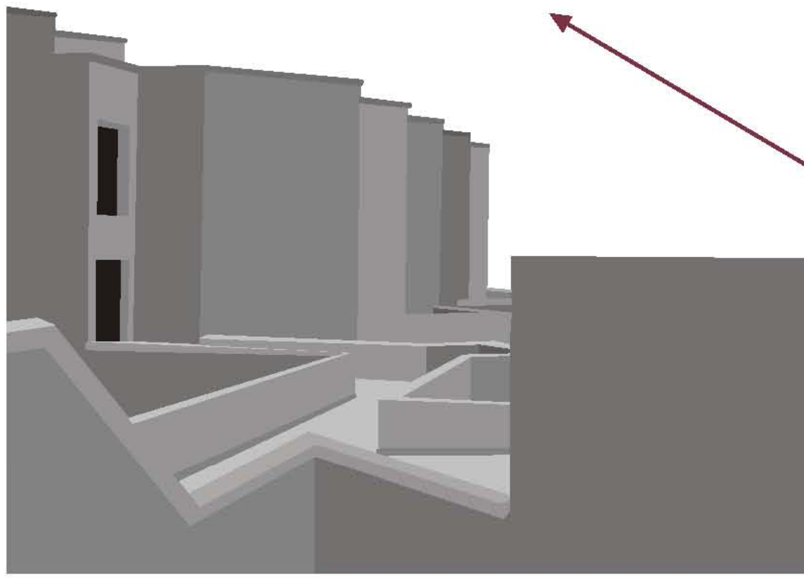

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Comunicación del CIMAT No I-08-20/19-11-2008
(PE/CIMAT)

CIMAT

The S-Transform of Symmetric Probability Measures with Unbounded Supports

Octavio Arizmendi E.*† Victor Pérez-Abreu ‡

Abstract

The Voiculescu S-transform is an analytic tool for computing free multiplicative convolutions of probability measures. It has been studied for probability measures with nonnegative support and for probability measures having all moments and zero mean. We extend the S-transform to symmetric probability measures with unbounded support and without moments. As an application, a representation of symmetric free stable measures is derived as a multiplicative convolution of the semicircle measure with a positive free stable measure.

AMS 2000 Subject Classification: 46L54, 15A52.
Keywords Multiplicative convolution, free stable distribution, random matrix.

1. Introduction

Let $\mathcal{M}$ be the class of all Borel probability measures on the real line $\mathbb{R}$ and let $\mathcal{M}_b$ and $\mathcal{M}^+$ be the subclasses of $\mathcal{M}$ consisting of probability measures with bounded support and of probability measures having support on $\mathbb{R}_+ = [0, \infty)$, respectively. A measure μ in $\mathcal{M}$ is *symmetric* if $\mu(-B) = \mu(B)$ for all Borel sets B, it has *all moments* if $m_k(\mu) = \int_{\mathbb{R}} |t|^k \mu(\mathrm{d}t) < \infty$, for each integer $k \geq 1$, and has *zero mean* when $m_1(\mu) = 0$.

The free multiplicative convolution $\boxtimes$ is a binary operation on $\mathcal{M}$ that was introduced by Voiculescu [16] to describe the multiplication of free (non-commuting) random variables. This operation is important in the study of free products of certain operator algebras [15] and has found one of its main applications in the theory of large dimensional random matrices, since it allows one to compute the asymptotic spectrum of the product of two independent random matrices from the individual asymptotic spectra [9], [17].

An important analytic tool for computing the free multiplicative convolution of two probability measures is the Voiculescu S-transform. It was introduced in [16] for non-zero mean distributions in $\mathcal{M}_b$ and further studied by Bercovici and Voiculescu [7] in the case of probability measures in $\mathcal{M}^+$ with unbounded support, see also [6]. In particular, this S-transform

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is important for computing free multiplicative convolutions of positive free stable distributions with themselves, as shown in [4].

Recently, Raj Rao and Speicher [14] extended the S-transform to the case of measures in $\mathcal{M}$ having zero mean and all moments. Their main tools are combinatorial arguments based on moment calculations. This allows them to compute interesting free multiplicative convolutions of measures with bounded support, like the ones of the Marchenko-Pastur distribution with the semicircle distribution and with the arcsine distribution, respectively, see [1] and [14].

The purpose of this paper is to extend the S-transform to general symmetric probability measures on $\mathbb{R}$ with unbounded support and without moments. We follow an analytic approach and use the known results for the nonnegative unbounded support case. This allows us to compute interesting examples of free multiplicative convolutions of probability measures in $\mathcal{M}^+$ with general symmetric probability measures in $\mathcal{M}$. As an important example of distributions without finite moments and unbounded supports we consider the free stable distributions studied in [4], [7], [13]. As an application, we show that any symmetric free stable distribution is the multiplicative convolution of the semicircle distribution with a positive free stable distribution. A reproducing property for these distributions is also found.

The paper is organized as follows. In Section 2 we review non-conmutative random variables and their free products, and collect known results on the S-transform that are used later on. Section 3 is dedicated to the study of the S-transform of symmetric probability measures on $\mathbb{R}$ as well as to the free multiplicative convolution of symmetric distributions in $\mathcal{M}$ with distributions in $\mathcal{M}^+$. Finally, Section 4 gives a description of the symmetric free stable distributions as the multiplicative convolution of the semicircle distribution with a positive free stable distribution.

2. Preliminaries on Free Convolutions

Following [17], we recall that a non-conmutative probability space $(\mathcal{A}, \varphi)$ is called a W^*-*probability space* if $\mathcal{A}$ is a non-commutative von Neumann algebra and φ is a normal faithful trace. A family of unital von Neumann subalgebras $\{\mathcal{A}_i\}_{i\in I} \subset \mathcal{A}$ in a W^*-probability space is said to be *free* if $\varphi(a_1 a_2 \cdots a_n) = 0$ whenever $\varphi(a_j) = 0, a_j \in \mathcal{A}_{i_j}$, and $i_1 \neq i_2 \neq \ldots \neq i_n$. A self-adjoint operator X *is said to be affiliated with* $\mathcal{A}$ if $f(X) \in \mathcal{A}$ for any bounded Borel function f on $\mathbb{R}$. In this case it is also said that X is a (non-commutative) *random variable*. Given a self-adjoint operator X affiliated with $\mathcal{A}$, the *distribution* of X is the unique measure μ_X in $\mathcal{M}$ satisfying

$$\varphi(f(X)) = \int_{\mathbb{R}} f(x)\mu_X(\mathrm{d}x)$$

for every Borel bounded function f on $\mathbb{R}$. If $\{\mathcal{A}_i\}_{i\in I}$ is a family of free unital von Neumann subalgebras and X_i is a random variable affiliated with $\mathcal{A}_i$ for each $i \in I$, then the *random variables* $\{X_i\}_{i\in I}$ are said to be *free*.

The *Cauchy transform* of a probability measure μ on $\mathbb{R}$ is defined, for $z \in \mathbb{C}\backslash\mathbb{R}$, by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\mu(\mathrm{d}x). \tag{1}$$

It is well known that G_μ is an analytic function in $\mathbb{C}\backslash\mathbb{R}$, $G_\mu : \mathbb{C}_+ \to \mathbb{C}_-$ and that G_μ determines uniquely the measure μ. The *reciprocal of the Cauchy transform* is the function $F_\mu(z) : \mathbb{C}_+ \to \mathbb{C}_+$

defined by $F_\mu(z) = 1/G_\mu(z)$. It was proved in [7] that there are positive numbers η and M such that F_μ has a right inverse F_μ^{-1} defined on the region

$$\Gamma_{\eta,M} := \{z \in \mathbb{C}; |\mathrm{Re}(z)| < \eta\,\mathrm{Im}(z), \quad \mathrm{Im}(z) > M\}. \tag{2}$$

The *Voiculescu transform of* μ is defined by

$$\phi_\mu(z) = F_\mu^{-1}(z) - z \tag{3}$$

on any region of the form $\Gamma_{\eta,M}$, where F_μ^{-1} is defined, see [4], [7]. The *free cumulant transform* is a variant of ϕ_μ defined as

$$\mathcal{C}_\mu^{\boxplus}(z) = z\phi_\mu(\frac{1}{z}) = zF_\mu^{-1}\left(\frac{1}{z}\right) - 1, \tag{4}$$

for z in a domain $D_\mu \subset \mathbb{C}_-$ such that $1/z \in \Gamma_{\eta,M}$ where F_μ^{-1} is defined, see [2].

The *free additive convolution* of two probability measures μ_1, μ_2 on $\mathbb{R}$ is defined as the probability measure $\mu_1 \boxplus \mu_2$ on $\mathbb{R}$ such that $\phi_{\mu_1 \boxplus \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z)$ or equivalently

$$\mathcal{C}_{\mu_1 \boxplus \mu_2}^{\boxplus}(z) = \mathcal{C}_{\mu_1}^{\boxplus}(z) + \mathcal{C}_{\mu_2}^{\boxplus}(z) \tag{5}$$

for $z \in D_{\mu_1} \cap D_{\mu_2}$. It turns out that $\mu_1 \boxplus \mu_2$ is the distribution of the sum $X_1 + X_2$ of two free random variables X_1 and X_2 having distributions μ_1 and μ_2 respectively.

On the other hand, the *free multiplicative operation* $\boxtimes$ on $\mathcal{M}$ is defined as follows, see [7]. Let μ_1, μ_2 be probability measures on $\mathbb{R}$, with $\mu_1 \in \mathcal{M}^+$ and let X_1, X_2 be free random variables such that $\mu_{X_i} = \mu_i$. Since μ_1 is supported on $\mathbb{R}_+$, X_1 is a positive self-adjoint operator and $\mu_{X_1^{1/2}}$ is uniquely determined by μ_1. Hence the distribution $\mu_{X_1^{1/2} X_2 X_1^{1/2}}$ of the self-adjoint operator $X_1^{1/2} X_2 X_1^{1/2}$ is determined by μ_1 and μ_2. This measure is called the *free multiplicative convolution* of μ_1 and μ_2 and it is denoted by $\mu_1 \boxtimes \mu_2$. This operation on $\mathcal{M}$ is associative and commutative.

The next result was proved in [7] for probability measures in $\mathcal{M}^+$ with unbounded support.

Proposition 1. *Let $\mu \in \mathcal{M}^+$ such that $\mu(\{0\}) < 1$. The function*

$$\Psi_\mu(z) = \int_0^\infty \frac{zx}{1 - zx}\mu(\mathrm{d}x), \quad z \in \mathbb{C}\backslash\mathbb{R}_+ \tag{6}$$

in univalent in the left-plane $i\mathbb{C}_+$ and $\Psi_\mu(i\mathbb{C}_+)$ is a region contained in the circle with diameter $(\mu(\{0\}) - 1, 0)$. Moreover, $\Psi_\mu(i\mathbb{C}_+) \cap \mathbb{R} = (\mu(\{0\}) - 1, 0)$.

Let $\chi_\mu : \Psi_\mu(i\mathbb{C}_+) \to i\mathbb{C}_+$ be the inverse function of Ψ_μ. The *S-transform* of μ is the function

$$S_\mu(z) = \chi(z)\frac{1+z}{z}.$$

The following result shows the role of the S-transform as an analytic tool for computing free multiplicative convolutions. It was shown in [15] for measures in $\mathcal{M}^+$ with bounded support and in [7] for measures in $\mathcal{M}^+$ with unbounded support.

We collect Corollaries 6.6 and 6.7 and Lemma 6.9 in [7] as follows.

Proposition 2. *Let μ_1 and μ_2 be probability measures in $\mathcal{M}^+$ with $\mu_i \neq \delta_0$, $i = 1, 2$. Then $\mu_1 \boxtimes \mu_2 \neq \delta_0$ and*

$$S_{\mu_1 \boxtimes \mu_2}(z) = S_{\mu_1}(z) S_{\mu_2}(z)$$

in that component of the common domain which contains $(-\varepsilon, 0)$ for small $\varepsilon > 0$. Moreover, $(\mu_1 \boxtimes \mu_2)(\{0\}) = \max\{\mu_1(\{0\}), \mu_2(\{0\})\}$.

Proposition 3. *Let $\{\mu_n\}_{n=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$ be sequences of probability measures in $\mathcal{M}^+$ converging to probability measures μ and ν in $\mathcal{M}^+$, respectively, in the weak* topology and such that $\mu \neq \delta_0 \neq \nu$. Then, the sequences $\{\mu_n \boxtimes \nu_n\}_{n=1}^{\infty}$ converges to $\mu \boxtimes \nu$ in the weak* topology.*

The next proposition is a particular case of a recent result proved in [14] for probability measures μ_1, μ_2 on $\mathbb{R}$ with all moments, when μ_1 has zero mean and $\mu_2 \in \mathcal{M}^+$.

Proposition 4. *Let μ_1 be a compactly supported symmetric probability measure on $\mathbb{R}$ and let $\mu_2 \in \mathcal{M}^+$ have compact support, with $\mu_i \neq \delta_0$, $i = 1, 2$. Then, $\mu_1 \boxtimes \mu_2 \neq \delta_0$ and*

$$S_{\mu_1 \boxtimes \mu_2}(z) = S_{\mu_1}(z) S_{\mu_2}(z).$$

From (4) and the fact that $\Psi_\mu(z) = \frac{1}{z} G_\mu(\frac{1}{z}) - 1$, one obtains the following relation observed in [10] between the free cumulant transform and the S-transform

$$z = \mathcal{C}_\mu^{\boxplus}(z S_\mu(z)). \tag{7}$$

This equation holds for measures in $\mathcal{M}^+$ or in $\mathcal{M}_b$ with zero mean. It was suggested in [14] that (7) may be used to define S-transforms of general probability measures on $\mathbb{R}$.

We finally mention that for free random variables having distributions with compact supports, a combinatorial approach to multiplicative convolutions is given in [12] and a Banach algebra approach to the S-transform is presented in [8].

3. Main Results

For a general probability measure μ on $\mathbb{R}$, let

$$\Psi_\mu(z) = \int_{\mathbb{R}} \frac{zt}{1 - zt} \mu(\mathrm{d}t) = \frac{1}{z} G_\mu\left(\frac{1}{z}\right) - 1, \quad z \in \mathbb{C} \backslash \mathbb{R}_+. \tag{8}$$

The function Ψ_μ determines the measure μ uniquely since the Cauchy transform G_μ does.

Given a symmetric probability measure μ on $\mathbb{R}$, let $Q(\mu) = \mu^2$ be the probability measure in $\mathcal{M}^+$ induced by the map $t \to t^2$. The transformation Q has been used, for example, in [11] to study a relation between multiplicative convolutions of positively supported probability measures with the commutator and the anticommutator of two symmetric probability measures. Likewise, the transformation μ^2 is called in [3] the push forward of the measure μ and it is used to study the so called rectangular free convolution.

We first prove two important relations between the Cauchy transforms and the Ψ functions of μ and μ^2.

Proposition 5. *Let μ be a symmetric probability measure μ on $\mathbb{R}$. Then*
a) $G_\mu(z) = zG_{\mu^2}(z^2)$, $z \in \mathbb{C}\backslash\mathbb{R}_+$.
b) $\Psi_\mu(z) = \Psi_{\mu^2}(z^2)$, $z \in \mathbb{C}\backslash\mathbb{R}_+$.

Proof a) Use the symmetry of μ twice to obtain

$$\begin{aligned} G_\mu(z) &= \int_{\mathbb{R}} \frac{1}{z-t}\mu(\mathrm{d}t) = \int_{\mathbb{R}_+} \frac{1}{z-t}\mu(\mathrm{d}t) + \int_{\mathbb{R}_+} \frac{1}{z+t}\mu(\mathrm{d}t) \\ &= 2z \int_{\mathbb{R}_+} \frac{1}{z^2-t^2}\mu(\mathrm{d}t) = z\int_{\mathbb{R}} \frac{1}{z^2-t^2}\mu(\mathrm{d}t) \\ &= z\int_{\mathbb{R}_+} \frac{1}{z^2-t}\mu^2(\mathrm{d}t) = zG_{\mu^2}(z^2). \end{aligned}$$

b) Use (8) twice and (a) to obtain

$$\begin{aligned} \Psi_\mu(z) &= \int_{\mathbb{R}} \frac{zx}{1-zx}\mu(\mathrm{d}x) = \frac{1}{z}G_\mu\left(\frac{1}{z}\right) - 1 \\ &= \frac{1}{z^2}G_{\mu^2}\left(\frac{1}{z^2}\right) - 1 = \Psi_{\mu^2}(z^2) \end{aligned}$$

which shows (b) ■

Theorem 6. *Let μ be a symmetric probability measure μ on $\mathbb{R}$.*
a) If $\mu \neq \delta_0$, the function Ψ_μ is univalent in each of the following two disjoint sets

$$H = \{z \in \mathbb{C}_+; \quad |\mathrm{Re}(z)| < \mathrm{Im}(z)\},$$

$$\widetilde{H} = \{z \in \mathbb{C}_-; \quad |\mathrm{Re}(z)| < |\mathrm{Im}(z)|\}.$$

Therefore Ψ_μ has a unique inverse $\chi_\mu : \Psi_\mu(H) \to H$ and a unique inverse $\widetilde{\chi}_\mu : \Psi_\mu(\widetilde{H}) \to \widetilde{H}$.
b) If $\mu \neq \delta_0$, the S-transforms

$$S_\mu(z) = \chi_\mu(z)\frac{1+z}{z} \text{ and } \widetilde{S}_\mu(z) = \widetilde{\chi}_\mu(z)\frac{1+z}{z} \tag{9}$$

are such that

$$S_\mu^2(z) = \frac{1+z}{z}S_{\mu^2}(z) \text{ and } \widetilde{S}_\mu^2(z) = \frac{1+z}{z}S_{\mu^2}(z) \tag{10}$$

for z in $\Psi_\mu(H)$ and $\Psi_\mu(\widetilde{H})$ respectively.

Proof a) Let $h : \mathbb{C} \to \mathbb{C}$ be the function $h(z) = z^2$. Then $h(H) = h(\widetilde{H}) = i\mathbb{C}_+$ and therefore h is univalent in H and in $\widetilde{H}$. On the other hand, since $\mu^2 \in \mathcal{M}^+$, by Proposition 1, $\Psi_{\mu^2}(z)$ is univalent in $i\mathbb{C}_+$ and therefore $\Psi_{\mu^2}(z^2)$ is univalent in H and in $\widetilde{H}$.

b) Since $\mu^2 \in \mathcal{M}^+$, from Proposition 1, the unique inverse χ_{μ^2} of Ψ_{μ^2} is such that $\chi_{\mu^2} : \Psi_{\mu^2}(i\mathbb{C}_+) \to i\mathbb{C}_+$. Thus, use (a) to obtain $\Psi_{\mu^2}(\chi_\mu^2(z)) = \Psi_\mu(\chi_\mu(z)) = z$ for $z \in \Psi_\mu(H)$ and the uniqueness of χ_{μ^2} gives $\chi_{\mu^2}(z) = \chi_\mu^2(z)$, $z \in \Psi_\mu(H)$. Hence

$$\begin{aligned} S_\mu^2(z) &= \chi_\mu^2(z)(\frac{1+z}{z})^2 = \chi_{\mu^2}(z)(\frac{1+z}{z})^2 \\ &= S_{\mu^2}(z)\frac{1+z}{z}, \quad z \in \Psi_\mu(H), \end{aligned}$$

and similarly for $\widetilde{S}_\mu(z)$. ∎

The following is the main result of this paper. It shows how to compute the multiplicative convolution of a symmetric probability measure on $\mathbb{R}$ with a probability measure on $\mathbb{R}_+$. No existence of moments or bounded supports for the probability measures are assumed.

Theorem 7. *Let μ and ν be probability measures on $\mathbb{R}$ such that μ is symmetric, $\nu \in \mathcal{M}^+$ and $\mu \neq \delta_0 \neq \nu$. Let S_μ and $\widetilde{S}_\mu$ be the two S-transforms of μ. Then*

$$S_{\mu\boxtimes\nu}(z) = S_\mu(z)S_\nu(z) \text{ and } \widetilde{S}_{\mu\boxtimes\nu}(z) = \widetilde{S}_\mu(z)S_\nu(z) \tag{11}$$

are the two S-transforms of the symmetric probability measure $\mu \boxtimes \nu$, where the functions in (11) are considered in the common domain which contains $(-\varepsilon, 0)$ for small $\varepsilon > 0$.

The key in proving this theorem is the following lemma, which is a result of independent interest.

Lemma 8. *Let μ and ν be probability measures on $\mathbb{R}$ such that μ is symmetric, $\nu \in \mathcal{M}^+$ and $\mu \neq \delta_0 \neq \nu$. Then*

$$\nu \boxtimes \mu^2 \boxtimes \nu = (\mu \boxtimes \nu)^2. \tag{12}$$

Proof We first prove the result when μ and ν have compact supports. In this case, since μ has zero mean, use Proposition 4 to obtain $S_{\mu\boxtimes\nu}(z) = S_\mu(z)S_\nu(z)$. Then, since $\mu \boxtimes \nu$ is a symmetric probability measure, the use of (10) gives

$$\frac{z}{z+1}S_{\mu\boxtimes\nu}^2(z) = S_{(\mu\boxtimes\nu)^2}(z).$$

Hence, use again (10) to obtain

$$S_{(\mu\boxtimes\nu)^2}(z) = \frac{z}{z+1}S_\mu^2(z)S_\nu^2(z) = S_{\mu^2}(z)S_\nu^2(z).$$

Since μ^2 and ν are in $\mathcal{M}^+$, by Proposition 2, $S_{\mu^2}(z)S_\nu^2(z) = S_{\nu\boxtimes\mu^2\boxtimes\nu}(z)$ and therefore (12) holds for μ and ν having compact supports.

Next, for μ and ν with unbounded supports, choose symmetric probability measures μ_n on $\mathbb{R}$ with compact support and ν_n in $\mathcal{M}^+$ also with compact support, $n \geq 1$, such that $\{\mu_n\}_{n=1}^\infty$ and $\{\nu_n\}_{n=1}^\infty$ converge to μ and ν, respectively, in the weak* topology. From the first part of the proof we obtain $S_{\nu_n\boxtimes\mu_n^2\boxtimes\nu_n} = S_{(\mu_n\boxtimes\nu_n)^2}$, $n \geq 1$. On the other hand, since μ_n^2, ν_n and $(\mu_n \boxtimes \nu_n)^2$ belong to $\mathcal{M}^+$, by Proposition 3, the sequences $\left\{\nu_n \boxtimes \mu_n^2 \boxtimes \nu_n\right\}_{n=1}^\infty$ and $\left\{(\mu_n \boxtimes \nu_n)^2\right\}_{n=1}^\infty$ converge to $\nu \boxtimes \mu^2 \boxtimes \nu$ and $(\mu \boxtimes \nu)^2$, respectively, in the weak* topology. This proves the result. ∎

Proof of Theorem 7 We obtain the result as a consequence of Theorem 6 (b) and the above lemma as follows

$$S_\mu^2(z)S_\nu^2(z)=\frac{z+1}{z}S_{\mu^2}(z)S_\nu^2(z)=\frac{z+1}{z}S_{\nu\boxtimes\mu^2\boxtimes\nu}(z)=\frac{z+1}{z}S_{(\mu\boxtimes\nu)^2}(z)=S_{\mu\boxtimes\nu}^2(z)$$

for $z\in\Psi_\mu(H)\cap\Psi_\nu(H)$ and similarly for $\widetilde{S}_\mu$. ■

We finally observe that (7) also holds for symmetric probability measures on $\mathbb{R}$.

Proposition 9. *Let μ be a symmetric probability measure on $\mathbb{R}$. Then*

$$z=\mathcal{C}_\mu^{\boxplus}\left(zS_\mu(z)\right) \text{ and } z=\mathcal{C}_\mu^{\boxplus}\left(z\widetilde{S}_\mu(z)\right), \tag{13}$$

whenever $1/[zS_\mu(z)]$ and $1/[z\widetilde{S}_\mu(z)]$ belong to domains $\Gamma_{\eta,M}$ where F_μ^{-1} is defined, respectively.

Proof Let χ_μ be the inverse of Ψ in H. Using (8) we have that for $z\in\Psi_\mu(H)$, $z=\Psi_\mu(\chi_\mu(z))=G_\mu(1/\chi_\mu(z))/\chi_\mu(z)-1$. Thus, $G_\mu(1/\chi_\mu(z))=(1+z)\chi_\mu(z)$ and

$$F_\mu\left(\frac{1}{\chi_\mu(z)}\right)=\frac{1}{(1+z)\chi_\mu(z)}.$$

Then,

$$\frac{1}{\chi_\mu(z)}=F_\mu^{-1}\left(\frac{1}{(1+z)\chi_\mu(z)}\right).$$

Finally, use (4) to obtain

$$z=(1+z)\chi_\mu(z)F_\mu^{-1}\left(\frac{1}{(1+z)\chi_\mu(z)}\right)-1=\mathcal{C}_\mu^{\boxplus}\left(zS_\mu(z)\right)$$

and similarly for $\widetilde{S}_\mu$. ■

4. Representation of Symmetric Free Stable Laws

Following [2], it is said that a probability measure μ on $\mathbb{R}$ is *stable with respect to the free additive convolution* $\boxplus$ defined by (5), or simply *free stable* or $\boxplus$-*stable*, if the class

$$\{\psi(\mu):\psi \text{ is an increasing affine transformation}\}$$

is closed under the free additive convolution $\boxplus$. We denote by $S^{\boxplus}(\mathbb{R})$ the class of all free stable distributions on $\mathbb{R}$. They are examples of probability measures on $\mathbb{R}$ with unbounded supports, without all moments and without atoms. These distributions and their domains of attractions have been studied in [4], [6] and [13]. Here we are interested in symmetric free stable distributions $S_s^{\boxplus}(\mathbb{R})$ and in free stable distributions $S^{\boxplus}(\mathbb{R}_+)$ with nonnegative support.

We first use the results in the Appendix in [4], to find the free cumulant transforms of distributions in $S_s^{\boxplus}(\mathbb{R})$ and $S^{\boxplus}(\mathbb{R}_+)$.

Lemma 10. *a) A probability measure σ on $\mathbb{R}_+$ belongs to $S^{\boxplus}(\mathbb{R}_+)$ if and only if there exist α, $0<\alpha<1$, $\theta>0$ and $\gamma_0 \geq 0$, such that*

$$\mathcal{C}_{\sigma}^{\boxplus}(z) = \gamma_0 z - \theta e^{i\alpha\pi} z^{\alpha}. \tag{14}$$

b) A probability measure ν on $\mathbb{R}_+$ belongs to $S_s^{\boxplus}(\mathbb{R})$ if and only if there exist α, $0<\alpha<2$ and $\theta>0$, such that when $1 \leq \alpha \leq 2$

$$\mathcal{C}_{\nu}^{\boxplus}(z) = \theta e^{i(\alpha-2)\frac{\pi}{2}} z^{\alpha} \tag{15}$$

and when $0<\alpha<1$

$$\mathcal{C}_{\nu}^{\boxplus}(z) = -\theta e^{i\alpha\frac{\pi}{2}} z^{\alpha}. \tag{16}$$

Proof We use the relation $\mathcal{C}_{\nu}^{\boxplus}(z) = z\phi_{\nu}(\frac{1}{z})$ given by (4) between the free cumulant transform $\mathcal{C}_{\mu}^{\boxplus}$ and the Voiculescu transform ϕ_{μ} defined by (3).

a) From Section A4 and (4) in the Appendix in [4] and Theorem 7.5 in [7], σ is a free stable distribution with support in $(0,\infty)$, if and only if there exist α, $0<\alpha<1$, and $\theta>0$, such $\phi_{\sigma}(z) = -\theta e^{i\alpha\pi} z^{-\alpha+1}$. Then, use (4) to obtain (14).

b) Similarly, from (2) in the Appendix [4] and Theorem 7.5 in [7], the measure μ is a symmetric free stable distribution (with asymmetry coefficient $\rho=1$ in the notation of [4]) if and only if there exist α, $0<\alpha\leq 2$, and $\theta>0$, such that $\phi_{\nu}(z) = \theta e^{i(\alpha-2)\frac{\pi}{2}} z^{-\alpha+1}$ when $1\leq\alpha\leq 2$ and $\phi_{\nu}(z) = -e^{i\alpha\frac{\pi}{2}} z^{-\alpha+1}$ when $0<\alpha<1$. In both cases (b) follows using (4). ■

For $0<\alpha<1$, we denote by σ_{α} an element in $S^{\boxplus}(\mathbb{R}_+)$ with free cumulant transform (14) and drift $\gamma_0=0$. In this case σ_{α} has support $(0,\infty)$ and we say that σ_{α} is a *positive free α-stable distribution.* In general, a distribution in $S^{\boxplus}(\mathbb{R}_+)$ has support (γ_0,∞) in which case we use the notation σ_{α,γ_0}. Similarly, ν_{α} is a *symmetric free α-stable distribution,* $0<\alpha<2$, if $\nu_{\alpha}\in S^{\boxplus}(\mathbb{R})$ and its free cumulant transform is given by (15) or (16). In what follows the power functions z^c are defined through their principal branches in $\mathbb{C}_+$.

The S-transforms of positive and symmetric free α-stable distributions are as follows.

Proposition 11. *a) If σ_{α} is a positive free α-stable distribution, $0<\alpha<1$, then there exists $\theta_{\alpha}>0$ such that*

$$S_{\sigma_{\alpha}}(z) = \theta_{\alpha} e^{i(1-\alpha)\frac{\pi}{\alpha}} z^{\frac{1-\alpha}{\alpha}}. \tag{17}$$

b) If ν_{α} is a symmetric free α-stable distribution, $0<\alpha\leq 2$, there exists $\theta_{\alpha}>0$ such that

$$S_{\nu_{\alpha}}(z) = \theta_{\alpha} e^{i(2-\alpha)\frac{\pi}{2\alpha}} z^{\frac{1-\alpha}{\alpha}}. \tag{18}$$

Proof a) Use (7) and (14) to obtain the equation

$$z = \mathcal{C}_{\sigma_{\alpha}}^{\boxplus}(zS_{\sigma_{\alpha}}(z)) = -\theta e^{i\alpha\pi}(zS_{\sigma_{\alpha}}(z))^{\alpha} = \theta e^{i(1+\alpha)\pi} z^{\alpha} S_{\sigma_{\alpha}}^{\alpha}(z),$$

from which (17) follows for $z\in\mathbb{C}_+$, since $\phi_{\sigma_{\alpha}}$ is analytic on $\mathbb{C}_+$.

b) Let $0<\alpha<1$. Use (13) and (16) to obtain the equation

$$z = \mathcal{C}_{\nu_{\alpha}}^{\boxplus}(zS_{\nu_{\alpha}}(z)) = -\theta e^{i\alpha\frac{\pi}{2}}(zS_{\nu_{\alpha}}(z))^{\alpha} = \theta e^{i(1+\frac{\alpha}{2})\pi} z^{\alpha} S_{\nu_{\alpha}}^{\alpha}(z),$$

from which (18) follows for $z\in\mathbb{C}_+$, since $\phi_{\nu_{\alpha}}$ is analytic on $\mathbb{C}_+$. Similarly for $1\leq\alpha\leq 2$. ■

The following result gives a representation of symmetric free stable distributions as the multiplicative convolution of the Wigner distribution with a positive free stable distribution. Recall that the standard *semicircle* or *Wigner distribution* is the probability measure

$$w(\mathrm{d}x) = \frac{1}{2\pi}\sqrt{4-x^2}\cdot 1_{[-2,2]}\mathrm{d}x. \tag{19}$$

It is a symmetric free stable distribution with $\alpha = 2$ and therefore

$$S_w(z) = \frac{1}{\sqrt{z}}. \tag{20}$$

Theorem 12. *For $0 < \alpha < 2$, the symmetric free α-stable distribution ν_α has the representation*

$$\nu_\alpha = \sigma_\beta \boxtimes w$$

where σ_β is a positive free β-stable distribution with index $\beta = 2\alpha/(2+\alpha)$.

Proof Use (20), (17) and (18) to obtain

$$S_{\nu_\beta}(z)S_w(z) = \theta_\beta e^{i(1-\beta)\frac{\pi}{\beta}} z^{\frac{1-\beta}{\beta}-\frac{1}{2}} = \theta_\alpha e^{i(2-\alpha)\frac{\pi}{2\alpha}} z^{\frac{1-\alpha}{\alpha}} = S_{\nu_\alpha}(z).$$

Hence, Theorem 7 gives the result. ■

For positive stable distribution the following reproducing property holds [4, Proposition A 4.3]: For all $s, t > 0$,

$$\sigma_{\frac{1}{1+s+t}} = \sigma_{\frac{1}{1+s}} \boxtimes \sigma_{\frac{1}{1+t}}.$$

As a final result, we obtain a similar reproducing property for symmetric free stable distributions.

Proposition 13. *For all $s, t > 0$*

$$\nu_{\frac{1}{1+s+t}} = \sigma_{\frac{1}{1+s}} \boxtimes \nu_{\frac{1}{1+t}} = \sigma_{\frac{1}{1+t}} \boxtimes \nu_{\frac{1}{1+s}}.$$

Proof It follows using (17), (18) and Theorem 12. ■

Acknowledgement 14. *The authors would like to thank Florent Benaych-Georges and Noriyoshi Sakuma for valuable comments in a preliminary version of this work.*

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