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Abstract. In this article we discuss analogues of the Artin and Yang-Baxter equations for dialgebras, obtain explicit solutions for a special class of dialgebras, and study the geometric properties of these solutions.

Introduction.

During the last twenty years or so, non-commutative versions of many classical theories have become increasingly important, both for mathematics and theoretical physics. Among the various constructs in this vein, J. L. Loday (see *e.g.*, [L1]) introduced the notion of a Leibniz algebra, which is a generalization of a Lie algebra, where the skew-symmetry of the bracket is dropped; these algebras have proved to be extremely useful, among other things, because they have a very nice cohomology theory. Loday also showed that the functorial relationship between Lie algebras and associative algebras, translates into an analogous functor between Leibniz algebras and the so-called *dialgebras*, which are a generalization of associative algebras possesing two operations.

On the other hand, the important role of the classical Yang-Baxter equation in the theory of Lie algebras is also well known; thus, in [FLO] we tried to define analogues of the classical Yang-Baxter equations for Leibniz algebras, and showed that one can rightfully state at least two versions of these equations.

In this paper, we use the relationship between Leibniz algebras and dialgebras to rederive one of the Yang-Baxter equations we had obtained in [FLO], namely

$$[T^{23}, T^{13}] + [T^{23}, T^{12}] - [T^{12}, T^{13}] = 0$$

(*cf*. equation (6)). We do this using a quite different approach, through the introduction of suitable versions for dialgebras of the so-called Artin and quantum Yang-Baxter equations.

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(And this certainly supports equation (6) as the "right" analogue for Leibniz algebras of the classical Yang-Baxter equation.)

Somewhat remarkably, we are also able to obtain explicit solutions in a special case, and for a simple but relatively ample class of dialgebras, where, moreover, the solutions exhibit very nice geometrical properties. It would be interesting to understand if this is related to the outstanding problem of integrating Leibniz algebras.

I. Yang-Baxter equations on matrix dialgebras.

Let $(\mathcal{U}, \dashv, \vdash)$ be a dialgebra, that is, a vector space with two bilinear, associative operations, $\dashv$, $\vdash$, satisfying the relations

$$\begin{aligned} x \dashv (y \dashv z) &= x \dashv (y \vdash z) \\ (x \vdash y) \dashv z &= x \vdash (y \dashv z) \\ (x \dashv y) \vdash z &= (x \vdash y) \vdash z. \end{aligned}$$

Then, it is well known that the space $M_n = M_n(\mathcal{U})$ of square $n \times n$ matrices with entries in $\mathcal{U}$, is also a dialgebra, with the operations defined entry-wise, and again denoted $\dashv$, $\vdash$ (*cf.* [L2]).

Also, recall that a dialgebra canonically defines a Leibniz algebra, with bracket

$$[x, y] = x \dashv y - y \vdash x.$$

Now, if we assume that $\mathcal{U}$ posseses a non-trivial bar unit e, *i.e.*, an element satisfying

$$e \vdash x = x = x \dashv e; \quad \forall x \in \mathcal{U};$$

then, since by bilinearity the operations in $\mathcal{U}$ satisfy

$$0 \vdash x = x \vdash 0 = 0 \dashv x = x \dashv 0 = 0,$$

M_n also has a non-trivial bar unit, namely

$$E = \begin{pmatrix} e & 0 & \dots & 0 \\ 0 & e & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & e \end{pmatrix}.$$

(By a "non-trivial bar unit" we mean here a bar unit that is not a unit from the *pointer* side, in which case it is well known that the two operations coincide, and the dialgebra is simply an associative algebra with unit.)

Let us pause for a moment to observe that, since $\mathcal{U}$ has two operations, M_n "acts" naturally on $\mathcal{U}^n$ from the left in two different ways: For $S = (s_{ij}) \in M_n$, and $u^T = (u_1, \dots, u_n) \in \mathcal{U}^n$, we can define:

$$S_{\vdash} u = \begin{pmatrix} \sum_i s_{1i} \vdash u_i \\ \vdots \\ \sum_i s_{ni} \vdash u_i \end{pmatrix}$$

and

$$S_{\dashv}u = \begin{pmatrix} \sum_i s_{1i} \dashv u_i \\ \vdots \\ \sum_i s_{ni} \dashv u_i \end{pmatrix}.$$

We will not have much use of this peculiarity of the operators in $\mathcal{M}_n$; but, in view of the axioms of the bar unit, and as we will shortly see, in what follows it is the former "action" that turns out to be the most interesting one, and in the remainder of this paper we will only consider it, dropping the subscript $\vdash$.

Remark: Evidently, we can similarly define right actions on row vectors; then we would simply interchange $\vdash$ by $\dashv$.

Now, let us derive the Artin and Yang-Baxter equations in this context, which we do following the presentation of [K]. (For simplicity in the writing, we will do the computations for the case $n = 2$; the general case $n = 2k$ would amount to little else than a block decomposition of the matrices.)

To begin with, given a fixed bar unit e, we can consider the matrix of $\vdash$-permutation

$$P = \begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix} \vdash \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} e \vdash u_2 \\ e \vdash u_1 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1 \end{pmatrix}.$$

(We observe at this point that there is no matrix of $\dashv$-permutation for the left actions, since

$$\begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix} \dashv \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} e \dashv u_2 \\ e \dashv u_1 \end{pmatrix} \neq \begin{pmatrix} u_2 \\ u_1 \end{pmatrix};$$

this is why we restrict our attention to the action $\vdash$. However, the same P obviously acts as a matrix of $\dashv$-permutation for the right action.)

Next, the existence of the bar unit determines three special embeddings $M_2 \to M_3$; that is, any $S \in M_2$ defines three matrices in M_3 via

$$S^{12} = \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{21} & s_{22} & 0 \\ 0 & 0 & e \end{pmatrix}; \quad S^{23} = \begin{pmatrix} e & 0 & 0 \\ 0 & s_{11} & s_{12} \\ 0 & s_{21} & s_{22} \end{pmatrix}; \quad S^{13} = \begin{pmatrix} s_{11} & 0 & s_{12} \\ 0 & e & 0 \\ s_{21} & 0 & s_{22} \end{pmatrix}.$$

It is then a simple computation to see that P satisfies the equation

$$P^{23} \vdash P^{12} \dashv P^{23} = P^{12} \vdash P^{23} \dashv P^{12}$$

(in view of the second axiom for dialgebras no parentheses are needed here). This motivates the following:

Definition 1. *Let E be a dialgebra, and e a fixed bar unit in E. The (dialgebra) Artin equation (relative to e) is*

$$S^{23} \vdash S^{12} \dashv S^{23} = S^{12} \vdash S^{23} \dashv S^{12}. \tag{1}$$

Remark. As defined, the Artin equation depends on the bar unit; however, the form of the equation is clearly independent of the choice of the bar unit.

The set of bar units is sometimes called the *halo* of the dialgebra, and, in general, when it exists it is an affine subspace of the dialgebra: Indeed, if we set

$$N_{\vdash} = \{x | x \vdash y = 0 \quad \forall y\} \; ; \quad {}_{\dashv}N = \{x | y \dashv x = 0 \quad \forall y\},$$

and e is a non-trivial bar unit, then the halo is the affine space modelled after $N_{\vdash} \cap {}_{\dashv}N$, and passing through e.

Now, to relate the Artin equation with the Yang-Baxter equations, we then consider perturbations of the known solution P to the Artin equation, and look first for solutions of the form

$$A = P \vdash R \quad \text{or} \quad B = R \dashv P. \tag{2}$$

(Only these types of perturbations need to be considered, since P acts as permutation only on the bar side. And although it is perhaps not clear *a priori* that both cases yield the same results—so one should in principle take both into account, nevertheless, the end result turns out to be the same, so we will consider only the former.)

Now, for A as above one clearly has

$$A^{ij} = P^{ij} \vdash R^{ij} \; ; \quad 1 \leq i < j \leq 3;$$

hence, upon substitution of this in Artin's equation, we get

$$\begin{aligned} &(P^{23} \vdash R^{23}) \vdash (P^{12} \vdash R^{12}) \dashv (P^{23} \vdash R^{23}) \\ &= (P^{12} \vdash R^{12}) \vdash (P^{23} \vdash R^{23}) \dashv (P^{12} \vdash R^{12}). \end{aligned} \tag{3}$$

But, since P is the permutation matrix, the P's and R's satisfy the commutation relations stated in the following lemma:

Lemma 1.

$$\begin{aligned} R^{12} \dashv P^{23} &= P^{23} \vdash R^{13} \; ; \quad & R^{12} \dashv P^{13} &= P^{13} \vdash R^{23} \; ; \\ R^{13} \dashv P^{12} &= P^{12} \vdash R^{23} \; ; \quad & R^{13} \dashv P^{23} &= P^{23} \vdash R^{12} \; ; \\ R^{23} \dashv P^{12} &= P^{12} \vdash R^{13} \; ; \quad & R^{23} \dashv P^{13} &= P^{13} \vdash R^{12}. \end{aligned}$$

Proof. This is again a rather straightforward computation; for the sake of completeness let us just verify the first of these relations:

$$R^{12} \dashv P^{23} = \begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & e \end{pmatrix} \dashv \begin{pmatrix} e & 0 & 0 \\ 0 & 0 & e \\ 0 & e & 0 \end{pmatrix} = \begin{pmatrix} r_{11} & 0 & r_{12} \\ r_{21} & 0 & r_{22} \\ 0 & e & 0 \end{pmatrix};$$

$$P^{23} \vdash R^{13} = \begin{pmatrix} e & 0 & 0 \\ 0 & 0 & e \\ 0 & e & 0 \end{pmatrix} \vdash \begin{pmatrix} r_{11} & 0 & r_{12} \\ 0 & e & 0 \\ r_{21} & 0 & r_{22} \end{pmatrix} = \begin{pmatrix} r_{11} & 0 & r_{12} \\ r_{21} & 0 & r_{22} \\ 0 & e & 0 \end{pmatrix}. \qquad \square$$

Remark. There is of course a nice pattern in the above commutation rules: $R^{ij} \dashv P^{kl} = P^{kl} \vdash R^{mn}$, where $i < j$, $k < l$ and $m < n$, and the three pairs of indices are different. The important point is that switching the order of the factors requires an interchange of the two operations in the dialgebra.

Using this lemma and the properties of the dialgebra operations (the so-called "dimonoid calculus"; [L2]), the left hand side of (1) can be rewritten as follows

$$\begin{aligned}(P^{23} \vdash R^{23}) \vdash (P^{12} \vdash R^{12}) \dashv (P^{23} \vdash R^{23}) &= P^{23} \vdash R^{23} \vdash P^{12} \vdash R^{12} \dashv P^{23} \dashv R^{23} \\ &= P^{23} \vdash R^{23} \vdash P^{12} \vdash P^{23} \vdash R^{13} \dashv R^{23} \\ &= P^{23} \vdash P^{12} \dashv R^{13} \vdash P^{23} \vdash R^{13} \dashv R^{23} \\ &= P^{23} \vdash P^{12} \dashv R^{13} \dashv P^{23} \vdash R^{13} \dashv R^{23} \\ &= (P^{23} \vdash P^{12} \dashv P^{23}) \vdash (R^{12} \vdash R^{13} \dashv R^{23});\end{aligned}$$

where, for convenience, we have inserted the parentheses in the last expression. Similarly, the right hand side becomes

$$(P^{12} \vdash P^{23} \dashv P^{12}) \vdash (R^{23} \vdash R^{13} \dashv R^{12}).$$

Thus, we have proved

Proposition 1. *A sufficient condition for $A = P \vdash R$ to be a solution to Artin's equation is*

$$R^{12} \vdash R^{13} \dashv R^{23} = R^{23} \vdash R^{13} \dashv R^{12}. \tag{4}$$

This is the quantum Yang-Baxter equation, or QYB, for dialgebras.

Remark. Had we used instead a deformation of the type $B = R \dashv P$, we would have arrived to the equality

$$\begin{aligned}(R^{23} \vdash R^{13} \dashv R^{12}) \vdash (P^{23} \vdash P^{12} \dashv P^{23}) = \\ (R^{12} \vdash R^{13} \dashv R^{23}) \vdash (P^{12} \vdash P^{23} \dashv P^{12}),\end{aligned}$$

also leading to (4).

Obviously, the bar unit $E \in M_2$ is a solution to the QYB equation, since E^{ij} is just the corresponding bar unit in M_3 (still denoted E), and thus, from the quantum equation we can now try to pass on to the classical equation in the usual way (*cf.* [K]): We consider again a deformation of this solution, this time involving a formal parameter h; without loss of generality for the following argument we can consider only first order deformations, $R = E + hT$, and so, substituting in both sides of (4) we get:

$$\begin{aligned}(E + hT^{12}) \vdash (E + hT^{13}) \dashv (E + hT^{23}) = \\ E + h[T^{12} \vdash E + T^{13} + E \dashv T^{23}] \\ +h^2[T^{13} \dashv T^{23} + T^{12} \vdash T^{13} + T^{12} \vdash E \dashv T^{23}] \\ +h^3[T^{12} \vdash T^{13} \dashv T^{23}]\end{aligned}$$

$$\begin{aligned}(E + hT^{23}) \vdash (E + hT^{13}) \dashv (E + hT^{12}) = \\ E + h[T^{23} \vdash E + T^{13} + E \dashv T^{12}] \\ +h^2[T^{13} \dashv T^{12} + T^{23} \vdash T^{13} + T^{23} \vdash E \dashv T^{12}] \\ +h^3[T^{23} \vdash T^{13} \dashv T^{12}]\end{aligned}$$

Here we have a difference with the standard case, because the linear terms do not automatically cancel out; this happens on the condition that $[E, T^{12}] = [E, T^{23}] = 0$, which clearly is the same as

$$[E, T] = 0, \tag{5}$$

and a short computation shows that this is the same as $[e, t_{ij}] = 0, \ \forall i, j$.

Nevertheless, assuming this is the case, and equating the terms quadratic in h, we get again as a necessary condition for R to satisfy QYB the following relation

$$[T^{23}, T^{13}] + [T^{23}, T^{12}] - [T^{12}, T^{13}] = 0, \tag{6}$$

which is a non-skew symmetric version of the classical Yang-Baxter equation (CYB).

Equation (6) exactly corresponds to the CYB equation we obtained in [FLO] (for the bracket called there $[\cdot, \cdot]_+$). Notice the minus sign in the third term, which implies that this equation is not equivalent to the usual CYB equation if the bracket is not alternating.

II. Solutions to the Yang-Baxter equations for φ-dialgebras.

Now, even in the standard setting it is not an easy task to find nontrivial solutions to the Yang-Baxter equations; in this section we construct explicit solutions for an interesting class of dialgebras with nontrivial bar units (related to the algebras with operators mentioned in [L1], see also [F]), as follows:

Let E be a vector space and fix $\varphi \in E'$ (the algebraic dual). Then one can define a dialgebra structure on E by setting

$$x \dashv y = \varphi(y)x \; ; \quad x \vdash y = \varphi(x)y. \tag{7}$$

Bilinearity of the operations is evident from the definition, and associativity and the dialgebra axioms are also quite straightforward: For instance, for the associativity of $\dashv$ we have

$$x \dashv (y \dashv z) = x\varphi(y \dashv z) = x\varphi(\varphi(y)z) = x\varphi(y)\varphi(z) = (x \dashv y)\varphi(z) = (x \dashv y) \dashv z;$$

and clearly the same holds for the $\vdash$ operation.

Similarly, for the first dialgebra axiom we have

$$x \dashv (y \dashv z) = x\varphi(y \dashv z) = x\varphi(y\varphi(z)) = x\varphi(y)\varphi(z)$$

while

$$x \dashv (y \vdash z) = x\varphi(z\varphi(y)) = x\varphi(z)\varphi(y).$$

The remaining dialgebra axioms are equally easy to prove.

We shall call such a dialgebra a φ-dialgebra, and denote it by E_φ.

The key point for us is that bar units for these dialgebras are also easily constructed: Indeed, if $\varphi \neq 0$, from the equation $x \dashv e = x$, for all $x \in E_\varphi$, we get that e is a bar unit in E_φ iff $\varphi(e) = 1$. So, if x_0 is any element in E_φ such that $\varphi(x_0) \neq 0$, $x_0/\varphi(x_0)$ is a bar unit.

Moreover, if e is any fixed bar unit, it is clear that another element e' will be a bar unit iff $\varphi(e - e') = 0$; in other words, $N_\vdash = {}_\dashv N = \ker\varphi$ in this case, and hence the bar units in E_φ form an affine space modelled after $\ker\varphi$. Thus we have:

Lemma 2. *Let E be any vector space, and fix $\varphi \in E'$, $\varphi \neq 0$. Then E_φ is a dialgebra, with non-trivial bar units. Moreover, its halo is an affine space modelled after the subspace $\ker\varphi$.*

In what follows, we consider a fixed φ-dialgebra E_φ, with $\varphi \neq 0$.

Remark. It is perhaps worthwhile to remark that, although these dialgebras give rise to abelian Leibniz algebras (hence in fact Lie algebras), as one easily checks, they are by no means uninteresting, since the corresponding matrix algebras do generate nontrivial Leibniz algebras. But, still more to the point, as we shall see the corresponding Yang-Baxter equations are also not trivial.

Now, to explicitly write down the Yang-Baxter equations for these dialgebras, we merely need to compute. Thus, for QYB we have:

$$R^{12} \vdash R^{13} \dashv R^{23} = \begin{pmatrix} x \vdash x & x \vdash y \dashv z + y \vdash e \dashv x & x \vdash y \dashv z + y \vdash e \dashv x \\ z \vdash x & x \vdash y \dashv z + y \vdash e \dashv x & x \vdash y \dashv z + y \vdash e \dashv x \\ z & w \dashv z & w \dashv w \end{pmatrix}$$
$$= \begin{pmatrix} \varphi(x)x & \varphi(x)\left(\varphi(z)x + \varphi(y)e\right) & \varphi(x)\varphi(w)y + \varphi(y)^2 e \\ \varphi(z)x & \varphi(z)^2 y + \varphi(x)\varphi(w)e & \varphi(w)\left(\varphi(z)y + \varphi(y)e\right) \\ z & \varphi(z)w & \varphi(w)w \end{pmatrix},$$

and similarly

$$R^{23} \vdash R^{13} \dashv R^{12} = \begin{pmatrix} \varphi(x)x & \varphi(y)x & y \\ \varphi(x)\left(\varphi(z)e + \varphi(y)z\right) & \varphi(y)^2 z + \varphi(x)\varphi(w)e & \varphi(y)w \\ \varphi(z)^2 e + \varphi(x)\varphi(w)z & \varphi(w)\left(\varphi(z)e + \varphi(y)z\right) & \varphi(w)w \end{pmatrix}$$

Hence, upon equating both expressions, we see that the upper left corner and the lower right corner give identities, and so we are left with the following set of seven equations for QYB, corresponding to each of the remaining entries:

(Q1) $y = \varphi(y)^2 e + \varphi(x)\varphi(w)y.$

(Q2) $z = \varphi(z)^2 e + \varphi(x)\varphi(w)z.$

(Q3) $\varphi(z)^2 y = \varphi(y)^2 z.$

(Q4) $\varphi(y)x = \varphi(x)\left(\varphi(z)y + \varphi(y)e\right).$

(Q5) $\varphi(z)x = \varphi(x)\left(\varphi(y)z + \varphi(z)e\right).$

(Q6) $\varphi(y)w = \varphi(w)\left(\varphi(z)y + \varphi(y)e\right).$

(Q7) $\varphi(z)w = \varphi(w)\left(\varphi(y)z + \varphi(z)e\right).$

Remark. We wrote the equations in an order that makes several symmetries of the equations apparent: indeed, we can plainly interchange x and w, and/or y and z, and get the same set of equations.

Similarly, if we work out the corresponding equations for the Artin equation on the φ-dialgebra we get

(A1) $x = \varphi(x)^2 e + \varphi(y)\varphi(z)x.$

(A2) $w = \varphi(w)^2 e + \varphi(y)\varphi(z)w.$

(A3) $\varphi(x)y = \varphi(y)\left(\varphi(w)x + \varphi(x)e\right).$

(A4) $\varphi(w)y = \varphi(y)\left(\varphi(x)w + \varphi(w)e\right).$

(A5) $\varphi(x)z = \varphi(z)\left(\varphi(w)x + \varphi(x)e\right).$

(A6) $\varphi(w)z = \varphi(z)\left(\varphi(x)w + \varphi(w)e\right).$

(A7) $\varphi(x)^2 w = \varphi(w)^2 x.$

By observing both sets of equations, we see an evident similarity between them (and this explains why we will only consider QYB): we merely have to interchange x and y, and z and w. The core of this assertion is the fact that, by construction, the "bar identity" E solves the Artin equations, while the permutation matrix P solves QYB. We have thus proved the following result:

Proposition 2. *Let e be a fixed bar unit in E_φ. Then, a matrix*

$$R = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

is a solution to the quantum Yang-Baxter equation if and only if its entries satisfy equations (Q1) - (Q7); and it is a solution to the Artin equation if and only if it satisfies (A1) - (A7).

Moreover, given R as above, the matrix

$$Q = \begin{pmatrix} y & x \\ w & z \end{pmatrix}$$

is a solution to the Artin equation if and only if the matrix R is a solution to the quantum Yang-Baxter equation.

Now, by the same type of reasoning, we can construct the following set of equations for the classical Yang-Baxter equation: Starting with equation (6)

$$[T^{23}, T^{13}] + [T^{23}, T^{12}] - [T^{12}, T^{13}] = 0,$$

and substituting

$$T = \begin{pmatrix} x & y \\ z & w \end{pmatrix},$$

we get a set of six equations, equivalent to CYB:

(C1) $\varphi(y)\,(e - x - z) + (\varphi(x) - 1)\,y = 0.$

(C2) $\varphi(y)\,(2e - x - y - w) = 0.$

(C3) $\varphi(z)\,(e - x - y) + (\varphi(x) - 1)\,z = 0.$

(C4) $\varphi(z)\,(2e - x - z - w) = 0.$

(C5) $\varphi(y)z = 0.$

(C6) $\varphi(z)y = 0.$

These equations also posses one of the symmetries of the QYB; namely, we can interchange y and z (but not x and w).

Let us now solve QYB for this dialgebra. Using the symmetry of the equations, we see that we can divide the analysis in three cases, according to whether $y, z \in \ker\varphi$ or not:

Case I: $y, z \in \ker\varphi$.

In this case, (Q3) - (Q7) are all reduced to $0 = 0$. Then, if $y = z = 0$ there are no restrictions upon x and w; while if $y \neq 0$ or $z \neq 0$, (Q1) or (Q2) respectively imply $\varphi(x)\varphi(w) = 1$.

Every matrix satisfying these conditions evidently solves QYB.

Case II: $y, z \notin \ker\varphi$.

Here we have $\varphi(y), \varphi(z) \neq 0$, by hypothesis. Then, applying φ, to (Q4), we have

$$\varphi(y)\varphi(x) = \varphi(x)\varphi(y)^2 + \varphi(x)\varphi(y), \tag{8}$$

from which we get $\varphi(x) = 0$; but then, (Q4) reduces to $\varphi(y)x = 0$, so that $x = 0$. Similarly, $w = 0$.

Finally, applying φ to (Q1) we get $\varphi(y) = \varphi(y)^2$, so that $\varphi(y) = 1$, and then necessarily $y = e$. Similarly, $z = e$, and so the only solution in this case is the permutation matrix P.

Case III: $z \in \ker\varphi$; $y \notin \ker\varphi$.

This case is in a sense the most interesting of the three:

First, from (Q3) we get that $z = 0$.

Next, from (Q4) we get $\varphi(y)x = \varphi(x)\varphi(y)e$, so that $x = \varphi(x)e$; which in particular means that x is in the space spanned by e. Similarly, $w = \varphi(w)e \in \langle e\rangle$.

Now, $y \in \langle e\rangle$ also; for, from (Q1),

$$(1 - \varphi(x)\varphi(w))\, y = \varphi(y)^2 e, \tag{9}$$

but necessarily $\varphi(x)\varphi(w) \neq 1$, since $e \neq 0$ and by hypothesis $\varphi(y) \neq 0$ also. Applying φ to this equation we get

$$\varphi(y) = 1 - \varphi(x)\varphi(w) \quad \text{and} \quad y = \varphi(y)e.$$

Thus, setting $a = \varphi(x)$, $b = \varphi(w)$, $c = \varphi(y)$, equation (9) (*i.e.*, (Q1)) reduces to the simple equation in a 3-dimensional space

$$1 = c + ab, \tag{10}$$

whose graph is a hyperbolic saddle. Thus, the solutions of this case are given by the points of this surface that satisfy $ab \neq 1$; however, the limit points, where $ab = 1$, are also solutions to QYB; they are included in case I.

By a similar analysis we can solve the classical Yang-Baxter equations:

Case I: $y, z \in \ker\varphi$. Here we are simply left with the relations

$$(\varphi(x) - 1)\, y = 0\,; \quad (\varphi(x) - 1)\, z = 0,$$

so that no condition is imposed on w; but if $y \neq 0$ or $z \neq 0$, x must satisfy $\varphi(x) = 1$, while if $y = z = 0$, x also becomes unrestricted.

Case II: $y, z \notin \ker\varphi$. Applying φ to (C1) we get $\varphi(y)\varphi(z) = 0$, so that necessarily $y \in \ker\varphi$ or $z \in \ker\varphi$, which immediately rules out this case. (This also follows immediately from (C5) and (C6).)

Case III: $y \notin \ker\varphi$, $z \in \ker\varphi$. In this case, (C5) implies $z = 0$, while C1) gives

$$\varphi(y)(e - x) = (1 - \varphi(x))y, \tag{11}$$

and C2) gives

$$2e = x + w + y. \tag{12}$$

III. A geometrical interpretation of the solutions.

Up to this point, the setting has been purely algebraic. Nevertheless, although in the absence of a topology in E such notions as "tangent to a hypersurface" are not meaningful in general, Case III of the solutions clearly shows that there is some kind of geometry behind, and we now turn our attention to this point.

To explicitly exhibit this geometry, we observe first that, formally, we might think of CYB as a kind of "lowest order approximation" to QYB. Now, in the context of dialgebras, this makes sense only if the deformations T, introduced in section I, satisfy equation (5): $[E, T] = 0$ (otherwise, this is a lower order equation).

For a φ-dialgebra E_φ this condition explicitly reads

$$x = \varphi(x)e\ ; \quad y = \varphi(y)e\ ; \quad w = \varphi(w)e\ ; \quad z = \varphi(z)e.$$

Hence, in particular $x, y, z, w \in \langle e \rangle$, and so CYB is a "good linear approximation" to QYB only if T lives in this 4-dimensional subspace of E_φ.

Now, let us assume this is the case, and, as we did in Case III of QYB, for simplicity write $a = \varphi(x)$, $b = \varphi(w)$, $c = \varphi(y)$, and $d = \varphi(z)$. Since $e \neq 0$, we can rewrite (Q1) - (Q7), and (C1) - (C6) as equations in these variables, and an easy computation then gives the following simplified sets of equations:

For QYB:

(Q1') $c(1 - c - ab) = 0$.

(Q2') $d(1 - d - ab) = 0$.

(Q3') $acd = 0$.

(Q4') $bcd = 0$.

And for CYB:

(C1') $c(2 - a - b - c) = 0$.

(C2') $d(2 - a - b - d) = 0$.

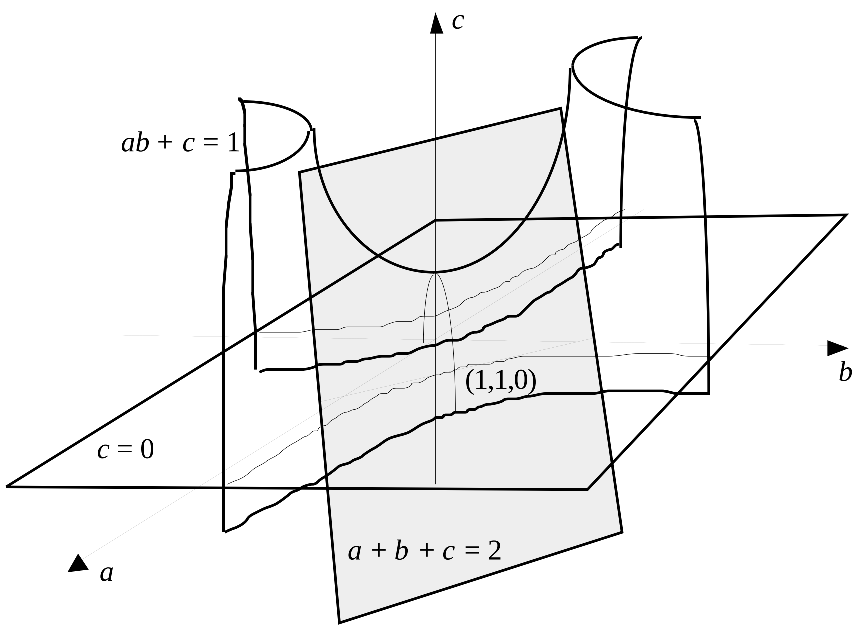
It is clear that, in both cases, each equation describes a 3-dimensional hypersurface with singularities, consisting in fact of the union of some simple figures (for instance, (C2') describes the union of the hyperplanes given by $d = 0$ and $2 = a + b + d$, and where c is a free parameter), and therefore the whole object is in fact an algebraic variety, given by the intersection of the corresponding hypersurfaces. Since we cannot picture these objects in a 4-dimensional space, we can try to visualize their intersections with 3-dimensional subspaces, which we do next, using d as a parameter.

Let us assume first $d = 0$.

In this case several equations become trivial, and in particular QYB reduces to the single equation (Q1'), which gives the union of the plane $c = 0$ with the hyperbolic paraboloid $1 = c + ab$; let us denote this surface S_0.

CYB, on the other hand, reduces to (C1'), which describes the union of the planes $c = 0$ and $a + b + c = 2$. Let us call this surface T_0.

So, we see that the plane $c = 0$ is a common part of both sets of solutions. But now, if we consider the tangent plane to hyperbolic saddle at an arbitrary point (a, b, c), we find that its normal vector is given by $\mathbf{n}(a, b, c) = (b, a, 1)$; this vector coincides with the normal vector to the plane $2 = a + b + c$ iff $a = b = 1$. Thus, we see that this plane gives the tangent plane to the saddle precisely at the point $(1, 1, 0)$, and, therefore, we can think of T_0 as representing the tangent (*i.e.* flat) surface to S_0 at $(1, 1, 0)$. (See the figure.)



For $d \neq 0$, the description is slightly more complicated, since the loci of solutions is given by further intersecting S_0 and T_0 with other surfaces:

For QYB, the additional surfaces are given by

$$ac = 0\ ; \quad bc = 0\ ; \quad ab = d - 1. \tag{14}$$

Each of the first two describes the union of two coordinate planes, whose intersection is the plane $c = 0$, already contained in the previous surface S_0. The last equation describes a vertical cylinder with a hyperbolic section, except for $d = 1$, where this hyperbolic surface degenerates into the union of the two coordinate planes defined by $b = 0$ and $a = 0$. Thus, this new surface intersects S_0 in two curves: for $d \neq 1$, $d \neq 2$,these are the two parallel hyperbolas $ab = d-1$, $c = 0$, and $1-ab = 0$; $c = d$; observe that for $d = 2$ both hyperbolas coincide, while for $d = 1$ both hyperbolas degenerate into the pairs of lines $a = b = c = 0$, and $a = b = 0$, $c = 1$.

On the other hand, for the solutions of CYB we have to intersect T_0 with the vertical plane $a + b = 2 - d$. Since $d \neq 0$, this plane again intersects T_0 in two separate parts, namely the lines $a + b = 2 - d$, $c = 0$, and $a + b = 2 - d$, $c = d$. These two lines could be thought of as describing the tangents at the apexes of the hyperbolas formed by the solutions to QYB.

There is a caveat, because, actually, only the tangent to the hyperbola contained in the plane $c = 0$ can be obtained this way; moreover, this cannot happen at the same values of d for both sets of solutions. Indeed, fixing d for the solutions of QYB, and denoting by $\tilde{d}$ the corresponding value for the solutions of CYB, elementary calculus shows that the tangents to the hyperbolas in S_0 contained in the plane $c = d$ are given by $a + b = 2\sqrt{|1-d|}$, and so we must have the relationship $2 - \tilde{d} = 2\sqrt{|1-d|}$, or

$$\tilde{d} = 2\left(1 - \sqrt{|1-d|}\right). \tag{15}$$

Setting $\tilde{d}$ to this value, and $c = 0$, this obviously gives the tangent to the hyperbola contained in the plane $c = 0$. However, the only hyperbola contained in the hyperbolic saddle, whose tangent is also contained in the plane given by $a + b + c = 2$ is the one at $c = 0$, since, if we have $c = d$ and $d = \tilde{d}$, necessarily $d = 0$.

As a final remark, notice that the sets of solutions analysed above correspond to a fixed value of the bar-unit e in E_φ. If we allow e to vary, we see that the posible solutions to the Yang-Baxter equations describe a sort of "bundle of algebraic varieties" over the halo of E_φ. It might be of interest to study the geometry of this construction.

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