A NOTE ON THE INFINITE DIVISIBILITY OF SKEW-SYMMETRIC DISTRIBUTIONS

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Comunicación Técnica No I-04-07/17-08-2004
(PE/CIMAT)

A Note on the Infinite Divisibility of Skew-Symmetric Distributions

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Abstract

Infinite divisibility of some of the most important symmetric distributions skewed by an additive component is investigated. We find in particular that the skew-normal distribution of Azzalini (1985) and the multivariate skew-normal distribution of Azzalini and Dalla Valle (1996) are not infinitely divisible.

1. Introduction

Skew-symmetric distributions have been developed as natural extensions of the skew-normal distribution introduced by Azzalini (1985). The aim of this note is to determine the infinite divisibility of skew-symmetric distributions.

There are several ways of skewing a symmetric distribution; see Arnold and Beaver (2002). Here we only consider those symmetric distributions skewed by an additive component.

Definition 1. *Let $c \geq 0$. A random variable Y is said to have skew-symmetric distribution if there exist constants a and $b \neq 0$; and independent random variables X and X_c such that*

$$Y \stackrel{d}{=} aX + bX_c, \tag{1}$$

where X is symmetric and X_c is a copy of X truncated below at c. For the special case $c = 0$ is said that X_0 has a half-distribution of X.

We discuss the infinite divisibility of a skew-symmetric distribution of the form (1) only when X is infinitely divisible. This leads to the problem of determining the infinite divisibility of X_c.

First we put attention to the case $c = 0$.

The skew-normal distribution of Azzalini has representation (1), see Henze (1986), with X standard normal, $a = 1/\sqrt{1+\delta^2}$, $b = \delta/\sqrt{1+\delta^2}$ and δ is a real skewness parameter. It is important to remark that it is not infinitely divisible due to the half-normal distribution is not infinitely divisible as is noted in Steutel and Van Harn (2003, p. 126).

Immediate examples of infinitely divisible skew-symmetric distributions are skew-Laplace and skew-Cauchy, since the half-Laplace is the exponential distribution and the half-Cauchy is infinitely divisible as is shown in Steutel and Van Harn (2003, p. 411).

2. The case $c > 0$

The skew-t distribution with ν degree of freedom and the skew-double Pareto are infinitely divisible since, in both cases, their corresponding X_c in (1) is infinitely divisible by a log-convexity argument.

Proposition 2. *The skew-Student t with ν degree of freedom is infinitely divisible if $c > \sqrt{\nu}$.*

Proof. We prove that X_c in (1) is infinitely divisible by showing that its density is log-convex, see Sato (1999, Th. 51.4). Let us consider the Student-t density f with ν degree of freedom truncated below at c

$$f(x) = K\left(1 + \frac{1}{\nu}x^2\right)^{-\frac{\nu+1}{2}} 1_{[c,\infty)}(x).$$

where K is the normalizing constant. Let $x > c$. Differentiating twice, we have

$$f'(x) = -K\frac{1+\nu}{\nu}x\left(1+\frac{1}{\nu}x^2\right)^{-\frac{1}{2}\nu-\frac{3}{2}},$$

$$f''(x) = -K\frac{1+\nu}{\nu}\left(\frac{x^2}{\nu}+1\right)^{-\frac{1}{2}\nu-\frac{3}{2}} + K\frac{3+4\nu+\nu^2}{\nu^2}x^2\left(\frac{x^2}{\nu}+1\right)^{-\frac{1}{2}\nu-\frac{5}{2}}$$

and

$$\left[f'f'' - f'^2\right](x) = K^2\frac{(\nu+1)}{\nu^2}\left(x^2-\nu\right)\left(1+\frac{1}{\nu}x^2\right)^{-3-\nu}.$$

Thus $ff'' - (f')^2 > 0$ if $x > \sqrt{\nu}$. Hence f is log-convex. ∎

In particular the skew-Cauchy distribution is infinitely divisible if $c > 1$ since the Student-t is Cauchy when $\nu = 1$. We are not able to give an answer in the truncation range $(0, 1]$ for Cauchy and $[0, \sqrt{\nu}]$ for Student-t.

Proposition 3. *Let $c > 0$. The skew-double Pareto distribution is infinitely divisible.*

Proof. We proceed similarly as the former proof. Consider the double-Pareto density truncated below at c

$$f(x) = K\frac{1}{(1+x)^r}1_{[c,\infty)}(x),$$

where $r > 1$ and K is the normalizing constant. We obtain $[f'f'' - f'^2](x) = K^2\frac{r}{(1+x)^{2(r+1)}} >$ 0 for any $x > c$. ∎

Using a tail behavior criterion for infinite divisibility we show that skew-normal distribution is not infinitely divisible.

Proposition 4. *Let $c > 0$. The skew normal distribution is not infinitely divisible.*

Proof. Let us consider X in model (1) be a standard normal random variable. We prove that X_c is not infinitely divisible by proving that it does not fulfill the necessary condition $-\log P(X_c > x) \leq \alpha x \log x$ for some $\alpha > 0$ and x sufficiently large, cf. Steutel (1979). Consider the standard normal density truncated below at c

$$f(x) = K\phi(x)\, 1_{[c,\infty)}(x),$$

where ϕ is the standard normal density K is the normalizing constant. Let $\Phi(x)$ denote the standard normal distribution. Observe that

$$\lim_{x\to\infty}\frac{-\log P(X_c > x)}{x\log x} = \lim_{x\to\infty}\frac{-\log K[1-\Phi(x)]}{x\log x}$$

and apply L'Hôpital Rule twice to lead the limit to

$$\lim_{x\to\infty}\left[\frac{x}{1+\log x} + \frac{1}{x(1+\log x)^2}\right] = \infty.$$

∎

We finally conclude that the multivariate skew-normal distribution of Azzalini and Dalla Valle (1996) is not infinitely divisible.

Let $Y = (Y_1, ..., Y_p)^T$ be a random vector with coordinates

$$Y_i = a_i X_i + b_i X_0, \quad i = 1, ..., p, \tag{2}$$

where $(X_1, ..., X_p)^T$ is a jointly normal vector independent of the half-normal random variable X_0. Any linear combination $\alpha^T Y$ is of the form

$$\sum_{i=1}^{p} \alpha_i a_i X_i + \left(\sum_{i=1}^{p} \alpha_i b_i \right) X_0,$$

which is not infinitely divisible by Proposition 4 and hence Y so is not.

The representation of the skew-normal random vector in Section 2.1 of Azzalini and Dalla Valle (1996) is a special case of the model (2).

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