

FINITE TIME BLOWUP AND LIFE SPAN OF A NONAUTONOMOUS SEMILINEAR EQUATION

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Finite time blowup and life span of a nonautonomous semilinear equation

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Abstract

Consider the semilinear nonautonomous equation $\frac{\partial}{\partial t}u(t) = k(t)\Delta_\alpha u(t) + u^{1+\beta}(t)$ with $u(0, x) = \lambda\varphi(x)$, $x \in \mathbb{R}^d$, where $\Delta_\alpha := -(-\Delta)^{\alpha/2}$, $0 < \alpha \leq 2$, $\lambda, \beta > 0$ are constants, $\varphi \geq 0$ is bounded, continuous and does not identically vanish, and $k : [0, \infty) \rightarrow [0, \infty)$ is a locally integrable function satisfying $\varepsilon_1 t^\rho \leq \int_0^t k(r) dr \leq \varepsilon_2 t^\rho$ for all t large enough, where $\varepsilon_1, \varepsilon_2, \rho > 0$ are given constants. We prove that any constellation of positive parameters d, α, ρ, β , obeying $0 < d\rho\beta/\alpha < 1$, yields finite time blow up of any nontrivial positive solution. Under suitable additional assumptions, we also obtain upper and lower bounds for the life span $T_{\lambda\varphi}$ of the above equation, which prove that $T_{\lambda\varphi} \sim \lambda^{-\frac{\alpha\beta}{\alpha-d\rho\beta}}$ near zero.

Key words: Semilinear partial differential equations, non-autonomous Cauchy problem, Feynman-Kac representation, critical exponent, finite time blow-up, nonglobal solution, life span.

Mathematics Subject Classification: 60H30, 35K55, 35K57, 35B35.

1 Introduction

We consider positive solutions of the semilinear non-autonomous Cauchy problem

$$\frac{\partial u(t, x)}{\partial t} = k(t)\Delta_\alpha u(t, x) + u^{1+\beta}(t, x), \quad u(0, x) = \varphi(x) \geq 0, \quad x \in \mathbb{R}^d, \quad (1.1)$$

where $\Delta_\alpha := -(-\Delta)^{\alpha/2}$ denotes the fractional power of the Laplacian, $0 < \alpha \leq 2$, $\beta \in (0, \infty)$ is a constant, and $k : [0, \infty) \rightarrow [0, \infty)$ is a locally integrable function satisfying

$$\varepsilon_1 t^\rho \leq \int_0^t k(r) dr \leq \varepsilon_2 t^\rho \quad (1.2)$$

for all t large enough, where $\varepsilon_1, \varepsilon_2$ and ρ are given positive constants. Solutions will be understood in the mild sense, so that (1.1) is meaningful for any bounded measurable initial value.

Recall that there exists a number $T_\varphi \in (0, \infty]$ such that (1.1) has a unique solution u on $[0, T_\varphi) \times \mathbb{R}^d$, which is given by

$$u(t, x) = U(t, 0)\varphi(x) + \int_0^t \left(U(t, s)u^{1+\beta}(s, \cdot) \right) (x) ds, \quad (1.3)$$

and is bounded on $[0, T] \times \mathbb{R}^d$ for any $0 < T < T_\varphi$. Moreover, if $T_\varphi < \infty$, then $\|u(t, \cdot)\|_\infty \rightarrow \infty$ as $t \uparrow T_\varphi$. Here $\{U(t, s), 0 \leq s \leq t\}$ denotes the evolution system corresponding to the family of generators $\{k(t)\Delta_\alpha, t \geq 0\}$. When $T_\varphi = \infty$ we say that u is a global solution, and when $T_\varphi < \infty$ we say that u blows up in finite time or that u is nonglobal. The extended real number T_φ is termed life span of Eq. (1.1).

In this paper we continue the investigation initiated in our previous article [11], to which we refer for motivations and additional references. In [11] we proved that $d > \frac{\alpha}{\rho\beta}$ implies existence of non-trivial global solutions of (1.1) for all sufficiently small initial values, and that, under the additional assumption $\beta \in \{2, 3, \dots\}$, the condition $d < \frac{\alpha}{\rho\beta}$ yields finite time blowup of any positive solution. Moreover, the case $\rho = 0$, which under condition (1.2) corresponds to an integrable k , yields finite time blow up of (1.1) for any non-trivial initial value, regardless of the spatial dimension and the stability exponent α . Here we consider the case $d < \frac{\alpha}{\rho\beta}$ with $\beta \in (0, \infty)$, and focus on the asymptotic behavior of the life span of (1.1) when the initial value is of the form $\lambda\varphi$, where $\lambda > 0$ is a parameter.

The life span asymptotics are an aspect of semilinear parabolic Cauchy problems which give insight about how the “size” of the initial value affects the blowup time of their positive solutions; see [6], [7], [9], [10], [13], [14] and the references therein. Given two functions $f, g : [0, \infty) \rightarrow [0, \infty)$, let us say that $f \sim g$ near $c \in \{0, \infty\}$ if there exist two positive constants C_1, C_2 such that $C_1 f(r) \geq g(r) \geq C_2 f(r)$ for all r which are sufficiently close to c . In [10] it was proved, initially for $k(t) \equiv 1$ and $\alpha = 2$, that $T_{\lambda\varphi} \sim \lambda^{-\beta}$ near ∞ provided $\varphi \geq 0$ is bounded, continuous and does not identically vanish. Later on, Gui and Wang [6] showed that, in fact, $\lim_{\lambda \rightarrow \infty} T_{\lambda\varphi} \cdot \lambda^\beta = \beta^{-1} \|\varphi\|_{L^\infty(\mathbb{R}^d)}^{-\beta}$. The behavior of $T_{\lambda\varphi}$ as λ approaches 0 was also investigated in [10], turning out that $T_{\lambda\varphi} \sim \lambda^{-\beta}$ near 0. Notice that these asymptotics are similar to those of the ordinary differential equation $df(t)/dt = f^{1+\beta}(t)$, $f(0) = \lambda\varphi$, where $\varphi > 0$.

In the present paper we obtain upper and lower bounds for the life span $T_{\lambda\varphi}$ of (1.1), and provide in this way a description of the behavior of $T_{\lambda\varphi}$ as $\lambda \rightarrow \infty$ and $\lambda \rightarrow 0$. Here is a brief outline.

We start by proving that any constellation of positive parameters d, α, ρ, β , obeying $0 < d\rho\beta/\alpha < 1$, yields finite time blow up of any nontrivial positive solution of (1.1). This is carried out by bounding from below the mild solution of (1.1) by a subsolution which locally grows to ∞ . Finite-time blowup of (1.1) is then inferred from a classical comparison procedure that dates back to [8] (see also [2], Sect. 3). The construction of our subsolution uses the Feynman-Kac representation of (1.1), and requires to control the decay of the bridge probabilities of $W \equiv \{W(t), t \geq 0\}$, where W is the \mathbb{R}^d -valued Markov process corresponding to the evolution system $\{U(t, s), t \geq s \geq 0\}$; see [2], [3] and [12] for the time-homogeneous case.

A further consequence of the Feynman-Kac representation of (1.1) is the inequality

$T_{\lambda\varphi} \leq \text{Const.}\lambda^{-\frac{\alpha\beta}{\alpha-d\rho\beta}}$, which holds for small positive λ when $0 < d\rho\beta/\alpha < 1$. Together with the lower bound of $T_{\lambda\varphi}$ given in Section 6, this implies (under the condition $0 < d\rho\beta/\alpha < 1$) that

$$T_{\lambda\varphi} \sim \lambda^{-\frac{\alpha\beta}{\alpha-d\rho\beta}}$$

near 0. Finally, we also provide an upper bound for $T_{\lambda\varphi}$ which is valid for all $\lambda > 0$, namely

$$T_{\lambda\varphi} \leq \left(C\lambda^{-\beta} + \left[(10\varepsilon_2/\varepsilon_1)^{\frac{1}{\rho}} \theta \right]^{\frac{\alpha-d\rho\beta}{\alpha}} \right)^{\frac{\alpha}{\alpha-d\rho\beta}} + \eta, \quad (1.4)$$

where C, θ and η are suitable positive constants. Notice that, even if $\lambda > 0$ is large, for small positive ρ the upper bound in (1.4) is big; in fact, it grows to ∞ as $\rho \rightarrow 0$. An intuitive explanation of this behavior is as follows. Consider the representative case $k(t) = \rho/t^{1-\rho}$, $t > 0$, which corresponds to $\varepsilon_1 = \varepsilon_2 = 1$. No matter how big is λ , if $\rho > 0$ is sufficiently close to 0, then, near $t = 0$, the mobility of the motion process $\{W(t), t \geq 0\}$ is so big that it smears out the initial value $\lambda\varphi$. Hence the growth of the upper bound in (1.4).

We remark that our bridge and semigroup bounds (see Section 3) seem to be not sharp enough to yield, using our present methods, a subsolution of (1.1) growing uniformly on a ball for the parameter configuration $d\rho\beta/\alpha = 1$. Therefore, the blowup behavior and life span asymptotics of (1.1) under such configuration remain to be investigated.

As this paper is partly aimed at the multidisciplinary reader, in the next section we recall some basic facts, including the Feynman-Kac representation which we prove there for the sake of completeness. In Section 3 we obtain semigroup and bridge estimates that we shall need in the sequel. Section 4 is devoted to prove that (1.1) does not admit nontrivial global solutions if $d < \frac{\alpha}{\rho\beta}$. In the remaining sections 5 and 6 we prove our bounds for the life span of (1.1).

2 The Feynman-Kac representation and subsolutions

Let us denote by $Z \equiv \{Z(t)\}_{t \geq 0}$ the symmetric α -stable process in \mathbb{R}^d , whose infinitesimal generator is Δ_α , $0 < \alpha \leq 2$. Recall that the case $\alpha = 2$ corresponds to standard Brownian motion with variance parameter 2.

For any $T > 0$ let us consider the initial value problem

$$\begin{aligned} \frac{\partial v(t, x)}{\partial t} &= k(t)\Delta_\alpha v(t, x) + \zeta(t, x)v(t, x), & 0 < t \leq T, \\ v(0, x) &= \varphi(x), & x \in \mathbb{R}^d, \end{aligned} \quad (2.5)$$

where $k : [0, \infty) \rightarrow [0, \infty)$ is integrable on any bounded interval, and ζ and φ are nonnegative bounded continuous functions on $[0, T] \times \mathbb{R}^d$ and \mathbb{R}^d , respectively. It is well known that, in the classical setting $k \equiv 1$, $\alpha = 2$, $\zeta(t, x) \equiv \zeta(x)$, the solution of (2.5) can be expressed via

the Feynman-Kac formula, see e.g. [4]. Here we shall prove the Feynman-Kac representation corresponding to (2.5).

Let $W \equiv \{W(t)\}_{t \geq 0}$ be the (time-inhomogeneous) càdlàg Feller process corresponding to the family of generators $\{k(t) \Delta_\alpha\}_{t \geq 0}$. We designate P_x the distribution of $\{W(t) + x\}_{t \geq 0}$, and write E_x for the expectation with respect to P_x , $x \in \mathbb{R}^d$.

Theorem 1 *Let k , ζ and φ be as above. Then, the solution of (2.5) admits the Feynman-Kac representation*

$$v(t, x) = E_x \left[\varphi(W(t)) \exp \left\{ \int_0^t \zeta(t-s, W(s)) ds \right\} \right], \quad (t, x) \in [0, T] \times \mathbb{R}^d. \quad (2.6)$$

Proof. Our method of proof is an adaptation to our time-inhomogeneous context of the approach used in [1]. We shall assume that $0 < \alpha < 2$; the case $\alpha = 2$ can be handled in a similar fashion.

Recall [15] that there exists a Poisson random measure $N(dt, dx)$ on $[0, \infty) \times \mathbb{R}^d \setminus \{0\}$ having expectation $EN(dt, dx) = dt \nu(dx)$, with

$$\nu(dx) = \frac{\alpha 2^{\alpha-1} \Gamma((\alpha+d)/2)}{\pi^{d/2} \Gamma(1-\alpha/2) \|x\|^{\alpha+d}} dx,$$

and such that the paths of Z admits the Lévy-Itô decomposition

$$Z(t) = \int_{|x| < 1} x \tilde{N}(t, dx) + \int_{|x| \geq 1} x N(t, dx), \quad t \geq 0, \quad (2.7)$$

where $N(t, dx) := \int_0^t N(dt, dx)$, and $\tilde{N}(t, dx)$ is the compensated Poisson random measure

$$\tilde{N}(t, B) = N(t, B) - t\nu(B), \quad t \geq 0, \quad B \in \mathcal{B}(\mathbb{R}^d);$$

here $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ -algebra in \mathbb{R}^d . W also admits a Lévy-Itô decomposition, with corresponding Poisson random measure $k(t) N(dt, dx)$.

Let us write $W(p^-)$ for the limit of W from the left of p . From the integration by parts formula we obtain

$$\begin{aligned} & d \left[v(t-s, W(s)) \exp \left\{ \int_0^s \zeta(t-r, W(r)) dr \right\} \right] \\ &= v(t-s, W(s^-)) \zeta(t-s, W(s^-)) \exp \left\{ \int_0^s \zeta(t-r, W(r^-)) dr \right\} ds \\ & \quad + \exp \left\{ \int_0^s \zeta(t-r, W(r^-)) dr \right\} dv(t-s, W(s)). \end{aligned}$$

Using Itô's formula to calculate $dv(t-s, W(s))$ yields

$$\begin{aligned}
& d \left[v(t-s, W(s)) \exp \left\{ \int_0^s \zeta(t-r, W(r)) dr \right\} \right] \\
&= \exp \left\{ \int_0^s \zeta(t-r, W(r^-)) dr \right\} \left\{ v(t-s, W(s^-)) \zeta(t-s, W(s^-)) ds - \frac{d}{ds} v(t-s, W(s^-)) \right. \\
&\quad + k(s) \int_{|x|<1} [v(t-s, W(s^-) + x) - v(t-s, W(s^-))] \tilde{N}(ds, dx) \\
&\quad + k(s) \int_{|x|\geq 1} [v(t-s, W(s^-) + x) - v(t-s, W(s^-))] N(ds, dx) \\
&\quad \left. + k(s) \int_{|x|<1} \left[v(t-s, W(s^-) + x) - v(t-s, W(s^-)) - \sum_i x_i \frac{d}{dx_i} v(t-s, W(s^-)) \right] \nu(dx) ds \right\}.
\end{aligned}$$

Integrating from 0 to t , and taking expectation with respect to P_x , we obtain

$$\begin{aligned}
& E_x \left[\varphi(W(t)) \exp \left\{ \int_0^t \zeta(t-s, W(s)) ds \right\} \right] - v(t, x) \\
&= E_x \int_0^t \exp \left\{ \int_0^s \zeta(t-r, W(r^-)) dr \right\} \left\{ v(t-s, W(s^-)) \zeta(t-s, W(s^-)) - \frac{d}{ds} v(t-s, W(s^-)) \right. \\
&\quad + k(s) \int_{|x|<1} \left[v(t-s, W(s^-) + x) - v(t-s, W(s^-)) - \sum_i x_i \frac{d}{dx_i} v(t-s, W(s^-)) \right] \nu(dx) \\
&\quad \left. + k(s) \int_{|x|\geq 1} [v(t-s, W(s^-) + x) - v(t-s, W(s^-))] \nu(dx) \right\} ds \\
&= 0,
\end{aligned}$$

where in the first equality we used $\tilde{N}(ds, dx) = N(ds, dx) - ds \nu(dx)$, and the fact that the stochastic integrals with respect to $\tilde{N}(ds, dx)$ are martingales, and therefore have expectation 0. \blacksquare

The Feynman-Kac representation is suitable to construct subsolutions of reaction-diffusion equations of the prototype

$$\frac{\partial w(t, y)}{\partial t} = k(t) \Delta_\alpha w(t, y) + w^{1+\beta}(t, y), \quad w(0, y) = \varphi(y), \quad y \in \mathbb{R}^d, \quad (2.8)$$

where $\beta > 0$ is a constant, and k, φ are as in (2.5). From Theorem 1 we know that

$$w(t, y) = E_y \left[\varphi(W(t)) \exp \int_0^t w^\beta(t-s, W(s)) ds \right], \quad (t, y) \in [0, T] \times \mathbb{R}^d,$$

for some $T \geq 0$. Hence, for every $y \in \mathbb{R}^d$,

$$w(t, y) \geq E_y [\varphi(W(t))] =: v_0(t, y), \quad t \geq 0,$$

so that v_0 is a subsolution of (2.8), i.e. $w(0, \cdot) = v_0(0, \cdot)$ and $w(t, \cdot) \geq v_0(t, \cdot)$ for every $t > 0$. A direct consequence of the Feynman-Kac representation is the next lemma, which we will need in the following section.

Lemma 2 Let k, φ be as in (2.5), and let $\zeta(\cdot, \cdot)$ be a subsolution of (2.8). Then, any solution of

$$\frac{\partial v(t, y)}{\partial t} = \Delta_\alpha v(t, y) + \zeta^\beta(t, y)v(t, y), \quad v(0, \cdot) = \varphi,$$

remains a subsolution of (2.8).

3 Bridge and semigroup bounds

Let us denote by $p(t, x)$, $t \geq 0$, $x \in \mathbb{R}^d$, the transition densities of the d -dimensional symmetric α -stable process $\{Z(t)\}_{t \geq 0}$. Recall that $p(t, \cdot)$, $t > 0$, are strictly positive, radially symmetric continuous functions that satisfy the following properties.

Lemma 3 For any $s, t > 0$, and $x, y \in \mathbb{R}^d$, $p(t, x)$ satisfies

- i) $p(ts, x) = t^{-\frac{d}{\alpha}} p\left(s, t^{-\frac{1}{\alpha}} x\right)$,
- ii) $p(t, x) \leq p(t, y)$ when $|x| \geq |y|$,
- iii) $p(t, x) \geq \left(\frac{s}{t}\right)^{\frac{d}{\alpha}} p(s, x)$ for $t \geq s$,
- iv) $p\left(t, \frac{1}{\tau}(x - y)\right) \geq p(t, x)p(t, y)$ if $p(t, 0) \leq 1$ and $\tau \geq 2$.

Proof. See [5] or [16]. ■

Let $\varphi : \mathbb{R}^d \rightarrow [0, \infty)$ be bounded and measurable, and let $k : [0, \infty) \rightarrow [0, \infty)$ be locally integrable. Notice that the transition probabilities of the Markov process $\{W(t), t \geq 0\}$ are given by

$$P(W(t) \in dy | W(s) = x) = p\left(\int_s^t k(r) dr, y - x\right) dy, \quad 0 \leq s \leq t, \quad x \in \mathbb{R}^d.$$

We define the function

$$v_0(t, x) = E_x[\varphi(W(t))] = E_x[\varphi(Z(K(t, 0)))] = \int p(K(t, 0), y - x) \varphi(y) dy, \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad (3.9)$$

where $K(t, s) := \int_s^t k(r) dr$, $0 \leq s \leq t$, and write $B(r) \equiv B_r \subset \mathbb{R}^d$ for the ball of radius r , centered at the origin.

Lemma 4 There exists a constant $c_0 > 0$ satisfying

$$v_0(t, x) \geq c_0 K^{-\frac{d}{\alpha}}(t, 0) 1_{B_1}\left(K^{-\frac{1}{\alpha}}(t, 0)x\right) \quad (3.10)$$

for all $x \in \mathbb{R}^d$, and all $t > 0$ such that $K^{\frac{1}{\alpha}}(t, 0) \geq 1$.

Proof. From Lemma 3 i), ii) and radial symmetry of $p(t, \cdot)$ we have, for $K^{\frac{1}{\alpha}}(t, 0) \geq 1$,

$x \in B_{K^{\frac{1}{\alpha}}(t,0)}$ and $z \in \partial B_2$, that

$$\begin{aligned}
v_0(t, x) &= E_0 [\varphi(Z(K(t, 0)) + x)] \\
&= E_0 \left[\varphi \left(K^{\frac{1}{\alpha}}(t, 0) (Z(1) + K^{-\frac{1}{\alpha}}(t, 0)x) \right) \right] \\
&\geq \int_{B_1} \varphi \left(K^{\frac{1}{\alpha}}(t, 0)y \right) P \left[Z(1) \in dy - K^{-\frac{1}{\alpha}}(t, 0)x \right] \\
&= \int_{B_1} \varphi \left(K^{\frac{1}{\alpha}}(t, 0)y \right) p \left(1, y - K^{-\frac{1}{\alpha}}(t, 0)x \right) dy \\
&\geq p(1, z) \int_{B_1} \varphi \left(K^{\frac{1}{\alpha}}(t, 0)y \right) dy \\
&= p(1, z) K^{-\frac{d}{\alpha}}(t, 0) \int_{B_{K^{\frac{1}{\alpha}}(t,0)}} \varphi(y) dy \\
&\geq p(1, z) K^{-\frac{d}{\alpha}}(t, 0) 1_{B_1} \left(K^{-\frac{1}{\alpha}}(t, 0)x \right) \int_{B_1} \varphi(y) dy.
\end{aligned}$$

Letting $c_0 = p(1, z) \int_{B_1} \varphi(y) dy$ yields (3.10). ■

Fix $\theta > 0$ such that (1.2) holds for all $t \geq \theta$ and such that $K^{\frac{1}{\alpha}}(\theta, 0) \geq 1$. Define $\delta_0 = \min \left\{ \left(\frac{\varepsilon_1}{2\varepsilon_2} \right)^{\frac{1}{p}}, 1 - \left(\frac{\varepsilon_1}{2\varepsilon_2} \right)^{\frac{1}{p}} \right\}$.

Lemma 5 *There exists $c > 0$ such that for all $x, y \in B_1$ and t large enough,*

$$P_x \left[W(s) \in B_{K^{\frac{1}{\alpha}}(t-s,0)} \mid W(t) = y \right] \geq c$$

for $s \in [\theta, \delta_0 t]$.

Proof. Using (1.2) and Lemma 3 i) we get

$$\begin{aligned}
&P_x \left[W(s) \in B_{K^{\frac{1}{\alpha}}(t-s,0)} \mid W(t) = y \right] \\
&= \int_{B_{K^{\frac{1}{\alpha}}(t-s,0)}} \frac{p(K(s, 0), x - z) p(K(t, s), z - y)}{p(K(t, 0), x - y)} dz \\
&\geq \int_{B_{\frac{r}{r_s \alpha}}} \frac{K^{-\frac{d}{\alpha}}(s, 0) p \left(1, K^{-\frac{1}{\alpha}}(s, 0)(x - z) \right) K^{-\frac{d}{\alpha}}(t, s) p \left(1, K^{-\frac{1}{\alpha}}(t, s)(z - y) \right)}{K^{-\frac{d}{\alpha}}(t, 0) p \left(1, K^{-\frac{1}{\alpha}}(t, 0)(x - y) \right)} dz,
\end{aligned} \tag{3.11}$$

with $r = \varepsilon_1^{\frac{1}{\alpha}} \left(\frac{1}{\delta_0} - 1 \right)^{\frac{p}{\alpha}}$. It is straightforward to verify that $K^{-\frac{1}{\alpha}}(s, 0)(x - z) \in B_{r_1}$, where $r_1 = 2 \left(\frac{\varepsilon_2}{\varepsilon_1} \right)^{\frac{1}{\alpha}} \delta_0^{\frac{p}{\alpha}}$ with $\delta = \max \left\{ 1, \frac{1}{\delta_0} - 1 \right\}$, hence Lemma 3 ii) and radial symmetry of $p(t, \cdot)$ imply

$$p \left(1, K^{-\frac{1}{\alpha}}(s, 0)(x, z) \right) \geq p(1, \varsigma) \equiv c_1$$

for any $\varsigma \in \partial B_{r_1}$.

Thus, the term in the right-hand side of (3.11) is bounded from below by

$$\int_{B_{rs\frac{\rho}{\alpha}}} \frac{c_1 K^{-\frac{d}{\alpha}}(s, 0) K^{-\frac{d}{\alpha}}(t, s) p\left(1, K^{-\frac{1}{\alpha}}(t, s)(z - y)\right)}{K^{-\frac{d}{\alpha}}(t, 0) p\left(1, K^{-\frac{1}{\alpha}}(t, 0)(x - y)\right)} dz.$$

Using (1.2), and the facts that $K(t, 0) \geq K(t, s)$ and $p(t, x) \leq p(t, 0)$ for all $t > 0$ and $x \in \mathbb{R}^d$, it follows that

$$P_x \left[W(s) \in B_{K\frac{1}{\alpha}(t-s, 0)} \mid W(t) = y \right] \geq \int_{B_{rs\frac{\rho}{\alpha}}} c_2 s^{-\frac{dp}{\alpha}} p\left(1, K^{-\frac{1}{\alpha}}(t, s)(z - y)\right) dz, \quad (3.12)$$

where $c_2 = \frac{c_1 \varepsilon_2^{-\frac{d}{\alpha}}}{p(1, 0)}$. Since $\theta \leq s \leq \delta_0 t$, we have from (1.2) and the definition of δ_0 that

$$K^{-\frac{1}{\alpha}}(t, s) = [K(t, 0) - K(s, 0)]^{-\frac{1}{\alpha}} \leq (\varepsilon_1 t^\rho - \varepsilon_2 \delta_0^\rho t^\rho)^{-\frac{1}{\alpha}} \leq c_3 t^{-\frac{\rho}{\alpha}},$$

where $c_3 = \left(\frac{2}{\varepsilon_1}\right)^{\frac{1}{\alpha}}$. Since $y \in B_1$, $z \in B_{rs\frac{\rho}{\alpha}}$ and $\theta \leq s \leq \delta_0 t$, we deduce that, for $t \geq 1$, $y \in B_{t\frac{\rho}{\alpha}}$ and $z \in B_{r\delta_0^\frac{\rho}{\alpha}t\frac{\rho}{\alpha}}$. Letting $\gamma = \max\left\{1, r\delta_0^\frac{\rho}{\alpha}\right\}$, it follows that $z - y \in B_{2\gamma t\frac{\rho}{\alpha}}$, and thus $K^{-\frac{1}{\alpha}}(t, s)(z - y) \in B_{2\gamma c_3}$. Therefore,

$$p\left(1, K^{-\frac{1}{\alpha}}(t, s)(z - y)\right) \geq p(1, \varsigma) \equiv c_4$$

for any $\varsigma \in \partial B_{2\gamma c_3}$. From (3.12) we conclude that

$$P_x \left[W(s) \in B_{K\frac{1}{\alpha}(t-s, 0)} \mid W(t) = y \right] \geq \int_{B_{rs\frac{\rho}{\alpha}}} c_5 s^{-\frac{dp}{\alpha}} dz \equiv c.$$

■

4 Nonexistence of positive global solutions

In this section we shall use the Feynman-Kac representation to construct a subsolution of (1.1) which grows to infinity uniformly on the unit ball. As we are going to prove afterward, this guarantees nonexistence of nontrivial positive solutions of (1.1).

Let v solve the semilinear nonautonomous equation

$$\begin{aligned} \frac{\partial v(t, x)}{\partial t} &= k(t) \Delta_\alpha v(t, x) + v_0^\beta(t, x) v(t, x), \\ v(0, x) &= \varphi(x), \quad x \in \mathbb{R}^d, \end{aligned} \quad (4.13)$$

where k and φ are as in (1.1), and v_0 is defined in (3.9). Since $v_0 \leq u$, where u is the solution of (1.1), it follows from Lemma 2 that $v \leq u$ as well. Without loss of generality we shall assume that φ does not a.s. vanish on the unit ball.

Proposition 6 *There exist $c', c'' > 0$ such that, for all $x \in B_1$ and all $t > 0$ large enough,*

$$v(t, x) \geq c' t^{-\frac{d\rho}{\alpha}} \exp\left(c'' t^{1-\frac{d\rho\beta}{\alpha}}\right).$$

Proof. Let c_0, c_1, \dots, c_8 denote suitable positive constants. From Theorem 1 we know that

$$v(t, x) = \int_{\mathbb{R}^d} \varphi(y) p(K(t, 0), x - y) E_x \left[\exp \int_0^t v_0^\beta(t - s, W(s)) ds \mid W(t) = y \right] dy.$$

Let θ and δ_0 be as in Lemma 5. For any $\theta \leq s \leq \delta_0 t$, we have $t - s \geq t - \delta_0 t = (1 - \delta_0)t \geq \delta_0 t \geq \theta$, and therefore $K^{\frac{1}{\alpha}}(t - s, 0) \geq 1$. From here, using (3.10) and Jensen's inequality, we get

$$\begin{aligned} v(t, x) &\geq \int_{\mathbb{R}^d} \varphi(y) p(K(t, 0), x - y) \\ &\quad \cdot E_x \left[\exp \int_\theta^{\delta_0 t} c_0^\beta K^{-\frac{d\beta}{\alpha}}(t - s, 0) 1_{B_{K^{\frac{1}{\alpha}}(t-s, 0)}}(W(s)) ds \mid W(t) = y \right] dy \\ &\geq \int_{B_1} \varphi(y) p(K(t, 0), x - y) \\ &\quad \cdot \exp \int_\theta^{\delta_0 t} c_0^\beta K^{-\frac{d\beta}{\alpha}}(t - s, 0) P_x \left[W(s) \in B_{K^{\frac{1}{\alpha}}(t-s, 0)} \mid W(t) = y \right] ds dy. \end{aligned}$$

It follows from Lemma 3 and Lemma 5 that

$$\begin{aligned} v(t, x) &\geq \int_{B_1} \varphi(y) p(K(t, 0), x - y) \exp \int_\theta^{\delta_0 t} c_6 K^{-\frac{d\beta}{\alpha}}(t - s, 0) ds dy \\ &= \int_{B_1} \varphi(y) K^{-\frac{d}{\alpha}}(t, 0) p\left(1, K^{-\frac{1}{\alpha}}(t, 0)(x - y)\right) dy \exp \int_\theta^{\delta_0 t} c_6 K^{-\frac{d\beta}{\alpha}}(t - s, 0) ds. \end{aligned}$$

Let $x, y \in B_1$. Then $K^{-\frac{1}{\alpha}}(t, 0)(x - y) \in B_2$. Radial symmetry of $p(t, \cdot)$ implies

$$p\left(1, K^{-\frac{1}{\alpha}}(t, 0)(x - y)\right) \geq p(1, \varsigma) \equiv c_7$$

for any $\varsigma \in \partial B_2$. Therefore

$$v(t, x) \geq \int_{B_1} c_7 \varphi(y) K^{-\frac{d}{\alpha}}(t, 0) dy \exp \int_\theta^{\delta_0 t} c_6 K^{-\frac{d\beta}{\alpha}}(t - s, 0) ds. \quad (4.14)$$

Let $c_8 = c_7 \int_{B_1} \varphi(y) dy$. Using (1.2) and the fact that $K(t, 0) \geq K(t - s, 0)$, the term in the right of (4.14) is bounded below by

$$c_8 K^{-\frac{d}{\alpha}}(t, 0) \exp\left(c_6 \int_\theta^{\delta_0 t} K^{-\frac{d\beta}{\alpha}}(t, 0) ds\right) \geq c_8 \varepsilon_2^{-\frac{d}{\alpha}} t^{-\frac{d\rho}{\alpha}} \exp\left[c_6 \varepsilon_2^{-\frac{d\beta}{\alpha}} \left(\delta_0 t^{1-\frac{d\beta\rho}{\alpha}} - \theta t^{-\frac{d\beta\rho}{\alpha}}\right)\right]$$

if $t > 0$ is large. It follows that

$$v(t, x) \geq c' t^{-\frac{d\rho}{\alpha}} \exp\left(c'' t^{1-\frac{d\beta\rho}{\alpha}}\right) \quad (4.15)$$

for all sufficiently large t , where $c' = c_8 \varepsilon_2^{-\frac{d}{\alpha}} \exp\left(-c_6 \theta \varepsilon_2^{-\frac{d\beta}{\alpha}}\right)$ and $c'' = c_6 \delta_0 \varepsilon_2^{-\frac{d\beta}{\alpha}}$. \blacksquare

As a consequence of Proposition 6, if $0 < \frac{d\beta\rho}{\alpha} < 1$, then $\inf_{x \in B_1} v(t, x) \rightarrow \infty$ when $t \rightarrow \infty$. As v is subsolution of Equation (1.1), this implies that

$$C(t) := \inf_{x \in B_1} u(t, x) \rightarrow \infty \text{ when } t \rightarrow \infty. \quad (4.16)$$

Now we are ready to prove that (4.16) is enough to guarantee finite-time blow up of (1.1).

Theorem 7 *If $0 < \frac{d\rho\beta}{\alpha} < 1$, then all nontrivial positive solutions of (1.1) are nonglobal.*

Proof. Let u be the solution of (1.1), and let $t_0 > 0$ be such that $\|u(t_0, \cdot)\|_\infty < \infty$. Then

$$\begin{aligned} u(t + t_0, x) &= \int_{\mathbb{R}^d} p(K(t + t_0, t_0), y - x) u(t_0, y) dy \\ &\quad + \int_0^t \int_{\mathbb{R}^d} p(K(t + t_0, s + t_0), y - x) u^{1+\beta}(s + t_0, y) dy ds \\ &\geq \int_{B_1} p(K(t + t_0, t_0), y - x) u(t_0, y) dy \\ &\quad + \int_0^t \int_{B_1} p(K(t + t_0, s + t_0), y - x) u^{1+\beta}(s + t_0, y) dy ds. \end{aligned}$$

Therefore $w(t, \cdot) := u(t_0 + t, \cdot)$ satisfies

$$\begin{aligned} w(t, x) &\geq C(t_0) \int_{B_1} p(K(t + t_0, t_0), y - x) dy \\ &\quad + \int_0^t \int_{B_1} p(K(t + t_0, s + t_0), y - x) \left(\min_{z \in B_1} w(s, z) \right)^{1+\beta} dy ds. \end{aligned}$$

Using that $ct^{-\frac{d}{\alpha}} \leq p(t, x)$ for all $t > 0$ and all $x \in B_{\frac{1}{t^\alpha}}$, where $c > 0$ is a suitable constant, it is easy to see that

$$\xi := \min_{x \in B_1} \min_{0 \leq r \leq K(t_0+1, 0)} P_x(W(r) \in B_1) > 0.$$

It follows that for all $t \in [0, 1]$,

$$\min_{x \in B_1} w(t, x) \geq \xi C(t_0) + \xi \int_0^t \left(\min_{z \in B_1} w(s, z) \right)^{1+\beta} ds.$$

We put $w(t) \equiv \min_{z \in B_1} w(t, z)$, $t \geq 0$, and consider the integral equation

$$v(t) = \xi C(t_0) + \xi \int_0^t v^{1+\beta}(s) ds,$$

whose solution satisfies

$$v^\beta(t) = \frac{[\xi C(t_0)]^\beta}{1 - \beta \xi^{1+\beta} C^\beta(t_0) t}. \quad (4.17)$$

Choosing t_0 so big that the blow up time of v is smaller than one, renders

$$w(1) = \min_{x \in B_1} w(1, x) \geq v(1) = \infty,$$

wich proves blow up of u . ■

5 Upper estimates of the life span

In this section we obtain two upper bounds for the life span of Equation (1.1) with initial value $u(0, \cdot) = \lambda \varphi(\cdot)$, where λ is a positive parameter. We first consider the case of small $\lambda > 0$.

Proposition 8 *If $0 < \frac{d\rho\beta}{\alpha} \leq \frac{n}{n+1}$, $n \in \mathbb{N}$, then there exists a constant $C_n > 0$ such that for all sufficiently small $\lambda > 0$,*

$$T_{\lambda\varphi} \leq C_n \lambda^{-\frac{\alpha\beta}{\alpha-d\beta\rho}}.$$

Proof. From (4.15) and (4.16) it follows that

$$C(t) \geq \lambda c' t^{-\frac{d\rho}{\alpha}} \exp\left(c'' t^{1-\frac{d\beta\rho}{\alpha}}\right)$$

for all $t \geq \frac{\theta}{\delta_0}$. Recall from (4.17) that $v(1) = \infty$ provided $\beta \xi^{1+\beta} C^\beta(t_0) = 1$, that is, when

$$\beta \xi^{1+\beta} \lambda^\beta (c')^\beta t_0^{-\frac{d\beta\rho}{\alpha}} \exp\left(\beta c'' t_0^{1-\frac{d\beta\rho}{\alpha}}\right) = 1.$$

Choosing $\theta > 0$ such that in addition to conditions required in Lemma 5 satisfies that $\frac{\theta}{\delta_0} \geq 1$, then from the inequality $e^x \geq \frac{x^{n+1}}{(n+1)!}$ and the fact that the condition $0 < \frac{d\rho\beta}{\alpha} \leq \frac{n}{n+1}$ implies $\frac{d\rho\beta}{n\alpha} \leq 1 - \frac{d\rho\beta}{\alpha}$, we have that $t_0 \leq t_1$, where t_1 is such that

$$\frac{1}{(n+1)!} \beta^{n+2} \xi^{1+\beta} (c')^\beta (c'')^{n+1} \lambda^\beta t_1^{1-\frac{d\beta\rho}{\alpha}} = 1,$$

which is the same as

$$t_1 = \left[\frac{(n+1)!}{\beta^{n+2} \xi^{1+\beta} (c')^\beta (c'')^{n+1}} \right]^{\frac{\alpha}{\alpha-d\beta\rho}} \lambda^{-\frac{\alpha\beta}{\alpha-d\beta\rho}}.$$

Choosing

$$C_n = \left[\frac{(n+1)!}{\beta^{n+2} \xi^{1+\beta} (c')^\beta (c'')^{n+1}} \right]^{\frac{\alpha}{\alpha-d\beta\rho}}$$

renders $t_0 \leq t_1 = C_n \lambda^{-\frac{\alpha\beta}{\alpha-d\beta\rho}}$. Hence $T_{\lambda\varphi} \leq C_n \lambda^{-\frac{\alpha\beta}{\alpha-d\beta\rho}}$ for all sufficiently small $\lambda > 0$. ■

Let us define

$$v(t) = \int_{\mathbb{R}^d} p(K(t, 0), x) u(t, x) dx,$$

where u is the solution of (1.1), and let $\theta > 0$ be such that (1.2) holds for all $t \geq \theta$.

Lemma 9 *If there exist $\tau_0 \geq \theta$ such that $v(t) = \infty$ for $t \geq \tau_0$, then the solution to (1.1) blows up in finite time.*

Proof. Due to (1.2) and Lemma 3 i), we can assume that $p(K(t, 0), 0) \leq 1$ for all $t \geq \tau_0$.

If $\tau_0 \leq \varepsilon_1^{\frac{1}{\rho}} t$ and $\varepsilon_1^{\frac{1}{\rho}} t \leq r \leq (2\varepsilon_1)^{\frac{1}{\rho}} t$, we have, from the conditions on $k(t)$, that

$$\begin{aligned} \tau &\equiv \left[\frac{K\left((10\varepsilon_2)^{\frac{1}{\rho}} t, r\right)}{K(r, 0)} \right]^{\frac{1}{\alpha}} = \left[\frac{K\left((10\varepsilon_2)^{\frac{1}{\rho}} t, 0\right) - K(r, 0)}{K(r, 0)} \right]^{\frac{1}{\alpha}} \geq \left[\frac{K\left((10\varepsilon_2)^{\frac{1}{\rho}} t, 0\right)}{K\left((2\varepsilon_1)^{\frac{1}{\rho}} t, 0\right)} - 1 \right]^{\frac{1}{\alpha}} \\ &\geq \left[\frac{\varepsilon_1 (10\varepsilon_2) t^\rho}{\varepsilon_2 (2\varepsilon_1) t^\rho} - 1 \right]^{\frac{1}{\alpha}} = 4^{\frac{1}{\alpha}} \geq 2. \end{aligned}$$

Using properties i) and iv) in Lemma 3, with $\tau = \left[\frac{K\left((10\varepsilon_2)^{\frac{1}{\rho}} t, r\right)}{K(r, 0)} \right]^{\frac{1}{\alpha}}$, yields

$$\begin{aligned} &p\left(K\left((10\varepsilon_2)^{\frac{1}{\rho}} t, r\right), x - y\right) \\ &= p\left(K(r, 0) \left[\frac{K\left((10\varepsilon_2)^{\frac{1}{\rho}} t, r\right)}{K(r, 0)} \right], x - y\right) \\ &= \left[\frac{K(r, 0)}{K\left((10\varepsilon_2)^{\frac{1}{\rho}} t, r\right)} \right]^{\frac{d}{\alpha}} p\left(K(r, 0), \left[\frac{K(r, 0)}{K\left((10\varepsilon_2)^{\frac{1}{\rho}} t, r\right)} \right]^{\frac{1}{\alpha}} (x - y)\right) \\ &\geq \left[\frac{K(r, 0)}{K\left((10\varepsilon_2)^{\frac{1}{\rho}} t, r\right)} \right]^{\frac{d}{\alpha}} p(K(r, 0), x) p(K(r, 0), y). \end{aligned}$$

Since $v(t) = \infty$ for all $t \geq \tau_0$, it follows that

$$\int_{\mathbb{R}^d} p\left(K\left((10\varepsilon_2)^{\frac{1}{\rho}} t, r\right), x - y\right) u(r, y) dy \geq \left[\frac{K(r, 0)}{K\left((10\varepsilon_2)^{\frac{1}{\rho}} t, r\right)} \right]^{\frac{d}{\alpha}} p(K(r, 0), x) v(r) = \infty.$$

The solution $u(t, x)$ of (1.1) satisfies

$$\begin{aligned} u(t, x) &= \lambda \int_{\mathbb{R}^d} p(K(t, 0), x - y) \varphi(y) dy + \int_0^t \left(\int_{\mathbb{R}^d} p(K(t, r), x - y) u^{1+\beta}(r, y) dy \right) dr \\ &\geq \int_0^t \left(\int_{\mathbb{R}^d} p(K(t, r), x - y) u^{1+\beta}(r, y) dy \right) dr. \end{aligned}$$

Thus,

$$u\left((10\varepsilon_2)^{\frac{1}{\rho}}t, x\right) \geq \int_0^{(10\varepsilon_2)^{\frac{1}{\rho}}t} \left(\int_{\mathbb{R}^d} p\left(K\left((10\varepsilon_2)^{\frac{1}{\rho}}t, r\right), x-y\right) u^{1+\beta}(r, y) dy \right) dr.$$

Jensen's inequality renders

$$u\left((10\varepsilon_2)^{\frac{1}{\rho}}t, x\right) \geq \int_{\varepsilon_1^{\frac{1}{\rho}}t}^{(2\varepsilon_1)^{\frac{1}{\rho}}t} \left(\int_{\mathbb{R}^d} p\left(K\left((10\varepsilon_2)^{\frac{1}{\rho}}t, r\right), x-y\right) u(r, y) dy \right)^{1+\beta} dr = \infty,$$

so that $u(t, x) = \infty$ for any $t \geq \left(10\frac{\varepsilon_2}{\varepsilon_1}\right)^{\frac{1}{\rho}}\tau_0$ and $x \in \mathbb{R}^d$. ■

Proposition 10 *Let $0 < \frac{d\rho\beta}{\alpha} < 1$. There exists a constant $C > 0$ depending on $\alpha, \beta, d, \varepsilon_1, \varepsilon_2, \theta, \rho$ and φ , such that*

$$T_{\lambda\varphi} \leq \left\{ C\lambda^{-\beta} + \left[\left(10\frac{\varepsilon_2}{\varepsilon_1}\right)^{\frac{1}{\rho}}\theta \right]^{\frac{\alpha-d\rho\beta}{\alpha}} \right\}^{\frac{\alpha}{\alpha-d\rho\beta}} + \eta, \quad \lambda > 0, \quad (5.18)$$

where η is any positive real number satisfying $p(K(\eta, 0), 0) \leq 1$.

Proof. From Lemma 3 we obtain

$$\begin{aligned} p(K(\eta, 0), x-y) &= p\left(K(\eta, 0), \frac{1}{2}(2x-2y)\right) \geq p(K(\eta, 0), 2x)p(K(\eta, 0), 2y) \\ &= 2^{-d}p(2^{-\alpha}K(\eta, 0), x)p(K(\eta, 0), 2y). \end{aligned}$$

Therefore

$$\begin{aligned} u(\eta, x) &\geq \lambda \int_{\mathbb{R}^d} p(K(\eta, 0), x-y)\varphi(y) dy \\ &\geq 2^{-d}\lambda p(2^{-\alpha}K(\eta, 0), x) \int_{\mathbb{R}^d} p(K(\eta, 0), 2y)\varphi(y) dy \\ &= \lambda N_0 p(2^{-\alpha}K(\eta, 0), x), \end{aligned}$$

where $N_0 = 2^{-d} \int_{\mathbb{R}^d} p(K(\eta, 0), 2y)\varphi(y) dy$. Thus, for any $\lambda > 0, t \geq 0$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} u(t+\eta, x) &= \int_{\mathbb{R}^d} p(K(t+\eta, \eta), x-y)u(\eta, y) dy \\ &\quad + \int_0^t \left(\int_{\mathbb{R}^d} p(K(t+\eta, r+\eta), x-y)u^{1+\beta}(r+\eta, y) dy \right) dr \\ &\geq \lambda N_0 \int_{\mathbb{R}^d} p(K(t+\eta, \eta), x-y)p(2^{-\alpha}K(\eta, 0), y) dy \\ &\quad + \int_0^t \left(\int_{\mathbb{R}^d} p(K(t+\eta, r+\eta), x-y)u^{1+\beta}(r+\eta, y) dy \right) dr \\ &\geq \lambda N_0 p(K(t+\eta, \eta) + 2^{-\alpha}K(\eta, 0), x) \\ &\quad + \int_0^t \left(\int_{\mathbb{R}^d} p(K(t+\eta, r+\eta), x-y)u^{1+\beta}(r+\eta, y) dy \right) dr \\ &\geq w(t, x), \end{aligned}$$

where w solves the equation

$$\begin{aligned} w(t, x) &= \lambda N_0 p(K(t + \eta, \eta) + 2^{-\alpha} K(\eta, 0), x) \\ &\quad + \int_0^t \left(\int_{\mathbb{R}^d} p(K(t + \eta, r + \eta), x - y) w^{1+\beta}(r, y) dy \right) dr, \quad t \geq 0, x \in \mathbb{R}^d. \end{aligned} \quad (5.19)$$

Hence, it is enough to prove that w is non-global, and, because of Lemma 9, it suffices to show finite time blowup of

$$v(t) = \int_{\mathbb{R}^d} p(K(t, 0), x) w(t, x) dx, \quad t \geq 0.$$

Multiplying both sides of (5.19) by $p(K(t, 0), x)$ and integrating, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^d} p(K(t, 0), x) w(t, x) dx \\ &= \lambda N_0 \int_{\mathbb{R}^d} p(K(t + \eta, \eta) + 2^{-\alpha} K(\eta, 0), x) p(K(t, 0), x) dx \\ &\quad + \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} p(K(t + \eta, r + \eta), x - y) p(K(t, 0), x) w^{1+\beta}(r, y) dy dr dx \\ &= \lambda N_0 p(K(t, 0) + K(t + \eta, \eta) + 2^{-\alpha} K(\eta, 0), 0) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} p(K(t + \eta, r + \eta) + K(t, 0), y) w^{1+\beta}(r, y) dy dr, \quad t \geq 0. \end{aligned}$$

Therefore

$$\begin{aligned} v(t) &= \lambda N_0 p(K(t, 0) + K(t + \eta, \eta) + 2^{-\alpha} K(\eta, 0), 0) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} p(K(t + \eta, r + \eta) + K(t, 0), y) w^{1+\beta}(r, y) dy dr. \end{aligned}$$

From Lemma 3 i), we have $p(t, 0) \leq p(s, 0)$ for all $0 < s \leq t$. Hence

$$\begin{aligned} v(t) &\geq \lambda N_0 p(2K(t + \eta, 0) + 2^{-\alpha} K(\eta, 0), 0) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} p(K(t + \eta, r + \eta) + K(t, 0), y) w^{1+\beta}(r, y) dy dr. \end{aligned}$$

Using now Lemma 3 iii) we have,

$$\begin{aligned} v(t) &\geq \lambda N_0 p(2K(t + \eta, 0) + 2^{-\alpha} K(\eta, 0), 0) \\ &\quad + \int_0^t \left(\frac{K(r, 0)}{K(t + \eta, r + \eta) + K(t, 0)} \right)^{\frac{d}{\alpha}} \int_{\mathbb{R}^d} p(K(r, 0), y) w^{1+\beta}(r, y) dy dr. \end{aligned}$$

Jensen's inequality together with Lemma 3 i) gives

$$\begin{aligned} v(t) &\geq \lambda N_0 p(2K(t + \eta, 0) + 2^{-\alpha} K(\eta, 0), 0) + \int_0^t \left(\frac{K(r, 0)}{2K(t + \eta, 0)} \right)^{\frac{d}{\alpha}} v^{1+\beta}(r) dr \\ &= \lambda N_0 [2K(t + \eta, 0) + 2^{-\alpha} K(\eta, 0)]^{-\frac{d}{\alpha}} p(1, 0) + \int_0^t \left(\frac{K(r, 0)}{2K(t + \eta, 0)} \right)^{\frac{d}{\alpha}} v^{1+\beta}(r) dr. \end{aligned}$$

Let $f_1(t) = K^{\frac{d}{\alpha}}(t + \eta, 0)v(t)$ and $t \geq \theta$. We have

$$f_1(t) \geq \lambda p(1, 0) N_0 \left[\frac{K(\theta + \eta, 0)}{2K(\theta + \eta, 0) + 2^{-\alpha}K(\eta, 0)} \right]^{\frac{d}{\alpha}} + 2^{-\frac{d}{\alpha}} \int_{\theta}^t K^{-\frac{d\beta}{\alpha}}(r, 0) f_1^{1+\beta}(r) dr,$$

and if $N := p(1, 0) N_0 \left[\frac{K(\theta + \eta, 0)}{2K(\theta + \eta, 0) + 2^{-\alpha}K(\eta, 0)} \right]^{\frac{d}{\alpha}}$, then

$$f_1(t) \geq \lambda N + 2^{-\frac{d}{\alpha}} \int_{\theta}^t K^{-\frac{d\beta}{\alpha}}(r, 0) f_1^{1+\beta}(r) dr, \quad t \geq \theta.$$

Let f_2 be the solution of the integral equation

$$f_2(t) = \lambda N + 2^{-\frac{d}{\alpha}} \int_{\theta}^t K^{-\frac{d\beta}{\alpha}}(r, 0) f_2^{1+\beta}(r) dr, \quad t \geq \theta,$$

which satisfies

$$f_2^\beta(t) = \frac{(\lambda N)^\beta}{1 - \beta(\lambda N)^\beta \left(\frac{1}{2}\right)^{\frac{d}{\alpha}} H(t)} \quad (5.20)$$

with

$$H(t) \equiv \int_{\theta}^t K^{-\frac{d\beta}{\alpha}}(r, 0) dr.$$

From (1.2) and the assumption $0 < \frac{d\rho\beta}{\alpha} < 1$, we get

$$H(t) \geq \varepsilon_2^{-\frac{d\beta}{\alpha}} \int_{\theta}^t r^{-\frac{d\rho\beta}{\alpha}} dr = \frac{\alpha}{\alpha - d\rho\beta} \varepsilon_2^{-\frac{d\beta}{\alpha}} \left[t^{\frac{\alpha - d\rho\beta}{\alpha}} - \theta^{\frac{\alpha - d\rho\beta}{\alpha}} \right] \rightarrow \infty \quad \text{as } t \rightarrow \infty.$$

Hence, there exist $\tau_0 \geq \theta$ such that $\beta \left(\frac{1}{2}\right)^{\frac{d}{\alpha}} (\lambda N)^\beta H(\tau_0) = 1$, and therefore, $\int_{\theta}^{\tau_0} K^{-\frac{d\beta}{\alpha}}(r, 0) dr = \frac{2^{\frac{d}{\alpha}}}{\beta} N^{-\beta} \lambda^{-\beta}$, which together with (1.2) gives $\int_{\theta}^{\tau_0} (\varepsilon_2 r^\rho)^{-\frac{d\beta}{\alpha}} dr \leq \frac{2^{\frac{d}{\alpha}}}{\beta} N^{-\beta} \lambda^{-\beta}$. Hence

$$\tau_0^{\frac{\alpha - d\rho\beta}{\alpha}} \leq \frac{2^{\frac{d}{\alpha}} [\alpha - d\rho\beta]}{\alpha\beta} N^{-\beta} \varepsilon_2^{\frac{d\beta}{\alpha}} \lambda^{-\beta} + \theta^{\frac{\alpha - d\rho\beta}{\alpha}},$$

or, equivalently,

$$\tau_0 \leq \left\{ \frac{2^{\frac{d}{\alpha}} [\alpha - d\rho\beta]}{\alpha\beta} N^{-\beta} \varepsilon_2^{\frac{d\beta}{\alpha}} \lambda^{-\beta} + \theta^{\frac{\alpha - d\rho\beta}{\alpha}} \right\}^{\frac{\alpha}{\alpha - d\rho\beta}}. \quad (5.21)$$

From (5.20), we deduce that $f_2(\tau_0) = \infty$. It follows that

$$K^{\frac{d}{\alpha}}(\tau_0 + \eta, 0)v(\tau_0) = f_1(\tau_0) \geq f_2(\tau_0) = \infty,$$

which implies (as in the proof of Lemma 9) that $w(t, x) = \infty$ if $t \geq \left(10 \frac{\varepsilon_2}{\varepsilon_1}\right)^{\frac{1}{\rho}} \tau_0$, and thus, $u(t, x) = \infty$ provided $t \geq \left(10 \frac{\varepsilon_2}{\varepsilon_1}\right)^{\frac{1}{\rho}} \tau_0 + \eta$. Therefore

$$T_{\lambda\varphi} \leq \left(10 \frac{\varepsilon_2}{\varepsilon_1}\right)^{\frac{1}{\rho}} \tau_0 + \eta.$$

We conclude from (5.21) that there exist a positive constant $C = C(\alpha, \beta, d, \varepsilon_1, \varepsilon_2, \theta, \rho, \varphi)$ satisfying (5.18). \blacksquare

6 A lower estimate for the life span

In order to bound from below the life span $T_{\lambda\varphi}$ of the initial value problem (1.1), we need to assume that (1.2) holds for any $t \geq 0$, and that φ is integrable.

Let $\{U(t, s)\}_{t \geq s \geq 0}$ be the evolution family on $C_b(\mathbb{R}^d)$ generated by the family of operators $\{k(t) \Delta_\alpha\}_{t \geq 0}$, which is given by

$$U(t, s) \varphi(x) = \int_{\mathbb{R}^d} \varphi(y) p(K(t, s), x - y) dy = S(K(t, s)) \varphi(x),$$

where $\{S(t)\}_{t \geq 0}$ is the semigroup with infinitesimal generator Δ_α .

Proposition 11 *Let $0 < \frac{d\rho\beta}{\alpha} < 1$. There exists a constant $c > 0$, depending on $\alpha, \beta, d, \varepsilon_1, \rho$ and φ , such that*

$$T_{\lambda\varphi} \geq c\lambda^{-\frac{\alpha\beta}{\alpha-d\rho\beta}}, \quad \lambda > 0. \quad (6.22)$$

Proof. The function

$$\bar{u}(t, x) := \left[\lambda^{-\beta} - \beta \int_0^t \|U(r, 0) \varphi\|_\infty^\beta dr \right]^{-\frac{1}{\beta}} U(t, 0) \varphi(x), \quad t \geq 0, \quad x \in \mathbb{R}^d.$$

is a supersolution of (1.1). In fact $\bar{u}(0, \cdot) = \lambda\varphi(\cdot)$, and

$$\begin{aligned} \frac{\partial \bar{u}(t, x)}{\partial t} &= -\frac{1}{\beta} \left[\lambda^{-\beta} - \beta \int_0^t \|U(r, 0) \varphi\|_\infty^\beta dr \right]^{-\frac{1}{\beta}-1} \left[-\beta \|U(t, 0) \varphi\|_\infty^\beta \right] U(t, 0) \varphi(x) \\ &\quad + \left[\lambda^{-\beta} - \beta \int_0^t \|U(r, 0) \varphi\|_\infty^\beta dr \right]^{-\frac{1}{\beta}} k(t) \Delta_\alpha U(t, 0) \varphi(x). \end{aligned}$$

Since $-\frac{1}{\beta} - 1 = -\frac{\beta+1}{\beta}$, we get

$$\begin{aligned} \frac{\partial \bar{u}(t, x)}{\partial t} &= \left\{ \left[\lambda^{-\beta} - \beta \int_0^t \|U(r, 0) \varphi\|_\infty^\beta dr \right]^{-\frac{1}{\beta}} \right\}^{\beta+1} \|U(t, 0) \varphi\|_\infty^\beta U(t, 0) \varphi(x) \\ &\quad + k(t) \Delta_\alpha \left[\lambda^{-\beta} - \beta \int_0^t \|U(r, 0) \varphi\|_\infty^\beta dr \right]^{-\frac{1}{\beta}} U(t, 0) \varphi(x). \end{aligned}$$

Using the inequality

$$\|U(t, 0) \varphi\|_\infty^\beta U(t, 0) \varphi(x) \geq [U(t, 0) \varphi(x)]^{1+\beta}$$

it follows that

$$\frac{\partial \bar{u}(t, x)}{\partial t} \geq k(t) \Delta_\alpha \bar{u}(t, x) + \bar{u}^{1+\beta}(t, x),$$

showing that \bar{u} is a supersolution of (1.1). Writing $\bar{L}(\lambda)$ for the life span of \bar{u} , it follows that

$$\bar{L}(\lambda) \leq T_{\lambda\varphi}, \quad \lambda \geq 0.$$

Now,

$$\bar{u}(t, x) = \left[\lambda^{-\beta} - \beta \int_0^t \|U(r, 0) \varphi\|_\infty^\beta dr \right]^{-\frac{1}{\beta}} U(t, 0) \varphi(x) = \infty$$

when $\lambda^{-\beta} = \beta \int_0^t \|U(r, 0) \varphi\|_\infty^\beta dr$. By definition of $\bar{L}(\lambda)$,

$$\beta^{-1} \lambda^{-\beta} = \int_0^{\bar{L}(\lambda)} \|U(r, 0) \varphi\|_\infty^\beta dr. \quad (6.23)$$

Notice that, by Lemma 3 i), ii),

$$\begin{aligned} U(t, 0) \varphi(x) &= S(K(t, 0)) \varphi(x) = \int_{\mathbb{R}^d} \varphi(y) p(K(t, 0), x - y) dy \\ &\leq p(1, 0) K^{-\frac{d}{\alpha}}(t, 0) \|\varphi\|_1, \quad t > 0, \quad x \in \mathbb{R}^d. \end{aligned}$$

Since, by assumption, (1.2) holds for any $t \geq 0$, we obtain

$$\|U(t, 0) \varphi\|_\infty \leq p(1, 0) (\varepsilon_1 t^\rho)^{-\frac{d}{\alpha}} \|\varphi\|_1.$$

Inserting this inequality in (6.23) and using that $0 < \frac{d\rho\beta}{\alpha} < 1$, we get

$$\begin{aligned} \beta^{-1} \lambda^{-\beta} &\leq (p(1, 0) \|\varphi\|_1)^\beta \varepsilon_1^{-\frac{d\beta}{\alpha}} \int_0^{\bar{L}(\lambda)} r^{-\frac{d\rho\beta}{\alpha}} dr \\ &= \frac{\alpha}{\alpha - d\rho\beta} (p(1, 0) \|\varphi\|_1)^\beta \varepsilon_1^{-\frac{d\beta}{\alpha}} \bar{L}(\lambda)^{\frac{\alpha - d\rho\beta}{\alpha}}, \end{aligned}$$

which gives

$$\bar{L}(\lambda)^{\frac{\alpha - d\rho\beta}{\alpha}} \geq \frac{\alpha - d\rho\beta}{\alpha\beta} (p(1, 0) \|\varphi\|_1)^{-\beta} \varepsilon_1^{\frac{d\beta}{\alpha}} \lambda^{-\beta}.$$

In this way we obtain the inequality

$$T_{\lambda\varphi} \geq \left[\frac{\alpha - d\rho\beta}{\alpha\beta} \right]^{\frac{\alpha}{\alpha - d\rho\beta}} (p(1, 0) \|\varphi\|_1)^{-\frac{\alpha\beta}{\alpha - d\rho\beta}} \varepsilon_1^{\frac{d\beta}{\alpha - d\rho\beta}} \lambda^{-\frac{\alpha\beta}{\alpha - d\rho\beta}},$$

which proves the existence of a constant $c \equiv c(\alpha, \beta, d, \varepsilon_1, \rho, \varphi) > 0$ that satisfies (6.22). ■

Summarizing both, upper and lower bounds for the life span of (1.1), we get the following

Theorem 12 *Let $0 < \frac{d\rho\beta}{\alpha} < 1$, and let $T_{\lambda\varphi}$ be the life span of the nonautonomous semi-linear equation*

$$\begin{aligned} \frac{\partial u(t, x)}{\partial t} &= k(t) \Delta_\alpha u(t, x) + u^{1+\beta}(t, x) \\ u(0, x) &= \lambda \varphi(x) \geq 0, \quad x \in \mathbb{R}^d, \end{aligned}$$

where $\lambda > 0$. Then

$$\lim_{\lambda \rightarrow 0} T_{\lambda\varphi} = \infty \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} T_{\lambda\varphi} \in \left[0, \left(10 \frac{\varepsilon_2}{\varepsilon_1} \right)^{\frac{1}{\rho}} \theta + \eta \right], \quad (6.24)$$

where θ and η are any positive numbers such that $\varepsilon_1 \theta^\rho \leq K(\theta, 0) \leq \varepsilon_2 \theta^\rho$ and $p(K(\eta, 0), 0) \leq 1$, respectively.

Proof. Due to (5.18) and (6.22),

$$c\lambda^{-\frac{\alpha\beta}{\alpha-d\rho\beta}} \leq T_{\lambda\varphi} \leq \left\{ C\lambda^{-\beta} + \left[\left(10 \frac{\varepsilon_2}{\varepsilon_1} \right)^{\frac{1}{\rho}} \theta \right]^{\frac{\alpha-d\rho\beta}{\alpha}} \right\}^{\frac{\alpha}{\alpha-d\rho\beta}} + \eta,$$

from which (6.24) follows directly using the fact that $0 < \frac{d\rho\beta}{\alpha} < 1$. ■

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