# SOME PROPERTIES OF INVARIANT POLYNOMIALS WITH MATRIX ARGUMENTS

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# Some properties of invariant polynomials with matrix arguments

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**Abstract**

Some extensions of the properties for invariant polynomials given by Davis (1980), Chikuse (1980), Chikuse and Davis (1986) and Ratnarajah *et al.* (2005) are proposed for symmetric and Hermitian matrices.

*Key words:* Zonal polynomials, invariant polynomials with several matrix arguments, Haar measure, unitary group, Hermitian matrices.
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## 1 Introduction

A number of integrals in distribution theory of random matrices are expanded in terms of zonal polynomials, see Constantine (1963) and Díaz-García and Gutiérrez-Jáimez (2001) among many others. However, important distributional problems can not be solved via zonal polynomials; such as, the distribution of the eigenvalues of a noncentral Wishart distribution or the doubly noncentral Beta type I and II distributions. Solutions of these kinds of problems have been provided via invariant polynomials with matrix arguments, see for example Davis (1980), Chikuse (1980), Chikuse (1981), Chikuse and Davis (1986), Díaz-García and Gutiérrez-Jáimez (2001) and Ratnarajah *et al.* (2005), among many others.

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The fundamental property of the real invariant polynomials is the following:

$$\int_{\mathcal{O}(m)} C_\kappa(AH'XH)C_\lambda(BH'YH)(dH) = \sum_{\phi\in\kappa\cdot\lambda} \frac{C_\phi^{\kappa,\lambda}(A,B)C_\phi^{\kappa,\lambda}(X,Y)}{C_\phi(I)}, \qquad (1)$$

see Davis (1980) or its generalization for $r$ matrices in Chikuse (1980, eq (2.2)) and Chikuse and Davis (1986, eq.(2.2)), where $A$, $B$, $X$ and $Y$ are $m \times m$ symmetric matrices, $(dH)$ is the invariant Haar measure over the group $\mathcal{O}(m)$ of $m\times m$ orthogonal matrices, $C_\phi^{\kappa,\lambda}$ is the invariant polynomial and $C_\phi$ denotes the zonal polynomial, see Davis (1980), Chikuse (1980) and Chikuse and Davis (1986).

An analogous property for invariant polynomials of Hermitian matrix arguments can be found in Ratnarajah *et al.* (2005, eq. (7)).

Many other characteristics of the invariant polynomials are a consequence of the following basic property

$$\int_{\mathcal{O}(m)} C_\phi^{\kappa,\lambda}(AH'XH, AH'YH)(dH) = \frac{C_\phi(A)C_\phi^{\kappa,\lambda}(X,Y)}{C_\phi(I)}, \qquad (2)$$

see Davis (1980), Chikuse and Davis (1986) (which corrects Chikuse (1980)) for the real case, and Ratnarajah *et al.* (2005) for the complex case.

In the present work we generalize the property (1) when the product of the zonal polynomials is replaced by the product of invariant polynomials. To obtain this fundamental property we need to generalize (2) by placing a new matrix $B \neq A$ in the second argument of the invariant polynomial. We emphasize that the generalization of (2) provides alternative derivations for most of the invariant polynomial properties given independently by Davis (1980), Chikuse and Davis (1986) and Chikuse (1980). The same generalizations and alternatives apply to the complex invariant polynomial expressions of Ratnarajah *et al.* (2005).

## 2 New properties of invariant polynomials.

The following theorem generalizes eq. (5.4) of Davis (1980).

**Theorem 1** *Let $A$, $B$, $X$ and $Y$ be $m\times m$ symmetric matrices, then we have*

$$\int_{\mathcal{O}(m)} C_\phi^{\kappa,\lambda}(AH'XH, BH'YH)(dH) = \frac{C_\phi^{\kappa,\lambda}(A,B)C_\phi^{\kappa,\lambda}(X,Y)}{\theta_\phi^{\kappa,\lambda}C_\phi(I)}, \qquad (3)$$

*with*

$$\theta_\phi^{\kappa,\lambda} = \frac{C_\phi^{\kappa,\lambda}(I,I)}{C_\phi(I)}.$$

*Proof.* Different approaches can be implemented to prove this result, for instance: we can assume that there exists a differential operator for the invariant polynomials, like the operator given in equation (3.41) of Chikuse (1980), and then we can generalize the proof of Theorem 7.2.5 in Muirhead (1982); we can proceed in a similar way as in the proof of Theorem 1, pp. 27-28 of Takemura (1984); or, we can follow the ideas of James (1961) and James (1960).

By extending the procedures of James and by using the results of Davis (1980), we find the following straightforward demonstration of the theorem:

From Muirhead (1982, eq. (3), p. 259) we have

$$\int_{\mathcal{O}(m)} \operatorname{etr}(AH'XH + BH'YH)(dH) =$$
$$= \sum_k^\infty \sum_\kappa \sum_l^\infty \sum_\lambda \frac{1}{k!\ l!} \int_{\mathcal{O}(m)} C_\kappa(AH'XH)C_\lambda(BH'YH)(dH), \qquad (4)$$

then from Davis (1980, eq. (5.8)) and by using the notation for the summations given in Davis (1980), we get

$$\int_{\mathcal{O}(m)} \operatorname{etr}(AH'XH + BH'YH)(dH)$$
$$= \sum_{\kappa,\lambda;\phi}^\infty \frac{\theta_\phi^{\kappa,\lambda}}{k!\ l!} \int_{\mathcal{O}(m)} C_\phi^{\kappa,\lambda}(AH'XH, BY'YH)(dH),$$

but from Davis (1980, eq. (5.12))

$$\int_{\mathcal{O}(m)} \operatorname{etr}(AH'Q'XQH + BH'Q'YQH)(dH) = \sum_{\kappa,\lambda;\phi}^\infty \frac{C_\phi^{\kappa,\lambda}(A,B)C_\phi^{\kappa,\lambda}(X,Y)}{k!\ l! C_\phi(I_m)},$$

then, the required result is obtained. ■

**Remark 2** *Observe that no recurrent facts are being used in the proof of Theorem 1, i.e., we do not require (3) for demonstrating (5.12) in Davis (1980), because the last expression is obtained by integrating (4) via (4.13) of Davis (1980).*

For $r$ matrices we have the following generalization of Chikuse (1980), eq. (3.14)

**Theorem 3** *Let $A_1, \ldots, A_r$ be $m \times m$ symmetric matrices, then*

$$\int_{\mathcal{O}(m)} C_{\phi}^{\kappa[r]}(A_1 H' X_1 H, A_2 H' X_2 H, \ldots, A_r H' X_r H)(dH) = \frac{C_{\phi}^{\kappa[r]}(A_1, A_2, \ldots, A_r) C_{\phi}^{\kappa[r]}(X_1, X_2, \ldots, X_r)}{\theta_{\phi}^{\kappa[r]} C_{\phi}(I)}, \quad (5)$$

*where we keep the notations of Chikuse and Davis (1986), i.e., $\alpha[s,r] = (\alpha_s, \ldots, \alpha_r)$, in particular $\alpha[1,r] \equiv \alpha[r] = (\alpha_1, \ldots, \alpha_r)$.*

It is important to note that from Theorems 1 and 3 most of the results in Davis (1980), Chikuse (1980) and Chikuse and Davis (1986) can be derived in an alternative way. For example, by taking $A = B$ and by using equation (5.1) in Davis (1980), we find equation (5.4) of Davis (1980).

Analogously, from (3) we have

$$\int_{\mathcal{O}(m)} C_{\phi}^{\kappa,\lambda}(A' H' X H A, B' H' Y H B)(dH) = \frac{C_{\phi}^{\kappa,\lambda}(AA', BB') C_{\phi}^{\kappa,\lambda}(X, Y)}{\theta_{\phi}^{\kappa,\lambda} C_{\phi}(I)}. \quad (6)$$

Thus, by replacing $X = I$, $A'A = R$ or $Y = I$, $B'B = S$ in (6) and by using Davis (1980, equation (5.2)), we obtain,

$$\int_{\mathcal{O}(m)} C_{\phi}^{\kappa,\lambda}(R, B' H' Y H B)(dH) = \frac{C_{\phi}^{\kappa,\lambda}(R, BB') C_{\lambda}(Y)}{C_{\lambda}(I)},$$

and

$$\int_{\mathcal{O}(m)} C_{\phi}^{\kappa,\lambda}(A' H' Y H A, S)(dH) = \frac{C_{\phi}^{\kappa,\lambda}(AA', S) C_{\kappa} Y)}{C_{\kappa}(I)},$$

respectively, see (Davis, 1980, equation 5.13).

Of course, a number of new properties can be established from Theorems 1 and 3. For instance, an interesting result can be derived when (5) is mixed with Chikuse and Davis (1986, equation (2.7)), then the following generalization of expression (1.1) in Davis (1980) is obtained, i.e, when the product of zonal polynomials is substituted by a product of invariant polynomials, this is

**Theorem 4**

$$\int_{\mathcal{O}(m)} C_{\sigma}^{\kappa[q]}(A_1 H' X_1 H, \ldots, A_q H' X_q H) C_{\tau}^{\kappa[q+1,r]}(A_{q+1} H' X_{q+1} H, \ldots, A_r H' X_r H)(dH)$$

$$= \sum_{\phi \in \sigma^* \cdot \tau^*} \pi_{\sigma,\tau}^{\kappa[r]:\phi} \frac{C_\phi^{\kappa[r]}(A_1, \ldots, A_r) C_\phi^{\kappa[r]}(X_1, \ldots, X_r)}{\theta^{\kappa[r]} C_\phi(I)}, \quad (7)$$

*where* $\pi_{\sigma,\tau}^{\kappa[r]:\phi}$ *is defined in Chikuse and Davis (1986, Lemma 2.2(iii)) and* $\sigma^*$ *denotes the partition* $\sigma$ *ignoring multiplicity.*

## 3 Complex case

Now, we can apply the above procedures to the similar expressions of complex invariant polynomials for deriving the extensions of Ratnarajah *et al.* (2005).

Let $Y, B, A_j$ and $X_j$, $j = 1, 2, \ldots, r$, be $m \times m$ Hermitian matrices, then we have

$$\int_{\mathcal{U}(m)} \widetilde{C}_\phi^{\kappa,\lambda}(AU^H XU, BU^H Y HU)(dU) = \frac{\widetilde{C}_\phi^{\kappa,\lambda}(A, B) \widetilde{C}_\phi^{\kappa,\lambda}(X, Y)}{\widetilde{\theta}_\phi^{\kappa,\lambda} \widetilde{C}_\phi(I)}, \quad (8)$$

$$\int_{\mathcal{U}(m)} \widetilde{C}_\phi^{\kappa[r]}(A_1 U^H X_1 U, A_2 U^H X_2 U, \ldots, A_r U^H X_r U)(dU)$$

$$= \frac{\widetilde{C}_\phi^{\kappa[r]}(A_1, A_2, \ldots, A_r) \widetilde{C}_\phi^{\kappa[r]}(X_1, X_2, \ldots, X_r)}{\widetilde{\theta}_\phi^{\kappa[r]} \widetilde{C}_\phi(I)}, \quad (9)$$

with

$$\widetilde{\theta}_\phi^{\kappa[r]} = \frac{\widetilde{C}_\phi^{\kappa[r]}(I, \cdots, I)}{\widetilde{C}_\phi(I)},$$

$$\int_{\mathcal{U}(m)} \widetilde{C}_\sigma^{\kappa[q]}(A_1 U^H X_1 U, \ldots, A_q U^H X_q U) \widetilde{C}_\tau^{\kappa[q+1,r]}(A_{q+1} U^H X_{q+1} U, \ldots, A_r U^H X_r U)(dU)$$

$$= \sum_{\phi \in \sigma^* \cdot \tau^*} \widetilde{\pi}_{\sigma,\tau}^{\kappa[r]:\phi} \frac{\widetilde{C}_\phi^{\kappa[r]}(A_1, \ldots, A_r) \widetilde{C}_\phi^{\kappa[r]}(X_1, \ldots, X_r)}{\theta_\phi^{\kappa[r]} \widetilde{C}_\phi(I)}, \quad (10)$$

where $\widetilde{\pi}_{\sigma,\tau}^{\kappa[r]:\phi}$ are the similar extensions of Chikuse and Davis (1986) to the complex case, $[dU]$ denotes the unit invariant Haar measure on the unitary group $\mathcal{U}(m)$ and $\widetilde{C}_\phi^{\kappa[r]}$ denotes the invariant polynomial with Hermitian matrix arguments, see Ratnarajah *et al.* (2005).

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