

# COMPLEX ZONAL POLYNOMIALS OF SECOND ORDER

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# Complex Zonal Polynomials of Second Order

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## Abstract

A formula for complex zonal polynomials of second order is derived by solving a particular partial differential equation.

*Key words:* Laplace-Beltrami operator, zonal polynomials, Hermitian matrix, Hypergeometric differential equation.

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## 1 Introduction

Recently Díaz-García and Caro (2005) computed the zonal polynomials of positive definite hermitian matrix by the use of the Laplace-Beltrami operator. In

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the same way as the real case, the zonal polynomials of positive definite and a semi-definite positive symmetric matrices were calculated by James (1968) and Díaz-García and Caro (2004), respectively. It is known that general formulas for complex and real zonal polynomials are not available, and solutions for the general partial differential equations for both polynomials are also unsolved. However, the differential equation for the zonal polynomials of positive definite symmetric matrix argument of the second order was solved by James (1968).

In this paper we reduce the partial differential equation for zonal polynomials of positive definite hermitian matrix of second order to a Hypergeometric differential equation type, which is analogous to the results obtained by James (1968) in the real case. By solving the ordinary differential equation, we get an explicit formula for the corresponding zonal polynomials, see Section 2.

## 2 Complex Zonal Polynomials of Second Order.

Let the partition  $\kappa = (k_1, \dots, k_m)$  of  $k$  be a decreasing sequence of nonnegative integers. In Díaz-García and Caro (2005) it was proved that the zonal polynomials  $\tilde{C}_\kappa(Y)$  of an  $m \times m$  positive definite hermitian matrix  $Y$  satisfy the partial differential equation

$$\sum_{i=1}^m y_i^2 \frac{\partial^2}{\partial y_i^2} \tilde{C}_\kappa(Y) + 2 \sum_{i=1}^m \sum_{j=1(j \neq i)}^m y_i^2 (y_i - y_j)^{-1} \frac{\partial}{\partial y_i} \tilde{C}_\kappa(Y) = [\tilde{\rho}_\kappa + k(2m - 1)] \tilde{C}_\kappa(Y), \quad (1)$$

where

$$\tilde{\rho}_\kappa = \sum_{i=1}^m k_i(k_i - 2i),$$

$y_1, \dots, y_m$  are the eigenvalues of the matrix  $Y$  and  $\kappa = (k_1, \dots, k_m)$  is a partition of  $k$ .

When  $m = 2$  in (1) we get the partial differential equation

$$y_1^2 \frac{\partial^2 \tilde{C}}{\partial y_1^2} + y_2^2 \frac{\partial^2 \tilde{C}}{\partial y_2^2} + 2y_1^2 (y_1 - y_2)^{-1} \frac{\partial \tilde{C}}{\partial y_1} - 2y_2^2 (y_1 - y_2)^{-1} \frac{\partial \tilde{C}}{\partial y_2} - [k_1(k_1 + 1) + k_2(k_2 - 1)] \tilde{C} = 0, \quad (2)$$

where we denote  $\tilde{C}_\kappa(Y)$  as  $\tilde{C}$ .

Let us replace  $u = y_1 + y_2$  and  $v = y_1 y_2$  in (2), then we find

$$(u^2 - 2v) \frac{\partial^2 \tilde{C}}{\partial u^2} + 2v^2 \frac{\partial^2 \tilde{C}}{\partial v^2} + 2uv \frac{\partial^2 \tilde{C}}{\partial u \partial v} + 2u \frac{\partial \tilde{C}}{\partial u} + 2v \frac{\partial \tilde{C}}{\partial v} - [k_1(k_1 + 1) + k_2(k_2 - 1)] \tilde{C} = 0.$$

Substituting  $z = \frac{u}{2\sqrt{v}}$  and  $t = \sqrt{v}$  we obtain

$$(1 - z^2) \frac{\partial^2 \tilde{C}}{\partial z^2} - t^2 \frac{\partial^2 \tilde{C}}{\partial t^2} - 3z \frac{\partial \tilde{C}}{\partial z} - t \frac{\partial \tilde{C}}{\partial t} + 2[k_1(k_1 + 1) + k_2(k_2 - 1)] \tilde{C} = 0.$$

It is easy to see that the last equation is homogeneous in  $t$ . Thus by taking

$$\tilde{C} = t^k f(z),$$

the next ordinary differential equation results

$$(1 - z^2) \frac{d^2 f}{dz^2} - 3z \frac{df}{dz} + [(k_1 - k_2)(k_1 - k_2 + 2)] f = 0.$$

Taking  $w = (1 - z)/2$  as the independent variable, the differential equation becomes

$$w(1 - w) \frac{d^2 f}{dw^2} + \frac{3}{2}(1 - 2w) \frac{df}{dw} + \rho(\rho + 2)f = 0, \quad (3)$$

with  $\rho = k_1 - k_2$ , a non negative integer, according to the definition of the partition  $\kappa$ .

Comparing with the general hypergeometric equation

$$w(1 - w) \frac{d^2 f}{dw^2} + [c - (a + b + 1)w] \frac{df}{dw} - abf = 0, \quad (4)$$

we see that the complex zonal polynomials are involved in the solution of an hypergeometric differential equation of parameters  $a = -\rho$ ,  $b = \rho + 2$  and  $c = 3/2$ .

Following Erdélyi *et al.* (1981), we know that a solution of (4) which is regular at  $w = 0$  is given by

$$f(w) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} w^n = {}_2F_1(a, b; c; w),$$

where  ${}_2F_1(a, b; c; w)$  is the classical hypergeometric function, which we will now denote as  $F(a, b; c; w)$ .

Thus a solution of (3) is

$$f(z) = F\left(-\rho, \rho + 2; \frac{3}{2}; \frac{1-z}{2}\right),$$

Let us refined the above solution by applying properties of the hypergeometric functions. From Erdélyi *et al.* (1981, Section 2.11, p.111), equation (2), we see that

$$F\left(2d, 2e; d + e + \frac{1}{2}; t\right) = F\left(d, e; d + e + \frac{1}{2}; 4t(1-t)\right),$$

then

$$\begin{aligned} f(z) &= F\left(-\rho, \rho + 2; \frac{3}{2}; \frac{1-z}{2}\right) \\ &= F\left(-\frac{\rho}{2}, \frac{\rho}{2} + 1; \frac{3}{2}; 1-z^2\right) \end{aligned} \quad (5)$$

By Erdélyi *et al.* (1981, Section 2.10, p.108), equation (1),

$$\begin{aligned} F(a, b; c; t) &= A_1 F(a, b; a + b - c + 1; 1-t) \\ &\quad + A_2 (1-t)^{c-a-b} F(c-a, c-b; c-a-b+1; 1-t), \end{aligned}$$

where

$$A_1 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{and} \quad A_2 = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}.$$

Then (5) can be written as follows

$$\begin{aligned} F\left(-\frac{\rho}{2}, \frac{\rho}{2} + 1; \frac{3}{2}; 1-z^2\right) &= A_1 F\left(-\frac{\rho}{2}, \frac{\rho}{2} + 1; \frac{1}{2}; z^2\right) \\ &\quad + A_2 z F\left(\frac{3}{2} + \frac{\rho}{2}, \frac{1}{2} - \frac{\rho}{2}; \frac{3}{2}; z^2\right), \end{aligned}$$

where

$$A_1 = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2} + \frac{\rho}{2})\Gamma(\frac{1}{2} - \frac{\rho}{2})} \quad \text{and} \quad A_2 = \frac{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2})}{\Gamma(-\frac{\rho}{2})\Gamma(\frac{\rho}{2} + 1)}.$$

If  $\rho = k_1 - k_2 = 2n$ ,  $n = 0, 1, 2, \dots$ , and using the fact that  $\Gamma(\frac{1}{2} + z)\Gamma(\frac{1}{2} - z) = \pi \sec(z\pi)$  and  $\Gamma(z)\Gamma(1-z) = \pi \csc(z\pi)$  which implies  $A_1 = \frac{(-1)^n}{2n+1}$  and  $A_2 = 0$ , respectively, then the even complex zonal polynomials of second order are given by

$$\tilde{C}_{(k_1, k_2)}(Y) / \tilde{C}_{(k_1, k_2)}(I_2) = (y_1 y_2)^{\frac{k}{2}} \frac{(-1)^n}{2n+1} F\left(-n, n+1; \frac{1}{2}; \frac{(y_1 + y_2)^2}{4y_1 y_2}\right)$$

If  $\rho = k_1 - k_2 = 2n + 1$ ,  $n = 0, 1, 2, \dots$  and using the same properties for the simplification of A's, then we find the odd complex zonal polynomials of

second order

$$\tilde{C}_{(k_1, k_2)}(Y)/\tilde{C}_{(k_1, k_2)}(I_2) = (y_1 y_2)^{\frac{k}{2}} (-1)^n \frac{y_1 + y_2}{2\sqrt{y_1 y_2}} F\left(n + 2, -n; \frac{3}{2}; \frac{(y_1 + y_2)^2}{4y_1 y_2}\right),$$

where  $|z^2| < 1$  and

$$\tilde{C}_{(k_1, k_2)}(I_2) = k! \frac{(k_1 - k_2 + 1)^2}{(k_1 + 1)!(k_2)!},$$

see Khatri (1970).

These formulas agree with the expressions found for the complex zonal polynomials in Díaz-García and Caro (2005) until the 10th degree and the general conjectures established there.

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