RECURRENT EXTENSIONS OF POSITIVE SELF SIMILAR MARKOV PROCESSES AND CRAMER´S CONDITION II

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Abstract

We prove that a positive self-similar Markov process $(X, \mathbb{P})$ that hits 0 in a finite time admits a self-similar recurrent extension that leaves 0 continuously if and only if the underlying Lévy process satisfies Cramér's condition.

Key words: Self-similar Markov processes, Lamperti's transformation, excursion theory, Lévy processes, exponential functionals of Lévy processes.

MSC: (60G18,60G51,60J25)

1 Introduction and main result

Let $\mathbb{P} = (\mathbb{P}_x, x \geq 0)$ be a family of probability measures on Skorohod's space $\mathbb{D}^+$, the space of càdlàg paths defined on $[0, \infty[$ with values in $\mathbb{R}^+$. The space $\mathbb{D}^+$ is endowed with the Skohorod topology and its Borel σ-field. We will denote by X the canonical process of the coordinates and $(\mathcal{G}_t, t \geq 0)$ will be the natural filtration generated by X. Assume that under $\mathbb{P}$ the canonical process X is a positive self-similar Markov process (pssMp), that is to say that $(X, \mathbb{P})$ is a $[0, \infty[$-valued strong Markov process and that it has the scaling property: there exists an $\alpha > 0$ such that for every $c > 0$,

$$(\{cX_{tc^{-1/\alpha}}, t \geq 0\}, \mathbb{P}_x) \stackrel{\text{Law}}{=} (\{X_t, t \geq 0\}, \mathbb{P}_{cx}) \qquad \forall x \geq 0.$$

We will assume furthermore that $(X, \mathbb{P})$ is a pssMp that hits 0 in a $\mathbb{P}$-a.s. finite time $T_0 = \inf\{t > 0 : X_t = 0\}$, and dies. So $\mathbb{P}_0$ is the law of the degenerated path equal to 0. According to Lamperti's transformation [16] the family of laws $\mathbb{P}$ can be obtained as the image law of the exponential of a $\mathbb{R} \cup \{-\infty\}$-valued Lévy process ξ with law $\mathbf{P}$, time changed by the inverse of the additive functional

$$t \to \int_0^t \exp\{\xi_s/\alpha\} \mathrm{d}s, \qquad t \geq 0. \tag{1}$$

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As usual, any function $f : \mathbb{R} \to \mathbb{R}$ is extended to $\mathbb{R} \cup \{-\infty\}$ by tacking $f(-\infty) = 0$. Thus, the state $\{-\infty\}$ will be taken as a cemetery state for ξ and denote by ζ its lifetime, viz $\zeta := \inf\{t > 0 : \xi_t = -\infty\}$, and by $\{\mathcal{F}_t, t \geq 0\}$, the filtration of ξ. A consequence of Lamperti's transformation is that the law of T_0 under $\mathbb{P}_x$ is equal to that $x^{1/\alpha} I$ under $\mathbf{P}$, where I denotes the exponential functional associated to ξ

$$I := \int_0^\zeta \exp\{\xi_s/\alpha\} \mathrm{d}s.$$

Lamperti proved the following characterization for pssMp that hit 0 in a finite time: either $(X, \mathbb{P})$ hits 0 by a jump and in a finite time

$$\mathbb{P}_x(T_0 < \infty, X_{T_0-} > 0, X_{T_0+t} = 0, \ \forall\, t \geq 0) = 1, \qquad \forall x > 0,$$

which happens if and only if $\mathbf{P}(\zeta < \infty) = 1$; or $(X, \mathbb{P})$ hits 0 continuously and in a finite time

$$\mathbb{P}_x(T_0 < \infty, X_{T_0-} = 0, X_{T_0+t} = 0, \ \forall\, t \geq 0) = 1, \qquad \forall x > 0,$$

and this is equivalent to $\mathbf{P}(\zeta = \infty, \lim_{t\to\infty} \xi_t = -\infty) = 1$. Reciprocally, the image law of the exponential of any $\mathbb{R} \cup \{-\infty\}$ valued Lévy processes time changed by the inverse of the functional defined in (1) is the law of a pssMp that dies at its first hitting time of 0. For more details, see [16] or [18].

The main purpose of this note is continue our study initiated in [18] on the existence and characterization of positive valued self-similar Markov processes, $\widetilde{X}$, that behave like $(X, \mathbb{P})$ before its first hitting time of 0 and for which the state 0 is a regular and recurrent state. A such process $\widetilde{X}$ will be called a recurrent extension of $(X, \mathbb{P})$. We refer to [18],[21] and the references therein for an introduction to this problem and for background on excursion theory for positive self-similar Markov processes.

We say that a σ-finite measure n on $(\mathbb{D}^+, \mathcal{G}_\infty)$ having infinite mass is an *excursion measure* compatible with $(X, \mathbb{P})$ if the following are satisfied:

(i) n is carried by

$$\{\omega \in \mathbb{D}^+ \mid T_0(\omega) > 0 \text{ and } X_t(\omega) = 0, \forall t \geq T_0\};$$

(ii) for every bounded $\mathcal{G}_\infty$–measurable H and each $t > 0$ and $\Lambda \in \mathcal{G}_t$

$$\mathrm{n}(H \circ \iota_t, \Lambda \cap \{t < T_0\}) = \mathrm{n}(\mathbb{E}_{X_t}(H), \Lambda \cap \{t < T_0\}),$$

where ι_t denotes the shift operator.

(iii) $\mathrm{n}(1 - e^{-T_0}) < \infty$;

Moreover, we will say that n is self-similar if it has the scaling property: there exists a $0 < \gamma < 1$, s.t. for all $a > 0$, the measure $H_a\,\mathrm{n}$, which is the image of n under the mapping $H_a : \mathbb{D}^+ \to \mathbb{D}^+$, defined by

$$H_a(\omega)(t) = a\omega(a^{-1/\alpha} t), \qquad t \geq 0,$$

is such that

$$H_a \, \mathrm{n} = a^{\gamma/\alpha} \, \mathrm{n} \, .$$

The parameter γ will be called the index of self-similarity of n. See Section 2 in [18] for equivalent definitions of self-similar excursion measure.

The entrance law associated to n is the family of finite measures $(\mathrm{n}_t, t > 0)$, defined by

$$\mathrm{n}(X_t \in \mathrm{d}y, t < T_0) = \mathrm{n}_t(\mathrm{d}y), \qquad t > 0.$$

It is known that there exists a one to one correspondence between recurrent extensions of $(X, \mathbb{P})$ and self-similar excursion measures compatible with $(X, \mathbb{P})$, see e.g. [21] and [18]. So determining the existence of the recurrent extensions of $(X, \mathbb{P})$ is equivalent to doing so it for self-similar excursion measures. We recall that the index of self-similarity of a self-similar excursion measure coincides with that of the stable subordinator which is the inverse of the local time at 0 of the associated recurrent extension of $(X, \mathbb{P})$.

We say that a positive self-similar Markov process for which 0 is a regular and recurrent state leaves 0 continuously (respectively, by a jump) whenever its excursion measure, n, is carried by the paths that leave 0 continuously (respectively, that leave 0 by a jump)

$$\mathrm{n}\,(X_{0+} > 0) = 0; \quad (\text{respectively, } \mathrm{n}(X_{0+} = 0) = 0.)$$

Vuolle-Apiala [21] proved, under some hypotheses, that any positive self-similar Markov process for which 0 is a regular and recurrent state either leaves 0 continuously or by jumps. In fact his result still holds true in the general setting as it is proved in the following Lemma.

Lemma 1. *Let* n *be a self-similar excursion measure compatible with* $(X, \mathbb{P})$, *and with index of self-similarity* $\gamma \in]0, 1[$. *Then*

$$\textit{either } \mathrm{n}(X_{0+} > 0) = 0 \quad \textit{or} \quad \mathrm{n}(X_{0+} = 0) = 0.$$

Proof. Assume that the claim of the Lemma does not hold. Let $\mathrm{n}^c = c^{(c)} \, \mathrm{n} \, |_{\{X_{0+}=0\}}$ and $\mathrm{n}^j = c^{(j)} \, \mathrm{n} \, |_{\{X_{0+}>0\}}$, be the restrictions of n to the set of paths $\{X_{0+} = 0\}$, and $\{X_{0+} > 0\}$, respectively, and $c^{(c)}$ and $c^{(j)}$ are normalizing constants such that

$$\mathrm{n}^c(1 - e^{-T_0}) = 1 = \mathrm{n}^j(1 - e^{-T_0}).$$

The measures n^c and n^j are self-similar excursion measures compatible with $(X, \mathbb{P})$, and with the same self-similarity index γ. According to Lemma 3 in [18] the potential measure of n^c and that of n^j is given by the same purely excessive measure

$$\mathrm{n}^c \left(\int_0^{T_0} 1_{\{X_t \in \mathrm{d}y\}} \mathrm{d}t \right) = C_{\alpha,\gamma} y^{(1-\alpha-\gamma)/\alpha} \mathrm{d}y = \mathrm{n}^j \left(\int_0^{T_0} 1_{\{X_t \in \mathrm{d}y\}} \mathrm{d}t \right), \quad y > 0, \tag{2}$$

where $C_{\alpha,\gamma} \in]0, \infty[$ is a constant. So by Theorem 5.25 in [13], on the uniqueness of purely excessive measures, the entrance laws associated to n^c and n^j are equal. So, by Theorem 4.7 in [7] the measures n^c and n^j are equal. Which lead to a contradiction to the fact that the supports of the measures n^c and n^j are disjoint. □

If n^β is a self-similar excursion measure with index $\gamma=\beta\alpha\in]0,1[$ and that is carried by the paths that leave 0 by a jump then the self-similarity implies that n^β has the form $\mathrm{n}^\beta=c_{\alpha,\beta}\,\mathbb{P}_{\eta_\beta}$, where $0<c_{\alpha,\beta}<\infty$ is a normalizing constant and the starting measure or jumping-in measure η_β is given by

$$\eta_\beta(\mathrm{d}x)=\beta x^{-1-\beta}\mathrm{d}x,\qquad x>0.$$

The choice of the constant $c_{\alpha,\beta}$ depends on the normalization of the local time at 0 of the recurrent extension of $(X,\mathbb{P})$.

In the work [18] we provided necessary and sufficient conditions on the underlying Lévy process for the existence of recurrent extensions of $(X,\mathbb{P})$ that leave 0 by a jump. For sake of completeness we include an improved version of that result.

Theorem 1. *Let $(X,\mathbb{P})$ be an α-self-similar Markov process that hits the cemetery point 0 in a finite time a.s. and $(\xi,\mathbf{P})$ the Lévy process associated to it via Lamperti's transformation. For $0<\beta<1/\alpha$, the following are equivalent*

(i) $\mathbf{E}(e^{\beta\xi_1},1<\zeta)<1$,

(ii) $\mathbf{E}(I^{\alpha\beta})<\infty$,

(iii) There exists a recurrent extension of $(X,\mathbb{P})$, say $X^{(\beta)}$, that leaves 0 by a jump and its associated excursion measure n^β is such that

$$\mathrm{n}^\beta(X_{0+}\in\mathrm{d}x)=c_{\alpha,\beta}\beta x^{-1-\beta}\mathrm{d}x,\qquad x>0,$$

where $c_{\alpha,\beta}$ is a constant.

In this case, the process $X^{(\beta)}$ is the unique recurrent extension of $(X,\mathbb{P})$ that leaves 0 by a jump distributed as $c_{\alpha,\beta}\eta_\beta$.

The equivalence between (ii) and (iii) in Theorem 1 is the content of Proposition 1 in [18] and the equivalence between (i) and (ii) is a consequence of Lemma 2 below.

Thus only the existence of recurrent extensions that leave 0 continuously remains to be established. In this vein, we proved in [18] that under the hypotheses:

(H2a) $(\xi,\mathbf{P})$ is not-arithmetic, i.e. its state space is not a subgroup of $r\,\mathbb{Z}$, for any $r\in\mathbb{R}$.

(H2b) **Cramér's condition is satisfied**, that is to say that there exists a $\theta>0$ s.t.

$$\mathbf{E}(e^{\theta\xi_1},1<\zeta)=1,$$

(H2c) for θ as in the hypothesis (H2b), $\mathbf{E}(\xi_1^+e^{\theta\xi_1},1<\zeta)<\infty$.

and provided $0<\alpha\theta<1$, there exists a recurrent extension of $(X,\mathbb{P})$ that leaves 0 continuously. In a previous work, Vuolle-Apiala [21] provided a sufficient condition on the resolvent of $(X,\mathbb{P})$ for the existence of recurrent extensions of $(X,\mathbb{P})$ that leave 0 continuously. Actually, in [18] we proved that in the case where the underlying Lévy process is not-arithmetic the conditions of Vuolle-Apiala are equivalent to the conditions (H2b-c) above. So, it is natural to ask if the conditions of Vuolle-Apiala and ours are also necessary for the existence of recurrent extensions of $(X,\mathbb{P})$ that leave 0 continuously? The following counterexample answers this question negatively.

Counterexample 1. Let σ be a subordinator with law P such that its law is not arithmetic and has some exponential moments of positive order, i.e.

$$\mathcal{E} := \{\lambda > 0,\ 1 < E(e^{\lambda\sigma_1}) < \infty\} \neq \emptyset.$$

Assume that the upper bound of $\mathcal{E}$, say q, belongs to $\mathcal{E}\cap]0,1[$ and that the function

$$m(x) := E\left(1_{\{\sigma_1 > x\}} e^{q\sigma_1}\right), \qquad x > 0,$$

is regularly varying at infinity with index $-\beta$, for some $\beta \in]1/2, 1[$. Let $(\xi, \mathbf{P})$ be the Lévy process with finite lifetime ζ, obtained by killing σ at an independent exponential time of parameter $\kappa = \log\left(E(e^{q\sigma_1})\right)$. By construction, it follows that Cramér's condition is satisfied

$$\mathbf{E}(e^{q\xi_1}, 1 < \zeta) = 1,$$

and by Karamata's Theorem the function

$$m^\natural(x) := \int_0^x \mathbf{E}(1_{\{\xi_1 > u\}} e^{q\xi_1}, 1 < \zeta)\mathrm{d}u, \qquad x \geq 0,$$

is regularly varying at infinity with index $1 - \beta$. As a consequence, the integral $\mathbf{E}(\xi_1 e^{q\xi_1}, 1 < \zeta)$ is not finite. We will denote by $\mathbf{P}^\natural$, the Girsanov type transformation of $\mathbf{P}$ via the martingale $(e^{q\xi_s}, s \geq 0)$, viz. $\mathbf{P}^\natural$ is the unique measure s.t.

$$\mathbf{P}^\natural = e^{q\xi_t}\,\mathbf{P}, \qquad \text{on } \mathcal{F}_t, \qquad t \geq 0.$$

Let $(X, \mathbb{P})$ be the 1-pssMp associated to $(\xi, \mathbb{P})$ via Lamperti's transformation and let V_λ denote its λ-resolvent, $\lambda > 0$. We claim that the following assertions are satisfied:

(P1) *For any* $\lambda > 0$,

$$\lim_{x\to 0+} m^\natural(\log(1/x))\frac{V_\lambda f(x)}{x^q} = \frac{1}{\Gamma(\beta)\Gamma(1-\beta)} \int_0^\infty f(y)\, \mathbf{E}^\natural\left(\exp\left\{-\lambda y \int_0^\infty e^{-\xi_s}\mathrm{d}s\right\}\right) y^{-q}\mathrm{d}y,$$

for every $f :]0,\infty[\to \mathbb{R}$, *continuous and with compact support;*

(P2) *the limit*

$$\lim_{x\to 0+} m^\natural(\log(1/x))\frac{\mathbb{E}_x(1 - e^{-T_0})}{x^q} := C_q,$$

exists and $C_q \in]0, \infty[$;

(P3) *there exists a recurrent extension of* $(X, \mathbb{P})$ *that leaves* 0 *continuously.*

That the properties (P1-3) are satisfied in the framework of Counterexample 1 will be proved in Section 3.

In the previous counterexample we have constructed a pssMp that does not satisfy the hypotheses of Vuolle-Apiala [21] nor all the hypotheses in [18], and that nonetheless admits a recurrent extension that leaves 0 continuously. This allowed us to realize that only Cramér's condition is relevant for the existence of recurrent extensions of pssMp. That is the content of the main theorem of this paper. To state the result we need further notation.

First, observe that if Cramér's condition is satisfied with index θ, then the process $M := (e^{\theta\xi_t}, t \geq 0)$ is a martingale under $\mathbf{P}$. In this case, we will denote by $\mathbf{P}^\natural$ the Girsanov type transform of $\mathbf{P}$ via the martingale M, as we did in the Counterexample 1. Under the law $\mathbf{P}^\natural$ the process ξ is a $\mathbb{R}$-valued Lévy process with infinite lifetime and that drift to ∞. We will denote by J, the exponential functional

$$J := \int_0^\infty \exp\{-\xi_s/\alpha\}\mathrm{d}s.$$

A straight consequence of Theorem 1 in [5] and the fact that $(\xi, \mathbf{P}^\natural)$ is a Lévy process that drift to ∞, is that $J < \infty$, $\mathbf{P}^\natural$-a.s. More details on the construction of the probability measure $\mathbf{P}^\natural$ and its properties can be found in Section (2.3) in [18]. This being said we have all the elements to state our main result.

Theorem 2. *Let $(X, \mathbb{P})$ be an α-self-similar Markov process that hits its cemetery state 0 in a finite time $\mathbb{P}$-a.s. and $(\xi, \mathbf{P})$ be the Lévy process associated to $(X, \mathbb{P})$ via Lamperti's transformation. The following are equivalent:*

(i) There exists a $0 < \theta < 1/\alpha$, such that $\mathbf{E}(e^{\theta\xi_1}, 1 < \zeta) = 1$.

(ii) There exists a recurrent extension of $(X, \mathbb{P})$ that leaves 0 continuously and such that its associated excursion measure from 0, say $\mathbf{n}$, is such that

$$\mathbf{n}(1 - e^{-T_0}) = 1.$$

In this case, the recurrent extension in (ii) is unique and the entrance law associated to the excursion measure $\mathbf{n}$ is given by, for any f positive and measurable

$$\mathbf{n}(f(X_t), t < T_0) = \frac{1}{t^{\alpha\theta}\Gamma(1-\alpha\theta)\, \mathbf{E}^\natural(J^{\alpha\theta-1})} \mathbf{E}^\natural\left(f\left(\frac{t^\alpha}{J^\alpha}\right) J^{\alpha\theta-1}\right), \qquad t > 0, \tag{3}$$

with θ as in the condition (i).

Observe that the condition (ii) in Theorem 2 implies that the inverse of the local time at 0 for the recurrent extension of $(X, \mathbb{P})$ is a stable subordinator of parameter $\alpha\theta$ for some $0 < \theta < 1/\alpha$. It is implicit in the Theorem 2 that this is the unique $\theta > 0$ that fulfills the condition (i), and viceversa. Moreover, the expression of the entrance law associated to $\mathbf{n}$ should be compared with the entrance law of Bertoin and Caballero [2] and Bertoin and Yor [4] for positive self-similar Markov processes that drift to ∞.

Besides, we can ask whether the recurrent extension in Theorem 2 is of the type obtained in Theorem 2 in [18].

Corollary 1. *Assume that there exists a recurrent extension of* $(X, \mathbb{P})$ *that leaves* 0 *continuously and let* $\widetilde{\mathbb{P}}$ *and* $\mathbf{n}$ *denote its law and excursion measure at* 0, *respectively. For* θ *as in Theorem 2 the integrability condition*

$$\mathbf{E}(\xi_1^+ e^{\theta\xi_1}, 1 < \zeta) < \infty, \tag{4}$$

is satisfied if and only if

$$\mathbf{n}(X_1^\theta, 1 < T_0) < \infty. \tag{5}$$

Furthermore, the latter holds if and only if

$$\widetilde{\mathbb{E}}_x(X_t^\theta) < \infty, \qquad \forall\, x \geq 0,\ \forall\, t \geq 0. \tag{6}$$

During the elaboration of this work we learned that in [12] P. Fitzsimmons essentially proved the equivalence between (i) and (ii) in Theorem 2. He proved that Cramér's condition and a moment condition for the exponential functional I are necessary and sufficient for the existence of a recurrent extension of $(X, \mathbb{P})$ that leaves 0 continuously. Actually, the moment condition of Fitzsimmons is a consequence of Cramér's condition, as it is proved in Lemma 2 below. Besides, Fitzsimmons' arguments and ours are completely different. He used arguments based on the theory of Kusnetzov measures and time change of processes with random birth and death, while our proof uses some general results on the excursions of pssMp obtained in our previous work [18] and some facts from the fluctuation theory of Lévy processes.

The rest of this note is organized as follows: Section 2 is mainly devoted to the proof of Theorem 2 and in Section 3 we establish the facts claimed in Counterexample 1.

2 Proofs

To tackle our task we need some notation. The Laplace exponent of $(\xi, \mathbf{P})$ is the function $\psi : [0, \theta] \to \mathbb{R} \cup \{\infty\}$ defined by

$$\mathbf{E}(e^{\lambda\xi_1}, 1 < \zeta) := e^{\psi(\lambda)}, \qquad \lambda \in \mathbb{R}\,.$$

Holder's inequality implies that ψ is a strictly convex function on the set $\mathcal{E} := \{\lambda \in \mathbb{R} : \psi(\lambda) < \infty\}$. So, if Cramer's condition is satisfied then the equation $\psi(\lambda) = 0$, $\lambda > 0$, has a unique root that we will denote hereafter by θ. Observe that $[0, \theta] \subseteq \mathcal{E}$, and that ψ is derivable from the right at 0 and from the left at θ and

$$\mathbf{E}(\xi_1, 1 < \zeta) = \psi'_+(0) \in [-\infty, 0[, \qquad \mathbf{E}(\xi_1 e^{\theta\xi_1}, 1 < \zeta) = \psi'_-(\theta) \in]0, \infty].$$

Our first purpose is to prove that (i) and (ii) in Theorem 1 are equivalent and that in Theorem 2, (i) implies (ii). To reach our end we will need the following Lemma.

Lemma 2. *Let* $(\xi, \mathbf{P})$ *be a Lévy process and let* $\beta > 0$ *be such that*

$$\mathbf{E}(e^{\beta\xi_1}, 1 < \zeta) \leq 1,$$

and assume that $\beta < 1/\alpha$. Then

$$\mathbf{E}\left(I^{\alpha\beta-1}\right) < \infty.$$

Furthermore,

$$\mathbf{E}(e^{\beta\xi_1}, 1<\zeta) < 1, \quad \textit{if and only if} \quad \mathbf{E}\left(I^{\alpha\beta}\right) < \infty.$$

Proof of Lemma 2. For $t>0$, let Q_t denote the random variable

$$Q_t := \int_0^t \exp\{\xi_u/\alpha\} 1_{\{u<\zeta\}} \mathrm{d}u.$$

The main argument of the proof uses that $\mathbf{E}(Q_t^{\alpha\beta}) < \infty$, for all $t>0$. Indeed, the strict convexity of the mapping $\lambda \to \mathbf{E}(e^{\lambda\xi_1}, 1<\zeta)$ implies that for any $p>1$, $\mathbf{E}\left(e^{(\beta/p)\xi_t}, t<\zeta\right) = e^{t\psi(\beta/p)} < 1$, $t>0$. Thus,

$$\begin{aligned}
\mathbf{E}(Q_t^{\alpha\beta}) &\leq t^{\alpha\beta}\, \mathbf{E}\left[\sup_{0<u\leq t}\left\{e^{\beta\xi_u}1_{\{u<\zeta\}}\right\}\right] \\
&= t^{\alpha\beta}\, \mathbf{E}\left[\left(\sup_{0<u\leq t}\left\{e^{(\beta/p)\xi_u}1_{\{u<\zeta\}}\right\}\right)^p\right] \\
&\leq t^{\alpha\beta}\, \mathbf{E}\left[\left(\sup_{0<u\leq t}\left\{e^{(\beta/p)\xi_u}e^{-u\psi(\beta/p)}1_{\{u<\zeta\}}\right\}\right)^p\right] \\
&\leq t^{\alpha\beta}\left(\frac{p}{p-1}\right)^p \mathbf{E}\left[\left\{e^{(\beta/p)\xi_t}e^{-t\psi(\beta/p)}1_{\{t<\zeta\}}\right\}^p\right] \\
&\leq t^{\alpha\beta}\left(\frac{p}{p-1}\right)^p e^{-tp\psi(\beta/p)},
\end{aligned}$$

owing that the process $e^{(\beta/c)\xi_u - u\psi(\beta/c)}$, $u\geq 0$, is a positive martingale and Doob's L_p inequality. We now prove the first claim in Lemma 2. On the one hand, using the well known inequality

$$\left||x|^{\alpha\beta} - |y|^{\alpha\beta}\right| \leq |x-y|^{\alpha\beta}, \qquad x,y\in\mathbb{R},$$

we get that

$$\mathbf{E}\left[\left(\int_0^\infty \exp\{\xi_s/\alpha\}1_{\{s<\zeta\}}\mathrm{d}s\right)^{\alpha\beta} - \left(\int_t^\infty \exp\{\xi_s/\alpha\}1_{\{s<\zeta\}}\mathrm{d}s\right)^{\alpha\beta}\right] \leq \mathbf{E}\left(Q_t^{\alpha\beta}\right) < \infty.$$

On the other hand, we have a.s.

$$\begin{aligned}
&\left(\int_0^\infty \exp\{\xi_s/\alpha\}1_{\{s<\zeta\}}\mathrm{d}s\right)^{\alpha\beta} - \left(\int_t^\infty \exp\{\xi_s/\alpha\}1_{\{s<\zeta\}}\mathrm{d}s\right)^{\alpha\beta} \\
&= \alpha\beta\int_0^t \exp\{\xi_u/\alpha\}1_{\{u<\zeta\}}\left(\int_u^\infty \exp\{\xi_s/\alpha\}1_{\{s<\zeta\}}\mathrm{d}s\right)^{\alpha\beta-1}\mathrm{d}u \\
&= \alpha\beta\int_0^t \exp\{\beta\xi_u\}1_{\{u<\zeta\}}\left(\int_0^\infty \exp\{\widetilde{\xi}_r/\alpha\}1_{\{r<\widetilde{\zeta}\}}\mathrm{d}r\right)^{\alpha\beta-1}\mathrm{d}u,
\end{aligned}$$

where $\widetilde{\xi}_r = \xi_{r+u} - \xi_u$, $r \geq 0$, and $\widetilde{\zeta} = \zeta - u$. Thus, by taking expectations, using Fubini's Theorem and the independence of the increments of ξ we get the identity

$$\begin{aligned}
&\mathbf{E}\left(\left(\int_0^\infty \exp\{\xi_s/\alpha\}1_{\{s<\zeta\}}\mathrm{d}s\right)^{\alpha\beta} - \left(\int_t^\infty \exp\{\xi_s/\alpha\}1_{\{s<\zeta\}}\mathrm{d}s\right)^{\alpha\beta}\right) \\
&= \alpha\beta \int_0^t \mathbf{E}\left(\exp\{\beta\xi_u\}1_{\{u<\zeta\}}\left(\int_0^\infty \exp\{\widetilde{\xi}_r/\alpha\}1_{\{r<\widetilde{\zeta}\}}\mathrm{d}r\right)^{\alpha\beta-1}\right)\mathrm{d}u \\
&= \alpha\beta\, \mathbf{E}(I^{\alpha\beta-1}) \int_0^t \mathbf{E}\left(\exp\{\beta\xi_u\}1_{\{u<\zeta\}}\right)\mathrm{d}u.
\end{aligned}$$

The first claim in Lemma 2 follows. To prove the second assertion, we assume first that $\mathbf{E}\left(e^{\beta\xi_1}, 1<\zeta\right) < 1$. Thus, by making t tend to infinity and integrating in the latter equation we get the identity

$$\mathbf{E}\left(I^{\alpha\beta}\right) = \frac{\alpha\beta}{\psi(\beta)}\mathbf{E}\left(I^{\alpha\beta-1}\right). \tag{7}$$

This relation is well known, see e.g. [5] and [17]. Which together with the first assertion of the Lemma implies that $\mathbf{E}(I^{\alpha\beta}) < \infty$. We now prove the reciprocal. If $\mathbf{E}(I^{\alpha\beta}) < \infty$, then we have that

$$\begin{aligned}
\infty &> \mathbf{E}\left(\left(\int_0^\zeta \exp\{\xi_s/\alpha\}\mathrm{d}s\right)^{\alpha\beta}\right) \\
&> \mathbf{E}\left(\left(\int_1^\zeta \exp\{\xi_s/\alpha\}\mathrm{d}s\right)^{\alpha\beta} 1_{\{1<\zeta\}}\right) \\
&= \mathbf{E}\left(e^{\beta\xi_1}\,\mathbf{E}\left(\left(\int_0^\infty \exp\{(\xi_{1+s}-\xi_1)/\alpha\}1_{\{1+s<\zeta\}}\mathrm{d}s\right)^{\alpha\beta}\right)1_{\{1<\zeta\}}\right) \\
&= \mathbf{E}\left(e^{\beta\xi_1}1_{\{1<\zeta\}}\right)\mathbf{E}\left(\left(\int_0^\zeta \exp\{\xi_s/\alpha\}\mathrm{d}s\right)^{\alpha\beta}\right),
\end{aligned} \tag{8}$$

owing to the fact that ξ is a Lévy process. So, we get that in this case $\mathbf{E}\left(e^{\beta\xi_1}, 1<\zeta\right) < 1$. □

Theorem 2, (i) implies (ii). The proof of this result is based on Theorem 3 in [18], but to use that result we first need to establish some weak-duality relations.

By assumption (i) and Lemma 2 we have that $\mathbf{E}(I^{\alpha\theta-1}) < \infty$. Moreover, let $(\xi, \widehat{\mathbf{P}}^\natural) := (-\xi, \mathbf{P}^\natural)$ denote the dual of $(\xi, \mathbf{P}^\natural)$. Then $(\xi, \widehat{\mathbf{P}}^\natural)$ drift to $-\infty$, because $(\xi, \mathbf{P}^\natural)$ drift to ∞, and as a consequence $I < \infty$, $\widehat{\mathbf{P}}^\natural$–a.s. Furthermore, $(\xi, \widehat{\mathbf{P}}^\natural)$ satisfies the hypotheses of Lemma 2 with $\beta = \theta$ due to the identity

$$\widehat{\mathbf{E}}^\natural\left(e^{\theta\xi_1}\right) = \mathbf{E}^\natural\left(e^{-\theta\xi_1}\right) = \mathbf{E}\left(e^{-\theta\xi_1}e^{\theta\xi_1}, 1<\zeta\right) \leq 1.$$

Thus we can also ensure that $\widehat{\mathbf{E}}^\natural\left(I^{\alpha\theta-1}\right) < \infty$. Now, let $\widehat{\mathbb{P}}^\natural$ be the law of the α-pssMp associated to $(\xi, \widehat{\mathbf{P}}^\natural)$ via Lamperti's transformation. Then $(X, \widehat{\mathbb{P}}^\natural)$ is an α-pssMp that hits 0 continuously and in a finite time $\widehat{\mathbb{P}}^{\natural}$-a.s. and according to Lemma 2 in [4], $(X, \mathbb{P}^{\natural})$ and $(X, \widehat{\mathbb{P}}^{\natural})$ are in weak duality with respect to the measure $\alpha^{-1}x^{1/\alpha-1}\mathrm{d}x, x > 0$, and given that the law $\mathbb{P}^{\natural}$ is the h-transform of the law $\mathbb{P}$ via the invariant function $h(x) = x^{\theta}$ for the semigroup of $(X, \mathbb{P})$ (see Proposition 5 in [18]) it then follows that $(X, \mathbb{P})$ and $(X, \widehat{\mathbb{P}}^{\natural})$ are in weak duality w.r.t. the measure $\alpha^{-1}x^{1/\alpha-1-\theta}\mathrm{d}x$, $x > 0$. Furthermore, we have that for any $\lambda > 0$,

$$\alpha^{-1}\int_0^{\infty} \mathrm{d}x x^{1/\alpha-1-\theta}\,\mathbb{E}_x(e^{-\lambda T_0}) < \infty, \qquad \alpha^{-1}\int_0^{\infty} \mathrm{d}x x^{1/\alpha-1-\theta}\,\widehat{\mathbb{E}}^{\natural}_x(e^{-\lambda T_0}) < \infty. \tag{9}$$

Indeed, for $\lambda > 0$

$$\begin{aligned}
\alpha^{-1}\int_0^{\infty} \mathrm{d}x x^{1/\alpha-1-\theta}\,\mathbb{E}_x\left(e^{-\lambda T_0}\right) &= \alpha^{-1}\int_0^{\infty} \mathrm{d}x x^{1/\alpha-1-\theta}\,\mathbf{E}\left(e^{-\lambda x^{1/\alpha} I}\right)\\
&= \mathbf{E}\left(\alpha^{-1}\int_0^{\infty} \mathrm{d}x x^{1/\alpha-1-\theta}e^{-\lambda x^{1/\alpha} I}\right)\\
&= \lambda^{\alpha\theta-1}\,\mathbf{E}(I^{\alpha\theta-1})\Gamma(1-\alpha\theta) < \infty.
\end{aligned}$$

The same calculation applies to verify the finiteness of the second integral in equation (9). This being said, Theorem 3 in [18] ensures that there exists a unique recurrent extension of $(X, \mathbb{P})$, such that the λ-resolvent of its excursion measure, say $\mathbf{n}$, is given by

$$\mathbf{n}\left(\int_0^{T_0} e^{-\lambda t} f(X_t)\mathrm{d}t\right) = \frac{1}{\alpha\Gamma(1-\alpha\theta)\,\widehat{\mathbf{E}}^{\natural}(I^{\alpha\theta-1})}\int_0^{\infty} f(x)x^{1/\alpha-1-\theta}\,\widehat{\mathbb{E}}^{\natural}_x(e^{-\lambda T_0})\mathrm{d}x, \tag{10}$$

for $\lambda \geq 0$, and any function f, positive and measurable on $[0, \infty[$. An easy calculation proves that the λ-resolvent of $\mathbf{n}$ satisfies the self-similarity property in Lemma 2 in [18] and therefore the excursion measure $\mathbf{n}$ is self-similar. In particular, $\mathbf{n}(1 - e^{-T_0}) = 1$, and the potential of $\mathbf{n}$ is given by

$$\mathbf{n}\left(\int_0^{T_0} f(X_t)\mathrm{d}t\right) = \frac{1}{\alpha\Gamma(1-\alpha\theta)\,\widehat{\mathbf{E}}^{\natural}(I^{\alpha\theta-1})}\int_0^{\infty} f(x)x^{1/\alpha-1-\theta}\mathrm{d}x.$$

Which compared with the result in Lemma 3 in [18] implies that

$$\widehat{\mathbf{E}}^{\natural}\left(I^{\alpha\theta-1}\right) = \mathbf{E}\left(I^{\alpha\theta-1}\right). \tag{11}$$

Actually, Theorem 3 cited above establishes also that there exists a recurrent extension of $(X, \widehat{\mathbb{P}}^{\natural})$ with excursion measure $\widehat{\mathbf{n}}$ such that

$$\widehat{\mathbf{n}}\left(\int_0^{T_0} e^{-\lambda t} f(X_t)\mathrm{d}t\right) = \frac{1}{\alpha\Gamma(1-\alpha\theta)\,\mathbf{E}(I^{\alpha\theta-1})}\int_0^{\infty} f(x)x^{1/\alpha-1-\theta}\,\mathbb{E}_x(e^{-\lambda T_0})\mathrm{d}x.$$

Moreover, the recurrent extensions of $(X, \mathbb{P})$ and $(X, \widehat{\mathbb{P}}^{\natural})$ associated to $\mathbf{n}$ and $\widehat{\mathbf{n}}$, respectively, still are in weak duality. To verify that $\mathbf{n}$ is carried by the paths that leave 0 continuously, we claim that the image under time reversal at time T_0 of $\mathbf{n}$ is $\widehat{\mathbf{n}}$. This follows from the fact that $\mathbf{n}$ and $\widehat{\mathbf{n}}$ both have the same potential and an application of a result for time reversal of Kusnetzov measures established in Dellacherie et al. [10] Section XIX.23. Thus, using the

Markov property and that $(X, \widehat{\mathbb{P}}^{\natural})$ is a pssMp that hits 0 continuously and in a finite time $\widehat{\mathbb{P}}^{\natural}$-a.s., given that the underlying Lévy process $(\xi, \widehat{\mathbf{P}}^{\natural})$ drifts to $-\infty$, we get that $\widehat{\mathbf{n}}$ is carried by the paths that hit 0 continuously and therefore

$$0 = \widehat{\mathbf{n}}(X_{T_0-} > 0) = \mathbf{n}(X_{0+} > 0).$$

$\square$

We will next prove that in Theorem 2, (ii) implies (i). The proof is long so we will give it in two main steps: with the first we will prove that if it is possible to construct recurrent extensions of $(X, \mathbb{P})$ that leave 0 by a jump then the condition in (i) in Theorem 2 is satisfied; and with the second step, we will prove that if (ii) holds then it is indeed possible to construct recurrent extensions of $(X, \mathbb{P})$ that leaves 0 by a jump.

2.1 Step 1

The Step 1 of the proof that in Theorem 2, (ii) implies (i) relies on the following Proposition.

Proposition 1. *Assume that there exists a recurrent extension, $\widetilde{X}$, of $(X, \mathbb{P})$ that leaves 0 continuously and let $\alpha\vartheta \in]0,1[$ be the index of self-similarity of the subordinator inverse of the local time at 0 of $\widetilde{X}$. We have that for any $0 < \beta < \vartheta$ there exists a recurrent extension $X^{(\beta)}$ of $(X, \mathbb{P})$ with associated excursion measure $\mathbf{n}^{\beta} = c_{\alpha,\beta}\,\mathbb{P}_{\eta_\beta}$, where η_β is the jumping-in measure defined in the Introduction and $c_{\alpha,\beta}$ is a normalizing constant.*

The proof of this Proposition will be given in Subsection 2.2. Roughly, the process $X^{(\beta)}$ will be obtained by erasing randomly the debut of all the excursions out from 0 of $\widetilde{X}$. Formally, this will be done using a time change associated to a fluctuating additive functional.

If we take for granted the existence of $X^{(\beta)}$, for all $\beta \in]0, \vartheta[$, the rest of the proof is rather elementary as we shall next explain. Owing to the Theorem 1 this has as a consequence that for any $0 < \beta < \vartheta$ the Lévy process ξ has positive exponential moments of order β,

$$\mathbf{E}(e^{\beta\xi_1}, 1 < \zeta) < 1, \qquad \forall \beta \in]0, \vartheta[.$$

So we have that

$$\lim_{\beta \to \vartheta-} \mathbf{E}\left(e^{\beta\xi_1}, 1 < \zeta\right) = \mathbf{E}\left(e^{\vartheta\xi_1}, 1 < \zeta\right) \leq 1.$$

Nevertheless, it does not happens that $\mathbf{E}\left(e^{\vartheta\xi_1}, 1 < \zeta\right) < 1$. Because, if this were indeed the case Theorem 1 would imply that $(X, \mathbb{P})$ admits a recurrent extension that leaves 0 by a jump and with jumping-in measure proportional to η_ϑ. Thus, that the measure $m = 2^{-1}\mathbf{n} + 2^{-1}c_{\alpha,\vartheta}\,\mathbb{P}_{\eta_\vartheta}$, is a self-similar excursion measure compatible with $(X, \mathbb{P})$, and with index of self-similarity $\alpha\vartheta$; as before $c_{\alpha,\vartheta}$ is a normalizing constant. Therefore, there exists a recurrent extension of $(X, \mathbb{P})$ with excursion measure m and that may leave 0 by a jump and continuously, which leads to a contradiction to the fact that any recurrent extension of $(X, \mathbb{P})$ either leaves 0 by a jump or continuously. Therefore, Cramér's condition is satisfied.

2.2 Step 2. Construction of the auxiliary processes

The main purpose of this section is to prove the Proposition 1.

Our first concern will be ensure that there exists a measure $\widetilde{\mathbb{P}}^*$ on Skorohod's space $\mathbb{D}$ of càdlàg real valued paths defined on $[0,\infty)$, such that under $\widetilde{\mathbb{P}}^*$ the canonical process Y is a strong Markov self-similar process that has the following properties:

- $(Y,\widetilde{\mathbb{P}}^*)$ is a real valued α–self-similar Markov process that leaves 0 continuously,
- 0 is a recurrent and regular state for $(Y,\widetilde{\mathbb{P}}^*)$,
- $(Y,\widetilde{\mathbb{P}}^*)$ is symmetric,
- $(Y,\widetilde{\mathbb{P}}^*)$ killed at its first hitting time of $]-\infty,0]$ has the same law as $(X,\mathbb{P})$,
- the measure of the excursions from 0 of $(Y,\widetilde{\mathbb{P}}^*)$, say W, is supported by

  $$\{\omega\in\mathbb{D}:T_0(\omega)>0,\omega(t)=0,\forall\ t\geq T_0(\omega)\}\cap\left(\mathbb{D}^+\cup\mathbb{D}^-\right)$$

  where

  $$\begin{aligned}\mathbb{D}^+&=\{\ \omega\in\mathbb{D}:\omega(t)>0,\ \forall\ t\in]0,T_0(\omega)[\ \},\\ \mathbb{D}^-&=\{\ \omega\in\mathbb{D}:\omega(t)<0,\ \forall\ t\in]0,T_0(\omega)[\ \},\end{aligned}$$

- the restriction of W to the set of positive càdlàg paths $\mathbb{D}^+$ is equal to $\mathbf{n}$.

This implies that there exists a local time $L=(L_t,t\geq 0)$ at 0 for $(Y,\widetilde{\mathbb{P}}^*)$ and that the inverse of L, say L^{-1}, is a stable subordinator of parameter $\alpha\vartheta$.

The spaces $\mathbb{D}^+,\mathbb{D}^-$ are endowed with the σ-algebras $\mathcal{G}^+_\infty,\mathcal{G}^-_\infty$, generated by the coordinate maps, respectively.

Proof. We first introduce some notations. Let $\mathbb{P}^-$ be the law on $(\mathbb{D}^-,\mathcal{G}^-_\infty)$ which is the image of $(-X,\mathbb{P})$. Under $\mathbb{P}^-$ the canonical process of coordinates X^- is a self-similar Markov process taking values in $]-\infty,0]$, that dies at its first hitting time of 0. We will denote by $\left(P^+_t,t\geq 0\right)$ and $\left(P^-_t,t\geq 0\right)$ the semigroups of $(X,\mathbb{P})$ and $(X^-,\mathbb{P}^-)$, respectively. We define a sub-Markovian semigroup on $\mathbb{R}\setminus\{0\}$ as follows, for any $f:\mathbb{R}\to\mathbb{R}^+$ measurable

$$P^*_t f(x)=P^+_t f(x)1_{\{x>0\}}+P^-_t f(x)1_{\{x<0\}}+f(0)1_{\{x=0\}},\qquad t\geq 0,\ x\in\mathbb{R}\,.$$

The semigroup $\{P^*_t,t\geq 0\}$ is Fellerian on $\mathbb{R}\setminus\{0\}$ and satisfies the hypothesis in [7] Chapter 5, since the semigroups $(P^+_t,t\geq 0)$ and $(P^-_t,t\geq 0)$ have those properties on $]0,\infty[$ and $]-\infty,0[$, respectively. Thus, there exists a unique strong Markov process on $\mathbb{R}$ whose law will be denoted by $\mathbb{P}^*$, having $(P^*_t,t\geq 0)$ as semi-group and for which 0 is a cemetery state. Observe that when started at some $x>0$ the process $(Y,\mathbb{P}^*)$ has the same law as $(X,\mathbb{P})$. Besides, let $\check{\mathbf{n}}$ be the image of $\mathbf{n}$ under the measurable mapping $\Phi:\mathbb{D}^+\to\mathbb{D}^-$ defined by $\Phi(\omega)=-\omega$. The measure $\check{\mathbf{n}}$ is an excursion measure compatible with the semigroup $(P^-_t,t\geq 0)$. Then there exists a unique measure W on $\mathbb{D}^+ \cup \mathbb{D}^-$ endowed with the σ-algebra $\widetilde{\mathcal{G}} = \mathcal{G}^+_\infty \bigvee \mathcal{G}^-_\infty$, such that $W 1_{\mathbb{D}^+} = \mathbf{n}$ and $W 1_{\mathbb{D}^-} = \check{\mathbf{n}}$, and is an excursion measure compatible with the semigroup $(P^*_t, t \geq 0)$. The version of the Itô extension Theorem of Blumenthal [7] Chapter 5, implies that there exists a unique recurrent extension of $(Y, \mathbb{P}^*)$, with law, say $\widetilde{\mathbb{P}}^*$, and with excursion measure W. By construction the process $(Y, \widetilde{\mathbb{P}}^*)$ has the required properties. □

Next, we introduce other ingredients that will be useful to prove the Proposition 1.

For $q > 0$, let $A^+, A^{-,q}$ be the additive functionals of Y defined by

$$A^+_t = \int_0^t 1_{\{Y_s > 0\}} \mathrm{d}s, \qquad A^{-,q}_t = \int_0^t q^{1/\alpha\vartheta} 1_{\{Y_s < 0\}} \mathrm{d}s, \qquad t \geq 0,$$

and we introduce the time change $\tau^{(q)}$, which is the generalized inverse of the fluctuating additive functional $A^+ - A^{-,q}$, that is

$$\tau^{(q)}(t) = \inf\{s > 0 : A^+_s - A^{-,q}_s > t\}, \qquad t \geq 0, \qquad \inf \emptyset = \infty.$$

Now, let $Y^{(+,q)}$ be the process Y time changed by $\tau^{(q)}$,

$$Y^{(+,q)}_t = \begin{cases} Y_{\tau^{(q)}(t)} & \text{if } \tau^{(q)}(t) < \infty \\ \Delta & \text{if } \tau^{(q)}(t) = \infty \end{cases};$$

where Δ is a cemetery or absorbing state. The proof of Proposition 1 is a straightforward consequence of the following Lemma.

Lemma 3. *Under $\widetilde{\mathbb{P}}^*$ the process $Y^{(+,q)}$, is a positive α-self-similar Markov process for which 0 is a regular and recurrent state and that leaves 0 by a jump according to the jumping-in measure $c_{\alpha,\rho\vartheta}\eta_{\rho\vartheta}$, with*

$$\eta_{\rho\vartheta}(\mathrm{d}x) = \rho\vartheta x^{-1-\rho\vartheta}\mathrm{d}x, \qquad x > 0,$$

where ρ is given by

$$\rho = \frac{1}{2} + \frac{1}{\pi\alpha\vartheta} \arctan\left(\frac{1-q}{1+q} \tan\left(\frac{\pi\alpha\vartheta}{2}\right)\right) \in]0,1[,$$

and $0 < c_{\alpha,\rho\vartheta} = \mathbf{n}(X^{\vartheta\rho}_1, 1 < T_0) < \infty$, is a constant.

That the process $Y^{(+,q)}$ is a pssMp follows from standard arguments. We should prove that the measure $\mathbf{n}^{+,q}$ of the excursions from 0 of $Y^{(+,q)}$, is such that $\mathbf{n}^{+,q}(Y_0 \in \mathrm{d}y) = c_{\alpha,\rho\vartheta}\eta_{\rho\vartheta}(\mathrm{d}y)$. This will be a consequence of the following auxiliary Lemma.

Lemma 4. *(i) The processes $Z^+, Z^{-,q}$ defined by*

$$Z^+ \equiv (Z^+_t = A^+_{L^{-1}_t}, t \geq 0); \qquad Z^{-,q} \equiv (Z^{-,q}_t = A^{-,q}_{L^{-1}_t}, t \geq 0)$$

are independent stable subordinators of parameter $\alpha\vartheta$, and their respective Lévy measures are given by $\pi^+(\mathrm{d}x) = cx^{-1-\alpha\vartheta}\mathrm{d}x$, and $\pi^{-,q}(\mathrm{d}x) = qcx^{-1-\alpha\vartheta}\mathrm{d}x$, on $]0,\infty[$, and $c \in]0,\infty[$ is a constant.

(ii) The process $Z = Z^+ - Z^{-,q}$ is a stable process with parameter $\alpha\vartheta$ and positivity parameter $\rho = \mathbf{P}(Z_1 > 0)$, with ρ as defined in Lemma 3

(iii) The upward and downward ladder height processes H and $\widehat{H}$ associated to Z are stable subordinators of parameter $\alpha\vartheta\rho$ and $\alpha\vartheta(1-\rho)$, respectively.

The proof of Lemmas 3 and 4 uses arguments from the fluctuation theory of Lévy processes, we refer to [1] for background on this topic.

Proof. That the processes $Z^+, Z^{-,q}$ are independent subordinators is a standard result in the theory of excursions of Markov processes and follows from the fact that they are defined in terms of the positive and negative excursions of $(Y, \widetilde{\mathbb{P}}^*)$, respectively. The self-similarity property for $(Z^+, Z^{-,q})$ follows from that of $(Y, \widetilde{\mathbb{P}}^*)$. The jump measure π^+ of Z^+ is given by

$$\pi^+(\mathrm{d}t) = \mathbf{n}(T_0 \in \mathrm{d}t),$$

while that of $Z^{-,q}$ is

$$\pi^{-,q}(\mathrm{d}t) = q\mathbf{n}(T_0 \in \mathrm{d}t).$$

And given that the jumps of the stable subordinator L^{-1} are given by the measure $\mathbf{n}(T_0 \in \mathrm{d}t)$, it follows that $\mathbf{n}(T_0 \in \mathrm{d}t) = ct^{-1-\alpha\vartheta}\mathrm{d}t$, $t > 0$, for some constant $0 < c < \infty$. The proof of the assertion in (ii) is straightforward. It is well known in the fluctuation theory of Lévy processes that the upward and downward ladder height subordinators associated to a stable Lévy process have the form claimed in (iii). □

Proof of Lemma 3. By construction, the closure of set of times at which the process $Y^{(+,q)}$ visits 0 is the regenerative set which is the closure of the image of the supremum of the stable Lévy process Z. So 0 is a regular and recurrent state for $Y^{(+,q)}$. The length of any excursion out of 0 for $Y^{(+,q)}$ is distributed as a jump of Z to reach a new supremum or equivalently as a jump of the upward ladder height process H associated to Z. Let N denotes the measure of the excursions from 0 of $\overline{Z} - Z$, the process Z reflected at its current supremum, viz.

$$(\overline{Z} - Z)_t := \sup_{0 \leq s \leq t} \{0 \vee Z_s\} - Z_t, \qquad t \geq 0;$$

and let R denote the lifetime of the generic excursion from 0 of $\overline{Z} - Z$. Let Π_Z be the Lévy measure of Z and $\widehat{V}$ be the renewal measure of the downward ladder height subordinator $\widehat{H}$, that is

$$\widehat{V}(\mathrm{d}y) = \mathbf{E}\left(\int_0^\infty 1_{\{\widehat{H}_s \in \mathrm{d}y\}} \mathrm{d}s\right).$$

It is known in the fluctuation theory for Lévy processes that under N the joint law of Z_{R-} and $Z_R - Z_{R-}$ is given by

$$N(Z_{R-} \in \mathrm{d}x, -(Z_R - Z_{R-}) \in \mathrm{d}y) = \widehat{V}(\mathrm{d}x)\Pi_Z(\mathrm{d}y)1_{\{0<x<y\}}.$$

Moreover, the Lévy measure of H, say $po(\mathrm{d}x)$, is such that

$$po]x, \infty[= \int\int_{\{0 \leq s \leq u\}} \widehat{V}(\mathrm{d}s)\Pi_Z(\mathrm{d}u)1_{]x,\infty[}(u - s), \qquad x > 0.$$

cf. [20]. In our framework, $-(Z_R - Z_{R-})$ denotes the length of the generic positive excursion from 0 for $(Y, \widetilde{\mathbb{P}}^*)$ and Z_{R-} is the length of the portion of the generic positive excursion from 0 of $(Y, \widetilde{\mathbb{P}}^*)$ that is not observed while observing a generic excursion from 0 of $Y^{(+,q)}$. Furthermore, $-(Z_R - Z_{R-}) - Z_{R-}$ is the length of the generic excursion from 0 for $Y^{(+,q)}$, so $\mathbf{n}^{+,q}(T_0 \in \mathrm{d}t) = po(\mathrm{d}t)$.

Let $\mathbf{n}(\cdot | T_0 = \cdot)$ denote a version of the regular conditional law of the generic excursion under $\mathbf{n}$ given the lifetime T_0. Similarly, the notation $\mathbf{n}^{+,q}(\cdot | T_0 = \cdot)$ will be used for the analogous conditional law under $\mathbf{n}^{+,q}$. These laws can be constructed using the method in [9].

Finally, it follows from the verbal description above, that for any positive and measurable function $f : \mathbb{R} \to \mathbb{R}^+$

$$
\begin{aligned}
\mathbf{n}^{+,q}(f(Y_0)) &= \int_0^\infty \mathbf{n}^{+,q}(T_0 \in \mathrm{d}u)\mathbf{n}^{+,q}\left(f(Y_0)|T_0 = u\right) \\
&= \int_{t\in]0,\infty[} \int_{s\in]t,\infty[} N(Z_{R-} \in \mathrm{d}t, -(Z_R - Z_{R-}) \in \mathrm{d}s)\mathbf{n}(f(Y_t)|T_0 = s) \\
&= \int_{t\in]0,\infty[} \widehat{V}(\mathrm{d}t) \int_{s\in]t,\infty[} \Pi_Z(\mathrm{d}s)\mathbf{n}(f(Y_t)|T_0 = s) \\
&= \int_{t\in]0,\infty[} \widehat{V}(\mathrm{d}t) \int_{s\in]t,\infty[} \mathbf{n}(T_0 \in \mathrm{d}s)\mathbf{n}(f(Y_t)|T_0 = s) \\
&= \int_{t\in]0,\infty[} \widehat{V}(\mathrm{d}t)\mathbf{n}(f(Y_t), t < T_0).
\end{aligned}
$$

Given that the downward ladder height subordinator $\widehat{H}$ is a stable process with index $\alpha\vartheta(1-\rho)$, it follows that

$$
\begin{aligned}
\mathbf{n}^{+,q}(f(Y_0)) &= \alpha\vartheta(1-\rho)\widehat{k} \int_{t\in]0,\infty[} \mathrm{d}t t^{\alpha\vartheta(1-\rho)-1}\mathbf{n}(f(Y_t), t < T_0) \\
&= \alpha\vartheta(1-\rho)\widehat{k} \int_{t\in]0,\infty[} \mathrm{d}t t^{\alpha\vartheta(1-\rho)-1} t^{-\alpha\vartheta}\mathbf{n}(f(t^\alpha Y_1), 1 < T_0) \\
&= \alpha\vartheta(1-\rho)\widehat{k} \int_{]0,\infty[} \mathbf{n}(Y_1 \in \mathrm{d}x, 1 < T_0) \int_{t\in]0,\infty[} \mathrm{d}t t^{-\alpha\vartheta\rho-1} f(t^\alpha x) \\
&= \vartheta(1-\rho)\widehat{k}\mathbf{n}\left(Y_1^{\vartheta\rho}, 1 < T_0\right) \int_{u\in]0,\infty[} \mathrm{d}u u^{-\vartheta\rho-1} f(u) \\
&= c_{\alpha,\rho\vartheta}\eta_\rho f,
\end{aligned}
$$

where $\widehat{k}$ is a constant that depends on the normalization of the local time at zero for the reflected process $\overline{Z} - Z$; which without loss of generality can be and is supposed to be $\widehat{k} = 1$. □

Remark 1. The previous proof is inspired by the work of Rogers [19].

We finally have all the elements to prove the Proposition 1.

Proof of Proposition 1. By construction the process $Y^{(+,q)}$ is a recurrent extension of $(X, \mathbb{P})$ that leaves 0 by a jump according to the jumping-in measure $c_{\alpha,\vartheta\rho}\eta_{\vartheta\rho}$. Thus, for any $\beta < \vartheta$ the process $X^{(\beta)}$ in Proposition 1 is the process $Y^{(+,q)}$ with $q > 0$ such that $\rho\vartheta = \beta$, recall that ρ and q are related by the formula in Lemma 3. Which finishes the proof of the Proposition because if q ranges in $]0, \infty[$, then ρ ranges in $]0, 1[$. □

We have so finished the proof of the equivalence between the assertions (i) and (ii) in Theorem 2. Observe that the θ in the proof of the implication (i) $\Longrightarrow$ (ii) is equal to the ϑ in the implication (ii) $\Longrightarrow$ (i).

We next prove the uniqueness and characterization of the entrance law associated to the excursion measure claimed in Theorem 2.

2.3 Uniqueness and characterization

Assume that there exist two recurrent extensions of $(X, \mathbb{P})$ that satisfy the conditions in (ii) in Theorem 2 and let $\mathbf{n}$ and $\mathbf{n}'$, be its associated excursions measures. Then there exist θ_1 and θ_2 such that Cramér's condition is satisfied. The strict convexity of the mapping $\lambda \to \mathbb{E}(e^{\lambda\xi_1}, 1 < \zeta)$ implies that $\theta_2 = \theta = \theta_1$. As a consequence, the potential of both excursion measures is given by equation (2) with γ replaced by $\alpha\theta$. Therefore, arguing as in the proof of Lemma 1 we show that $\mathbf{n} = \mathbf{n}'$. Which finishes the proof of the unicity.

The characterization of the entrance law follows from our proof of the fact that (i) implies (ii) in Theorem 2. On the one hand, by construction the resolvent of the excursion measure $\mathbf{n}$ is given by equation (10). On the other hand,

$$\begin{aligned}\mathbf{n}\left(\int_0^{T_0} e^{-\lambda t} f(X_t)\mathrm{d}t\right) &= \int_0^{\infty} e^{-\lambda t} t^{-\alpha\theta}\mathbf{n}(f(t^{\alpha}X_1), 1 < T_0)\mathrm{d}t \\ &= \mathbf{n}\left(\int_0^{\infty} \mathrm{d}u\alpha^{-1}u^{1/\alpha-1-\theta}f(u)X_1^{\theta-1/\alpha}e^{-\lambda u^{1/\alpha}X_1^{-1/\alpha}}1_{\{1<T_0\}}\right) \qquad (12) \\ &= \int_0^{\infty} \mathrm{d}u\alpha^{-1}u^{1/\alpha-1-\theta}f(u)\mathbf{n}\left(X_1^{\theta-1/\alpha}e^{-\lambda u^{1/\alpha}X_1^{-1/\alpha}}1_{\{1<T_0\}}\right),\end{aligned}$$

where we used thee times Fubini's Theorem combined with the scaling property of $\mathbf{n}$ and a change of variables. Comparing the results in equations (10) and (12) we get the identity

$$\mathbf{n}\left(X_1^{\theta-1/\alpha}\exp\left\{-\lambda u^{1/\alpha}X_1^{-1/\alpha}\right\}1_{\{1<T_0\}}\right) = \frac{1}{\Gamma(1-\alpha\theta)\,\widehat{\mathbf{E}}^{\natural}\left(I^{\alpha\theta-1}\right)}\widehat{\mathbb{E}}^{\natural}_u\left(e^{-\lambda T_0}\right),$$

for all $\lambda \geq 0$ and a.e. $u > 0$. As a consequence,

$$\mathbf{n}(X_1^{\theta-1/\alpha}1_{\{1<T_0\}}) < \infty.$$

By the dominated convergence theorem, the latter identity holds for all $\lambda \geq 0$ and all $u > 0$. Recall that by Lamperti's transformation, T_0 under $\widehat{\mathbb{P}}^{\natural}_u$ has the same law as $u^{1/\alpha}I$ under $\widehat{\mathbf{P}}^{\natural}$. So by the uniqueness of Laplace transforms it follows that

$$\mathbf{n}\left(X_1^{\theta-1/\alpha}f\left(X_1^{-1/\alpha}\right)1_{\{1<T_0\}}\right) = \frac{1}{\Gamma(1-\alpha\theta)\,\widehat{\mathbf{E}}^{\natural}\left(I^{\alpha\theta-1}\right)}\widehat{\mathbf{E}}^{\natural}\left(f\left(I\right)\right),$$

The claim in Theorem 2 follows from this identity using the scaling property of $\mathbf{n}$, and that the law of I under $\widehat{\mathbf{P}}^{\natural}$ is equal to that of J under $\mathbf{P}^{\natural}$.

We have finished the proof of Theorem 2 and we next prove the Corollary 1.

2.4 Proof of Corollary 1

Owing that the left derivative of ψ at θ exists, it follows from Proposition 3.1 in [8] that

$$\mathbf{E}^{\natural}\left(J^{-1}\right) = \widehat{\mathbf{E}}^{\natural}\left(I^{-1}\right) = -\widehat{\mathbf{E}}^{\natural}(\xi_1) = \mathbf{E}(\xi_1 e^{\theta\xi_1}, 1 < \zeta),$$

and the leftmost quantity is finite if and only if $\mathbf{E}(\xi_1^+ e^{\theta\xi_1}, 1 < \zeta) < \infty$. So, that (4) and (5) are equivalent is an easy consequence of the representation of the entrance law obtained in Theorem 2. We will next prove that (5) is equivalent to (6). Let V_λ denote the λ-resolvent of $(X, \mathbb{P})$ and U_λ be the λ-resolvent of the unique recurrent extension of $(X, \mathbb{P})$ that leaves 0 continuously. The invariance of $h(x) = x^\theta, x > 0$ implies that

$$\mathbf{n}(X_1^\theta, 1 < T_0) = \mathbf{n}(X_t^\theta, t < T_0), \qquad t > 0.$$

Using a well known decomposition formula and Fubini's theorem we get

$$\begin{aligned}
\lambda U_\lambda h(x) &= \lambda V_\lambda h(x) + \mathbb{E}_x(e^{-\lambda T_0})\lambda U_\lambda h(0) \\
&= h(x) + \mathbb{E}_x(e^{-\lambda T_0})\frac{\lambda \mathbf{n}\left(\int_0^{T_0} e^{-\lambda t} h(X_t)\mathrm{d}t\right)}{\mathbf{n}(1 - e^{-\lambda T_0})} \\
&= h(x) + \mathbb{E}_x(e^{-\lambda T_0})\lambda^{-\alpha\theta}\int_0^\infty \lambda e^{-\lambda t}\mathbf{n}\left(h(X_t), t < T_0\right) \\
&= h(x) + \mathbb{E}_x(e^{-\lambda T_0})\lambda^{-\alpha\theta}\mathbf{n}\left(h(X_1), 1 < T_0\right).
\end{aligned}$$

Thus $\lambda U_\lambda h(x) < \infty$ for all x if and only if $\mathbf{n}(X_1^\theta, 1 < T_0) < \infty$. From where we get that if (5) holds then $\widetilde{\mathbb{E}}_x\left(X_t^\theta\right) < \infty$ for all $x > 0$, and a.e. $t > 0$. The self-similarity implies that in this case the latter holds for all $x > 0$, and all $t > 0$. Which finishes the proof of Corollary 1.

3 Proof of Counterexample 1

A key tool in the establishment of (P1) and (P2) is the following version of Erickson's renewal theorem [11].

Lemma 5 (Erickson's renewal theorem [11]). *Let G be a non-arithmetic probability distribution function on $\mathbb{R}^+$ such that $1 - G$ is a regularly varying function at infinity with index $\gamma \in]1/2, 1]$, U the renewal measure associated to G, and $m(x) := \int_0^x (1 - G(u))\,\mathrm{d}u$, $x \geq 0$. Then*

(i) for any directly Riemann integrable function $g : \mathbb{R}^+ \to \mathbb{R}^+$,

$$\lim_{t\to\infty} m(t)\int_0^t g(t - y)U(\mathrm{d}y) = \frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)}\int_0^\infty g(y)\mathrm{d}y;$$

(ii) For any directly Riemann integrable function $g:\mathbb{R}\to\mathbb{R}^+$,

$$\lim_{t\to\infty} m(t)\int_{-\infty}^{\infty} g(y-t)U(\mathrm{d}y) = \frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)}\int_{-\infty}^{\infty} g(y)\mathrm{d}y.$$

The statement in (i) in Lemma 5 is the content of Erickson's renewal theorem 3 and so only (ii) requires a proof, which is postponed to the end of this Section. Next, we proceed to prove the claims in Counterexample 1. To that end, observe that the law $\mathbf{P}^{\natural}$ is that of a subordinator with infinite lifetime such that the tail probability $\mathbf{P}^{\natural}(\xi_1 > x)$ is a regularly varying function with index $\beta \in]1/2, 1[$. Let $U^{\natural}$ be the renewal measure of the subordinator with law $\mathbf{P}^{\natural}$, that is to say

$$U^{\natural}(\mathrm{d}y) = \int^{\infty} \mathbf{P}^{\natural}(\xi_t \in \mathrm{d}y)\mathrm{d}t, \qquad y \geq 0.$$

According to Bertoin and Doney [3], the measure $U^{\natural}$ is the renewal measure associated to the probability distribution function given by $F(\cdot) = \mathbf{P}^{\natural}(\xi_{\mathbf{e}} \leq \cdot)$, where $\mathbf{e}$ is a standard exponential r.v. independent of ξ under $\mathbf{P}^{\natural}$. Let $\mathbb{P}^{\natural}$ be the law of the 1-pssMp associated to $(\xi, \mathbf{P}^{\natural})$ via Lamperti's transformation. The measure $\mathbb{P}^{\natural}$ is such that

$$\mathbb{P}^{\natural} = X_t^q\, \mathbb{P} \quad \text{on } \mathcal{G}_t,\ t \geq 0.$$

It follows that the resolvents of $(X, \mathbb{P}^{\natural})$ and $(X, \mathbb{P})$ are related by

$$V_{\lambda}^{\natural} f(x) = \frac{V_{\lambda} f h_q(x)}{h_q(x)}, \qquad x \in]0,\infty[, \tag{13}$$

with $h_q(x) := x^q, x > 0$. Moreover, we have that for any function $f:\mathbb{R}^+\to\mathbb{R}^+$, such that the application $y \to f(e^y)e^y$, is directly Riemann integrable

$$\lim_{x\to 0+} m^{\natural}(\log(1/x))V_0^{\natural} f(x) = \frac{1}{\Gamma(1-\gamma)\Gamma(\gamma)}\int_0^{\infty} f(y)\mathrm{d}y. \tag{14}$$

Indeed, by applying Lamperti's representation and (ii) in Lemma 5 we get

$$\begin{aligned} m^{\natural}(\log(1/x))V_0^{\natural} f(x) &= m^{\natural}(\log(1/x))\, \mathbf{E}^{\natural}\left[\int_0^{\infty} f(xe^{\xi_t})xe^{\xi_t}\mathrm{d}t\right] \\ &= m^{\natural}(\log(1/x)) \int_{\mathbb{R}} f(e^{y-log(1/x)})e^{y-log(1/x)}U^{\natural}(\mathrm{d}y) \\ &\xrightarrow[x\to 0+]{} \frac{1}{\Gamma(1-\gamma)\Gamma(\gamma)}\int_{\mathbb{R}} f(e^y)e^y\mathrm{d}y. \end{aligned}$$

And by making a change of variables in the rightmost quantity we obtain (14). Moreover, repeating the arguments in [4], we prove that for every $f:]0,\infty[\to\mathbb{R}$, continuous and with compact support and $\lambda > 0$,

$$\lim_{x\to 0+} m^{\natural}(\log(1/x))V_{\lambda}^{\natural} f(x) = \frac{1}{\Gamma(1-\gamma)\Gamma(\gamma)}\int_0^{\infty} f(y)\, \mathbf{E}^{\natural}\left[\exp\left\{-\lambda y\int_0^{\infty} e^{-\xi_s}\mathrm{d}s\right\}\right]\mathrm{d}y. \tag{15}$$

Therefore, the claim in (P1) is a straightforward consequence of (13) and (15).

Besides, in [18] Lemma 4, we proved that in general the exponential functional I satisfies the equation in law

$$I \stackrel{Law}{=} Q + M\widetilde{I}, \qquad \text{with } (Q, M) := \left(\int_0^1 \exp\{\xi_s/\alpha\}1_{\{s<\zeta\}}\mathrm{d}s, e^{\alpha^{-1}\xi_1}1_{\{1<\zeta\}}\right), \text{ and } I \stackrel{Law}{=} \widetilde{I},$$

and the pair (Q, M) is independent of $\widetilde{I}$. Moreover, under the hypotheses (H2) in Lemma 4 in [18] we obtained, as a consequence of Goldie's Theorems 2.3 and 4.1 in [15], an estimate of the tail probability of I. A perusal of the proofs provided by Goldie to those theorems allows us to ensure that the arguments can be extended, using Erickson's renewal theorem instead of the classical renewal theorem, to prove the following Lemma.

Lemma 6. *Under the hypothesis of Counterexample 1 we have that*

$$\begin{aligned}
&\lim_{t\to\infty} m^{\natural}(\log(t))t^{q}\,\mathbb{P}_1(T_0 > t)\\
&= \frac{1}{\Gamma(1-\gamma)\Gamma(\gamma)}\,\mathbf{E}\left(\left(\int_0^{\infty}\exp\{\xi_s\}1_{\{s<\zeta\}}\mathrm{d}s\right)^{q} - \left(\int_1^{\infty}\exp\{\xi_s\}1_{\{s<\zeta\}}\mathrm{d}s\right)^{q}\right)\\
&= \frac{1}{\Gamma(1-\gamma)\Gamma(\gamma)}q\,\mathbf{E}(I^{q-1}) \in]0,\infty[.
\end{aligned}$$

Therefore, Lemma 6 and Karamata's Tauberian Theorem imply that the property (P2) is satisfied.

Remark 2. The expression of the value of the limit in Lemma 6 is a consequence of the proof of Lemma 2.

Finally, to prove that the condition (P3) is satisfied, we argue as in [21] page 556-557 to ensure that there exists a family of finite measures on $]0,\infty[$, say $(\mathbf{n}_\lambda, \lambda > 0)$, such that

$$\mathbf{n}_\lambda f = \lim_{x\to 0+}\frac{V_\lambda f(x)}{\mathbb{E}_x(1-e^{-T_0})},$$

for any f, continuous and with compact support on $]0,\infty[$, and for $\lambda > 0$. Moreover, the family $(\mathbf{n}_\lambda, \lambda > 0)$ satisfies the resolvent type equation, for $\lambda, \mu > 0$

$$\mathbf{n}_\lambda V_\mu f = \frac{\mathbf{n}_\mu f - \mathbf{n}_\lambda f}{\lambda - \mu},$$

for any f continuous and with bounded support on $]0,\infty[$. Thus, Theorem 6.9 in [14] and Theorem 4.7 in [7] imply that there exist a unique excursion measure $\mathbf{n}$ such that its λ-potential is equal to $\mathbf{n}_\lambda$,

$$\mathbf{n}\left(\int_0^{T_0} e^{-\lambda t}1_{\{X_t\in \mathrm{d}y\}}\mathrm{d}t\right) = \mathbf{n}_\lambda(\mathrm{d}y).$$

for any $\lambda > 0$. In fact, all the results of Vuolle-Apiala [21] are still valid if we replace the power function that gives the normalization in his hypotheses (Aa) and (Ab), by a regularly varying function. Therefore, Theorem 1.2 therein ensures that $\mathbf{n}(X_{0+} > 0) = 0$. According to Blumenthal's [6] theorem, associated to this excursion measure $\mathbf{n}$ there exists a unique recurrent extension of $(X, \mathbb{P})$ that leaves 0 continuously. Which finishes the proof of Counterexample 1. Now, we just have to prove that (ii) in Lemma 5 holds.

Proof of Lemma 5. The claim in (i) is Theorem 3 of Erickson [11] and that (ii) holds is a consequence of the latter. We next prove the result for step functions and the general case follows by a standard argument. Let $(a_k, k \in \mathbb{Z})$ be a sequence of positive real numbers such that $\sum_{k\in\mathbb{Z}} a_k < \infty$, and $h > 0$ a constant. A consequence of Theorem 1 of Erickson [11] is that for any $k \in \mathbb{N}$,

$$m(t+kh)\int_{\mathbb{R}} 1_{\{[kh,(k+1)h[\}}(y-t)U(\mathrm{d}y) \xrightarrow[t\to\infty]{} C_\gamma \int_{\mathbb{R}} 1_{\{[0,h[\}}(y)\mathrm{d}y = C_\gamma \int_{\mathbb{R}} 1_{\{[kh,(k+1)h[\}}(y)\mathrm{d}y,$$

with $C_\gamma = (\Gamma(\gamma)\Gamma(1-\gamma))^{-1}$, and uniformly in k. Thus, given that m is an increasing function, we get that

$$\begin{aligned} &m(t)\int_{\mathbb{R}} \sum_{k\in\mathbb{N}} a_k 1_{\{[kh,(k+1)h[\}}(y-t)U(\mathrm{d}y) \\ &\leq \sum_{k\in\mathbb{N}} a_k m(t+kh)\int_{\mathbb{R}} 1_{\{[kh,(k+1)h[\}}(y-t)U(\mathrm{d}y). \end{aligned}$$

Therefore,

$$\begin{aligned} \limsup_{t\to\infty} m(t)\int_{\mathbb{R}} \sum_{k\in\mathbb{N}} a_k 1_{\{[kh,(k+1)h[\}}(y-t)U(\mathrm{d}y) &\leq C_\gamma \sum_{k\in\mathbb{N}} a_k \int_{\mathbb{R}} 1_{\{[kh,(k+1)h[\}}(y)\mathrm{d}y \\ &\leq C_\gamma \int_{\mathbb{R}} \sum_{k\in\mathbb{N}} a_k 1_{\{[kh,(k+1)h[\}}(y)\mathrm{d}y. \end{aligned}$$

Owing that m is regularly varying with positive index the following limit

$$\lim_{t\to\infty} \frac{m(t)}{m(t+kh)} = 1,$$

holds uniformly in $k \in \mathbb{N}$. A standard application of Fatou's Theorem and an easy manipulation gives that

$$\liminf_{t\to\infty} m(t)\int_{\mathbb{R}} \sum_{k\in\mathbb{N}} a_k 1_{\{[kh,(k+1)h[\}}(y-t)U(\mathrm{d}y) \geq C_\gamma \int_{\mathbb{R}} \sum_{k\in\mathbb{N}} a_k 1_{\{[kh,(k+1)h[\}}(y)\mathrm{d}y.$$

Let g be the step function defined by

$$g(t) = \sum_{k\in\mathbb{Z}} a_k 1_{[kh,(k+1)h[}(t), \qquad t \in \mathbb{R}.$$

It follows from the arguments above that

$$\lim_{t\to\infty} m(t)\int_{\mathbb{R}} g(y-t)1_{\{y-t\geq 0\}}U(\mathrm{dy}) = C_\gamma \int_{\mathbb{R}} g(y)1_{\{y\geq 0\}}\mathrm{d}y.$$

Moreover, the assertion in (i) implies that

$$\lim_{t\to\infty} m(t)\int_{\mathbb{R}} g(y-t)1_{\{y-t<0\}}U(\mathrm{dy}) = \lim_{t\to\infty} m(t)\int_0^t g(-(t-y))U(\mathrm{dy}) = C_\gamma \int_0^\infty g(-y)\mathrm{d}y.$$

From where the result follows. $\square$

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References

[1] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[2] J. Bertoin and M.-E. Caballero. Entrance from 0+ for increasing semi-stable Markov processes. *Bernoulli*, 8(2):195–205, 2002.

[3] J. Bertoin and R. A. Doney. Cramér's estimate for Lévy processes. *Statist. Probab. Lett.*, 21(5):363–365, 1994.

[4] J. Bertoin and M. Yor. The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes. *Potential Anal.*, 17(4):389–400, 2002.

[5] J. Bertoin and M. Yor. Exponential functionals of Lévy processes. *Probab. Surv.*, 2:191–212 (electronic), 2005.

[6] R. M. Blumenthal. On construction of Markov processes. *Z. Wahrsch. Verw. Gebiete*, 63(4):433–444, 1983.

[7] R. M. Blumenthal. *Excursions of Markov processes*. Probability and its Applications. Birkhäuser Boston Inc., Boston, MA, 1992.

[8] P. Carmona, F. Petit, and M. Yor. On the distribution and asymptotic results for exponential functionals of Lévy processes. In *Exponential functionals and principal values related to Brownian motion*, Bibl. Rev. Mat. Iberoamericana, pages 73–130. Rev. Mat. Iberoamericana, Madrid, 1997.

[9] L. Chaumont. Excursion normalisée, méandre et pont pour les processus de Lévy stables. *Bull. Sci. Math.*, 121(5):377–403, 1997.

[10] C. Dellacherie, B. Maisonneuve, and P.-A. Meyer. *Probabilités et potentiel: Processus de Markov (fin). Compléments du Calcul Stochastique*. Hermann, Paris, 1992.

[11] K. B. Erickson. Strong renewal theorems with infinite mean. *Trans. Amer. Math. Soc.*, 151:263–291, 1970.

[12] P. Fitzsimmons. On the existence of recurrent extensions of self-similar Markov processes. Preprint, University of California San Diego, March 2006. http://math.ucsd.edu/ pfitz/preprints/selfsimilar.html.

[13] R. K. Getoor. *Excessive measures*. Probability and its Applications. Birkhäuser Boston Inc., Boston, MA, 1990.

[14] R. K. Getoor and M. J. Sharpe. Last exit times and additive functionals. *Ann. Probability*, 1:550–569, 1973.

[15] C. M. Goldie. Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.*, 1(1):126–166, 1991.

[16] J. Lamperti. Semi-stable Markov processes. I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 22:205–225, 1972.

[17] K. Maulik and B. Zwart. Tail asymptotics for exponential functionals of Lévy processes. *Stochastic Process. Appl.*, 116(2):156–177, 2006.

[18] V. Rivero. Recurrent extensions of self-similar Markov processes and Cramér's condition. *Bernoulli*, 11(3):471–509, 2005.

[19] L. C. G. Rogers. Wiener-Hopf factorization of diffusions and Lévy processes. *Proc. London Math. Soc. (3)*, 47(1):177–191, 1983.

[20] V. Vigon. *Simplifiez vos Lévy en titillant la factorisation de Wiener-Hopf.* PhD thesis, Université Louis Pasteur, 2002.

[21] J. Vuolle-Apiala. Itô excursion theory for self-similar Markov processes. *Ann. Probab.*, 22(2):546–565, 1994.