GENERALIZED LODAY ALGEBRAS AND DIGROUPS

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This paper is dedicated to the memory of my cousin Ana Margarita.

ABSTRACT

In this paper we have given a method of constructing generalized Loday algebras with bar-unit. It turns out that it is very intimately related with functional analysis. In order to make progress in the search for an analogous concept of Lie groups for Liebniz algebras we introduce the notion of digroup. The basic results of the digroups are proved and some open problems are enunciated.

Contents

1. Introduction

The Leibniz algebras and Loday algebras (dialgebras) first arose in K-theory and are objects of current interest. They were introduced by J.L. Loday and these constitute an extension of the concepts of Lie algebra and associative algebra respectively. More exactly, the Leibniz algebras are a generalization of Lie algebras, for which the antisymmetry condition of the bracket is dropped and only the Jacobi identity is retained. On the other hand the definition of Loday algebra is the following:

Definition 1.1. A Loday algebra is a vector space V together with two associative and bilinear operators, $\vdash$, $\dashv$, satisfying the following requirements

$$
\begin{aligned}
x \dashv (y \dashv z) &= x \dashv (y \vdash z)\,, \\
(x \vdash y) \dashv z &= x \vdash (y \dashv z)\,, \\
(x \dashv y) \vdash z &= (x \vdash y) \vdash z,
\end{aligned}
$$

for all x, y and z of V. These operators are called respectively, right and left products. In our paper we will modify this notion in the following way: a generalized Loday algebra is a vector space V together with two associative operators, $\vdash$, $\dashv$, that satisfy the above properties but these operators are not necessarily bilinear.

Clearly all Loday algebra is also a generalized Loday algebra. We say that e is a bar-unit of a generalized Loday algebra $(V, \vdash, \dashv)$ if $e \vdash v = v = v \dashv e$ for any $v \in V$.

It is well known that if a Loday algebra is given then it gives rise to a Leibniz algebra which is obtained by defining the bracket as

$$[x, y] = x \dashv y - y \vdash x$$

see [7] and [8] for more detail.

A basic problem in this context is the construction of Loday algebras with bar-unit which are not associative algebras. A few very special examples of Loday algebras with bar-identity were given in [7]. In this paper, we present a huge number of Loday algebras by means of the dual space of a vector space, more exactly we consider a certain natural construction that associates a Loday algebra structure on this vector space with a given nonzero linear functional on this vector space . Thus, the Loday algebra structure is really very rich and from our point of view it 's principal feature, as will be seen below, is that this leads us to extend the group theory in a certain direction. Another very important problem is the construction of an analogous of Lie group for Liebniz algebras. The apparent difficulty is the absence of a similar concept of group for sets with two special products. In the present paper we attempt to prepare the way to solve this problem. We introduce the notion of digroup. Many questions arise around this concept that it should be considered in forthcoming papers. Some of them are:

i) Sets which are both a digroup and a manifold.

ii) Abstract harmonic analysis on topology digroup and construction of new Loday algebras.

iii) Cohomology of digroups.

iv) To construct Hopf dialgebras and quantum digroups.

2. New Loday algebras with bar-unit

The main object of this section is the construction of new examples of Loday algebras. As we will see below, associated to all nonzero elements belonging to the dual of a vector space, are given a structure of Loday algebras on this vector space. Starting this construction other very interesting structures are defined on Loday algebras possibility related with a new trend in the functional analysis and that has its origins in the paper by the author [3].

Throughout this section any vector space will always be finite dimensional. However, most of the definitions and results can be extended without this restriction.

Let V be a finite dimensional vector space over $\mathbb{C}$ and V^* its dual space. If φ is a nonzero element of V^* then we define the following bilinear operators:

$$x \vdash y = \varphi(x)\, y, \qquad\qquad w \dashv z = \varphi(z)\, w$$

where x, y, z and w are elements of V. We start this section with the following

THEOREM 2.1. *The operators* $\vdash$ *and* $\dashv$ *obey the properties of definition* 1.1.

Proof. Let x, y and z be elements of V

$$x \dashv (y \dashv z) = x \dashv (\varphi(z)\, y) = \varphi(\varphi(z)\, y)\, x = \varphi(z)\, \varphi(y)\, x,$$

on the other hand

$$x \dashv (y \vdash z) = x \dashv (\varphi(y)\, z) = \varphi(\varphi(y)\, z)\, x = \varphi(y)\, \varphi(z)\, x,$$

from the two last equations it now follows that $x \dashv (y \dashv z) = x \dashv (y \vdash z)$. Next we must prove that $(x \vdash y) \dashv z = x \vdash (y \dashv z)$

$$(x \vdash y) \dashv z = (\varphi(x)\, y) \dashv z = \varphi(z)\, \varphi(x)\, y$$

and

$$x \vdash (y \dashv z) = x \vdash (\varphi(z)\, y) = \varphi(x)\, \varphi(z)\, y$$

then, as was claimed, the equality holds. Finally we have

$$(x \dashv y) \vdash z = (\varphi(y)\, x) \vdash z = \varphi(\varphi(y)\, x)\, z = \varphi(y)\, \varphi(x)\, z,$$

also we have

$$(x \vdash y) \vdash z = (\varphi(x)\, y) \vdash z = \varphi(\varphi(x)\, y)\, z = \varphi(x)\, \varphi(y)\, z,$$

so $(x \dashv y) \vdash z = (x \vdash y) \vdash z$. ■

COROLLARY 2.1. $(V, \vdash, \dashv)$ *is a Loday algebra.*

Proof. Since it is readily verified that the operators $\vdash$ and $\dashv$ are both associative. ■

From now on, we will denote this Loday algebra by V_φ. Let $x \in V_\varphi$ such that $\varphi(x) \neq 0$ then $e = \frac{x}{\varphi(x)}$ is a bar-unit in V_φ, In fact, for any $z \in V_\varphi$ we have $e \vdash z = \varphi\left(\frac{x}{\varphi(x)}\right) z = z \dashv e = z$. Thus, in this Loday algebra the bar-unit is not unique.

DEFINITION 2.1. An element x in a generalized Loday algebra $(\mathcal{L}, \vdash, \dashv)$ is said to be $(\vdash)$ −regular $((\dashv)$ − regular) with respect to a bar-unit e provided there exists $x_\vdash \in \mathcal{L}\,(x_\dashv \in \mathcal{L})$, such that $x \vdash x_\vdash = (e - x) + (x \vdash e)$ $(x_\dashv \dashv x = (e - x) + (e \dashv x))$. The element $x_\vdash$ $(x_\dashv)$is called a $(\vdash)$-inverse $((\dashv)$-inverse) for x with respect to e. An element which is both $(\vdash)$ −regular and $(\dashv)$ −regular with respect to e, is called regular if it has a $(\vdash)$ −inverse that is also a $(\dashv)$ −inverse, both with respect to e.

Observe that if a Loday algebra has a bar-unit e which satisfies also that $e \dashv x = x = x \vdash e$ for any x, then $\vdash\, = \,\dashv$ and $\mathcal{L}$ is an associative algebra with unit. In this case the definition 4 coincides with the usual one. On the other hand e is regular while the null element θ of V is not regular with respect to e.

PROPOSITION 2.1. *In V_φ the regular elements are those vectors x such that $x \notin V^0$, where $V^0 = \{z \in V \mid \varphi(z) = 0\}$.*

Proof. First according to Definition 2.1 we must prove that if x satisfies the condition $\varphi(x) \neq 0$ then it is $(\vdash)$ −regular and also $(\dashv)$ −regular and has a $(\vdash)$ −inverse that is also a $(\dashv)$ −inverse with respect to e. To begin, let $x_\vdash$ be a $(\vdash)$ −inverse of x then we must have

$$\varphi(x)\, x_\vdash = x \vdash x_\vdash = (e - x) + (x \vdash e) = (e - x) + \varphi(x)\, e \qquad (1)$$

since $\varphi(x) \neq 0$, it follows that $x_\vdash$ is necessarily of the following form:

$$x_{\vdash} = \frac{(e-x)}{\varphi(x)} + e.$$

As we will show this vector $x_{\vdash}$ is also a $(\dashv)$ $-$inverse of x that is $x_{\dashv} = x_{\vdash}$; in fact

$$\begin{aligned} x_{\vdash} \dashv x &= \left(\frac{(e-x)}{\varphi(x)} + e \right) \dashv x \\ &= \varphi(x) \left(\frac{(e-x)}{\varphi(x)} + e \right) \\ &= (e-x) + \varphi(x)\, e \\ &= (e-x) + (e \dashv x)\,. \end{aligned}$$

Reciprocally, let x be a regular element with respect to e in V_{φ}. If $x = e$ then $\varphi(e) = 1 \neq 0$. If $x \neq e$ is regular then from (1) it follows that $\varphi(x) \neq 0$ since in the other case we will have $\varphi(e) = 0$ but it is impossible. Thus, we have proved that the set of the regular elements of V_{φ} with respect to e consists of all the vectors don't belong to V^0. ■

Next we introduce the concept of ideals in a generalized Loday algebra with a bar-unit e.

Definition 2.2. A subset I of a generalized Loday algebra $\mathcal{L}$, is said to be a $(\vdash)-$ ideal provided it is a linear subspace of $\mathcal{L}$ and $x \vdash y$, $y \vdash x \in I$ for all $x \in I$ and all $y \in \mathcal{L}$. It is a $(\dashv)-$ ideal if the latter condition is replaced by $y \dashv x$, $x \dashv y \in I$ for all $x \in I$ and all $y \in \mathcal{L}$. If I is both a $(\vdash)-$ ideal and a $(\dashv)-$ ideal, then it is called a two-sided ideal of $\mathcal{L}$. Any ideal of some type, different from $\mathcal{L}$ is called proper.

We note that if e is a bar-unit and I is an ideal of some type such that $e \in I$ then $I = \mathcal{L}$.

Lemma 2.1. *Let $(\mathcal{L}, \vdash, \dashv)$ a Loday algebra with a bar-unit e, let $x \in \mathcal{L}$ be a $(\vdash)$ $-$regular element with respect to e, then for all $z \in \mathcal{L}$*

$$(x \vdash x_{\vdash}) \vdash z = z \tag{2}$$

where $x_{\vdash}$ is a $(\vdash)$ $-$inverse of x. Also in the same context, if $x \in \mathcal{L}$ is $(\dashv)$ $-$regular with respect to e then we have

$$z \dashv (x_{\dashv} \dashv x) = z \tag{3}$$

for any $z \in \mathcal{L}$. In (3)*, $x_{\dashv}$ is some $(\dashv)$ $-$inverse of x.*

Proof. Let $x \in \mathcal{L}$ be $(\vdash)$ −regular with respect to a bar-unit e, as we have already seen this means, that

$$x \vdash x_{\vdash} = (e - x) + (x \vdash e),$$

for some $x_{\vdash} \in \mathcal{L}$ and hence, for all $z \in \mathcal{L}$

$$\begin{aligned}(x \vdash x_{\vdash}) \vdash z &= ((e - x) + (x \vdash e)) \vdash z \\ &= (e - x) \vdash z + (x \vdash e) \vdash z \\ &= (e \vdash z) - (x \vdash z) + (x \dashv e) \vdash z \\ &= z,\end{aligned}$$

hence the equality (2) holds. The proof of (3) is very similar. ■

COROLLARY 2.2. *Let $(\mathcal{L}, \vdash, \dashv)$ be a Loday algebra. If x is $(\vdash)$ −regular with respect to a bar-unit e, then it can't belong to a proper $(\vdash)$ −ideal.*

Proof. Since, if x is an element $(\vdash)$ −regular with respect to a bar-unit e and I a $(\vdash)$ −ideal, such that $x \in I$, we have that for all $z \in \mathcal{L}$, $z = (x \vdash x_{\vdash}) \vdash z = x \vdash (x_{\vdash} \vdash z) \in I$, where $x_{\vdash}$ is a $(\vdash)$ −inverse of x with respect to the bar-unit e. ■

A similar statement holds for $(\dashv)$ −regular elements of a Loday algebra $\mathcal{L}$ and proper $(\dashv)$ −ideals, to be more precise: if x is $(\dashv)$ −regular with respect to a bar-unit e, then it can't belong to a proper $(\dashv)$ −ideal.

EXAMPLE 2.1. Having in mind that $V^* = Mor(V, \mathbb{C})$ is also a finite dimensional vector space and its dual $(V^*)^*$ can be identified with V, then we can choose a nonzero element $x \in V$ and define the following Loday algebra structure in V^*

$$\varphi_1 \vdash \varphi_2 = \Phi_x(\varphi_1)\varphi_2, \qquad \varphi_3 \dashv \varphi_4 = \Phi_x(\varphi_4)\varphi_3 \tag{4}$$

where φ_1, φ_2, φ_3 and φ_4 are in V^*, moreover Φ_x is defined in the following form: $\Phi_x(\varphi) = \varphi(x)$ for any $\varphi \in V^*$. This Loday algebra will be denoted by V_x^*. Notice that if $x \neq \theta$ there exits $\varphi \in V^*$ such that $\varphi(x) \neq 0$, then $e(.) = \frac{\varphi(.)}{\varphi(x)}$ is a bar-unit of V_x^*. Let $V_0^* = \{\varphi \in V^* \mid \varphi(x) = 0\} \subset V^*$ the annihilator of the set $\{x\} \subset V$. It is easy to show that $\varphi \in V_x^*$ is regular if and only if $\varphi \notin V_0^*$

We have

THEOREM 2.2. *If $x \neq \theta$ then V_0^* is a two-sided proper ideal in V_x^*.*

Proof. It is clear that V_0^* is a subspace of V_x^*. Now let $\varphi \in V_0^*$ and $\gamma \in V_x^*$ then $(\varphi \vdash \gamma)(.) = \Phi_x(\varphi)\gamma(.) = \varphi(x)\gamma(.) = 0$ that is $(\varphi \vdash \gamma)(.) = \mathcal{O}(.)$ is the null functional. On the other hand $(\gamma \vdash \varphi)(.) = \Phi_x(\gamma)\varphi(.) = \gamma(x)\varphi(.)$ but $\varphi \in V_0^*$ hence $\gamma \vdash \varphi$ is zero in x and $\gamma \vdash \varphi \in V_0^*$. The following step is to see that $\varphi \dashv \gamma$ and $\gamma \dashv \varphi$ belong to V_0^*. Indeed $(\varphi \dashv \gamma)(.) = \Phi_x(\gamma)\varphi(.)$ that as we already have seen belong to V_0^*, finally observe that $(\gamma \dashv \varphi)(.) = \mathcal{O}(.)$. It remains to be proved that V_0^* is a proper subspace, but this assertion is trivial. ■

Definition 2.3. Let $(\mathcal{L}_1, \vdash, \dashv)$ and $(\mathcal{L}_2, \rhd, \lhd)$ be two generalized Loday algebras over $\mathbb{C}$; a linear mapping T: $\mathcal{L}_1 \to \mathcal{L}_2$ is called a homomorphism of generalized Loday algebras if

$$T(x \vdash y) = Tx \rhd Ty, \qquad T(w \dashv z) = Tw \lhd Tz, \tag{5}$$

for all x, y, w, $z \in \mathcal{L}_1$. In this case, one can say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are homomorphic; if T is invertible, T is said to be an isomorphism of generalized Loday algebras moreover we say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic.

Apparently this definition for Loday algebras was first introduced in [4] and [5].

Let T be a homomorphism of $\mathcal{L}_1$ into $\mathcal{L}_2$ where $(\mathcal{L}_1, \vdash, \dashv)$ and $(\mathcal{L}_2, \rhd, \lhd)$ are two generalized Loday algebras over $\mathbb{C}$. It is called unital if the image of any bar-unit is a bar-unit. Note that if e is a bar-unit of $\mathcal{L}_1$ and $T\mathcal{L}_1 = \mathcal{L}_2$, then Te is a bar-unit in $\mathcal{L}_2$. On the other hand, if T is unital and if $x \in \mathcal{L}_1$ is $(\vdash)-$regular with respect to the bar-unit e, then Tx is $(\rhd)-$regular with respect to Te. In fact, since x is $(\vdash)-$regular with respect to e, there is $x_\vdash$ in $\mathcal{L}_1$ such that $x \vdash x_\vdash = (e - x) + (x \vdash e)$ hence $T(x \vdash x_\vdash) = Tx \rhd Tx_\vdash = T((e-x) + (x \vdash e)) = T(e-x) + T(x \vdash e) = (Te - Tx) + (Tx \rhd Te)$. Also it is easy to see that if x is $(\dashv)-$regular with respect to e, the vector Tx is $(\lhd)-$regular with respect to Te. It shows that if x is a regular vector with respect to the bar-unit e then Tx is regular with respect to Te.

We recall that if $u \in L(E, V)$, that is, if u is a linear mapping of E into V, where E and V are finite dimensional vector spaces then the adjoint mapping u^t associated to u is defined in the following form u^t: $V^* \to E^*$ and $u^t(f) = f \circ u$ for any $f \in V^*$. We have

Theorem 2.3. *Let E and V be finite dimensional vector spaces. Let $u \in L(E, V)$ and $x \in E$ nonzero such that $u(x) \neq \theta$. Then the mapping T defined as $Tf = u^t(f)$ for all $f \in V_{u(x)}^*$ is a homomorphism of Loday algebras between $V_{u(x)}^*$ and E_x^*.*

Proof. It is obvious that T is a linear mapping. Let f, $g \in V_{u(x)}^*$ then

$$
\begin{aligned}
T\left(f \vdash g\right) &= T\left(\Phi_{u(x)}\left(f\right)g\left(.\right)\right) \\
&= T\left(f\left(u\left(x\right)\right)g\left(.\right)\right) \\
&= f\left(u\left(x\right)\right)Tg \\
&= \left(f \circ u\right)\left(x\right)\left(g \circ u\right)\left(.\right) \\
&= u^{t}\left(f\right)\left(x\right)u^{t}\left(g\right)\left(.\right) \\
&= Tf \vdash Tg,
\end{aligned}
$$

on the other hand for $h,\, l \in V^{*}_{u(x)}$

$$
\begin{aligned}
T\left(h \dashv l\right) &= T\left(\Phi_{u(x)}\left(l\right)h\left(.\right)\right) \\
&= T\left(l\left(u\left(x\right)\right)h\left(.\right)\right) \\
&= l\left(u\left(x\right)\right)Th \\
&= \left(l \circ u\right)\left(x\right)\left(h \circ u\right)\left(.\right) \\
&= u^{t}\left(l\right)\left(x\right)u^{t}\left(h\right)\left(.\right) \\
&= Th \dashv Tl.
\end{aligned}
$$

$\blacksquare$

Example 2.2. Let V be a vector space of dimension n over $\mathbb{C}$ and let $\{e_{1,\cdots,}e_n\}$ be an ordered basis of V; the corresponding dual ordered basis will be denoted by $\left\{e^{*}_{1,\cdots,}e^{*}_{n}\right\}$ that is if $x \in V$ and $x = x_1e_1 + \cdots + x_ne_n$ then $e^{*}_{i}\left(x\right) = x_i$ for $i = i, \ldots, n$. It is well known that

$$
\left\langle x, y\right\rangle = \sum_{i=1}^{n} e^{*}_{i}\left(x\right)\overline{e^{*}_{i}\left(y\right)} \tag{6}
$$

for any $x,\, y \in V$ defines a scalar product in V, let e be an element of V such that $\left\langle e, e\right\rangle = \|e\| = 1$. It follows that the operators $\vdash$ and $\dashv$ defined in the following way: $x \vdash y = \left\langle x, e\right\rangle y$, $z \dashv w = \left\langle w, e\right\rangle z$ convert V in a Loday algebra that will be denoted by $V\left(e\right)$.

This particular example of Loday algebra gives rise to the following

Definition 2.4. A normed Loday algebra is a Loday algebra $(\mathcal{L},\vdash,\dashv)$ over the field $\mathbb{C}$ together with a norm $x \to \|x\|$, such that,

$$
\|x \vdash y\| \leq \|x\|\,\|y\|\,, \qquad \|x \dashv y\| \leq \|x\|\,\|y\| \qquad \forall \;\; x, y \in \mathcal{L} \tag{7}
$$

Hence we have

PROPOSITION 2.2. *$V(e)$ a normed Loday algebra*

Proof. Obviously it remains to be proved (7) but it follows of Cauchy-Schwarz inequality. ■

Observe that $V(e)$ is complete, hence it is a Banach-Loday algebra in the sense introduced by the author in [3]. Thus, now it is justified to introduce the following definition

DEFINITION 2.5. Let $(\mathcal{L}, \vdash, \dashv)$ be a complex Loday algebra. A mapping $x \to x^*$ of $\mathcal{L}$ onto itself is called an involution of classic type provided the following conditions are satisfied:

$$\begin{array}{ll} (i) & (x^*)^* = x, \\ (ii) & (x+y)^* = x^* + y^*, \\ (iii) & (x \vdash y)^* = y^* \dashv x^*, \\ (iv) & (\alpha x)^* = \overline{\alpha} x^*, \end{array}$$

note that from (iii) it follows the following equality

$$(x \dashv y)^* = y^* \vdash x^*,$$

A complex Loday algebra with an involution of classic type is called a Loday $*-$algebra of classic type.

We shall write $L(V(e))$ for the space of bounded linear transformations from $V(e)$ to $V(e)$; with each linear transformation T of this space is associated a linear transformation $T^* \in L(V(e))$ called the adjoint.

DEFINITION 2.6. The adjoint of $T \in L(V(e))$ is the bounded linear transformation $T^*{:}V(e) \to V(e)$ defined by the equation

$$\langle Tx, y\rangle = \langle x, T^* y\rangle \tag{8}$$

It is well known that this definition makes sense and

THEOREM 2.4. *$L(V(e))$ is a Loday $*-$algebra of classic type.*

Proof. In the first place observe that $L(V(e))$ can be converted in a Loday algebra in the following way, we introduce in $L(V(e))$ the following associative operators

$$(A \vdash B)(x) = (Ae) \vdash (Bx), \qquad (C \dashv D)(y) = (Cy) \dashv (De) \tag{9}$$

for all A, B, C and D in $L(V(e))$ and any $x, y \in V(e)$. Then $(L(V(e)), \vdash, \dashv)$ is a Loday algebra. Let us see it in detail, first of all note that $A \vdash B$ and $C \dashv D$ are linear operators. On the other hand, since $V(e)$ is a normed Loday algebra it follows that $A \vdash B \in L(V(e))$ for any A and B in $L(V(e))$ and moreover $C \dashv D \in L(V(e))$ for all C and D also in $L(V(e))$.

By (9), we have that for A, B and C in $L(V(e))$

$$\begin{aligned}(A \dashv (B \dashv C))(x) &= Ax \dashv (B \dashv C)(e) \\ &= Ax \dashv (Be \dashv Ce),\end{aligned}$$

on the other hand

$$\begin{aligned}(A \dashv (B \vdash C))(x) &= Ax \dashv (B \vdash C)(e) \\ &= Ax \dashv (Be \vdash Ce),\end{aligned}$$

since $V(e)$ is a Loday algebra then $Ax \dashv (Be \dashv Ce) = Ax \dashv (Be \vdash Ce)$. Therefore we have $A \dashv (B \dashv C) = (A \dashv (B \vdash C))$.

In a similar way one can show that $(A \vdash B) \dashv C = A \vdash (B \dashv C)$. In fact

$$\begin{aligned}((A \vdash B) \dashv C)(x) &= (A \vdash B)(x) \dashv Ce \\ &= (Ae \vdash Bx) \dashv Ce,\end{aligned}$$

and we check

$$\begin{aligned}(A \vdash (B \dashv C))(x) &= Ae \vdash (B \dashv C)(x) \\ &= Ae \vdash (Bx \dashv Ce),\end{aligned}$$

using now the fact that $V(e)$ is a Loday algebra we have $((A \vdash B) \dashv C) = (A \vdash (B \dashv C))$. The reader easily examines that also $(A \dashv B) \vdash C = (A \vdash B) \vdash C$.

Finally let A and B be two bounded linear transformations of $L(V(e))$ and $x, y \in V(e)$ then we have

$$\langle (A \vdash B)\, x, y \rangle = \langle (Ae \vdash Bx), y \rangle$$

$$\begin{aligned}
&= \langle \langle Ae, e\rangle\, Bx, y\rangle \\
&= \langle Ae, e\rangle\, \langle Bx, y\rangle \\
&= \langle e, A^{*}e\rangle\, \langle x, B^{*}y\rangle \\
&= \left\langle x, \overline{\langle e, A^{*}e\rangle} B^{*}y \right\rangle \\
&= \langle x, \langle A^{*}e, e\rangle\, B^{*}y\rangle \\
&= \langle x, B^{*}y \dashv A^{*}e\rangle \\
&= \langle x, (B^{*} \dashv A^{*})\, y\rangle
\end{aligned}$$

from this it follows that $(A \vdash B)^{*} = B^{*} \dashv A^{*}$. The rest of the conditions for a Loday $*-$algebra of classic type clearly hold. This proves the Theorem. ■

Observe also that $L\,(V\,(e))$ is a normed Loday algebra.

3. Generalized Loday algebras and digroups.

We begin this section with a very interesting result

Theorem 3.1. *All vector space can be equipped with a generalized Loday algebra structure.*

Proof. Let X be a vector space. We define $x \vdash y = y$ for any x, $y \in X$ and $z \dashv w = z$ for all z, $w \in X$. Observe that in general $y = x \vdash y$ $\neq x \dashv y = x$. Then

$$\begin{aligned}
x \vdash (y \vdash z) &= x \vdash z = z, \\
(x \vdash y) \vdash z &= y \vdash z = z, \\
(x \dashv y) \vdash z &= x \vdash z = z, \\
x \dashv (y \dashv z) &= x \dashv y = x, \\
(x \dashv y) \dashv z &= x \dashv z = x, \\
x \dashv (y \vdash z) &= x \dashv z = x, \\
(x \vdash y) \dashv z &= y \dashv z = y, \\
x \vdash (y \dashv z) &= x \vdash y = y,
\end{aligned}$$

these equalities show that X is a generalized Loday algebra. The Theorem is proved. ■

Let us denote this generalized Loday algebra by X_{l}.

Theorem 3.2. *Let V be a finite dimensional vector space and $\Pi : V \to V$ a linear operator such that $\Pi^{2} = \Pi$ and*

$$x \vdash y = \Pi x + y, \qquad w \dashv z = w + \Pi z$$

where x, y, z and w are elements in V. Then $(V, \vdash, \dashv)$ is a generalized Loday algebra.

Proof. It must be shown that $x \dashv (y \dashv z) = x \dashv (y \vdash z)$. We have

$$\begin{aligned} x \dashv (y \dashv z) &= x \dashv (y + \Pi z) \\ &= x + \Pi (y + \Pi z) \\ &= x + \Pi y + \Pi^2 z = x + \Pi y + \Pi z, \end{aligned}$$

now

$$\begin{aligned} x \dashv (y \vdash z) &= x \dashv (\Pi y + z) \\ &= x + \Pi (\Pi y + z) \\ &= x + \Pi^2 y + \Pi z = x + \Pi y + \Pi z, \end{aligned}$$

this proves the equality required. On the other hand we have that $(x \vdash y) \dashv z = \Pi x + y + \Pi z$ and $x \vdash (y \dashv z) = \Pi x + y + \Pi z$, that is $(x \vdash y) \dashv z = x \vdash (y \dashv z)$. The last property of these operators is $(x \dashv y) \vdash z = (x \vdash y) \vdash z$. Let us prove it

$$\begin{aligned} (x \dashv y) \vdash z &= (x + \Pi y) \vdash z \\ &= \Pi (x + \Pi y) + z \\ &= \Pi x + \Pi^2 y + z \\ &= \Pi x + \Pi y + z, \end{aligned}$$

and

$$\begin{aligned} (x \vdash y) \vdash z &= (\Pi x + y) \vdash z \\ &= \Pi (\Pi x + y) + z \\ &= \Pi^2 x + \Pi y + z \\ &= \Pi x + \Pi y + z, \end{aligned}$$

the last two equalities show that $(x \dashv y) \vdash z = (x \vdash y) \vdash z$ as required. The proof the that each operator is an associative mapping is straightforward. ■

This generalized Loday algebra will be denoted by $V(\Pi)$. Observe that θ is a bar-unit of $V(\Pi)$ and if moreover $x \in V(\Pi)$ such that $x \neq \theta$ then x is a regular element of $V(\Pi)$ for which $x_{\vdash} = x_{\dashv} = -x$.

Historically the groups have been one of concepts more studied and it is one of the most fundamental concepts of contemporary mathematics, however after almost two century of development this theory has not been extended to sets with two special operators. A deeper analysis of the spaces X_l, V_φ and $V(\Pi)$ lead us to extend the group theory to sets with two particular products. We hope that this concept helps to construct an analogous of Lie group for Liebniz algebras.

Definition 3.1. A digroup is a pair (G, e) where G is a nonempty set and $e \in G$, such that, the set G is equipped with two associative maps called respectively right product and left product:

$$\vdash : G \times G \to G,$$
$$\dashv : G \times G \to G,$$

satisfying the following requirements:

a) For all $x,\ y,\ z \in G$

$$\begin{aligned} x \dashv (y \dashv z) &= x \dashv (y \vdash z)\,, \\ (x \vdash y) \dashv z &= x \vdash (y \dashv z)\,, \\ (x \dashv y) \vdash z &= (x \vdash y) \vdash z. \end{aligned}$$

b) For any $g \in G$ it holds that

$$e \vdash g = g = g \dashv e,$$

the element e is called the bar-unit of G.

c) For all $g \in G$ there exists an unique element $g^{-1} \in G$ such that with respect to e we have

$$g \vdash g^{-1} = e = g^{-1} \dashv g,$$

we say that g^{-1} is the inverse of g.

It must be noted that from this definition it does not follow that e is the unique identity in G; in fact in general the digroup can have many identities (that is several elements $\widetilde{e}$ such that $\widetilde{e} \vdash g = g = g \dashv \widetilde{e}$ for all $g \in G$). The notation (G, e) only suggests that between all the identities we have chosen e as the bar-unit of (G, e) with respect to which have means the point c) of the Definition 3.1.

The Theorem 3.1 allows us construct finite digroups, for instance: Let $G = \{x, y\}$ be an arbitrary set of two elements. We can introduce a 2×2 $(\vdash)-$multiplication table and a 2×2 $(\dashv)-$multiplication table in G of the following form:

$\vdash$	x	y
x	x	y
y	x	y

$\dashv$	x	y
x	x	x
y	y	y

,

it is easy to show that (G, x) is a digroup.

Below we establish the basic properties of a digroup.

EXAMPLE 3.1. Let X be an arbitrary vector space and x_0 an element in X. Then (X_l, x_0) is a digroup. In fact $x_0 \vdash z = z = z \dashv x_0$ for all $z \in X_l$, moreover $z \vdash x_0 = x_0 = x_0 \dashv z$ for any $z \in X_l$.

EXAMPLE 3.2. The set $V(\Pi)$ is a digroup. Here $e = \theta$ and $g^{-1} = -\Pi g$ for all $g \in G$.

EXAMPLE 3.3. Let V be a finite dimensional vector space and let φ be a nonzero element of V^* then the set $\widehat{V_\varphi} = \{x \in V \mid \varphi(x) \neq 0\} \subset V_\varphi$ is a digroup with bar-unit $e = \frac{x_0}{\varphi(x_0)}$ for some $x_0 \in \widehat{V_\varphi}$. If $z \in \widehat{V_\varphi}$ then $z^{-1} = \frac{1}{\varphi(z)} e$.

REMARK 3.1. Let (G, e) be a digroup and $f \in G$ some unit of G then $f^{-1} = e$.

REMARK 3.2. It is easy to show that if (G, e) is a digroup and $g = e \dashv g$ for any $g \in G$ then $\vdash = \dashv$ and G is a group, thus all groups are digroups.

A digroup (G, e) is called abelian if $x \dashv y = y \vdash x$ for all x, $y \in G$. A nonempty subset $H \subset G$ is said to be a subdigroup of (G, e), provided that (H, e) is a digroup for the same products that (G, e).

LEMMA 3.1. *Let (G, e) be a digroup then for all $g \in G$ we have $\left(g^{-1}\right)^{-1} = g \vdash e = e \dashv g$.*

Proof. Let $g \in G$ then $g^{-1} \vdash (g \vdash e) = \left(g^{-1} \vdash g\right) \vdash e = \left(g^{-1} \dashv g\right) \vdash e = e$. On the other hand $(e \dashv g) \dashv g^{-1} = e \dashv \left(g \dashv g^{-1}\right) = e \dashv \left(g \vdash g^{-1}\right) = e$. Now since the inverse of any element is unique it follows that $g \vdash e = e \dashv g$ moreover $\left(g^{-1}\right)^{-1} = g \vdash e = e \dashv g$. ■

THEOREM 3.3. *In order that (H, e) can be a subdigroup of a digroup (G, e) it is necessary and sufficient that for all f, g, l, m, $n \in H$ the elements $f \vdash e$, $g^{-1} \vdash l$ and $m \dashv n^{-1}$ belong to H.*

Proof. The conditions are clearly necessary. Let $x \in H$ then $\left(x^{-1} \vdash x\right) \vdash e = \left(x^{-1} \dashv x\right) \vdash e = e \in H$. Since now we know that e is an element of H, $x^{-1} \vdash e = x^{-1} \vdash \left(x \vdash x^{-1}\right) = \left(x^{-1} \vdash x\right) \vdash x^{-1} = \left(x^{-1} \dashv x\right) \vdash x^{-1} = e \vdash x^{-1} = x^{-1} \in H$, thus for all $x \in H$ also $x^{-1} \in H$. It follows from the Lemma 3.1 that $f \vdash g = (f \dashv e) \vdash g = (f \vdash e) \vdash g = \left(f^{-1}\right)^{-1} \vdash g \in H$ for all f, $g \in H$. Finally, also using the referred Lemma 3.1 we have $m \dashv l = m \dashv (e \vdash l) = m \dashv (e \dashv l) = m \dashv \left(l^{-1}\right)^{-1} \in H$ for any m, $l \in H$. Hence H is closed under the products $\vdash$ and $\dashv$. Consequently the theorem is proved. ■

The intersection of two subdigroups (H, e) and (K, e) of a group (G, e) is not an empty set, since all subdigroups contain the element e. The intersection is really a subdigroup of G: if $D = H \cap K$ and if the elements f, g, l, m, n belong to D then the elements $f \vdash e$, $g^{-1} \vdash l$ and $m \dashv n^{-1}$ belong to H as well as to K and hence to D, that is (D, e) is a subdigroup.

Let (H_1, e), (H_2, e), $\cdots$, (H_n, e),$\cdots$ be subdigroups of a digroup (G, e) which form an ascending sequence, that is, $H_n \subset H_{n+1}$, $n = 1, 2, \cdots$. We show that $(\cup H_n, e)$ is a subdigroup of (G, e). It is clear that $e \in \cup H_n$. Each $g \in \cup H_n$ lies in some subdigroup H_s then also $g^{-1} \in H_s$ and hence $g^{-1} \in \cup H_n$. Now if f, g, l, m, n belonging to $\cup H_n$ are chosen lying in H_{k_1}, H_{k_2}, H_{k_3}, H_{k_4} and H_{k_5} respectively, then if $k \geq k_i$ $i = 1, \cdots, 5$ we have that f, g, l, m, $n \in H_k$ hence the elements $f \vdash e$, $g^{-1} \vdash l$ and $m \dashv n^{-1}$ belong to H_k and also to $\cup H_n$.

From now on all operators in any digroup will be denoted by $\vdash$ and $\dashv$, this should cause no confusion.

DEFINITION 3.2. A mapping γ of a digroup (G, e) into a digroup (G', e') is called a digroup-homomorphism (or homomorphism) if $\gamma\,(a \vdash b) = \gamma\,(a) \vdash \gamma\,(b)$ and also $\gamma\,(c \dashv d) = \gamma\,(c) \dashv \gamma\,(d)$ for all a, b, c, $d \in G$. A homomorphism one-to-one correspondence is called a digroup-isomorphism (or isomorphism).

Let γ be a digroup-homomorphism of (G, e) into (G', e'), if $\gamma\,(G) = G'$, then $\gamma\,(e)$ is a unit of G'. Observe that $\gamma\,(e) \neq e'$ can happen. We now assume that $\gamma\,(e) = e'$, we shall show that $(\gamma\,(x))^{-1} = \gamma\left(x^{-1}\right)$ for all $x \in G$, in fact $e' = \gamma\,(e) = \gamma\left(x \vdash x^{-1}\right) = \gamma\,(x) \vdash \gamma\left(x^{-1}\right)$, on the other hand we have $e' = \gamma\,(e) = \gamma\left(x^{-1} \dashv x\right) = \gamma\left(x^{-1}\right) \dashv \gamma\,(x)$. Hence, $(\gamma\,(x))^{-1} = \gamma\left(x^{-1}\right)$.

Example 3.4. If (G, e) and (G', e') are digroups, the direct product of G with G', denoted $G \times G'$ is the set of all ordered pairs (g, g') , where $g \in G$ and $g' \in G$, with the two operators $(g, g') \vdash (f, f') = (g \vdash f, g' \vdash f')$ and $(g, g') \dashv (f, f') = (g \dashv f, g' \dashv f')$. It is easy to check that $(G \times G', (e, e'))$ is a digroup containing homomorphic copies of G and G' namely, $G \times \{e'\}$ and $\{e\} \times G'$.

Now we have

Theorem 3.4. *Let γ be a digroup-homomorphism of (G, e) into (G', e') such that $\gamma(e) = e'$. Then, if we define $N = \{g \in G \mid \gamma(g) = e'\}$, (N, e) is a subdigroup of (G, e) and is called the kernel of γ.*

Proof. Note that $e \in N$. Let us assume that $x \in N$ then $e' = \gamma\left(x^{-1} \dashv x\right) = \gamma\left(x^{-1}\right) \dashv e' = \gamma\left(x^{-1}\right)$. Thus, $x^{-1} \in N$. It is now obvious that if a, b, c, d and f are arbitrary elements of N then $a \vdash e$, $b^{-1} \vdash c$ and $d \dashv f^{-1}$ belong to N. The Theorem is proved. ∎

Theorem 3.5. *Let γ be a digroup-homomorphism of (G, e) into (G', e') such that $\gamma(e) = e'$. We define $I' = \{\gamma(g) \mid g \in G\} \subset G'$, then (I', e') is a subdigroup of (G', e').*

Proof. First let us note that $e' \in I'$. Suppose that $h' \in I'$ then $h' = \gamma(h)$ for some $h \in G$. Hence, we have $(h')^{-1} = (\gamma(h))^{-1} = \gamma\left(h^{-1}\right)$. It is then clear that $(h')^{-1} \in I'$. Finally, it is a simple matter to verify that if a', b', c', d' and f' are arbitrary elements of I' then $a' \vdash e'$, $(b')^{-1} \vdash c'$ and $d' \dashv (f')^{-1}$ are also elements of I'. ∎

A subdigroup (H, e) of the digroup (G, e) is called invariant or normal if $\left(a^{-1} \vdash x\right) \dashv a = a^{-1} \vdash (x \dashv a) \in H$ for all $a \in G$ and any $x \in H$. Then we have

Proposition 3.1. *Under the conditions of the Theorem 3.4, (N, e) is an invariant or normal subdigroup.*

Proof. Let $z = \left(a^{-1} \vdash x\right) \dashv a$ where $x \in N$ and $a \in G$ then from Lemma 3.1 it follows that

$$\begin{aligned}
\gamma(z) &= \gamma\left(\left(a^{-1} \vdash x\right) \dashv a\right) \\
&= \gamma\left(a^{-1} \vdash x\right) \dashv \gamma(a) \\
&= \left(\gamma\left(a^{-1}\right) \vdash \gamma(x)\right) \dashv \gamma(a) \\
&= \left(\gamma\left(a^{-1}\right) \vdash e'\right) \dashv \gamma(a) \\
&= (\gamma(a))^{-1} \vdash (e' \dashv \gamma(a)) \\
&= e',
\end{aligned}$$

this establishes that $z \in N$. Hence N is a normal subdigroup. ■

Lemma 3.2. *Let (G,e) be a digroup and let (H,e) be a normal subdigroup of it. Then $\left(a^{-1} \vdash H\right) \dashv a = H$ for any $a \in G$.*

Proof. Since (H,e) is an invariant subdigroup we have that

$$\left(a^{-1} \vdash H\right) \dashv a \subset H \tag{10}$$

for all $a \in G$. Let $b \in G$ arbitrary then taking $a = b^{-1}$ in (10) we have $((b \vdash e) \vdash H) \dashv b^{-1} \subset H$, now multiplying this inequality by b to the left and by b^{-1} to the right we obtain that

$$b^{-1} \vdash \left(((b \vdash e) \vdash H) \dashv b^{-1}\right) \dashv b \subset \left(b^{-1} \vdash H\right) \dashv b, \tag{11}$$

but $b^{-1} \vdash \left(((b \vdash e) \vdash H) \dashv b^{-1}\right) = H$, Hence from (11) it follows that

$$H \subset \left(b^{-1} \vdash H\right) \dashv b \tag{12}$$

for all $b \in G$. Thus (10) and (12) show that $\left(c^{-1} \vdash H\right) \dashv c = H$ for any $c \in G$. ■

Corollary 3.1. *If (H,e) is a normal subdigroup of (G,e), then also we have $(b \vdash H) \dashv b^{-1} = H$ for all $b \in H$.*

Proof. By the preceding Lemma $\left(a^{-1} \vdash H\right) \dashv a = H$ for any $a \in G$. Let $b \in G$ and $a = b^{-1}$ then we have $\left(\left(b^{-1}\right)^{-1} \vdash H\right) \dashv b^{-1} = H$ hence $((b \vdash e) \vdash H) \dashv b^{-1} = H$ that is $(b \vdash (e \vdash H)) \dashv b^{-1} = H$. ■

The notion of equivalence plays an important role in almost all branches of mathematics, since it is related to the partitioning of a set, that is, to its representation in the form of a union of mutually disjoint subsets. Let (H,e) be a subdigroup of (G,e). We say that a and b in G are right relationed and we write $a \backsim b$ if $a^{-1} \vdash b \in H$. Let us see that $\backsim$ is an equivalent relation in (G,e) under appropriate conditions. It will be called the right equivalent relation determined by H, but first we prove some results.

Lemma 3.3. *The following two results: $(a \vdash b)^{-1} = b^{-1} \vdash a^{-1}$ and $(a \dashv b)^{-1} = b^{-1} \dashv a^{-1}$hold for any a, $b \in G$.*

Proof. In fact, let a, $b \in G$ then

$$\begin{aligned}(a \vdash b) \vdash (b^{-1} \vdash a^{-1}) &= a \vdash \left(b \vdash (b^{-1} \vdash a^{-1})\right) \\ &= a \vdash \left(\left(b \vdash b^{-1}\right) \vdash a^{-1}\right) \\ &= a \vdash \left(e \vdash a^{-1}\right) \\ &= e\end{aligned}$$

on the other hand from Lemma 3.1 it follows that

$$\begin{aligned}(b^{-1} \vdash a^{-1}) \dashv (a \vdash b) &= (b^{-1} \vdash a^{-1}) \dashv (a \dashv b)\\ &= \big((b^{-1} \vdash a^{-1}) \dashv a\big) \dashv b\\ &= \big(b^{-1} \vdash \big(a^{-1} \dashv a\big)\big) \dashv b\\ &= \big(b^{-1} \vdash e\big) \dashv b = b^{-1} \vdash (e \dashv b)\\ &= e,\end{aligned}$$

thus, it shows that $(a \vdash b)^{-1} = b^{-1} \vdash a^{-1}$. We complete the proof by noting that

$$\begin{aligned}\big(b^{-1} \dashv a^{-1}\big) \dashv (a \dashv b) &= b^{-1} \dashv \big(a^{-1} \dashv (a \dashv b)\big)\\ &= b^{-1} \dashv \big(\big(a^{-1} \dashv a\big) \dashv b\big)\\ &= b^{-1} \dashv (e \dashv b)\\ &= b^{-1} \dashv (e \vdash b)\\ &= e\end{aligned}$$

and moreover using Lemma 3.1 again we have

$$\begin{aligned}(a \dashv b) \vdash \big(b^{-1} \dashv a^{-1}\big) &= (a \vdash b) \vdash \big(b^{-1} \dashv a^{-1}\big)\\ &= a \vdash \big(b \vdash \big(b^{-1} \dashv a^{-1}\big)\big)\\ &= a \vdash \big(\big(b \vdash b^{-1}\big) \dashv a^{-1}\big)\\ &= a \vdash \big(e \dashv a^{-1}\big)\\ &= (a \vdash e) \dashv a^{-1}\\ &= e,\end{aligned}$$

hence $(a \dashv b)^{-1} = b^{-1} \dashv a^{-1}$. The Lemma is proved. ■

From the Lemma proved, now holds the following

Corollary 3.2. *Let (G, e) be a digroup then for all $g \in G$ we have $\big(g^{-1} \vdash e\big) = \big(e \dashv g^{-1}\big) = g^{-1}$*

Proof. We recall that $e^{-1} = e$. Then from Lemma 3.3 it follows that $\big(g^{-1} \vdash e\big) = (e \vdash g)^{-1} = g^{-1}$ and in the same way $\big(e \dashv g^{-1}\big) = (g \dashv e)^{-1} = g^{-1}$. ■

Let (G, e) be a digroup. We now introduce the following set $\widetilde{G} = \big\{x \in G \mid x^{-1} = e\big\}$, notice that if $\widetilde{e}$ is an identity then $\widetilde{e} \in \widetilde{G}$ (since $\widetilde{e} \vdash$

$e = e = e \dashv \widetilde{e})$. We are going to prove that $\left(\widetilde{G}, e\right)$ is a subdigroup. Let x, $y \in \widetilde{G}$ then, clearly $(x \vdash y)^{-1} = y^{-1} \vdash x^{-1} = e \vdash e = e$ and if z, $w \in \widetilde{G}$ we also have $(z \dashv w)^{-1} = w^{-1} \dashv z^{-1} = e \dashv e = e$. From these considerations it follows that $\widetilde{G}$ is closed under the operations $\vdash$ and $\dashv$. On the other hand observe that if $x \in \widetilde{G}$ then $x^{-1} \in \widetilde{G}$. Lastly, it is easy to show that if x, y, z, $w \in \widetilde{G}$ then $x \vdash e$, $x^{-1} \vdash y$ and $z \dashv w^{-1}$ belong to $\widetilde{G}$. Thus $\left(\widetilde{G}, e\right)$ is a subdigroup.

We say that (G, e) is a reduced digroup if $\widetilde{G} = \{e\}$.

We turn to the proof that $\backsim$ is an equivalent relation in (G, e) if $\widetilde{G} = \{e\}$, that is when (G, e) is a reduced digroup. Assume that (H, e) is a given subdigroup of (G, e) with respect to which is defined the right relation $\backsim$. We begin with the proof of the reflexivity of $\backsim$. Let $a \in G$ then $\left(a^{-1} \vdash a\right)^{-1} = \left(a^{-1} \vdash (a \vdash e)\right) = \left(\left(a^{-1} \vdash a\right) \vdash e\right) = \left(\left(a^{-1} \dashv a\right) \vdash e\right) = e \vdash e = e$. Since $\widetilde{G} = \{e\}$ we have $a^{-1} \vdash a = e \in H$. Hence $a^{-1} \vdash a \in H$ and $a \backsim a$. The symmetry of $\backsim$ follows from the following facts: if $a \backsim b$ then $a^{-1} \vdash b \in H$ and since (H, e) is a subdigroup from Corollary 3.2 it follows that $\left(a^{-1} \vdash b\right)^{-1} = b^{-1} \vdash (e \dashv a) = \left(b^{-1} \vdash e\right) \dashv a = b^{-1} \dashv a \in H$. Hence $\left(b^{-1} \dashv a\right) \vdash e \in H$ then $\left(b^{-1} \dashv a\right) \vdash \left(a^{-1} \vdash a\right) \in H$, but

$$\begin{aligned}
\left(\left(b^{-1} \dashv a\right) \vdash \left(a^{-1} \vdash a\right)\right) &= \left(b^{-1} \vdash a\right) \vdash \left(a^{-1} \vdash a\right) \\
&= b^{-1} \vdash \left(a \vdash \left(a^{-1} \vdash a\right)\right) \\
&= b^{-1} \vdash \left((a \vdash a^{-1}) \vdash a\right) \\
&= b^{-1} \vdash (e \vdash a) = b^{-1} \vdash a
\end{aligned}$$

it shows that $b \backsim a$.

Suppose now that $a \backsim b$ and $b \backsim c$ then $a^{-1} \vdash b$, $b^{-1} \vdash c \in H$. Hence we must have $\left(a^{-1} \vdash b\right) \vdash \left(b^{-1} \vdash c\right) \in H$, but

$$\begin{aligned}
\left(\left(a^{-1} \vdash b\right) \vdash \left(b^{-1} \vdash c\right)\right) &= \left(\left(a^{-1} \vdash b\right) \vdash b^{-1}\right) \vdash c \\
&= \left(a^{-1} \vdash (b \vdash b^{-1})\right) \vdash c \\
&= \left(a^{-1} \vdash e\right) \vdash c = a^{-1} \vdash c \\
&= a^{-1} \vdash c,
\end{aligned}$$

again by Corollary 3.2. Thus $a \backsim c$ and hence the relation is transitive.

Lemma 3.4. *Let (G, e) be a reduced digroup and (H, e) a subdigroup of it. Then, all right equivalent classes determined by H are of the form $a \vdash H$ for some $a \in G$. A set of this type is called the right dicoset of H determined by a.*

Proof. Let A be a right equivalent class determined by H and $a \in A$ then for any $x \in A$ we know that $a^{-1} \vdash x \in H$. Hence, $a \vdash \left(a^{-1} \vdash x\right) = \left(a \vdash a^{-1}\right) \vdash x = e \vdash x = x$. Thus $x \in a \vdash H$. We now proceed to prove that $a \vdash H$ is a right equivalent class determined by H. Indeed, if x, $y \in H$, having in mind Theorem 3.3 and Lemma 3.3, we have

$$\begin{aligned}
(a \vdash x)^{-1} \vdash (a \vdash y) &= \left(x^{-1} \vdash a^{-1}\right) \vdash (a \vdash y) \\
&= x^{-1} \vdash \left(a^{-1} \vdash (a \vdash y)\right) \\
&= x^{-1} \vdash \left(\left(a^{-1} \vdash a\right) \vdash y\right) \\
&= x^{-1} \vdash \left(\left(a^{-1} \dashv a\right) \vdash y\right) \\
&= x^{-1} \vdash y
\end{aligned}$$

that is $(a \vdash x)^{-1} \vdash (a \vdash y) \in H$ in other words $(a \vdash x) \backsim (a \vdash y)$. Since $a = a \vdash \left(a^{-1} \vdash a\right) \in a \vdash H$ then $A = a \vdash H$. Here we have made use of the fact that $a \backsim a$ thus $\left(a^{-1} \vdash a\right) \in H$. ■

Proposition 3.2. *Let (G, e) be a reduced digroup and let (H, e) be a normal subdigroup of it. Let a, $b \in G$ arbitrary. We have $(a \vdash H) \vdash (b \vdash H) = (a \vdash b) \vdash H$ and $(a \vdash H) \dashv (b \vdash H) = (a \dashv b) \vdash H$.*

Proof. Since that (H, e) is a normal subdigroup, by Corollary 3.1 it follows that

$$\begin{aligned}
(a \vdash H) \vdash (b \vdash H) &= \left(a \vdash ((b \vdash H) \dashv b^{-1})\right) \vdash (b \vdash H) \\
&= \left((a \vdash (b \vdash H)) \dashv b^{-1}\right) \vdash (b \vdash H) \\
&= \left(\left((a \vdash b) \vdash H\right) \dashv b^{-1}\right) \vdash (b \vdash H) \\
&= \left(\left((a \vdash b) \vdash H\right) \vdash b^{-1}\right) \vdash (b \vdash H) \\
&= ((a \vdash b) \vdash H) \vdash (b^{-1} \vdash (b \vdash H)) \\
&= ((a \vdash b) \vdash H) \vdash ((b^{-1} \vdash b) \vdash H) \\
&= ((a \vdash b) \vdash H) \vdash ((b^{-1} \dashv b) \vdash H) \\
&= ((a \vdash b) \vdash H) \vdash H \\
&= (a \vdash b) \vdash (H \vdash H) \\
&= (a \vdash b) \vdash H,
\end{aligned}$$

here we have taken into account that $(H \vdash H) = H$. We now give the proof of the second equality. Using Corollary 3.1 again , we have

$$(a \vdash H) \dashv (b \vdash H) = \left(a \vdash ((b \vdash H) \dashv b^{-1})\right) \dashv (b \vdash H)$$

$$
\begin{aligned}
&= \left(\big(a \vdash (b \vdash H)\big) \dashv b^{-1}\right) \dashv (b \vdash H) \\
&= \left(\big((a \vdash b) \vdash H\big) \dashv b^{-1}\right) \dashv (b \vdash H) \\
&= \left(\big((a \dashv b) \vdash H\big) \dashv b^{-1}\right) \dashv (b \vdash H) \\
&= ((a \dashv b) \vdash H) \dashv \left(b^{-1} \dashv (b \vdash H)\right) \\
&= ((a \dashv b) \vdash H) \dashv \left(b^{-1} \dashv (b \dashv H)\right) \\
&= ((a \dashv b) \vdash H) \dashv \left(\left(b^{-1} \dashv b\right) \dashv H\right) \\
&= ((a \dashv b) \vdash H) \dashv (e \dashv H) \\
&= ((a \dashv b) \vdash H) \dashv (e \vdash H) \\
&= ((a \dashv b) \vdash H) \dashv H = (a \dashv b) \vdash (H \dashv H) \\
&= (a \dashv b) \vdash H,
\end{aligned}
$$

since $(H \dashv H) = H$. ■

Note that this Proposition holds even when (G, e) is not reduced.

We turn attention now to the set G/H of the right dicosets determined for the normal subdigroup H of G when (G, e) is a reduced digroup.

THEOREM 3.6. *Let (G, e) be a reduced digroup and let (H, e) be a normal subdigroup of (G, e). In this case G/H is a digroup.*

Proof. From Proposition 3.2, to prove the Theorem, it remains to establish the existence of the identity element in G/H and moreover we must prove that all right dicosets admit an inverse right dicoset. It is easy to show that $H = e \vdash H$ is the unit of G/H. Now by the previous Proposition it follows that $(a \vdash H) \vdash \left(a^{-1} \vdash H\right) = \left(a \vdash a^{-1}\right) \vdash H = e \vdash H = H$, and $\left(a^{-1} \vdash H\right) \dashv (a \vdash H) = \left(a^{-1} \dashv a\right) \vdash H = e \vdash H = H$. Observe that this inverse is clearly unique. ■

Let (G, e) be a digroup, a homomorphism $f\colon (G, e) \to (G, e)$ is called an endomorphism of (G, e); an isomorphism $f\colon (G, e) \to (G, e)$ is called an automorphism of (G, e).

PROPOSITION 3.3. *Let (G, e) be a digroup and the mapping $u_a\colon G \to G$ defined by the following form $u_a g = (a^{-1} \vdash g) \dashv a$ for $a \in G$. Then u_a is an automorphism of (G, e). It is called an inner automorphism of (G, e).*

Proof. In fact, if $a \in G$ then

$$
\begin{aligned}
u_a (g \vdash f) &= (a^{-1} \vdash (g \vdash f)) \dashv a \\
&= (a^{-1} \vdash (g \vdash ((a \vdash a^{-1}) \vdash f))) \dashv a \\
&= (\left(a^{-1} \vdash g\right) \vdash ((a \vdash a^{-1}) \vdash f)) \dashv a \\
&= \left(a^{-1} \vdash g\right) \vdash \left(((a \vdash a^{-1}) \vdash f) \dashv a\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(a^{-1} \vdash g\right) \vdash \left(\left(a \vdash \left(a^{-1} \vdash f\right)\right) \dashv a\right) \\
&= \left(a^{-1} \vdash g\right) \vdash \left(a \vdash \left(\left(a^{-1} \vdash f\right) \dashv a\right)\right) \\
&= \left(\left(a^{-1} \vdash g\right) \vdash a\right) \vdash \left(\left(a^{-1} \vdash f\right) \dashv a\right) \\
&= \left(\left(a^{-1} \vdash g\right) \dashv a\right) \vdash \left(\left(a^{-1} \vdash f\right) \dashv a\right),
\end{aligned}
$$

hence $u_a\left(g \vdash f\right) = \left(u_a g\right) \vdash \left(u_a f\right)$. On the other hand

$$
\begin{aligned}
u_a\left(g \dashv f\right) &= \left(a^{-1} \vdash \left(g \dashv f\right)\right) \dashv a \\
&= \left(a^{-1} \vdash \left(\left(g \dashv \left(a \vdash a^{-1}\right)\right) \dashv f\right)\right) \dashv a \\
&= \left(a^{-1} \vdash \left(g \dashv \left(\left(a \vdash a^{-1}\right) \dashv f\right)\right)\right) \dashv a \\
&= \left(a^{-1} \vdash \left(g \dashv \left(\left(a \vdash a^{-1}\right) \vdash f\right)\right)\right) \dashv a \\
&= a^{-1} \vdash \left(\left(g \dashv \left(\left(a \vdash a^{-1}\right) \vdash f\right)\right) \dashv a\right) \\
&= a^{-1} \vdash \left(g \dashv \left(\left(\left(a \vdash a^{-1}\right) \vdash f\right) \dashv a\right)\right) \\
&= a^{-1} \vdash \left(g \dashv \left(\left(a \vdash a^{-1}\right) \vdash \left(f \dashv a\right)\right)\right) \\
&= a^{-1} \vdash \left(g \dashv \left(a \vdash \left(a^{-1} \vdash \left(f \dashv a\right)\right)\right)\right) \\
&= \left(a^{-1} \vdash g\right) \dashv \left(a \vdash \left(a^{-1} \vdash \left(f \dashv a\right)\right)\right) \\
&= \left(a^{-1} \vdash g\right) \dashv \left(a \dashv \left(a^{-1} \vdash \left(f \dashv a\right)\right)\right) \\
&= \left(\left(a^{-1} \vdash g\right) \dashv a\right) \dashv \left(a^{-1} \vdash \left(f \dashv a\right)\right) \\
&= \left(\left(a^{-1} \vdash g\right) \dashv a\right) \dashv \left(\left(a^{-1} \vdash f\right) \dashv a\right),
\end{aligned}
$$

therefore $u_a\left(g \dashv f\right) = \left(u_a g\right) \dashv \left(u_a f\right)$. In order to prove that u_a is an automorphism of (G, e), we must show that u_a is one-to-one. Assume that $u_a f = u_a g$ then $(a^{-1} \vdash f) \dashv a = (a^{-1} \vdash g) \dashv a$, but it implies that $(a^{-1} \vdash f) = (a^{-1} \vdash g)$ and hence $f = g$. ■

The following Lemma will be useful.

LEMMA 3.5. *Let x, $g \in G$, being (G, e) a digroup, then*

$$
\left(\left(g^{-1} \vdash x\right) \dashv g\right)^{-1} = \left(g^{-1} \vdash x^{-1}\right) \dashv g
$$

.

Proof. Because of Lemma 3.1 we have

$$
\begin{aligned}
\left(\left(g^{-1} \vdash x\right) \dashv g\right) \vdash \left(\left(g^{-1} \vdash x^{-1}\right) \dashv g\right) &= \left(\left(g^{-1} \vdash x\right) \vdash g\right) \vdash \left(\left(g^{-1} \vdash x^{-1}\right) \dashv g\right) \\
&= \left(g^{-1} \vdash x\right) \vdash \left(g \vdash \left(\left(g^{-1} \vdash x^{-1}\right) \dashv g\right)\right) \\
&= \left(g^{-1} \vdash x\right) \vdash \left(\left(g \vdash \left(g^{-1} \vdash x^{-1}\right)\right) \dashv g\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(g^{-1} \vdash x\right) \vdash \left(\left(\left(g \vdash g^{-1}\right) \vdash x^{-1}\right) \dashv g\right) \\
&= \left(g^{-1} \vdash x\right) \vdash \left(\left(e \vdash x^{-1}\right) \dashv g\right) \\
&= \left(g^{-1} \vdash x\right) \vdash \left(x^{-1} \dashv g\right) \\
&= \left(\left(g^{-1} \vdash x\right) \vdash x^{-1}\right) \dashv g \\
&= \left(g^{-1} \vdash \left(x \vdash x^{-1}\right)\right) \dashv g \\
&= g^{-1} \vdash (e \dashv g) \\
&= e,
\end{aligned}
$$

on the other hand

$$
\begin{aligned}
\left(\left(g^{-1} \vdash x^{-1}\right) \dashv g\right) \dashv \left(g^{-1} \vdash (x \dashv g)\right) &= \left(\left(g^{-1} \vdash x^{-1}\right) \dashv g\right) \dashv \left(g^{-1} \dashv (x \dashv g)\right) \\
&= \left(\left(\left(g^{-1} \vdash x^{-1}\right) \dashv g\right) \dashv g^{-1}\right) \dashv (x \dashv g) \\
&= \left(\left(g^{-1} \vdash x^{-1}\right) \dashv \left(g \dashv g^{-1}\right)\right) \dashv (x \dashv g) \\
&= \left(\left(g^{-1} \vdash x^{-1}\right) \dashv \left(g \vdash g^{-1}\right)\right) \dashv (x \dashv g) \\
&= \left(\left(g^{-1} \vdash x^{-1}\right) \dashv e\right) \dashv (x \dashv g) \\
&= \left(g^{-1} \vdash x^{-1}\right) \dashv (x \dashv g) \\
&= g^{-1} \vdash \left(x^{-1} \dashv (x \dashv g)\right) \\
&= g^{-1} \vdash \left(\left(x^{-1} \dashv x\right) \dashv g\right) \\
&= g^{-1} \vdash (e \dashv g) \\
&= e,
\end{aligned}
$$

here, again we have used Lemma 3.1. ∎

Let (G, e) be a digroup, the center $Z(G)$ of (G, e) is the set of all $x \in G$ such that $\left(g^{-1} \vdash x\right) \dashv g = x$ for all $g \in G$. It is routine to check that it implys the equality $x \dashv g = g \vdash x$.

THEOREM 3.7. *Let (G, e) be a digroup. Then $(Z(G), e)$ is a subdigroup of (G, e)*

Proof. It follows from Lemma 3.1 that $\left(g^{-1} \vdash e\right) \dashv g = g^{-1} \vdash (e \dashv g) = e$, thus $e \in Z(G)$. Suppose now that $x \in Z(G)$ then $\left(g^{-1} \vdash x\right) \dashv g = x$ for all $g \in G$, taking the inverse in the last equality we have $\left(\left(g^{-1} \vdash x\right) \dashv g\right)^{-1} = x^{-1}$ and by Lemma 3.5 it follows that $\left(g^{-1} \vdash x^{-1}\right) \dashv g = x^{-1}$, that is $x^{-1} \in Z(G)$. Let $x \in Z(G)$, then for all $g \in G$

$$
\begin{aligned}
\left(g^{-1} \vdash (x \vdash e)\right) \dashv g &= \left(g^{-1} \vdash \left(\left(x \dashv \left(g \vdash g^{-1}\right)\right) \vdash e\right)\right) \dashv g \\
&= \left(\left(g^{-1} \vdash \left(x \dashv \left(g \vdash g^{-1}\right)\right)\right) \vdash e\right) \dashv g
\end{aligned}
$$

$$
\begin{aligned}
&= (((g^{-1} \vdash x) \dashv \left(g \vdash g^{-1}\right)) \vdash e) \dashv g \\
&= (((g^{-1} \vdash x) \dashv \left(g \dashv g^{-1}\right)) \vdash e) \dashv g \\
&= (((g^{-1} \vdash x) \dashv g) \dashv g^{-1}) \vdash e) \dashv g \\
&= (((g^{-1} \vdash x) \dashv g) \vdash g^{-1}) \vdash e) \dashv g \\
&= (((g^{-1} \vdash x) \dashv g) \vdash (g^{-1} \vdash e)) \dashv g \\
&= ((g^{-1} \vdash x) \dashv g) \vdash \left((g^{-1} \vdash e) \dashv g\right) \\
&= x \vdash e,
\end{aligned}
$$

hence $x \vdash e \in Z(G)$. On the other hand if $x,\, y \in Z(G)$ we have

$$
\begin{aligned}
(g^{-1} \vdash (x^{-1} \vdash y)) \dashv g &= (g^{-1} \vdash ((x^{-1} \dashv \left(g \vdash g^{-1}\right)) \vdash y)) \dashv g \\
&= (g^{-1} \vdash ((x^{-1} \dashv \left(g \dashv g^{-1}\right)) \vdash y)) \dashv g \\
&= (g^{-1} \vdash (((x^{-1} \dashv g) \dashv g^{-1}) \vdash y)) \dashv g \\
&= (g^{-1} \vdash (((x^{-1} \dashv g) \vdash g^{-1}) \vdash y)) \dashv g \\
&= (g^{-1} \vdash ((x^{-1} \dashv g) \vdash (g^{-1} \vdash y))) \dashv g \\
&= ((g^{-1} \vdash (x^{-1} \dashv g)) \vdash (g^{-1} \vdash y)) \dashv g \\
&= (((g^{-1} \vdash x^{-1}) \dashv g) \vdash (g^{-1} \vdash y)) \dashv g \\
&= ((g^{-1} \vdash x^{-1}) \dashv g) \vdash \left((g^{-1} \vdash y) \dashv g\right) \\
&= x^{-1} \vdash y,
\end{aligned}
$$

thus $x^{-1} \vdash y \in Z(G)$. Finally, it is easy to show that if $z,\, w \in Z(G)$ then $z \dashv w^{-1} \in Z(G)$. This concludes the proof of the Theorem. ■

THEOREM 3.8. *Let (G, e) be a digroup. Then $(Z(G), e)$ is an invariant subdigroup.*

Proof. In fact, we must prove that $\left((a^{-1} \vdash x) \dashv a\right) \in Z(G)$ for any $x \in Z(G)$ and all $a \in G$. Now if g is an arbitrary element of G we have

$$
\begin{aligned}
\left(g^{-1} \vdash \left((a^{-1} \vdash x) \dashv a\right)\right) \dashv g &= \left(g^{-1} \vdash x\right) \dashv g \\
&= x \\
&= \left(a^{-1} \vdash x\right) \dashv a,
\end{aligned}
$$

it implys that $\left((a^{-1} \vdash x) \dashv a\right) \in Z(G)$. ■

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