A NOTE ON CERTAIN SINGULAR TRANSFORMATIONS

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A note on certain singular transformations

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Abstract

In this paper we find the Jacobians of the transforms relating to matrix variate beta types I and II in the singular case. We also study the joint density function of the nonnull eigenvalues of matrix variate beta types I and II obtained from the singular distribution, and the approach proposed by Khatri (1970).

Key words: Random matrices, matrix variate beta, singular distribution, Hausdorff measure.
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1 Introduction

The nonsingular matrix variate beta type I and II distributions for doubly noncentral, noncentral and central cases are of great interest in multivariate analysis and in shape theory, see Olkin and Rubin (1964), Khatri (1970), Muirhead (1982), Cadet (1996), Gupta and Nagar (2000), Díaz-García and Gutiérrez-Jáimez (2001), Chikuse (1980) and Goodall and Mardia (1992).

Different relations have been described between matrix variate beta type I and II distributions, according to whether they are central, noncentral or doubly

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noncentral, see Srivastava and Khatri (1979) and Gupta and Nagar (2000), among many other authors. Moreover, as there exist various definitions for each of these, and these definitions, in turn contain a subdivision, different transforms from one type of beta distribution to another have been studied, see Greenacre (1973), Roux (1975) and Gupta and Nagar (2000).

The study of singular distributions has been limited to a few recent papers, and even fewer studies have been made of singular matrix variate beta distributions, see Khatri (1970), Uhlig (1994) and Díaz-García and Gutiérrez (1997). Thus, transforms from one type of beta distribution to another have been little studied. This is mainly because in the singular case such distributions exist with respect to the Hausdorff measure, and not with respect to the Lebesgue measure, as is the case with nonsingular distributions. As observed by Billingsley (1986), the problem in determining the corresponding Jacobians with respect to the Hausdorff measure is not a simple one. Interest in this question has increased, with the appearance of studies in which these singular distributions, and others, play an important role, both from the theoretical standpoint and from an applied one, see Ratnarajah and Vaillancourt (2005) and Ratnarajah and Vaillancourt (2005).

In the present study, we examine the Jacobians relating to the central singular matrix variate beta types I and II. Moreover, we study the equivalence between the joint distributions of the nonnull eigenvalues of singular matrix variate beta types I and II, obtained via singular distributions and applying the approach proposed by Khatri (1970).

2 Jacobianos

For the singular case, note that if A and B have a central Pseudo-Wishart and Wishart distribution, respectively, i.e. independent $A \sim \mathcal{PW}_m(r, I)$ and $B \sim \mathcal{W}_m(s, I)$, then the singular matrix beta type I U is defined as

$$U = (A + B)^{-1/2} A((A + B)^{-1/2})' \tag{1}$$

where $C^{1/2}(C^{1/2})' = C$ is a reasonable nonsingular factorization of C, see Gupta and Nagar (2000), Srivastava and Khatri (1979) and Muirhead (1982); and its density function is given by (see Díaz-García and Gutiérrez (1997))

$$dF_U(U) = c|L|^{(r-m-1)/2}|I_m - U|^{(s-m-1)/2}(dU), \quad 0 \le U < I_m \tag{2}$$

denoting as $U \sim \mathcal{B}I_m(q, r/2, s/2)$, $s \ge m$; where $U = H_1 L H_1'$, with $H_1 \in \mathcal{V}_{q,m}$; $\mathcal{V}_{q,m} = \{H_1 \in \Re^{m \times q} | H_1' H_1 = I_q\}$ denotes the Stiefel manifold; $L =$

$\mathrm{diag}(l_1, \dots, l_q)$, $1 > l_1 > \cdots > l_q > 0$; $q = m$ (nonsingular case) or $q = r < m$ (singular case);

$$c = \frac{\pi^{(-mr+rq)/2}\Gamma_m[(r+s)/2]}{\Gamma_q[r/2]\Gamma_m[s/2]} \tag{3}$$

and (dU) denotes the Hausdorff measure on $\mathcal{S}_m^+(q)$, the $(mq - q(q-1)/2)$-dimensional manifold of rank-q positive semidefinite $m \times m$ matrices U with q distinct nonnull eigenvalues, given by (see Uhlig (1994) and Díaz-García and Gutiérrez (1997))

$$(dU) = 2^{-q} \prod_{i=1}^{q} l_i^{m-q} \prod_{i<j}(l_i - l_j) \left(\bigwedge_{i=1}^{q} dl_i \right) \wedge (H_1' dH_1), \tag{4}$$

where $(H_1' dH_1)$ denotes the invariant measure on $\mathcal{V}_{q,m}$ and where finality, $\Gamma_m[a]$ denotes the multivariate gamma function and is defined as

$$\Gamma_m[a] = \int_{R>0} \mathrm{etr}(-R)|R|^{a-(m+1)/2}(dR),$$

$\mathrm{Re}(a) > (m-1)/2$ and $\mathrm{etr}(\cdot) \equiv \exp(\mathrm{tr}(\cdot))$.

In analogous fashion, the singular matrix variate beta type II is defined as

$$F = B^{-1/2}A(B^{-1/2})'. \tag{5}$$

This fact is denoted by $F \sim \mathcal{B}II_m(q, r/2, s/2)$. Moreover, if $F = H_1 G H_1'$, with $H_1 \in \mathcal{V}_{q,m}$ and $G = \mathrm{diag}(g_1, \dots, g_q)$; $g_1 > \cdots > g_q > 0$, in this case the central matrix variate beta type II density is given by (see Díaz-García and Gutiérrez (1997))

$$dG_F(F) = c|G|^{(r-m-1)/2}|I + F|^{-(r+s)/2}(dF), \quad F \geq 0. \tag{6}$$

where c is given by (3) and (dF) is given in analogous form to (4).

In the non-singular case, we have the following transforms to obtain the density of F with respect to the distribution of U:

$$F = \begin{cases} (I-U)^{-1} - I \\ (U^{-1} - I)^{-1} \\ (I-U)^{-1}U \\ U(I-U)^{-1} \\ U^{1/2}(I-U)^{-1}U^{1/2} \\ (I-U)^{-1/2}U(I-U)^{-1/2} \end{cases} \tag{7}$$

This is obtained straightforwardly from the first expression, taking into account that F, U and $(I-U)$ are symmetrical matrices. Similarly, the transforms to obtain matrix U from matrix F are:

$$U = \begin{cases} I - (I+F)^{-1} \\ (I + F^{-1})^{-1} \\ (I+F)^{-1}F \\ F(I+F)^{-1} \\ F^{1/2}(I+F)^{-1}F^{1/2} \\ (I+F)^{-1/2}F(I+F)^{-1/2} \end{cases} \tag{8}$$

In the singular case, not all the transforms are applicable, as $U \geq 0$ and $F \geq 0$. Thus, we obtain the following result.

Theorem 1 ***i)*** *Assume* $F \sim \mathcal{B}II_m(q, r/2, s/2)$ *and let* $U = (I+F)^{-1}F$. *Then* $U \sim \mathcal{B}I_m(q, r/2, s/2)$.
ii) *Assume* $U \sim \mathcal{B}I_m(q, r/2, s/2)$ *and let* $F = (I-U)^{-1}U$. *Then* $F \sim \mathcal{B}II_m(q, r/2, s/2)$.

To prove this result, we must determine the Jacobians, with respect to the Hausdorff measure, of the singular transforms $U = (I+F)^{-1}F$ and $F = (I-U)^{-1}U$, where $F,\ U \in \mathcal{S}_m^+(r)$.

Lemma 2 *Let* $F,\ U \in \mathcal{S}_m^+(r)$. *Then*

i) *if* $U = (I+F)^{-1}F$

$$(dU) = |I_m + F|^{(m+1-r)/2}|I_r + Q|^{-(m+1-r)/2}(dF),$$

where $F = H_1 Q H_1'$ is the nonsingular part of the spectral decomposition of the matrix F, with $H_1 \in \mathcal{V}_{r,m}$ and $Q = \operatorname{diag}(f_1, \dots, f_r)$, $f_1 > \cdots > f_r > 0$.

ii) *if $F = (I - U)^{-1}U$*

$$(dF) = |I_m - U|^{(m+1-r)/2} |I_r - L|^{-(m+1-r)/2} (dU),$$

with $U = H_1 L H_1'$, $H_1 \in \mathcal{V}_{r,m}$ and $L = \operatorname{diag}(u_1, \dots, u_r)$, $1 > u_1 > \cdots > u_r > 0$.

Proof. To prove this result, we proceed as in Díaz-García and Gutiérrez (1997).

Let h and t be the density functions of F and U, respectively.

i) Let $U = (I + F)^{-1}F$. Then, by the change of variable theorem

$$\begin{aligned} dH_F(F) &= t_U((I+F)^{-1}F)(d(I+F)^{-1}F) \\ &= t_U((I+F)^{-1}F)|J|(dF), \end{aligned} \tag{9}$$

where $|J|$ denotes the Jacobian. Now considerer the complete spectral decomposition

$$U = H \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix} H',$$

with $H = (H_1 H_2)$ orthogonal matrix, where $H_2 \in \mathcal{V}_{m-r,m}$ a function of H_1. Thus

$$\begin{aligned} \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix} &= H'UH = H'(I+F)^{-1}FH = H'(I+F)^{-1}HH'FH \\ &= (HH' + HFH')^{-1}H'FH = \left(I + \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}\right)^{-1} \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} \\ &= \left(I + \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}\right)^{+} \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (I+Q^{-1})^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

where C^+ denotes the Moore-Penrose inverse of C. From this, $L = (I+Q^{-1})^{-1}$. Thus by (2) and (9) we have

$$\begin{aligned} dH_F(F) &= c|(I+Q^{-1})^{-1}|^{(r-m-1)/2}|I - (I+F)^{-1}F|^{(s-m-1)/2}|J|(dF) \\ &= c|Q|^{(r-m-1)/2}|I+F|^{-(r+s)/2}|I+F|^{-(m+1-r)/2} \\ &\qquad |I+Q|^{-(r-m-1)/2}|J|(dF). \end{aligned}$$

But by (6)

$$dG_F(F) = c|Q|^{(r-m-1)/2}|I + F|^{-(r+s)/2}(dF).$$

Thus

$$1 = \frac{dG_F(F)}{dH_F(F)} = |I + F|^{-(m+1-r)/2}|I + Q|^{-(r-m-1)/2}|J|(dF).$$

from which the required result is obtained.

ii) The proof is parallel to that given in i), simply by proceeding as in i), $Q = (I - L)^{-1}LQ = (I - L)^{-1}L$. □

Now, the proof of Theorem 1 is immediate by applying Lemma 2.

3 Other results

An alternative definition of the matrix variate beta type I distribution was proposed by Khatri (1970), see also Srivastava and Khatri (1979, pp. 94-95), as follows. Assume that $B \sim \mathcal{W}_m(s, I)$ and that $A = Y'Y$ where $Y \sim \mathcal{N}_{r\times m}(0, I_r \otimes I_m)$, $m > r$, independently of B. Then $U_1 = Y(Y'Y+B)^{-1}Y' = Y(A+B)^{-1}Y'$. Moreover, $U_1 \sim \mathcal{B}I_r(m/2, (s+r-m)/2)$. However, observe that in the central case its properties and associated distributions can be obtained from Definition (1) by replacing m by r, r by m and s by $s+r-m$, i.e., by making the substitutions

$$m \to r, \quad r \to m, \quad s \to s + r - m, \tag{10}$$

see Srivastava and Khatri (1979, p. 96) or Muirhead (1982, eq. (7), p. 455). Note that in this definition, we are considering the singular case, as $r < m$; however, in this case the density is found with respect to the Lebesgue measure (dU_1), defined with respect to the space of dimension r of the positive defined matrices $U_1 : r \times r$. Without a doubt, the matrices U and U_1 are not the same, indeed they are not even of the same order, but where they do coincide is in their eigenvalues, and thus the density of the eigenvalues under both matrices must be the same. An analogous definition has been proposed by James (1964) for the case of the matrix variate beta type II, see also Muirhead (1982).

Recently, the noncentral singular and doubly noncentral singular cases of the matrix variate beta types I and II have been studied by Díaz-García and Gutiérrez-Jáimez (2006). Thus, we can now compare the corresponding joint densities of the eigenvalues obtained via the singular densities and via the ideas of Khatri (1970) and James (1964) for matrix variate beta types I and II, respectively.

To determine such an equivalence (equality) between the joint densities of the eigenvalues of the matrix variate beta type I and II in the noncentral case, see Khatri (1970), James (1964) and Díaz-García and Gutiérrez-Jáimez (2006); we need only consider the following result, as the only difference between these densities is in the constant, and specifically, in the quotient of the multivariate gamma functions.

Lemma 3 *Assume that $m < k < n$, then*

i) $\Gamma_m[n/2] = \pi^{k(m-k)/2}\Gamma_m[(n-m+k)/2]\Gamma_{m-k}[n/2]$.

ii) $\Gamma_m[n/2] = \pi^{k(m-k)/2}\Gamma_k[n/2]\Gamma_{m-k}[(n-k)/2]$.

Proof.

i) Given in Muirhead (1982, p. 95).
ii) Note, simply, that

$$\begin{aligned}
\Gamma_m[n/2] &= \pi^{m(m-1)/4}\prod_{i=1}^{m}\Gamma[(n-i+1)/2] \\
&= \pi^{k(k-1)/4}\left(\prod_{i=1}^{k}\Gamma[(n-i+1)/2]\right)\pi^{(m-k)(m-k-1)/4} \\
&\qquad \left(\prod_{i=k+1}^{m}\Gamma[(n-i+1)/2]\right)\pi^{k(m-k)/2} \\
&= \pi^{k(k-1)/4}\left(\prod_{i=1}^{k}\Gamma[(n-i+1)/2]\right)\pi^{(m-k)(m-k-1)/4} \\
&\qquad \left(\prod_{j=1}^{m-k}\Gamma[(n-k-j+1)/2]\right)\pi^{k(m-k)/2} \\
&= \Gamma_m[n/2] = \pi^{k(m-k)/2}\Gamma_k[n/2]\Gamma_{m-k}[(n-k)/2] \qquad \square
\end{aligned}$$

4 Conclusions

In this study we establish the transforms and the corresponding Jacobians with respect to the Hausdorff measure relating the matrix variate beta type I and II distributions.

The equality between the joint densities of the eigenvalues of the matrix variate beta types I and II obtained via the theory of singular distributions and the approaches proposed by Khatri (1970) and James (1964) is established.

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