

GENERALISED BIRNBAUM-SAUNDERS AND SINH-NORMAL DISTRIBUTIONS

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Generalised Birnbaum-Saunders and sinh-normal Distributions

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Abstract

This work proposes the univariate and multivariate generalisations of the following distributions: the Birnbaum-Saunders, the three parameter Birnbaum-Saunders and the sinh-normal of spherical type. Similarly, when the stochastic representation of a spherical random vector is assumed, we propose alternative definitions for the above-mentioned distributions including the log-elliptical distribution. We emphasize that all the distributions here derived belong to the family of the spherical distributions.

Key words: Birnbaum-Saunders distribution, life distributions, distributions, sinh-normal, stochastic representation, multivariate log-elliptical distribution.
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1 Introduction

Recently, the family of elliptical contour distributions have been a vertiginous development in many areas of the statistics; the main topic to research goes

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around the extension of the models and techniques of the statistical literature when the distributions are elliptical. The books of Gupta and Varga (1993), Fang et al. (1990) and Fang and Zhang (1990) give an excellent summary of that literature development. Topics in the context of distribution theory and sensibility analysis have been treated by Díaz-García and González-Farías (2004) and Díaz-García and González-Farías (2005b), among many others. Specifically, in statistical theory of reliability Díaz-García and Leiva-Sánchez (2005) generalise the Birnbaum-Saunders distribution under the spherical case. More recently, Díaz-García and Domínguez (2005) propose a family of distributions based on spherical distributions for a dependent sample of life data. Another life distribution is the log-normal distribution, see Nelson and Hann (1972), this was generalised by Fang et al. (1990, p. 55) to the multivariate elliptical case.

In the present work: the multivariate generalisation of the Birnbaum-Saunders distribution, under an elliptical distribution, is given in Section 3. In Section 4 we propose the univariate and multivariate generalisations of the three parameter Birnbaum-Saunders distribution (see Owen (2004)). In Section 5, the univariate and multivariate sinh-spherical distributions are derived (they are known as the generalised log-Birnbaum-Saunders distributions). Section 6 extends the results of Section 5 when the three parameter Birnbaum-Saunders distribution is considered. In Section 7 we propose alternative definitions for the Birnbaum-Saunders, the log-elliptical and the sinh-spherical multivariate distributions, in particular, when the stochastic representation of a random vector with spherical distribution is considered and noting that such representation coincides with some algebraic factorizations of that vector; we emphasized that the new family of distributions continue belonging the family of spherical distributions; the section ends with the extension of the new families to the elliptical case. Finally, in order to check the parameter estimations of the new families, we took some sets of data given in the literature for the independent case, and we simulated a set of life data for the dependent case; then the maximum likelihood estimators of the parameters under particular distributions are obtained, see Section 8.

2 Preliminary considerations

In this section some preliminary results are given as a necessary background for the development of the paper.

Suppose that $Z \sim \mathcal{N}(0, 1)$, and let $\alpha > 0$ and $\beta > 0$. The random variable

$$S = \beta \left[\frac{\alpha}{2} Z + \sqrt{\left(\frac{\alpha}{2} Z\right)^2 + 1} \right]^2 \quad (1)$$

has a distribution known as the Birnbaum-Saunders distribution, which is denoted by $S \sim \mathcal{BS}(\alpha, \beta)$. More over, its density function is given by

$$f_S(s) = \frac{s^{-3/2}(s + \beta)}{2\alpha(2\pi\beta)^{1/2}} \exp \left\{ -\frac{1}{2\alpha^2} \left(\frac{s}{\beta} + \frac{\beta}{s} - 2 \right) \right\}, \quad s > 0$$

where α is the shape parameter and β is the scale parameter and the median of the distribution, see Birnbaum and Saunders (1969a).

A generalisation of the Birnbaum-Saunders distribution was proposed by Owen (2004); in that case, the inverse transformation of (1) is defined by

$$Z = \frac{1}{\alpha} \left[\left(\frac{v}{\beta} \right)^\lambda - \left(\frac{\beta}{v} \right)^\lambda \right], \quad \lambda > 0 \quad (2)$$

Note that when $\lambda = 0.5$ we get as a particular case the Birnbaum-Saunders distribution.

The generalisation of the Birnbaum-Saunders distribution is termed the three parameter Birnbaum-Saunders distribution and its density is given by

$$f_V(v) = \frac{\lambda}{\alpha v (2\pi)^{1/2}} \left[\left(\frac{v}{\beta} \right)^\lambda + \left(\frac{\beta}{v} \right)^\lambda \right] \exp \left\{ -\frac{1}{2\alpha^2} \left[\left(\frac{v}{\beta} \right)^{2\lambda} + \left(\frac{\beta}{v} \right)^{2\lambda} - 2 \right] \right\},$$

$v > 0$; and we shall denoted by $V \sim \mathcal{GB-S}(\alpha, \beta, \lambda)$.

A very close distribution to the Birnbaum-Saunders is the sinh-normal distribution, and it plays an important role in the log-linear models for life data with Birnbaum-Saunders distribution. The density of a sinh-normal distribution is given by

$$f_Y(y) = \frac{2}{\alpha\sigma\sqrt{2\pi}} \cosh \left(\frac{y - \gamma}{\sigma} \right) \exp \left\{ -\frac{2}{\alpha^2} \sinh^2 \left(\frac{y - \gamma}{\sigma} \right) \right\}, \quad y \in \mathfrak{R},$$

and it is denoted by $Y \sim \mathcal{SN}(\alpha, \gamma, \sigma)$, where $\gamma \in \mathfrak{R}$ is a location parameter, $\sigma > 0$ is the scale parameter and $\alpha > 0$ is the shape parameter. This distribution and some properties are studied in Rieck and Nedelman (1991).

By other side, we say that a p -dimensional random vector $\mathbf{Y} = (Y_1, \dots, Y_p)'$ has an elliptical distribution with position parameter $\boldsymbol{\mu} : p \times 1$ and scale parameter

$\Sigma : p \times p$, $\Sigma > 0$, if its density function is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = c|\Sigma|^{-1/2}h\{(\mathbf{y} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{y} - \boldsymbol{\mu})\}, \quad \mathbf{y} \in \mathfrak{R}^p, \quad (3)$$

where the function $h : \mathfrak{R} \rightarrow [0, \infty)$ is termed the function generator and it is such that $\int_0^\infty u^{p-1}h(u^2)du < \infty$ with c a proportionality constant, so that $f_{\mathbf{Y}}(\mathbf{y})$ is a density. It shall be denoted by $\mathbf{Y} \sim \mathcal{E}l_p(\boldsymbol{\mu}, \Sigma; h)$. When the vector \mathbf{Y} has finite moments, we have that $E(\mathbf{Y}) = \boldsymbol{\mu}$ and $\text{Var}(\mathbf{Y}) = c_h\Sigma$, where c_h is a positive constant, see for example Fang and Zhang (1990) or Fang et al. (1990). In the particular case when $\boldsymbol{\mu} = \mathbf{0}$ and $\Sigma = \mathbf{I}_p$, we get the family of spherical distributions, they shall denote by $\mathbf{Y} \sim \mathcal{E}_p(\mathbf{0}, \mathbf{I}; h)$. These distributions include as particular cases the following distributions: the Normal, the t -Student, the Pearson type VII, the Logistic, the Bessel y Kotz, among many others.

Let \mathfrak{R}_+^p be the positive part of \mathfrak{R}^p and $\ln(\mathbf{W}) = (\ln(w_1), \dots, \ln(w_p))'$. Suppose that $\mathbf{w} \in \mathfrak{R}_+^p$ be a random vector such that $\log(\mathbf{W}) \sim \mathcal{E}l_p(\boldsymbol{\mu}, \Sigma; h)$; then we say that the vector \mathbf{W} has a *multivariate log-elliptical distribution Type I* if its density function is given by

$$f_{\mathbf{W}}(\mathbf{W}) = c|\Sigma|^{-1/2} \left(\prod_{i=1}^p w_i^{-1} \right) h\{(\log(\mathbf{w}) - \boldsymbol{\mu})'\Sigma^{-1}(\log(\mathbf{w}) - \boldsymbol{\mu})\}, \quad \mathbf{w} \in \mathfrak{R}_+^p,$$

and it shall be denoted by $\mathbf{W} \sim \mathcal{LE}l_p^I(\boldsymbol{\mu}, \Sigma; h)$.

Now let us consider the transformation (1) when the normality is substituted by a spherical law, this is, define

$$T = \beta \left[\frac{\alpha}{2}U + \sqrt{\left(\frac{\alpha}{2}U\right)^2 + 1} \right]^2$$

by assuming that $U \sim \mathcal{E}_1(0, 1; h)$, then T the following density function

$$f_T(t) = \frac{c t^{-3/2}(t + \beta)}{2\alpha\beta^{1/2}} h \left\{ \frac{1}{\alpha^2} \left(\frac{t}{\beta} + \frac{\beta}{t} - 2 \right) \right\}, \quad t > 0.$$

This is termed the generalised Birnbaum-Saunders distribution and it shall be denoted by $T \sim \mathcal{GBS}(\alpha, \beta; h)$, Díaz-García and Leiva-Sánchez (2005).

3 Birnbaum-Saunders distribution

In this section we give some generalisations of the Birnbaum-Saunders distribution to the multivariate case.

Theorem 1 Suppose that $\mathbf{U} \sim \mathcal{E}_n(\mathbf{0}, \mathbf{I}; h)$ and define the transformation

$$T_i = \beta_i \left(\frac{1}{2} \alpha_i U_i + \sqrt{\left(\frac{1}{2} \alpha_i U_i \right)^2 + 1} \right)^2, \quad \alpha_i > 0, \quad \beta_i > 0, \quad i = 1, \dots, n;$$

the distribution of the vector $\mathbf{T} = (T_1, \dots, T_n)'$ is termed the multivariate generalised Birnbaum-Saunders distribution type I, and its density function is given by

$$f_{\mathbf{T}}(\mathbf{t}) = \frac{c}{2^n} \left(\prod_{i=1}^n \frac{t_i^{-3/2} (t_i + \beta_i)}{\alpha_i \sqrt{\beta_i}} \right) h \left\{ \sum_{i=1}^n \frac{1}{\alpha_i^2} \left(\frac{t_i}{\beta_i} + \frac{\beta_i}{t_i} - 2 \right) \right\}, \quad \mathbf{t} \in \mathfrak{R}_+^n, \quad (4)$$

and it is denoted by $\mathbf{T} \sim \mathcal{GBS}_n^I(\boldsymbol{\alpha}, \boldsymbol{\beta}; h)$, with $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)'$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)'$.

Proof: Given that $\mathbf{U} \sim \mathcal{E}_n(0, \mathbf{I}; h)$, the density of \mathbf{U} is given by

$$f_{\mathbf{U}}(\mathbf{u}) = c h(\|\mathbf{u}\|^2).$$

The proof follows easily observing that:

$$\begin{aligned} u_i &= \frac{1}{\alpha_i} \left(\sqrt{\frac{t_i}{\beta_i}} - \sqrt{\frac{\beta_i}{t_i}} \right) \quad i = 1, \dots, n, \\ \left| \frac{\partial u_i}{\partial t_i} \right| &= \frac{1}{2^n} \prod_{i=1}^n \frac{t_i^{-3/2} (t_i + \beta_i)}{\alpha_i \sqrt{\beta_i}} \\ \|\mathbf{u}\|^2 &= \sum_{i=1}^n \frac{1}{\alpha_i^2} \left(\frac{t_i}{\beta_i} + \frac{\beta_i}{t_i} - 2 \right). \blacksquare \end{aligned}$$

Remark 2 An interesting particular case of the distribution (4) appears when $\alpha_1 = \dots = \alpha_n = \alpha$ and $\beta_1 = \dots = \beta_n = \beta$, which implies that $\boldsymbol{\alpha} = \alpha \mathbf{1}$ $\boldsymbol{\beta} = \beta \mathbf{1}$ with $\mathbf{1} = (1, \dots, 1)'$; because if the random vector T denotes a random sample of a univariate population, then the density of (4) defines a likelihood function when there is stochastic dependency in the random sample T . This idea and other particular cases and the estimation of the multivariate generalised Birnbaum-Saunders distribution type I can be found in Díaz-García and Domínguez (2005).

4 Three parameter Birnbaum-Saunders distribution

Now we propose the univariate and multivariate extensions of the three parameter Birnbaum-Saunders distribution under a spherical model.

Theorem 3 Suppose that $U \sim \mathcal{E}(0, 1; h)$ and consider the transformation

$$U = \frac{1}{\alpha} \left[\left(\frac{v}{\beta} \right)^\lambda - \left(\frac{\beta}{v} \right)^\lambda \right], \quad \lambda > 0, \quad \beta > 0, \quad \alpha > 0; \quad (5)$$

the distribution of the random variable V is termed generalised three parameter Birnbaum-Saunders distribution, and its density function is the following expression:

$$f_V(v) = \frac{c \lambda}{\alpha v} \left[\left(\frac{v}{\beta} \right)^\lambda + \left(\frac{\beta}{v} \right)^\lambda \right] h \left\{ \frac{1}{\alpha^2} \left[\left(\frac{v}{\beta} \right)^{2\lambda} + \left(\frac{\beta}{v} \right)^{2\lambda} - 2 \right] \right\}, \quad v > 0;$$

this shall be denoted by $V \sim \mathcal{GGB} - \mathcal{S}(\alpha, \beta, \lambda; h)$.

Proof: The demonstration follows by noting that

$$du = \frac{\lambda}{\alpha v} \left(\frac{v^{2\lambda} + \beta^{2\lambda}}{(\beta v)^\lambda} \right) \quad (6)$$

■

Also, observe that this distribution keeps the reciprocal property: $V^{-1} \sim \mathcal{GGB} - \mathcal{S}(\alpha, 1/\beta, \lambda; h)$

The generalisation of that distribution to the multivariate case is immediate:

Theorem 4 Suppose that $\mathbf{U} \sim \mathcal{E}_n(\mathbf{0}, \mathbf{I}; h)$, and consider the transformation (5), for u_i , and $\alpha_i > 0$, $\beta_i > 0$ and $\lambda_i > 0$, $i = 1, \dots, n$; the density function of the random vector $\mathbf{V} = (v_1, \dots, v_n)'$ is given by

$$f_{\mathbf{V}}(v) = c \prod_{i=1}^n \frac{\lambda_i}{\alpha_i v_i} \left[\left(\frac{v_i}{\beta_i} \right)^{\lambda_i} + \left(\frac{\beta_i}{v_i} \right)^{\lambda_i} \right] h \left\{ \sum_{i=1}^n \frac{1}{\alpha_i^2} \left[\left(\frac{v_i}{\beta_i} \right)^{2\lambda_i} + \left(\frac{\beta_i}{v_i} \right)^{2\lambda_i} - 2 \right] \right\},$$

$\mathbf{v} \in \mathfrak{R}_+^n$. Writing $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)'$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)'$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)'$; we shall call it the multivariate generalised three parameter Birnbaum-Saunders distribution type I and it shall denote by $\mathbf{V} \sim \mathcal{GGB} - \mathcal{S}_n^I(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\lambda}; h)$.

Remark 5 An analogous observation to Remark 2 can be established for the multivariate generalised three parameter Birnbaum-Saunders distribution type I, seen its role as a likelihood function in the case of a dependent sample.

5 Sinh-spherical distribution

Let $G_U(u)$ be denotes the distribution function of the random variable $U \sim \mathcal{E}(0, 1; h)$. By analogy to the case of a sinh-normal distribution(see Rieck and Nedelman (1991)), the distribution function $F_Y(y)$, of the random variable Y with sinh-spherical distribution, is given by

$$F_Y(y) = G_U\left(\frac{2}{\alpha} \sinh\left(\frac{y - \gamma}{\sigma}\right)\right);$$

where $\alpha > 0$, $\sigma > 0$ and $\gamma \in \mathfrak{R}$. So, deriving with respect to y we obtain

$$f_Y(y) = \frac{2c}{\alpha\sigma} \cosh\left(\frac{y - \gamma}{\sigma}\right) h\left\{\frac{4}{\alpha^2} \sinh^2\left(\frac{y - \gamma}{\sigma}\right)\right\}, \quad y \in \mathfrak{R}; \quad (7)$$

this distribution shall termed the *generalised sinh-spherical distribution* and it shall be denoted by $Y \sim \mathcal{SE}(\alpha, \gamma, \sigma)$.

As in the case of the sinh-normal distribution, the sinh-spherical distribution is related with the generalised Birnbaum-Saunders distribution.

Theorem 6 *Suppose that $T \sim \mathcal{GBS}(\alpha, \beta; h)$, then $Y = \frac{\sigma}{2} \ln T$ has a sinh-spherical distribution with shape, location and scale parameters given by α , $\gamma = \frac{\sigma}{2} \ln \beta$ and σ , respectively.*

Next, we find the first version of a multivariate generalised sinh-spherical distribution.

Theorem 7 *Consider $\mathbf{T} \sim \mathcal{GBS}_n(\boldsymbol{\alpha}, \boldsymbol{\beta}; h)$ and let $\mathbf{Y} = \frac{1}{2} \boldsymbol{\Sigma} \ln \mathbf{T}$, where $\boldsymbol{\Sigma} : n \times n$ is a non-random positive definite matrix. We say that Y has a multivariate sinh-spherical distribution type I of parameters $\boldsymbol{\alpha} \in \mathfrak{R}_+^n$, $\boldsymbol{\gamma} = \frac{1}{2} \boldsymbol{\Sigma} \ln \boldsymbol{\beta} \in \mathfrak{R}^n$ and $\boldsymbol{\Sigma} > 0$, and its density function is given by*

$$f_{\mathbf{Y}}(\mathbf{y}) = 2^n c \prod_{i=1}^n \left[\frac{\mathbf{e}'_i \boldsymbol{\Sigma}^{-1} \mathbf{e}_i}{\alpha_i} \cosh\left(\mathbf{e}'_i \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \boldsymbol{\gamma})\right) \right] h\left\{ \sum_{i=1}^n \frac{4}{\alpha_i^2} \sinh^2\left(\mathbf{e}'_i \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \boldsymbol{\gamma})\right) \right\}, \quad (8)$$

with $\mathbf{y} \in \mathfrak{R}^n$, and \mathbf{e}_i is the i -th vector of the canonical base in \mathfrak{R}^n . This fact shall be written as $\mathbf{Y} \sim \mathcal{SE}_n^I(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\Sigma}; h)$.

Proof: Define Σ such that

$$\Sigma^{-1} = (\sigma_{ij}^*) = \begin{pmatrix} \Sigma_1^{*'} \\ \vdots \\ \Sigma_n^{*'} \end{pmatrix}, \quad \Sigma_i^* \in \mathfrak{R}^n, \quad i = 1, \dots, n$$

and noting that $\sigma_{ij}^* = \mathbf{e}_i \Sigma^{-1} \mathbf{e}'_j$ and $\Sigma_i^* = \Sigma^{-1} \mathbf{e}_i$. Thus, if $\mathbf{T} = \exp \{2\Sigma^{-1} \mathbf{Y}\}$,

$$(d\mathbf{T}) = 2^n \prod_{i=1}^p \mathbf{e}_i \Sigma^{-1} \mathbf{e}'_i \exp \{2\mathbf{e}'_i \Sigma^{-1} \mathbf{Y}\} (d\mathbf{Y}),$$

where $(d\mathbf{T}) = \bigwedge_{i=1}^p dt_i$ denotes the exterior product of the differential elements of the vector $d\mathbf{T}$, see Muirhead (1982, pp. 52-53). The proof follows by making the change of variable $\boldsymbol{\gamma} = \frac{1}{2}\Sigma \ln \boldsymbol{\beta}$ in the expression (4). ■

Interesting particular cases of the distribution (8) are obtained when:

(1) $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$, in this occasion

(a) $\Sigma^{-1} = \text{diag}(\sigma_1^{-1}, \dots, \sigma_n^{-1})$

(b) $\mathbf{e}'_i \Sigma^{-1} \mathbf{e}_i = \sigma_i^{-1}$

(c) $\mathbf{e}'_i \Sigma^{-1} (\mathbf{y} - \boldsymbol{\gamma}) = \frac{y_i - \gamma_i}{\sigma_i}$

then

$$f_{\mathbf{Y}}(\mathbf{y}) = c 2^n \left(\prod_{i=1}^n \frac{\cosh \left(\frac{y_i - \gamma_i}{\sigma_i} \right)}{\sigma_i \alpha_i} \right) h \left\{ 4 \sum_{i=1}^n \frac{1}{\alpha_i^2} \sinh^2 \left(\frac{y_i - \gamma_i}{\sigma_i} \right) \right\},$$

(2) $\Sigma = \sigma \mathbf{I}_n$, $\gamma_i = \gamma$ y $\alpha_i = \alpha$ for every $i = 1, \dots, n$, then

(a) $\Sigma^{-1} = \sigma^{-1} \mathbf{I}_n$

(b) $\mathbf{e}'_i \Sigma^{-1} \mathbf{e}_i = \sigma^{-1}$

(c) $\mathbf{e}'_i \Sigma^{-1} (\mathbf{y} - \boldsymbol{\gamma}) = \frac{y_i - \gamma}{\sigma}$

thus

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{2^n c}{\sigma^n \alpha^n} \left(\prod_{i=1}^n \cosh \left(\frac{y_i - \gamma}{\sigma} \right) \right) h \left\{ \frac{4}{\alpha^2} \sum_{i=1}^n \sinh^2 \left(\frac{y_i - \gamma}{\sigma} \right) \right\}.$$

6 Sinh-spherical distribution under the three parameter Birnbaum-Saunders distribution

Now we propose the univariate and multivariate versions of the sinh-spherical distribution under the three parameter Birnbaum-Saunders distribution.

Theorem 8 If $V \sim \mathcal{GGB} - \mathcal{S}(\alpha, \beta, \lambda; h)$, then $Y = \frac{\sigma}{2} \ln V$ has a generalised sinh-spherical distribution with shape, location and scale parameters given by α , $\gamma = \frac{\sigma}{2} \ln \beta$ and σ , respectively, where the parameter $\lambda > 0$. Its density function is the following:

$$f_Y(y) = \frac{4 c \lambda}{\alpha \sigma} \cosh \left[2\lambda \left(\frac{y - \gamma}{\sigma} \right) \right] h \left\{ \frac{4}{\alpha^2} \sinh^2 \left[2\lambda \left(\frac{y - \gamma}{\sigma} \right) \right] \right\}, \quad y \in \mathfrak{R}. \quad (9)$$

This shall be denoted by $Y \sim \mathcal{GSE}(\alpha, \gamma, \sigma, \lambda; h)$.

Proof. This is straightforward from Theorem 3 and noting that $dv = \frac{2}{\sigma} \exp(2y/\sigma) dy$.

■

Next we write the multivariate extension:

Theorem 9 Consider $\mathbf{V} \sim \mathcal{GB} - \mathcal{S}_n^I(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\lambda}; h)$ and let $\mathbf{Y} = \frac{1}{2} \boldsymbol{\Sigma} \ln \mathbf{V}$, where $\boldsymbol{\Sigma} : n \times n$ is a non-random positive definite matrix, then, we say \mathbf{Y} has a multivariate generalised sinh-spherical distribution type I of parameters $\boldsymbol{\alpha}$, $\boldsymbol{\lambda} \in \mathfrak{R}_+^n$, $\boldsymbol{\gamma} = \frac{1}{2} \boldsymbol{\Sigma} \ln \boldsymbol{\beta} \in \mathfrak{R}^n$ and $\boldsymbol{\Sigma} > 0$; and its density function is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = 2^{2n} c \prod_{i=1}^n \left[\frac{\mathbf{e}'_i \boldsymbol{\Sigma}^{-1} \mathbf{e}_i \lambda_i}{\alpha_i} \cosh \left(2\lambda_i \mathbf{e}'_i \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \boldsymbol{\gamma}) \right) \right] h \left\{ \sum_{i=1}^n \frac{4}{\alpha_i^2} \sinh^2 \left(2\lambda_i \mathbf{e}'_i \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \boldsymbol{\gamma}) \right) \right\}, \quad (10)$$

where $\mathbf{y} \in \mathfrak{R}^n$, and \mathbf{e}_i is the i -th vector of the canonical base in \mathfrak{R}^n . Here we write this fact by $\mathbf{Y} \sim \mathcal{GSE}_n^I(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}; h)$.

Once again we note that if $\lambda_i = 0.5$, for every $i = 1, \dots, n$, it is obtained the multivariate sinh-spherical distribution type I, see Theorem 7.

7 Some extensions

Let $\mathbf{A} \in \mathfrak{R}^{m \times m}$ be a positive definite matrix with spectral decomposition $\mathbf{A} = \mathbf{H} \mathbf{D} \mathbf{H}'$, where \mathbf{H} is an orthogonal matrix, and $\mathbf{D} = \text{diag}(\lambda_1, \dots, \lambda_m)$ is a diagonal matrix such that $\lambda_1 > \dots > \lambda_m > 0$; if $f(x)$ is a differentiable function of x and $\mathbf{F}(\mathbf{D}) = \text{diag}(f(\lambda_1), \dots, f(\lambda_m))$; then we define $\mathbf{F}(\mathbf{A}) = \mathbf{H} \mathbf{F}(\mathbf{D}) \mathbf{H}'$. For example, if $f(x) = \sqrt{x}$, then $\mathbf{F}(\mathbf{A}) = \mathbf{A}^{1/2}$ defines the non-negative definite square root of \mathbf{A} . Similarly, if $f(x) = x^{-1}$, then $\mathbf{F}(\mathbf{A}) = \mathbf{A}^{-1}$ defines the inverse of the matrix \mathbf{A} , see Srivastava and Khatri (1979, p. 39), Mathai (1997, pp. 95-96) and Díaz-García and Gutiérrez-Jáimez (2005). In function of the spectral decomposition and the singular

value decomposition, SVD, this result was generalised by Díaz-García and Gutiérrez-Jáimez (2005) to semi-definite positive matrices and general rectangular matrices, respectively. With these generalisations we can define the function of a random matrix in an alternative way: instead of the definition of the logarithm of a matrix $\mathbf{A} = (a_{ij})$ in the classical way $\ln(\mathbf{A}) = (\ln(a_{ij}))$, we can define it like $\ln(\mathbf{A}) = \mathbf{H} \ln(\mathbf{D}) \mathbf{H}'$. In particular, these results are applicable to the case of a random vector. But, if $\mathbf{X} \in \Re^m$, the SVD of the vector \mathbf{X} is given simply by $\mathbf{X} = \rho \mathbf{W}$, where $\mathbf{W} \in \Re^m$ is $\|\mathbf{W}\|^2 = 1$ and $\rho = \|\mathbf{X}\|$, and here the vectorial cases for the SVD, the QR decomposition and the polar decomposition coincide, see Díaz-García and González-Farías (2005a). Moreover, the decomposition $\mathbf{X} = \rho \mathbf{W}$, for a random vector with spherical (or elliptical) distribution, is known in the literature as the stochastic representation of the vector \mathbf{X} , see Fang and Zhang (1990, Section 2.5.1) or Fang et al. (1990, Theorem 2.3, p. 30). Also, it is known that $(d\mathbf{X}) = \rho^{m-1} d\rho (\mathbf{W}' d\mathbf{W})$, see Díaz-García and González-Farías (2005a) or Muirhead (1982); then we have the following result:

Theorem 10 *Let $\mathbf{X} \in \Re^m$ be a vector such that its SVD is given by $\mathbf{X} = \rho \mathbf{W}$, with $\rho > 0$ y $\mathbf{W} \in \Re^m$ con $\|\mathbf{W}\|^2 = 1$. Let $f(x)$ be a differentiable function in x and define $\mathbf{F}(\mathbf{X}) = f(\rho) \mathbf{W}$. Then*

$$(d\mathbf{F}(\mathbf{X})) = \left(\frac{f(\rho)}{\rho} \right)^{m-1} \left(\frac{df(\rho)}{d\rho} \right) (d\mathbf{X}), \quad (11)$$

where, as before, $(d\mathbf{X})$ denotes the exterior product, see Muirhead (1982, pp. 52-53).

Proof: The demonstration is similar to that one given in Díaz-García and Gutiérrez-Jáimez (2005) for the general case.

As consequences of the present section we can establish the following results:

Corollary 11 *Let $\mathbf{U} \sim \mathcal{E}_n(\mathbf{0}, \mathbf{I}; h)$ be such that $\mathbf{U} = \mathbf{F}(\mathbf{T}) = f(\rho) \mathbf{W}$, where $\mathbf{T} = \rho \mathbf{W}$ is the SVD of the random vector \mathbf{T} , with $\rho > 0$, $\mathbf{W} \in \Re^m$ and $\|\mathbf{W}\|^2 = 1$; so $f(\rho)$ is given by*

$$f(\rho) = \frac{1}{\alpha} \left[\sqrt{\frac{\rho}{\beta}} - \sqrt{\frac{\beta}{\rho}} \right], \quad \alpha > 0, \quad \beta > 0.$$

The distribution of the vector $\mathbf{T} = (T_1, \dots, T_n)'$ shall be termed the multivariate generalised Birnbaum-Saunders distribution type II, with density function:

$$g_{\mathbf{T}}(\mathbf{t}) = \frac{c (\|\mathbf{t}\| - \beta)^{n-1} (\|\mathbf{t}\| + \beta)}{2(\alpha\sqrt{\beta})^n \|\mathbf{t}\|^{3n/2}} h \left\{ \frac{(\|\mathbf{t}\| - \beta)^2}{\alpha^2 \beta \|\mathbf{t}\|} \right\}, \quad (12)$$

which shall be denoted by $\mathbf{T} \sim \mathcal{GBS}_n^{II}(\alpha\mathbf{1}, \beta\mathbf{1}; h)$, with $\mathbf{1} : n \times 1 = (1, \dots, 1)'$.

Similarly, for a generalised three parameter Birnbaum-Saunders distribution we have the following result:

Corollary 12 Suppose that $\mathbf{U} \sim \mathcal{E}_n(\mathbf{0}, \mathbf{I}; h)$, and consider the transformation

$$f(\epsilon) = \frac{1}{\alpha} \left[\left(\frac{\epsilon}{\beta} \right)^\lambda - \left(\frac{\beta}{\epsilon} \right)^\lambda \right], \quad \lambda > 0, \quad \beta > 0, \quad \alpha > 0,$$

with $\mathbf{U} = \mathbf{F}(\mathbf{V}) = f(\epsilon)\mathbf{W}$, where $\mathbf{V} = \epsilon\mathbf{W}$ is the SVD of the random vector \mathbf{V} , for $\epsilon > 0$ and $\mathbf{W} \in \mathfrak{R}^m$ with $\|\mathbf{W}\|^2 = 1$. The density of the vector \mathbf{V} is

$$g_{\mathbf{V}}(\mathbf{v}) = \frac{c \lambda}{(\alpha \|\mathbf{v}\|)^n} \left[\left(\frac{\|\mathbf{v}\|}{\beta} \right)^\lambda - \left(\frac{\beta}{\|\mathbf{v}\|} \right)^\lambda \right]^{n-1} \left[\left(\frac{\|\mathbf{v}\|}{\beta} \right)^\lambda + \left(\frac{\beta}{\|\mathbf{v}\|} \right)^\lambda \right] h \left\{ \frac{1}{\alpha^2} \left[\left(\frac{\|\mathbf{v}\|}{\beta} \right)^{2\lambda} + \left(\frac{\beta}{\|\mathbf{v}\|} \right)^{2\lambda} - 2 \right] \right\},$$

This shall be termed the multivariate generalised three parameter Birnbaum-Saunders distribution type II and it shall be denoted by $\mathbf{V} \sim \mathcal{GGBS}_n^{II}(\alpha\mathbf{1}, \beta\mathbf{1}, \lambda\mathbf{1}; h)$.

Next we give an extension of the multivariate sinh-spherical distribution:

Corollary 13 Suppose that $\mathbf{T} \sim \mathcal{GBS}_n^{II}(\alpha\mathbf{1}, \beta\mathbf{1}; h)$ and let $\mathbf{T} = \mathbf{F}(\mathbf{Y}) = f(\epsilon)\mathbf{P}$, with $f(\epsilon) = \exp\{2\epsilon/\sigma\}$; where $\mathbf{Y} = \epsilon\mathbf{P}$ is the SVD of \mathbf{Y} , for $\epsilon > 0$, $\mathbf{P} \in \mathfrak{R}^m$ and $\|\mathbf{P}\|^2 = 1$. We say that \mathbf{Y} has a multivariate sinh-spherical distribution type II of parameters $\alpha\mathbf{1}$, $\gamma\mathbf{1}$ with $\beta = \exp\{2\gamma/\sigma\}$ and $\sigma\mathbf{I}$. Moreover, its density function is given by

$$g_{\mathbf{Y}}(\mathbf{y}) = \frac{2^{-n}c}{\sigma^n \alpha \|\mathbf{y}\|^{n-1}} \left(\cosh^2 \left(\frac{\|\mathbf{y}\| - \gamma}{\sigma} \right) - 1 \right)^{(n-1)/2} \cosh \left(\frac{\|\mathbf{y}\| - \gamma}{\sigma} \right) h \left\{ \frac{4}{\alpha^2} \sinh^2 \left(\frac{\|\mathbf{y}\| - \gamma}{\sigma} \right) \right\}, \quad (13)$$

with $\mathbf{y} \in \mathfrak{R}^n$. We write this fact like $\mathbf{Y} \sim \mathcal{SE}_n^{II}(\alpha\mathbf{1}, \gamma\mathbf{1}, \sigma\mathbf{I}; h)$.

Now, the result of Theorem 13 is extended to the generalised three parameter Birnbaum-Saunders distribution.

Corollary 14 If $\mathbf{V} \sim \mathcal{GGB} - \mathcal{S}_n^{II}(\alpha\mathbf{1}, \beta\mathbf{1}, \lambda\mathbf{1}; h)$ and $\mathbf{V} = \mathbf{F}(\mathbf{Y}) = f(\epsilon)\mathbf{N}$, con $f(\epsilon) = \exp\{2\epsilon/\sigma\}$; where $\mathbf{Y} = \epsilon\mathbf{N}$ is the SVD of \mathbf{Y} ; $\epsilon > 0$ and $\mathbf{N} \in \mathfrak{R}^m$ with $\|\mathbf{N}\|^2 = 1$; then we say \mathbf{Y} has a multivariate generalised sinh-spherical

distribution type II of parameters $\alpha\mathbf{1}$, $\gamma\mathbf{1}$ with $\beta = \exp\{2\gamma/\sigma\}$ and $\sigma\mathbf{I}$. And its density function is given by

$$g_{\mathbf{Y}}(\mathbf{y}) = \frac{2^{1-n}c\lambda}{\sigma\alpha^n\|\mathbf{y}\|^{n-1}} \left\{ \cosh^2 \left[2\lambda \left(\frac{\|\mathbf{y}\| - \gamma}{\sigma} \right) \right] - 1 \right\}^{(n-1)/2} \\ \times \cosh \left[2\lambda \left(\frac{\|\mathbf{y}\| - \gamma}{\sigma} \right) \right] h \left\{ \frac{4}{\alpha^2} \sinh^2 \left[2\lambda \left(\frac{\|\mathbf{y}\| - \gamma}{\sigma} \right) \right] \right\}, \quad (14)$$

with $\mathbf{y} \in \mathfrak{R}^n$. This fact shall be denoted by $Y \sim \mathcal{GSE}_n^{II}(\alpha\mathbf{1}, \gamma\mathbf{1}, \sigma\mathbf{I}; h)$.

An alternative expression for a log-elliptical distribution is the following:

Corollary 15 Suppose that $\mathbf{Y} = \ln(\mathbf{W}) \sim \mathcal{E}l_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}; h)$. Then we say \mathbf{W} has a multivariate log-elliptical distribution type II and its density function is given by:

$$g_{\mathbf{W}}(\mathbf{W}) = \frac{c \ln^{n-1}(\|\mathbf{w}\|)}{|\boldsymbol{\Sigma}|^{1/2} \|\mathbf{w}\|^n} h \{ (\log(\mathbf{w}) - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\log(\mathbf{w}) - \boldsymbol{\mu}) \}, \quad \mathbf{w} \in \mathfrak{R}_+^n \quad (15)$$

with $\log(\mathbf{w}) = \ln(\kappa)\mathbf{M}$, and $\mathbf{W} = \kappa\mathbf{M}$ is the SVD of the vector \mathbf{W} ; where $\kappa > 0$, $\mathbf{M} \in \mathfrak{R}^m$ and $\|\mathbf{M}\|^2 = 1$. This shall denote like $\mathbf{W} \sim \mathcal{LEl}_p^{II}(\boldsymbol{\mu}, \boldsymbol{\Sigma}; h)$.

Note that the results based on Lemma 10 (corollaries 11 - 14) can be generalised still more. For example, in Theorem 11, now suppose that $\mathbf{U} \sim \mathcal{E}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}; h)$, $\boldsymbol{\mu} \in \mathfrak{R}^n$ and $\boldsymbol{\Sigma} \in \mathfrak{R}^{n \times n}$ with $\boldsymbol{\Sigma} > 0$, then we obtain

$$g_{\mathbf{T}}(\mathbf{t}) = \frac{c \|\mathbf{t}\|^{-3n/2} (\|\mathbf{t}\| - \beta)^{n-1} (\|\mathbf{t}\| + \beta)}{2(\alpha\sqrt{\beta})^n |\boldsymbol{\Sigma}|^{1/2}} h \{ (\mathbf{F}(\mathbf{t}) - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{F}(\mathbf{t}) - \boldsymbol{\mu}) \} \quad (16)$$

with $\mathbf{F}(\mathbf{t}) = f(\rho)\mathbf{W}$, where $\mathbf{T} = \rho\mathbf{W}$ is the SVD of the random vector \mathbf{T} , for $\rho > 0$, $\mathbf{W} \in \mathfrak{R}^m$ and $\|\mathbf{W}\|^2 = 1$. Here $f(\rho)$ is given by

$$f(\rho) = \frac{1}{\alpha} \left[\sqrt{\frac{\rho}{\beta}} - \sqrt{\frac{\beta}{\rho}} \right], \quad \alpha > 0, \quad \beta > 0.$$

An interesting point to highlight comes from the fact that the families of distributions derived in Corollaries 11- 14 keep being families of spherical distributions in \mathfrak{R}^n and \mathfrak{R}_+^n ; then many of their properties: moments, marginals, conditionals, etc. can be obtained as particular cases of the general results derived for the families of spherical distributions, see Fang and Zhang (1990) or Fang et al. (1990). Moreover, note that from those distributions could be generate the corresponding families of elliptical distributions just by defining

$\mathbf{T} = \Sigma^{-1/2}(\mathbf{X} - \boldsymbol{\mu})$, with $\Sigma > \mathbf{0}$ and $\boldsymbol{\mu} \in \Re^n$. For example, the corresponding elliptical expression for the density (12) is given by

$$g_{\mathbf{X}}(\mathbf{x}) = \frac{c \left([(\mathbf{x} - \boldsymbol{\mu})' \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})]^{1/2} - \beta \right)^{n-1} \left([(\mathbf{x} - \boldsymbol{\mu})' \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})]^{1/2} + \beta \right)}{2(\alpha\sqrt{\beta})^n |\Sigma|^{1/2} [(\mathbf{x} - \boldsymbol{\mu})' \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})]^{3n/4}} \times h \left\{ \frac{\left([(\mathbf{x} - \boldsymbol{\mu})' \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})]^{1/2} - \beta \right)^2}{\alpha^2 \beta [(\mathbf{x} - \boldsymbol{\mu})' \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})]^{1/2}} \right\}. \quad (17)$$

We finish this section noting that for a sample of dependent life data X_1, \dots, X_n , the expressions of the densities given in Corollaries 11-15, can be consider with their respective likelihood functions, according to the case.

8 Some applications

This sections starts showing some graphics of different densities like: the generalised three parameter Birnbaum-Saunders distribution (see Figure 1), the log-elliptical distribution (see Figure 2) and the sinh-spherical distribution (see Figure 3).

Next we explain, in an example, the way to estimate some of the distributions derived in the preceding sections: we took the first set of independent data given in Birnbaum and Saunders (1969b) and we simulated a set of dependent data, for them the parameters are estimated and the corresponding criterion of information of Shewarz is calculated, see Schwarz (1978).

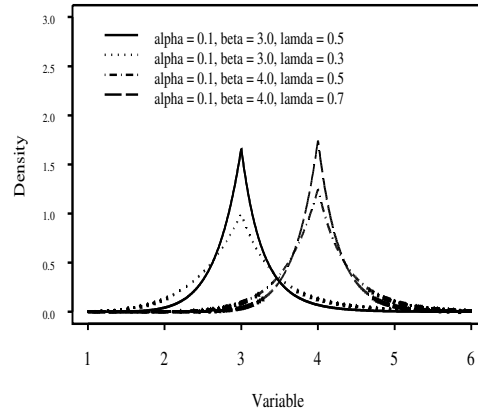
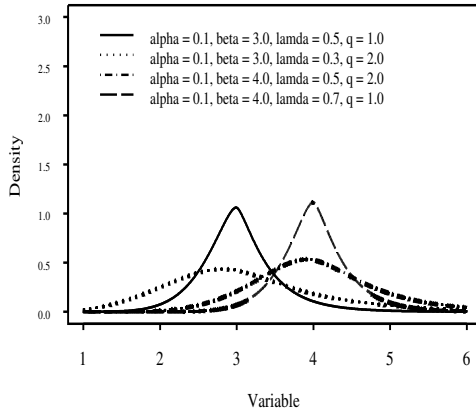
Distribution	<i>SIC</i>	LogVer	NP	μ	s^2	ν
Special Case	922.5638	-456.6668	2	4.882796	0.038613230	
Laplace	923.0040	-456.8869	2	4.890352	0.016497691	
Normal	923.4683	-457.1190	2	4.881763	0.028737664	
Pearson VII	925.2058	-455.6802	3	4.886503	0.182448431	4.676901
T	925.2058	-455.6802	3	4.886503	0.021840419	8.353855
Cauchy	947.8048	-469.2873	2	4.897028	0.008428758	

Table 1: Fit of the Log-Elliptical distributions for the independent case, ordered according to the *SIC* criterion from lowest to highest. NP denotes the number of parameters considered in the optimisation.

Generalised Three Parameter Birnbaum-Saunders Distributions

(a) Bessel

(b) Laplace



(c) Normal

(d) Pearson VII

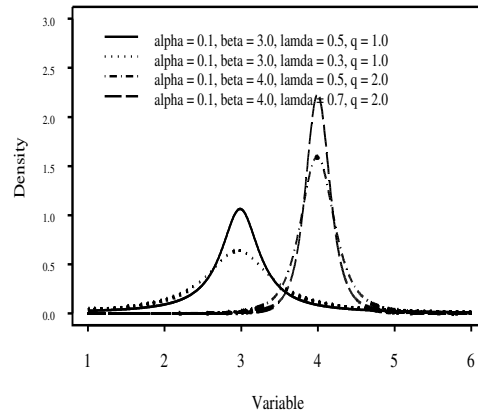
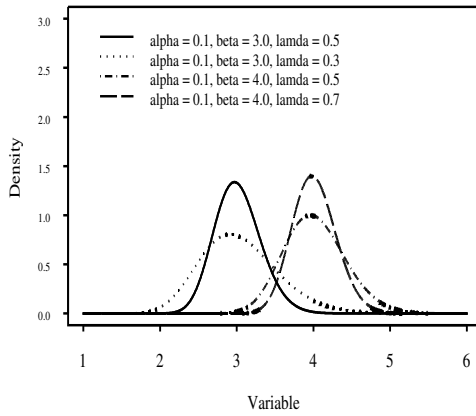


Fig. 1. Densities of generalised three parameter Birnbaum-Saunders distribution type I

Distribution	SIC	LogVer	NP	μ	s^2	ν
Normal	867.20886	-428.99926	2	5.3069933	0.007658998	
Laplace	867.90034	-429.34500	2	5.3069934	0.000076589	
Special Case	870.26121	-430.52544	2	5.3069933	0.132657078	
Cauchy	872.12753	-431.45859	2	5.3069934	0.007658263	
T	872.77665	-429.48057	3	5.3069949	0.007658495	62.115737
Pearson VII	873.99107	-430.08778	3	5.3069934	0.100452931	56.557866

Table 2: Fit of the Log-Elliptical distributions for the dependent case, ordered according to the SIC criterion from lowest to highest. NP denotes the number

Log-Elliptical Distributions

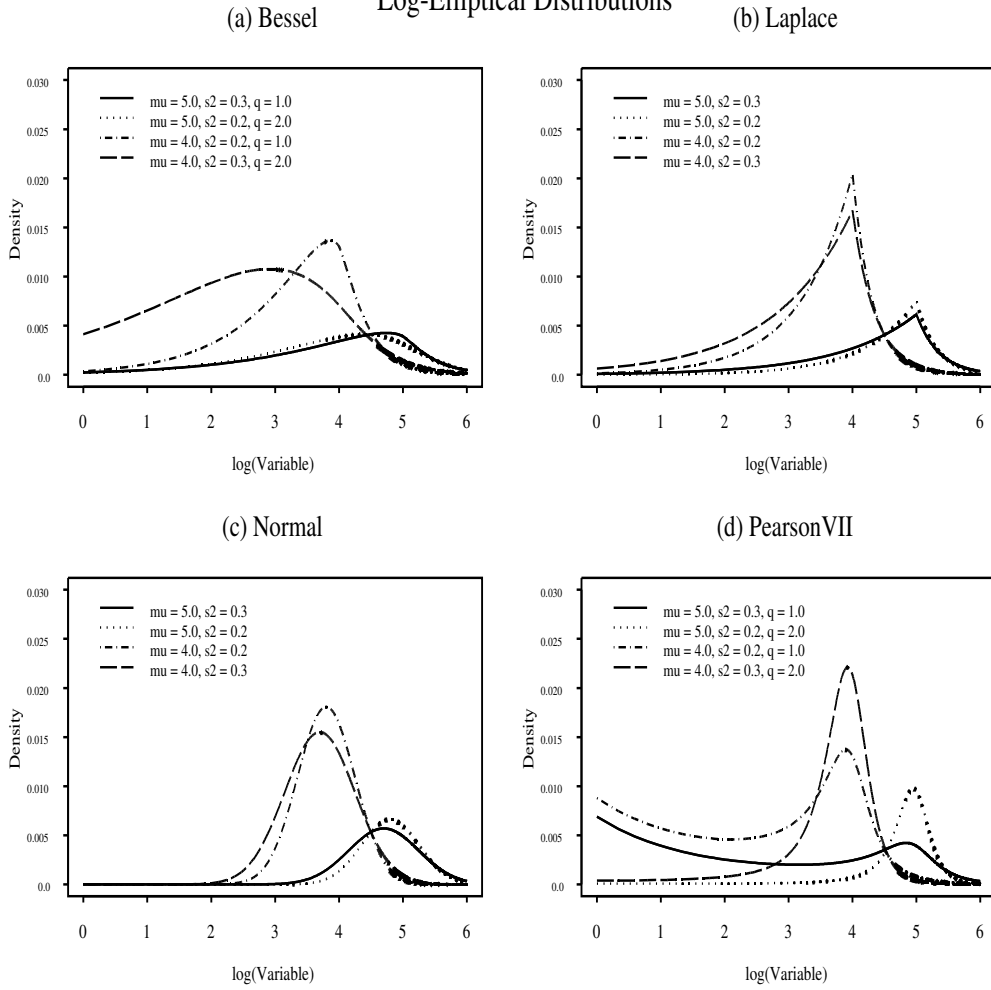


Fig. 2. Densities of generalised log-elliptical distribution type I of parameters considered in the optimisation.

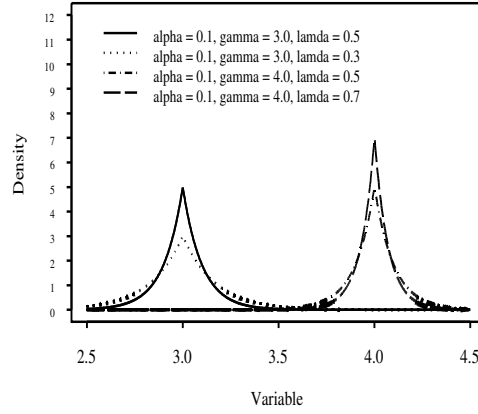
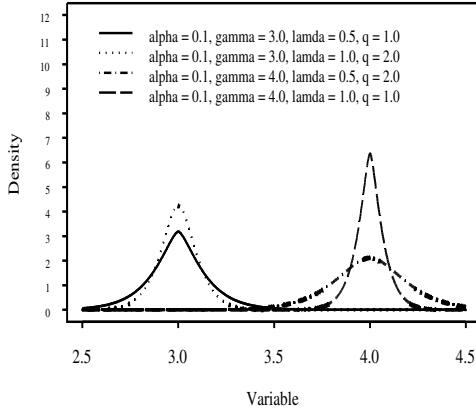
Distribution	SIC	LogVer	NP	α	β	ν
Special Case	922.30800	-456.53888	2	0.39571929	131.95014	
Laplace	922.49944	-456.63460	2	0.26014664	133.00000	
Normal	924.75657	-457.76316	2	0.34605813	131.67771	
T	925.21941	-455.68702	3	0.29346658	132.49943	7.1484897
Pearson VII	925.21941	-455.68702	3	0.78461501	132.49952	4.0741357
Cauchy	946.59686	-468.68331	2	0.18426419	133.89997	

Table 3: Fit of the generalised three parameters Birnbaum-Saunders distribu-

Sinh-Spherical Distributions Under Three Parameter Birnbaum-Saunders Distributions

(a) Bessel

(b) Laplace



(c) Normal

(d) PearsonVII

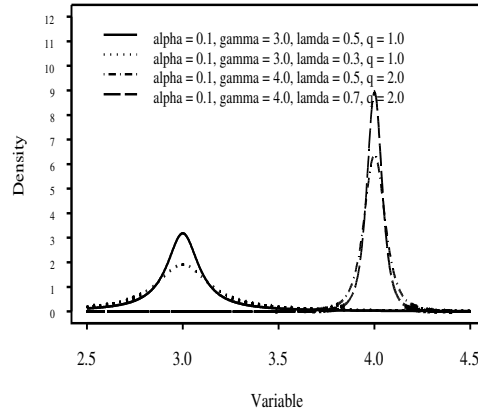
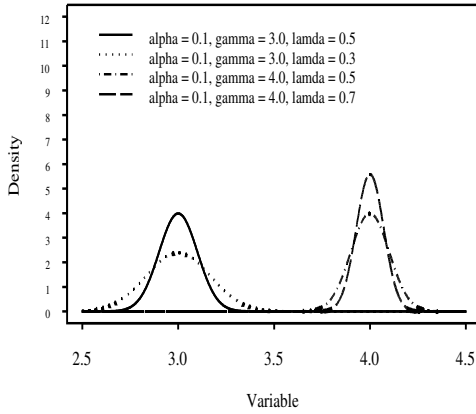


Fig. 3. Densities of generalised sinh-spherical distribution type I

tions for the independent case, ordered according to the *SIC* criterion from lowest to highest. NP denotes the number of parameters considered in the optimisation.

Distribution	<i>SIC</i>	LogVer	NP	α	β	ν
Normal	867.42236	-429.10601	2	0.175886865	201.77284	
Laplace	868.11384	-429.45175	2	0.017588691	201.77315	
Special Case	870.47471	-430.63219	2	0.732003896	201.77314	
Cauchy	872.34103	-431.56534	2	0.175881535	201.77315	
T	872.99746	-429.59097	3	0.175886986	201.77314	61.390864
Pearson VII	874.25650	-430.22049	3	0.619121931	201.77314	56.195175

Table 4: Fit of the generalised three parameters Birnbaum-Saunders distributions for the dependent case, ordered according to the *SIC* criterion from lowest to highest. NP denotes the number of parameters considered in the optimisation.

Distribution	<i>SIC</i>	LogVer	NP	α	γ	ν
Special Case	-383.10426	196.16725	2	0.080578881	1.5853847	
Laplace	-382.73764	195.98394	2	0.052799946	1.5872637	
Normal	-380.98375	195.10700	2	0.070163281	1.5848798	
T	-380.23087	197.03812	3	0.060009231	1.5862995	7.5079185
Pearson VII	-380.23087	197.03812	3	0.164429417	1.5862995	4.2539823
Cauchy	-358.54091	183.88558	2	0.037500503	1.5888055	

Table 5: Fit of the sinh-spherical distribution under generalised three parameters Birnbaum-Saunders distributions for the independent case, ordered according to the *SIC* criterion from lowest to highest. NP denotes the number of parameters considered in the optimisation.

Distribution	<i>SIC</i>	LogVer	NP	α	γ	ν
Normal	-528.44805	268.82920	2	0.032911007	1.6688907	
Laplace	-527.75657	268.48346	2	0.003291101	1.6688907	
Special Case	-525.39570	267.30302	2	0.136968769	1.6688907	
Cauchy	-523.52938	266.36986	2	0.032913891	1.6688907	
T	-522.88956	268.35253	3	0.032911013	1.6688907	63.054749
Pearson VII	-521.72838	267.77194	3	0.123381789	1.6688907	57.027312

Table 6: Fit of the sinh-spherical distribution under generalised three parameters Birnbaum-Saunders distributions for the dependent case, ordered according to the *SIC* criterion from lowest to highest. NP denotes the number of parameters considered in the optimisation.

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