CRITICAL EXPONENTS OF SEMILINEAR EQUATIONS VIA THE FEYNMAN-KAC FORMULA

Ekaterina T. Kolkovska, José Alfredo López-Mimbela

Comunicación Técnica No I-06-17/18-12-2006
(PE/CIMAT)

Critical exponents of semilinear equations via the Feynman-Kac formula

Ekaterina T. Kolkovksa* José Alfredo López-Mimbela*

Key words: Semilinear partial differential equations, Feynman-Kac representation, critical exponent, finite time blow-up, non-global solution.

Mathematics Subject Classification: 60H30, 35K55, 35K57, 35B35.

1 Introduction and backgrownd

This note is a continuation of the paper [7] on branching particle representations of semilinear equations, to which we refer for background and previous work. We consider a semilinear Cauchy problem of the form

$$\frac{\partial u_t}{\partial t}(x) = Lu_t(x) + u_t^{1+\beta}(x), \qquad u_0(x) = \varphi(x), \qquad x \in \mathbb{R}^d, \tag{1}$$

where $\beta > 0$ is constant, $\varphi \geq 0$ is bounded and measurable, and L is the generator of a strong Markov process in $\mathbb{R}^d$. Solutions will be understood in the mild sense, so that (1) is meaningful for any bounded, measurable initial value.

Recall that for any non-trivial $\varphi \geq 0$, there exists a number $T_\varphi \in (0, \infty]$ such that (1) has a unique solution $u \equiv u_t(x)$ on $[0, T_\varphi) \times \mathbb{R}^d$, which is bounded on $[0, T] \times \mathbb{R}^d$ for any $0 < T < T_\varphi$, and if $T_\varphi < \infty$, then

*Department of Probability and Statistics, Centro de Investigación en Matemáticas, Apartado Postal 402, 36000 Guanajuato, México.

$\|u(t,\cdot)\|_\infty \to \infty$ as $t \uparrow T_\varphi$. When $T_\varphi = \infty$ we say that u is a global solution, and when $T_\varphi < \infty$ we say that u blows up in finite time or that u is nonglobal.

Our purpose in this work is to complement the review paper [7], where we focussed on certain branching particle representations of (1), and on how such representations can be used to investigate finite time blowup and existence of global positive solutions of (1). Our interest here is to give a brief account of the Feynman-Kac representation of (1) and its applications to obtain critical parameters for finite-time blowup of (1). Our main sources are the papers [3] and [4], where the approach was introduced, and also [9] and [8]. Though we are going to provide full proofs of some of the main results, the reader is remitted to the above references for more insights and further extensions of results.

One of our goals is to explain the approach to a multidisciplinary reader. Therefore, we shall describe thoroughly the method in the case of the generator $L = \Delta_\alpha$ of the spherically symmmetric α-stable process, $\alpha \in (0,2]$, and then consider equations with other generators —specifically the gamma generator and the Laplacian perturbed by a bounded potential—, explaining the changes and adaptations of arguments needed to cover those cases. Moreover, for the sake of completeness, we include below a deduction of the Feynman-Kac formula.

Let us start by describing briefly the main line of the Feynman-Kac approach. Recall that the mild solution of (1) is given by

$$u_t(x) = T_t\varphi(x) + \int_0^t (T_s u_{t-s}^{1+\beta})(x)\, ds, \qquad t \in [0, T_\varphi), \tag{2}$$

where $\{T_t,\, t \geq 0\}$ is the semigroup with generator L. Since $\varphi \geq 0$ and T_t is a positive operator for any $t \geq 0$, it follows that

$$f_t(y) := T_t\varphi(y), \quad t \geq 0, \quad x \in \mathbb{R}^d,$$

is a subsolution of (1), meaning that

$$f_0 = u_0 \quad \text{ and } \quad f_t(x) \leq u_t(x) \quad \text{ for all } (t,x) \in [0,T_\varphi) \times \mathbb{R}^d.$$

Let now g_t be the mild solution of

$$\frac{\partial g_t}{\partial t} = L g_t + f_t^\beta g_t, \quad g_0 = \varphi.$$

Then, as a consequence of the Feynman-Kac formula, g_t is a subsolution of (1) satisfying $0 \le f_t \le g_t$. Letting h_t be the mild solution of

$$\frac{\partial h_t}{\partial t} = Lh_t + g_t^\beta h_t, \quad h_0 = \varphi,$$

we get a subsolution of (1) satisfying $0 \le f_t \le g_t \le h_t$, and so on. Of course, if a positive subsolution of (1) exhibited finite-time blowup, then u_t would also explode in finite time. However, as we shall see, a slightly weaker condition, namely, that a subsolution of (1) grows to ∞ uniformly on a ball as $t \to \infty$, is sufficient to guarantee finite time blow of u_t. Thus, we are lead to estimating the growth as $t \to \infty$ of subsolutions of (1), such as g_t and h_t, and this can be done efficiently using the Feynman-Kac representation.

1.1 The Feynman-Kac formula

Let $T > 0$ be fixed, and consider the linear equation

$$\begin{aligned} \frac{\partial v(t,x)}{\partial t} &= \Delta_\alpha v(t,x) + k(t,x)v(t,x), \quad 0 < t \le T, \\ v_0(x) &= \varphi(x), \quad x \in \mathbb{R}^d, \end{aligned} \tag{3}$$

where φ and $k(t,x)$ are bounded continuous functions on $\mathbb{R}^d$ and $[0,T] \times \mathbb{R}^d$, respectively. It is well-known that, when $\alpha = 2$ and $k(t,x) \equiv k(x)$, the solution of (3) is given by the Feynman-Kac formula. However, it is not common to find in the literature the formula corresponding to the above equation for more general α and k. For such reason, in this section we will prove that the solution of (3) admits the Feynman-Kac representation

$$v(t,y) = E_y \left[\varphi(W_t) \exp \int_0^t k(t-s, W_s)\, ds \right], \quad (t,y) \in [0,T] \times \mathbb{R}^d, \tag{4}$$

where $\{W_t,\ t \ge 0\}$ is the d-dimensional spherically symmetric α-stable process, and E_y denotes expectation with respect to the distribution of $\{y + W_t,\ t \ge 0\}$.

In order to prove (4) we shall adapt the method of proof of [1]. From the integration by parts formula we obtain

$$d\left[v(t-s, W_s) \exp \int_0^s k(t-r, W_r)\, dr \right]$$

$$
\begin{aligned}
&= v(t-s,W_s)k(t-s,W_s)\exp\left\{\int_0^s k(t-r,W_r)\,dr\right\}\,ds \\
&\quad+\exp\left\{\int_0^s k(t-r,W_r)\,dr\right\}dv(t-s,W_s).
\end{aligned}
$$

Applying Itô's formula to the last term (see e.g. [1]) gives

$$
\begin{aligned}
&d\left[v(t-s,W_s)\exp\int_0^s k(t-r,W_r)\,dr\right] \\
&= \exp\left\{\int_0^s k(t-r,W_{r-})\,dr\right\} \\
&\quad\cdot\left\{v(t-s,W_{s-})k(t-s,W_{s-})\,ds-\frac{d}{ds}v(t-s,W_{s-})\right. \\
&\quad+\int_{|x|<1}\left[v(t-s,W_{s-}+x)-v(t-s,W_{s-})\right]\widetilde{N}(ds,dx) \\
&\quad+\int_{|x|\geq 1}\left[v(t-s,W_{s-}+x)-v(t-s,W_{s-})\right]N(ds,dx) \\
&\quad+\int_{|x|<1}\left[v(t-s,W_{s-}+x)-v(t-s,W_{s-})\right. \\
&\qquad\qquad\left.\left.-\sum_i x_i\frac{d}{dx_i}v(t-s,W_{s-})\right]\nu(dx)\,ds\right\};
\end{aligned}
$$

here $N\,(dt,dx)$ is the Poisson random measure on $[0,\infty)\times\mathbb{R}^d\backslash\{0\}$ with intensity $EN\,(dt,dx)=dt\,\nu\,(dx)$, where

$$
\nu(dx)=\frac{\alpha 2^{\alpha-1}\Gamma((\alpha+d)/2)}{\pi^{d/2}\Gamma(1-\alpha/2)\|x\|^{\alpha+d}}\,dx,
$$

and $\widetilde{N}$ is the compensated Poisson random measure $\widetilde{N}=N-EN$. Integrating over $[0,t]$ and taking expectation with respect to P_y, we obtain

$$
\begin{aligned}
&E_y\left[\varphi(W_t)\exp\left\{\int_0^t k(t-s,W_s)\,ds)\right\}\right]-v_t(y) \\
&= E_y\int_0^t\exp\left\{\int_0^s k(t-r,W_{r-})\,dr\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \cdot \Bigg\{ v(t-s,W_{s-})k(t-s,W_{s-}) - \frac{d}{ds}v(t-s,W_{s-}) \\
& + \int_{|x|<1} \Bigg[v(t-s,W_{s-}+x) - v(t-s,W_{s-}) \\
& \qquad\qquad - \sum_i x_i \frac{d}{dx_i} v(t-s,W_{s-}) \Bigg] \nu(dx) \\
& + \int_{|x|\geq 1} [v(t-s,W_{s-}+x) - v(t-s,W_{s-})\nu(dx)] \Bigg\} ds \\
= {} & 0,
\end{aligned}
$$

where we have used the equality $\widetilde{N}(ds,dx) = N(ds,dx) - ds\,\nu(dx)$, and the fact that the stochastic integrals with respect to $\widetilde{N}(ds,dx)$ are martingales, hence they have expectation 0. This proves (4).

Using the symmetry and positivity of stable densities, we shall deduce below a variant of (4) which is more useful for our purposes. For fixed $t > 0$ we consider the Lévy process $\{W_s,\ 0 \leq s \leq t\}$. We know that $\{W_s,\ 0 \leq s \leq t\}$ has symmetric and strictly positive transition densities. For $x \in \mathbb{R}^d$ we denote by P_x^t the distribution of $\{x + W_s,\ 0 \leq s \leq t\}$ in the Skorokhod space $D([0,t],\mathbb{R}^d)$, where $D([0,t],\mathbb{R}^d)$ is given the canonical filtration $\{\mathcal{F}_s\}_{0\leq s\leq t}$. The following results are from [13].

Proposition 1 *Let $t > 0$ be fixed.*

a) For every $x, y \in \mathbb{R}^d$, there exists a unique probability measure $P_{x,y}^t$ on $D([0,t],\mathbb{R}^d)$ such that, if $s \in [0,t)$ and $A \in \mathcal{F}_s$, then

$$
P_{x,y}^t(A) = \frac{1}{p(t,x,y)} E_x^t \left[p(t-s,W_s,y) 1_A \right], \tag{5}
$$

where E_x^t denotes expectation with respect to P_x^t.

b) We have

$$
P_{x,y}^t[W_t = y] = 1. \tag{6}
$$

c) $y \to P_{x,y}^t$ is a conditional probability of P_x^t, given that $\{W_t = y\}$.

d) The image of $P_{x,y}^t$ under the mapping $\omega(\cdot) \mapsto \omega(t - \cdot)$ is $P_{y,x}^t$.

Proof. Let $x, y \in \mathbb{R}^d$. It is clear that (5) determines a unique probability measure $P^t_{x,y}$ on $\mathcal{F}_t$, hence, in order to prove a) it is sufficient to show existence of $P^t_{x,y}$. First we shall prove that $\{p(t-s, W_s, y),\ s \in [0,t)\}$, is an $\mathcal{F}_s$-martingale. Indeed, for $0 \le s < r < t$ and $A \in \mathcal{F}_s$, we have

$$\begin{aligned} E^t_x\left[1_A p(t-r, W_r, y)\right] &= E^t_x\left[1_A E_{W_s}\left(p(t-r, W_r, y)\right)\right] \\ &= E^t_x\left[1_A \int p(t-r, z, y) p(r-s, W_s, z)\, dz\right] \\ &= E^t_x\left[1_A p(t-s, W_s, y)\right], \end{aligned}$$

where we have used the Markov property and the Chapman-Kolmogorov equation. From Kolmogorov's consistency theorem (see e.g. [5]), it follows that there exists a unique probability measure $\widetilde{P}^t_{x,y}$ on $D([0,t), R^d)$ such that

$$\widetilde{P}^t_{x,y}(A) = \frac{1}{p(t,x,y)} E^t_x\left[1_A p(t-s, W_s, y)\right], \quad s \in [0,t), \quad A \in \mathcal{F}_s.$$

Writing $\widetilde{E}^t_{x,y}$ for the expectation with respect to $\widetilde{P}^t_{x,y}$, it follows that for $t > r > s \ge 0$ and $A \in \mathcal{F}_s$,

$$\begin{aligned} \widetilde{E}^t_{x,y}\left[\frac{1}{p(t-r, W_r, y)} 1_A\right] &= \frac{1}{p(t,x,y)} E^t_x\left[\frac{p(t-r, W_r, y)}{p(t-r, W_r, y)} 1_A\right] \\ &= \frac{1}{p(t,x,y)} E^t_x\left[\frac{p(t-s, W_s, y)}{p(t-s, W_s, y)} 1_A\right] \\ &= \widetilde{E}^t_{x,y}\left[\frac{1}{p(t-s, W_s, y)} 1_A\right], \end{aligned}$$

implying that $\{p^{-1}(t-s, W_s, y),\ 0 \le s < t\}$ is a non-negative martingale under $\widetilde{P}^t_{x,y}$. The martingale convergence theorem renders that $\dfrac{1}{p(t-s, W_s, y)}$ converges $\widetilde{P}^t_{x,y}$-almost surely to a finite limit when $s \to t$, hence $W_s \to y$ as $s \to t$. It follows that $\widetilde{P}^t_{x,y}$ is concentrated on $D([0,t], R^d)$, and we can define $P^t_{x,y}$ as the restriction of $\widetilde{P}^t_{x,y}$ to $D([0,t], R^d)$. This proves a) and b).

Let $0 \le s < t$ and $A \in \mathcal{F}_s$. Because of the Markov property,

$$E^t_x\left[1_{W_t \in B}\, 1_A\right] = \int_B E^t_x\left[p(t-s, W_s, y) 1_A\right] dy$$

$$
\begin{aligned}
&= \int_B p(t,x,y) P^t_{x,y}(A)\, dy \\
&= E^t_x \left[1_{\{W_t \in B\}} E^t_{x,W_t}(A) \right].
\end{aligned}
$$

From the monotone class lemma we deduce that the last equality holds for every $A \in \mathcal{F}_t$, thus proving c). Part d) follows directly from the symmetry of stable densities and the equality

$$
\begin{aligned}
& P^t_{x,y}\left[W_{s_1} \in dz_1, \ldots, W_{s_n} \in dz_n\right] \\
&= \frac{p(s_1,x,z_1)p(s_2-s_1,z_1,z_2)\cdots p(t-s_n,z_n,y)}{p(t,x,y)}\, dz_1 \cdots dz_n, \\
& x, z_1, \ldots, z_n, y \in R^d, \quad 0 \le s_1 \le \cdots \le s_n \le t.
\end{aligned}
$$

■

A consequence of Proposition 1 is the following alternative form of the Feynman-Kac formula, which will be useful in the sequel. From the Markov property,

$$
\begin{aligned}
v(t,y) &= E_y \left[\varphi(W_t) E^t_{y,W_t} \exp\left\{ \int_0^t k(t-s,W_s)\, ds \right\} \right] \\
&= \int \varphi(x) p(t,y,x) E^t_{y,x} \exp\left\{ \int_0^t k(t-s,W_s)\, ds \right\} dx \\
&= \int \varphi(x) p(t,x,y) E^t_{x,y} \exp\left\{ \int_0^t k(s,W_s)\, ds \right\} dx \\
&= \int E_x \left[\exp\left\{ \int_0^t k(s,W_s)\, ds \right\} \middle| W_t = y \right] \varphi(x) p(t,x,y)\, dx,
\end{aligned}
$$

where we used the symmetry of $p(t,x,y)$ and Part d) of Proposition 1. Hence,

$$
v(t,y) = \int E_x \left[\exp\left\{ \int_0^t k(s,W_s)\, ds \right\} \middle| W_t = y \right] \varphi(x) p(t,x,y)\, dx. \tag{7}
$$

1.2 The Feynman-Kac formula and subsolutions

Let w be the solution of the semilinear equation

$$
\frac{\partial w_t}{\partial t}(y) = \Delta_\alpha w_t(y) + \gamma w_t^{1+\beta}(y), \qquad w_0(y) = \varphi(y), \quad y \in \mathbb{R}^d, \tag{8}
$$

where $\gamma, \beta > 0$ and $\varphi \geq 0$ is bounded, measurable and greater than zero on a set of positive volume. Due to the Feynman-Kac formula (4), w_t admits the representation

$$w_t(y) = E_y\left[\varphi(W_t)\exp\int_0^t \gamma w_{t-s}^\beta(W_s)\,ds\right]$$

and hence,

$$w_t(y) \geq E_y\left[\varphi(W_t)\right] = f_t(y), \quad t \geq 0.$$

It results that f_t is a subsolution of (8), that is, $w_0 = f_0$ and $w_t \geq f_t$ for every $t > 0$. By linearity we obtain the following lemma, see [8].

Lemma 2 *Let $\varphi \geq 0$ be bounded and measurable. If u_t, v_t, respectively, are solutions of*

$$\frac{\partial u_t}{\partial t}(y) = \Delta_\alpha u_t(y) + \zeta_t(y)u_t(y) \quad \textit{and} \quad \frac{\partial v_t}{\partial t}(y) = \Delta_\alpha v_t(y) + \xi_t(y)v_t(y),$$

with $u_0 \geq v_0$ and $\zeta_t \geq \xi_t$, then $u_t \geq v_t$.

In the following, we shall use without further explanation the fact that, if u_t is a positive subsolution of (8), then each solution of

$$\frac{\partial v_t}{\partial t}(y) = \Delta_\alpha v_t(y) + \gamma u_t^\beta(y)v_t(y), \qquad v_0 = \varphi,$$

with $\varphi \geq 0$, is also a subsoluction of (8).

2 Bounds for bridges and semigroups

Let B_r denote the ball in $\mathbb{R}^d$ with radius $r > 0$ centered at the origin. The following two lemmas are from [3]; for the sake of completeness we include their proofs here.

Lemma 3 *There exists a constant $c > 0$ such that for all $t \geq 2$, $y \in B_{t^{1/\alpha}}$, $x \in B_1$ and $s \in [1, t/2]$, there holds*

$$P_x\left\{W_s \in B_{s^{1/\alpha}} \middle|\, W_t = y\right\} \;\geq\; c. \tag{9}$$

Proof. Using self-similarity, continuity and strict positivity of stable densities, we have that for all $s \in [1, t/2]$,

$$\begin{aligned}
&\int_{B_{s^{1/\alpha}}} \frac{p_s(z-x)p_{t-s}(y-z)}{p_t(y-x)}\,dz \\
&= \int_{B_{s^{1/\alpha}}} \frac{s^{-d/\alpha}p_1(s^{-1/\alpha}(z-x))(t-s)^{-d/\alpha}p_1((t-s)^{-1/\alpha}(y-z))}{t^{-d/\alpha}p_1(t^{-1/\alpha}(y-x))}\,dz \\
&\geq \frac{s^{-d/\alpha}(t-s)^{-d/\alpha}}{t^{-d/\alpha}} \cdot \frac{\left(\inf_{w\in B_2} p_1(w)\right)^2}{p_1(0)} \int_{B_{s^{1/\alpha}}} dz \\
&\geq s^{-d/\alpha}\mathrm{Vol}(B_{s^{1/\alpha}})\frac{\left(\inf_{w\in B_2} p_1(w)\right)^2}{2p_1(0)},
\end{aligned}$$

which proves (9). ■

Lemma 4 *For any bounded measurable $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ let*

$$f_t(y) := T_t\varphi(y) = \mathbb{E}_y\left[\varphi(W_t)\right], \quad t \geq 0. \tag{10}$$

For all $t \geq 1$ there holds

$$f_t(y) \;\geq\; c_0 t^{-d/\alpha}\mathbf{1}_{B_1}(t^{-1/\alpha}y)\int_{B_1}\varphi(x)\,dx \tag{11}$$

for some constant $c_0 > 0$.

Proof. Let $y \in B_{t^{1/\alpha}}$. Then $t^{-1/\alpha}y \in B_1$, and by self-similarity of W we have

$$\begin{aligned}
f_t(y) &= \mathbb{E}_0\left[\varphi(W_t + y)\right] \\
&= \mathbb{E}_0\left[\varphi\left(t^{1/\alpha}(W_1 + t^{-1/\alpha}y)\right)\right] \\
&\geq \int_{B_1} p_1(x - t^{-1/\alpha}y)\varphi(t^{1/\alpha}x)\,dx \\
&\geq \left(\inf_{x\in B_2} p_1(x)\right)\int_{B_1}\varphi(t^{1/\alpha}x)\,dx \\
&= c_0 t^{-d/\alpha}\int_{B_{t^{1/\alpha}}}\varphi(x)\,dx.
\end{aligned}$$

■

Let g_t solve

$$\frac{\partial g_t}{\partial t} = \Delta_\alpha g_t + \gamma g_t f_t^\beta, \quad g_0 = \varphi, \tag{12}$$

where f_t is defined in (10). Since f_t is a subsolution of (8), g_t is a subsolution of (8) as well.

Proposition 5 *Let $d < \alpha/\beta$. Then g_t grows to ∞ uniformly on the unit ball as $t \to \infty$, i.e.,*

$$\lim_{t\to\infty} \inf_{x\in B_1} g_t(x) = \infty.$$

Proof. From the Feynman-Kac representation we know that g_t is given by

$$g_t(y) = \int \varphi(x)\, p_t(y-x)\, \mathbb{E}_x\left[\exp \int_0^t \gamma f_s(W_s)^\beta\, ds \,\middle|\, W_t = y\right]\, dx.$$

Using (11) and Jensen's inequality, it follows that for $y \in B_{t^{1/\alpha}}$,

$$\begin{aligned} & g_t(y) \\ \geq\ & \int \varphi(x)\, p_t(y-x)\, \mathbb{E}_x\left[\exp \int_1^{t/2} c_1 s^{-\beta d/\alpha} \mathbf{1}_{B_{s^{1/\alpha}}}(W_s)\, ds \,\middle|\, W_t = y\right] dx \\ \geq\ & \int \varphi(x)\, p_t(y-x) \exp\left(c_2 \int_1^{t/2} s^{-\beta d/\alpha} P_x\left\{W_s \in B_{s^{1/\alpha}} \middle|\, W_t = y\right\}\, ds\right) dx \\ \geq\ & c_3 t^{-d/\alpha} \exp\left(c_4 \int_1^{t/2} s^{-\beta d/\alpha} ds\right), \end{aligned} \tag{13}$$

where we have used Lemma 3 to obtain the last inequality, and where c_i, $i = 1, 2, 3, 4$, are positive constants. The result follows from the condition $d < \alpha/\beta$. ■

3 Explosion in subcritical dimensions

Theorem 6 *Let $d < \alpha/\beta$. Each positive non-trivial solution of (8) is non-global.*

Proof. Let $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ be bounded, measurable and positive on a set of positive Lebesgue measure. Let w_t and g_t solve, respectively, equations (8) and (12). Since $w_t \geq g_t$, from Proposition 5 it follows that

$$K(t) := \inf_{x\in B_1} w_t(x) \to \infty \quad \text{as} \quad t \to \infty, \tag{14}$$

which is sufficient to guarantee explosion in finite time due to a classical argument which goes back to Kobayashi, Sirao and Tanaka [6], see also [3]. In fact, putting $u_t := w_{t+t_0}$ with $t_0 > 0$, we obtain

$$u_t(x) = \int p_t(y-x)u_0(y)\,dy + \int_0^t ds \int p_{t-s}(y-x)u_s(y)^{1+\beta}\,dy. \tag{15}$$

Noting that

$$\zeta := \min_{x\in B_1} \min_{0\leq s\leq 1} P_x\left\{W_s \in B_1\right\} > 0,$$

we deduce from (15) that, for every $t \in [0,1]$,

$$\min_{x\in B_1} u_t(x) \;\geq\; \zeta K(t_0) + \zeta \int_0^t \left(\min_{y\in B_1} u_s(y)\right)^{1+\beta} ds. \tag{16}$$

Due to (14) we can choose t_0 so big that the explosion time of the equation

$$v(t) = \zeta K(t_0) + \zeta \int_0^t v(s)^{1+\beta}\,ds \tag{17}$$

is less 1. Then we have $\min_{x\in B_1} u_1(x) \geq v(1) = \infty$, which proves that w_t is nonglobal. $\blacksquare$

Remark 1 By a second application of the Feynman-Kac formula, in [3] it is proved that the subsolution h_t of (8) given by

$$\frac{\partial h_t}{\partial t} = \Delta_\alpha h_t + \gamma g_t^\beta h_t, \quad h_0 = \varphi,$$

where g_t is the subsolution of (8) obtained in the previous section, is such that

$$\inf_{x\in B_1} h_t(x) \to \infty \quad \text{as} \quad t \to \infty$$

even if $d = \alpha/\beta$. In this way it is proved in [3] that (8) has no positive global solutions if $d \leq \alpha/\beta$.

Remark 2 One can prove (see e.g. [10]) that, for $d > \alpha/\beta$, Equation (1) admits both, global and nonglobal positive solutions. In this sense the number α/β is the critical dimension (or equivalently, $1+\alpha/d$ is the critical exponent) for blowup of (1).

4 Explosion of a semilinear equation with generator Γ

In the previous section we used the Feynman-Kac formula to prove that a subsolution g_t of (8) grows uniformly to infinity in the unit ball. A crucial step to get (7) are the lemmas 3 and 4, which provide with lower bounds, respectively, of the bridge and semigroup corresponding to the α-stable process, and in which the self-similarity and symmetry of the α-stable process are essential.

In this section we describe briefly another application of the Feynman-Kac representation, this time to determine the critical exponents for blowup of the semilinear equation

$$\frac{\partial w_t}{\partial t} = \Gamma w_t + \nu t^\sigma w_t^{1+\beta}, \quad w_0(x) = \varphi(x), \quad x \in \mathbb{R}_+, \tag{18}$$

where ν, σ, β are positive constants, φ is non-negative and Γ is the pseudodifferential operator

$$\Gamma f(x) = \int_0^\infty (f(x+y) - f(x)) \frac{e^{-y}}{y}\, dy,$$

i.e. Γ is the infinitesimal generator of the standard Gamma process, which we denote by $X^\Gamma \equiv \{X_t^\Gamma,\ t \geq 0\}$.

Let us recall that the Gamma process belongs to a special family of Lévy processes called subordinatores (see e.g. [2] or [11]), which are purely non-Gaussian processes in $\mathbb{R}$ whose Lévy measure κ is such that $\kappa((-\infty, 0)) = 0$ and $\int_{(0,1]} x\, \kappa(dx) < \infty$. In particular the test functions $t \mapsto X_t^\Gamma(\omega)$ are almost surely increasing, and the transition probabilities $P[X_t^\Gamma \in dy | X_s^\Gamma = x]$ have support in $[x, \infty)$ for each $x \geq 0$.

In contrast with the α-stable processes, subordinators are not in general self-similar nor symmetric, and these circumstances pose serious troubles to apply the method of the previous section, which depends substantially on the symmetry and self-similarity of the Gaussian and stable distributions. In order to get the analogues of lemmas 3 and 4 in this case, it is necessary to impose conditions on the decay of the initial value $\varphi(x)$ as $x \to \infty$. The precise statements are given in the following two lemmas, where $\{T_t^\Gamma, t \geq 0\}$ denotes the semigroup of generator Γ.

Lemma 7 [8] *Let $\varphi : \mathbb{R} \to \mathbb{R}_+$ be bounded and measurable. Assume that there exist $c_1 \in [0, \infty)$, $c_2 \in (0, \infty]$, and $a_1 \geq a_2 > 0$ such that for all x large enough,*

$$c_1 x^{-a_1} \leq \varphi(x) \leq c_2 x^{-a_2}.$$

Then, for all $\eta \geq 0$ and $0 < \varepsilon \leq 1$, there exists $t_0 = t_0(\varepsilon, \eta) > 0$ such that

1. *For all $t > t_0$ and all $y \geq 0$,*

$$\left(\frac{1-\varepsilon}{3}\right)^{a_1} \frac{c_1}{2} t^{-a_1} 1_{[0,t+\eta]}(y) \leq T_t^\Gamma \varphi(y) \leq c_2(1+\varepsilon) t^{-a_2}.$$

2. *For all $t > t_0$ and any $0 \leq y \leq \eta + t/2$,*

$$(1-\varepsilon)\frac{c_1}{2^{1+a_1}} t^{-a_1} 1_{[0,\eta+t/2]}(y) \leq T_t^\Gamma (1_{[t-1/3,2t]}\varphi)(y) \leq c_2(1+\varepsilon) t^{-a_2}. \quad (19)$$

3. *For all $t > t_0$ and any $0 \leq y \leq \eta \leq 1$,*

$$(1-\varepsilon)\frac{\eta c_1}{\sqrt{2\pi}} t^{-a_1-1/2} 1_{[0,\eta]}(y) \leq T_t^\Gamma (1_{[t-\eta,t]}\varphi)(y) \leq (1+\varepsilon)\frac{\eta c_2}{\sqrt{2\pi}} t^{-a_2-1/2}.$$

The bridge bounds are given below, where P_x denotes the distribution of $\{x + X_t^\Gamma,\ t \geq 0\}$.

Lemma 8 [8] *Let $\eta > 0$. We have*

$$P_y \left[0 < X_s^\Gamma < s + \eta | X_t^\Gamma = x\right] \ \geq \ \frac{1}{2}$$

for all $0 < s < t/2$, $0 < y < \eta$, $0 < t - 2\eta < t - \eta < x < t$, *and*

$$P_y\left[0 < X_s^\Gamma < 2s + \frac{t}{2} \,\middle|\, X_t^\Gamma = x\right] \;\geq\; \frac{1}{2} \tag{20}$$

for all $0 < s < t/2$, $0 < y < t/2$ *and* $0 < t/2 < x < 2t$.

Existence of subsolutions of (18) growing locally to infinity follows from the next proposition.

Proposition 9 [8] *Assume that* $\varphi \geq 0$ *is such that* $\varphi(x) \geq cx^{-a}$ *for all* x *large enough, where* $a, c \geq 0$. *Let* $\nu > 0$, $\beta > 0$ *and* $\sigma > -1$. *Let* g_t *be the solution of*

$$\frac{\partial g_t}{\partial t}(y) = \Gamma g_t(y) + \nu t^\sigma (T_t^\Gamma \varphi)^\beta(y) g_t(y), \qquad g_0 = \varphi. \tag{21}$$

If $a\beta < 1 + \sigma$, *then*

$$\lim_{t\to\infty} \inf_{0\leq x\leq 1} g_t(x) = \infty.$$

Proof. Let $0 < \eta < 1$. The Feynman-Kac formula and the first statement in Lemma 7 yield, for $0 < y < \eta + t/2$, $t > 6t_0$, and some $c_0 > 0$:

$$\begin{aligned}
g_t(y) &= \int_y^\infty \varphi(x)\gamma_t(x-y) E_y\left[e^{\nu\int_0^t (t-s)^\sigma (T_{t-s}^\Gamma \varphi(X_s^\Gamma))^\beta\,ds} \,\middle|\, X_t^\Gamma = x\right] dx \\
&\geq \int_y^\infty \varphi(x)\gamma_t(x-y) E_y\left[e^{c_0\nu\int_{t_0}^{t/2} 1_{[0,\eta+t-s]}(X_s^\Gamma)(t-s)^{\sigma-a\beta}\,ds} \,\middle|\, X_t^\Gamma = x\right] dx \\
&\geq \int_{t-1/3}^{2t} \varphi(x)\gamma_t(x-y) e^{c_0\nu\int_{t_0}^{t/2}(t-s)^{\sigma-a\beta} P_y[0<X_s^\Gamma<\eta+t-s|X_t^\Gamma=x]\,ds} dx \\
&\geq \int_{t-1/3}^{2t} \varphi(x)\gamma_t(x-y) e^{c_0\nu\int_{t_0}^{t/6}(t-s)^{\sigma-a\beta} P_y[0<X_s^\Gamma<2s+t/2|X_t^\Gamma=x]\,ds} dx \\
&\geq c_1 1_{[0,\eta+t/2]}(y) t^{-a} e^{\frac{c_0\nu}{2}\int_{t_0}^{t/6}(t-s)^{\sigma-a\beta}\,ds},
\end{aligned}$$

where we used (19) and (20) to obtain the last inequality. Hence

$$\begin{aligned}
g_t(y) &\geq 1_{[0,\eta+t/2]}(y) c_1 t^{-a} e^{\frac{c_0\nu}{2}\int_{t_0}^{t/6}(t-s)^{\sigma-a\beta}\,ds} \\
&= 1_{[0,\eta+t/2]}(y) c_1 t^{-a} e^{\frac{c_0\nu}{2(1+\sigma-a\beta)}((t-t_0)^{1+\sigma-a\beta}-(5t/6)^{1+\sigma-a\beta})},
\end{aligned}$$

and it suffices that $a\beta < 1+\sigma$ in order to get $\inf_{0<y<1} g_t(y) \to \infty$ as $t \to \infty$. ■

¿From here, following the line of proof of Theorem 6, one can obtain critical exponents for blowup of (18). More precisely, every bounded, measurable initial condition $\varphi \geq 0$ satisfying

$$c_1 x^{-a_1} \leq \varphi(x), \quad x > x_0$$

for some positive constants x_0, c_1, a_1, where $a_1\beta < 1+\sigma$, produces a non-global solution of (18). On the other side, if φ fulfils

$$\varphi(x) \leq c_2 x^{-a_2}, \quad x > x_0,$$

where x_0, c_2, a_2 are positive numbers and $a_2\beta > 1+\sigma$, then the solution w_t of (18) is global and, moreover,

$$0 \leq w_t(x) \leq Ct^{-a_2}, \quad x \geq 0,$$

for some constant $C > 0$.

In the special case of $\sigma = 0$ with $\varphi(x) \sim_{x\to\infty} cx^{-a}$ for some constants $c > 0$ and $a > 0$, we have explosion in finite time if $a\beta < 1$, while if $a\beta > 1$, (18) admits global solutions. Hence, if $\sigma = 0$ and

$$\liminf_{x\to\infty} x^{-\varepsilon+1/\beta}\varphi(x) > 0,$$

for some $\varepsilon > 0$, then the solution of (18) blows up in finite time, whereas it is global provided that

$$\liminf_{x\to\infty} x^{\varepsilon+1/\beta}\varphi(x) = 0.$$

Detailed proofs of the above statements are given in [8].

5 Explosion of a semilinear equation with a quadratic potential

In this last section we obtain critical exponents for blow-up of the semilinear equation

$$\frac{\partial u_t}{\partial t}(x) = \Delta u_t(x) - V(x)u_t(x) + t^{\zeta} u_t^{1+\beta}(x), \quad u_0(x) = \varphi(x), \quad x \in \mathbb{R}^d, \tag{22}$$

where β and ζ are positive constants, $\varphi \geq 0$ and V is a bounded potential. In the special case $\zeta = 0$, the above equation has been studied by Souplet and Zhang [12, 14, 15]. They proved that if $d \geq 3$ and

$$0 \leq V(x) \leq \frac{a}{1+|x|^b}, \quad x \in \mathbb{R}^d, \tag{23}$$

for some $a > 0$ and $b \in [2, \infty)$, then $b > 2$ implies finite time blow-up of (22) for all $0 < \beta < 2/d$, whereas if $b = 2$, then there exists $\beta_*(a) < 2/d$ such that blow-up occurs if $0 < \beta < \beta_*(a)$. It is also proved that if

$$V(x) \geq \frac{a}{1+|x|^b}, \quad x \in \mathbb{R}^d, \tag{24}$$

for some $a > 0$ and $0 \leq b < 2$, then (22) admits a global solution for all $\beta > 0$ and all non-negative initial values satisfying $\varphi(x) \leq c/(1+|x|^\sigma)$ for a sufficiently small constant $c > 0$ and all σ obeying $\sigma \geq b/\beta$.

The above results were recovered in [9] by means of the Feynman-Kac formula. Moreover, two critical exponents $\beta_*(a)$, $\beta^*(a)$ were found for the quadratic decay case

$$V(x) \sim_{+\infty} a(1+|x|^2)^{-1}, \quad a > 0.$$

These exponents obey $0 < \beta_*(a) \leq \beta^*(a) < 2/d$, and are such that any nontrivial positive solution is nonglobal provided $0 < \beta < \beta_*(a)$, whereas if $\beta^*(a) < \beta$, then nontrivial positive global solutions may exist.

In order to set up the scenario of [9], let us write $\{S_t,\ t \geq 0\}$ for the semigroup generated by $\Delta - V$, i.e.

$$S_t\varphi(y) = \int_{\mathbb{R}^d} \varphi(x) p_t(x,y)\, dx = f_t(y),$$

where f_t denotes the solution of

$$\frac{\partial f_t}{\partial t}(x) = \Delta f_t(x) - V(x) f_t(x), \qquad f_0(x) = \varphi(x),$$

and $\{p_t(x,y),\ t > 0\}$, are the transition densities of the Markov process in $\mathbb{R}^d$ having $\Delta - V$ as its generator. To get started the Feynman-Kac approach, we require appropriate lower bounds for the bridge and semigroup corresponding to the Markov process with generator $\Delta - V$. We start by recalling the following basic estimates of $p_t(x,y)$ which we borrowed from [14].

Theorem 10 *Let $d \geq 3$, and assume that Condition (23) holds for some $b \geq 0$ and $a > 0$. There exist constants $c_4, c_5, c_6 > 0$, and $\alpha_2(a) > 0$, such that for all $t > 0$ and $x, y \in \mathbb{R}^d$,*

$$p_t(x,y) \geq \begin{cases} c_6 e^{-2c_5 t} G_t(c_4(x-y)) & \text{if } b < 2, \\ \\ c_6 t^{-\alpha_2(a)} G_t(c_4(x-y)) & \text{if } b = 2, \\ \\ c_6 G_t(c_4(x-y)) & \text{if } b > 2. \end{cases}$$

Theorem 10 together with self-similarity of Gaussian densities immediately yield

Lemma 11 *Let $d \geq 3$, $b \geq 2$, and let $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ be bounded and measurable. Assume that*

$$0 \leq V(x) \leq \frac{a}{1+|x|^b}.$$

Then, for all $t \geq 1$ and $y \in \mathbb{R}^d$ we have

$$S_t\varphi(y) \geq c_0 t^{-\alpha_2 - d/2} \mathbf{1}_{B_{t^{1/2}}}(y) \int_{B_{t^{1/2}}} \varphi(x)\, dx,$$

where $\alpha_2 = 0$ if $b > 2$, and $\alpha_2(a) = ca$ for some $c > 0$ when $b = 2$.

The next lemma provides lower bounds on certain balls for the distributions of the bridges of the Markov process $(X_t)_{t \in \mathbb{R}_+}$ generated by $\Delta - V$.

Lemma 12 [9] *Assume $d \geq 3$, and let $(X_t)_{t \in \mathbb{R}_+}$ denote the Markov process with generator $\Delta - V$. If (23) holds for some $b \geq 2$, then there exists $c_8 > 0$ such that for all $t \geq 2$, $y \in B_{t^{1/2}}$, $x \in B_1$ and $s \in [1, t/2]$,*

$$P_x(X_s \in B_{s^{1/2}} \mid X_t = y) \geq c_8 t^{-2\alpha_2(a)},$$

where $\alpha_2(a) = 0$ when $b > 2$ and $\alpha_2(a) = ca$ when $b = 2$.

Proof. Recall that from the Feynman-Kac formula,

$$p_t(x,y) = G_t(x-y) E_x \left[e^{-\int_0^t V(Z_s)\, ds} \middle| Z_t = y \right],$$

where $\{Z_t,\ t \geq 0\}$ denotes Brownian motion in R^d, and $\{G_t,\ t > 0\}$ its transition densities. Since $V(x) \geq 0$, the above expression renders

$$p_t(x,y) \leq G_t(y-x), \qquad t > 0, \quad x, y \in \mathbb{R}^d.$$

An application of Theorem 10 and of the Markov property of $(X_s)_{s \in \mathbb{R}_+}$ give

$$\begin{aligned}
&P_x(X_s \in B_{s^{1/2}} \mid X_t = y) \\
&\geq \int_{B_{s^{1/2}}} \frac{p_{t-s}(y,z)p_s(z,x)}{p_t(y,x)}\, dz \\
&= \frac{1}{c_6^2 s^{\alpha_2(a)}(t-s)^{\alpha_2(a)}} \int_{B_{s^{1/2}}} \frac{G_{t-s}(c_4(y-z))G_s(c_4(z-x))}{G_t(c_4(y-x))}\, dz \\
&\geq c_8 t^{-2\alpha_2(a)},
\end{aligned}$$

where we used Lemma 2.2 of [3] to obtain the last inequality. ■

Proceeding as in previous sections, using the Feynman-Kac representation and the bounds provided in the above lemmas, after some effort one can prove the following, see [9].

Let $d \geq 3$, and assume condition (23) holds with $b > 2$. If

$$0 < \beta < 2(1+\zeta)/d,$$

then any nontrivial positive solution of (22) is nonglobal. When $0 \leq b < 2$, Equation (22) admits nontrivial global positive solutions.

In the critical case $b = 1$, it is proved, assuming again $d \geq 3$, that there exist strictly positive numbers $\beta_*(a) \leq \beta^*(a)$, both decreasing in $a > 0$, given by

$$\begin{aligned}
\beta_*(a) &= \frac{2(1+\zeta) - 4ac}{d + 2ac}, \\
\beta^*(a) &= \frac{2(1+\zeta)}{d + \min(1, a(d+4)^{-2}/64)},
\end{aligned}$$

where $c > 0$ is independent of a, and such that

a) If $0 \leq V(x) \leq \dfrac{a}{1+|x|^2}$, then (22) blows up in finite time for all $0 < \beta < \beta_*(a)$.

b) If $V(x) \geq \dfrac{a}{1+|x|^2}$, then (22) admits a global solution for all $\beta > \beta^*(a)$.

The blowup behavior of (22) when $\beta \in [\beta_*(a), \beta^*(a)]$, remains to be investigated.

References

[1] Applebaum, David. Lévy processes and stochastic calculus. Cambridge Studies in Advanced Mathematics, 93. Cambridge University Press, Cambridge, 2004.

[2] Bertoin, J. Subordinators: examples and applications, in: Lectures on Probability Theory and Statistics (Saint-Flour, 1997), in: *Lecture Notes in Math.* vol. 1717, Springer, Berlin, 1999, pp. 1-91.

[3] Birkner, M., López-Mimbela, J. A. and Walkonbinger, A., Blow-up of Semilinear PDE's at the Critical Dimension. A Probabilistic Approach, *Proc. Amer. Math. Soc.* **130** (2002), 2431-2442.

[4] Birkner, M., López-Mimbela, J. A. and Walkonbinger, A., Comparison Results and Steady States for the Fujita Equation with Fractional Laplacian, *Annales de L'Institute Henri Poincare -Analyse non Linéare* **22** (2005), 83-97.

[5] Chow Y.S. y Teicher, H. *Probability Theory*. Springer, 1988.

[6] Kobayashi, Kusuo; Sirao, Tunekiti; Tanaka, Hiroshi. On the growing up problem for semilinear heat equations. *J. Math. Soc. Japan* **29** (1977), no. 3, 407–424.

[7] López-Mimbela, J.A. Branching particle representation of a class of semilinear equations. *Pliska Stud. Math. Bulgar.* **16** (2004), 101–119.

[8] López-Mimbela, J. A. and Privault, N., Blow-up and Stability of Semilinear PDEs with Gamma Generators, *J. Math. Anal. Appl.* **307** (2005), 181-205.

[9] López-Mimbela, J. A. and Privault, N., Critical Exponents for Semilinear PDEs with Bounded Potentials, preprint (2005).

[10] López-Mimbela, J. A. and Wakolbinger, A., A Probabilistic Proof of Non-explosion of a Non-linear PDE System, *J. Appl. Probab.* **37** (2000), no. 3, 635–641.

[11] Sato, Ken-iti, *Lévy Processes and Infinitely Divisible Distributions,* Cambridge University Press, 1999.

[12] P. Souplet and Q.S. Zhang. Stability for semilinear parabolic equations with decaying potentials in $\mathbb{R}^n$ and dynamical approach to the existence of ground states. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 19(5):683–703, 2002.

[13] Sznitman, Alain-Sol. Brownian motion, obstacles and random media. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[14] Q.S. Zhang. Large time behavior of Schrödinger heat kernels and applications. *Comm. Math. Phys.*, 210(2):371–398, 2000.

[15] Q.S. Zhang. The quantizing effect of potentials on the critical number of reaction-diffusion equations. *J. Differential Equations*, 170(1):188–214, 2001.