1,2j-1}, \qquad a_{2i+1,2j} = -a_{2i,2j-1}.$$

Thus A is the image of a complex matrix under the inclusion $i : Gl_n(\mathbb{C}) \to Gl_{2n}(\mathbb{R})$, which implies $\det(A) > 0$[1]. Hence the structure group of M is reduced from $Gl_{2n}(\mathbb{R})$ to $Gl_n(\mathbb{C})$ and if M is a Riemannian manifold then, taking ortogonal bases, the structure group reduces from $SO(2n)$ to $U(n)$.

Not all the even dimensional orientable manifolds have an almost complex structure.

Example 1.6.2. *The sphere S^n has an almost complex structure if and only if $n = 2$ or $n = 6$. This can be proved using characteristic classes. We refer to [18, p.211] for the proof and describe the usual almost complex structure (see [19, p.119] for more details of this example).*

For $p \in S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ and $v \in T_pS^2 = \{y \in \mathbb{R}^3 : \langle p, y\rangle = 0\}$ define $J_p(v) = p \times v$. Using the properties of the the cross product one has

$$J_p(J_p(v)) = p \times (p \times v) = \langle p, v\rangle\, p - \langle p, p\rangle\, v = -v.$$

The octonions $\mathbb{O}$ is a normed division algebra. Every $p \in \mathbb{O}$ can be written as $p = a + b_1e_1 + \cdots + b_7e_7 = a + b$ where $a, b_i \in \mathbb{R}$ and $1, e_i$ are the generators of the algebra; b is called the imaginary part of p. We can identify $\mathbb{R}^7 = \{p \in \mathbb{O} : a = 0\}$ and $S^6 = \{p \in \mathbb{O} : a = 0,\ |b| = 1\}$.

With the product of the octonios a cross product in $\mathbb{R}^7$ can be given: $p \times v = Im(p \cdot v) = \frac{1}{2}(p \cdot v - v \cdot p)$. This product satisfies the following identities involving the standard inner product of $\mathbb{R}^7$:

$$\langle u \times v, w\rangle = \langle u, v \times w\rangle$$

and

$$(u \times v) \times w + u \times (v \times w) = 2\,\langle u, w\rangle\, v - \langle v, w\rangle\, u - \langle v, u\rangle\, w.$$

For $p \in S^6$ and $v \in T_pS^6 = \{y \in \mathbb{R}^7 : \langle p, y\rangle = 0\}$ define $J_p(v) = p \times v$. From the definition of the cross product and the first identity $p \times v$ is orthogonal to p, i.e. $J_p(v) \in T_pS^6$. Using the second

$$J_p(J_p(v)) = 2\,\langle p, v\rangle\, p - \langle p, v\rangle\, p - \langle p, p\rangle\, v = -v.$$

The natural question of an almost complex manifold having a complex structure has some equivalences.

For the complexification of the tangent bundle we have

$$T \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)} \oplus T^{(0,1)}$$

where

$$T^{(1,0)} = \{X - iJX : X \in TM\}$$

[1] This can be proved as follows. Let $A \in Gl_n(\mathbb{C})$, as $Gl_n(\mathbb{C})$ is path connected and the composition of the inclusion with det is continuous then $\det(A)$ must be in the connected component of $\det(I) = 1$.

is the eigenspace of J corresponding to the eigenvalue i and $T^{(0,1)} = \overline{T^{(1,0)}}$ is the eigenspace corresponding to the eigenvalue $-i$.

We say that the distribution $T^{(1,0)}$ is integrable if it is closed under the Lie bracket:

$$[T^{(1,0)}, T^{(1,0)}] \subset T^{(1,0)}. \tag{1.5}$$

Let $X - iJX,\ Y - iJY \in T^{(1,0)}$, the bracket

$$[X - iJX, Y - iJY] = [X, Y] - [JX, JY] - i([X, JY] + [JX, Y])$$

belongs to $T^{(1,0)}$ if and only if

$$[X, Y] - [JX, JY] + J([X, JY] + [JX, Y]) = 0.$$

Thus $T^{(1,0)}$ is integrable if and only if the tensor

$$N_J(X, Y) = [X, Y] - [JX, JY] + J([X, JY] + [JX, Y]) \tag{1.6}$$

vanishes for all $X, Y \in \Gamma(TM)$. We also call the structure J integrable if this tensor vanishes.

Theorem 1.6.1 (Newlander-Nirenberg). *Let (M, J) be an almost complex manifold. The almost complex structure J is induced by a complex structure on M if and only if the Nijenhuis tensor* (1.6) *vanishes.*

If M is complex it is immediate that $N_J \equiv 0$ with J the almost complex structure of Example 1.6.1. The other direction is nontrivial.

If the components of J are real analytic functions then the theorem is less difficult to prove, the proof in this case is given in [15, vol II p. 321]. B. Malgrange reduced the smooth case to the analytic case in [17], i.e. assuming that the almost complex structure J has smooth components and the integrability condition (1.5) holds, Malgrange proved that the there exist a change of coordinates in which the components of J are real analytic functions.

We prove that the Newlander-Nirenberg Theorem is a local result, i.e. if for every point $p \in M$ there exist local complex coordinates $f = (f_1, f_2, \ldots, f_n)$ such that the forms df_j, $j = 1 \ldots, n$, are $(1, 0)$ forms then the transition functions of these local coordinates is holomorphic.

We explain the last paragraph. A complex valued form $\omega : TM \to \mathbb{C}$ is of type $(1, 0)$ if $\omega(Z) = 0$ for all $Z \in T^{(0,1)}$. Now, with the description of the complex space $T^{(0,1)}$ that we have made, $df_j = du_j + idv_j$ being of type $(1, 0)$ means that

$$\begin{aligned} 0 &= Zf_j \\ &= df_j(Z) \\ &= df_j(X + iJX) \\ &= (du_j(X + iJX) + idv_j(X + iJX) \\ &= du_j(X) - dv_j(JX) + i(dv_j(X) + du_j(JX)), \end{aligned}$$

for all $X \in TM$.

Thus, df_j is of type $(1,0)$ if and only if

$$du_j(X) \quad = \quad dv_j(JX), \tag{1.7}$$

for all $X \in TM$.

On the other hand a map $f : M \to M'$ between almost complex manifolds (M, J) and (M', J') is said to be pseudo-holomorphic if $df \circ J = J' \circ df$. With $M' = \mathbb{C}^n$ and J' the standard almost complex structure (which, in this case, is just multiplication by the complex number i), we have that $f : M \to \mathbb{C}^n$ is pseudo-holomorphic if and only if the components of f satisfy

$$df_j(JX) = i(df_j(X)), \qquad X \in TM.$$

Putting $f_j = u_j + iv_j$, the last equation is equivalent to

$$du_j(JX) + dv_j(X) + i(dv_j(JX) - du_j(X)) = 0$$

which gives us the equation (1.7). Hence $f : M \to \mathbb{C}^n$ is pseudo-holomorphic if and only if the forms df_j are of type $(1,0)$.

Now, if $f : M \to M'$ is pseudo-holomorphic then $df^{-1} : M' \to M$ is pseudo-holomorphic: from $w = df(v) \in TM'$ we have $v = df^{-1}(w)$ and

$$df^{-1}(J'w) = df^{-1}(J'df(v)) = df^{-1}(df(Jv)) = Jdf^{-1}(w).$$

Also, the composition of pseudo-holomorphic maps is pseudo-holomorphic

$$d(g \circ f) \circ J = dg \circ (df \circ J) = dg \circ (J' \circ df) = J'' \circ dg \circ df = J'' \circ d(g \circ f).$$

Let us summarize the last lines: if U, V are open sets with corresponding local complex coordinates $f = (f_1, f_2, \ldots, f_n)$ and $g = (g_1, g_2, \ldots, g_n)$, such that the df_j and dg_j are of type $(1,0)$, then, on the overlapping, the transition function $h = f \circ g^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ is pseudo-holomorphic. This in turn means that the components $h_j = u_j + iv_j$, $j = 1 \ldots, n$, satisfy the equation (1.7) with J the usual almost complex structure, i.e:

$$\begin{aligned} \frac{\partial u_j}{\partial x_k} &= \frac{\partial v_j}{\partial y_k}, \\ \frac{\partial v_j}{\partial x_k} &= -\frac{\partial u_j}{\partial y_k}. \end{aligned}$$

These are the Cauchy-Riemman equations that we have already presented in Example 1.6.1. Hence the transition function is holomorphic, proving that the Newlander-Nirenberg Theorem is a local result. For the rest of the details of the proof see [17, 23, 25].

1.7 Pure Spinors

We end this chapter giving a relation between spinors and almost complex structures.

Definition 1.7.1. *Let M be a Spin manifold. A nonzero spinor $\psi \in \Gamma(S)$ is pure if for every $p \in M$ and $X \in T_pM$ there exists $Y \in T_pM$ such that*

$$X \cdot \psi = iY \cdot \psi. \tag{1.8}$$

Proposition 1.7.1. *If the Riemannian manifold (M, g) has a Spin structure with a pure spinor ψ then M has an orthogonal almost complex structure.*

Proof. Let $Y = J(X)$ be the unique vector corresponding to X in the equation (1.8). The corresponding vector for $J(X)$ is $J(J(X)) = -X$.

From (1.8)

$$(X - iJ_\psi(X)) \cdot \psi = 0, \qquad X \in TM.$$

Thus

$$(X - iJ_\psi(X)) \cdot (X - iJ_\psi(X)) \cdot \psi = 0$$

if and only if

$$g(X, X) = g(J(X), J(X)) \qquad \text{and} \qquad X \cdot J(X) + J(X) \cdot X = 2g(X, J(X)) = 0.$$

□

Example 1.7.1. *The spinor $\psi = u_+ \otimes u_+ \otimes \cdots \otimes u_+ \in \Delta_{2n}$ is a pure spinor. With Clifford multiplication we can verify explicitly than $e_{2i-1} \cdot \psi = ie_{2i} \cdot \psi$. Thus, this spinor ψ gives the standard almost complex structure in $\mathbb{R}^{2n}$ which is orthogonal with respect to the standard inner product. Recall that for $n = 2$ we did the explicit calculation in example 1.1.1.*

An orthogonal almost complex structure J is called almost Hermitian. If it is integrable then we call it Hermitian.

Let ∇ be the Levi-Civita connection associated to g and ∇^S the conection induced by ∇ on the bundle of spinors $S = P_{Spin(n)} \times_{\kappa_n} \Delta_n$. The connection ∇^S is compatible with Clifford product.

Proposition 1.7.2. *Let J be an almost Hermitian structure coming from a pure spinor ψ. The structure J is Hermitian if and only if*

$$W \cdot \nabla^S_Z \psi = 0, \qquad Z, W \in T^{(1,0)}.$$

Proof. Taking covariant derivative of $g(JY, Z) + g(Y, JZ) = 0$ we have

$$g((\nabla_X J)Y, Z) + g(Y, (\nabla_X J)Z) = 0. \tag{1.9}$$

Observe that $N_J = 0$ if and only if

$$(\nabla_{JX} J)Y = J((\nabla_X J)Y) \tag{1.10}$$

If (1.10) holds then

$$\begin{aligned} N_J(X, Y) &= [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] \\ &= J(\nabla_X J)Y - J(\nabla_Y J)X - (\nabla_{JX} J)Y + (\nabla_{JY} J)X \\ &= 0. \end{aligned}$$

If $N_J = 0$ then

$$A(X, Y, Z) = g(J(\nabla_X J)Y - (\nabla_{JX} J)Y, Z)$$

satisfies $A(X, Y, Z) = A(Y, X, Z)$. Using $(\nabla_X J)(JY) = -J((\nabla_X J)Y)$ and (1.9) we deduce $A(X, Y, Z) = -A(X, Z, Y)$. Thus $A(X, Y, Z) = -A(X, Z, Y) = -A(Z, X, Y)$ and

$A(X,Y,Z) = -A(Y,Z,X) = A(Z,X,Y) = -A(X,Y,Z)$. Hence $A = 0$.

Now, put $(\nabla_X J)Y$ instead of X in (1.8) and use (1.10) to obtain

$$\begin{aligned}
0 =&((\nabla_X J)Y - i(\nabla_{JX} J)Y) \cdot \psi \\
=&(\nabla_X JY - J\nabla_X Y - i\nabla_{JX} JY + iJ\nabla_{JX} Y) \cdot \psi \\
=&(\nabla_X JY + i\nabla_X Y - i\nabla_{JX} JY + \nabla_{JX} Y) \cdot \psi \\
=&i(\nabla_{X-iJX}(Y - iJY) \cdot \psi.
\end{aligned}$$

Hence integrability of J is equivalent to $(\nabla_W Z) \cdot \psi = 0$ for all $Z, W \in T^{(1,0)}$.

Taking covariant derivative of $Z \cdot \psi = 0$ and using the compatibility of ∇^S with Clifford product we get

$$0 = \nabla^S_W(Z \cdot \psi) = Z \cdot \nabla^S_W \psi + (\nabla_W Z) \cdot \psi.$$

Concluding $Z \cdot \nabla_W \psi = 0$ if and only if $(\nabla_W Z) \cdot \psi = 0$, for all $Z, W \in T^{(1,0)}$ $\square$

Chapter 2

Twisted Partially Pure Spinors

2.1 Characterization of Almost Complex Structures

Motivated by the relationship between orthogonal complex structures and pure spinors, showed in Proposition 1.7.1, we define twisted partially pure spinors in order to characterize spinorially subspaces of Euclidean space endowed with a complex structure.

Recall that a classical pure spinor $\phi \in \Delta_{2m}$ is a spinor such that the (isotropic) subspace of complexified vectors $X - iY \in \mathbb{R}^{2m} \otimes \mathbb{C}$, $X, Y \in \mathbb{R}^{2m}$, which annihilate ϕ under Clifford multiplication

$$(X - iY) \cdot \phi = 0$$

is of maximal dimension, where $m \in \mathbb{N}$ and Δ_{2m} is the standard complex representation of the Spin group $Spin(2m)$ (cf. [16]). This means that for every $X \in \mathbb{R}^{2m}$ there exists a $Y \in \mathbb{R}^{2m}$ satisfying

$$X \cdot \phi = iY \cdot \phi.$$

By setting $Y = J(X)$, one can see that a pure spinor determines a complex structure on $\mathbb{R}^{2m}$. Geometrically, the two subspaces $TM \cdot \phi$ and $iTM \cdot \phi$ of Δ_{2m} coincide, which means $TM \cdot \phi$ is a complex subspace of Δ_{2m}, and the effect of multiplication by the number $i = \sqrt{-1}$ is transferred to the tangent space TM in the form of J.

The authors of [4, 30] investigated (the classification of) non-pure classical spinors by means of their isotropic subspaces. In [30], the authors noted that there may be many spinors (in different orbits under the action of the Spin group) admitting isotropic subspaces of the same dimension, and that there is a gap in the possible dimensions of such isotropic subspaces.

In our Euclidean/Riemannian context, such isotropic subspaces correspond to subspaces of Euclidean space endowed with orthogonal complex structures. We define twisted partially pure spinors (cf. Definition 2.2.1) in order to establish a one-to-one correspondence between subspaces of Euclidean space (of a fixed codimension) endowed with orthogonal complex structures (and oriented orthogonal complements), and orbits of such spinors under a particular subgroup of the twisted Spin group (cf. Theorem 2.2.1). By using spinorial twists we avoid having different orbits under the full twisted Spin group and also the aforementioned gap in the dimensions.

The need to establish such a correspondence arises from our interest in developing a spinorial setup to study the geometry of manifolds admitting (almost) CR structures (of arbitrary codimension) and elliptic structures. Since such manifolds are not necessarily Spin nor Spin^c, we are led to consider spinorially twisted Spin groups, representations, structures, etc. Geometric and topological considerations regarding such manifolds will be presented in the next chapter.

This chapter is organized as follows. First we define the anti-symmetric 2-forms and endomorphims associated to twisted spinors; we also present some results on subgroups and branching of representations. In Section 2.2, we define partially pure spinors, deduce their basic properties and prove the main theorem, Theorem 2.2.1, which establishes the aforementioned one-to-one correspondence.

2.1.1 Skew-symmetric 2-Forms and Endomorphisms Associated to Twisted Spinors

In this section, we define the antisymmetric 2-forms and endomorphisms associated to a twisted spinor, and describe various inclusions of groups into (twisted) Spin groups.

Recall from chapter 1 that $\Delta_n = (\mathbb{C}^2)^{\otimes[n/2]}$, the space of spinors, is a complex representation of $Spin(n)$.

We will make the following convention. Consider the involution

$$\begin{array}{rcl} F_{2m}: \Delta_{2m} & \longrightarrow & \Delta_{2m} \\ \phi & \mapsto & (-i)^m e_1 e_2 \cdots e_{2m} \cdot \phi, \end{array}$$

and let

$$\Delta_{2m}^{\pm} = \{\phi \mid F_{2m}(\phi) = \pm\phi\}.$$

This definition, of positive and negative Weyl spinors, differs from the one in [9] by a factor $(-1)^m$. Nevertheless, we have chosen this convention so that the spinor

$$u_{1,\ldots,1} = u_+ \otimes u_+ \otimes \cdots \otimes u_+,$$

of example 1.7.1, is always positive and corresponds to the standard complex structure on $\mathbb{R}^{2m}$.

On the twisted representation $\Delta_r \otimes \Delta_n$, we extend Clifford multiplication by

$$\begin{array}{rcl} \mu_r \otimes \mu_n : \left(\bigwedge^* \mathbb{R}^r \otimes_{\mathbb{R}} \bigwedge^* \mathbb{R}^n\right) \otimes_{\mathbb{R}} (\Delta_r \otimes \Delta_n) & \longrightarrow & \Delta_r \otimes \Delta_n \\ (w_1 \otimes w_2) \otimes (\psi \otimes \varphi) & \mapsto & (w_1 \otimes w_2) \cdot (\psi \otimes \varphi) = (w_1 \cdot \psi) \otimes (w_2 \cdot \varphi). \end{array}$$

As in the untwisted case, $\mu_r \otimes \mu_n$ is an equivariant homomorphism of $Spin^{c,r}(n)$ representations.

From now on, we will often write f_{kl} for the Clifford product $f_k \cdot f_l$.

Definition 2.1.1. [8] *Let $r \geq 2$, $\phi \in \Delta_r \otimes \Delta_n$, $X, Y \in \mathbb{R}^n$, $(f_1 \ldots, f_r)$ an orthonormal basis of $\mathbb{R}^r$ and $1 \leq k, l \leq r$.*

- *Define the real 2-forms associated to the spinor ϕ by*

$$\eta_{kl}^{\phi}(X,Y) = \mathrm{Re}\left\langle X\wedge Y\cdot f_k f_l\cdot\phi,\phi\right\rangle.$$

- *Define the antisymmetric endomorphisms $\hat{\eta}_{kl}^{\phi}\in\mathrm{End}^{-}(\mathbb{R}^n)$ by*

$$X\mapsto\hat{\eta}_{kl}^{\phi}(X) := (X\lrcorner\eta_{kl}^{\phi})^{\sharp},$$

where $X\in\mathbb{R}^n$, $\lrcorner$ denotes contraction and $\sharp$ denotes metric dualization from 1-forms to vectors.

An explicit expression of these forms and endomorphisms can be given. Let $\{e_1,\ldots,e_{2m+r}\}$ be an orthonormal basis with respect to the inner product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^n$ and let w_j be the 1-form $w_j(X)=\langle e_j,X\rangle$ so that $w_j^{\sharp}=e_j$.

Contracting

$$\eta_{kl}^{\phi}=\sum_{i<j}\eta_{kl}^{\phi}(e_i,e_j)w_i\wedge w_j,$$

we obtain

$$X\lrcorner\eta_{kl}^{\phi}=\sum_{i<j}\eta_{kl}^{\phi}(e_i,e_j)(w_i(X)w_j-w_j(X)w_i).$$

Thus

$$\begin{aligned}\hat{\eta}_{kl}^{\phi}(X) &= (X\lrcorner\eta_{kl}^{\phi})^{\sharp}\\ &= \sum_{i<j}\eta_{kl}^{\phi}(e_i,e_j)(w_i(X)e_j-w_j(X)e_i)\\ &= \sum_{i<j}\eta_{kl}^{\phi}(e_i,e_j)(\langle e_i,X\rangle e_j-\langle e_j,X\rangle e_i).\end{aligned}$$

From where

$$\begin{aligned}(\hat{\eta}_{kl}^{\phi})_{ij} &= \left\langle e_i,\hat{\eta}_{kl}^{\phi}(e_j)\right\rangle\\ &= \sum_{r<s}\eta_{kl}^{\phi}(e_r,e_s)(\delta_{jr}\delta_{si}-\delta_{js}\delta_{ri})\\ &= \eta_{kl}^{\phi}(e_j,e_i).\end{aligned}$$

Now it is clear that $\hat{\eta}_{kl}^{\phi}$ is an antisymmetric endomorphisms.

Lemma 2.1.1. *Let $r\geq 2$, $\phi\in\Delta_r\otimes\Delta_n$, $X,Y\in\mathbb{R}^n$, $(f_1\ldots,f_r)$ an orthonormal basis of $\mathbb{R}^r$ and $1\leq k,l\leq r$. Then*

$$\begin{aligned}\mathrm{Re}\left\langle f_kf_l\cdot\phi,\phi\right\rangle &= 0,\\ \mathrm{Re}\left\langle X\wedge Y\cdot\phi,\phi\right\rangle &= 0, && (2.1)\\ \mathrm{Im}\left\langle X\wedge Y\cdot f_kf_l\cdot\phi,\phi\right\rangle &= 0, && (2.2)\\ \mathrm{Re}\left\langle X\cdot\phi,Y\cdot\phi\right\rangle &= \langle X,Y\rangle\,|\phi|^2, && (2.3)\end{aligned}$$

Proof. By using (1.2) twice

$$\langle f_k f_l \cdot \phi, \phi\rangle \quad = \quad -\overline{\langle f_k f_l \phi, \phi\rangle}.$$

For identity (2.1), recall that for $X, Y \in \mathbb{R}^n$

$$X \wedge Y = X \cdot Y + \langle X, Y\rangle \,.$$

Thus

$$\langle X \wedge Y \cdot \phi, \phi\rangle \quad = \quad -\overline{\langle X \wedge Y \cdot \phi, \phi\rangle}.$$

Identities (2.2) and (2.3) follow similarly. □

Note that

$$\eta^\phi_{kl} = (\delta_{kl} - 1)\eta^\phi_{lk}$$

and by (2.2), if $k \neq l$,

$$\eta^\phi_{kl}(X, Y) = \langle X \wedge Y \cdot f_k f_l \cdot \phi, \phi\rangle \,.$$

Lemma 2.1.2. [8] *Any spinor $\phi \in \Delta_r \otimes \Delta_n$, $r \geq 2$, defines two maps (extended by linearity)*

$$\begin{array}{ccc} \bigwedge^2 \mathbb{R}^r & \longrightarrow & \bigwedge^2 \mathbb{R}^n \\ f_{kl} & \mapsto & \eta^\phi_{kl} \end{array}$$

and

$$\begin{array}{ccc} \bigwedge^2 \mathbb{R}^r & \longrightarrow & \mathrm{End}(\mathbb{R}^n) \\ f_{kl} & \mapsto & \hat{\eta}^\phi_{kl}, \end{array}$$

□

2.1.2 Subgroups, Isomorphisms and Decompositions

In this section we will describe various inclusions of groups into (twisted) Spin groups.

Lemma 2.1.3. *There exists a monomorphism $h : Spin(2m) \times_{\mathbb{Z}_2} Spin(r) \to Spin(2m + r)$ such that the following diagram commutes*

$$\begin{array}{ccc} Spin(2m) \times_{\mathbb{Z}_2} Spin(r) & \xrightarrow{h} & Spin(2m+r) \\ \downarrow{\scriptstyle \lambda_n \times \lambda_r} & & \downarrow{\scriptstyle \lambda_{2m+r}} \\ SO(2m) \times SO(r) & \xhookrightarrow{i} & SO(2m+r) \end{array}$$

Proof. Consider the decomposition

$$\mathbb{R}^{2m+r} = \mathbb{R}^{2m} \oplus \mathbb{R}^r,$$

and let

$$Spin(2m) \quad = \quad \left\{ \prod_{i=1}^{2s} x_i \in Cl_{2m+r} \,|\, x_i \in \mathbb{R}^{2m}, |x_i| = 1, s \in \mathbb{N} \right\} \quad \subset \quad Spin(2m + r),$$

$$Spin(r) \quad = \quad \left\{ \prod_{j=1}^{2t} y_j \in Cl_{2m+r} \,|\, y_j \in \mathbb{R}^r, |y_j| = 1, t \in \mathbb{N} \right\} \quad \subset \quad Spin(2m+r).$$

It is easy to prove that

$$Spin(2m) \cap Spin(r) = \{1, -1\}.$$

Define the homomorphism

$$\begin{array}{rcl} h : Spin(2m) \times_{\mathbb{Z}_2} Spin(r) & \longrightarrow & Spin(2m+r) \\ [g, g'] & \mapsto & gg'. \end{array}$$

If $[g, g'] \in Spin(2m) \times_{\mathbb{Z}_2} Spin(r)$ is such that

$$gg' = 1 \in Spin(2m+r),$$

then

$$g' = g^{-1} \in Spin(2m) \subset Spin(2m+r),$$

so that

$$g, g' \in Spin(2m) \cap Spin(r) = \{1, -1\}.$$

Hence $[g, g'] = [1, 1]$ and h is injective. $\square$

Lemma 2.1.4. *Let $r \in \mathbb{N}$. There exists a monomorphism $h : U(m) \times SO(r) \hookrightarrow Spin^{c,r}(2m+r)$ such that the following diagram commutes*

$$\begin{tikzcd} & Spin^{c,r}(2m+r) \arrow[d] \\ U(m) \times SO(r) \arrow[r] \arrow[ur] & SO(2m+r) \times SO(r) \times U(1) \end{tikzcd}$$

Proof. Suppose we have an orthogonal complex structure on $\mathbb{R}^{2m} \subset \mathbb{R}^{2m+r}$

$$J : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^{2m}, \quad J^2 = -\mathrm{Id}_{2m}, \quad \langle \cdot, \cdot \rangle = \langle J\cdot, J\cdot \rangle.$$

The subgroup of $SO(2m+r)$ that respects both the orthogonal decomposition $\mathbb{R}^{2m+r} = \mathbb{R}^{2m} \oplus \mathbb{R}^r$ and J is

$$U(m) \times SO(r) \quad \subset \quad SO(2m) \times SO(r) \quad \subset \quad SO(2m+r).$$

There exists a lift [9]

$$\begin{tikzcd} & Spin^c(2m) \arrow[d] \\ U(m) \arrow[r] \arrow[ur] & SO(2m) \times U(1) \end{tikzcd}$$

where the bottom arrow is $A \mapsto (A_{\mathbb{R}}, \det_{\mathbb{C}}(A))$.

Recall the commutative diagram [8]

$$\begin{tikzcd}
& Spin(r) \times_{\mathbb{Z}_2} Spin(r) \arrow[d] \\
SO(r) \arrow[ur] \arrow[r, "\text{diagonal}"] & SO(r) \times SO(r).
\end{tikzcd}$$

We can put them together

$$\begin{tikzcd}
& Spin^c(2m) \times_{\mathbb{Z}_2} Spin^r(r) \arrow[d] \\
U(m) \times SO(r) \arrow[ur] \arrow[r] & SO(2m) \times U(1) \times SO(r) \times SO(r).
\end{tikzcd}$$

The nautural isomorphism

$$Spin^c(2m) \times_{\mathbb{Z}_2} Spin^r(r) \cong Spin^r(2m) \times_{\mathbb{Z}_2} Spin^c(r)$$

and the inclusion

$$Spin^r(2m) \times_{\mathbb{Z}_2} Spin^c(r) \hookrightarrow Spin(2m+r) \times_{\mathbb{Z}_2} Spin^c(r)$$

given by Lemma 2.1.3, gives the inclusion

$$Spin^c(2m) \times_{\mathbb{Z}_2} Spin^r(r) \hookrightarrow Spin(2m+r) \times_{\mathbb{Z}_2} Spin^c(r)$$

that completes the diagram

$$\begin{tikzcd}
& Spin^c(2m) \times_{\mathbb{Z}_2} Spin^r(r) \arrow[r, hook] \arrow[d] & Spin(2m+r) \times_{\mathbb{Z}_2} Spin^c(r) \arrow[d] \\
U(m) \times SO(r) \arrow[ur] \arrow[r] & SO(2m) \times U(1) \times SO(r) \times SO(r) \arrow[r] & SO(2m+r) \times SO(r) \times U(1).
\end{tikzcd}$$

□

Lemma 2.1.5. *Let $r \in \mathbb{N}$. The standard representation Δ_{2m+r} of $Spin(2m+r)$ decomposes as follows*

$$\Delta_{2m+r} = \Delta_r \otimes \Delta_{2m}^+ \ \oplus \ \Delta_r \otimes \Delta_{2m}^-,$$

with respect to the subgroup $Spin(2m) \times_{\mathbb{Z}_2} Spin(r) \subset Spin(2m+r)$.

Proof. Consider the restriction of the standard representation of $Spin(2m+r)$ to

$$Spin(2m) \times_{\mathbb{Z}_2} Spin(r) \subset Spin(2m+r) \longrightarrow Gl(\Delta_{2m+r}).$$

By using the explicit description of a unitary basis of Δ_{2m+r}, described in Section 1.1, we see that the elements of $Spin(2m)$ act on the last m factors of

$$\Delta_{2m+r} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{[r/2] \text{ times}} \otimes \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{m \text{ times}},$$

as they do on $\Delta_{2m} = \Delta_{2m}^+ \oplus \Delta_{2m}^-$. The elements of $Spin(r)$ act as usual on the first $[r/2]$ factors of Δ_r, act trivially on Δ_{2m}^+, and act by multiplication by (-1) on the factor Δ_{2m}^-. □

2.2 Twisted Partially Pure Spinors

In order to simplify the statements, we will consider the twisted spin representation

$$\Sigma_r \otimes \Delta_n \subseteq \Delta_r \otimes \Delta_n.$$

where

$$\Sigma_r = \begin{cases} \Delta_r & \text{if } r \text{ is odd,} \\ \Delta_r^+ & \text{if } r \text{ is even,} \end{cases}$$

$n, r \in \mathbb{N}$.

Now we define our special type of spinor.

Definition 2.2.1. *Let $(f_1, \ldots, f_r)$ be an orthonormal frame of $\mathbb{R}^r$. A unit-length spinor $\phi \in \Sigma_r \otimes \Delta_n$, $r < n$, is called a* twisted partially pure spinor *if*

- *there exists a $(n-r)$-dimensional subspace $V^\phi \subset \mathbb{R}^n$ such that for every $X \in V^\phi$, there exists a $Y \in V^\phi$ such that*
$$X \cdot \phi = i\, Y \cdot \phi.$$
- *it satisfies the equations*
$$\begin{aligned} (\eta_{kl}^\phi + f_k f_l) \cdot \phi &= 0, \\ \langle f_k f_l \cdot \phi, \phi \rangle &= 0, \end{aligned}$$
for all $1 \leq k < l \leq r$.
- *If $r = 4$, it also satisfies the condition*
$$\langle f_1 f_2 f_3 f_4 \cdot \phi, \phi \rangle = 0.$$

Remarks.

1. The requirement $|\phi| = 1$ is made in order to avoid renormalizations later on.
2. The extra condition for the case $r = 4$ is fulfilled for all other ranks.
3. From now on we will drop the adjective twisted since it will be clear from the context.

2.2.1 Example of Partially Pure Spinor

We will write an example of a partially pure spinor using real or quaternionic structures. A *real* structure on a complex vector space V is an $\mathbb{R}$-linear map $\alpha : V \to V$ satisfying $\alpha^2(v) = Id_V$ and $\alpha(iv) = -i\alpha(v)$. A *quaternionic* structure on a complex vector space V is an $\mathbb{R}$-linear map $\alpha : V \to V$ satisfying $\alpha^2(v) = -Id_V$ and $\alpha(iv) = -i\alpha(v)$.

The maps

$$\alpha \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -\overline{z}_2 \\ \overline{z}_1 \end{pmatrix}, \qquad \beta \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \overline{z}_1 \\ \overline{z}_2 \end{pmatrix},$$

define quaternionic and real structures, respectively, on $\mathbb{C}^2$. Using α and β, real or quaternionic structures γ_n are built on Δ_n, for $n \geq 2$, as follows

$$\begin{array}{rclll} \gamma_n & = & (\alpha\otimes\beta)^{\otimes 2k} & \text{if } n = 8k, 8k+1 & \text{(real)},\\ \gamma_n & = & \alpha\otimes(\beta\otimes\alpha)^{\otimes 2k} & \text{if } n = 8k+2, 8k+3 & \text{(quaternionic)},\\ \gamma_n & = & (\alpha\otimes\beta)^{\otimes 2k+1} & \text{if } n = 8k+4, 8k+5 & \text{(quaternionic)},\\ \gamma_n & = & \alpha\otimes(\beta\otimes\alpha)^{\otimes 2k+1} & \text{if } n = 8k+6, 8k+7 & \text{(real)}. \end{array}$$

Lemma 2.2.1. *Given $r, m \in \mathbb{N}$, there exists a partially pure spinor in $\Sigma_r \otimes \Delta_{2m+r}$.*

Proof. Let $(e_1, \ldots, e_{2m}, e_{2m+1}, \ldots, e_{2m+r})$ and $(f_1, \ldots, f_r)$ be orthonormal frames of $\mathbb{R}^{2m+r}$ and $\mathbb{R}^r$ respectively. Consider the decomposition of Lemma 2.1.5

$$\Delta_{2m+r} = \Delta_r \otimes \Delta_{2m}^+ \ \oplus\ \Delta_r \otimes \Delta_{2m}^-,$$

corresponding to the decomposition

$$\mathbb{R}^{2m+r} = \mathrm{span}\{e_1, \ldots, e_{2m}\} \oplus \mathrm{span}\{e_{2m+1}, \ldots, e_{2m+r}\}.$$

Let

$$\varphi_0 = u_{1,\ldots,1} \in \Delta_{2m}^+,$$

and

$$\{v_{\varepsilon_1,\ldots,\varepsilon_{[r/2]}} \,|\, (\varepsilon_1, \ldots, \varepsilon_{[r/2]}) \in \{\pm 1\}^{[r/2]}\}$$

be the unitary basis of the twisting factor $\Delta_r = \Delta(\mathrm{span}(f_1, \ldots, f_r))$ which contains Σ_r. Let us define the standard twisted partially pure spinor $\phi_0 \in \Sigma_r \otimes \Delta_r \otimes \Delta_{2m}^+$ by

$$\phi_0 = \begin{cases} \frac{1}{\sqrt{2^{[r/2]}}} \left(\sum_{I\in\{\pm1\}^{\times[r/2]}} v_I \otimes \gamma_r(u_I)\right) \otimes \varphi_0 & \text{if } r \text{ is odd},\\ \frac{1}{\sqrt{2^{[r/2]-1}}} \left(\sum_{I\in\left[\{\pm1\}^{\times[r/2]}\right]_+} v_I \otimes \gamma_r(u_I)\right) \otimes \varphi_0 & \text{if } r \text{ is even}, \end{cases} \tag{2.4}$$

where the elements of $\left[\{\pm1\}^{\times[r/2]}\right]_+$ contain an even number of (-1).

Checking the conditions in the definition of partially pure spinor for ϕ_0 is done by a direct calculation as in [8].

Example 2.2.1. *For instance, taking $n = 7$, $r = 3$, we have*

$$\phi_0 = \frac{1}{\sqrt{2}}(v_1 \otimes \gamma_3(u_1) \otimes u_1 \otimes u_1 + v_{-1} \otimes \gamma_3(u_{-1}) \otimes u_1 \otimes u_1)$$

where γ_3 is a quaternionic structure. We check that this ϕ_0 is a partially pure spinor. Remembering that $\gamma_3(u_\epsilon) = -i\epsilon u_{-\epsilon}$, we get

$$\phi_0 = i\frac{1}{\sqrt{2}}(v_{-1} \otimes u_1 \otimes u_1 \otimes u_1 - v_1 \otimes u_{-1} \otimes u_1 \otimes u_1),$$

which has unit length. Let $\{e_i\}$ be the standard basis of $\mathbb{R}^7$, so that

$$\begin{aligned} e_1 \cdot \phi_0 &= i\frac{1}{\sqrt{2}}(v_{-1} \otimes u_1 \otimes u_1 \otimes g_1(u_1) - v_1 \otimes u_{-1} \otimes u_1 \otimes g_1(u_1))\\ &= ie_2 \cdot \phi_0, \end{aligned}$$

and, similarly,

$$e_3 \cdot \phi_0 = ie_4 \cdot \phi_0.$$

So, ϕ_0 induces the standard complex structure on $V^{\phi_0} = \langle e_1, e_2, e_3, e_4 \rangle$. Let $\{f_i\}$ be the standard basis of $\mathbb{R}^3$. Similar calculations give

$$\eta_{kl}^{\phi_0} = e_{4+k} \wedge e_{4+l},$$

$$(\eta_{kl}^{\phi_0} + f_{kl}) \cdot \phi_0 = 0,$$

and

$$\langle f_{kl} \cdot \phi_0, \phi_0 \rangle = 0.$$

□

2.2.2 Properties of Partially Pure Spinors

Lemma 2.2.2. *The definition of partially pure spinor does not depend on the choice of orthonormal basis of $\mathbb{R}^r$.*

Proof. If $r = 0, 1$, a partially pure spinor is a classical pure spinor for n even or the straightforward generalization of pure spinor for n odd [16, p. 336]. Suppose $(f_1', \ldots, f_r')$ is another orthonormal frame of $\mathbb{R}^r$, then

$$f_i' = \alpha_{i1} f_1 + \cdots + \alpha_{ir} f_r,$$

so that the matrix $A = (\alpha_{ij}) \in SO(r)$. Let us denote

$$\eta_{kl}'^{\phi}(X, Y) := \mathrm{Re}\left\langle X \wedge Y \cdot f_k' f_l' \cdot \phi, \phi \right\rangle$$

Thus,

$$\begin{aligned}
\eta_{kl}'^{\phi} \cdot \phi &= \sum_{1 \le a < b \le n} \eta_{kl}'^{\phi}(e_a, e_b) e_a e_b \cdot \phi \\
&= \sum_{1 \le a < b \le n} \sum_{s=1}^{r} \sum_{t=1}^{r} \alpha_{ks} \alpha_{lt} \mathrm{Re} \left\langle e_a e_b \cdot f_s f_t \cdot \phi, \phi \right\rangle e_a e_b \cdot \phi \\
&= \sum_{1 \le a < b \le n} \sum_{s=1}^{r} \sum_{t=1}^{r} \alpha_{ks} \alpha_{lt} \eta_{st}^{\phi}(e_a, e_b) e_a e_b \cdot \phi \\
&= \sum_{s=1}^{r} \sum_{t=1}^{r} \alpha_{ks} \alpha_{lt} \eta_{st}^{\phi} \cdot \phi \\
&= -\sum_{s=1}^{r} \sum_{t=1}^{r} \alpha_{ks} \alpha_{lt} f_s f_t \cdot \phi \\
&= -\left(\sum_{s=1}^{r} \alpha_{ks} f_s \right) \left(\sum_{t=1}^{r} \alpha_{lt} f_t \right) \cdot \phi \\
&= -f_k' f_l' \cdot \phi.
\end{aligned}$$

For the third part of the definition, note that

$$\begin{aligned}\left\langle f'_k f'_l \cdot \phi, \phi \right\rangle &= \left\langle \left(\sum_{s=1}^{r} \alpha_{ks} f_s\right) \left(\sum_{t=1}^{r} \alpha_{lt} f_t\right) \cdot \phi, \phi \right\rangle \\ &= \sum_{s=1}^{r}\sum_{t=1}^{r} \alpha_{ks}\alpha_{lt} \left\langle f_s f_t \cdot \phi, \phi \right\rangle \\ &= 0.\end{aligned}$$

For $r = 4$, the volume form is invariant under $SO(4)$, $f'_1 f'_2 f'_3 f'_4 = f_1 f_2 f_3 f_4$, and

$$\left\langle f'_1 f'_2 f'_3 f'_4 \cdot \phi, \phi \right\rangle = \left\langle f_1 f_2 f_3 f_4 \cdot \phi, \phi \right\rangle = 0.$$

□

Lemma 2.2.3. *Given a partially pure spinor $\phi \in \Sigma_r \otimes \Delta_n$, there exists an orthogonal complex structure on V^ϕ and $n - r \equiv 0 \pmod 2$.*

Proof. By definition, for every $X \in V^\phi$, there exists $Y \in V^\phi$ such that

$$X \cdot \phi = iY \cdot \phi,$$

and

$$Y \cdot \phi = i(-X) \cdot \phi.$$

If we set

$$J^\phi(X) := Y,$$

we get a linear transformation $J^\phi : V^\phi \to V^\phi$, such that $(J^\phi)^2 = -\mathrm{Id}_{V^\phi}$, i.e. J^ϕ is a complex structure on the vector space V^ϕ and $\dim_{\mathbb{R}}(V^\phi)$ is even. Furthermore, this complex structure is orthogonal. Indeed, for every $X \in V^\phi$,

$$\begin{aligned} X \cdot JX \cdot \phi &= -i|X|^2\phi, \\ JX \cdot X \cdot \phi &= i|JX|^2\phi, \end{aligned}$$

and

$$(-2\left\langle X, JX\right\rangle + i(|JX|^2 - |X|^2))\,\phi = 0,$$

i.e.

$$\begin{aligned} \left\langle X, JX \right\rangle &= 0 \\ |X| &= |JX|. \end{aligned}$$

□

Lemma 2.2.4. *Let $r \geq 2$ and $\phi \in \Sigma_r \otimes \Delta_n$ be a partially pure spinor. The forms η^ϕ_{kl} are non-zero, $1 \leq k < l \leq r$.*

Proof. Since $(f_k f_l)^2 = -1$, the equation

$$\eta^\phi_{kl} \cdot \phi = -f_k f_l \cdot \phi \tag{2.5}$$

implies

$$\eta^{\phi}_{kl} \cdot f_k f_l \cdot \phi = \phi. \tag{2.6}$$

By taking an orthonormal frame $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$ we can write

$$\eta^{\phi}_{kl} = \sum_{1 \le i < j \le n} \eta^{\phi}_{kl}(e_i, e_j) e_i e_j.$$

By (2.6), and taking hermitian product with ϕ

$$\begin{aligned}
1 &= |\phi|^2 \\
&= \left\langle \eta^{\phi}_{kl} \cdot f_k f_l \cdot \phi, \phi \right\rangle \\
&= \left\langle \sum_{1 \le i < j \le n} \eta^{\phi}_{kl}(e_i, e_j) e_i e_j \cdot f_k f_l \cdot \phi, \phi \right\rangle \\
&= \sum_{1 \le i < j \le n} \eta^{\phi}_{kl}(e_i, e_j) \left\langle e_i e_j \cdot f_k f_l \cdot \phi, \phi \right\rangle \\
&= \sum_{1 \le i < j \le n} \eta^{\phi}_{kl}(e_i, e_j)^2.
\end{aligned}$$

□

Lemma 2.2.5. *Let $r \ge 2$. The image of the map associated to a partially pure spinor $\phi \in \Sigma_r \otimes \Delta_n$,*

$$\begin{aligned}
\textstyle\bigwedge^2 \mathbb{R}^r &\longrightarrow \mathrm{End}(\mathbb{R}^n) \\
f_{kl} &\mapsto \hat{\eta}^{\phi}_{kl},
\end{aligned}$$

forms a Lie algebra of endomorphisms isomorphic to $\mathfrak{so}(r)$.

Proof. Let $(e_1, \ldots, e_n)$ be an orthonormal frame of $\mathbb{R}^n$. First, let us consider the following calculation for $i \ne j$, $k \ne l$, $s \ne t$:

$$\begin{aligned}
\mathrm{Re}\left\langle e_s e_t \cdot \eta^{\phi}_{ij} \cdot f_k f_l \cdot \phi, \phi \right\rangle &= \mathrm{Re}\left\langle e_s e_t \cdot \left(\sum_{a<b} \eta^{\phi}_{ij}(e_a, e_b) e_a e_b \right) \cdot f_k f_l \cdot \phi, \phi \right\rangle \\
&= -\sum_b \eta^{\phi}_{ij}(e_s, e_b) \eta^{\phi}_{kl}(e_b, e_t) + \sum_b \eta^{\phi}_{kl}(e_s, e_b) \eta^{\phi}_{ij}(e_b, e_t) \\
&= -\sum_b [\hat{\eta}^{\phi}_{kl}]_{tb} [\hat{\eta}^{\phi}_{ij}]_{bs} + \sum_b [\hat{\eta}^{\phi}_{ij}]_{tb} [\hat{\eta}^{\phi}_{kl}]_{bs} \\
&= [\hat{\eta}^{\phi}_{ij}, \hat{\eta}^{\phi}_{kl}]_{ts}
\end{aligned} \tag{2.7}$$

is the entry in row t and column s of the matrix $[\hat{\eta}^{\phi}_{ij}, \hat{\eta}^{\phi}_{kl}]$.

Secondly, we prove that the endomorphisms $\hat{\eta}^{\phi}_{kl}$ satisfy the commutation relations of $\mathfrak{so}(r)$:

1. If $1 \le i, j, k, l \le r$ are all different,

$$[\hat{\eta}^{\phi}_{kl}, \hat{\eta}^{\phi}_{ij}] = 0. \tag{2.8}$$

2. If $1 \leq i, j, k \leq r$ are all different,

$$[\hat{\eta}^\phi_{ij}, \hat{\eta}^\phi_{jk}] = -\hat{\eta}^\phi_{ik}. \tag{2.9}$$

To prove (2.8), use (2.5) to obtain

$$\eta^\phi_{ij} \cdot f_k f_l \cdot \phi \;=\; \eta^\phi_{kl} \cdot f_i f_j \cdot \phi. \tag{2.10}$$

Now, using (2.7) we get

$$\begin{aligned} \mathrm{Re}\left\langle e_s e_t \cdot \eta^\phi_{ij} \cdot f_k f_l \cdot \phi, \phi \right\rangle &= [\hat{\eta}^\phi_{ij}, \hat{\eta}^\phi_{kl}]_{ts}, \\ \mathrm{Re}\left\langle e_s e_t \cdot \eta^\phi_{kl} \cdot f_i f_j \cdot \phi, \phi \right\rangle &= [\hat{\eta}^\phi_{kl}, \hat{\eta}^\phi_{ij}]_{ts}, \end{aligned}$$

and by (2.10) and the anticommutativity of the bracket,

$$[\hat{\eta}^\phi_{ij}, \hat{\eta}^\phi_{kl}] = 0.$$

To prove (2.9), note that by (2.5)

$$f_i f_j \cdot \eta^\phi_{jk} \cdot \phi \;=\; f_i f_k \cdot \phi$$

and

$$f_j f_k \cdot \eta^\phi_{ij} \cdot \phi \;=\; -f_i f_k \cdot \phi$$

so that

$$f_j f_k \cdot \eta^\phi_{ij} \cdot \phi \;=\; f_i f_j \cdot \eta^\phi_{jk} \cdot \phi - 2 f_i f_k \cdot \phi.$$

Thus,

$$\mathrm{Re}\left\langle e_s e_t \cdot \eta^\phi_{ij} \cdot f_j f_k \cdot \phi, \phi \right\rangle \;=\; \mathrm{Re}\left\langle e_s e_t \cdot \eta^\phi_{jk} \cdot f_i f_j \cdot \phi, \phi \right\rangle - 2\eta^\phi_{ik}(e_s, e_t)$$

and by (2.7)

$$[\hat{\eta}^\phi_{ij}, \hat{\eta}^\phi_{jk}] \;=\; [\hat{\eta}^\phi_{jk}, \hat{\eta}^\phi_{ij}] - 2\hat{\eta}^\phi_{ik},$$

i.e.

$$[\hat{\eta}^\phi_{ij}, \hat{\eta}^\phi_{jk}] \;=\; -\hat{\eta}^\phi_{ik}.$$

Thirdly, we will prove, in five separate cases, that the set of endomorphisms $\{\hat{\eta}^\phi_{kl}\}$ is linearly independent. For $r = 0, 1$ there are no endomorphisms. For $r = 2$ it is obvious since there is only one non-zero endomorphism. For $r = 3$, suppose

$$0 \;=\; \alpha_{12}\hat{\eta}^\phi_{12} + \alpha_{13}\hat{\eta}^\phi_{13} + \alpha_{23}\hat{\eta}^\phi_{23},$$

where $\alpha_{12} \neq 0$. Take the Lie bracket with $\hat{\eta}^\phi_{13}$ to get

$$0 \;=\; \alpha_{12}\hat{\eta}^\phi_{23} - \alpha_{23}\hat{\eta}^\phi_{12},$$

i.e.

$$\hat{\eta}^{\phi}_{23} = \frac{\alpha_{23}}{\alpha_{12}}\hat{\eta}^{\phi}_{12}.$$

We can also consider the bracket with $\hat{\eta}^{\phi}_{23}$,

$$0 = -\alpha_{12}\hat{\eta}^{\phi}_{13} + \alpha_{13}\hat{\eta}^{\phi}_{12},$$

so that

$$\hat{\eta}^{\phi}_{13} = \frac{\alpha_{13}}{\alpha_{12}}\hat{\eta}^{\phi}_{12}.$$

By substituting in the original equation we get

$$0 = (\alpha_{12}^2 + \alpha_{13}^2 + \alpha_{23}^2)\hat{\eta}^{\phi}_{12},$$

which gives a contradiction.

Now suppose $r \geq 5$ and that there is a linear combination

$$0 = \sum_{k<l} \alpha_{kl}\hat{\eta}^{\phi}_{kl}.$$

Taking succesive brackets with $\hat{\eta}^{\phi}_{13}$, $\hat{\eta}^{\phi}_{12}$, $\hat{\eta}^{\phi}_{34}$ and $\hat{\eta}^{\phi}_{45}$ we get the identity

$$\alpha_{12}\hat{\eta}^{\phi}_{15} = 0,$$

i.e. $\alpha_{12} = 0$. Similar arguments give the vanishing of every α_{kl}.

For $r = 4$, suppose there is a linear combination

$$0 = \alpha_{12}\eta^{\phi}_{12} + \alpha_{13}\eta^{\phi}_{13} + \alpha_{14}\eta^{\phi}_{14} + \alpha_{23}\eta^{\phi}_{23} + \alpha_{24}\eta^{\phi}_{24} + \alpha_{34}\eta^{\phi}_{34}.$$

Multiply by $-\phi$

$$0 = (\alpha_{12}f_{12} + \alpha_{13}f_{13} + \alpha_{14}f_{14} + \alpha_{23}f_{23} + \alpha_{24}f_{24} + \alpha_{34}f_{34}) \cdot \phi.$$

Multiply by $-f_{12}$

$$0 = (\alpha_{12} - \alpha_{13}f_{23} - \alpha_{14}f_{24} + \alpha_{23}f_{13} + \alpha_{24}f_{14} - \alpha_{34}f_{1234}) \cdot \phi.$$

Now, take hermitian product with ϕ

$$\begin{aligned} 0 &= \langle(\alpha_{12} - \alpha_{13}f_{23} - \alpha_{14}f_{24} + \alpha_{23}f_{13} + \alpha_{24}f_{14} - \alpha_{34}f_{1234}) \cdot \phi, \phi\rangle \\ &= \alpha_{12}|\phi|^2 - \alpha_{34}\langle f_{1234} \cdot \phi, \phi\rangle \\ &= \alpha_{12}. \end{aligned}$$

Similar arguments give the vanishing of the other coefficients. □

Lemma 2.2.6. *Let $r \geq 2$ and $\phi \in \Sigma_r \otimes \Delta_n$ be a partially pure spinor. Then, $u \in (V^{\phi})^{\perp}$ if and only if for every $X \in V^{\phi}$*

$$X \cdot u \cdot \phi = i\,J(X) \cdot u \cdot \phi.$$

Proof. If $u \in (V^\phi)^\perp$ then

$$\begin{aligned} X \cdot u \cdot \phi &= -u \cdot X \cdot \phi \\ &= -iu \cdot J(X) \cdot \phi \\ &= iJ(X) \cdot u \cdot \phi, \end{aligned}$$

since X and u are orthogonal to each other.

Conversely, assume that

$$X \cdot u \cdot \phi = iJX \cdot u \cdot \phi$$

for every $X \in V^\phi$. But

$$X \cdot \phi = iJX \cdot \phi,$$

implies

$$u \cdot X \cdot \phi = iu \cdot JX \cdot \phi,$$

so that

$$(-2\langle u, X\rangle + 2i\langle u, JX\rangle)\phi = 0.$$

□

Lemma 2.2.7. *Let $r \geq 2$ and $\phi \in \Sigma_r \otimes \Delta_n$ be a partially pure spinor. Then*

$$V^\phi \subseteq \bigcap_{1\leq k<l\leq r} \ker \hat{\eta}^\phi_{kl}.$$

Proof. Let $1 \leq k < l \leq r$ be fixed and $X \in V^\phi$. Since $\mathbb{R}^n = V^\phi \oplus (V^\phi)^\perp$ and J^ϕ is a complex structure on V^ϕ, there exists a basis $\{e_1, e_2, \ldots, e_{2m-1}, e_{2m}\} \cup \{e_{2m+1}, \ldots, e_{2m+r}\}$ such that

$$\begin{aligned} V^\phi &= \mathrm{span}(e_1, e_2, \ldots, e_{2m-1}, e_{2m}), \\ (V^\phi)^\perp &= \mathrm{span}(e_{2m+1}, \ldots, e_{2m+r}), \\ J^\phi(e_{2j-1}) &= e_{2j}, \\ J^\phi(e_{2j}) &= -e_{2j-1}, \end{aligned}$$

where $m = (n-r)/2$ and $1 \leq j \leq m$. Note that

$$\begin{aligned} \hat{\eta}^\phi_{kl}(e_{2j-1}) &= \sum_{a=1}^n \mathrm{Re}\,\langle e_{2j-1} \wedge e_a \cdot f_{kl} \cdot \phi, \phi\rangle\, e_a \\ &= -\sum_{a\neq 2j-1}^n \mathrm{Re}\,\langle f_{kl} \cdot e_a e_{2j-1} \cdot \phi, \phi\rangle\, e_a \\ &= -\sum_{a\neq 2j-1}^n \mathrm{Re}\left\langle f_{kl} \cdot e_a(iJ^\phi(e_{2j-1})) \cdot \phi, \phi\right\rangle e_a \\ &= \sum_{a\neq 2j-1}^n \mathrm{Im}\,\langle e_a e_{2j} \cdot f_{kl} \cdot \phi, \phi\rangle\, e_a \\ &= -\mathrm{Im}\,\langle f_{kl} \cdot \phi, \phi\rangle\, e_{2j} \\ &= 0. \end{aligned}$$

□

Lemma 2.2.8. *Let $r \geq 2$ and $\phi \in \Sigma_r \otimes \Delta_n$ be a partially pure spinor. Then $(V^\phi)^\perp$ carries a standard representation of $\mathfrak{so}(r)$, and an orientation.*

Proof. By Lemma 2.2.5, $\mathfrak{so}(r)$ is represented non-trivially on $\mathbb{R}^n = V^\phi \oplus (V^\phi)^\perp$ and, by Lemma 2.2.7, it acts trivially on V^ϕ. Thus $(V^\phi)^\perp$ is a nontrivial representation of $\mathfrak{so}(r)$ of dimension r. □

Remark. The existence of a partially pure spinor implies $r \equiv n \pmod 2$. In this case, let $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_r)$ be orthonormal frames for $\mathbb{R}^n$ and $\mathbb{R}^r$ respectively,

$$\mathrm{vol}_n = e_1 \cdots e_n, \qquad \mathrm{vol}_r = f_1 \cdots f_r,$$

and

$$\begin{array}{rcl} F : \Sigma_r \otimes \Delta_n & \longrightarrow & \Sigma_r \otimes \Delta_n \\ \phi & \mapsto & (-i)^{n/2} i^{r/2} \mathrm{vol}_n \cdot \mathrm{vol}_r \cdot \phi. \end{array}$$

Note that $i^{r/2}\mathrm{vol}_r$ acts as $(-1)^{r/2}\mathrm{Id}_{\Sigma_r}$ on Σ_r and that $(-i)^{n/2}\mathrm{vol}_n$ determines the decomposition $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$. Thus we have that

$$\Sigma_r \otimes \Delta_n = (\Sigma_r \otimes \Delta_n)^+ \oplus (\Sigma_r \otimes \Delta_n)^-,$$

and we will call elements in $(\Sigma_r \otimes \Delta_n)^+$ and $(\Sigma_r \otimes \Delta_n)^-$ positive and negative twisted spinors respectively.

Definition 2.2.2. *Let n be even, $\mathbb{R}^n$ be endowed with the standard inner product and orientation, and vol_n denote the volume form. Let V, W be two orthogonal oriented subspaces such that $\mathbb{R}^n = V \oplus W$. Furthermore, assume V admits a complex structure inducing the given orientation on V. The oriented triple (V, J, W) will be called* positive *if given (oriented) orthonormal frames $(v_1, J(v_1), \ldots, v_m, J(v_m))$ and $(w_1, \ldots, w_r)$ of V and W respectively,*

$$v_1 \wedge J(v_1) \wedge \ldots \wedge v_m \wedge J(v_m) \wedge w_1 \wedge \ldots \wedge w_r = \mathrm{vol}_n,$$

and negative *if*

$$v_1 \wedge J(v_1) \wedge \ldots \wedge v_m \wedge J(v_m) \wedge w_1 \wedge \ldots \wedge w_r = -\mathrm{vol}_n.$$

Lemma 2.2.9. *If r is even, a partially pure spinor ϕ is either positive or negative. Furthermore, a partialy pure spinor ϕ is positive (resp. negative) if and only if the corresponding oriented triple $(V^\phi, J^\phi, (V^\phi)^\perp)$ is positive (resp. negative).*

Proof. We must prove that either $\phi \in (\Sigma_r \otimes \Delta_n)^+$ or $\phi \in (\Sigma_r \otimes \Delta_n)^-$. Since ϕ is a partially pure spinor, there exist frames $(e'_1, \ldots, e'_{2m})$ and $(e'_{2m+1}, \ldots, e'_{2m+r})$ of V^ϕ and $(V^\phi)^\perp$ respectively such that

$$e'_{2j} = J(e'_{2j-1}) \quad \text{and} \quad \eta^\phi_{kl} = e'_{2m+k} \wedge e'_{2m+l},$$

where $1 \leq j \leq m$ and $1 \leq k < l \leq r$. Now, if

$$e'_1 \wedge e'_2 \wedge \ldots \wedge e'_{2m} \wedge e'_{2m+1} \wedge \ldots \wedge e'_{2m+r} = \pm\mathrm{vol}_n,$$

then

$$(-i)^{n/2} i^{r/2} \mathrm{vol}_n \cdot \mathrm{vol}_r \cdot \phi = \pm(-i)^{n/2} i^{r/2} e'_1 e'_2 \cdots e'_{2m} e'_{2m+1} \cdots e'_{2m+r} \cdot f_1 \cdots f_r \cdot \phi.$$

Using (2.6), the last terms simplify to

$$\begin{aligned} e'_{2m+1}\cdots e'_{2m+r}\cdot f_1\cdots f_r\cdot\phi &= \eta^{\phi}_{12}\cdots\eta^{\phi}_{r-1,r}f_1\cdots f_r\phi \\ &= \eta^{\phi}_{12}f_{1,2}\cdots\eta_{r-1,r}f_{r-1,r}\cdot\phi \\ &= \phi. \end{aligned}$$

In the terms left

$$\begin{aligned} e'_1e'_2\cdots e'_{2m-1}e'_{2m}\phi &= e'_1e'_2\cdots e'_{2m-1}(-ie'_{2m-1})\phi \\ &= ie'_1e'_2\cdots e'_{2m-3}e'_{2m-2}\phi \\ &= i^m\phi. \end{aligned}$$

Thus

$$(-i)^{n/2}i^{r/2}\mathrm{vol}_n\cdot\mathrm{vol}_r\cdot\phi = \pm(-i)^{n/2}i^{r/2}i^m\phi = \pm\phi$$

i.e. $\phi\in(\Sigma_r\otimes\Delta_n)^{\pm}$. □

2.2.3 Orbit of a Partially Pure Spinor

Lemma 2.2.10. *Let $\phi\in\Sigma_r\otimes\Delta_n$ be a partially pure spinor. If $g\in Spin^{c,r}(n)$, then $g(\phi)$ is also a partially pure spinor.*

Proof. Let $g\in Spin^{c,r}(n)$ and $\lambda^{c,r}_n(g)=(g_1,g_2,g_3)\in SO(n)\times SO(r)\times U(1)$.
First, suppose $X,Y\in V^{\phi}$,

$$X\cdot\phi = i\,Y\cdot\phi.$$

Applying g on both sides and using equivariance of Clifford multiplication

$$g_1(X)\cdot g(\phi) = i\,g_1(Y)\cdot g(\phi). \tag{2.11}$$

This means that $g_1(V^{\phi})\subset V^{g(\phi)}$, and g_1 maps V^{ϕ} into $V^{g(\phi)}$ injectively. On the other hand, any pair of vectors $\tilde{X},\tilde{Y}\in V^{g(\phi)}$ such that

$$\tilde{X}\cdot g(\phi) = i\,\tilde{Y}\cdot g(\phi),$$

are the image under g_1 of some vectors $X,Y\in\mathbb{R}^n$, i.e.

$$g_1(X)\cdot g(\phi) = i\,g_1(Y)\cdot g(\phi).$$

Apply g^{-1} on both sides to get

$$X\cdot\phi = i\,Y\cdot\phi,$$

so that $X,Y\in V^{\phi}$, i.e. $V^{g(\phi)}=g_1(V^{\phi})$.

Equation (2.11) implies that for every $X\in V^{\phi}$

$$J^{g_1(\phi)}(g_1X) = g_1(JX)$$

i.e.

$$J^{g(\phi)} = g_1|_{V^{\phi}}\circ J^{\phi}\circ(g_1|_{V^{\phi}})^{-1}. \tag{2.12}$$

Now, let $e'_a = g_1^{-1}(e_a)$ and $f'_k = g_2^{-1}(f_k)$, so that

$$\begin{aligned}
\eta_{kl}^{g(\phi)} \cdot g(\phi) &= \sum_{1\leq a<b\leq n} \eta_{kl}^{g(\phi)}(e_a,e_b) e_a e_b \cdot g(\phi) \\
&= \sum_{1\leq a<b\leq n} \left\langle g_1(e'_a)g_1(e'_b)\cdot g_2(f'_k)g_2(f'_l)\cdot g(\phi), g(\phi)\right\rangle g_1(e'_a)g_1(e'_b)\cdot g(\phi) \\
&= \sum_{1\leq a<b\leq n} \left\langle e'_a e'_b \cdot f'_k f'_l \cdot \phi, \phi\right\rangle g(e'_a e'_b \cdot \phi) \\
&= g\left(\sum_{1\leq a<b\leq n} \eta_{kl}^{\prime\phi}(e'_a,e'_b) e'_a e'_b \cdot \phi\right) \\
&= g\left(\eta_{kl}^{\prime\phi}\cdot\phi\right) \\
&= g\left(-f'_k f'_l \cdot \phi\right) \\
&= -f_k f_l \cdot g(\phi),
\end{aligned}$$

and

$$\begin{aligned}
\langle f_k f_l \cdot g(\phi), g(\phi)\rangle &= \left\langle g(f'_k f'_l \cdot \phi), g(\phi)\right\rangle \\
&= \left\langle f'_k f'_l \cdot \phi, \phi\right\rangle \\
&= 0.
\end{aligned}$$

For $r=4$, note that the volume form is invariant under $SO(4)$

$$\langle f_1 f_2 f_3 f_4 \cdot g(\phi), g(\phi)\rangle = \langle f_1 f_2 f_3 f_4 \cdot \phi, \phi\rangle = 0.$$

□

Lemma 2.2.11. *Let $\phi \in \Sigma_r \otimes \Delta_n$ be a partially pure spinor. The stabilizer of ϕ is isomorphic to $U(m)\times SO(r)$.*

Proof. Let $g \in Spin^{c,r}(n)$ be such that $g(\phi)=\phi$ and

$$\lambda_n^{c,r}(g) = (g_1,g_2,g_3) \in SO(n)\times SO(r)\times U(1).$$

First, let us see that $g_1 = \lambda_1(g) \in SO(n)$ preserves V^ϕ and $(V^\phi)^\perp$.

If $X\in V^\phi$ then from $X\cdot\phi = i\,J(X)\cdot\phi$ we obtain

$$X\cdot g(\phi) = i\,J(X)\cdot g(\phi),$$

which implies $V^\phi \subset V^{g(\phi)}$. Applying g to the equation $X\cdot\phi = i\,J(X)\cdot\phi$ we have

$$g_1(X)\cdot\phi = i\,g_1(J(X))\cdot\phi,$$

thus $g_1(V^\phi) = V^{g(\phi)} \subset V^\phi$ and in consequence $g_1(V^\phi) = V^{g(\phi)} = V^\phi$.

This shows that $g_1\in SO(n)$ preserves V^ϕ, the same happens with g_1^{-1} (which can be proved changing g by g^{-1} in the previous lines). Moreover, from (2.12) we have

$$g_1(J(X)) = J(g_1(X)),$$

so $g_1|_{V^\phi} \in U(V^\phi, J^\phi) \cong U(m)$.

Now, by Lemma 2.2.6, $u \in (V^\phi)^\perp$ if and only if

$$X \cdot u \cdot \phi = i J^\phi(X) \cdot u \cdot \phi, \qquad X \in V^\phi,$$

thus

$$X \cdot g_1(u) \cdot \phi = g(g_1^{-1}(X) \cdot u \cdot \phi) = g(i J(g_1^{-1}(X)) \cdot u \cdot \phi) = i J(X) \cdot g_1(u) \cdot \phi.$$

Hence g_1 preserves $(V^\phi)^\perp$ also. The subgroup of $SO(n)$ that preserves the splitting $\mathbb{R}^n = V^\phi \oplus (V^\phi)^\perp$ and the orthogonal complex structure on V^ϕ is $U(m) \times SO(r)$, i.e.

$$g_1 \in U(V^\phi, J^\phi) \times SO((V^\phi)^\perp) \cong U(m) \times SO(r).$$

Now, let $g_1 = (h_1, h_2) \in U(m) \times SO(r)$. We will show that $g_2 = h_2$. First, as in Lemma 2.2.7, one can prove $\eta^\phi_{kl} \in \bigwedge^2 (V^\phi)^\perp$. Form this we have $g_1(\eta^\phi_{kl}) = h_2(\eta^\phi_{kl})$.

Applying g to $(\eta^\phi_{kl} + f_k f_l) \cdot \phi = 0$ and using equivariance of Clifford multiplication we get

$$0 = (g_1(\eta^\phi_{kl}) + g_2(f_k f_l)) \cdot g(\phi) = (h_2(\eta^\phi_{kl}) + g_2(f_k f_l)) \cdot \phi = h_2(\eta^\phi_{kl}) \cdot \phi - g_2(\eta^\phi_{kl}) \cdot \phi.$$

But the action of $SO(r)$ on $\bigwedge^2 (V^\phi)^\perp$, the adjoint action, is faithful. Thus $h_2 = g_2$.

Finally we will prove $g_3 = \det_{\mathbb{C}}(h_1)$. Since h_1 is unitary with respect to J, there is a frame $(e_1, \ldots, e_{2m})$ of V^ϕ such that

$$e_{2j} = J(e_{2j-1})$$

and h_1 is diagonal with respect to the unitary basis $\{e_{2j-1} - ie_{2j} \,|\, j = 1, \ldots, m\}$, i.e.

$$h_1(e_{2j-1} - ie_{2j}) = e^{i\theta_j}(e_{2j-1} - ie_{2j})$$

where $0 \leq \theta_j < 2\pi$. On the other hand, there is a frame $(f_1, \ldots, f_r)$ of $\mathbb{R}^r$ such that

$$g_2 = R_{\varphi_1} \circ \cdots \circ R_{\varphi_{[r/2]}}$$

where R_{φ_k} is a rotation by an angle φ_k on the plane generated by f_{2k-1} and f_{2k}, $1 \leq k \leq [r/2]$. Now, since the endomorphisms $\hat{\eta}^\phi_{kl}$ span an isomorphic copy of $\mathfrak{so}(r)$, there is a frame $(e_{2m+1}, \ldots, e_{2m+r})$ of $(V^\phi)^\perp$ such that

$$\eta^\phi_{kl} = e_{2m+k} \wedge e_{2m+l},$$

$1 \leq k < l \leq r$. Since the adjoint representation of $SO(r)$ is faithful

$$h_2 = R'_{\varphi_1} \circ \cdots \circ R'_{\varphi_{[r/2]}}$$

where R'_{φ_k} is a rotation by an angle φ_k on the plane generated by $e_{2m+2k-1}$ and e_{2m+2k}, $1 \leq k \leq [r/2]$. Thus,

$$g = \pm[\tilde{g_1}, \tilde{h}, e^{i\theta/2}]$$

where

$$\tilde{g_1} \;=\; \prod_{j=1}^{m} (\cos(\theta_j/2) - \sin(\theta_j/2) e_{2j-1} e_{2j}) \cdot \prod_{k=1}^{[r/2]} (\cos(\varphi_k/2) - \sin(\varphi_k/2) \eta^\phi_{2k-1,2k}),$$

$$\tilde{h} = \prod_{k=1}^{[r/2]} (\cos(\varphi_k/2) - \sin(\varphi_k/2) f_{2k-1} f_{2k}).$$

Let us compute $g(\phi)$. First, note that for every k

$$\begin{aligned} & (\cos(\varphi_k/2) - \sin(\varphi_k/2) f_{2k-1} f_{2k}) \cdot (\cos(\varphi_k/2) - \sin(\varphi_k/2) \eta^{\phi}_{2k-1,2k}) \cdot \phi \\ = \; & (\cos(\varphi_k/2) - \sin(\varphi_k/2) f_{2k-1} f_{2k}) \cdot (\cos(\varphi_k/2) + \sin(\varphi_k/2) f_{2k-1} f_{2k}) \cdot \phi \\ = \; & \phi. \end{aligned}$$

Thus

$$\begin{aligned} \phi &= g(\phi) \\ &= \pm e^{i\theta/2} \prod_{j=1}^{m} (\cos(\theta_j/2) - \sin(\theta_j/2) e_{2j-1} e_{2j}) \cdot \phi \\ &= \pm e^{i\theta/2} \prod_{j=1}^{m} (\cos(\theta_j/2) + i \sin(\theta_j/2) e_{2j-1} e_{2j-1}) \phi \\ &= \pm e^{i\theta/2} \prod_{j=1}^{m} (\cos(\theta_j/2) - i \sin(\theta_j/2)) \phi \\ &= \pm e^{i\theta/2} \prod_{j=1}^{m} e^{-i\theta_j/2} \phi \\ &= \pm e^{\frac{i}{2}(\theta - \sum_{j=1}^{m} \theta_j)} \phi. \end{aligned}$$

This means

$$e^{\frac{i}{2}(\theta - \sum_{j=1}^{m} \theta_j)} = \pm 1$$

i.e.

$$\sum_{j=1}^{m} \theta_j \equiv \theta \pmod{2\pi}.$$

Hence

$$\det_{\mathbb{C}}(h_1) = e^{i \sum_{j=1}^{m} \theta_j} = e^{i\theta} = g_3.$$

We conclude

$$\lambda_n^{c,r}(g) = ((h_1, h_2), h_2, \det_{\mathbb{C}}(h_1)),$$

which is in the image of the horizontal row in the diagram of Lemma 2.1.4

$$\begin{tikzcd} & Spin^{c,r}(2m+r) \arrow[d] \\ U(m) \times SO(r) \arrow[ur] \arrow[r] & SO(2m+r) \times SO(r) \times U(1) \end{tikzcd}$$

$\square$

Remark. Note that for any spinor $\phi \in \Sigma_r \otimes \Delta_n$, $g \in Spin^{c,r}(n)$, $\lambda_n^{c,r}(g) \in SO(n) \times SO(r) \times U(1)$,

$$\begin{aligned}
\eta_{kl}^{g(\phi)}(X,Y) &= \langle X \wedge Y \cdot f_k f_l \cdot g(\phi), g(\phi)\rangle \\
&= \left\langle g_1(X') \wedge g_1(Y') \cdot g_2(f_k')g_2(f_l') \cdot g(\phi), g(\phi)\right\rangle \\
&= \left\langle g(X' \wedge Y' \cdot f_k' f_l' \cdot \phi), g(\phi)\right\rangle \\
&= \left\langle X' \wedge Y' \cdot f_k' f_l' \cdot \phi, \phi\right\rangle \\
&=: \eta_{kl}'^{\phi}(X', Y'),
\end{aligned}$$

for $X' = g_1^{-1}(X), Y' = g_1^{-1}(Y) \in \mathbb{R}^n$, $f_k' = g_2^{-1}(f_k)$. Thus, the matrices representing $\eta_{kl}^{g(\phi)}$ (with respect to some basis) are conjugate to the matrices representing $\eta_{kl}'^{\phi}$.

Lemma 2.2.12. *Let $\phi, \psi \in \Sigma_r \otimes \Delta_n$ be partially pure spinors and $Spin^c(r)$ the standard copy of this group in $Spin^{c,r}(n)$. Then, $\psi \in Spin^c(r) \cdot \phi$ if and only if they generate the same oriented tiple $(V^\phi, J^\phi, (V^\phi)^\perp) = (V^\psi, J^\psi, (V^\psi)^\perp)$.*

Proof. Suppose $\psi = g(\phi)$ for some $g \in Spin^c(r) \subset Spin^{c,r}(n)$, and let $\lambda_n^{c,r}(g) = (1, g_2, e^{i\theta})$. With such an element

$$\langle X \wedge Y \cdot f_k f_l \cdot g(\phi), g(\phi)\rangle = \left\langle X \wedge Y \cdot f_k' f_l' \cdot \phi, \phi\right\rangle$$

for $f_k' = g_2^{-1}(f_k)$. This implies

$$\eta_{kl}^{g(\phi)}(X,Y) = \eta_{kl}'^{\phi}(X,Y).$$

So that ϕ and $g(\phi)$ span the same copy of $\mathfrak{so}(r)$ in $\mathrm{End}^-(\mathbb{R}^n)$,

$$\mathrm{span}(\eta_{kl}^{g(\phi)}) = \mathrm{span}(\eta_{kl}'^{\phi}) \cong \mathfrak{so}(r) \subset \mathrm{End}^-(\mathbb{R}^n).$$

Thus $(V^{g(\phi)})^\perp = (V^\phi)^\perp$ and, under such a g, we have as in Lemma 2.2.10: $V^{g(\phi)} = V^\phi$ and $J^{g(\phi)} = J^\phi$.

Conversely, assume $(V^\phi, J^\phi, (V^\phi)^\perp) = (V^\psi, J^\psi, (V^\psi)^\perp)$ and consider the subalgebras

$$\begin{aligned}
\mathfrak{so}(r)^\phi &= \mathrm{span}(\eta_{kl}^\phi + f_{kl}), \\
\mathfrak{so}(r)^\psi &= \mathrm{span}(\eta_{kl}^\psi + f_{kl}).
\end{aligned}$$

We will show that there exist $g \in \mathrm{Spin}^c(r)$ such that $g(\psi)$ and ϕ share the same stabilizer, for this we will seek g for which $\mathfrak{so}(r)^\psi = \mathfrak{so}(r)^{g(\phi)}$

There exist frames $(e_{2m+1}, \ldots, e_{2m+r})$ and $(e_{2m+1}', \ldots, e_{2m+r}')$ of $(V^\phi)^\perp$ and $(V^\psi)^\perp$ respectively, such that

$$\begin{aligned}
\eta_{kl}^\phi &= e_{2m+k} \wedge e_{2m+l}, \\
\eta_{kl}^\psi &= e_{2m+k}' \wedge e_{2m+l}'.
\end{aligned}$$

Let $A = (a_{kl}) \in SO(r)$ the matrix such that

$$A: \, e_{2m+k}' \mapsto a_{k1}e_{2m+1}' + \cdots + a_{kr}e_{2m+r}' = e_{2m+k}, \qquad 1 \leq k \leq r.$$

The induced transformation on $A : \bigwedge^2 (V^\psi)^\perp \to \bigwedge^2 (V^\phi)^\perp$ is given by

$$A:\ e'_{2m+k} \wedge e'_{2m+l} \mapsto e_{2m+k} \wedge e_{2m+l}.$$

Set

$$A^T:\ f_k \mapsto a_{1k} f_1 + \cdots + a_{rk} f_r = f'_k,$$

and

$$A^T:\ f_k \wedge f_l \mapsto f'_k \wedge f'_l.$$

Putting $e_{2m+p} = \sum_{s=1}^r a_{ps} e'_{2m+s}$ and $e_{2m+q} = \sum_{t=1}^r a_{qt} e'_{2m+t}$ we get

$$\begin{aligned}
& \eta'^{\psi}_{kl}(e_{2m+p}, e_{2m+q}) \\
= & \left\langle e_{2m+p} \wedge e_{2m+q} \cdot f'_k f'_l \cdot \psi, \psi \right\rangle \\
= & \left\langle \left(\sum_{s=1}^r a_{ps} e'_{2m+s} \right) \wedge \left(\sum_{t=1}^r a_{qt} e'_{2m+t} \right) \cdot f'_k f'_l \cdot \psi, \psi \right\rangle \\
= & \left\langle \left(\sum_{s<t} (a_{ps} a_{qt} - a_{pt} a_{qs}) e'_{2m+s} \wedge e'_{2m+t} \right) \cdot f'_k f'_l \cdot \psi, \psi \right\rangle \\
= & \sum_{s<t} (a_{ps} a_{qt} - a_{pt} a_{qs}) \left\langle e'_{2m+s} \wedge e'_{2m+t} \cdot f'_k f'_l \cdot \psi, \psi \right\rangle \\
= & \sum_{s<t} (a_{ps} a_{qt} - a_{pt} a_{qs}) \left\langle e'_{2m+s} \wedge e'_{2m+t} \cdot \left(\sum_{i=1}^r a_{ik} f_i \right) \left(\sum_{j=1}^r a_{jl} f_j \right) \cdot \psi, \psi \right\rangle \\
= & \sum_{s<t} (a_{ps} a_{qt} - a_{pt} a_{qs}) \left\langle e'_{2m+s} \wedge e'_{2m+t} \cdot \left(\sum_{i<j} (a_{ik} a_{jl} - a_{il} a_{jk}) f_i f_j \right) \cdot \psi, \psi \right\rangle \\
= & \sum_{s<t} \sum_{i<j} (a_{ps} a_{qt} - a_{pt} a_{qs})(a_{ik} a_{jl} - a_{il} a_{jk}) \left\langle e'_{2m+s} \wedge e'_{2m+t} \cdot f_i f_j \cdot \psi, \psi \right\rangle \\
= & \sum_{s<t} \sum_{i<j} (a_{ps} a_{qt} - a_{pt} a_{qs})(a_{ik} a_{jl} - a_{il} a_{jk}) \delta_{si} \delta_{tj} \\
= & \sum_{s<t} (a_{ps} a_{qt} - a_{pt} a_{qs})(a_{sk} a_{tl} - a_{sl} a_{tk}) \\
= & \delta_{pk} \delta_{ql},
\end{aligned}$$

since the induced tranformation by A on $\bigwedge^2 \mathbb{R}^r$ is orthogonal. This means

$$\eta'^{\psi}_{kl} = \eta^{\phi}_{kl} = e_{2m+k} \wedge e_{2m+l}.$$

Now if $g \in Spin^c(r) \subset Spin^{c,r}(n)$ is such that $\lambda_n^{c,r}(g) = (1, A, 1)$ then

$$\begin{aligned}
\delta_{pk} \delta_{ql} &= \left\langle e_{2m+p} \wedge e_{2m+q} \cdot f'_k f'_l \cdot \psi, \psi \right\rangle \\
&= \left\langle g(e_{2m+p} \wedge e_{2m+q} \cdot f'_k f'_l \cdot \psi), g(\psi) \right\rangle \\
&= \left\langle e_{2m+p} \wedge e_{2m+q} \cdot A(f'_k) A(f'_l) \cdot g(\psi), g(\psi) \right\rangle \\
&= \left\langle e_{2m+p} \wedge e_{2m+q} \cdot f_k f_l \cdot g(\psi), g(\psi) \right\rangle,
\end{aligned}$$

i.e.

$$\eta^{g(\psi)}_{kl} = e_{2m+k} \wedge e_{2m+l} = \eta^{\phi}_{kl},$$

so that
$$\mathfrak{so}(r)^{g(\psi)} = \mathrm{span}(\eta_{kl}^{g(\psi)} + f_{kl}) = \mathrm{span}(\eta_{kl}^{\phi} + f_{kl}) = \mathfrak{so}(r)^{\phi}.$$
This implies that $g(\psi)$ and ϕ share the same stabilizer
$$U(V^{\phi}, J^{\phi}) \times \exp(\mathfrak{so}(r)^{\phi}) = U(V^{g(\psi)}, J^{g(\psi)}) \times \exp(\mathfrak{so}(r)^{g(\psi)}) \cong U(m) \times SO(r).$$
But there is only a 1-dimensional summand in the decomposition of $\Sigma_r \otimes \Delta_n$ under this subgroup. More precisely, under this subgroup
$$\Sigma_r \otimes \Delta_n \quad = \quad \Sigma_r \otimes \Delta_r \otimes \Delta_{2m},$$
where Δ_{2m} decomposes under $U(m)$ and contains only a 1-dimensional trivial summand [9], while $\Sigma_r \otimes \Delta_r$ is isomorphic to a subspace of the complexified space of alternating forms on $\mathbb{R}^r$ which also contains only a 1-dimensional trivial summand. Thus, $g(\psi) = e^{i\theta}\phi$ for some $\theta \in [0, 2\pi) \subset \mathbb{R}$. $\Box$

Lemma 2.2.13.

- *If r is odd, the group $Spin^{c,r}(n)$ acts transtitively on the set of partially pure spinors in $\Sigma_r \otimes \Delta_n$.*
- *If r is even, the group $Spin^{c,r}(n)$ acts transtitively on the set of positive partially pure spinors in $(\Sigma_r \otimes \Delta_n)^+$.*

Proof. Suppose that r is odd. Note that the standard partially pure spinor ϕ_0 satisfies the conditions
$$\left\{\begin{array}{rcl} e_{2j-1}e_{2j} \cdot \phi_0 & = & i\phi_0, \\ e_{2m+k}e_{2m+l} \cdot \phi_0 & = & -f_{kl} \cdot \phi_0, \\ \langle f_{kl} \cdot \phi_0, \phi_0 \rangle & = & 0, \end{array}\right. \tag{2.13}$$
where $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_r)$ are the standard oriented frames of $\mathbb{R}^n$ and $\mathbb{R}^r$ respectively.

Let ϕ be a partially pure spinor. There exist frames $(e'_1, \ldots, e'_{2m})$ and $(e'_{2m+1}, \ldots, e'_{2m+r})$ of V^{ϕ} and $(V^{\phi})^{\perp}$ respectively such that
$$e'_{2j} = J(e'_{2j-1}) \quad \text{and} \quad \eta_{kl}^{\phi} = e'_{2m+k} \wedge e'_{2m+l}, \qquad 1 \le k < l \le r, \;\; 1 \le j \le m.$$
Call $g'_1 \in O(n)$ the transformation of $\mathbb{R}^n$ taking the new frame to the standard one. Define $g_1 \in SO(n)$ as follows
$$\left\{\begin{array}{ll} g_1 = g'_1, & \text{if } e'_1 \wedge \ldots \wedge e'_{2m+r} = \mathrm{vol}_n, \\ g_1 = -g'_1, & \text{if } e'_1 \wedge \ldots \wedge e'_{2m+r} = -\mathrm{vol}_n. \end{array}\right.$$
Then $(g_1, 1, 1) \in SO(n) \times SO(r) \times U(1)$ has two preimages $\pm\tilde{g} \in Spin^{c,r}(n)$. By Lemma 2.2.10, $\tilde{g}(\phi)$ is a partially pure spinor. We will check that $\tilde{g}(\phi)$ satisfies (2.13) as ϕ_0 does. Indeed,
$$\begin{array}{rcl} e_{2j-1}e_{2j} \cdot \tilde{g}(\phi) & = & g'_1(e'_{2j-1})g'_1(e'_{2j}) \cdot \tilde{g}(\phi) \\ & = & (\pm g_1(e'_{2j-1}))(\pm g_1(e'_{2j})) \cdot \tilde{g}(\phi) \\ & = & g_1(e'_{2j-1})g_1(e'_{2j}) \cdot \tilde{g}(\phi) \\ & = & \tilde{g}(e'_{2j-1}e'_{2j} \cdot \phi) \end{array}$$

$$\begin{aligned} &= \tilde{g}(i\phi) \\ &= i\tilde{g}(\phi), \end{aligned}$$

and

$$\begin{aligned} e_{2m+k}e_{2m+l}\cdot\tilde{g}(\phi) &= g_1'(e_{2m+k}')g_1'(e_{2m+l}')\cdot\tilde{g}(\phi) \\ &= (\pm g_1(e_{2m+k}'))(\pm g_1(e_{2m+l}'))\cdot\tilde{g}(\phi) \\ &= g_1(e_{2m+k}')g_1(e_{2m+l}')\cdot\tilde{g}(\phi) \\ &= \tilde{g}(e_{2m+k}'e_{2m+l}'\cdot\phi) \\ &= \tilde{g}(-f_kf_l\cdot\phi) \\ &= -\lambda_r(\tilde{g})(f_k)\lambda_r(\tilde{g})(f_l)\cdot\tilde{g}(\phi) \\ &= -f_kf_l\cdot\tilde{g}(\phi), \end{aligned}$$

since $\lambda_r(\tilde{g})=1$. Similarly,

$$\begin{aligned} \langle f_kf_l\cdot\tilde{g}(\phi),\tilde{g}(\phi)\rangle &= \langle\lambda_r(\tilde{g})(f_k)\lambda_r(\tilde{g})(f_l)\cdot\tilde{g}(\phi),\tilde{g}(\phi)\rangle \\ &= \langle\tilde{g}(f_kf_l\cdot\phi),\tilde{g}(\phi)\rangle \\ &= \langle f_kf_l\cdot\phi,\phi\rangle \\ &= 0. \end{aligned}$$

Thus, $\tilde{g}(\phi)$ generates the same oriented triple $(V^{\tilde{g}(\phi)},J^{\tilde{g}(\phi)},(V^{\tilde{g}(\phi)})^\perp)=(V^{\phi_0},J^{\phi_0},(V^{\phi_0})^\perp)$ as ϕ_0 which, by Lemma 2.2.12, concludes the proof for r odd. The case for r even is similar. □

Theorem 2.2.1. *Let $\mathbb{R}^n$ be endowed with the standard inner product and orientation. Given $r\in\mathbb{N}$ such that $r<n$, the following objects are equivalent:*

1. *A (positive) triple consisting of a codimension r vector subspace endowed with an orthogonal complex structure and an oriented orthogonal complement.*

2. *An orbit $Spin^c(r)\cdot\phi$ for some (positive) twisted partially pure spinor $\phi\in\Sigma_r\otimes\Delta_n$.*

Proof. Given a codimension r vector subspace D endowed with an orthogonal complex structure, $\dim_{\mathbb{R}}(D)=2m$, $n=2m+r$. By Lemma 2.1.5

$$\Delta_n\cong\Delta(D^\perp)\otimes\Delta(D).$$

Let us define

$$\Sigma_r \cong \begin{cases} \Delta(D^\perp) & \text{if } r \text{ is odd,} \\ \Delta(D^\perp)^+ & \text{if } r \text{ is even,} \end{cases}$$

so that

$$\Sigma_r\otimes\Delta_n$$

contains the standard twisted partially pure spinor ϕ_0 of Lemma 2.2.1. The proof of the converse is the content of Subsection 2.2.2. □

Let $\tilde{\mathcal{S}}$ denote the set of all partially pure spinors of rank r

$$\tilde{\mathcal{S}} = \frac{Spin^{c,r}(n)}{U(m) \times SO(r)}.$$

Consider

$$\mathcal{S} = \frac{\tilde{\mathcal{S}}}{Spin^{c}(r)}$$

where $Spin^c(r)$ is the canonical copy of such a group in $Spin^{c,r}(n)$. Thus we have the following expected result.

Corollary 2.2.1. *The space parametrizing subspaces with orthogonal complex structures of codimension r in $\mathbb{R}^n$ with oriented orthogonal complements is*

$$\mathcal{S} \cong \frac{SO(n)}{U(m) \times SO(r)}.$$

□

Chapter 3

Spin^{c,r} Structures

In this chapter we introduce doubly twisted Spin structures, which we call Spin^{c,r} structures, by means of the twisted Spin group $Spin^{c,r}(n)$. We characterize the existence of such structures with the existence of a Spin structure. On a simply connected manifold we give a result concerning the existence of Spin^{c,r} structures which do not restrict to Spin, Spin^c or Spin^r structures in terms of homotopy groups.

We also write the corresponding covariant derivatives on spinor fields associated to these structures and the corresponding twisted differential operators, as the Dirac operator and the twisted Laplacian operator. We deduce the corresponding Schrödinger-Lichnerowicz formula and deduce some corollaries of this result using Bochner type arguments.

3.1 Preliminaries

3.1.1 The Group $Spin^{c,r}(n)$

Remember that $Spin^{c,r}(n)$ is the group

$$\begin{aligned} Spin^{c,r}(n) &= (Spin(n)\times Spin^c(r))/\{\pm(1,1)\} \\ &= Spin(n)\times_{\mathbb{Z}_2} Spin^c(r), \end{aligned}$$

where $r\in\mathbb{N}$ and whose Lie algebra is

$$\mathfrak{spin}^c(n)=\mathfrak{spin}(n)\oplus\mathfrak{spin}(r)\oplus i\mathbb{R}.$$

It fits into the exact sequence

$$1\longrightarrow\mathbb{Z}_2\longrightarrow Spin^{c,r}(n)\xrightarrow{\lambda_{n,r,2}} SO(n)\times SO(r)\times U(1)\longrightarrow 1,$$

where

$$(\lambda_{n,r,2})[g,[h,z]] \;=\; (\lambda_n(g),\lambda_r(h),z^2).$$

In what follows we will use additive notation for subgroups which contain a copy of $\mathbb{Z}_2=\{0,1\}$, where 0 denotes the trivial element and 1 is the element whose square is trivial. We also use the notation $\langle g_1,\ldots,g_k\rangle$ to denote the subgroup generated by the elements $g_1,\ldots,g_k$.

Lemma 3.1.1. *For $n \geq 3$ the following holds: $\pi_1(Spin^{c,r}(n)) = \mathbb{Z}_2 \oplus \mathbb{Z}$, for $r \geq 3$, and $\pi_1(Spin^{c,2}(n)) = \mathbb{Z} \oplus \mathbb{Z}$. Moreover*

$$\ker(i_{\#}) = (\lambda_n \times \lambda_r \times \lambda_2)_{\#}(\pi_1(Spin^{c,r}(n))),$$

where i is the inclusion of $SO(n) \times SO(r) \times U(1)$ in $SO(n+r+2)$.

Proof. As $Spin^{c,r}(n)$ is a 2:1 cover of $SO(n) \times SO(r) \times U(1)$, we have that $\pi_1(Spin^{c,r}(n))$ is isomorphic to an index 2 subgroup of $\pi_1(SO(n) \times SO(r) \times U(1))$. Using additive notation, we see that the image of $\lambda^{r,2}_{n\#}$ contains the elements $(1,1,0)$ and $(1,0,1)$.

The subgroup $H = \langle (1,1,0), (1,0,1) \rangle$ is an index 2 subgroup of $\pi_1(SO(n) \times SO(r) \times U(1))$, thus $\pi_1(Spin^{c,r}(n)) \cong H$. If $r = 2$ then $H \cong \mathbb{Z} \oplus \mathbb{Z}$ and if $r \geq 3$ then $H \cong \mathbb{Z}_2 \oplus \mathbb{Z}$. Finally, using Example 1.5.1, we have $H = \ker(i_{\#})$. $\square$

3.1.2 Certain Homogeneous Spaces

In this subsection, we present certain homogeneous spaces which will provide examples for various results in the following sections.

Consider the partial flag manifold

$$\mathcal{G}_{m,s,r} = \frac{SO(2m+s+r)}{U(m) \times SO(s) \times SO(r)}$$

We will decompose the Lie algebra $\mathfrak{so}(2m+s+r)$ according to the natural inclusions

$$U(m) \times SO(s) \times SO(r) \subset SO(2m) \times SO(s) \times SO(r) \subset SO(2m+s+r).$$

Note that

$$\begin{aligned}
\mathfrak{so}(2m+s+r) &= \textstyle\bigwedge^2 \mathbb{R}^{2m+s+r} \\
&= \textstyle\bigwedge^2 (\mathbb{R}^{2m} \oplus \mathbb{R}^s \oplus \mathbb{R}^r) \\
&= \textstyle\bigwedge^2 \mathbb{R}^{2m} \oplus \bigwedge^2 \mathbb{R}^s \oplus \bigwedge^2 \mathbb{R}^r \oplus \mathbb{R}^{2m} \otimes \mathbb{R}^s \oplus \mathbb{R}^{2m} \otimes \mathbb{R}^r \oplus \mathbb{R}^s \otimes \mathbb{R}^s \\
&= \mathfrak{so}(2m) \oplus \mathfrak{so}(s) \oplus \mathfrak{so}(r) \oplus \mathbb{R}^{2m} \otimes \mathbb{R}^s \oplus \mathbb{R}^{2m} \otimes \mathbb{R}^r \oplus \mathbb{R}^s \otimes \mathbb{R}^s \\
\mathfrak{so}(2m) \otimes \mathbb{C} &= \textstyle\bigwedge^2 (\mathbb{C}^m \oplus \overline{\mathbb{C}^m}) \\
&= \textstyle\bigwedge^2 \mathbb{C}^m \oplus \mathbb{C}^m \otimes \overline{\mathbb{C}^m} \oplus \bigwedge^2 \overline{\mathbb{C}^m} \\
&= [[\textstyle\bigwedge^2 \mathbb{C}^m]] \otimes \mathbb{C} \oplus \mathfrak{u}(m) \otimes \mathbb{C}, \\
\mathbb{R}^{2m} &= [[\mathbb{C}^m]],
\end{aligned}$$

where the symbol $[[\mathbb{C}^m]]$ denotes the underlying real vector space $\mathbb{R}^{2m}$ of $\mathbb{C}^m$ carrying a complex structure. Thus

$$\mathfrak{so}(2m+s+r) = \mathfrak{u}(m) \oplus \mathfrak{so}(s) \oplus \mathfrak{so}(r) \oplus \left([[\textstyle\bigwedge^2 \mathbb{C}^m]] \oplus [[\mathbb{C}^m]] \otimes \mathbb{R}^s \oplus [[\mathbb{C}^m]] \otimes \mathbb{R}^r \oplus \mathbb{R}^s \otimes \mathbb{R}^r \right)$$

and the tangent space of $\mathcal{G}_{m,s,r}$ decomposes as follows

$$T_{\mathrm{Id}}\mathcal{G}_{m,s,r} \cong [[\textstyle\bigwedge^2 \mathbb{C}^m]] \oplus [[\mathbb{C}^m]] \otimes \mathbb{R}^s \oplus [[\mathbb{C}^m]] \otimes \mathbb{R}^r \oplus \mathbb{R}^s \otimes \mathbb{R}^r.$$

This gives the isotropy representation

$$\begin{array}{rcl} U(m)\times SO(s)\times SO(r) & \longrightarrow & SO(T_{\mathrm{Id}}\mathcal{G}_{m,s,r}) \\ (A,B,C) & \mapsto & \begin{pmatrix} [[\bigwedge^2 A]] & & & \\ & [[A]]\otimes B & & \\ & & [[A]]\otimes C & \\ & & & B\otimes C \end{pmatrix}, \end{array}$$

where $\bigwedge^2 A$ denotes the linear transformation induced by A on $\bigwedge^2\mathbb{C}^m$, $[[A]]$ the transformation A viewed as a real linear transformation on $[[\mathbb{C}^m]]=\mathbb{R}^{2m}$, and $B\otimes C$ the induced transformation on $\mathbb{R}^s\otimes\mathbb{R}^r$ (i.e. the Kronecker product of B and C).

3.2 Doubly Twisted Spin Structures

In this section, we introduce the (doubly) twisted Spin structures we need to carry out our spinorial characterization of CR structures, and the corresponding twisted Dirac operator and Laplacian. We deduce some topological conditions on manifolds that support such structures, a Schrödinger-Lichnerowicz type formula, and give some Bochner-type arguments.

Definition 3.2.1. *Let M be an oriented n-dimensional Riemannian manifold, $P_{SO(M)}$ be its principal bundle of orthonormal frames and $r\in\mathbb{N}$. A $Spin^{c,r}(n)$ structure on M consists of an auxiliary principal $SO(r)$ bundle $P_{SO(r)}$, an auxiliary principal $U(1)$ bundle $P_{U(1)}$ and a principal $Spin^{c,r}(n)$ bundle $P_{Spin^{c,r}(n)}$ together with an equivariant $2:1$ covering map*

$$\Lambda : P_{Spin^{c,r}(n)} \longrightarrow P_{SO(M)}\tilde{\times}P_{SO(r)}\tilde{\times}P_{U(1)},$$

where $\tilde{\times}$ denotes the fibered product, such that $\Lambda(pg)=\Lambda(p)(\lambda_{n,r,2})(g)$ for all $p\in P_{Spin^{c,r}(n)}$ and $g\in Spin^{c,r}(n)$.

A n-dimensional Riemannian manifold M admitting a $Spin^{c,r}(n)$ structure will be called a Spin^{c,r} manifold.

Remark. A Spin^{c,r} manifold with trivial $P_{SO(r)}$ and $P_{U(1)}$ auxiliary bundles is a Spin manifold. On the other hand, we have the following:

- Any Spin manifold admits a Spin^{c,r} structure with trivial $P_{SO(r)}$ and $P_{U(1)}$ auxiliary bundles via the inclusion $Spin(n)\subset Spin^{c,r}(n)$.
- Any Spin^c manifold admits a Spin^{c,r} structure with trivial $P_{SO(r)}$ auxiliary bundle via the inclusion $Spin^c(n)\subset Spin^{c,r}(n)$.
- Any Spin^r manifold (cf. [8]) admits a Spin^{c,r} structure with trivial $P_{U(1)}$ auxiliary bundle via the inclusion $Spin^r(n)\subset Spin^{c,r}(n)$.

3.2.1 Existence of Spin^{c,r} Structures

We will characterize the existence of a Spin^{c,r} structure in terms of a Spin structure. By setting $N=n+r+2$, $G=SO(n)\times SO(r)\times U(1)$, $Q=P_{SO(M)}\tilde{\times}P_{SO(r)}\tilde{\times}P_{U(1)}$ and considering the natural inclusion of $SO(n)\times SO(r)\times U(1)\subset SO(n+r+2)$ we have that

$$\pi_1\left(\frac{SO(n+r+2)}{SO(n)\times SO(r)\times U(1)}\right)=0.$$

Using proposition 1.5.1 we obtain

Corollary 3.2.1. *The bundle* $Q = P_{SO(M)} \tilde{\times} P_{SO(r)} \tilde{\times} P_{U(1)}$ *over* M *has a Spin structure if and only if there exists a homomorphism* $f : \pi_1(Q) \to \pi_1(SO(n+r+2))$ *for which the diagram*

$$\begin{tikzcd} \pi_1(SO(n) \times SO(r) \times U(1)) \arrow[r, "i_\#"] \arrow[d, "h"'] & \pi_1(SO(n+r+2)) \\ \pi_1(Q) \arrow[ur, "f"'] & \end{tikzcd}$$

commutes. □

Proposition 3.2.1. M *admits an* $SO(r) \times SO(2)$*-principal bundle* $P_{SO(r)\times SO(2)}$ *such that the fibre product* $Q = P_{SO(n)} \tilde{\times} P_{SO(r)\times SO(2)}$ *has a Spin structure if and only if* M *has a Spin*^{c,r} *structure.*

Proof. If M has a Spin^{c,r} structure then there exist $P_{SO(2)}$ and $P_{SO(r)}$, $SO(2)$ and $SO(r)$ principal bundles respectively, such that the bundle $P_{SO(r)\times SO(2)} := P_{SO(r)} \tilde{\times} P_{SO(2)}$ is an $SO(r) \times SO(2)$ principal bundle over M. Now, there exists an injective homomorphism $\tilde{\iota}$ which makes the diagram

$$\begin{tikzcd} Spin^{c,r}(n) \arrow[r, "\tilde{\iota}"] \arrow[d, "\lambda_n^{c,r}"'] & Spin(n+r+2) \arrow[d, "\lambda_{n+r+2}"] \\ SO(n) \times SO(r) \times SO(2) \arrow[r, "i"'] & SO(n+r+2) \end{tikzcd}$$

commute. From this we obtain a Spin structure for Q in the sense of Corollary 3.2.1 (analogously to Proposition 1.5.3).

Conversely, let $\lambda = \lambda_n \times \lambda_r \times \lambda_2$ and $F = SO(n) \times SO(r) \times U(1)$. According to Corollary 3.2.1, due to the existence of f, $H = \ker(f) \subset \pi_1(Q)$ is a subroup of index 2. Therefore, there exists a double covering space $\Lambda : P_{Spin^{c,r}(n)} \to Q$ corresponding to H. Let $\mu : Q \times F \to Q$ be the action of F in Q and consider the composition of induced maps on fundamental groups

$$\pi_1(P_{Spin^{c,r}(n)} \times Spin^{c,r}(n)) \xrightarrow{(\Lambda\times\lambda)_\#} \pi_1(Q \times F) \xrightarrow{\mu_\#} \pi_1(Q).$$

If $(\sigma, \tau) \in \pi_1(P_{Spin^{c,r}(n)}) \times \pi_1(Spin^{c,r}(n))$, by means of the inclusion h,

$$\begin{aligned} \mu_\# \circ (\Lambda \times \lambda)_\#(\sigma, \tau) &= \Lambda_\#(\sigma)\lambda_\#(\tau) \\ &= \Lambda_\#(\sigma) * h(\lambda_\#(\tau)) \end{aligned}$$

where $*$ denotes product in the relevant fundamental group. We know that

$$\Lambda_\#(\sigma) \in H \qquad \text{and} \qquad f(h(\lambda_\#(\tau))) = i_\#(\lambda_\#(\tau)) = 0$$

by Lemma 3.1.1 and Corollary 3.2.1. Thus, $h(\lambda_\#(\tau)) \in H$ and $\Lambda_\#(\sigma) * h(\lambda_\#(\tau)) \in H$. Hence, there exists a lift $\tilde{\mu} : P_{Spin^{c,r}(n)} \times Spin^{c,r}(n) \to P_{Spin^{c,r}(n)}$ which gives the equivariance in Definition 3.2.1. □

Now, we will derive a condition for a simply connected manifold to have a "non-reducible" $\mathrm{Spin}^{c,r}$ structure, i.e. a $\mathrm{Spin}^{c,r}$ structure which does not come from a Spin, nor a Spin^c, nor Spin^r structure.

Proposition 3.2.2. *Let M be simply connected and Q its $SO(n)$-principal bundle of orthonormal frames. The following are equivalent*

1. *Q has a $\mathrm{Spin}^{c,r}$ structure but does not have a Spin, nor a Spin^c, nor a Spin^r structure.*
2. *There exists a $SO(r)\times SO(2)$ bundle P_1 over M such that in the long exact sequence*

$$\cdots \longrightarrow \pi_2(M) \xrightarrow{\partial} \pi_1(SO(n)\times SO(r)\times SO(2)) \xrightarrow{h} \pi_1(Q\tilde{\times}P_1) \longrightarrow \pi_1(M)=0,$$

$Im(\partial)\cong\langle(1,0,p),(0,1,p)\rangle\subset\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}$ with p odd.

Proof. If (P,Λ) is a $\mathrm{Spin}^{c,r}$ structure on Q then there exist $P_{SO(2)}$ and $P_{SO(r)}$, $SO(2)$ and $SO(r)$ principal bundles respectively, so that the bundle $P_1 := P_{SO(r)}\tilde{\times}P_{SO(2)}$ is a $SO(r)\times SO(2)$ principal bundle over M. Now, by Proposition 3.2.1, the fibre product $Q\tilde{\times}P_1$ has a Spin structure. By Corollary 3.2.1, this means that there exists a map $f:\pi_1(Q\tilde{\times}P_1)\to\pi_1(SO(n+r+2))$ such that the diagram

$$\begin{tikzcd}
\pi_1(SO(n)\times SO(r)\times SO(2)) = \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z} \arrow[r, "i_\#"] \arrow[d, "h"'] & \pi_1(SO(n+r+2)) = \mathbb{Z}_2 \\
\pi_1(Q\tilde{\times}P_1) \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes. Now, if Q does not have a Spin structure then we have $\pi_1(Q)=0$ in the following commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \\
 & \pi_2(M) \arrow[d, "\partial"] & & \\
 & \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z} \arrow[d, "h"] & & \\
\mathbb{Z}_2\oplus\mathbb{Z} = \pi_1(SO(r)\times SO(2)) \arrow[ur, "j_\#"] \arrow[r, "k"] & \pi_1(Q\tilde{\times}P_1) \arrow[r] \arrow[d] & \pi_1(Q) \arrow[r] & 0 \\
 & \pi_1(M)=0. & &
\end{tikzcd} \tag{3.1}$$

Thus, k is onto and

$$(\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z})/\mathrm{Im}(\partial)\cong\pi_1(Q\tilde{\times}P_1)=h(\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z})=k(\mathbb{Z}_2\oplus\mathbb{Z}).$$

Now, we will describe the group $K=\pi_1(Q\tilde{\times}P_1)$. It depends on the nontrivial elements $h(1,0,0)=\alpha$, $h(0,1,0)=\beta$ and $h(0,0,1)=\gamma$. First, we have $K=\langle\beta,\gamma\rangle$, so that $\alpha=a\beta+b\gamma$ for some integers a,b. Since α and β have order two in K

$$0=2\alpha=2a\beta+2b\gamma=2b\gamma,$$

and K is a finite group. Now,

(i) If $\beta \in \langle\gamma\rangle$ then γ has order $2p$, $K \cong \mathbb{Z}_{2p}$ and $\alpha = \beta = p\gamma$. Since there is only one nontrivial map $f : \mathbb{Z}_{2p} \to \mathbb{Z}_2$, $f \circ h = i_\#$ if and only if p is odd.

(ii) If $\beta \notin \langle\gamma\rangle$ then $K \cong \mathbb{Z}_2 \oplus \mathbb{Z}_d$. If d is odd, $f : \mathbb{Z}_d \to \mathbb{Z}_2$ must be trivial and $(f \circ h)(0,0,1) = 0$ which gives us no Spin^{c,r} structure. If $d = 2p$, then

$$\alpha = \beta, \quad \text{or} \quad \alpha = p\gamma \quad \text{or} \quad \alpha = \beta + p\gamma.$$

In order to have $i_\# = f \circ h$ and, therefore, the existence of the Spin^{c,r} structure, if $\alpha = p\gamma$ then p must be odd, and if $\alpha = \beta + p\gamma$ then p must be even.

Now, we are going to rule out the three options in (ii). Note that

$$K' = \pi_1(Q\tilde{\times}P_{SO(r)}\tilde{\times}P_{SO(2)})$$

and the $SO(2)$ fibre bundle $Q\tilde{\times}P_{SO(r)}\tilde{\times}P_{SO(2)} \to Q\tilde{\times}P_{SO(r)}$ gives the commutative diagram

$$\begin{tikzcd}
 & & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & & \pi_2(M) \arrow[d] & \pi_2(M) \arrow[d] & \\
\cdots \arrow[r] & \mathbb{Z} = \pi_1(SO(2)) \arrow[r] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \mathbb{Z} = \pi_1(SO(2)) \arrow[r] \arrow[ur] & \pi_1(Q\tilde{\times}P_{SO(r)}\tilde{\times}P_{SO(2)}) \arrow[r] \arrow[d] & \pi_1(Q\tilde{\times}P_{SO(r)}) \arrow[r] \arrow[d] & 0 \\
 & & \pi_1(M) = 0 & \pi_1(M) = 0 &
\end{tikzcd} \tag{3.2}$$

- If $K' = \mathbb{Z}_2 \oplus \mathbb{Z}_{2p} = \langle\beta,\gamma\rangle$ with $\alpha = \beta$ then, by exactness of the diagram, $\pi_1(Q\tilde{\times}P_{SO(r)}) = (\mathbb{Z}_2 \oplus \mathbb{Z}_{2p})/\langle\gamma\rangle = \mathbb{Z}_2 \cong \langle\beta\rangle$, which gives us a Spin^r structure.

- The same happens if $\alpha = \beta + p\gamma$. The quotient is isomorphic to $\mathbb{Z}_2$, whose equivalence classes are

$$\{(0,0),(0,1),\ldots,(0,2p-1)\} \quad \text{and} \quad \{(1,0),\ldots,(1,2p-1)\},$$

where $\gamma = (0,1)$ belongs to the first one, and $\alpha = (1,p)$ and $\beta = (1,0)$ belong to the second one. In other words, α and β are mapped to the nontrivial class and we have a Spin^r structure.

- Now if we have the group $K' = \mathbb{Z}_2 \oplus \mathbb{Z}_{2p}$ with p odd and $\alpha = p\gamma$, the $SO(r)$ fibre bundle

$Q\tilde{\times}P_{SO(r)}\tilde{\times}P_{SO(2)} \to Q\tilde{\times}P_{SO(2)}$ gives the commutative diagram

$$\begin{tikzcd}
 & & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & & \pi_2(M) \arrow[d] & \pi_2(M) \arrow[d] & \\
\cdots \arrow[r] & \mathbb{Z}_2 = \pi_1(SO(r)) \arrow[r] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}_2 \oplus \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \mathbb{Z}_2 = \pi_1(SO(r)) \arrow[r] \arrow[ur] & \pi_1(Q\tilde{\times}P_{SO(r)}\tilde{\times}P_{SO(2)}) \arrow[r] \arrow[d] & \pi_1(Q\tilde{\times}P_{SO(2)}) \arrow[r] \arrow[d] & 0 \\
 & & \pi_1(M) = 0 & \pi_1(M) = 0 &
\end{tikzcd}$$

so that $\pi_1(Q\tilde{\times}P_{SO(2)}) = \mathbb{Z}_2 \oplus \mathbb{Z}_{2p}/\langle\beta\rangle = \mathbb{Z}_{2p}$ with p odd, which implies the existence of a Spin^c structure.

Now we know that $K = \mathbb{Z}_{2p} = \langle\gamma\rangle$ with p odd and $\alpha = \beta = p\gamma$. This should be the same as the quotient $(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z})/\mathrm{Im}(\partial)$ where, by extacness, $\mathrm{Im}(\partial) = \ker(h)$. We see that the map h is given by $h(a, b, c) = ((a+b)p+c)\gamma$. The kernel of this map is given by the $(a, b, c) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$ such that $(a + b)p + c \equiv 0 \pmod{2p}$, i.e. $(a, b, c) \in \langle(0, 1, p), (1, 0, p)\rangle$ where p is odd.

Conversely, assume $\mathrm{Im}(\partial) = \langle(1, 0, p), (0, 1, p)\rangle$, p odd, and put this in the diagram (3.1). By exactness of the column, $\pi_1(Q\tilde{\times}P_1) \cong \mathbb{Z}_{2p}$. This group is generated by the non trivial element $\gamma = h(0, 0, 1)$, and we have $h(1, 0, 0) = h(0, 1, 0) = p\gamma$.

Thus, $k = h \circ j_\#$ is onto, we have no Spin structure and the only nonzero homomorphism $f : \mathbb{Z}_{2p} \to \mathbb{Z}_2$ gives us $i_\# = f \circ h$, i.e. the existence of a Spin^{c,r} structure.

The $SO(r)$ bundle $P_{SO(r)} = P_1/SO(2)$ fits into a commutative diagram similar to (3.2). By exactness, we have $\pi_1(Q\tilde{\times}P_{SO(r)}) = \{0\}$ and we cannot have a Spin^r structure. Similarly, the $SO(2)$ bundle $P_{SO(2)} = P_1/SO(r)$ fits into a similar diagram so that $\pi_1(Q\tilde{\times}P_{SO(2)}) = \mathbb{Z}_p$, and there is not map f as in Corollary 3.2.1 to have a Spin^c structure. Note that in the case $p = 1$ this last group is zero. □

Example 3.2.1. *Now we will give an example of a manifold satisfying the conditions of the previous proposition. Let* $M = G/H$ *with* $G = SO(2m+2+r)$ *and* $H = U(m)\times U(1)\times SO(r)$, $r \geq 3$. *Since* H *is a compact connected subgroup of* G *and the inclusion map induces a map of fundamental groups which is onto,* $\pi_1(M) = \{0\}$.

Consider the bundle of orthonormal frames $Q = G \times_\rho SO(n)$ *where* $n = m^2+2mr+3m+2r = \dim(M)$ *and*

$$\rho : H \hookrightarrow G \times SO(n)$$

is given by the inclusion of H *into the first factor and the isotropy representation in the second which is given by*

$$\xi : H \quad \longrightarrow \quad SO(n)$$

$$(A, e^{i\theta}, B) \quad \mapsto \quad [[\Lambda^2 A]] \oplus ([[A]] \otimes R_\theta) \oplus ([[A]] \otimes B) \oplus (R_\theta \otimes B),$$

where R_θ is the rotation in $\mathbb{R}^2$ by an angle of θ. This gives a fibration

$$\begin{array}{ccc} H & \hookrightarrow & G \times SO(n) \\ & & \downarrow \\ & & Q \end{array}$$

which induces the long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_1(H) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{\rho_\#} \pi_1(G \times SO(n)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \pi_1(Q) \longrightarrow 0.$$

First note that the isotropy representation induces the map

$$\begin{array}{rcl} \pi_1(H) & \longrightarrow & \pi_1(SO(n)) \\ (a, b, c) & \mapsto & (m-1+2+r)a + (2m+r)b + (2m+2)c \ \textit{(mod 2)} \\ & = & (m-1+r)a + rb \ \textit{(mod 2)}. \end{array}$$

Thus,

$$\rho_\#(a, b, c) = ((a+b+c) \ \textit{(mod 2)}, ((m-1+r)a + br) \ \textit{(mod 2)}).$$

Note that $(m-1+r)a+br \equiv 0$ (mod 2) if and only if r is even and m is odd. So, by exactness, $\pi_1(Q) = \mathbb{Z}_2$ (Q has a Spin structure) if and only if r is even and m is odd.

Let m and r be even and consider

$$\begin{array}{rcl} \sigma : H & \longrightarrow & (G \times SO(n)) \times U(1) \times SO(r) \\ (A, e^{i\theta}, B) & \mapsto & (\rho(A, e^{i\theta}, B), e^{i\theta}, B) \end{array}$$

We have the fibration $H \hookrightarrow G \times SO(n) \times U(1) \times SO(r) \xrightarrow{\nu} G \times_\sigma SO(n) \times U(1) \times SO(r) = Q \tilde{\times} P_{U(1)} \tilde{\times} P_{SO(r)}$ which gives

$$\cdots \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{\sigma_\#} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{\nu_\#} \pi_1(Q \tilde{\times} P_{U(1)} \tilde{\times} P_{SO(r)}) \longrightarrow 0,$$

where

$$\sigma_\#(a, b, c) = ((a+b+c) \ \textit{(mod 2)}, a \ \textit{(mod 2)}, b, c \ \textit{(mod 2)}).$$

We see that $Im(\sigma_\#) = \langle (1,1,0,0), (1,0,1,0), (1,0,0,1) \rangle = L$ is a subgroup of index two because $(1,0,0,0) \notin L$ and $((1,0,0,0)+L) \cup L = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}_2$. By exactness, $\pi_1(Q \tilde{\times} P_{U(1)} \tilde{\times} P_{SO(r)}) \cong \mathbb{Z}_2$.

Consider $f : \pi_1(Q \tilde{\times} P_{U(1)} \tilde{\times} P_{SO(r)}) \to \pi_1(SO(n+2+r))$ to be the only nontrivial homomorphism between these groups. Now, the inclusion of the fiber $SO(n) \times Un(1) \times SO(r)$ into the fiber bundle $Q \tilde{\times} P_{U(1)} \tilde{\times} P_{SO(r)}$ is given by the inclusion j into the last three factors of $G \times SO(n) \times U(1) \times SO(r)$, followed by the projection ν. Thus, the map h in Proposition 3.2.2 is given by $h = \nu_\# \circ j_\#$.
Consider, for simplicity, $\pi_1(Q \tilde{\times} P_{U(1)} \tilde{\times} P_{SO(r)}) = \{0, 1\}$. From the explicit description of L, we can see

$$h(a, b, c) = \begin{cases} 0 & \textit{if } a+b+c \equiv 0 \ \textit{(mod 2)}, \\ 1 & \textit{if } a+b+c \equiv 1 \ \textit{(mod 2)}. \end{cases}$$

This means that $f \circ h$ is the same map as the inclusion of $\pi_1(SO(n) \times U(1) \times SO(r)) \hookrightarrow \pi_1(SO(n+2+r))$. By Proposition 3.2.2, M has a Spin^{c,r} structure which does not come from either a Spin, nor a Spin^c, nor a Spin^r structure.

3.2.2 Covariant Derivatives and Twisted Differential Operators

Let M be a Spin^{c,r} n-dimensional manifold, ω the Levi-Civita connection 1-form on its principal bundle of orthonormal frames $P_{SO(n)}$, θ and iA chosen connection 1-forms on the auxiliary bundles $P_{SO(r)}$ and $P_{U(1)}$ respectively. These connections forms give rise to covariant derivatives ∇, ∇^θ and ∇^A on the associated vector bundles

$$\begin{aligned} TM &= P_{Spin^{c,r}(n)} \times_{\lambda_{n,r,2}} (\mathbb{R}^n \times \{0\} \times \{0\}), \\ F &= P_{Spin^{c,r}(n)} \times_{\lambda_{n,r,2}} (\{0\} \times \mathbb{R}^r) \times \{0\}), \\ L &= P_{Spin^{c,r}(n)} \times_{\lambda_{n,r,2}} (\{0\} \times \{0\} \times \mathbb{C}), \end{aligned}$$

Furthermore, the three connections help define a connection on the twisted spinor bundle

$$S = P_{Spin^{c,r}(n)} \times_{\kappa_n^{c,r}} (\Sigma_r \otimes \Delta_n)$$

given (locally) as follows

$$\begin{aligned} \nabla^{\theta,A} &: \Gamma(S) \longrightarrow \Gamma(T^*M \otimes S) \\ \nabla^{\theta,A}(\varphi \otimes \psi) &= d(\varphi \otimes \psi) + \varphi \otimes \left[\frac{1}{2} \sum_{1 \le i < j \le n} \omega_{ij} \otimes e_i e_j \cdot \psi \right] \\ &\quad + \left[\frac{1}{2} \sum_{1 \le k < l \le r} \theta_{kl} \otimes \kappa_{r*}(f_k f_l) \cdot \varphi \right] \otimes \psi + \frac{i}{2} \varphi \otimes (A \cdot \psi), \end{aligned}$$

where $\varphi \otimes \psi \in \Gamma(S)$, $(e_1, \dots, e_n)$ and $(f_1, \dots, f_r)$ are local orthonormal frames of TM and F respectively. ω_{ij}, θ_{kl} and A are the corresponding local connection 1-forms for TM, F and L respectively.

From now on, we will omit the upper and lower bounds on the indices, by declaring i and j to be the indices for the frame vectors of TM, and k and l to be the indices for the frame sections of F.

Now, for any tangent vectors $X, Y \in T_xM$, the spinorial curvature is defined by

$$\begin{aligned} R^{\theta,A}(X,Y)(\varphi \otimes \psi) &= \varphi \otimes \left[\frac{1}{2} \sum_{i<j} \Omega_{ij}(X,Y) e_i e_j \cdot \psi \right] \\ &\quad + \left[\frac{1}{2} \sum_{k<l} \Theta_{kl}(X,Y) \kappa_{r*}(f_k f_l) \cdot \varphi \right] \otimes \psi \\ &\quad + \frac{i}{2} \varphi \otimes (dA(X,Y)\psi), \end{aligned} \tag{3.3}$$

where

$$\Omega_{ij}(X,Y) = \langle R^M(X,Y)(e_i), e_j \rangle \qquad \text{and} \qquad \Theta_{kl}(X,Y) = \langle R^F(X,Y)(f_k), f_l \rangle.$$

Here R^M (resp. R^F) denotes the curvature tensor of M (resp. of F).

For X, Y vector fields and $\phi \in \Gamma(S)$ a spinor field, we have compatibility of the covariant derivative with Clifford multiplication,

$$\nabla_X^{\theta,A}(Y \cdot \phi) = (\nabla_X Y) \cdot \phi + Y \cdot \nabla_X^{\theta,A} \phi.$$

Definition 3.2.2. *The* twisted Dirac operator *is the first order differential operator* $\not{\partial}^{\theta,A}$: $\Gamma(S) \longrightarrow \Gamma(S)$ *defined by*

$$\not{\partial}^{\theta,A}(\phi) \quad = \quad \sum_{i=1}^{n} e_i \cdot \nabla^{\theta,A}_{e_i}(\phi).$$

Remark. The twisted Dirac operator $\not{\partial}^{\theta,A}$ is well-defined and formally self-adjoint on compact manifolds. Moreover, if $h \in C^\infty(M)$, $\phi \in \Gamma(S)$, we have

$$\not{\partial}^{\theta,A}(h\phi) = \mathrm{grad}(h) \cdot \phi + h\not{\partial}^{\theta,A}(\phi).$$

The proofs of these facts are analogous to the ones for the Spin^c Dirac operator [9].

Definition 3.2.3. *The* twisted Spin connection Laplacian *is the second order differential operator* $\Delta^{\theta,A} : \Gamma(S) \to \Gamma(S)$ *defined as*

$$\Delta^{\theta,A}(\phi) = -\sum_{i=1}^{n} \nabla^{\theta,A}_{e_i}\nabla^{\theta,A}_{e_i}(\phi) - \sum_{i=1}^{n} \mathrm{div}(e_i)\nabla^{\theta,A}_{e_i}(\phi).$$

3.2.3 A Schrödinger-Lichnerowicz-type Formula

Just as in [8, 9], we have the following

Proposition 3.2.3. *For $X \in \Gamma(TM)$ and $\phi \in \Gamma(S)$, we have*

$$\begin{aligned}\sum_{i=1}^{n} e_i \cdot R^{\theta,A}(X,e_i)(\phi) \quad &= \quad -\frac{1}{2}\mathrm{Ric}(X)\cdot\phi + \frac{1}{2}\sum_{k<l}(X\lrcorner\Theta_{kl})\cdot\kappa_{r*}(f_kf_l)\cdot\phi \\ &\quad +\frac{i}{2}X\lrcorner dA\cdot\phi, \end{aligned} \tag{3.4}$$

where Ric *denotes the Ricci tensor of M and $R^{\theta,A}$ the curvature operator of the twisted spinorial connection.*

Proof. For $\phi = \varphi\otimes\psi$, by (3.3),

$$\begin{aligned} R^{\theta,A}(X,e_\alpha)(\varphi\otimes\psi) \quad &= \quad \varphi\otimes\left[\frac{1}{2}\sum_{i<j}\Omega_{ij}(X,e_\alpha)e_ie_j\cdot\psi\right] \\ &\quad + \left[\frac{1}{2}\sum_{k<l}\Theta_{kl}(X,e_\alpha)\kappa_{r*}(f_kf_l)\cdot\varphi\right]\otimes\psi + \frac{i}{2}\varphi\otimes(dA(X,e_\alpha)\psi). \end{aligned}$$

Multiply by e_α and sum over α

$$\begin{aligned} \sum_\alpha e_\alpha\cdot R^{\theta,A}(X,e_\alpha)(\varphi\otimes\psi) \quad &= \quad \varphi\otimes\left[\frac{1}{2}\sum_\alpha\sum_{i<j}\Omega_{ij}(X,e_\alpha)e_\alpha e_ie_j\cdot\psi\right] \\ &\quad +\frac{1}{2}[\kappa_{r*}(f_kf_l)\cdot\varphi]\otimes\sum_{k<l}\left[\sum_\alpha\Theta_{kl}(X,e_\alpha)e_\alpha\cdot\psi\right] \\ &\quad +\frac{i}{2}\varphi\otimes\left[\sum_\alpha dA(X,e_\alpha)e_\alpha\cdot\psi\right]. \end{aligned}$$

Now,

$$\begin{aligned}\frac{1}{2}\sum_{\alpha}\sum_{i<j}\Omega_{ij}(X,e_\alpha)e_\alpha e_i e_j &= -\frac{1}{2}\mathrm{Ric}(X),\\ \kappa_{r*}(f_k f_l)\cdot\varphi\otimes\frac{1}{2}\sum_{k<l}\left[\sum_{\alpha}\Theta_{kl}(X,e_\alpha)e_\alpha\cdot\psi\right], &= \frac{1}{2}\sum_{k<l}(X\lrcorner\Theta_{kl})\cdot\kappa_{r*}(f_k f_l)\cdot(\varphi\otimes\psi),\\ \frac{i}{2}\varphi\otimes\left[\sum_{\alpha}dA(X,e_\alpha)e_\alpha\cdot\psi\right] &= \frac{i}{2}\varphi\otimes X\lrcorner dA\cdot\psi.\end{aligned}$$

□

Proposition 3.2.4. *Let $\phi\in\Gamma(S)$. Then*

$$\sum_{i,j}e_i e_j\cdot R^{\theta,A}(e_i,e_j)(\phi)=\frac{\mathrm{R}}{2}\phi+\sum_{k<l}\Theta_{kl}\cdot\kappa_{r*}(f_k f_l)\cdot\phi+i\,dA\cdot\phi,$$

where $\Theta_{kl}=\sum_{i<j}\Theta_{kl}(e_i,e_j)e_i\wedge e_j$ and R is the scalar curvature of M.

Proof. By (3.4),

$$\sum_{j=1}^{n}e_j\cdot R^{\theta,A}(e_i,e_j)(\phi)=-\frac{1}{2}\mathrm{Ric}(e_i)\cdot\phi+\frac{1}{2}\sum_{j}\sum_{k<l}\Theta_{kl}(e_i,e_j)e_j\cdot\kappa_{r*}(f_k f_l)\cdot\phi+\frac{i}{2}e_i\lrcorner dA\cdot\phi.$$

Multiplying with e_i and summing over i, we get

$$\begin{aligned}\sum_{i,j}e_i e_j\cdot R^{\theta,A}(e_i,e_j)(\phi) &= -\frac{1}{2}\sum_{i}e_i\cdot\mathrm{Ric}(e_i)\cdot\phi\\ &\quad+\frac{1}{2}\sum_{k<l}\left[\sum_{i,j}\Theta_{kl}(e_i,e_j)e_i e_j\right]\cdot\kappa_{r*}(f_k f_l)\cdot\phi\\ &\quad+\frac{i}{2}\sum_{i}e_i\cdot e_i\lrcorner dA\cdot\phi.\end{aligned}$$

Now,

$$-\sum_{i}e_i\cdot\mathrm{Ric}(e_i) = \mathrm{R},$$

where R denotes the scalar curvature of M. For k and l fixed,

$$\begin{aligned}\sum_{i,j}\Theta_{kl}(e_i,e_j)e_i e_j &= 2\sum_{i<j}\Theta_{kl}(e_i,e_j)e_i e_j\\ &= 2\Theta_{kl},\\ \frac{i}{2}\sum_{i}e_i\cdot e_i\lrcorner dA\cdot\psi &= \frac{i}{2}\sum_{i,\alpha}dA(e_i,e_\alpha)e_i\cdot e_\alpha\cdot\psi\\ &= i\sum_{i<\alpha}dA(e_i,e_\alpha)e_i\cdot e_\alpha\cdot\psi\\ &= i\,dA\cdot\psi.\end{aligned}$$

□

Let us define

$$\begin{aligned}
\Theta &= \sum_{k<l} \Theta_{kl} \otimes f_k f_l \in \bigwedge^2 T^*M \otimes \bigwedge^2 F, \\
\hat{\Theta} &= \sum_{k<l} \hat{\Theta}_{kl} \otimes f_k f_l \in \mathrm{End}^-(TM) \otimes \bigwedge^2 F, \\
\eta^\phi &= \sum_{k<l} \eta^\phi_{kl} \otimes f_k f_l \in \bigwedge^2 T^*M \otimes \bigwedge^2 F, \\
\hat{\eta}^\phi &= \sum_{k<l} \hat{\eta}^\phi_{kl} \otimes f_k f_l \in \mathrm{End}^-(TM) \otimes \bigwedge^2 F,
\end{aligned}$$

where $\hat{\Theta}_{kl}$ denotes the skew-symmetric endomorphism associated to Θ_{kl} via the metric.

Denote by

$$\tilde{\Theta} = (\mu_n \otimes \kappa_{r*})(\Theta),$$

the corresponding operator on twisted spinor fields. In order to simplify notation, we also define

$$\begin{aligned}
\left\langle \Theta, \eta^\phi \right\rangle_0 &= \sum_{k<l} \sum_{i<j} \Theta_{kl}(e_i, e_j) \eta^\phi_{kl}(e_i, e_j), \\
\left\langle \hat{\Theta}, \hat{\eta}^\phi \right\rangle_1 &= \sum_{k<l} \mathbf{tr}(\hat{\Theta}_{kl} (\hat{\eta}^\phi_{kl})^T).
\end{aligned}$$

Theorem 3.2.1 (Twisted Schrödinger-Lichnerowicz Formula)**.** *Let* $\phi \in \Gamma(S)$*. Then*

$$\partial\!\!\!/^{\theta,A}(\partial\!\!\!/^{\theta,A}(\phi)) = \Delta^{\theta,A}(\phi) + \frac{\mathrm{R}}{4}\phi + \frac{1}{2}\tilde{\Theta}\cdot\phi + \frac{i}{2} dA \cdot \phi \tag{3.5}$$

where R *is the scalar curvature of the Riemannian manifold* M*.*

Proof. Consider the difference

$$\partial\!\!\!/^{\theta,A}(\partial\!\!\!/^{\theta,A}(\phi)) - \Delta^{\theta,A}(\phi) = \sum_i \sum_{j\neq k} \langle \nabla_{e_i} e_j, e_k \rangle \, e_i e_k \cdot \nabla^{\theta,A}_{e_i}\phi + \sum_{i\neq j} e_i e_j \cdot \nabla^{\theta,A}_{e_j} \nabla^{\theta,A}_{e_j}\phi),$$

since

$$\sum_j \sum_{i=k} \langle \nabla_{e_i} e_j, e_k \rangle \, e_i e_k \nabla^{\theta,A}_{e_j}\phi = -\sum_j \mathrm{div}(e_j) \nabla^{\theta,A}_{e_j}\phi.$$

Thus,

$$\begin{aligned}
\partial\!\!\!/^{\theta,A}(\partial\!\!\!/^{\theta,A}(\phi)) - \Delta^{\theta,A}(\phi) &= \sum_j \sum_{i<k} \langle e_j, [e_k, e_i] \rangle \, e_i e_k \cdot \nabla^{\theta,A}_{e_i}\phi \\
&\quad + \sum_{i<j} e_i e_j \cdot (\nabla^{\theta,A}_{e_i} \nabla^{\theta,A}_{e_j} - \nabla^{\theta,A}_{e_j} \nabla^{\theta,A}_{e_i})\phi \\
&= \frac{1}{2} \sum_{i,j} e_i e_j R^{\theta,A}(e_i, e_j)\phi.
\end{aligned}$$

The result follows from Proposition 3.2.4. $\square$

3.2.4 Bochner-type Arguments

In this subsection we will prove some corollaries of the Schrödinger-Lichnerowicz-type formula and Bochner type arguments (cf. [9]). For the rest of the section, let us assume that the n-dimensional Riemannian Spin^{c,r} manifold M is compact (without boundary) and connected.

Harmonic spinors

A twisted spinor field $\phi \in \Gamma(S)$ such that

$$\partial\!\!\!/^{\theta,A}\phi = 0$$

will be called a *harmonic spinor.*

Corollary 3.2.2. *If* $\mathrm{R} \geq 2|\tilde{\Theta}| + 2|dA|$ *everywhere (in pointwise operator norm), then a harmonic spinor is parallel. Furthermore, if the inequality is strict at a point, then there are no non-trivial harmonic spinors*

$$\ker(\partial\!\!\!/^{\theta,A}) = \{0\}.$$

Proof. If $\phi \neq 0$ is a solution of

$$\partial\!\!\!/^{\theta,A}(\phi) = 0,$$

by the twisted Schrödinger-Lichnerowicz formula (3.5)

$$0 = \Delta^{\theta,A}(\phi) + \frac{\mathrm{R}}{4}\phi + \frac{1}{2}\tilde{\Theta}\cdot\phi + \frac{i}{2}dA\cdot\phi.$$

By taking hermitian product with ϕ and integrating over M we get

$$0 \quad \geq \quad \int_M |\nabla^{\theta,A}\phi|^2 + \frac{1}{4}\int_M \left(\mathrm{R} - 2|\tilde{\Theta}| - 2|dA|\right)|\phi|^2.$$

Since

$$\mathrm{R} - 2|\tilde{\Theta}| - 2|dA| \geq 0,$$

then

$$|\nabla^{\theta,A}\phi| = 0,$$

so that ϕ is parallel, has non-zero constant length and no zeroes.

Now, if

$$\mathrm{R} - 2|\tilde{\Theta}| - 2|dA| > 0$$

at some point,

$$0 \geq |\phi|^2 \int_M \left(\mathrm{R} - 2|\tilde{\Theta}| - 2|dA|\right) > 0.$$

∎

Now notice that

$$\begin{aligned}\left\langle \tilde{\Theta}\cdot\phi, \phi\right\rangle &= \left\langle \sum_{k<l}\left[\sum_{i<j}\Theta_{kl}(e_i,e_j)e_ie_j\right]\cdot\kappa_{r*}(f_kf_l)\cdot\phi, \phi\right\rangle \\ &= \sum_{k<l}\sum_{i<j}\Theta_{kl}(e_i,e_j)\eta^{\phi}_{kl}(e_i,e_j)\end{aligned}$$

$$= \left\langle \Theta, \eta^{\phi} \right\rangle_0,$$

which is a real number dependent on the curvature of the connection θ and the specific spinor ϕ.

Corollary 3.2.3. *If ϕ is such that*

$$\mathrm{R}|\phi|^2 + 2\left\langle \Theta, \eta^{\phi} \right\rangle_0 + 2i\left\langle dA\cdot\phi, \phi\right\rangle \geq 0$$

everywhere, and the inequality is strict at a point, then

$$\partial\!\!\!/^{\theta}(\phi) \neq 0.$$

Proof. Suppose $\phi \neq 0$ is such that

$$\partial\!\!\!/^{\theta}(\phi) = 0.$$

Then, by (3.5)

$$0 = \int_M |\nabla^{\theta}\phi|^2 + \frac{1}{4}\int_M \left(\mathrm{R}|\phi|^2 + 2\left\langle \Theta, \eta^{\phi} \right\rangle_0 + 2i\left\langle dA\cdot\phi, \phi\right\rangle\right) \geq 0,$$

so that ϕ is parallel, has non-zero constant length and no zeroes. Since

$$\mathrm{R}|\phi|^2 + 2\left\langle \Theta, \eta^{\phi} \right\rangle_0 + 2i\left\langle dA\cdot\phi, \phi\right\rangle > 0$$

at some point,

$$0 \geq \int_M \left(\mathrm{R}|\phi|^2 + 2\left\langle \Theta, \eta^{\phi} \right\rangle_0 + 2i\left\langle dA\cdot\phi, \phi\right\rangle\right) > 0.$$

□

Killing spinors

A twisted spinor field $\phi \in \Gamma(S)$ is called a *Killing spinor* if

$$\nabla^{\theta,A}_X \phi = \mu X \cdot \phi$$

for all $X \in \Gamma(TM)$, and μ a complex constant.

Corollary 3.2.4. *Suppose $\phi \neq 0$ is a Killing spinor with Killing constant μ. Then μ is either real or imaginary, and*

$$\mu^2 \geq \frac{1}{4n^2}\min_M(\mathrm{R} - 2|\tilde{\Theta}| - 2|dA|).$$

If the inequality is attained, then ϕ is parallel, i.e. $\mu = 0$.

Proof. Recall that

$$\begin{aligned} \partial\!\!\!/^{\theta,A}(\phi) &= \sum_{i=1}^{n} e_i \cdot \nabla^{\theta,A}_{e_i}\phi \\ &= -n\mu\phi. \end{aligned}$$

Then, by the twisted Schrödinger-Lichnerowicz formula (3.5)

$$n^2\mu^2\phi = \Delta^{\theta,A}(\phi) + \frac{\mathrm{R}}{4}\phi + \frac{1}{2}\tilde{\Theta}\cdot\phi + \frac{i}{2}dA\cdot\phi.$$

By taking hermitian product with ϕ and integrating over M we get

$$\begin{aligned} n^2\mu^2\int_M|\phi|^2 &= \int_M|\nabla^{\theta,A}\phi|^2+\int_M\frac{\mathrm{R}}{4}|\phi|^2+\int_M\frac{1}{2}\left\langle\tilde{\Theta}\cdot\phi,\phi\right\rangle+\frac{i}{2}\int_M\langle dA\cdot\phi,\phi\rangle\\ &\geq \frac{1}{4}\min_M(\mathrm{R}-2|\tilde{\Theta}|-2|dA|)\int_M|\phi|^2, \end{aligned}$$

and the inequality follows. Since the right hand side of the equality above is a real number, μ must be either real or imaginary. Now, if the inequality is attained,

$$\int_M|\nabla^{\theta,A}\phi|^2=0 \qquad \text{and} \qquad \nabla^{\theta,A}\phi=0.$$

□

Corollary 3.2.5. *Suppose $\phi\in\Gamma(S)$ is a Dirac eigenspinor*

$$\partial\!\!\!/^{\theta,A}\phi=\lambda\phi.$$

Then

$$\lambda^2 \geq \frac{n}{4(n-1)}\left(\min_M(\mathrm{R}-2|\tilde{\Theta}|-2|dA|)\right).$$

If the lower bound is non-negative and is attained, the spinor ϕ is a real Killing spinor with Killing constant

$$\mu=\pm\frac{1}{2}\sqrt{\frac{1}{n(n-1)}\min_M(\mathrm{R}-2|\tilde{\Theta}|-2|dA|)}.$$

Proof. Let $h:M\to\mathbb{R}$ be a fixed smooth function. Consider the following metric connection on the twisted Spin bundle

$$\nabla^h_X\phi=\nabla^{\theta,A}_X\phi+hX\cdot\phi.$$

Let

$$\Delta^h(\phi)=-\sum_{i=1}^n\nabla^h_{e_i}\nabla^h_{e_i}\phi-\sum_{i=1}\mathrm{div}(e_i)\nabla^h_{e_i}\phi,$$

be the Laplacian for this connection and recall that

$$|\nabla^h\phi|^2=\sum_{i=1}^n|\nabla^{\theta,A}_{e_i}\phi+he_i\cdot\phi|^2.$$

Then, by (3.5)

$$\begin{aligned} (\partial\!\!\!/^{\theta,A}-h)\circ(\partial\!\!\!/^{\theta,A}-h)(\phi) &= \partial\!\!\!/^{\theta,A}(\partial\!\!\!/^{\theta,A}\phi)-2h\partial\!\!\!/^{\theta,A}\phi-\mathrm{grad}(h)\cdot\phi+h^2\phi\\ &= \Delta^{\theta,A}(\phi)+\frac{\mathrm{R}}{4}\phi+\frac{1}{2}\tilde{\Theta}\cdot\phi+\frac{i}{2}dA\cdot\phi-2h\partial\!\!\!/^{\theta,A}\phi\\ &\quad -\mathrm{grad}(h)\cdot\phi+h^2\phi. \end{aligned}$$

On the other hand,

$$\Delta^h\phi=\Delta^{\theta,A}\phi-2h\partial\!\!\!/^{\theta,A}\phi-\mathrm{grad}(h)\cdot\phi+nh^2\phi.$$

Thus

$$(\not{\partial}^{\theta,A} - h) \circ (\not{\partial}^{\theta,A} - h)(\phi) = \Delta^h(\phi) + \frac{\mathrm{R}}{4}\phi + \frac{1}{2}\tilde{\Theta}\cdot\phi + \frac{i}{2}dA\cdot\phi + (1-n)h^2\phi$$

By using $\not{\partial}^{\theta,A}\phi = \lambda\phi$, setting $h = \frac{\lambda}{n}$, taking hermitian product with ϕ and integrating over M we get

$$\begin{aligned} \lambda^2\left(\frac{n-1}{n}\right)^2 \int_M |\phi|^2 &= \int_M |\nabla^{\lambda/n}\phi|^2 + \lambda^2\frac{1-n}{n^2}\int_M |\phi|^2 + \int_M \frac{\mathrm{R}}{4}|\phi|^2 \\ &\quad + \int_M \frac{1}{2}\left\langle \tilde{\Theta}\cdot\phi, \phi\right\rangle + \frac{i}{2}\int_M \langle dA\cdot\phi,\phi\rangle \end{aligned}$$

so that

$$\lambda^2\left(\frac{n-1}{n}\right)\int_M |\phi|^2 \;\geq\; \frac{1}{4}\min_M(\mathrm{R} - 2|\tilde{\Theta}| - 2|dA|)\int_M |\phi|^2.$$

If the lower bound is attained,

$$\int_M |\nabla^{\lambda/n}\phi|^2 = 0,$$

i.e.

$$\nabla^{\lambda/n}\phi = 0.$$

□

Chapter 4

CR Structures

In this chapter we develop a spinorial description of CR structures of arbitrary codimension. More precisely, we characterize almost CR structures of arbitrary codimension on (Riemannian) manifolds by the existence of a $\mathrm{Spin}^{c,r}$ structure carrying a partially pure spinor field. We study various integrability conditions of the almost CR structure in our spinorial setup, including the classical integrability of a CR structure as well as those implied by Killing-type conditions on the partially pure spinor field.

4.1 CR Structures of Arbitrary Codimension

In this section we will explore the twisted spinorial geometry associated to almost CR structures. We carry out the spinorial characterization and explore some integrability conditions of almost CR structures implied by assuming the typical conditions on spinors, such as being parallel or Killing, but just in prescribed directions.

4.1.1 Spinorial Characterization of Almost CR (Hermitian) Structures

Definition 4.1.1. *Let M be a smooth $(2m+r)$-dimensional smooth manifold.*

- *An* almost CR structure *on a manifold M consists of a sub-bundle $D \subset TM$ and a bundle automorphism J of D such that $J^2 = -\mathrm{Id}_D$.*
- *An* almost CR hermitian structure *on M is an almost CR structure whose almost complex structure is orthogonal with respect to the metric.*

Remark. Given an almost CR structure on M we can introduce an (auxiliary) compatible metric as follows. Take any Riemannian metric g_0 on M and consider the orthogonal complement $D^\perp$ of D with respect to this metric. Let g_1 and g_2 denote the restrictions of g_0 to D and $D^\perp$ respectively. Average g_1 with respect to J and call it g_3. Finally, consider the metric $g = g_3 \oplus g_2$.

Definition 4.1.2. *Let M be an oriented Riemannian $Spin^{c,r}(n)$ manifold and S the associated twisted spinor bundle. A (nowhere zero) spinor field $\phi \in \Gamma(S)$ is called* partially pure *if $\phi_x \in S_x$ is partially pure at each point $x \in M$.*

Theorem 4.1.1. *Let M be an oriented n-dimensional Riemannian manifold. Then the following two statements are equivalent:*

(a) M admits a twisted $Spin^{c,r}$ structure carrying a partially pure spinor field $\phi \in \Gamma(S)$, where S denotes the associated twisted spinor bundle.

(b) M admits an almost CR hermitian structure of codimension r.

Proof. If the manifold M admits a partially pure spinor field $\phi \in \Gamma(S)$, the subspaces V_x^ϕ determine a smooth distribution of even rank $n-r$ carrying an almost complex structure. Conversely, if M has an orthogonal almost CR hermitian structure of codimension r, the tangent bundle decomposes orthogonally as

$$TM = D \oplus D^\perp,$$

where D has real rank $2m = n-r$ and admits an almost complex structure, and $D^\perp$ is the oriented orthogonal complement. The structure group of the Riemannian manifold M reduces to $U(m) \times SO(r)$ and, by Lemma 2.1.4, there is a monomorphism

$$U(m) \times SO(r) \hookrightarrow Spin^{c,r}(2m+r)$$

with image $\widehat{U(m) \times SO}(r)$, which allows us to associate a $Spin^{c,r}(n)$ principal bundle P on M, i.e. a $\mathrm{Spin}^{c,r}$ structure similarly as we did in Propositions 1.5.3 and 3.2.1 . Note that the corresponding twisted spinor bundle S decomposes under $\widehat{U(m) \times SO}(r)$ as follows

$$\begin{aligned} S &= \left[\kappa_D^{-1/2} \otimes \Sigma(D^\perp)\right] \otimes \Delta(M) \\ &= \left[\kappa_D^{-1/2} \otimes \Sigma(D^\perp)\right] \otimes \Delta(D^\perp) \otimes \Delta(D) \\ &= \left[\kappa_D^{-1/2} \otimes \Sigma(D^\perp)\right] \otimes \Delta(D^\perp) \otimes \left[(\textstyle\bigwedge^* D^{0,1}) \otimes \kappa_D^{1/2}\right] \\ &= \left[\Sigma(D^\perp) \otimes \Delta(D^\perp)\right] \otimes \left[\textstyle\bigwedge^* D^{0,1}\right], \end{aligned}$$

where $\kappa_D = \bigwedge^m D^{1,0}$. We see that it contains a rank 1 trivial subbundle generated by the partially pure spinor given in (2.4) with stabilizer $\widehat{U(m) \times SO}(r)$, i.e. M admits a global partially pure spinor field. $\square$

Example 4.1.1. *Recall from Subsection 3.1.2 that*

$$T_{\mathrm{Id}}\mathcal{G}_{m,1,r} \cong [[\textstyle\bigwedge^2 \mathbb{C}^m]] \oplus [[\mathbb{C}^m]] \otimes \mathbb{R}^r \oplus [[\mathbb{C}^m]] \oplus \mathbb{R}^r$$

For the sake of clarity, consider $m=2$, $r=2$ and $\mathbb{R}^7 = \mathbb{R}^4 \oplus \mathbb{R}^2 \oplus \mathbb{R}^1$, where the first summand $\mathbb{R}^4$ is endowed with the standard complex structure

$$\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}.$$

The different summands in the decomposition

$$T_{\mathrm{Id}}\mathcal{G} \cong [[\textstyle\bigwedge^2 \mathbb{C}^2]] \oplus [[\mathbb{C}^2]] \otimes \mathbb{R}^2 \oplus [[\mathbb{C}^2]] \oplus \mathbb{R}^2.$$

correspond to the following skew-symmetric matrices

$$
\begin{aligned}
[[\bigwedge^2\mathbb{C}^2]] &= \left\{ \begin{pmatrix} 0 & 0 & b_1 & b_2 & & & \\ & 0 & b_2 & -b_1 & & & \\ -b_1 & -b_2 & 0 & & & & \\ -b_2 & b_1 & & 0 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix} : \, b_1, b_2 \in \mathbb{R} \right\}, \\
[[\mathbb{C}^2]] \otimes \mathbb{R}^2 &= \left\{ \begin{pmatrix} 0 & & & & c_1 & c_2 & \\ & 0 & & & c_3 & c_4 & \\ & & 0 & & c_5 & c_6 & \\ & & & 0 & c_7 & c_8 & \\ -c_1 & -c_3 & -c_5 & -c_7 & 0 & & \\ -c_2 & -c_4 & -c_6 & -c_8 & & 0 & \\ & & & & & & 0 \end{pmatrix} : \, c_j \in \mathbb{R}, \, j = 1, \ldots, 8 \right\}, \\
[[\mathbb{C}^2]] &= \left\{ \begin{pmatrix} 0 & & & & & & d_1 \\ & 0 & & & & & d_2 \\ & & 0 & & & & d_3 \\ & & & 0 & & & d_4 \\ & & & & 0 & & \\ & & & & & 0 & \\ -d_1 & -d_2 & -d_3 & -d_4 & & & 0 \end{pmatrix} : \, d_j \in \mathbb{R}, \, j = 1, \ldots, 4 \right\}, \\
\mathbb{R}^2 &= \left\{ \begin{pmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & 0 & & \delta_1 \\ & & & & & 0 & \delta_2 \\ & & & & -\delta_1 & -\delta_2 & 0 \end{pmatrix} : \, \delta_1, \delta_2 \in \mathbb{R} \right\}.
\end{aligned}
$$

The induced complex structure on $[[\bigwedge^2\mathbb{C}^2]] \oplus [[\mathbb{C}^2]] \otimes \mathbb{R}^2 \oplus [[\mathbb{C}^2]]$*, which respects each summand, is*

$$
J \begin{pmatrix} 0 & 0 & b_1 & b_2 & c_1 & c_2 & d_1 \\ & 0 & b_2 & -b_1 & c_3 & c_4 & d_2 \\ & & 0 & 0 & c_5 & c_6 & d_3 \\ & & & 0 & c_7 & c_8 & d_4 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -b_2 & b_1 & -c_3 & -c_4 & -d_2 \\ & 0 & b_1 & b_2 & c_1 & c_2 & d_1 \\ & & 0 & 0 & -c_7 & -c_8 & -d_4 \\ & & & 0 & c_5 & c_6 & d_3 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}
$$

where we have only written the upper triangle part for notational simplicity.

Thus, this example gives us several candidates of distributions carrying an almost complex structure, as well as their orthogonal complements (with respect to the natural metric):

$$
\begin{cases} D_1 &= [[\bigwedge^2\mathbb{C}^m]] \\ D_1^\perp &= [[\mathbb{C}^m]] \otimes \mathbb{R}^r \oplus [[\mathbb{C}^m]] \oplus \mathbb{R}^r \end{cases}
$$

$$\begin{cases} D_2 &= [[\mathbb{C}^m]] \otimes \mathbb{R}^r \\ D_2^\perp &= [[\bigwedge^2 \mathbb{C}^m]] \oplus [[\mathbb{C}^m]] \oplus \mathbb{R}^r \end{cases}$$
$$\begin{cases} D_3 &= [[\mathbb{C}^m]] \\ D_3^\perp &= [[\bigwedge^2 \mathbb{C}^m]] \oplus [[\mathbb{C}^m]] \otimes \mathbb{R}^r \oplus \mathbb{R}^r \end{cases}$$
$$\begin{cases} D_4 &= [[\bigwedge^2 \mathbb{C}^m]] \oplus [[\mathbb{C}^m]] \otimes \mathbb{R}^r \\ D_4^\perp &= [[\mathbb{C}^m]] \oplus \mathbb{R}^r \end{cases}$$
$$\begin{cases} D_5 &= [[\bigwedge^2 \mathbb{C}^m]] \oplus [[\mathbb{C}^m]] \\ D_5^\perp &= [[\mathbb{C}^m]] \otimes \mathbb{R}^r \oplus \mathbb{R}^r \end{cases}$$
$$\begin{cases} D_6 &= [[\mathbb{C}^m]] \otimes \mathbb{R}^r \oplus [[\mathbb{C}^m]] \\ D_6^\perp &= [[\bigwedge^2 \mathbb{C}^m]] \oplus \mathbb{R}^r \end{cases}$$
$$\begin{cases} D_7 &= [[\bigwedge^2 \mathbb{C}^m]] \oplus [[\mathbb{C}^m]] \otimes \mathbb{R}^r \oplus [[\mathbb{C}^m]] \\ D_7^\perp &= \mathbb{R}^r \end{cases}$$

By computing the Lie brackets at the Lie algebra level, we see that the distributions $D_1, D_4, D_6^\perp$ and $D_7^\perp$ are involutive with their foliations corresponding to the fibers of the following four fibrations

$$\begin{array}{ccc} \frac{SO(2m)}{U(m)} & \hookrightarrow & \frac{SO(2m+r+1)}{U(m)\times SO(r)} \\ & & \downarrow \\ & & \frac{SO(2m+r+1)}{SO(2m)\times SO(r)}, \end{array}$$

$$\begin{array}{ccc} \frac{SO(2m+r)}{U(m)\times SO(r)} & \hookrightarrow & \frac{SO(2m+r+1)}{U(m)\times SO(r)} \\ & & \downarrow \\ & & S^{2m+r}, \end{array}$$

$$\begin{array}{ccc} \frac{SO(2m)}{U(m)} \times S^r & \hookrightarrow & \frac{SO(2m+r+1)}{U(m)\times SO(r)} \\ & & \downarrow \\ & & \frac{SO(2m+r+1)}{SO(2m)\times SO(r+1)}, \end{array}$$

$$\begin{array}{ccc} S^r & \hookrightarrow & \frac{SO(2m+r+1)}{U(m)\times SO(r)} \\ & & \downarrow \\ & & \frac{SO(2m+r+1)}{U(m)\times SO(r+1)}, \end{array}$$

respectively.

4.1.2 Adapted Connection for Almost CR-Hermitian Manifolds

Before we proceed with the characterizations of integrability conditions, we need to give (at least) a choice of connection on the relevant bundles of an almost CR hermitian manifold, or equivalently, on a $\mathrm{Spin}^{c,r}$ manifold carrying a partially pure spinor.

As we mentioned earlier, we can adapt a metric on an almost CR manifold M in order to make it an almost CR hermitian manifold. Let us fix one such metric and its Levi-Civita connection 1-form ω and covariant derivative ∇. The metric determines the orthogonal complement $D^\perp$ and gives us a covariant derivative as follows

$$\nabla_X^{D^\perp} : \Gamma(D^\perp) \longrightarrow \Gamma(D^\perp)$$

$$W \mapsto \mathrm{proj}_{D^\perp}(\nabla_X W)$$

for $X \in \Gamma(TM)$, whose local connection 1-forms and curvature 2-forms will be denoted by $\theta_{kl}^{D^\perp}$ and $\Theta_{kl}^{D^\perp}$ respectively, $1 \leq k < l \leq r$. The analogous connection on D is given by the covariant derivative

$$\begin{array}{rcl} \nabla_X^D : \Gamma(D) & \longrightarrow & \Gamma(D) \\ W & \mapsto & \mathrm{proj}_D(\nabla_X W) \end{array}$$

for $X \in \Gamma(TM)$. However, we need to induce a connection on κ_D^{-1}. Thus, we consider the hermitian connection for $(D, \langle , \rangle, J)$ defined by

$$\tilde{\nabla}_X^D Y = \nabla_X^D Y + \frac{1}{2}(\nabla_X^D J)(JY).$$

so that

$$\tilde{\nabla}^D J = 0.$$

The conection $\tilde{\nabla}^D$ induces a covariant derivative $\tilde{\nabla}^{\kappa_D^{-1}}$ on the anticanonical bundle κ_D^{-1} of D, whose local connection 1-form will be denoted by $i\tilde{A}^D$. More precisely, if $(e_1, \ldots, e_n)$ is a local orthonormal frame of TM such that

$$\begin{array}{rcl} D & = & \mathrm{span}(e_1, \ldots, e_{2m}), \\ e_{2s} & = & J(e_{2s-1}), \\ D^\perp & = & \mathrm{span}(e_{2m+1}, \ldots, e_{2m+r}), \end{array}$$

for $1 \leq s \leq m$ and $1 \leq k < l \leq r$, and the matrix of connection 1-forms of $\tilde{\nabla}^D$ is

$$\begin{pmatrix} 0 & \tilde{\omega}_{1,2} & & \tilde{\omega}_{1,2m-1} & \tilde{\omega}_{1,2m} \\ -\tilde{\omega}_{12} & 0 & & -\tilde{\omega}_{1,2m} & \tilde{\omega}_{1,2m-1} \\ & & \ddots & & \\ -\tilde{\omega}_{1,2m-1} & \tilde{\omega}_{1,2m} & & 0 & \tilde{\omega}_{2m-1,2m} \\ -\tilde{\omega}_{1,2m} & -\tilde{\omega}_{1,2m-1} & & -\tilde{\omega}_{2m-1,2m} & 0 \end{pmatrix},$$

the induced connection on $\kappa_D^{-1} = \bigwedge^m D^{0,1}$ is

$$i\tilde{A}^D = -i[\tilde{\omega}_{1,2} + \cdots + \tilde{\omega}_{2m-1,2m}].$$

By using ∇, $\nabla^{D^\perp}$ and the unitary connection $i\tilde{A}^D$, we can define a connection ∇^S on the globally defined twisted spinor vector bundle $S = \left[\kappa_D^{-1/2} \otimes \Sigma(D^\perp)\right] \otimes \Delta(M)$ which is compatible with Clifford multiplication.

4.1.3 Spinorial Characterization of Integrability

Definition 4.1.3. *Let M be a smooth $2m+r$ dimensional smooth manifold. An almost CR structure is called a* CR structure *if for every $X, Y \in \Gamma(D)$*

- $[X, Y] - [J(X), J(Y)] \in \Gamma(D)$,
- $[J(X), Y] + [X, J(Y)] \in \Gamma(D)$,
- $J([X, Y] - [J(X), J(Y)]) = [J(X), Y] + [X, J(Y)]$.

Example. By computing the relevant combinations of brackets one can check that the distributions D_1, D_4, D_5 and D_7 on $\mathcal{G}_{m,1,r}$ are CR-integrable.

Theorem 4.1.2. *Let M be an oriented n-dimensional Riemannian manifold. The following are equivalent:*

(i) *M is endowed with a CR hermitian structure of codimension r.*

(ii) *M admits a twisted $Spin^{c,r}(n)$ structure and a twisted spinor bundle S carrying a partially pure spinor field $\phi \in \Gamma(S)$ which satisfies*

$$(X - iJ^\phi(X)) \cdot \nabla^S_{(Y-iJ^\phi(Y))}\phi = (Y - iJ^\phi(Y)) \cdot \nabla^S_{(X-iJ^\phi(X))}\phi,$$

for every $X, Y \in \Gamma(V^\phi)$, where ∇^S is the covariant drivative described in subsection 4.1.2.

Proof. First, let us assume (i), i.e. M admits a CR hermitian structure. By Theorem 4.1.1, M admits a twisted spinor vector bundle $S = \left[\kappa_D^{-1/2} \otimes \Sigma(D^\perp)\right] \otimes \Delta(M)$ carrying a partially pure spinor field $\phi \in \Gamma(S)$ such that $V^\phi = D$, $J^\phi = J$ and

$$(X - iJX) \cdot \phi = 0$$

for every $X \in \Gamma(D)$. By differentiating this identity

$$(\nabla_Y X - i\nabla_Y(JX)) \cdot \phi + (X - iJX) \cdot \nabla^S_Y \phi = 0, \tag{4.1}$$

and similarly

$$(\nabla_X Y - i\nabla_X(JY)) \cdot \phi + (Y - iJY) \cdot \nabla^S_X \phi = 0, \tag{4.2}$$

By subtracting (4.1) from (4.2)

$$([X,Y] - i\nabla_X(JY) + i\nabla_Y(JX)) \cdot \phi = (X - iJX) \cdot \nabla^S_Y \phi - (Y - iJY) \cdot \nabla^S_X \phi, \tag{4.3}$$

By substituting X with JX, and Y with JY in (4.3)

$$([JX,JY] + i\nabla_{JX}(Y) - i\nabla_{JY}(X)) \cdot \phi = -(X - iJX) \cdot \nabla^S_{-iJY}\phi + (Y - iJY) \cdot \nabla^S_{-iJX}\phi, \tag{4.4}$$

Subtract (4.4) from (4.3)

$$\begin{aligned} &([X,Y] - [JX,JY] - i([X,JY] + [JX,Y])) \cdot \phi \\ &= (X - iJX) \cdot \nabla^S_{Y-iJY}\phi - (Y - iJY) \cdot \nabla^S_{X-iJX}\phi. \end{aligned} \tag{4.5}$$

Since $[X,Y] - [JX,JY] \in \Gamma(D)$

$$\begin{aligned} ([X,Y] - [JX,JY]) \cdot \phi &= iJ([X,Y] - [JX,JY]) \cdot \phi \\ &= i([J(X),Y] + [X,J(Y)]) \cdot \phi, \end{aligned}$$

so that the left hand side of (4.5) vanishes.

Conversely, let us assume (ii). Then, the subbundle V^ϕ together with its endomorphism J^ϕ provide an almost CR hermitian structure on M. By considering the equation

$$(X - iJ^\phi X) \cdot \phi = 0$$

for all $X \in V^\phi$, and performing the same calculations as before, we arrive at

$$\begin{aligned}&\Big([X,Y]-[J^\phi X,J^\phi Y]-i([X,J^\phi Y]+[J^\phi X,Y])\Big)\cdot\phi\\ &=\Big(X-iJ^\phi X\Big)\cdot\nabla^S_{Y-iJ^\phi Y}\phi-\Big(Y-iJ^\phi Y\Big)\cdot\nabla^S_{X-iJ^\phi X}\phi\\ &=0,\end{aligned}$$

i.e.

$$([X,Y]-[J^\phi X,J^\phi Y])\cdot\phi=i([X,J^\phi Y]+[J^\phi X,Y])\cdot\phi,$$

which implies

- $[X,Y]-[J^\phi(X),J^\phi(Y)]\in\Gamma(V^\phi)$,
- $[J^\phi(X),Y]+[X,J^\phi(Y)]\in\Gamma(V^\phi)$,
- $J^\phi([X,Y]-[J^\phi(X),J^\phi(Y)])=[J^\phi(X),Y]+[X,J^\phi(Y)]$,

since ϕ is a partially pure spinor. □

4.1.4 D-parallel Partially Pure Spinor

The following theorem is motivated by the condition

$$\nabla^S_X\phi=0$$

for all $X\in\Gamma(D)$, i.e. ϕ being D-parallel.

Theorem 4.1.3. *Let M be an oriented n-dimensional Riemannian manifold. The following are equivalent:*

(i) *M admits a twisted $Spin^{c,r}(n)$ structure and a twisted spinor bundle S carrying a partially pure spinor field $\phi\in\Gamma(S)$ satisfying*

$$(Y-iJ^\phi(Y))\cdot\nabla^S_X\phi=0$$

for every $X,Y\in\Gamma(V^\phi)$, where ∇^S is the covariant derivative described in subsection 4.1.2.

(ii) *M is endowed with an almost CR hermitian structure of codimension r, where D and J are D-parallel. (In particular, J restricts to a Kähler structure on each leaf of the integral foliation of D, and $D^\perp$ is D-parallel.)*

Proof. Let us assume (i) and $D=V^\phi$, $D^\perp=(V^\phi)^\perp$, $J=J^\phi$. Since

$$(Y-iJ^\phi Y)\cdot\phi=0$$

for every $Y\in\Gamma(V^\phi)$, if $X\in\Gamma(V^\phi)$

$$\begin{aligned}0&=\nabla^S_X((Y-iJ^\phi Y)\cdot\phi)\\ &=(\nabla_XY-i\nabla_X(J^\phi Y))\cdot\phi+(Y-iJ^\phi Y)\cdot\nabla^S_X\phi\\ &=(\nabla_XY-i\nabla_X(J^\phi Y))\cdot\phi,\end{aligned}$$

which means

$$\begin{aligned} \nabla_X Y &\in D, \\ \nabla_X (JY) &= J(\nabla_X Y). \end{aligned}$$

i.e. D and J are D-parallel so that the leaves of this totally geodesic foliation are Kähler manifolds. If $u \in \Gamma(D^\perp)$

$$\langle Y, u \rangle = 0$$

for every $Y \in \Gamma(D)$, so that for every $X \in \Gamma(D)$

$$\begin{aligned} 0 &= X \langle Y, u \rangle \\ &= \langle Y, \nabla_X u \rangle \end{aligned}$$

since D is D-parallel, thus showing that $\nabla_X u \in \Gamma(D^\perp)$.

Conversely, if M admits an almost CR hermitian structure. By Theorem 4.1.1, M admits a twisted Spin structure and a twisted spinor bundle S endowed with a connection ∇^S, carrying a partially pure spinor field $\phi \in \Gamma(S)$ such that $V^\phi = D$, $J^\phi = J$ and

$$(Y - iJY) \cdot \phi = 0$$

for every $Y \in \Gamma(D)$. Thus, for $X \in \Gamma(D)$,

$$\begin{aligned} 0 &= \nabla_X((Y - iJY) \cdot \phi) \\ &= (\nabla_X Y - iJ(\nabla_X Y)) \cdot \phi + (Y - iJ(Y) \cdot \nabla^S_X \phi \\ &= (Y - iJ(Y)) \cdot \nabla^S_X \phi \end{aligned}$$

since J is D-parallel. As before, $D^\perp$ is D-parallel. □

Example 4.1.2. *The space $\mathcal{G}_{m,1,r}$ admits the CR distribution D_1 satisfying the hypotheses of Theorem 4.1.3, as can be seen from the fibration:*

$$\begin{array}{ccc} \frac{SO(2m)}{U(m)} & \hookrightarrow & \frac{SO(2m+r+1)}{U(m)\times SO(r)} \\ & & \downarrow \\ & & \frac{SO(2m+r+1)}{SO(2m)\times SO(r)}. \end{array}$$

When the partially pure spinor is parallel, we can actually say more about the foliation leaves' Ricci curvature.

Theorem 4.1.4. *Let M be a Spin^{c,r} n-dimensional Riemannian manifold such that its twisted spinor bundle S admits a partially pure spinor field $\phi \in \Gamma(S)$ satisfying*

$$\nabla^S_X \phi = 0$$

for every $X \in \Gamma(V^\phi)$, where ∇^S is the covariant derivative described in subsection 4.1.2. Then

1. *The Ricci tensor of V^ϕ satisfies*

$$\mathrm{Ric}^{V^\phi} = \left[\mathrm{proj}_{V^\phi} \circ \widehat{dA}|_{V^\phi}\right] \circ J^\phi, \tag{4.6}$$

where $\widehat{dA}$ denotes the skew-symmetric endomorphism determined by dA (the curvature of the connection 1-form on the auxiliary principal $U(1)$ bundle) and metric dualization.

2. *The scalar curvature is given by*

$$\mathrm{R}^{V^\phi} = \mathbf{tr}\left(\left[\mathrm{proj}_{V^\phi} \circ \widehat{dA}|_{V^\phi}\right] \circ J^\phi\right).$$

3. *If the connection A on the auxiliary bundle L is flat along an integral leaf of V^ϕ, then the leaf is Calabi-Yau.*

Remark. The identity (4.6) tells us that $\mathrm{proj}_{V^\phi} \circ \widehat{dA}|_{V^\phi}$, restricted to the leaves of the corresponding foliation, equals their Ricci form.

Proof. Since ϕ is partially pure, $n = 2m + r$ where $\mathrm{rank}(V^\phi) = 2m$ and $\mathrm{rank}((V^\phi)^\perp) = r$. Let $(e_1, \ldots, e_n)$ and $(f_1, \ldots f_r)$ be local orthonormal frames of TM and F respectively, such that

$$\begin{aligned} V^\phi &= \mathrm{span}(e_1, \ldots, e_{2m}), \\ e_{2j} &= J^\phi(e_{2j-1}), \\ (V^\phi)^\perp &= \mathrm{span}(e_{2m+1}, \ldots, e_{2m+r}), \\ \eta^\phi_{kl} &= e_{2m+k} \wedge e_{2m+l}, \end{aligned}$$

for $1 \leq j \leq m$ and $1 \leq k < l \leq r$. If $X \in \Gamma(V^\phi)$ and $1 \leq \alpha \leq 2m$ then, by Theorem 4.1.3, $[X, e_\alpha] \in \Gamma(V^\phi)$ and

$$\begin{aligned} R^M(X, e_\alpha)e_i &\in \Gamma(V^\phi) && \text{if } 1 \leq i \leq 2m, \\ R^M(X, e_\alpha)e_i &\in \Gamma((V^\phi)^\perp) && \text{if } 2m+1 \leq i \leq 2m+r. \end{aligned}$$

so that

$$\begin{aligned} \left\langle R^M(X, e_\alpha)e_i, e_j\right\rangle &= 0 && \text{if } 1 \leq i \leq 2m,\ 2m+1 \leq j \leq 2m+r, \\ \left\langle R^M(X, e_\alpha)e_i, e_j\right\rangle &= 0 && \text{if } 2m+1 \leq i \leq 2m+r,\ 1 \leq j \leq 2m. \end{aligned}$$

For ϕ,

$$\begin{aligned} 0 &= R^{\theta,A}(X, e_\alpha)\phi \\ &= \frac{1}{2}\sum_{1\leq i<j\leq n} \left\langle R^M(X, e_\alpha)e_i, e_j\right\rangle e_i e_j \cdot \phi \\ &\quad + \frac{1}{2}\sum_{1\leq k<l\leq r} \Theta_{kl}(X, e_\alpha)\kappa_{r*}(f_k f_l) \cdot \phi + \frac{i}{2} dA(X, e_\alpha)\phi \\ &= \frac{1}{2}\sum_{1\leq i<j\leq 2m} \left\langle R^M(X, e_\alpha)e_i, e_j\right\rangle e_i e_j \cdot \phi \\ &\quad + \frac{1}{2}\sum_{1\leq k<l\leq r} \left\langle R^M(X, e_\alpha)e_{2m+k}, e_{2m+l}\right\rangle e_{2m+k}e_{2m+l} \cdot \phi \\ &\quad + \frac{1}{2}\sum_{1\leq k<l\leq r} \Theta_{kl}(X, e_\alpha)\kappa_{r*}(f_k f_l) \cdot \phi + \frac{i}{2} dA(X, e_\alpha)\phi \\ &= \frac{1}{2}\sum_{1\leq i<j\leq 2m} \left\langle R^M(X, e_\alpha)e_i, e_j\right\rangle e_i e_j \cdot \phi \\ &\quad + \frac{1}{2}\sum_{1\leq k<l\leq r} \left\langle R^M(X, e_\alpha)e_{2m+k}, e_{2m+l}\right\rangle \kappa_{r*}(f_k f_l) \cdot \phi \end{aligned}$$

$$+\frac{1}{2}\sum_{1\le k<l\le r}\Theta_{kl}(X,e_\alpha)\kappa_{r*}(f_kf_l)\cdot\phi+\frac{i}{2}dA(X,e_\alpha)\phi,$$

where Θ_{kl} denote the local curvature 2-forms of the auxiliary connection on $P_{SO(r)}$. Multiply by e_α and sum over α, $1\le\alpha\le 2m$,

$$\begin{aligned}
0 &= \sum_{\alpha=1}^{2m}\sum_{1\le i<j\le 2m}\left\langle R^M(X,e_\alpha)e_i,e_j\right\rangle e_\alpha e_i e_j\cdot\phi\\
&\quad+\sum_{\alpha=1}^{2m}\sum_{1\le k<l\le r}\left\langle R^M(X,e_\alpha)e_{2m+k},e_{2m+l}\right\rangle e_\alpha\cdot\kappa_{r*}(f_kf_l)\cdot\phi\\
&\quad+\sum_{\alpha=1}^{2m}\sum_{1\le k<l\le r}\Theta_{kl}(X,e_\alpha)e_\alpha\cdot\kappa_{r*}(f_kf_l)\cdot\phi+i\sum_{\alpha=1}^{2m}dA(X,e_\alpha)e_\alpha\cdot\phi\\
&= -\mathrm{Ric}^{V^\phi}(X)\cdot\phi+\sum_{\alpha=1}^{2m}\sum_{1\le k<l\le r}\left\langle R^M(X,e_\alpha)e_{2m+k},e_{2m+l}\right\rangle e_\alpha\cdot\kappa_{r*}(f_kf_l)\cdot\phi\\
&\quad+\sum_{\alpha=1}^{2m}\sum_{1\le k<l\le r}\Theta_{kl}(X,e_\alpha)e_\alpha\cdot\kappa_{r*}(f_kf_l)\cdot\phi+i\sum_{\alpha=1}^{2m}dA(X,e_\alpha)e_\alpha\cdot\phi.
\end{aligned}$$

By taking the real part of the hermitian inner product with $e_i\cdot\phi$, $1\le i\le 2m$,

$$\begin{aligned}
\mathrm{Re}\left\langle\mathrm{Ric}^{V^\phi}(e_j)\cdot\phi,e_i\cdot\phi\right\rangle &= \left\langle\mathrm{Ric}^{V^\phi}(e_j),e_i\right\rangle|\phi|^2\\
&= \mathrm{Ric}^{V^\phi}_{ij},
\end{aligned}$$

since $|\phi|=1$, where now $1\le j\le 2m$. On the other hand,

$$\begin{aligned}
&\mathrm{Re}\left\langle\mathrm{Ric}^{V^\phi}(e_j)\cdot\phi,e_i\cdot\phi\right\rangle\\
&=\mathrm{Re}\left\langle\sum_{\alpha=1}^{2m}\sum_{1\le k<l\le r}\left\langle R^M(e_j,e_\alpha)e_{2m+k},e_{2m+l}\right\rangle e_\alpha\cdot\kappa_{r*}(f_kf_l)\cdot\phi,e_i\cdot\phi\right\rangle\\
&+\mathrm{Re}\left\langle\sum_{\alpha=1}^{2m}\sum_{1\le k<l\le r}\Theta_{kl}(e_j,e_\alpha)e_\alpha\cdot\kappa_{r*}(f_kf_l)\cdot\phi,e_i\cdot\phi\right\rangle\\
&+\mathrm{Re}\left\langle i\sum_{\alpha=1}^{2m}dA(e_j,e_\alpha)e_\alpha\cdot\phi,e_i\cdot\phi\right\rangle\\
&=\sum_{\alpha=1}^{2m}dA(e_j,e_\alpha)\mathrm{Re}\left\langle ie_\alpha\cdot\phi,e_i\cdot\phi\right\rangle\\
&=-\sum_{\alpha=1}^{2m}dA(e_j,e_\alpha)\mathrm{Re}\left\langle J^\phi(e_\alpha)\cdot\phi,e_i\cdot\phi\right\rangle\\
&=\sum_{\alpha=1}^{2m}dA(e_j,e_\alpha)\left\langle J^\phi(e_\alpha),e_i\right\rangle|\phi|^2\\
&=\sum_{\alpha=1}^{2m}dA(e_j,e_\alpha)J^\phi_{i\alpha}.
\end{aligned}$$

Thus

$$\mathrm{Ric}^{V^\phi} = \left[\mathrm{proj}_{V^\phi} \circ \widehat{dA}|_{V^\phi}\right] \circ J^\phi,$$

where $\widehat{dA}$ denotes the skew-symmetric endomorphism determined by dA and metric dualization. □

Remark. On each Kähler leaf, the spinor ϕ restricts to a parallel pure Spin^c spinor field.

4.1.5 $D^\perp$-parallel Partially Pure Spinor

The following theorem is motivated by the condition

$$\nabla^S_u \phi = \lambda u \cdot \phi$$

for all $u \in \Gamma(D^\perp)$, $\lambda \in \mathbb{R}$, i.e. ϕ being a real $D^\perp$-Killing spinor.

Theorem 4.1.5. *Let M be an oriented n-dimensional Riemannian manifold. The following are equivalent:*

(i) *M admits a twisted Spin$^{c,r}(n)$ structure and a twisted spinor bundle S carrying a partially pure spinor field $\phi \in \Gamma(S)$ satisfying*

$$(Y - iJ^\phi(Y)) \cdot \nabla^S_u \phi = 0$$

for every $Y \in \Gamma(V^\phi)$ and $u \in \Gamma((V^\phi)^\perp)$, where ∇^S is the covariant derivative described in subsection 4.1.2.

(ii) *M is endowed with an almost CR hermitian structure of codimension r, where D and J are $D^\perp$-parallel. (In particular, the integral foliation of $D^\perp$ is totally geodesic.)*

Proof. Let us assume (i). For $X \in \Gamma(V^\phi)$,

$$X \cdot \phi = iJ^\phi X \cdot \phi$$

Differentiate with respect to $u \in \Gamma((V^\phi)^\perp)$

$$\nabla_u X \cdot \phi + X \cdot \nabla^S_u \phi = i\nabla_u(J^\phi X) \cdot \phi + iJ^\phi X \cdot \nabla^S_u \phi,$$

so that

$$\nabla_u X \cdot \phi = i\nabla_u(J^\phi X) \cdot \phi.$$

Since ϕ is a partially pure spinor

$$\begin{aligned} \nabla_u X &\in V^\phi \\ \nabla_u(J^\phi X) &= J^\phi(\nabla_u X), \end{aligned}$$

i.e. D and J are $D^\perp$ parallel, and so is $D^\perp$.

Conversely, if M admits a CR hermitian structure, by Theorem 4.1.1, M admits a twisted Spin structure, a twisted spinor bundle S endowed with a connection ∇^S as described in subsection 4.1.2, and a partially pure spinor field $\phi \in \Gamma(S)$ such that $V^\phi = D$, $J^\phi = J$ and

$$(X - iJX) \cdot \phi = 0$$

for every $X \in \Gamma(D)$. Let $u \in \Gamma(D^\perp)$ and differentiate

$$X \cdot \phi = iJX \cdot \phi$$

so that

$$\nabla_u X \cdot \phi + X \cdot \nabla^S_u \phi = i\nabla_u(JX) \cdot \phi + iJX \cdot \nabla^S_u \phi.$$

Since J is $D^\perp$-parallel

$$\nabla_u(JX) = J(\nabla_u X),$$

and

$$X \cdot \nabla^S_u \phi = iJX \cdot \nabla^S_u \phi.$$

i.e.

$$(X - iJX) \cdot \nabla^S_u \phi = 0.$$

□

Example 4.1.3. *The almost CR distribution D_7 on $\mathcal{G}_{m,1,r}$ gives the following example for Theorem 4.1.5*

$$\begin{array}{ccc} S^r & \hookrightarrow & \frac{SO(2m+r+1)}{U(m)\times SO(r)} \\ & & \downarrow \\ & & \frac{SO(2m+r+1)}{U(m)\times SO(r+1)}, \end{array}$$

Remark. A generalized $D^\perp$-Killing partially pure spinor field ϕ is a spinor such that

$$\nabla^S_u \phi = E(u) \cdot \phi,$$

where E is a symmetric endomorphism of $D^\perp$. Such a spinor also satisfies the hypotheses of Theorem 4.1.5.

From Theorems 4.1.3 and 4.1.5 we obtain the following.

Corollary 4.1.1. *Let M be an oriented n-dimensional Riemannian manifold. The following are equivalent:*

(i) *M is locally the Riemannian product of a Kähler manifold and a Riemannian manifold.*

(ii) *M admits a twisted $Spin^{c,r}(n)$ structure and a twisted spinor bundle S carrying a partially pure spinor field $\phi \in \Gamma(S)$ satisfying*

$$(Y - iJ^\phi(Y)) \cdot \nabla^S_Z \phi = 0$$

for every $Y \in \Gamma(V^\phi)$ and $Z \in \Gamma(TM)$, where ∇^S is the covariant derivative described in subsection 4.1.2.

□

In the case of a real $D^\perp$-Killing partially pure spinor, we can say a little more about the foliation leaves' curvature.

Theorem 4.1.6. *Let M be a $Spin^{c,r}$ n-dimensional Riemannian manifold such that its twisted spinor bundle S admits a partially pure spinor field $\phi \in \Gamma(S)$ satisfying*

$$\nabla^S_u \phi = \mu u \cdot \phi$$

for every $u \in \Gamma((V^\phi)^\perp)$, where ∇^S is the connection described in subsection 4.1.2 and $\mu \in \mathbb{R}$, i.e. ϕ is real Killing in the directions of $(V^\phi)^\perp$. Then

- *The Ricci tensor decomposes as follows*

$$\mathrm{Ric}^{(V^\phi)^\perp} = 4(r-1)\mu^2 \mathrm{Id}_{(V^\phi)^\perp} + \sum_{1\leq k<l\leq r} \left[\mathrm{proj}_{(V^\phi)^\perp} \circ \hat{\Theta}_{kl}|_{(V\phi)^\perp}\right] \circ \hat{\eta}^\phi_{kl},$$

  where Θ_{kl} denote the local curvature 2-forms corresponding to the auxiliary connection on the $SO(r)$ principal bundle.

- *The scalar curvature of each leaf tangent to $(V^\phi)^\perp$ is given by*

$$\mathrm{R}^{(V^\phi)^\perp} = 4r(r-1)\mu^2 + \sum_{1\leq k<l\leq r} \mathbf{tr}\left(\left[\mathrm{proj}_{(V^\phi)^\perp} \circ \hat{\Theta}_{kl}|_{(V\phi)^\perp}\right] \circ \hat{\eta}^\phi_{kl}\right).$$

- *If*

$$\sum_{1\leq k<l\leq r} \left[\mathrm{proj}_{(V^\phi)^\perp} \circ \hat{\Theta}_{kl}|_{(V\phi)^\perp}\right] \circ \hat{\eta}^\phi_{kl} = \lambda \mathrm{Id}_{(V^\phi)^\perp}$$

  along a leaf of the foliation tangent to $(V^\phi)^\perp$ for some constant $\lambda \in \mathbb{R}$, then the leaf is Einstein.

Proof. Since ϕ is partially pure, $n = 2m + r$ where $\mathrm{rank}(V^\phi) = 2m$ and $\mathrm{rank}((V^\phi)^\perp) = r$. Let $(e_1, \ldots, e_n)$ and $(f_1, \ldots f_r)$ be local orthonormal frames of TM and F respectively, such that

$$\begin{aligned} V^\phi &= \mathrm{span}(e_1, \ldots, e_{2m}), \\ e_{2j} &= J^\phi(e_{2j-1}), \\ (V^\phi)^\perp &= \mathrm{span}(e_{2m+1}, \ldots, e_{2m+r}), \\ \eta^\phi_{kl} &= e_{2m+k} \wedge e_{2m+l}, \end{aligned}$$

for $1 \leq j \leq m$ and $1 \leq k < l \leq r$. First, if $u, v \in \Gamma((V^\phi)^\perp)$,

$$R^{\theta,A}(u,v)\phi = \mu^2(v \cdot u - u \cdot v) \cdot \phi$$

Now, for $2m+1 \leq i, j \leq 2m+r$,

$$\sum_{i=2m+1}^{2m+r} e_i \cdot R^{\theta,A}(e_j, e_i)(\phi) = -2(r-1)\mu^2 e_j \cdot \phi.$$

By taking the real part of the hermitian product with $e_t \cdot \phi$ we get

$$\mathrm{Re}\left[-2(r-1)\mu^2 \langle e_j \cdot \phi, e_t \cdot \phi\rangle\right] = -2(r-1)\mu^2 \delta_{jt}.$$

If $u, v \in \Gamma((V^\phi)^\perp)$ then, by Theorem 4.1.5, $[u, v] \in \Gamma((V^\phi)^\perp)$, and

$$R^M(u,v)e_i \in \Gamma(V^\phi) \qquad \text{if } 1 \leq i \leq 2m,$$

$$R^M(u,v)e_i \in \Gamma((V^\phi)^\perp) \qquad \text{if } 2m+1 \le i \le 2m+r.$$

so that

$$\begin{aligned}
\left\langle R^M(u,v)e_i, e_j \right\rangle = 0 &\qquad \text{if } 1 \le i \le 2m,\ 2m+1 \le j \le 2m+r,\\
\left\langle R^M(u,v)e_i, e_j \right\rangle = 0 &\qquad \text{if } 2m+1 \le i \le 2m+r,\ 1 \le j \le 2m.
\end{aligned}$$

Now, if $1 \le \alpha \le r$,

$$\begin{aligned}
R^{\theta,A}(u,e_{2m+\alpha})\phi &= \frac{1}{2}\sum_{1\le i<j\le n} \left\langle R^M(u,e_{2m+\alpha})e_i, e_j \right\rangle e_ie_j\cdot\phi\\
&\quad +\frac{1}{2}\sum_{1\le k<l\le r} \Theta_{kl}(u,e_{2m+\alpha})\kappa_{r*}(f_kf_l)\cdot\phi + \frac{i}{2}dA(u,e_{2m+\alpha})\phi\\
&= \frac{1}{2}\sum_{1\le i<j\le 2m} \left\langle R^M(u,e_{2m+\alpha})e_i, e_j \right\rangle e_ie_j\cdot\phi\\
&\quad +\frac{1}{2}\sum_{1\le k<l\le r} \left\langle R^M(u,e_{2m+\alpha})e_{2m+k}, e_{2m+l} \right\rangle e_{2m+k}e_{2m+l}\cdot\phi\\
&\quad +\frac{1}{2}\sum_{1\le k<l\le r} \Theta_{kl}(u,e_{2m+\alpha})\kappa_{r*}(f_kf_l)\cdot\phi + \frac{i}{2}dA(u,e_{2m+\alpha})\phi,
\end{aligned}$$

where dA denotes the curvature 2-form of the auxiliary connection on the $U(1)$-principal bundle. Multiply by $e_{2m+\alpha}$ and sum over α, $1 \le \alpha \le r$,

$$\begin{aligned}
&\sum_{\alpha=1}^{r} e_{2m+\alpha}\cdot R^{\theta,A}(u,e_{2m+\alpha})\phi\\
&=\sum_{\alpha=1}^{r}\sum_{1\le i<j\le 2m} \left\langle R^M(u,e_{2m+\alpha})e_i, e_j \right\rangle e_{2m+\alpha}\cdot e_ie_j\cdot\phi\\
&+\sum_{\alpha=1}^{r}\sum_{1\le k<l\le r} \left\langle R^M(u,e_{2m+\alpha})e_{2m+k}, e_{2m+l} \right\rangle e_{2m+\alpha}\cdot e_{2m+k}e_{2m+l}\cdot\phi\\
&+\sum_{\alpha=1}^{r}\sum_{1\le k<l\le r} \Theta_{kl}(u,e_{2m+\alpha})e_{2m+\alpha}\cdot\kappa_{r*}(f_kf_l)\cdot\phi\\
&+i\sum_{\alpha=1}^{r} dA(u,e_{2m+\alpha})e_{2m+\alpha}\cdot\phi.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\mathrm{Re}\left\langle \sum_{\alpha=1}^{r} e_{2m+\alpha}\cdot R^{\theta,A}(e_{2m+\gamma},e_{2m+\alpha})(\phi), e_{2m+\beta}\cdot\phi \right\rangle\\
=\ &\mathrm{Re}\left\langle \sum_{\alpha=1}^{r}\sum_{1\le i<j\le 2m} \left\langle R^M(e_{2m+\gamma},e_{2m+\alpha})e_i, e_j \right\rangle e_{2m+\alpha}\cdot e_ie_j\cdot\phi, e_{2m+\beta}\cdot\phi \right\rangle\\
&+\mathrm{Re}\left\langle \sum_{\alpha=1}^{r}\sum_{1\le k<l\le r} \left\langle R^M(e_{2m+\gamma},e_{2m+\alpha})e_{2m+k}, e_{2m+l} \right\rangle e_{2m+\alpha}\cdot e_{2m+k}e_{2m+l}\cdot\phi, e_{2m+\beta}\cdot\phi \right\rangle
\end{aligned}$$

$$+\mathrm{Re}\left\langle \sum_{\alpha=1}^{r}\sum_{1\le k<l\le r}\Theta_{kl}(e_{2m+\gamma},e_{2m+\alpha})e_{2m+\alpha}\cdot\kappa_{r*}(f_kf_l)\cdot\phi, e_{2m+\beta}\cdot\phi\right\rangle$$
$$+\mathrm{Re}\left\langle i\sum_{\alpha=1}^{r} dA(e_{2m+\gamma},e_{2m+\alpha})e_{2m+\alpha}\cdot\phi, e_{2m+\beta}\cdot\phi\right\rangle$$
$$= -\left\langle \mathrm{Ric}^{(V^\phi)^\perp}(e_{2m+\alpha}), e_{2m+\beta}\right\rangle |\phi|^2$$
$$+\sum_{\alpha=1}^{r}\sum_{1\le k<l\le r}\Theta_{kl}(e_{2m+\gamma},e_{2m+\alpha})\mathrm{Re}\left\langle e_{2m+\alpha}\cdot\kappa_{r*}(f_kf_l)\cdot\phi, e_{2m+\beta}\cdot\phi\right\rangle$$
$$= -\mathrm{Ric}^{(V^\phi)^\perp}_{2m+\beta,2m+\gamma}+\sum_{\alpha=1}^{r}\sum_{1\le k<l\le r}\Theta_{kl}(e_{2m+\gamma},e_{2m+\alpha})\eta^\phi_{kl}(e_{2m+\alpha},e_{2m+\beta}),$$

i.e.

$$\mathrm{Ric}^{(V^\phi)^\perp} = 4(r-1)\mu^2\mathrm{Id}_{(V^\phi)^\perp}+\sum_{1\le k<l\le r}\left[\mathrm{proj}_{(V^\phi)^\perp}\circ\hat{\Theta}_{kl}|_{(V^\phi)^\perp}\right]\circ\hat{\eta}^\phi_{kl}.$$

□

4.1.6 CR Foliation with Equidistant Leaves

Theorem 4.1.7. *Let M be a Spin^{c,r} Riemannian manifold such that its twisted spinor bundle S admits a partially pure spinor field $\phi\in\Gamma(S)$. Let V^ϕ denote the almost-CR distribution and $(V^\phi)^\perp$ its orthogonal distribution. If*

$$\begin{aligned}(X-iJ^\phi X)\cdot\nabla^S_u\psi &= 0,\\ (X-iJ^\phi X)\cdot\nabla^S_X\psi &= 0,\end{aligned}$$

for all $X\in\Gamma(V^\phi)$ and $u\in\Gamma((V^\phi)^\perp)$, where ∇^S the covariant derivative described in subsection 4.1.2, then

- *V^ϕ, J^ϕ and $(V^\phi)^\perp$ are $(V^\phi)^\perp$-parallel;*
- *the totally geodesic foliation tangent to $(V^\phi)^\perp$ has equidistant leaves.*

Furthermore, if the complex structure J descends to the space of leaves N at regular points, such a complex structure is nearly-Kähler structure.

Proof. The first statement follows from Theorem 4.1.5. Recall the condition for a foliation to have equidistant leaves [7, Proposition 7]

$$\left\langle\nabla_X u, Y\right\rangle+\left\langle X,\nabla_Y u\right\rangle = 0, \tag{4.7}$$

for every $X,Y\in\Gamma(D)$ and $u\in\Gamma(D^\perp)$. Since $\left\langle u,Y\right\rangle=\left\langle u,X\right\rangle=0$,

$$\left\langle\nabla_X u, Y\right\rangle+\left\langle u,\nabla_X Y\right\rangle = 0,$$

$$\left\langle\nabla_Y u, X\right\rangle+\left\langle u,\nabla_Y X\right\rangle = 0,$$

so that (4.7) becomes

$$\left\langle\nabla_X Y+\nabla_Y X, u\right\rangle = 0.$$

We must prove $\nabla_X Y + \nabla_Y X \in \Gamma(D)$. Taking covariant derivative with respect to Y on

$$X \cdot \psi = iJ(X) \cdot \psi,$$

and with respect to X on

$$Y \cdot \psi = iJ(Y) \cdot \psi.$$

we get

$$\begin{aligned} \nabla_Y X \cdot \psi + X \cdot \nabla^S_Y \psi &= i\nabla_Y (J(X)) \cdot \psi + iJ(X) \cdot \nabla^S_Y \psi, \\ \nabla_X Y \cdot \psi + Y \cdot \nabla^S_X \psi &= i\nabla_X (J(Y)) \cdot \psi + iJ(Y) \cdot \nabla^S_X \psi. \end{aligned}$$

Rearranging terms

$$\begin{aligned} \nabla_Y X \cdot \psi + (X - iJ(X)) \cdot \nabla^S_Y \psi &= i\nabla_Y (J(X)) \cdot \psi, \\ \nabla_X Y \cdot \psi + (Y - iJ(Y)) \cdot \nabla^S_X \psi &= i\nabla_X (J(Y)) \cdot \psi. \end{aligned}$$

Adding up the last two equations and using

$$\begin{aligned} 0 &= ((X+Y) - iJ(X+Y)) \cdot \nabla^S_{X+Y} \psi \\ &= (X - iJ(X)) \cdot \nabla_X \psi + (Y - iJ(Y)) \cdot \nabla^S_Y \psi \\ &\quad + (X - iJ(X)) \cdot \nabla_Y \psi + (Y - iJ(Y)) \cdot \nabla^S_X \psi \\ &= (X - iJ(X)) \cdot \nabla_Y \psi + (Y - iJ(Y)) \cdot \nabla^S_X \psi, \end{aligned}$$

we get

$$(\nabla_Y X + \nabla_X Y) \cdot \psi = i(\nabla_Y (J(X)) + \nabla_X (J(Y))) \cdot \psi.$$

This means that $\nabla_Y X + \nabla_X Y \in \Gamma(D)$, i.e. the foliation has equidistant leaves. Furthermore, we have

$$\nabla_Y (J(X)) + \nabla_X (J(Y)) = J(\nabla_Y X + \nabla_X Y).$$

By setting $X = Y$,

$$\nabla_X (J(X)) = J(\nabla_X X),$$

i.e.

$$(\nabla_X J)(X) = 0.$$

Since the leaves of the foliation are equidistant, by [7, Lemma 19] the leaf space N inherits a Riemannian metric at regular points and the quotient map π is a Riemannian submersion. The Levi-Civita connection at a regular point of N is given by

$$\nabla^*_x y = \pi_*(\nabla_X Y)$$

where $\pi_*(X) = x$ and $\pi_*(Y) = y$.

Finally, if we also have that J descends to a complex structure $\mathcal{J}$ on N in the form $\mathcal{J}(\pi_*(X)) = \pi_*(JX)$ and

$$\begin{aligned} \nabla^*_x(\mathcal{J}(x)) - \mathcal{J}(\nabla^*_x x) &= \pi_* \nabla_X (J(X)) - \mathcal{J}(\pi_*(\nabla_X X)) \\ &= \pi_* \nabla_X (J(X)) - \pi_*(J(\nabla_X X)) \\ &= \pi_*(\nabla_X (J(X)) - J(\nabla_X X)) \\ &= \pi_*(0) \\ &= 0. \end{aligned}$$

i.e. $\mathcal{J}$ is a nearly-Kähler structure at regular points of N. □

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