RANDOM TIME-CHANGED EXTREMAL PROCESSES

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Abstract

The point process $= \{(T_k, X_k) : k \geq 1\}$ we deal here with is assumed Bernoulli point process with independent random vectors X_k in $[0, \infty)^d$ and with random time points T_k in $[0, \infty)$, independent of X. For normalizing we use a regular sequence $\xi_n(t, x) = (\tau_n(t), u_n(x))$ of time-space changes of $[0, \infty)^{1+d}$. We consider the sequence of the associated extremal processes

$$\tilde{Y}_n(t) = \{\vee u_n^{-1}(X_k) : T_k \leq \tau_n(t), \}$$

where the max-operation " $\vee$ " is defined in R^d componentwise. We assume further that there exist a stochastically continuous time process $\theta = \{\theta(t) : t \geq 0\}$, strictly increasing and independent of $\{X_k\}$ and an integer-valued deterministic counting function k on $[0, \infty)$, so that the counting process N of $\mathcal{N}$ has the form $N(s) = k(\theta(s))$ a.s.

In this framework we prove a Functional Transfer Theorem which claims in general that if $\tau_n^{-1} \circ \theta \circ \tau_n \Longrightarrow \Lambda$, where Λ is strictly increasing and stochastically continuous and if $\bigvee_{k=1}^{k(\tau_n(.))} u_n^{-1}(X_k) \Longrightarrow Y(.)$, then $\tilde{Y}_n \Longrightarrow \tilde{Y} = Y \circ \Lambda$ where Y is a self-similar extremal process. We call such limit processes random time-changed or compound. They are stochastically continuous and self-similar with respect to the same one-parameter norming group as Y. We show that the compound process is an extremal process (i.e. a process with independent max-increments) if and only if Λ has independent increments and Y has homogeneous max-increments. At the end we apply random time-changed extremal processes to find a lower bound for the ruin probability in an associated with $\mathcal{N}$ insurance model. We give also a upper bound using an α-stable Levy motion.

Key words: Extremal processes; Weak limit theorems; Ruin probability.

1 Introduction

In this Section we recall some definitions and basic facts used in the paper.

Time-space *Bernoulli point processes* $\mathcal{N} = \{(T_k, X_k) : k \geq 1\}$ were introduced in [2], Section 7. They are point processes defined on a locally compact metric space $\mathcal{S}$ and satisfying the conditions :

i) their mean measure μ is a Radon measure on $\mathcal{S}$ (i.e. finite on compact subsets of $\mathcal{S}$);

ii) they are simple in time: $T_k \neq T_j$ a.s. for $k \neq j$;

iii) restrictions of $\mathcal{N}$ to slices over disjoint time intervals are independent.

Such point processes are important in Extreme Value Theory: In fact, any extremal process $Y : [0, \infty) \to [0, \infty)^d$ is generated by an increasing right continuous curve C, the lower curve of Y (see [2] for details), and a Bernoulli point process $\mathcal{N}$ on $\mathcal{S} = [0, C]^c$ by

$$Y(t) = C(t) \vee \{\vee X_k : T_k \leq t\} \tag{1.1}$$

Here $\{T_k\}$ are distinct random time points and $\{X_k\}$ are independent random vectors in $[0, \infty)^d$. The operation maximum $\vee$ as well as equalities and inequalities in R^d we understand componentwise, and A^c denotes the complement of the set A in $[0, \infty)^{d+1}$.

An extremal process $Y : [0, \infty) \to [0, \infty)^d$ is *max-id* (cf. [1]) if and only if the associated Bernoulli point process is Poisson. In this case there is a simple connection between the df f of Y, where $f(t, x) = P(Y(t) < x)$, and the mean measure μ of $\mathcal{N}$, where $\mu(A) = E\mathcal{N}(A),\ A \subset \mathcal{S}$, namely

$$f(t, x) = \exp\{-\mu([0, t] \times A_x^c)\}, \quad x > C(t), \quad t > 0$$

with $A_x = \{y \in [0, \infty)^d : y < x\}$. Note, $\mu(A) < \infty$ for all A of the form $[0, t] \times A_x^c$ as long as $x > C(t)$, whereas $[0, C]$ is the explosion area of μ.

Let $\mathcal{M}^*([0, \infty))$ be the space of all right continuous increasing functions $y : [0, \infty) \to [0, \infty)^d$, $\quad y(t) < \infty$, $\quad y(t) \to \vec{\infty}$ for $t \to \infty$, $\quad \vec{\infty} = (\infty, ..., \infty)$. So, the sample paths of any extremal process belong to $\mathcal{M}^*$ a.s. Given a sequence of extremal processes $\{Y_n\}$, $\quad Y_n : [0, \infty) \to [0, \infty)^d$, we denote the df of $\{Y_n\}$ and the probability distribution (pd) of $\{Y_n\}$ on $\mathcal{M}^*$ by f_n and π_n, respectively. For fixed $t > 0$, let $F_{n,t}(.) := f_n(t, .)$. We say the sequence $\{Y_n\}$ is *weakly convergent* to an extremal process $Y : [0, \infty) \to [0, \infty)^d$ with df f and pd π, briefly $Y_n \Longrightarrow Y$, if one of the following equivalent statements are met ([2], Th.6.1.):

1) $f_n \to f$ at all continuity points of f;

2) $F_{n,t} \to F_t := f(t, .)$ weakly for each t in a dense subset of $(0, \infty)$;

3) $\int \phi d\pi_n \to \int \phi d\pi$ for bounded $\phi : \mathcal{M}^*([0, \infty)) \to R$ which are continuous in the weak topology of $\mathcal{M}^*$.

Recall that the univariate marginals of an extremal process determine its finite dimensional distributions. If additionally to $Y_n \stackrel{fdd}{\longrightarrow} Y$ we assume that the limit extremal process Y is stochastically continuous, then $Y_n \Longrightarrow Y$ also in the Skorohod topology of $\mathcal{M}^*$ (e.g.[3], Th.3).

A mapping $\xi(t,x) = (\tau(t), u(x)), \quad t \in [0,\infty), x \in [0,\infty)^d$, strictly increasing and continuous in each coordinate is called *time-space change* of $[0,\infty)^{1+d}$. An increasing in n sequence of time-space changes $\{\xi_n\}$ is refered to as *regular* if for any $s > 0$ there exists a time-space change $\eta_s(t,x) = (\sigma_s(t), U_s(x))$ so that

$$\begin{aligned} \tau_n^{-1} \circ \tau_{[ns]}(t) &\rightarrow \sigma_s(t) \\ u_n^{-1} \circ u_{[ns]}(x) &\rightarrow U_s(x) \end{aligned}$$

pointwise and the correspondence $s \leftrightarrow \eta_s$ is one-to-one. Then the family $\mathcal{L} = \{\eta_s : s > 0\}$ forms a continuous one-parameter group w.r.t. composition (cf. [9] and also [10] for details).

Consider the following model (A): let $X : [0,\infty) \rightarrow [0,\infty)^d$ be an extremal process with lower curve C and associated Bernoulli point process $\mathcal{N} = \{(t_k, X_k) : k \geq 1\}$, t_k distinct and non-random, X_k independent with df which does not have defect at $+\infty$. We assume that there is a regular norming sequence $\xi_n(t,x) = (\tau_n(t), u_n(x))$ of time-space changes of $[0,\infty)^{1+d}$ so that the sequence of extremal processes

$$\begin{aligned} Y_n(t) &:= u_n^{-1} \circ X \circ \tau_n(t) \\ &= C_n(t) \vee \{\vee u_n^{-1}(X_k) : t_k \leq \tau_n(t)\} \end{aligned}$$

is weakly convergent to a non-degenerate extremal process Y with df f and $Y(0) = 0$ a.s. Then the limit extremal process is stochastically continuous for all $t \geq 0$ and self-similar w.r.t. $\mathcal{L}$, i.e. Y satisfies

$$U_s \circ Y(t) \stackrel{d}{=} Y \circ \sigma_s(t), \quad \forall s > 0 \tag{1.2}$$

or equivalently

$$f(t,x) = f(\sigma_s(t), U_s(x)), \quad \forall s > 0.$$

The paper [9] is devoted to studying the properties of self-similar extremal processes. One of them is the fact that the univariate marginals of Y are max-selfdecomposable. If additionally the initial extremal process X has homogeneous max-increments, then the limit process Y is max-stable. Another important property of a self-similar extremal process is that its lower curve is continuous.

As known, the self-similar extremal processes form a special subclass of the semi-selfsimilar extremal processes studied in [10]. The latter processes satisfy characteristic equation (1.2) for only one fixed $s_0 \in (0,\infty)$ rather than for all $s > 0$.

In the present paper we change the previous model (A) by the model (B) where we assume that the time points T_k of a given Bernoulli point process $\tilde{\mathcal{N}} = \{(T_k, X_k) : k \geq 1\}$ are random variables in $[0,\infty)$ and the space points X_k are iid random vectors in $[0,\infty)^d$. Note, since X_k are iid, the lower curve $C(t)$ of the associated with $\tilde{\mathcal{N}}$ extremal process

$$X(t) = C(t) \vee \{\vee X_k : T_k \leq t\}$$

is constant. Hence we may and do assume $C(t) \equiv 0$. Consequently, $\forall n \geq 1$ the lower curve C_n of $\tilde{Y}_n(t) = u_n^{-1} \circ X \circ \tau_n(t)$ is zero too. In Section 2 we are concerned with the questions:

1. Under which conditions on the Bernoulli point process $\tilde{\mathcal{N}}$ is the sequence of the associated extremal processes

$$\tilde{Y}_n(t) = \{\vee u_n^{-1}(X_k) : T_k \leq \tau_n(t)\}$$

weakly convergent to a non-degenerate process $\tilde{Y}$?

2. Which class does $\tilde{Y}$ belong to ?

We assume further that there exists a stochastically continuous time process $\theta(t), t \geq 0$, independent of $\{X_k\}$, such that the counting process N of $\tilde{\mathcal{N}}$, $\quad N(t) = \sum_k I_{[0,t]}(T_k)$, is of the form $k(\theta(t))$. Here I_A is the indicator of the set A and $k(t)$ is a deterministic counting function. In this framework our main result in Section 2 is the Functional Transfer Theorem. It claims in general that if $Y_n \Longrightarrow Y$ and $\tau_n^{-1} \circ \theta \circ \tau_n \Longrightarrow \Lambda$ in $\mathcal{M}([0,\infty))$, then the limit process $\tilde{Y}$ is of the form $\tilde{Y} = Y \circ \Lambda$. Here Y is a self-similar extremal process and $\Lambda : [0,\infty) \to [0,\infty)$ is a stochastically continuous time process independent of Y and with a.s. strictly increasing sample paths. We call such processes $\tilde{Y}$ *random time-changed* or *compound* self-similar extremal processes. We study their properties in Section 3. In Section 4 we use the compound extremal process to find a lower bound for the ruin probability in a particular insurance model. Furthermore, using a similar technics as in Furrer H., Michna Zb. and Weron A. (1997), we find an upper bound too.

Another aspects of random time-changed extremal processes can be found in S.Satheesh (2002).

2 Compound Extremal Process as Limiting

Let us denote by $\mathcal{M}([0,\infty))$ the set of all strictly increasing right-continuous functions $\tau : [0,\infty) \to [0,\infty)$, $\tau(0) = 0$, $\tau(u) \to \infty$ for $u \to \infty$.

In this section we consider the following model (B): The point process $\tilde{\mathcal{N}} = \{(T_k, X_k) : k \geq 1\}$ we deal with is assumed to be Bernoulli with iid r.v's

X_k in $[0,\infty)^d$ and with random time points T_k in $[0,\infty)$, independent of X. We suppose the latter to be ordered, $T_1 < T_2 < ...,\quad T_k \to \infty$ a.s., and defined on the same probability space $[\Omega, \mathcal{A}, \mathcal{P}]$ as X. Denote by $N(t)$ the counting function of $\tilde{\mathcal{N}}$. We assume further that there exist a stochastically continuous time process $\theta : [0,\infty) \to [0,\infty)$ with sample paths in $\mathcal{M}([0,\infty))$ and a deterministic counting function k on $[0,\infty)$ such that for $s > 0$ and a.a. $\omega \in \Omega$ it holds

$$N(\omega, s) = k(\theta(\omega, s)) \tag{2.1}$$

Let $\{t_k : k \geq 1\},\ t_1 < t_2 < ...$, be the non-random time point process whose counting function $k(t) = \sum_k I_{[0,t]}(t_k)$ coincides with the function $k(.)$ of (2.1). Then

$$N(t) = \sum_k I_{[0,t]}(T_k) \overset{a.s.}{=} \sum_k I_{[0,\theta(t)]}(t_k) = k(\theta(t)) \tag{2.2}$$

and

$$P(N(t) \geq k) = P(T_k \leq t) = P(t_k \leq \theta(t)).$$

Example 1. Assume that $\{T_k\}$ is a simple Poisson point process on $[0,\infty)$ with mean measure $E(N(t)) = \lambda t$, $\lambda > 0$. One can interpret T_k as the arrival time of the kth claim X_k in a certain insurance model. Assume further that there is a deterministic counting process $k(t)$ such that the accumulated claim process $S(t) = \sum_{k=1}^{k(t)} X_k$, properly normalized, has a non-degenerate weak limit. Let $\{t_0 = 0,\ t_1, t_2, ...\}$, $t_k \to \infty$, be the point process associated with $k(t)$. We show that there exists a stochastically continuous time process $\theta(t)$ with sample paths in $\mathcal{M}$ such that $N(t) = k(\theta(t))$. Let us denote $Q_t(s) = P(\theta(t) < s)$.

Indeed, since for every $t > 0$

$$P(k(\theta(t)) = n) = \sum_{k=0}^{\infty} P(k(\theta(t)) = n, t_k \leq \theta(t) < t_{k+1})$$

$$= P(t_n \leq \theta(t) < t_{n+1}) = Q_t(t_{n+1}) - Q_t(t_n),$$

we obtain the values of $Q_t(s)$ for $s \in \{t_0,\ t_1, t_2, ...\}$ by the iteration formula

$$Q_t(t_{n+1}) - Q_t(t_n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

For $s \notin \{t_0,\ t_1, t_2, ...\}$ one can interpolate $Q_t(s)$ by preserving the properties required above.

■

Let us use a regular sequence $\xi_n(t,x) = (\tau_n(t), u_n(x))$ of time-space changes of $[0,\infty)^{1+d}$ for normalizing, so that the sequence of the associated with $\tilde{\mathcal{N}}$ extremal processes

$$\tilde{Y}_n(t) = \{\vee u_n^{-1}(X_k) : T_k \le \tau_n(t)\} \tag{2.3}$$

is weakly convergent to a non-degenerate increasing process $\tilde{Y}$. We ask here which class does $\tilde{Y}$ belong to ?

Consider the point process $\mathcal{N} = \{(t_k, X_k) : k \ge 1\}$ associated with $\tilde{\mathcal{N}}$ by (2.1). Using the same norming sequence as in (2.3), we form the sequence of point processes

$$\mathcal{N}_n = \{(\tau_n^{-1}(t_k), u_n^{-1}(X_k)) : k \ge 1\},$$

with counting function

$$k_n(t) = \sum_k I_{[0,t]}(\tau_n^{-1}(t_k)),$$

and

$$\tilde{\mathcal{N}}_n = \{(\tau_n^{-1}(T_k), u_n^{-1}(X_k)) : k \ge 1\}$$

with a random counting function

$$N_n(t) = \sum_k I_{[0,t]}(\tau_n^{-1}(T_k)).$$

Since $P(\tau_n^{-1}(T_k) \le t) = P(\tau_n^{-1}(t_k) \le \tau_n^{-1} \circ \theta \circ \tau_n(t))$ we see that $N_n(t) \stackrel{d}{=} k_n(\theta_n(t))$ with $\theta_n(t) = \tau_n^{-1} \circ \theta \circ \tau_n(t))$.

The extremal process

$$Y_n(t) = \{\vee_k u_n^{-1}(X_k) : \tau_n^{-1}(t_k) \le t\}, \tag{2.4}$$

associated with $\mathcal{N}_n$, and the extremal process $\tilde{Y}_n$ associated with $\tilde{\mathcal{N}}_n$, are connected by the relation $\tilde{Y}_n(t) \stackrel{d}{=} Y_n(\theta_n(t))$. In this way we have reduced the convergence problem of $\tilde{Y}_n$ to both the convergence of Y_n , considered in the previous section, and the convergence of θ_n. To solve it we use the continuity of the composition in the weak topology in $D([0,\infty))$. (See D.Silvestrov and J.Teugels (1998), Theorem 3 and the comments following the theorem; also consult W.Whitt (1980)).

Proposition 2.1 *Let $\{Y_n, n \ge 1\}$ be a sequence of extremal processes weakly convergent to a stochastically continuous extremal process Y. Let $\{\theta_n, n \ge 1\}$ and Λ be processes with sample paths in $\mathcal{M}([0,\infty))$ such that $\theta_n \Longrightarrow \Lambda$. Assume that Y_n is independent of θ_n for all n. Then*

$$Y_n \circ \theta_n \Longrightarrow Y \circ \Lambda\ .$$

Proof :

For $y \in \mathcal{M}([0,\infty))$ let $D(y) := \{t \geq 0 : y(t-0) \neq y(t)\}$. Since Λ is strictly increasing and Y is stochastically continuous, $D(Y) \cap D(\Lambda) = \emptyset$ a.s. Hence by Theorem 3 in Silvestrov and Teugels (1998),

$$\{\tilde{Y_n}(t) = Y_n(\theta_n(t)) : t > 0\} \Longrightarrow \{\tilde{Y}(t) = Y(\Lambda(t)) : t > 0\} .$$

■

At the end of this Section we consider the iid case and give a Functional Transfer Theorem (FTT) in analogy to the famous Gnedenko's Transfer Theorem (cf. B. V. Gnedenko and Kh. Fakhim (1969), B. V. Gnedenko and D. B. Gnedenko (1982)). But let us first see the meaning of weak convergence $\theta_n = \tau_n^{-1} \circ \theta \circ \tau_n \Longrightarrow \Lambda$ by considering three examples.

Example 2. Let $\tau_n(t) = nt$ and $\theta(t) = ct^a, 0 < a < 1$. Then $\theta_n = ct^a/n^{1-a} \to 0, n \to \infty$.

Example 3. Let $\tau_n(t) = nt$ and $\theta(t) = e^t$. Then $\theta_n(t) = e^{nt}/n \to \infty, n \to \infty$.

Example 4. Let θ possess the scaling property: for all $t > 0$ there exists a subsequence $m_n = [nt]$ such that for $n \to \infty$

$$\theta \circ \tau_n(t) \sim \tau_{[nt]} \circ \theta(1) . \tag{2.5}$$

Then, using the regularity of $\{\tau_n\}$ (i.e. $\tau_n^{-1} \circ \tau_{[nt]}$ converges pointwise to a continuous strictly increasing mapping $\sigma_t : [0,\infty) \to [0,\infty)$) we get

$$\tau_n^{-1} \circ \theta \circ \tau_n(t) \sim \tau_n^{-1} \circ \tau_{[nt]} \circ \theta(1) \Longrightarrow \sigma_t \circ \theta(1) = \Lambda(t) . \tag{2.6}$$

Obviously in this case $\Lambda(t)$ is continuous and increasing in t but does not have independent increments.

Thus, we see that τ_n and θ must have "comparable" behaviour at infinity in order for Λ to be finite and non-degenerate.

Theorem 2.1 *(FTT) Let $\tilde{\mathcal{N}} = \{(T_k, X_k)\}$ and $\mathcal{N} = \{(t_k, X_k)\}$ be Bernoulli point processes with counting functions $N(t)$ and $k(t)$, resp. Let $\{X_k\}$ be iid r.v.'s and $\xi_n(t,x) = (\tau_n(t), u_n(x))$ be a regular norming sequence. Suppose there is a stochastically continuous time process θ in $\mathcal{M}([0,\infty))$, independent of $\{X_k\}$, satisfying (2.5) and such that $N(t) = k(\theta(t))$. Denote the d.f. of $\theta(1)$ by Q, and set $N(\tau_n(t)) = N_n(t)$ and $k(\tau_n(t)) = k_n(t)$. Assume further the weak convergence as $n \to \infty$*

$$P(\bigvee_{k=1}^{k_n(t)} u_n^{-1}(X_k) < x) \longrightarrow f(t,x) := P(Y(t) < x) .$$

Then there exists a time change $\tau(s)$ such that

$$P(\bigvee_{k=1}^{N_n(t)} u_n^{-1}(X_k) < x) \longrightarrow \int_0^\infty f^{\tau(s)}(1,x)dQ(\sigma_t^{-1}(s))$$

weakly as $n \to \infty$. Here σ_t is the time change from (2.6).

Proof: Let Y_n be the extremal process associated with

$$\mathcal{N}_n = \{(\tau_n^{-1}(t_k), u_n^{-1}(X_k)) : k \geq 1\}$$

and let F be the d.f. of X_1. By assumption

$$f_n(t,x) := P(Y_n(t) < x) = P(\bigvee_{k=1}^{k_n(t)} X_k < u_n(x)) = F^{k_n(t)}(u_n(x)) \longrightarrow f(t,x).$$

On the other hand, since

$$f_n(t,x) = \left[F^{k_n(1)}(u_n(x))\right]^{\frac{k_n(t)}{k_n(1)}}$$

and since $f(1,x) \in (0,1)$, we conclude that for all $t > 0$ there exists (perhaps up to a subsequence)

$$\lim_n \frac{k_n(t)}{k_n(1)} =: \tau(t) \; ; \tag{2.7}$$

hence $f(t,x) = f^{\tau(t)}(1,x)$. The limit extremal process Y is stochastically continuous and one can see that the conditions of Proposition 2.1 are satisfied. Applying Proposition 2.1 and (2.5) we have

$$\begin{aligned} P(Y_n(\theta_n(t)) < x) \quad &\longrightarrow \quad P(Y(\Lambda(t)) < x) = \\ \int_0^\infty f(s,x)dP(\Lambda(t) < s) \quad &= \quad \int_0^\infty f^{\tau(s)}(1,x)dQ(\sigma_t^{-1}(s)). \end{aligned}$$

■

Remark 1. Note that (2.7) is a consequence of the weak convergence $Y_n \Longrightarrow Y$. If we do suppose (2.7), then we need only assume $Y_n(1) \xrightarrow{d} Y(1)$ instead of $Y_n \Longrightarrow Y$ in order to get $\tilde{Y}_n \Longrightarrow \tilde{Y}$.

Remark 2. From the proof one can see that FTT remains true if condition (2.5) is replaced by the more general convergence condition $\theta_n \Longrightarrow \Lambda$ of Proposition 2.1. In this case

$$P(\bigvee_{1}^{N_n(t)} u_n^{-1}(X_k) < x) \longrightarrow \int_0^\infty f^{\tau(s)}(1,x)dQ_t(s) \; , \tag{2.8}$$

where $Q_t(s) := P(\Lambda(t) < s)$.

Remark 3. One can see also that the FTT remains true if X_k are assumed independent but not identically distributed. Then

$$P(\bigvee_{1}^{N_n(t)} u_n^{-1}(X_k) < x) \longrightarrow \int_0^\infty f(s,x)dQ_t(s).$$

Corollary 1. Let $\tau_n^{-1} \circ \theta \circ \tau_n \Longrightarrow \Lambda$. Then the time process Λ satisfies $\sigma_s \circ \Lambda \circ \sigma_s^{-1}(t) \stackrel{d}{=} \Lambda(t), \quad \forall s > 0$; hence Λ is stochastically continuous.

Proof:

Let us assume that there is a $t_0 > 0$ with $\Lambda(t_0 - 0) < \Lambda(t_0)$. We can choose $s > 0$ so that $\sigma_s^{-1}(t_0)$ is a continuity point of Λ. Hence

$$\Lambda(t_0) \stackrel{d}{=} \sigma_s \circ \Lambda \circ \sigma_s^{-1}(t_0) \stackrel{a.s.}{=} \sigma_s \circ \Lambda \circ \sigma_s^{-1}(t_0 - 0) \stackrel{d}{=} \Lambda(t_0 - 0)\ ,$$

which contradicts the above assumption.

■

Corollary 2. The limit process $\tilde{Y}$ in (2.8) has the properties :

i) it is stochastically continuous;

ii) it is self-similar wrt $\mathcal{L} = \{(\sigma_s, U_s) : s > 0\}$;

iii) it does not have stationary increments;

iv) it is not max-stable.

Proof:

i) $\tilde{Y}$ is composition of two stochastically continuous processes.

ii) We have to show that $\tilde{Y}(\sigma_s(t)) \stackrel{d}{=} U_s \circ \tilde{Y}(t) \quad \forall s > 0$. For a fixed $s > 0, \quad N_{[ns]}(t) = N(\tau_{[ns]}(t)) = N_n(\tau_n^{-1} \circ \tau_{[ns]}(t))$ and by the FTT and continuity of the composition we conclude the weak convergence

$$\bigvee_{k=1}^{N_{[ns]}(t)} u_n^{-1}(X_k) \Longrightarrow \tilde{Y}(\sigma_s(t)).$$

On the other hand

$$\bigvee_{k=1}^{N_{[ns]}(t)} u_n^{-1}(X_k) = u_n^{-1} \circ u_{[ns]}(\bigvee_{k=1}^{N_{[ns]}(t)} u_{[ns]}^{-1}(X_k)) \Longrightarrow U_s \circ \tilde{Y}(t)$$

which entails the self-similarity of $\tilde{Y}$.

iii) and iv) are easily seen from the RHS of (2.8).

■

In Pancheva (1998), Propositions 2.1 and 2.3 , it is shown that the limit extremal process Y (resp. d.f. f) is self-similar. So we refer to the process $\tilde{Y}(t) = Y(\Lambda(t))$ as random time-changed or compound self-similar extremal process. In the next section we study its properties.

3 Properties of a Compound Extremal Process

In this section we consider the composition $\tilde{Y} = Y \circ \Lambda$, $\quad \tilde{Y} : [0,\infty) \to [0,\infty)^d$, of an extremal process $Y : [0,\infty) \to [0,\infty)^d$, $\quad Y(0) = C(0) = 0$ a.s., and a stochastically continuous time-process $\Lambda : [0,\infty) \to [0,\infty)$ independent of Y and with a.a. sample paths in the functional space $\mathcal{M}([0,\infty))$.

In general, the compound extremal process $Y \circ \Lambda$ may have dependent max-increments, cf. Example 4 of the previous section.

Property 3.1 *Let Y be self-similar extremal process w.r.t. the norming group $\{\eta_s(t,x) = (\sigma_s(t), U_s(x)) : s > 0\}$, i.e. $L_s \circ Y(t) = Y \circ \sigma_s(t)$ and let the random time-change be of the form $\Lambda(t) = \sigma_t(\theta)$, where θ is a positive r.v. Then the compound extremal process is self-similar w.r.t. the group $\{\eta^*_s(t,x) = (ts, U_s(x)) : s > 0\}$.*

Proof. Indeed

$$\begin{aligned} U_s \circ \tilde{Y}(t) &= U_s \circ Y(\Lambda(t)) = Y \circ \sigma_s(\Lambda(t)) = \\ Y \circ \sigma_s(\sigma_t(\theta)) &= Y \circ \sigma_{st}(\theta) = Y \circ \Lambda(st) = \tilde{Y}(st) \ . \end{aligned}$$

■

The first question naturally arising in our framework is: "under what conditions on Λ and Y is the composition $Y \circ \Lambda$ an extremal process in the sense of (1.1)."

Denote by $\mathcal{N}_0 = \{(\Gamma_k, Z_k) : k \geq 1\}$ the Bernoulli point process associated with Y. Let $U_Y(s,t]$ (resp. $U_{\tilde{Y}}(s,t]$) be the max-increment of Y (resp. of $\tilde{Y}$) over a time interval $(s,t]$. Then

$$\begin{aligned} U_{\tilde{Y}}(0,s] &= \{\vee Z_k : 0 < \Gamma_k \leq \Lambda(s)\} \\ U_{\tilde{Y}}(s,t] &= \{\vee Z_k : \Lambda(s) < \Gamma_k \leq \Lambda(t)\} \ . \end{aligned}$$

The intervals $(0,\Lambda(s)]$ and $(\Lambda(s),\Lambda(t)]$ are a.s. disjoint since the time-process Λ has a.s. strictly increasing sample paths.

Theorem 3.1 *Assume that Y is an extremal process and Λ is a time-process independent of Y and with a.a. sample paths in $\mathcal{M}([0,\infty))$. In this framework the compound process $Y \circ \Lambda$ is an extremal process iff*

i) Λ has independent additive increments,

ii) Y has homogeneous max-increments.

Proof.

1. Sufficiency . In view of (1.1) we have to check that $\tilde{Y}$
a) has right continuous increasing sample paths,
b) for arbitrary $s > 0, t > s$, the random vectors $U_{\tilde{Y}}(0, s]$ and $U_{\tilde{Y}}(s, t]$ are independent.

In our framework condition a) is obviously satisfied. Recall that the lower curve of an extremal process with homogeneous increments is constant. Thus we may assume for the lower curve C of Y that $C(t) \equiv 0$.

Let us consider in detail the probability $P(U_{\tilde{Y}}(0, s] < x, U_{\tilde{Y}}(s, t] < y)$ for $0 < s < t$ and arbitrary $x, y \in [0, \infty)^d$. Using step by step the assumptions: Y and Λ are independent, Y has independent max-increments, Λ has independent increments and Y has homogeneous increments, we get :

$$P(U_{\tilde{Y}}(0, s] < x, U_{\tilde{Y}}(s, t] < y)$$

$$\begin{aligned}
&= \iint_{\{(p,q):0<p<q\}} P(U_Y(0, \Lambda(s)] < x, U_Y(\Lambda(s), \Lambda(t)] < y | \Lambda(s) = p, \Lambda(t) = q) \\
&\quad . \quad dP(\Lambda(s) < p, \Lambda(t) < q) \\
&= \iint_{\{(p,q):0<p<q\}} P(U_Y(0, p] < x)P(U_Y(p, q] < y).dP(\Lambda(s) < p, \Lambda(t) < q) \\
&= \int_0^\infty \int_0^\infty P(U_Y(0, p] < x)P(U_Y(p, p + h] < y) \\
&\quad . \quad dP(\Lambda(s) < p)dP(\Lambda(t) - \Lambda(s) < h) \\
&= \int_0^\infty P(U_Y(0, p] < x)dP(\Lambda(s) < p) \int_0^\infty P(U_Y(0, h] < y)dP(\Lambda(t) - \Lambda(s) < h)
\end{aligned}$$

$$\begin{aligned}
&= P(U_{\tilde{Y}}(0, s] < x)P(U_Y(0, \Lambda(t) - \Lambda(s)] < y) \\
&= P(U_{\tilde{Y}}(0, s] < x)P(U_Y(\Lambda(s), \Lambda(t)] < y) \\
&= P(U_{\tilde{Y}}(0, s] < x)P(U_{\tilde{Y}}(s, t] < y)
\end{aligned}$$

2. Necessity. Now we assume that the composition $\tilde{Y} = Y \circ \Lambda$ is an extremal process. Then, necessarily, Λ has independent increments, i.e. $\Lambda(s) \perp \Lambda(t) - \Lambda(s) \quad \forall 0 \le s < t$. We have to show only that Y has homogeneous increments, or equivalently that the counting measure N of the associated Bernoulli point process $\mathcal{N}_0$ is homogeneous.

Indeed, the independence $U_{\tilde{Y}}(0, s] \perp U_{\tilde{Y}}(s, t]$ implies

$$\bigvee_0^{N(\Lambda(s))} Z_k \perp \bigvee_{N(\Lambda(s))+1}^{N(\Lambda(t))} Z_k$$

implies

$$N(0,\Lambda(s)] \perp N(\Lambda(s),\Lambda(s)-(\Lambda(t)-\Lambda(s))].$$

This is possible only if $N(\Lambda(s),\Lambda(t)]$ does not depend on $\Lambda(s)$ but on $\Lambda(t)-\Lambda(s)$ only. For the counting measure N this means that N is homogeneous in the sense that $N(s,t] \stackrel{d}{=} N(0,t-s]$.

■

Let us now simplify the model and assume additionally that Y is a self-similar extremal process with homogeneous max-increments. Then the univariate marginal d.f. $f_t(x) = P(Y(t) < x)$ is max-stable and satisfies

$$f_t(x) = f_1^t(x) = f_1(U_t^{-1}(x)), \ \forall t > 0$$

(cf. Pancheva (1998)). Without loss of generality we may, and do assume, that $f_1(x)$ has Frechet marginals, i.e. $f_1(x_i) = e^{-x_i^{-\alpha}}, \alpha > 0, i = 1,...,d$. So $f_t(x) = \exp\{-t\nu_\alpha(A_x^c)\}$. Here ν_α is the exponent measure of $Y(1)$ and $A_x^c = \{y \in R_+^d : y < x\}^c$. The exponent measure ν_α bears the dependence structure of $Y(1)$, see e.g. S.Resnick (1987). Now observe that

$$s^H Y(t) = Y(st), \ \forall t > 0, \ s > 0, \ H = 1/\alpha,$$

i.e. $Y(t)$ is self-similar w.r.t. the multiplicative group $\{\eta_s(t,x) = (st, s^H x) : s > 0\}$.

Property 3.2 *Denote the Laplace transform of the time process $\Lambda(t)$ by $l_t(r) = E\exp\{-\Lambda(t)r\}, \ r > 0$ and its d.f. by $P(\Lambda(t) < s) = G_t(s)$. The compound extremal process is then distributed by $P(\tilde{Y}(t) < x) = l_t(\nu_\alpha(A_x^c))$.*

Indeed,

$$\begin{aligned} P(\tilde{Y}(t) < x) &= \int_0^\infty P(Y(s) < x | \Lambda(t) = s) dG_t(s) \\ &= \int_0^\infty f_1^s(x) dG_t(s) \\ &= E\exp\{-\Lambda(t).\nu_\alpha(A_x^c)\} = l_t(\nu_\alpha(A_x^c)) \ . \end{aligned}$$

Property 3.3 *Assume $\Lambda(t)$ has independent increments. Then the compound process $\tilde{Y}$ is max-id with mean measure*

$$\tilde{\mu}([0,t] \times A_x^c) = \int_0^\infty \left(1 - e^{-u\nu_\alpha(A_x^c)}\right) d\mathcal{L}_t(u) \quad x > 0 \ ,$$

where $\mathcal{L}_t$ is the Levy measure of $\Lambda(t)$.

Proof. As a stochastically continuous process with independent increments the time process Λ is infinitely divisible. Since $\Lambda(t)$ is positive and increasing, its characteristic function ϕ_t has the form

$$\phi_t(s) = \exp\{\int_0^\infty (e^{isu} - 1) d\mathcal{L}_t(u)\}$$

By Property 3.2, and since $l_t(s) = \phi_t(is)$, we can further write:

$$\begin{aligned} - & \quad \log P(\tilde{Y}(t) < x) = -\log l_t(\nu_\alpha(A_x^c)) = \\ = & \quad -\log \phi_t(-i \log f_1(x)) = \int_0^\infty \left(1 - e^{-u\nu_\alpha(A_x^c)}\right) d\mathcal{L}_t(u) \ . \end{aligned}$$

The measure $\tilde{\mu}$ defined by

$$\tilde{\mu}([0,t] \times A_x^c) := \int_0^\infty \left(1 - e^{-u\nu_\alpha(A_x^c)}\right) d\mathcal{L}_t(u)$$

has all the properties of an exponent measure on $[0,\infty)^{1+d}$ (cf Balkema and Resnick (1977), also Resnick (1987)). Since $P(\tilde{Y}(t) < x) = \exp\{-\tilde{\mu}([0,t] \times A_x^c)\}$, the compound extremal process is max-id.

■

In the case when Λ has homogeneous increments, i.e.

$$\tilde{\mu}([0,t] \times A_x^c) = t \int_0^\infty \left(1 - e^{-u\nu_\alpha(A_x^c)}\right) d\mathcal{L}_1(u) \ ,$$

one gets the following asymptotic for $t \to 0$ and for x far away from 0 as a by-product of the proof:

$$\lim_{t\to 0} \frac{1}{t} P(\tilde{Y}(t) \in A_x^c) \sim \int_0^\infty \left(1 - e^{-u\nu_\alpha(A_x^c)}\right) d\mathcal{L}_1(u) \quad x \to \infty.$$

For any Borel set $B \subset [0,\infty)^d$ and $x \in (0,\infty)^d$ let $\frac{B}{x}$ denote the set $\{s \in [0,\infty) : sx \in B\}$. Define the measure Q_t on $[0,\infty)$ corresponding to the d.f. G_t of $\Lambda(t)$ by $Q_t(A) := \int_0^\infty I_A(s) dG_t(s), \quad A \subset [0,\infty)$ and put $T_H Q_t(A) := Q_t(\{s^H : s \in A\})$. Then the d.f. of the compound process can be expressed as follows (cf. Maejima & Sato & Watanabe (1997)).

Property 3.4 $P(\tilde{Y}(t) < x) = E(T_H Q_t)(\frac{A_x}{Y(1)})$.

Proof. For $x \in (0,\infty)^d$ we have

$$\begin{aligned} P(\tilde{Y}(t) < x) &= P(\Lambda^H(t).Y(1) \in A_x) = \int P(\Lambda^H(t) \in \frac{A_x}{y}) df_1(y) \\ &= \int df_1(y) \int_0^\infty I_{A_x/y}(s^H) G_t(ds) = E(T_H Q_t(\frac{A_x}{Y(1)})) \ , \end{aligned}$$

where the integral is taken over $[0,\infty)^d \setminus \{0\}$.

$\blacksquare$

Remark. By Corollary 1 in the previous section if $\tau_n^{-1} \circ \theta \circ \tau_n \Longrightarrow \Lambda$ then $\sigma_s \circ \Lambda \circ \sigma_s^{-1}(t) \stackrel{d}{=} \Lambda(t), \quad \forall s > 0$. For $\sigma_s^{-1}(t) = st$ the latter equation means

$$\Lambda(t) \stackrel{d}{=} t\Lambda(1) \ . \tag{3.1}$$

In this section we do not assume the above limit relation. Yet (3.1) has to hold if both processes $\tilde{Y}$ and Y are assumed self-similar w.r.t. the same multiplicative group $\mathcal{L} = \{\eta_s(t,x) = (st, s^H x) : s > 0\}$.

Property 3.5 *a) If $\tilde{Y}$ and Y are self-similar w.r.t. the multiplicative group $\mathcal{L}$, then Λ has stationary increments.*

b) If (3.1) holds and if Y is self-similar w.r.t. $\mathcal{L}$, then $\tilde{Y}$ is also self-similar w.r.t. the same $\mathcal{L}$.

Proof. a) Indeed, one can check that both

$$\tilde{Y}(t) \stackrel{d}{=} Y(\Lambda(t)) \stackrel{d}{=} \Lambda^H(t).Y(1) \quad \text{and} \quad \tilde{Y}(t) \stackrel{d}{=} t^H.\tilde{Y}(1)$$

entail (3.1), or equivalently

$$\Lambda(t+h) \stackrel{d}{=} \Lambda(t) + \Lambda(h) \ ,$$

i.e. Λ has stationary increments.

b) Indeed,

$$\tilde{Y}(st) = Y(\Lambda(st)) \stackrel{d}{=} Y(s\Lambda(t)) \stackrel{d}{=} s^H.\tilde{Y}(t).$$

$\blacksquare$

Finally, note that the compound extremal process $\tilde{Y}$ considered in this section can be decomposed in a product of two independent random processes, namely

$$\begin{aligned} \tilde{Y}(t) &= Y(\Lambda(t)) = \Lambda^H(t).Y(1) = \left(\Lambda^H(t)t^{-H}\right).\left(t^H Y(1)\right) = \\ &= M(t).Y(t) \ , \end{aligned}$$

where $M(t) := \left(\frac{\Lambda(t)}{t}\right)^H$. So, the stability character of $\tilde{Y}$ is governed by the self-similar process $Y(t)$ and the volatility of $\tilde{Y}$ is borne by the random time $M(t)$.

4 Application to Ruin Probability

The basic Bernoulli point process $\mathcal{N} = \{(T_k, X_k) : k \geq 1\}$, we are dealing with here, can be interpreted as describing a particular insurance model with

a) claim size process: the claim sizes $\{X_k\}$ are positive iid random variables which df F has a regularly varying tail, namely $\bar{F} \in RV_{-\alpha}$, $\alpha \in (0, 1)$;

b) claim times: the claims occur at times $\{T_k\}$, where $T_1 < T_2 < ..., \quad T_k \to \infty$ a.s., and the number of claims in the interval $[0, t]$, $N(t) = \sum_k I_{[0,t]}(T_k)$, satisfies the condition (2.1), i.e. there exists a time process $\theta : [0, \infty) \to [0, \infty)$ such that $N(t) = k(\theta(t))$, k deterministic counting function whose asymptotic property we specify below ;

c) both sequences $\{X_k\}$ and $\{T_k\}$ are independent.

With the point process $\mathcal{N}$ we associate three random processes:

- the accumulated claim process $S(t) = \sum_{k=1}^{N(t)} X_k$;
- the extremal claim process $Y(t) = \bigvee_{k=1}^{N(t)} X_k$;
- the risk process $R(t) = c(t) - S(t)$, where $u := c(0) \geq 0$ is the initial capital and $c(t)$ is the premium income up to time t (hence it is an increasing curve). We assume $c(t)$ right continuous.

In order to estimate the ruin probability in our framework we follow the idea of a stable Levy motion approximation of the risk process $R(t)$ developed in H. Furrer, Zb. Michna and A. Weron (1997). To this end we transform time and space properly and get a sequence of risk processes weakly convergent to a risk process whose accumulated claim process is an α - stable Levy motion.

Let $\xi_n(t, x) = (\tau_n(t), u_n(x))$ be a norming sequence of time-space changes and let $L(.)$ be a slowly varying function. We suppose that $\tau_n(t)$ satisfy the following condition:

d) $\frac{k_n(t)}{k_n} \to t$ with $k_n(t) = k(\tau_n(t))$ and $k_n := k_n(1)$.

The choice of $u_n(x) = k_n^{1/\alpha} L(k_n) x$ is determined by the regularly varying tail of the claim size df F. Denote $\tau_n^{-1}(T_k) =: T_{nk}$ and $u_n^{-1}(X_k) = \frac{X_k}{k_n^{1/\alpha} L(k_n)} =: X_{nk}$. Now, the sequence of point processes

$$\mathcal{N}_n = \{(T_{nk}, X_{nk}) : k \geq 1\}$$

generates associated sequences of:

- the counting functions $N_n(t) = k_n(\theta_n(t))$, where as before $\theta_n = \tau_n^{-1} \circ \theta \circ \tau_n$;
- the accumulated claims $\sum_{k=1}^{N_n(t)} X_{nk} = S_n(\theta_n(t))$. Here $S_n(t)$ is an abbreviation for $\sum_{k=1}^{k_n(t)} X_{nk}$;

- the extremal claims $\bigvee_{k=1}^{N_n(t)} X_{nk} = Y_n(\theta_n(t))$ where $Y_n(t)$ is the extremal process $\bigvee_{k=1}^{k_n(t)} X_{nk}$;
- the risk process $R_n(t) = c^{(n)}(t) - S_n(\theta_n(t))$, where $c^{(n)}(t) = u_n^{-1} \circ c \circ \tau_n(t)$.

In our insurance model described by a) - d) let us assume additionally that

e) $\theta_n \Longrightarrow \Lambda$, Λ in $\mathcal{M}([0,\infty))$.

In fact the last two assumptions are implicit conditions on the claim times process $\{T_k\}$ (through $\theta(.)$), e.g. they imply that

$$\frac{1}{k_n} N_n(t) \longrightarrow \Lambda(t), \quad n \to \infty.$$

The following examples illustrate the impact of d) for the time change $\tau_n(t)$.

Example 5. For large n

$$\begin{array}{lrcll} a) & k(t) & = & [at+b] \quad \text{implies} & \tau_n(t) = \dfrac{nt-b}{a} \ ; \\ b) & k(t) & = & [\log t] \quad \text{implies} & \tau_n(t) = e^{nt} \ ; \\ c) & k(t) & = & [a^t] \quad \text{implies} & \tau_n(t) = \dfrac{\log nt}{\log a} \ . \end{array}$$

Assume that $\sum_{k=1}^{k_n} X_{nk}$ converges to an α-stable rv Z_α. Using the stable FCLT for sum and maxima of positive iid rv one can see that conditions a) - d) imply the convergences

$$S_n(t) = \sum_{k=1}^{k_n(t)} X_{nk} \Longrightarrow Z_\alpha(t), \quad Z_\alpha(1) = Z_\alpha, \tag{4.1}$$

where $Z_\alpha(t)$ is an one-sided α-stable Levy motion, and

$$Y_n(t) = \bigvee_{k=1}^{k_n(t)} X_{nk} \Longrightarrow Y_\alpha(t) \ , \tag{4.2}$$

where the univariate marginals of the limit extremal process are Frechet distributed, i.e.

$$P(Y_\alpha(t) < x) = \Phi_\alpha^t(x) = \exp\{-tx^{-\alpha}\} \quad x \geq 0, \quad t \geq 0.$$

Let us observe that the conditions of Proposition 2.1. are satisfied, hence we conclude

$$Y_n \circ \theta_n \Longrightarrow Y_\alpha \circ \Lambda \qquad S_n \circ \theta_n \Longrightarrow Z_\alpha \circ \Lambda$$

By Proposition 3.21. in Resnick (1987) convergence (4.2) is equivalent to the weak convergence of the associated point processes. Denote the limit Poisson point process by $\mathcal{N}_0$, say $\mathcal{N}_0 = \{(\Gamma_k, Z_k) : k \geq 1\}$. Its mean measure is

$$\mu_0([0,t] \times [x, \infty)) = t(-\log \Phi_\alpha(x)) = tx^{-\alpha} .$$

Moreover, since the time process Λ is independent of the space points $\{Z_k\}$, Λ is independent of Y_α and Z_α.

Let us come back to the sequence of the risk processes $R_n(t)$. To reach the weak convergence

$$R_n(t) = c^{(n)}(t) - S_n(\theta_n(t)) \Longrightarrow c_0(t) - Z_\alpha(\Lambda(t)) =: R_\alpha(t)$$

we need also the asymptotic relation :

f) $c^{(n)} \xrightarrow{w} c_0, \quad n \to \infty, \quad c_0$ increasing curve.

Now we are ready to obtain lower and upper bounds of the ruin probability associated with the limit risk process $R_\alpha(t)$. Below we make use of the self-similarity of Z_α and of the reflection principle proved in H.Furrer, Zb.Michna and A.Weron (1997, Th. 5). Denote $Q_t(s) := P(\Lambda(t) < s)$, $G_\alpha(y) := P(Z_\alpha(1) < y) \quad u_0 := c_0(0)$. Note, the constant $\rho := P(Z_\alpha(1) > 0)$ from [4, Th. 5] is equal to 1 here. We have

$$\begin{aligned}
\psi(u_0, t) &= P(\inf_{0 \leq s \leq t} R_\alpha(s) < 0) \\
&= P(\sup_{0 \leq s \leq t} \{[\sum Z_k : \Gamma_k \leq \Lambda(s)] - c_0(s)\} > 0) \\
&\leq P(\sup_{0 \leq s \leq t} Z_\alpha(\Lambda(s)) > u_0) \leq P(\sup_{0 \leq s \leq \Lambda(t)} Z_\alpha(s) > u_0) \\
&= \int_0^\infty P(\sup_{0 \leq s \leq v} Z_\alpha(s) > u_0) dQ_t(v) \\
&\leq \int_0^\infty P(v^{1/\alpha} Z_\alpha(1) > u_0) dQ_t(v) = P(\Lambda^{1/\alpha}(t) Z_\alpha(1) > u_0) \\
&= \int_0^\infty \bar{Q}_t(\left(\frac{u_0}{y}\right)^\alpha) dG_\alpha(y) =: \bar{\psi}(u_0, t) .
\end{aligned}$$

Here $\bar{Q}_t = 1 - Q_t$. On the other hand

$$\begin{aligned}
\psi(u_0, t) &\geq P(\sup_{0 \leq s \leq t} \{[\bigvee Z_k : \Gamma_k \leq \Lambda(s)] - c_0(s)\} > 0) \\
&\geq P(Y_\alpha \circ \Lambda(t) > c_0(t)) = P(\Lambda^{1/\alpha}(t) Y_\alpha(1) > c_0(t)) \\
&= \int_0^\infty P(\Lambda(t) > \left(\frac{c_0(t)}{x}\right)^\alpha) d\Phi_\alpha(x) \\
&= \int_0^\infty \bar{Q}_t(\left(\frac{c_0(t)}{x}\right)^\alpha) d\Phi_\alpha(x) =: \underline{\psi}(u_0, t) .
\end{aligned}$$

Here we have used again the self-similarity of the extremal process Y_α. Thus, we get finally

$$\underline{\psi}(u_0, t) \leq \psi(u_0, t) \leq \bar{\psi}(u_0, t) \ .$$

Remember, our initial insurance model was described by the point process $\mathcal{N}$ with the associated risk process $R(t)$. Let us denote the corresponding ruin probability by $\Psi(u,t)$ with $u = c(0)$. Then

$$\Psi(u,t) = P(\inf_{0 \leq s \leq t} \{c(s) - \sum_{k=1}^{N(s)} X_k\} < 0) \tag{4.3}$$

$$\begin{aligned} &= P(\inf_{0 \leq s \leq t} \{u_n^{-1} \circ c(s) - \sum_{k=1}^{N(s)} X_{nk}\} < 0) \\ &= P(\inf_{0 \leq s \leq \tau_n^{-1}(t)} \{u_n^{-1} \circ c \circ \tau_n(s) - \sum_{k=1}^{k_n(\theta_n(s))} X_{nk}\} < 0) \ . \end{aligned}$$

Now let the initial capital u and time t increase with $n \to \infty$ in such a way that $\frac{u}{k_n^{1/\alpha} L(k_n)} = u_0, \quad \tau_n^{-1}(t) = t_0$. Recall $c^{(n)}(t) = u_n^{-1} \circ c \circ \tau_n(t)$. Observe that under conditions a) - f) we may approximate

$$\Psi(u,t) \approx \psi(u_0, t_0)$$

and consequently for u and t "large enough"

$$\underline{\psi}(u_0, t_0) \leq \Psi(u,t) \leq \bar{\psi}(u_0, t_0) \ . \tag{4.4}$$

Example 6. Assume our insurance model is characterized by

a) the claim size df $F \in NDA(S_{1/2}(1,1,0))$. This means that $G_\alpha(x) = 2(1 - \Phi(\sqrt{1/x}))$ is the Levy df. Here Φ is the standard normal df. Hence we have to choose $u_n(x) = n^2 x$;

b) the claim times process is determined by $k(t) = [t]$ and $\theta(t) \stackrel{d}{=} t.\theta(1)$ with $\theta(1)$ uniformly distributed in $[0,1]$.

Then we have to choose $\tau_n(t) = nt$ and consequently $Q_t(x) = \frac{x}{t}$. Furthermore, since $c^{(n)}(t) = \frac{1}{n^2} c(nt) \to c_0(t)$ we take $c_0(t) = u_0 + t^2$. Then

$$\underline{\psi}(u_0, t_0) = \int_{\frac{u_0}{t_0^2}+1}^{\infty} (1 - \frac{1}{t_0} \sqrt{\frac{u_0 + t_0^2}{x}}) d(e^{-\frac{1}{\sqrt{x}}}) \tag{4.5}$$

and

$$\bar{\psi}(u_0, t_0) = \int_{\frac{u_0}{t_0^2}}^{\infty} (1 - \frac{1}{t_0} \sqrt{\frac{u_0}{x}}) dG_\alpha(x) \ . \tag{4.6}$$

5 Appendix

Here we give numerical results related to the computation of (4.5) and (4.6) in the last example. Let us consider first (4.5). For the numerical computation, it is necessary to find a sufficiently large constant K_1 such that:

$$\underline{\psi}_\varepsilon(u_0,t_0)=\int_{\frac{u_0}{t_0^2}+1}^{K_1(\varepsilon)}(1-\frac{1}{t_0}\sqrt{\frac{u_0+t_0^2}{x}})d(e^{-\frac{1}{\sqrt{x}}})$$

and $|\underline{\psi}(u_0,t_0)-\underline{\psi}_\varepsilon(u_0,t_0)|<\varepsilon$, where $\varepsilon>0$ is arbitrary small. We start by observing that

$$|\underline{\psi}(u_0,t_0)-\underline{\psi}_\varepsilon(u_0,t_0)|<\int_{K_1}^{\infty}d(e^{-\frac{1}{\sqrt{x}}})=1-e^{-\frac{1}{\sqrt{K_1}}}\ .$$

Hence, if $K_1(\varepsilon)=(1/\ln(1-\varepsilon))^2$, we achieve the desired accuracy. Moreover, since

$$|\underline{\psi}(u_0,t_0)-\underline{\psi}_\varepsilon(u_0,t_0)|>(1-\frac{1}{t_0}\sqrt{\frac{u_0+t_0^2}{K_1}})\int_{K_1}^{\infty}d(e^{-\frac{1}{\sqrt{x}}})\ ,$$

we can find upper and lower bounds of the approximation error:

$$(1-\frac{1}{t_0}\sqrt{\frac{u_0+t_0^2}{K_1(\varepsilon)}})(1-e^{-\frac{1}{\sqrt{K_1(\varepsilon)}}})<|\underline{\psi}(u_0,t_0)-\underline{\psi}_\varepsilon(u_0,t_0)|<1-e^{-\frac{1}{\sqrt{K_1(\varepsilon)}}}\ .$$

Concerning (4.6), we arrive at similar results using the same reasoning and the asymptotic behavior of the tail of α-stable rv X (see e.g. [11]):

$$\lim_{\lambda\to\infty}\lambda^\alpha P(X>\lambda)=C_\alpha\frac{1+\beta}{2}\sigma^\alpha\ .$$

The lower and upper bounds of the approximation error in the particular case of Levy distribution G_α are the following:

$$(1-\frac{1}{t_0}\sqrt{\frac{u_0}{K_2(\varepsilon)}})(1-G_\alpha(K_2(\varepsilon)))<|\bar{\psi}(u_0,t_0)-\bar{\psi}_\varepsilon(u_0,t_0)|<1-G_\alpha(K_2(\varepsilon))\ ,$$

where $K_2(\varepsilon)=(\frac{C_{1/2}}{\varepsilon})^2$ and

$$\bar{\psi}_\varepsilon(u_0,t_0)=\int_{\frac{u_0}{t_0^2}}^{K_2(\varepsilon)}(1-\frac{1}{t_0}\sqrt{\frac{u_0}{x}})dG_\alpha(x)\ .$$

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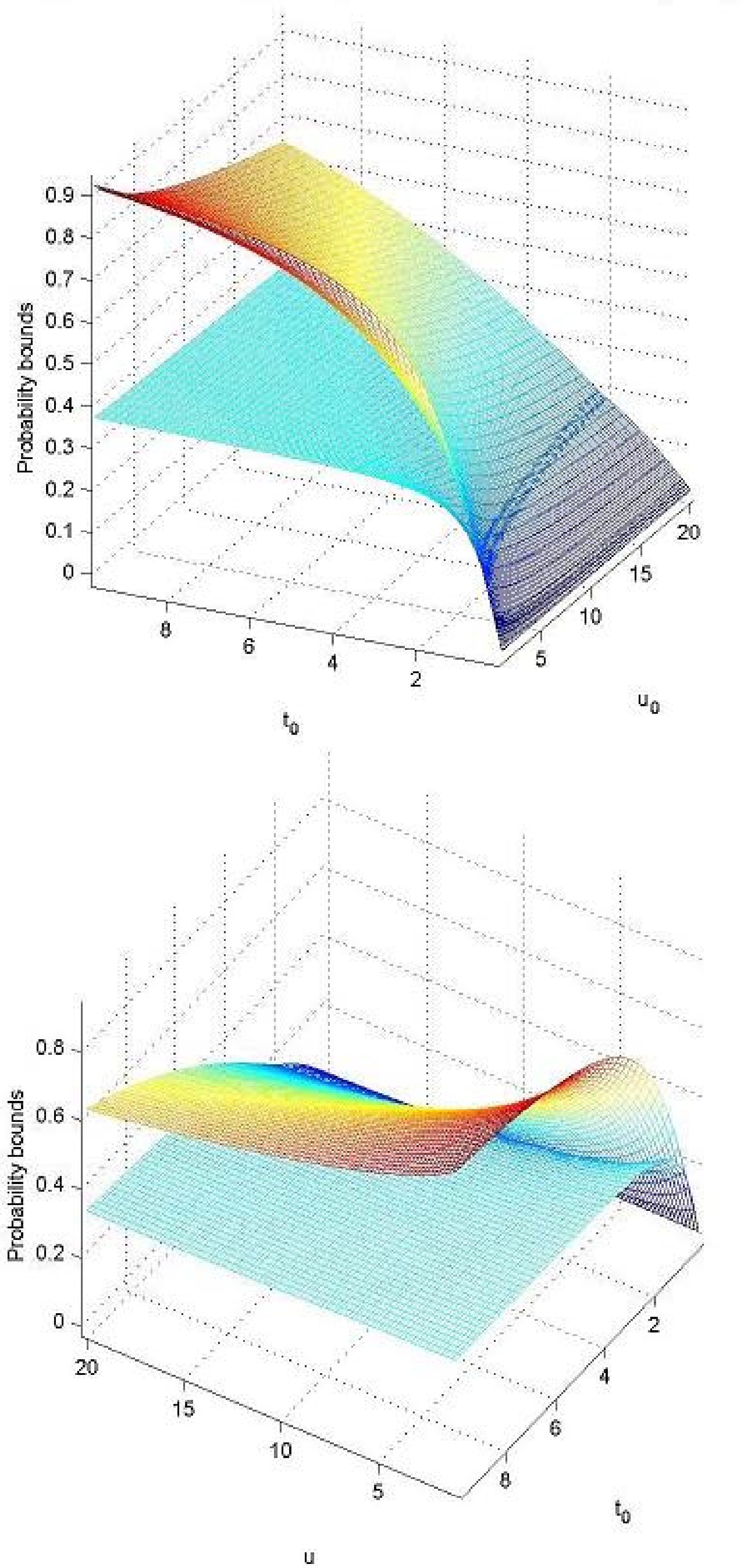
		t_0							
		2.3	2.8	3.3	4.8	5.8	6.8	7.8	8.8
u_0	1	0.3454	0.3522	0.3564	0.3623	0.3640	0.3651	0.3657	0.3662
	1.61	0.3337	0.3436	0.3499	0.3590	0.3617	0.3634	0.3644	0.3652
	2.84	0.3137	0.3283	0.3380	0.3527	0.3573	0.3601	0.3619	0.3631
	4.06	0.2971	0.3151	0.3273	0.3468	0.3530	0.3568	0.3594	0.3612
	5.29	0.2831	0.3034	0.3177	0.3411	0.3489	0.3537	0.3570	0.3592
	6.52	0.2710	0.2931	0.3090	0.3358	0.3449	0.3507	0.3546	0.3573
	7.74	0.2604	0.2838	0.3009	0.3307	0.3411	0.3477	0.3522	0.3554
	8.97	0.2511	0.2754	0.2936	0.3259	0.3374	0.3449	0.3500	0.3536
	10.19	0.2427	0.2677	0.2867	0.3213	0.3339	0.3421	0.3477	0.3517
	11.42	0.2352	0.2607	0.2804	0.3169	0.3304	0.3394	0.3455	0.3500
	12.65	0.2284	0.2543	0.2745	0.3127	0.3271	0.3367	0.3434	0.3482
	13.87	0.2222	0.2483	0.2690	0.3087	0.3239	0.3341	0.3413	0.3465
	15.1	0.2164	0.2428	0.2638	0.3048	0.3208	0.3316	0.3393	0.3448
	16.32	0.2112	0.2376	0.2589	0.3011	0.3178	0.3292	0.3372	0.3431
	17.55	0.2063	0.2328	0.2544	0.2975	0.3148	0.3268	0.3353	0.3415
	18.77	0.2017	0.2283	0.2500	0.2941	0.3120	0.3244	0.3333	0.3398
	20	0.1975	0.2240	0.2459	0.2908	0.3092	0.3222	0.3314	0.3382

Table 1: $\underline{\psi_\epsilon}(u_0, t_0)$, $\epsilon = 10^{-6}$

		t_0							
		2.3	2.8	3.3	4.8	5.8	6.8	7.8	8.8
u_0	1	0.6563	0.7156	0.7583	0.8338	0.8624	0.8827	0.8977	0.9093
	1.61	0.5748	0.6425	0.6941	0.7889	0.8253	0.8510	0.8701	0.8848
	2.84	0.4735	0.5440	0.6023	0.7204	0.7683	0.8023	0.8277	0.8472
	4.06	0.4115	0.4796	0.5386	0.6673	0.7231	0.7635	0.7938	0.8172
	5.29	0.3687	0.4335	0.4912	0.6241	0.6851	0.7304	0.7648	0.7915
	6.52	0.3370	0.3985	0.4543	0.5880	0.6523	0.7014	0.7391	0.7686
	7.74	0.3122	0.3707	0.4246	0.5574	0.6237	0.6755	0.7159	0.7479
	8.97	0.2922	0.3480	0.4000	0.5310	0.5984	0.6521	0.6948	0.7288
	10.19	0.2756	0.3291	0.3792	0.5080	0.5759	0.6310	0.6754	0.7112
	11.42	0.2615	0.3129	0.3613	0.4876	0.5556	0.6117	0.6574	0.6947
	12.65	0.2494	0.2988	0.3458	0.4695	0.5373	0.5940	0.6408	0.6793
	13.87	0.2388	0.2866	0.3321	0.4533	0.5206	0.5776	0.6253	0.6649
	15.1	0.2295	0.2757	0.3198	0.4386	0.5054	0.5626	0.6108	0.6513
	16.32	0.2212	0.2659	0.3089	0.4252	0.4914	0.5486	0.5973	0.6385
	17.55	0.2137	0.2572	0.2990	0.4130	0.4785	0.5356	0.5846	0.6263
	18.77	0.2069	0.2492	0.2900	0.4017	0.4666	0.5234	0.5726	0.6148
	20	0.2008	0.2419	0.2817	0.3914	0.4555	0.5120	0.5613	0.6039

Table 2: $\overline{\psi_\epsilon}(u_0, t_0)$, $\epsilon = 10^{-6}$

Upper and lower default probability bounds for different values of u_0 and t_0



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