AN EXAMPLE OF A COQUECIGRUE EMBEDDED IN R^4
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ABSTRACT. In this note I will describe a simple, but non-trivial, explicit example of a solution to the "coquecigrue" problem of Loday. More precisely, a Leibniz algebra structure will be defined on $\mathbb{R}^4$, and therefrom a digroup contained in this space, whose tangent space at the unit element inherits the original Leibniz algebra structure. This illustrates both, the properties that might be expected from the coquecigrues, and the difficulties still remaining in the full understanding of the problem.

Introduction.

The "coquecigrue" problem of Loday may be thought of as the analog in the context of Leibniz algebras to Lie's third problem. That is to say, given a Leibniz algebra, one wishes to construct a manifold possessing also an algebraic structure, in such a way that its tangent space at some distinguished point inherits the structure of the given Leibniz algebra; such an object was called by Loday a "coquecigrue." (Recall that a Leibniz algebra is a generalization of a Lie algebra, where the bracket is bilinear and satisfies a form of the Jacobi identity, but is not assumed to be anticommutative, see [L1].)

Although this problem is still unsolved in full generality, perhaps the best step towards its solution was taken by M. Kinyon in [K]. There he showed that, for a significant class of Leibniz algebras, this can be accomplished through the use of

Lie digroups. Essentially, his strategy was to consider the action of conjugation in the digroup (which yields a structure sometimes referred to as a "rack"), and then, by differentiating this action, a Leibniz bracket can be defined in the tangent space at the identity of the digroup.

Rather independently, but along the same line of ideas, in [O] I described some more restricted examples of digroups, determined by a linear functional φ on a Leibniz algebra, and discussed in what sense they can be regarded as explicit solutions to the coquecigrue problem. The interesting point here is the following: As showed in [K], digroups have a natural splitting as the product of the subgroup of inverses by the semigroup of bar units; for the digroups considered in [O] the subgroup of inverses is a (real) general linear group, and moreover, these digroups have the geometric structure of a flat trivial principal bundle. But surprisingly enough—and contrary to what might be thought of at first, the structure group is not the group of inverses: this is rather the base manifold of the bundle, while the structure group is the vector space that models the affine space of bar units.

In this note, by "toying" with the quaternionic structure of $\mathbb{R}^4$, I will construct a coquecigrue of a different type, embedded in $\mathbb{R}^4$. This coquecigrue has obvious relations to those in [O] (*e.g.*, it still has the structure of a bundle with connection), but it presents some differences, so I believe this provides an additional nice illustration of both, the properties that might be expected from the coquecigrues, and the difficulties still remaining in the full understanding of the problem.

I. A non-trivial dialgebra structure on $\mathbb{R}^4$.

Let us begin by recalling some basic facts.

A disemigroup is a set G together with two associative operations, $\dashv, \vdash$, satisfying the relations

$$x \dashv (y \dashv z) = x \dashv (y \vdash z)$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z)$$
$$(x \dashv y) \vdash z = (x \vdash y) \vdash z.$$

A bar unit in a disemigroup is an element, e, satisfying

$$e \vdash x = x = x \dashv e; \quad \forall x \in G.$$

Bar units may or may not exist in a disemigroup; but a bar unit is said to be non-trivial if the corresponding relations from the pointer side (*i.e.*, $e \dashv x = x = x \vdash e$) do not in general hold; for in that case it is well known—and easy to see—that the two operations coincide, and the bar unit is unique. The set of bar units will be denoted by E.

If the disemigroup does possess a non-trivial bar unit, say 1, an element $x \in G$ is (pointer) invertible (relative to the fixed bar unit) if there exists $y \in G$ such that $y \dashv x = x \vdash y = 1$. Such an inverse is necessarily unique, but several elements may have the same inverse. A digroup is a disemigroup in which every element has an inverse. Given a disemigroup G with a bar unit 1, we shall denote the set of elements of G that are inverses of some element by J. It is also well known that both operations restricted to J coincide, and J thus becomes a group. As usual, when the sets are manifolds and the operations involved are differentiable, we speak of Lie digroups, etc., and hereafter we place us in this setting.

A dialgebra is a vector space, V, which is also a disemigroup, with the disemigroup operations bilinear. If the dialgebra posseses a non-trivial bar unit 1, then the set of bar units, sometimes called the *halo* of the dialgebra, is an affine subspace of the dialgebra, modelled after the subspace $N = N_\vdash \cap {}_\dashv N$, where

$$N_\vdash = \{x | x \vdash y = 0 \quad \forall y\} ; \quad {}_\dashv N = \{x | y \dashv x = 0 \quad \forall y\},$$

and passing through 1.

Finally, to each dialgebra one associates a Leibniz algebra, by defining the bracket as

$$[x, y] = x \vdash y - y \dashv x$$

(we follow here the convention of [K] of using *left* Leibniz algebras, as this seems better suited to the analysis of the coquecigrue problem).

We are now ready to describe our construction: Let $V = \mathbb{R}^4$, with the standard basis $1, i, j, k$ as in the quaternions. Let $V = \mathbb{C} \oplus N$, with $N = \langle j, k \rangle$ (the linear span of j and k); essentially we want to modify the quaternionic operations, so that 1 will be a bar unit, but **not** a unit in the resulting dialgebra, and the halo of this dialgebra will be $E = 1 + N$.

Now, in view of the bilinearity of the operations, obviously it suffices to determine the operations on the basis elements, and they imply

$$1 \vdash x = x \dashv 1 = x \; ; \quad j \vdash x = x \dashv j = k \vdash x = x \dashv k = 0 \; ; \; \forall x \in V.$$

If moreover—as implied by our notation—we also want $\mathbb{C}$ to be embedded in V, we need also define

$$i \vdash 1 = 1 \dashv i = i \; ; \quad i \vdash i = i \dashv i = -1,$$

and hence the above requests determine most of the structure constants of the dialgebra and we are left with the problem of determining only the following products by i:

$$i \vdash j \; ; \; i \vdash k \; ; \quad j \dashv i \; ; \; k \dashv i,$$

and writing the associativity conditions in terms of the basis then yields a system of algebraic equations for the solutions of these relations. Unfortunately, this system is somewhat 'messy', and although it might be of interest to classify all the possible dialgebra structures so obtained, I will not go into this, and will only analyse the following simple case:

$$i \vdash j = k \; ; \; i \vdash k = -j \; ; \quad j \dashv i = -k \; ; \; k \dashv i = j.$$

The main reason why this structure is of interest is that it has a nice and easy geometric interpretation: $\vdash$ multiplication on the left by i acts as a rotation on the plane N by $\pi/2$ in the positive direction—just as in the quaternions—while $\dashv$ multiplication on the right acts as a rotation by the same amount, but in the *opposite* direction (and notice that the products involved are not the same). More generally, it follows easily that $\vdash$ multiplication on the left by $e^{i\theta} \in \mathbb{C}$ acts as a rotation in the positive sense by an angle θ, while $\dashv$ multiplication on the right acts as a rotation by the same amount but in the opposite direction.

Actually, the products can be described as follows: let $x = u + a$, $y = v + b$, where $u, v \in \mathbb{C}$, and $a, b \in N$; then

$$(1) \qquad x \vdash y = uv + u \vdash y = uv + ub \; ; \quad x \dashv y = uv + x \dashv v = uv + \bar{v}a,$$

where the bar denotes complex conjugate, and with the product that of the quaternions. (However, one must keep in mind that the products are not in general the quaternionic ones; *e.g.*, $b \vdash u = 0$.)

It also follows easily from these considerations that this is indeed a dialgebra structure: associativity is clear for the $\mathbb{C}$ and N summands separately (being the standard structure in the former case and trivial in the latter); moreover, N is invariant under both products, and whenever an element of N appears on the bar side, the product is zero. Thus, only a few cases need to be checked (such as $(i \vdash i) \vdash j$, or $(i \vdash k) \dashv i$), and the computations are straightforward. Also this shows that the dialgebra is not trivial, and neither is the associated Leibniz algebra, since for instance, $[i, k] = 2ik = -2j$, while $[k, i] = 0$. Incidentally, the only other non-zero bracket of basis elements is $[i, j] = 2k$.

We collect these remarks:

Proposition 1. *The products (1) give $V = \mathbb{R}^4$ the structure of a dialgebra, with 1 as a bar unit. The associated Leibniz algebra is not a Lie algebra.*

II. The digroup of invertible elements and its geometry.

To describe the invertible elements, let x and y be as in (1), and suppose we want y to be the inverse of x; then we must have

$$x \vdash y = uv + ub = 1$$

Since $ub \in N$, necessarily $v = u^{-1}$ and $b = 0$. Clearly, y so defined also satisfies $y \dashv x = 1$, and so $y = x^{-1}$. Let us denote by G this digroup of invertible elements.

Equally clearly, we have that $G = V \setminus N$, so it is an open subset and therefore its tangent space at 1 is $T_1 G \cong \mathbb{R}^4$, and it also follows immediately from these considerations that the subgroup of inverses is precisely $J = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Finally, it is also plain that, as a set and even as a manifold, $G = E \times J \cong J \times E$.

Remark. As mentioned, the last remark shows that the structure of G is what should be, *cf.* [K]; but incidentally, this also shows that this digroup is **not** of the same type as those discussed in [O], since for those the group of inverses is a real general linear group. (It is also not a complex version of that construction, since some of the operations involve taking complex conjugates, but I shall not dwell more on these matters, since they are not relevant to what follows.)

Next, we want to compute the adjoint action of G on $T_1 G$, which also gives the rack structure associated to the digroup. By the linearity of the operations we need

only compute the conjugation for $x \in G$ and $y \in V$:

$$(2) \quad x \vdash y \dashv x^{-1} = (u + a) \vdash (v + b) \dashv u^{-1} = uvu^{-1} + u \vdash b \dashv u^{-1} = v + \frac{u^2}{|u|^2} b.$$

Notice how this acts on y: it fixes the complex part v, and rotates the direct complement b by *twice* the angle of the vector u.

Finally, by differentiating this formula the Leibniz algebra structure in the tangent space is obtained. This is of course a quite standard argument, but let us actually carry it over: Thus, consider a curve $x(t)$ in G such that $x(0) = 1$; then we have to compute

$$\frac{d}{dt}|_{t=0}(x \vdash y \dashv x^{-1}) \; ; \; y \in V.$$

Since the operations are bilinear, it is plain that

$$\frac{d}{dt}|_{t=0}(x \vdash y \dashv x^{-1}) = x'(0) \vdash y \dashv 1 + 1 \vdash y \dashv (x^{-1})'(0)$$
$$= x'(0) \vdash y + y \dashv (x^{-1})'(0),$$

and so we need to compute $(x^{-1})'(0)$.

We know that $(x^{-1})'(t) = (u^{-1})'(t) = -u^{-2}(t)u'(t)$, so $(x^{-1})'(0) = -u'(0)$. Notice that this is **not** $-x'(0)$; but we also know that $x \vdash y = u \vdash y$, so in the end we are left with

$$\frac{d}{dt}|_{t=0}(x \vdash y \dashv x^{-1}) = u'(0) \vdash y - y \dashv u'(0) = [u'(0), y] = [x'(0), y].$$

Thus, although the argument is slightly trickier than for Lie groups, the Leibniz algebra structure in the tangent space is indeed that of V, as should be.

Let us summarize what we have up until now:

Proposition 2. *The set of invertible elements in V is $G = \mathbb{C}^* \times N \cong J \times E$, and its tangent space at 1 naturally inherits the Leibniz algebra structure of V.*

Proposition 2 tells us that the digroup G can be regarded as a solution to the coquecigrue problem.

Now, we know that as a manifold $G = J \times E$; the point is that it possesses a differential geometric structure determined by the set of units (*cf.*, [O]). More

precisely, we want to regard G as a fiber bundle over J with fiber E (or, what is the same, N), and we want to determine what kind of bundle structure can we get from the dialgebra operations. To this end, we notice that any bar unit can be written in the form $x \dashv x^{-1}$; since $x \dashv x^{-1} = 1 + \bar{u}^{-1}a$, this always has a component in $\mathbb{C}$, and thus gives a direction transversal to N. We claim that this can be used to define a connection on G.

Indeed, let $x \in G$ be fixed, and $y \in V \cong T_xG$, and consider $y \vdash x \dashv x^{-1}$; this operation is $\mathbb{R}$-linear in y, so its image is a subspace of T_xG, and indeed a direct computation gives

$$y \vdash x \dashv x^{-1} = y \vdash \left(1 + \frac{a}{\bar{u}}\right) = v + v\left(\frac{a}{\bar{u}}\right);$$

since $a/\bar{u}$ is fixed, we actually get at each point x a real 2-dimensional subspace H_x, complementary to N in T_xG. Therefore we get a distribution $\mathcal{H} \subset TG$, and we want this to be the horizontal distribution of a connection.

In order for this to make sense, we consider the abelian group N acting on the right on G, via

$$R_bx = x + x \vdash b = x + ub \; ; \; x \in G \; ; \; b \in N.$$

This action trivially preserves the fibers, since if $x = a \in N$, $R_ba = a$, so let us see how the distribution $\mathcal{H}$ behaves under this action:

First, since the action is linear, the same formula will give the action on tangent vectors, $\mathbb{R}_{b*}$. As we have seen, any horizontal tangent vector, say $\xi \in T_xG$, can be written as $\xi = v + v(a/\bar{u})$; therefore we have

$$\mathbb{R}_{b*}\xi = \xi + \xi \vdash b = v + v\left(\frac{a}{\bar{u}}\right) + vb = v + v\left(\frac{a + ub}{\bar{u}}\right),$$

which is a horizontal tangent vector at $R_bx = u + (a + ub)$. We have thus proved

Proposition 3. *G has the structure of a principal N-bundle under the action R, and $\mathcal{H}$ is equivariant under this action. Thus, the distribution $\mathcal{H}$ is a connection on G. The horizontal part of a vector $y \in V$, regarded as an element of T_xG, is given by $y \vdash (x \dashv x^{-1})$*

Remark. Alternatively, one could make N act according to $x \mapsto x + b \dashv x$, and then an equivariant distribution will be determined by $x \dashv x^{-1} \dashv y$.

III. Subdigroups of G.

To conclude, let us briefly discuss the restrictions of the bundle G to the two distinguished 1-dimensional subgroups of $\mathbb{C}^*$, namely $\mathbb{R}^+ = (0, \infty)$ and $U(1)$, corresponding to the one parameter subgroups generated by 1 and i, as this will illustrate some of the striking differences in the analysis of digroups as compared to the case of Lie groups.

First, let us denote these restrictions by G_1 and G_i respectively; then, recalling formula (1), if $x = u + a$ and $y = v + b$, then

$$x \vdash y = uv + ub ; \quad x \dashv y = uv + \bar{v}a.$$

We easily see from here that G_1 and G_i are closed under both operations; moreover, the bar unit 1 belongs to both, so they are subdigroups of G. Also clearly none of them is trivial, in the sense that they are not Lie groups.

As a group $\mathbb{R}^+$ is isomorphic to $\mathbb{R}$ via the logarithm, therefore, for G_1, what this gives is a non-trivial digroup structure on $\mathbb{R}^3$. The Leibniz algebra associated to G_1 is however not very interesting, since it is easily seen to be abelian, being generated by $1, j, k$.

In contrast, formula (2) shows that the rack structure of G is completely determined by G_i; it follows that the Leibniz algebra associated to G_i is also non-trivial, and indeed, it is generated by i, j, k. This is probably not surprising, since the topological properties of $\mathbb{C}^*$ are encoded in $U(1)$.

Now, a very interesting situaion arises if we assume further that $|a| = |b| = 1$, so that $x, y \in G_s = U(1) \times S^1$, where the second factor is the unit cirle in N. Indeed, the formulas above show that then also $x \vdash y; x \dashv y \in G_s$, since both sumands in the decomposition of the products are also of norm 1. Thus G_s, which is a torus, is closed under both operations.

However, it is **not** a subdigroup of G, because $1 \notin G_s$, and this in spite of the fact that G_s has many bar units (so we might say that it is a *subdimonoid*, in the terminology of [K]). Indeed, any element in G_s can be written as

$$x = e^{i\theta} + e^{i\eta} j,$$

and such an element is a bar unit, iff it has the form $1 + e^{i\eta} j$.

We could therefore try to consider any of these, say $e_0 = 1 + e^{i\eta_0}j$ as the new distinguished bar unit, and define inverses with respect to it. But there is actually no way to fix things up and give this torus a subdigroup structure: In fact, if we try to solve the equations for $y = e^{i\tau} + e^{i\zeta}j$ to be the inverse of x with respect to e_0, we get:

$$x \vdash y = e_0 \iff \tau = -\theta \; ; \; \zeta = \eta_0 - \theta;$$

but then

$$y \dashv x = 1 + e^{i(\eta_0 - 2\theta)}j = e_0 \iff \theta = 0,$$

and so the only invertible element with respect to e_0 is e_0 itself.

The failure of G_s to form a subdigroup can perhaps be explained by the fact that the tangent spaces to G_s are not Leibniz subalgebras of V. For instance, the tangent space at $1+j$ is generated by i and k, but since $[i, k] = -2j$, it is not closed under the Leibniz bracket.

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