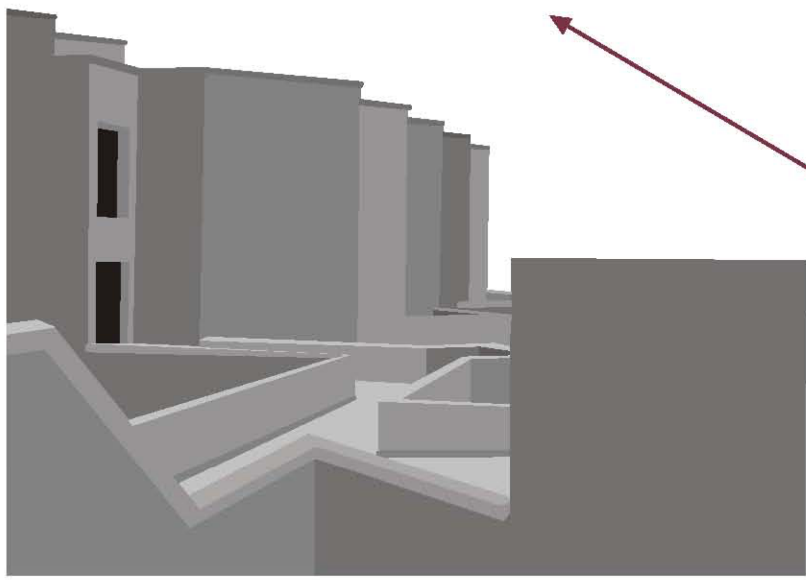

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ELLIPTICAL CONFIGURATION DISTRIBUTION

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Elliptical Configuration Distribution

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Abstract

The noncentral configuration density under an elliptical model is derived; it generalizes and corrects the Gaussian configuration and some Pearson results. Then by using partition theory a number of explicit configuration densities are obtained; i.e. configuration densities associated with the matrix variate symmetric Kotz type distributions (it includes normal), the matrix variate Pearson type VII distributions (it includes t and Cauchy distributions), the matrix variate symmetric Bessel distribution (it includes Laplace distribution) and the matrix variate symmetric logistic distribution.

Key words: Statistical shape theory, configuration density, elliptically contoured distributions, zonal and invariant polynomials, matrix variate Kotz, Pearson, Bessel and Logistic distributions.
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1 Introduction

When the statistical theory of shape was placed in the setting of the non-central multivariate analysis (Goodall and Mardia (1993)), a wide gamma of standard theories developed in the last 60 years were available to solve the new distributional problems.

As usual the first works assumed Gaussian distributions for the landmark components (Goodall and Mardia (1993), Díaz-García *et al.* (2003)), and integration over Euclidean and affine transformations provided the required shape and configuration distributions, respectively, in terms of a well study theory, the zonal polynomials of matrix argument.

Theory of integration over orthogonal and positive definite matrices involving zonal polynomials led exact distributions, but the problem for large computations remained open for years, and approximations were needed for applications. Recently, with the appearance of efficient algorithms for zonal and hypergeometric functions, the exact distributions can be studied in the corresponding inference problem, and then the applications can be potentially improved (Koev and Edelman (2006)).

However, the normal constraint stands ideal, so new enriched distributions, for example elliptically contoured distributions, could be considered for the landmark components, but the corresponding new integrals under the Euclidean or affine transformations demands new developments. The Euclidean case was solved with the usual multivariate analysis and classical integration formulae for zonal and invariant polynomials of several matrix arguments (Goodall and Mardia (1993), Díaz-García *et al.* (2003), etc.). But the configuration distribution (based on affine transformation) for any elliptical model, has not been studied in literature.

Two motivations seem reasonable for solving the configuration problem, one, the geometrical meaning for applications and second, the involved distributional problem. The first one, it is clearly the most important for users of shape theory, the transformation refers problems which are not equally deformed in all the directions (as in the Euclidean transformation models), but uniformly deformed component by component. It is specially useful in growth theory, mechanical deformations, non rigid evolution, electrophoretic gel studies, etc. but even the classical applications studied by Euclidean transformations and Gaussian distributions, can be research again by guessing an elliptical model previously ratified by a Schwarz's dimensional criterion (Schwarz (1978)), for example, and under an affine transformation. The second topic, is a proba-

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bilistic problem, and it considers fundamentally advances in integration over positive definite matrices via zonal polynomials. Noncentral multivariate analysis gives the key for studying one by one the particular elliptical models by solving the corresponding multiple integral as in the normal case (Goodall and Mardia (1993),Díaz-García *et al.* (2003)) and some integrals given for Pearson models (Xu and Fang (1989)), for example. But, this technique provides no solution for any general model and certainly, some multiple integrals seem prohibitive. So, an interesting problem could go in that general direction, by simplifying the integration problem.

Fortunately, some general results for integration on positive definite matrices are available (Teng *et al* (1989)) and they can be used in the setting we need for the configuration distributions . Those results lead that our distributions for any elliptical model are easily integrable. However, the simplicity in the integration will have a price, k-th derivative expressions for the elliptical model functions. Then the second tool for solving the problem appears, a partitional treatment for expressing the derivative in a fashion that the single integral can be easily computed (Caro-Lopera (2008)).

With the distributional problem solved, another important question appears, the computation, but as we mentioned before, series of zonal polynomials can be now computed efficiently, then the inference problem based on the exact configuration density is set in this work as a solvable numerical aspect.

This work is distributed as follows: first, the main integral which supports all the distributional results and the corresponding corollaries are listed in section 2; as a consequence, a subsection is devoted to correct some Pearson published results which are necessary for the corresponding Pearson configuration density; finally, section 3 studies the configuration densities corresponding to the classical matrix variate elliptical contoured distributions including, Pearson, Kotz, Bessel, Logistic, etc. (See Caro-Lopera (2008).)

2 The main integral

Noncentral elliptical multivariate distributions involve a number of general integrals to study, it depends on the transformations under consideration, but all of them are founded in an important fact, the elliptically contoured distribution are characterized by a symmetric function, say, $h(\mathbf{U})$, i.e, $h(\mathbf{AB}) = h(\mathbf{BA})$, for any squared matrix $\mathbf{A}$ and $\mathbf{B}$. The simplest function we can consider is the trace of a positive definite matrix and then the zonal polynomials arise naturally, then euclidian and affine transformations of the random matrix will precise of integration over the orthogonal group and the positive definite space, etc. In the case of positive definite matrices, we find in Constantine

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(1963) the source for all the posterior works, in fact it inspired the following general result for elliptical integration which can be seen as a combination of Xu and Fang (1989) and Teng *et al* (1989), see Caro-Lopera (2008) for a detailed proof:

Theorem 1 *Let* $\mathbf{Z}$ *be a complex symmetric* $m \times m$ *matrix with* $Re(\mathbf{Z}) > \mathbf{0}$ *and let* $\mathbf{Y}$ *be a symmetric* $m \times m$ *matrix. Then*

$$\int_{\mathbf{X}>\mathbf{0}} h(\operatorname{tr}\mathbf{XZ})|\mathbf{X}|^{a-(m+1)/2} C_\kappa(\mathbf{XY})(d\mathbf{X}) = \frac{|\mathbf{Z}|^{-a}(a)_\kappa \Gamma_m(a) C_\kappa(\mathbf{YZ}^{-1})}{\Gamma(ma+k)} S, \quad (1)$$

where

$$S = \int_0^\infty h(w) w^{ma+k-1} dw < \infty, \quad (2)$$

and $(a)_\kappa$ *and* $\Gamma_m(a)$ *are the generalized hypergeometric coefficient and the multivariate gamma function respectively (see Muirhead (1982) p.62, 247,248).*

As a convention, in this work we always assume that the integrals we meet with exist.

A simple consequence is the following.

Corollary 2 *Let* $\mathbf{Z}$ *be a complex symmetric* $m \times m$ *matrix with* $Re(\mathbf{Z}) > \mathbf{0}$*. Then*

$$\int_{\mathbf{X}>\mathbf{0}} h(\operatorname{tr}\mathbf{XZ})|\mathbf{X}|^{a-(m+1)/2}(d\mathbf{X}) = \frac{|\mathbf{Z}|^{-a}\Gamma_m(a)}{\Gamma(ma)} S,$$

where

$$S = \int_0^\infty h(y) y^{ma-1} dy < \infty.$$

Note that if we take $\mathbf{Z} = \mathbf{\Sigma}^{-1}$ and $h(y) = e^{-y/2}$ which implies $S = 2^{ma}\Gamma(ma)$; we have the classical result for multivariate gamma function proved in detail by Muirhead (1982), pp. 61-63.

2.1 On some Pearson VII type published results

Now, some consequences of theorem 1 for Pearson VII type aspects can be listed.

Corollary 3

$$\int_{\mathbf{W}>\mathbf{0}} \left(1 + m^{-1}\operatorname{tr}\mathbf{W}\right)^{-(m+np)/2} |\mathbf{W}|^{\frac{n}{2}-\frac{(p+1)}{2}} C_\kappa(\mathbf{WU})(d\mathbf{W})$$
$$= \frac{m^{\frac{np}{2}+k}\left(\frac{n}{2}\right)_\kappa \Gamma_p\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}-k\right)}{\Gamma\left(\frac{m}{2}+\frac{np}{2}\right)} C_\kappa(\mathbf{U}).$$

Consider the density of $\mathbf{B} = \mathbf{Y}'\mathbf{Y}$ given in eq. (1.5) of Runze (1997):

$$\frac{\pi^{np/2} c_{n,p}}{\Gamma_p\left(\frac{n}{2}\right)} |\mathbf{\Sigma}|^{-n/2} |\mathbf{B}|^{(n-p-1)/2} h(\operatorname{tr} \mathbf{\Sigma}^{-1}\mathbf{B}), \tag{3}$$

where $c_{n,p}$ is a normalization constant (see Runze (1997)). Then another consequence of Theorem 1 is the expectation of a zonal polynomial respect to $\mathbf{B}$ with the above defined density.

Corollary 4 *Suppose that* $\mathbf{B}$ *has density (3) and* $\mathbf{A}$ *is an arbitrary symmetric* $p \times p$ *constant matrix, then*

$$E_{\mathbf{B}}[C_\kappa(\mathbf{BA})] = \frac{\pi^{np/2} c_{n,p} \left(\frac{n}{2}\right)_\kappa}{\Gamma\left(\frac{np}{2} + k\right)} C_\kappa(\mathbf{A\Sigma}) J,$$

where

$$J = \int_0^\infty h(y) y^{\frac{np}{2}+k-1} dy < \infty.$$

Corollary 5 *Under the conditions of Corollary 4 and by using Muirhead (1982), p.251, we have*

$$E_{\mathbf{B}}[r_j(\mathbf{B})] = \frac{\pi^{np/2} c_{n,p} \left(\frac{n}{2}\right)_\kappa}{\Gamma\left(\frac{np}{2} + k\right)}\, r_j(\mathbf{\Sigma}) J,$$

where $j = 1, \ldots, p$.

Corollary 6 *Suppose that* $\mathbf{B}$ *has density (3) and* $\mathbf{A}$ *and* $\mathbf{C}$ *are arbitrary symmetric* $p \times p$ *constant matrices, then*

$$E_{\mathbf{B}}[C_\phi^{\kappa,\lambda}(\mathbf{BC}, \mathbf{BU})] = \frac{\pi^{np/2} c_{n,p} \left(\frac{n}{2}\right)_\phi}{\Gamma\left(\frac{np}{2} + k + l\right)} C_\phi^{\kappa,\lambda}(\mathbf{C\Sigma}, \mathbf{U\Sigma}) J_1,$$

where

$$J_1 = \int_0^\infty h(y) y^{\frac{np}{2}+k+l-1} dy < \infty,$$

κ, λ *and* ϕ *are partitions of* k, l *and* $k+l$, *respectively; see Davis (1980) for the theory of invariant polynomials.*

Now, if we take $h(\mathbf{B}) = (1 + m^{-1} \operatorname{tr} \mathbf{B})^{-(m+np)/2}$ in corollary 4 we have that $I = m^{k+np/2} \mathrm{B}(k + np/2, -k + m/2)$, $c_{n,p} = \Gamma[(m+np)/2]/[\pi^{np/2} m^{np/2} \Gamma(m/2)]$ and

$$E_{\mathbf{B}}[C_\kappa(\mathbf{BA})] = \frac{m^k \Gamma[\frac{m}{2} - k] \left(\frac{n}{2}\right)_\kappa}{\Gamma\left(\frac{m}{2}\right)}\, C_\kappa(\mathbf{A\Sigma}), \tag{4}$$

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and the corollary 6 in this case turns

$$E_{\mathbf{B}}[C_{\phi}^{\kappa,\lambda}(\mathbf{BC},\mathbf{BU})] = \frac{m^{k+l}\Gamma[\frac{m}{2}-k-l]\left(\frac{n}{2}\right)_{\phi}}{\Gamma(m/2)} C_{\phi}^{\kappa,\lambda}(\mathbf{C\Sigma},\mathbf{U\Sigma}). \qquad (5)$$

Finally, we can check again (4) by using properties of invariant polynomials in (5). In fact, just take $\mathbf{C} = \mathbf{0}$, then $C_{\phi}^{\kappa,\lambda}(\mathbf{0},\mathbf{BU}) = 0$ for $k > 0$ and $C_{\phi}^{\kappa,\lambda}(\mathbf{0},\mathbf{BU}) = C_{\lambda}(\mathbf{BU})$ for $k = 0$, then

$$E_{\mathbf{B}}[C_{\lambda}(\mathbf{BU})] = E_{\mathbf{B}}[C_{\phi}^{\kappa,\lambda}(\mathbf{0},\mathbf{BU})] = \frac{m^{l}\Gamma[\frac{m}{2}-l]\left(\frac{n}{2}\right)_{\lambda}}{\Gamma(m/2)} C_{\lambda}(\mathbf{U\Sigma}),$$

which corresponds with (4).

Recall that if we take $m = 1$ in the family of Pearson VII distributions we obtain the multivariate Cauchy distribution; thus by replacing this parameter in (4) we must have that $\frac{1}{2} \geq k$, which means that the Cauchy distribution has no moments.

Now, corollaries 3-6 were derived by Xu and Fang (1989) and Runze (1997), but as the reader can check the published results are incorrect. Note that the discrepancy must be clarified because our Pearson configuration density involves the corrected integrals.

3 Configuration Density

First, we recall a definition given by Goodall and Mardia (1993).

Definition 7 *Two figures* $\mathbf{X} : N \times K$ *and* $\mathbf{X}_1 : N \times K$ *have the same configuration, or affine shape, if* $\mathbf{X}_1 = \mathbf{XE} + \mathbf{1}_N e'$*, for some translation* $e : K \times 1$ *and a nonsingular* $\mathbf{E} : K \times K$*.*

The configuration coordinates are constructed in two steps summarized in the expression

$$\mathbf{LX} = \mathbf{Y} = \mathbf{UE}. \qquad (6)$$

The matrix $\mathbf{U} : N-1 \times K$ contains configuration coordinates of $\mathbf{X}$. Let $\mathbf{Y}_1 : K \times K$ be nonsingular and $\mathbf{Y}_2 : q = N-K-1 \geq 1 \times K$, such that $\mathbf{Y} = (\mathbf{Y}_1' \mid \mathbf{Y}_2')'$. Define also $\mathbf{U} = (\mathbf{I} \mid \mathbf{V}')'$, then $\mathbf{V} = \mathbf{Y}_2\mathbf{Y}_1^{-1}$ and $\mathbf{E} = \mathbf{Y}_1$. Where $\mathbf{L}$ is an $N-1 \times N$ Helmert sub-matrix.

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Now we establish the jacobian of the configuration transformation:

Lemma 8 *Let* $(\mathbf{F}^{1/2})^2 = \mathbf{F} > \mathbf{0}$, $\mathbf{H}$ *a* $K \times K$ *orthogonal matrix, and* $\mathbf{E} = \mathbf{F}^{1/2}\mathbf{H}$ *so for* $\mathbf{Y} = \mathbf{U}\mathbf{F}^{1/2}\mathbf{H}$ *then*

$$(d\mathbf{Y}) = 2^{-K}|\mathbf{F}|^{(q-1)/2}(d\mathbf{V})(d\mathbf{F})(\mathbf{H}d\mathbf{H}').$$

where $(\mathbf{H}'d\mathbf{H})$ *denotes the Haar measure, see Muirhead (1982), p. 72.*

Proof. Let $\mathbf{E} = \mathbf{F}^{1/2}H$, with $\mathbf{E}$ a $K \times K$ invertible matrix, $\mathbf{H}$ orthogonal and $\mathbf{F}^{1/2} > \mathbf{0}$. So $\mathbf{E}'\mathbf{E} = \mathbf{H}'\mathbf{F}\mathbf{H}$, because $\mathbf{E}'\mathbf{E}$ and $\mathbf{F}$ are symmetric and $\mathbf{H}$ non singular, then $(\mathbf{E}'\mathbf{E}) = |\mathbf{H}|^{K+1}(d\mathbf{F}) = (d\mathbf{F})$. But by theorem 2.1.14 of Muirhead (1982): $(d\mathbf{E}) = 2^{-K}|\mathbf{E}'\mathbf{E}|^{-1/2}(\mathbf{E}'\mathbf{E})(\mathbf{H}'d\mathbf{H})$. Then we obtain: $(d\mathbf{E}) = 2^{-K}|\mathbf{H}'\mathbf{F}\mathbf{H}|^{-1/2}(d\mathbf{F})(\mathbf{H}'d\mathbf{H}) = 2^{-K}|\mathbf{F}|^{-1/2}(d\mathbf{F})(\mathbf{H}'d\mathbf{H})$. Summarizing we get

$$\mathbf{E} = \mathbf{F}^{1/2}\mathbf{H} \Rightarrow (d\mathbf{E}) = 2^{-K}|\mathbf{F}|^{-1/2}(d\mathbf{F})(\mathbf{H}'d\mathbf{H}). \tag{7}$$

Now,

$$\mathbf{Y} = \begin{pmatrix} \mathbf{I} \\ \mathbf{V} \end{pmatrix} \mathbf{E} = \begin{pmatrix} \mathbf{E} \\ \mathbf{V}\mathbf{E} \end{pmatrix}.$$

Differentiating and computing the exterior product, we get $(d\mathbf{Y}) = |\mathbf{E}|^q(d\mathbf{V})(d\mathbf{E})$, but $|\mathbf{E}| = |\mathbf{F}^{1/2}\mathbf{H}| = |\mathbf{F}|^{1/2}$, so

$$(d\mathbf{Y}) = |\mathbf{F}|^{q/2}(d\mathbf{V})(d\mathbf{E}). \tag{8}$$

Replacing (7) in (8) we obtain the required result. □

Now we can state with the help of Theorem 1 the main statistical result of this work, the general case of the configuration density under a non-isotropic noncentral elliptical model.

Theorem 9 *If* $\mathbf{Y} \sim \mathcal{E}_{N-1\times K}(\mu_{N-1\times K}, \mathbf{\Sigma}_{N-1\times N-1} \otimes \mathbf{I}_K, h)$, *for* $\mathbf{\Sigma}$ *positive definite* $(\mathbf{\Sigma} > \mathbf{0})$, $\mu \neq 0_{N-1\times K}$, *then the configuration density is given by*

$$\frac{\pi^{K^2/2}\Gamma_K\left(\frac{N-1}{2}\right)}{|\mathbf{\Sigma}|^{\frac{K}{2}}|\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}|^{\frac{N-1}{2}}\Gamma_K\left(\frac{K}{2}\right)}\sum_{t=0}^{\infty}\frac{1}{t!\Gamma\left(\frac{K(N-1)}{2}+t\right)}\sum_{r=0}^{\infty}\frac{1}{r!}\left[\operatorname{tr}\left(\mu'\mathbf{\Sigma}^{-1}\mu\right)\right]^r$$
$$\times\sum_{\tau}\frac{\left(\frac{N-1}{2}\right)_\tau}{\left(\frac{K}{2}\right)_\tau}C_\tau(\mathbf{U}'\mathbf{\Sigma}^{-1}\mu\mu'\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U})^{-1})S, \tag{9}$$

where

$$S = \int_0^\infty h^{(2t+r)}(y)y^{\frac{K(N-1)}{2}+t-1}dy < \infty, \tag{10}$$

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Proof. The density of $\mathbf{Y}$ is given by

$$\frac{1}{|\mathbf{\Sigma}|^{\frac{K}{2}}} h\{\operatorname{tr}[(\mathbf{Y}-\mu)'\mathbf{\Sigma}^{-1}(\mathbf{Y}-\mu)]\}.$$

If we factorize $\mathbf{Y}$ according to Lemma 8, then the joint density of $\mathbf{U}$, $\mathbf{F}$ and $\mathbf{H}$ is

$$\frac{|\mathbf{F}|^{(q-1)/2}}{2^K|\mathbf{\Sigma}|^{\frac{K}{2}}} h\left[\operatorname{tr}\left(\mu'\mathbf{\Sigma}^{-1}\mu+\mathbf{F}\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}\right)+\operatorname{tr}\left(-2\mu'\mathbf{\Sigma}^{-1}\mathbf{U}\mathbf{F}^{1/2}\mathbf{H}\right)\right](\mathbf{H}'d\mathbf{H})(d\mathbf{F})(d\mathbf{V}).$$

Assuming that h admits a Taylor expansion (see Fang (1990a,b)), the joint density of $\mathbf{U}$, $\mathbf{F}$ and $\mathbf{H}$ becomes:

$$\frac{|\mathbf{F}|^{(q-1)/2}}{2^K|\mathbf{\Sigma}|^{\frac{K}{2}}}\sum_{t=0}^{\infty}\frac{1}{t!}\frac{h^{(t)}\left(\operatorname{tr}\left(\mathbf{F}\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}\right)+\operatorname{tr}\left(\mu'\mathbf{\Sigma}^{-1}\mu\right)\right)}{\left[\operatorname{tr}\left(-2\mu'\mathbf{\Sigma}^{-1}\mathbf{U}\mathbf{F}^{1/2}\mathbf{H}\right)\right]^{-t}}(\mathbf{H}'d\mathbf{H})(d\mathbf{F})(d\mathbf{V}).$$

Now from James (1964) equation (22), then integration with respect to $\mathbf{H}$ gives the joint density of $\mathbf{F}$ and $\mathbf{U}$ as follows

$$\frac{\pi^{K^2/2}|\mathbf{F}|^{(q-1)/2}}{|\mathbf{\Sigma}|^{\frac{K}{2}}\Gamma_K\left(\frac{K}{2}\right)}\sum_{t=0}^{\infty}\frac{1}{t!}h^{(2t)}\left(\operatorname{tr}\left(\mathbf{F}\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}\right)+\operatorname{tr}\left(\mu'\mathbf{\Sigma}^{-1}\mu\right)\right)$$
$$\times\sum_{\tau}\frac{1}{\left(\frac{K}{2}\right)_{\tau}}C_{\tau}\left(\mathbf{U}'\mathbf{\Sigma}^{-1}\mu\mu'\mathbf{\Sigma}^{-1}\mathbf{U}\mathbf{F}\right)(d\mathbf{F})(d\mathbf{V}).$$

Noting that $h^{2t}(\cdot)$ admits a Taylor expansion, then the joint density of $\mathbf{F}$ and $\mathbf{U}$ finally takes the form :

$$\frac{\pi^{K^2/2}|\mathbf{F}|^{(q-1)/2}}{|\mathbf{\Sigma}|^{\frac{K}{2}}\Gamma_K\left(\frac{K}{2}\right)}\sum_{t=0}^{\infty}\frac{1}{t!}\sum_{r=0}^{\infty}\frac{1}{r!}h^{(2t+r)}\left(\operatorname{tr}\left(\mathbf{F}\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}\right)\right)\left[\operatorname{tr}\left(\mu'\mathbf{\Sigma}^{-1}\mu\right)\right]^{r}$$
$$\times\sum_{\tau}\frac{1}{\left(\frac{K}{2}\right)_{\tau}}C_{\tau}\left(\mathbf{U}'\mathbf{\Sigma}^{-1}\mu\mu'\mathbf{\Sigma}^{-1}\mathbf{U}\mathbf{F}\right)(d\mathbf{F})(d\mathbf{V}).$$

Thus, integration over $\mathbf{F}>\mathbf{0}$ gives the configuration density as follows

$$\frac{\pi^{K^2/2}}{|\mathbf{\Sigma}|^{\frac{K}{2}}\Gamma_K\left(\frac{K}{2}\right)}\sum_{t=0}^{\infty}\frac{1}{t!}\sum_{r=0}^{\infty}\frac{1}{r!}\left[\operatorname{tr}\left(\mu'\mathbf{\Sigma}^{-1}\mu\right)\right]^{r}\sum_{\tau}\frac{1}{\left(\frac{K}{2}\right)_{\tau}}$$
$$\times\int_{\mathbf{F}>\mathbf{0}}h^{(2t+r)}\left(\operatorname{tr}\left(\mathbf{F}\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}\right)\right)|\mathbf{F}|^{(q-1)/2}C_{\tau}\left(\mathbf{U}'\mathbf{\Sigma}^{-1}\mu\mu'\mathbf{\Sigma}^{-1}\mathbf{U}\mathbf{F}\right)(d\mathbf{F}). \quad (11)$$

Here, the integral in (11) is reduced by (1) to

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$$\int_{\mathbf{F}>\mathbf{0}} h^{(2t+r)}\left(\operatorname{tr}\left(\mathbf{F}\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}\right)\right)|\mathbf{F}|^{(q-1)/2}C_\tau\left(\mathbf{U}'\mathbf{\Sigma}^{-1}\mu\mu'\mathbf{\Sigma}^{-1}\mathbf{U}\mathbf{F}\right)(d\mathbf{F})$$
$$= \frac{|\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}|^{-\frac{N-1}{2}}\left(\frac{N-1}{2}\right)_\tau \Gamma_K\left(\frac{N-1}{2}\right)C_\tau(\mathbf{U}'\mathbf{\Sigma}^{-1}\mu\mu'\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U})^{-1})}{\Gamma\left(\frac{K(N-1)}{2}+t\right)}S,$$

where
$$S=\int_0^\infty h^{(2t+r)}(y)y^{\frac{K(N-1)}{2}+t-1}dy<\infty,$$
and the required result follows. □

Finally, the central configuration density can be obtained.

Corollary 10 *If* $\mathbf{Y}\sim\mathcal{E}_{N-1\times K}(0_{N-1\times K},\mathbf{\Sigma}_{N-1\times N-1}\otimes\mathbf{I}_K,h)$, $\mathbf{\Sigma}>\mathbf{0}$, *then the central configuration density is invariant under the elliptical contoured distributions and it is given by*
$$\frac{\Gamma_K\left(\frac{K(N-1)}{2}\right)}{\pi^{\frac{Kq}{2}}|\mathbf{\Sigma}|^{\frac{K}{2}}\Gamma_K\left(\frac{K}{2}\right)}|\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}|^{-\frac{N-1}{2}}.$$

Proof. A detailed proof of this result is given by Caro-Lopera (2008). □

The preceding proposition generalizes Theorem 3.2 of Díaz-García *et al.* (2003) which concerned the isotropic case, $\mathbf{\Sigma}=\sigma^2 I_{N-1}$, (recall that $|\mathbf{U}'\mathbf{U}|=|\mathbf{I}_K+\mathbf{V}'\mathbf{V}|$); in this case both expression coincide excepting the factor 2^K, since Goodall and Mardia (1993) and Díaz-García *et al.* (2003) did not described the Haar measure $(\mathbf{H}'d\mathbf{H})$ employed in the computation of the jacobian of $\mathbf{Y}=\mathbf{U}\mathbf{F}^{1/2}\mathbf{H}$, see our Lemma 8 for solving this discrepancy.

Most of the applications in statistical theory of shape resides on the isotropic model (see Dryden and Mardia (1998)), so in the case of the noncentral elliptical configuration density if we take $\mathbf{\Sigma}=\sigma^2 I_{N-1}$ in Theorem 9 we obtain

Corollary 11 *If* $\mathbf{Y}\sim\mathcal{E}_{N-1\times K}(\mu_{N-1\times K},\sigma^2 I_{N-1}\otimes\mathbf{I}_K,h)$, *then the isotropic noncentral configuration density is given by*

$$\frac{\pi^{K^2/2}\Gamma_K\left(\frac{N-1}{2}\right)}{|\mathbf{I}_K+\mathbf{V}'\mathbf{V}|^{\frac{N-1}{2}}\Gamma_K\left(\frac{K}{2}\right)}\sum_{t=0}^\infty\frac{1}{t!\Gamma\left(\frac{K(N-1)}{2}+t\right)}\sum_{r=0}^\infty\frac{1}{r!}\left[\operatorname{tr}\left(\frac{1}{\sigma^2}\mu'\mu\right)\right]^r$$
$$\times\sum_\tau\frac{\left(\frac{N-1}{2}\right)_\tau}{\left(\frac{K}{2}\right)_\tau}C_\tau\left(\frac{1}{\sigma^2}\mathbf{U}'\mu\mu'\mathbf{U}(\mathbf{U}'\mathbf{U})^{-1}\right)S,$$

where
$$S=\int_0^\infty h^{(2t+r)}(y)y^{\frac{K(N-1)}{2}+t-1}dy<\infty,$$

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4 Families of elliptical configuration densities

In this section we derive explicit configuration densities for matrix variate symmetric Kotz type distributions (it includes normal), matrix variate Pearson type VII distributions (it includes t and Cauchy distributions), matrix variate symmetric Bessel distribution (it includes Laplace distribution) and matrix variate symmetric logistic distribution.

4.1 Pearson type VII configuration density

We already discussed some facts of this distributions in subsection 2.1.

Now the corresponding configuration is derived.

Corollary 12 *If* $\mathbf{Y} \sim \mathcal{E}_{N-1\times K}(\mu_{N-1\times K}, \mathbf{\Sigma}_{N-1\times N-1} \otimes \mathbf{I}_K, h)$*,* $\mathbf{\Sigma} > \mathbf{0}$*, then the non-isotropic noncentral Pearson type VII configuration density is given by*

$$\frac{\Gamma_K\left(\frac{N-1}{2}\right)}{\pi^{Kq/2}|\mathbf{\Sigma}|^{\frac{K}{2}}|\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}|^{\frac{N-1}{2}}\Gamma_K\left(\frac{K}{2}\right)}$$
$$ {}_1P_1\left(\left(s-\frac{K(N-1)}{2}\right)_t\left(1+\frac{\operatorname{tr}(\mu'\mathbf{\Sigma}^{-1}\mu)}{R}\right)^{-s+\frac{K(N-1)}{2}-t}: \frac{N-1}{2};\frac{K}{2};\frac{1}{R}\mathbf{U}'\mathbf{\Sigma}^{-1}\mu\mu'\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U})^{-1}\right),$$

where

$$ {}_1P_1(f(t):a;b;\mathbf{X}) = \sum_{t=0}^{\infty}\frac{f(t)}{t!}\sum_{\tau}\frac{(a)_\tau}{(b)_\tau}C_\tau(\mathbf{X}).$$

Proof. In this case we take

$$h(y) = \frac{\Gamma(s)}{(\pi R)^{\frac{K(N-1)}{2}}\Gamma\left(s-\frac{K(N-1)}{2}\right)}\left(1+\frac{y}{R}\right)^{-s},$$

see Gupta and Varga (1993), then

$$h(y)^{(2t+r)} = \frac{\Gamma(s)(-1)^r(s)_{2t+r}}{(\pi R)^{\frac{K(N-1)}{2}}\Gamma\left(s-\frac{K(N-1)}{2}\right)R^{2t+r}}\left(1+\frac{y}{R}\right)^{-(s+2t+r)},$$

and after some simplification (10) becomes

$$\mathcal{S} = \frac{(-1)^r\Gamma\left(\frac{K(N-1)}{2}+t\right)\left(s-\frac{K(N-1)}{2}\right)_{t+r}}{\pi^{\frac{K(N-1)}{2}}R^{t+r}}.$$

Thus (9) takes the form

$$\frac{\Gamma_K\left(\frac{N-1}{2}\right)}{\pi^{Kq/2}|\mathbf{\Sigma}|^{\frac{K}{2}}|\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}|^{\frac{N-1}{2}}\Gamma_K\left(\frac{K}{2}\right)}$$
$$\times\sum_{t=0}^{\infty}\frac{1}{t!}\left\{\sum_{r=0}^{\infty}\frac{\left(s-\frac{K(N-1)}{2}\right)_{t+r}}{r!}\left[\mathrm{tr}\left(-\frac{1}{R}\mu'\mathbf{\Sigma}^{-1}\mu\right)\right]^r\right\}$$
$$\times\sum_{\tau}\frac{\left(\frac{N-1}{2}\right)_\tau}{\left(\frac{K}{2}\right)_\tau}C_\tau\left(\frac{1}{R}\mathbf{U}'\mathbf{\Sigma}^{-1}\mu\mu'\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U})^{-1}\right).$$

The term in braces preserves the core of the distribution, in this case it is $(a)_t(1+u/R)^{-a-t}$, with $a=s-K(N-1)/2>0$, $u=\mathrm{tr}(\mu'\mathbf{\Sigma}^{-1}\mu)$, so we get the non-isotropic noncentral Pearson type VII configuration density. □

If we take $s=(K(N-1)+R)/2$ in corollary 12 we obtain the configuration density associated to a matrix variate t-distribution with R degrees of freedom. And if we replace $R=1$ in the above density, we obtain the respective Cauchy configuration density. So

Corollary 13 *If $\mathbf{Y}\sim\mathcal{E}_{N-1\times K}(\mu_{N-1\times K},\mathbf{\Sigma}_{N-1\times N-1}\otimes\mathbf{I}_K,h)$, with $\mathbf{\Sigma}>\mathbf{0}$, then:*

(1) the non-isotropic noncentral t configuration density is given by

$$\frac{\Gamma_K\left(\frac{N-1}{2}\right)}{\pi^{Kq/2}|\mathbf{\Sigma}|^{\frac{K}{2}}|\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}|^{\frac{N-1}{2}}\Gamma_K\left(\frac{K}{2}\right)}{}_1P_1\left(\left(\frac{R}{2}\right)_t\left(1+\frac{\mathrm{tr}(\mu'\mathbf{\Sigma}^{-1}\mu)}{R}\right)^{-\frac{R}{2}-t}:\frac{N-1}{2};\frac{K}{2};\frac{1}{R}\mathbf{U}'\mathbf{\Sigma}^{-1}\mu\mu'\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U})^{-1}\right).$$

(2) then the non-isotropic noncentral Cauchy configuration density is given by

$$\frac{\Gamma_K\left(\frac{N-1}{2}\right)}{\pi^{Kq/2}|\mathbf{\Sigma}|^{\frac{K}{2}}|\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}|^{\frac{N-1}{2}}\Gamma_K\left(\frac{K}{2}\right)}{}_1P_1\left(\left(\frac{1}{2}\right)_t\left(1+\mathrm{tr}(\mu'\mathbf{\Sigma}^{-1}\mu)\right)^{-\frac{1}{2}-t}:\frac{N-1}{2};\frac{K}{2};\mathbf{U}'\mathbf{\Sigma}^{-1}\mu\mu'\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U})^{-1}\right).$$

4.2 Kotz type configuration density

For finding in this case a closed form of (9) we need some additional results, in this case the partition theory provides suitable expressions for derivatives, by Caro-

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Lopera (2008) we have that

Lemma 14 *Let* $h(y) = y^{T-1}\,\mathrm{e}^{-Ry^s}$*, where,* $R > 0$*,* $s > 0$*,* $2T + K(N-1) > 2$ *and* $N - 1 - K \geq 1$*; if* $w^{(k)}$ *denotes* $\frac{d^k w}{dy^k}$ *and* $\sum_{\kappa \in P_r}$ *is the summation over all the partitions* $\kappa = (k^{\nu_k}, (k-1)^{\nu_{k-1}}, \cdots, 3^{\nu_3}, 2^{\nu_2}, 1^{\nu_1})$ *of* r*, then we have*

$$h^{(k)} = y^{T-1}\,\mathrm{e}^{-Ry^s}\left\{\sum_{\kappa\in P_k}\frac{k!(-R)^{\sum_{i=1}^{k}\nu_i}\prod_{j=0}^{k-1}(s-j)^{\sum_{i=j+1}^{k}\nu_i}}{\prod_{i=1}^{k}\nu_i!(i!)^{\nu_i}}y^{\sum_{i=1}^{k}(s-i)\nu_i}\right.$$
$$+\sum_{m=1}^{k}\binom{k}{m}\left[\prod_{i=0}^{m-1}(T-1-i)\right]$$
$$\left.\times\sum_{\kappa\in P_{k-m}}\frac{(k-m)!(-R)^{\sum_{i=1}^{k-m}\nu_i}\prod_{j=0}^{k-m-1}(s-j)^{\sum_{i=j+1}^{k-m}\nu_i}}{\prod_{i=1}^{k-m}\nu_i!(i!)^{\nu_i}}y^{\sum_{i=1}^{k-m}(s-i)\nu_i-m}\right\}.(12)$$

So from Gupta and Varga (1993),

$$h(y) = \frac{sR^{\frac{2T+K(N-1)-2}{2s}}\Gamma\left(\frac{K(N-1)}{2}\right)}{\pi^{K(N-1)/2}\Gamma\left(\frac{2T+K(N-1)-2}{2s}\right)}y^{T-1}\,\mathrm{e}^{-Ry^s},$$

and from (12) we have the required derivative; then the integral (10) becomes

$$S = \frac{\Gamma\left(\frac{K(N-1)}{2}\right)}{\pi^{K(N-1)/2}\Gamma\left(\frac{2T+K(N-1)-2}{2s}\right)}$$
$$\times\left\{\sum_{\kappa\in P_{2t+r}}\frac{(2t+r)!\prod_{j=0}^{2t+r-1}(s-j)^{\sum_{i=j+1}^{2t+r}\nu_i}\Gamma\left(\frac{2\sum_{i=1}^{2t+r}(s-i)\nu_i+2T-2+K(N-1)+2t}{2s}\right)}{(-1)^{\sum_{i=1}^{2t+r}\nu_i}R^{\frac{-\sum_{i=1}^{2t+r}i\nu_i+t}{s}}\prod_{i=1}^{2t+r}\nu_i!(i!)^{\nu_i}}\right.$$
$$+\sum_{m=1}^{2t+r}\binom{2t+r}{m}\left[\prod_{i=0}^{m-1}(T-1-i)\right]$$
$$\times\sum_{\kappa\in P_{2t+r-m}}\frac{(2t+r-m)!\prod_{j=0}^{2t+r-m-1}(s-j)^{\sum_{i=j+1}^{2t+r-m}\nu_i}}{(-1)^{\sum_{i=1}^{2t+r-m}\nu_i}R^{\frac{-\sum_{i=1}^{2t+r-m}i\nu_i-m+t}{s}}\prod_{i=1}^{2t+r-m}\nu_i!(i!)^{\nu_i}}$$
$$\left.\times\,\Gamma\left(\frac{2\sum_{i=1}^{2t+r-m}(s-i)\nu_i-2m+2T-2+K(N-1)+2t}{2s}\right)\right\}.$$

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Finally the corresponding configuration density results

Corollary 15 *If* $\mathbf{Y} \sim \mathcal{E}_{N-1\times K}(\mu_{N-1\times K}, \mathbf{\Sigma}_{N-1\times N-1} \otimes \mathbf{I}_K, h)$*, with* $\mathbf{\Sigma} > \mathbf{0}$*, then the Kotz type III non-isotropic noncentral configuration density is given by*

$$\frac{\Gamma_K\left(\frac{N-1}{2}\right)}{\pi^{Kq/2}|\mathbf{\Sigma}|^{\frac{K}{2}}|\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}|^{\frac{N-1}{2}}\Gamma_K\left(\frac{K}{2}\right)}\sum_{t=0}^{\infty}\frac{\Gamma\left(\frac{K(N-1)}{2}\right)}{t!\Gamma\left(\frac{K(N-1)}{2}+t\right)\Gamma\left(\frac{2T+K(N-1)-2}{2s}\right)}$$

$$\times\sum_{r=0}^{\infty}\frac{1}{r!}\left[\operatorname{tr}\left(\mu'\mathbf{\Sigma}^{-1}\mu\right)\right]^r\sum_{\tau}\frac{\left(\frac{N-1}{2}\right)_\tau}{\left(\frac{K}{2}\right)_\tau}C_\tau\left(\mathbf{U}'\mathbf{\Sigma}^{-1}\mu\mu'\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U})^{-1}\right)$$

$$\times\left\{\sum_{\kappa\in P_{2t+r}}\frac{(2t+r)!\prod_{j=0}^{2t+r-1}(s-j)^{\sum_{i=j+1}^{2t+r}\nu_i}\Gamma\left(\frac{2\sum_{i=1}^{2t+r}(s-i)\nu_i+2T-2+K(N-1)+2t}{2s}\right)}{(-1)^{\sum_{i=1}^{2t+r}\nu_i}R^{\frac{-\sum_{i=1}^{2t+r}i\nu_i+t}{s}}\prod_{i=1}^{2t+r}\nu_i!(i!)^{\nu_i}}\right.$$

$$+\sum_{m=1}^{2t+r}\binom{2t+r}{m}\left[\prod_{i=0}^{m-1}(T-1-i)\right]$$

$$\times\sum_{\kappa\in P_{2t+r-m}}\frac{(2t+r-m)!\prod_{j=0}^{2t+r-m-1}(s-j)^{\sum_{i=j+1}^{2t+r-m}\nu_i}}{(-1)^{\sum_{i=1}^{2t+r-m}\nu_i}R^{\frac{-\sum_{i=1}^{2t+r-m}i\nu_i-m+t}{s}}\prod_{i=1}^{2t+r-m}\nu_i!(i!)^{\nu_i}}$$

$$\left.\times\,\Gamma\left(\frac{2\sum_{i=1}^{2t+r-m}(s-i)\nu_i-2m+2T-2+K(N-1)+2t}{2s}\right)\right\}.$$

Recall that if the above densities exists, then the arguments in gamma's must be positive and this suggest a careful election of the Kotz parameters for doing inference, more over, given the complexity of the expressions it is necessary to truncate the series for some t and r in such way that the parameters T, s and R can be selected.

Some particular cases simplify substantially the above density. If $s = 1$, then we have.

Corollary 16 *If* $\mathbf{Y} \sim \mathcal{E}_{N-1\times K}(\mu_{N-1\times K}, \mathbf{\Sigma}_{N-1\times N-1} \otimes \mathbf{I}_K, h)$*, with* $\mathbf{\Sigma} > \mathbf{0}$*, then the Kotz type* $s = 1$ *non-isotropic noncentral configuration density is given by*

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$$
\frac{\Gamma_K\left(\frac{N-1}{2}\right)}{\pi^{Kq/2}|\mathbf{\Sigma}|^{\frac{K}{2}}|\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}|^{\frac{N-1}{2}}\Gamma_K\left(\frac{K}{2}\right)}\sum_{t=0}^{\infty}\frac{\Gamma\left(\frac{K(N-1)}{2}\right)}{t!\Gamma\left(\frac{K(N-1)}{2}+t\right)\Gamma\left(T-1+\frac{K(N-1)}{2}\right)}
$$
$$
\times\sum_{r=0}^{\infty}\frac{1}{r!}\left[\operatorname{tr}\left(-R\mu'\mathbf{\Sigma}^{-1}\mu\right)\right]^r\sum_{\tau}\frac{\left(\frac{N-1}{2}\right)_\tau}{\left(\frac{K}{2}\right)_\tau}C_\tau\left(R\mathbf{U}'\mathbf{\Sigma}^{-1}\mu\mu'\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U})^{-1}\right)
$$
$$
\times\left\{\Gamma\left(T-1+\frac{K(N-1)}{2}+t\right)\right.
$$
$$
\left.+\sum_{m=1}^{2t+r}\binom{2t+r}{m}\left[\prod_{i=0}^{m-1}(T-1-i)\right](-1)^m\Gamma\left(T-1-m+\frac{K(N-1)}{2}+t\right)\right\}. \quad (13)
$$

Now, consider $T=1$ in (13), then a confluent hypergeometric class of densities indexed by R are obtained, i.e.

Corollary 17 *If* $\mathbf{Y}\sim\mathcal{E}_{N-1\times K}(\mu_{N-1\times K},\mathbf{\Sigma}_{N-1\times N-1}\otimes\mathbf{I}_K,h)$, *with* $\mathbf{\Sigma}>\mathbf{0}$ *and* $T=1$, *then the Kotz type I non-isotropic noncentral configuration density simplifies to*

$$
\frac{\Gamma_K\left(\frac{N-1}{2}\right)\operatorname{etr}\left(R\mathbf{U}'\mathbf{\Sigma}^{-1}\mu\mu'\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U})^{-1}-R\mu'\mathbf{\Sigma}^{-1}\mu\right)}{\pi^{Kq/2}|\mathbf{\Sigma}|^{\frac{K}{2}}|\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}|^{\frac{N-1}{2}}\Gamma_K\left(\frac{K}{2}\right)}
$$
$$
\times\ {}_1F_1\left(-\frac{q}{2};\frac{K}{2};-R\mathbf{U}'\mathbf{\Sigma}^{-1}\mu\mu'\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U})^{-1}\right).
$$

Finally the normal configuration density can be derived by taking $R=\frac{1}{2}$. In this case we obtained a result of Díaz-García *et al.* (2003) (proposed by Goodall and Mardia (1993) with some errors) except for the factor 2^k which comes from their anonymous jacobian computation.

Another simplification comes from $T=1$, then the particular configuration density is:

Corollary 18 *If* $\mathbf{Y}\sim\mathcal{E}_{N-1\times K}(\mu_{N-1\times K},\mathbf{\Sigma}_{N-1\times N-1}\otimes\mathbf{I}_K,h)$, *with* $\mathbf{\Sigma}>\mathbf{0}$, *then the Kotz type* $T=1$ *non-isotropic noncentral configuration density is given by*

$$
\frac{\Gamma_K\left(\frac{N-1}{2}\right)}{\pi^{Kq/2}|\mathbf{\Sigma}|^{\frac{K}{2}}|\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}|^{\frac{N-1}{2}}\Gamma_K\left(\frac{K}{2}\right)}\sum_{t=0}^{\infty}\frac{\Gamma\left(\frac{K(N-1)}{2}\right)}{t!\Gamma\left(\frac{K(N-1)}{2}+t\right)\Gamma\left(\frac{K(N-1)}{2s}\right)}
$$
$$
\times\sum_{r=0}^{\infty}\frac{1}{r!}\left[\operatorname{tr}\left(\mu'\mathbf{\Sigma}^{-1}\mu\right)\right]^r\sum_{\tau}\frac{\left(\frac{N-1}{2}\right)_\tau}{\left(\frac{K}{2}\right)_\tau}C_\tau\left(\mathbf{U}'\mathbf{\Sigma}^{-1}\mu\mu'\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U})^{-1}\right)
$$
$$
\times\sum_{\kappa\in P_{2t+r}}\frac{(2t+r)!\prod_{j=0}^{2t+r-1}(s-j)^{\sum_{i=j+1}^{2t+r}\nu_i}\Gamma\left(\frac{2\sum_{i=1}^{2t+r}(s-i)\nu_i+K(N-1)+2t}{2s}\right)}{(-1)^{\sum_{i=1}^{2t+r}\nu_i}R^{-\frac{\sum_{i=1}^{2t+r}i\nu_i-t}{s}}\prod_{i=1}^{2t+r}\nu_i!(i!)^{\nu_i}}. \quad (14)
$$

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We can get again corollary 17, and so the normal case, by taking $s = 1$ in (14) and noting that in this trivial case $\nu_1 = 2t + r$, $\nu_i = 0$ for $i = 2, \ldots, 2t + r$, and $\prod_{j=0}^{2t+r-1}(s-j)^{\sum_{i=j+1}^{2t+r}\nu_i} = (1-0)^{\sum_{i=0+1}^{2t+r}\nu_i} = 1^{2t+r} = 1$. Then the last summation of (14) becomes $(-1)^r R^{t+r}\Gamma\left(\frac{K(N-1)}{2} + t\right)$ and the result follows easily by exponential series, hypergeometric function definition and Kummer formula.

Recall that if the above densities exists, then the arguments in gamma's must be positive and this suggest a careful election of the Kotz parameters for doing inference, more over, given the complexity of the expressions it is necessary to truncate the series for some t and r in such way that the parameters T, s and R can be selected (Caro-Lopera (2008)).

4.3 Bessel configuration density

Another elliptical distribution is the so called Bessel distribution, and after some correction in Gupta and Varga (1993), we can define that, the $p \times n$ random matrix is said to have a matrix variate symmetric Bessel distribution with parameters $q, r \in \mathbb{R}$, $\mu : p \times n$, $\mathbf{\Sigma} : p \times p$, $\Phi : n \times n$ with $r > 0$, q is integer such that $q > -\frac{np}{2}$, $\mathbf{\Sigma} > \mathbf{0}$, and $\Phi > 0$ if its probability density function is

$$\frac{[\operatorname{tr}(\mathbf{Y}-\mu)'\mathbf{\Sigma}^{-1}(\mathbf{Y}-\mu)\Phi^{-1}]^{\frac{q}{2}} K_q\left(\frac{[\operatorname{tr}(\mathbf{Y}-\mu)'\mathbf{\Sigma}^{-1}(\mathbf{Y}-\mu)\Phi^{-1}]^{\frac{1}{2}}}{r}\right)}{2^{q+np-1}\pi^{\frac{np}{2}} r^{np+q}\Gamma\left(q+\frac{np}{2}\right)|\mathbf{\Sigma}|^{\frac{n}{2}}|\Phi|^{\frac{p}{2}}},$$

where $K_q(z)$ is the modified Bessel function of the third kind; that is

$$K_q(z) = \frac{\pi}{2}\frac{I_{-q}(z) - I_q(z)}{\sin(q\pi)}, \quad |\arg(z)| < \pi,$$

and

$$I_q(z) = \sum_{k=0}^{\infty}\frac{1}{k!\Gamma(k+q+1)}\left(\frac{z}{2}\right)^{q+2k}, \ |z| < \infty, |\arg(z)| < \pi.$$

In this case the function h takes the form

$$h(y) = \frac{y^{\frac{q}{2}} K_q\left(\frac{1}{r}y^{\frac{1}{2}}\right)}{2^{q+K(N-1)-1}\pi^{\frac{K(N-1)}{2}} r^{K(N-1)+q}\Gamma\left(q+\frac{K(N-1)}{2}\right)}, \tag{15}$$

and the required derivatives for the modified Bessel function are given by

$$K_q^{(k)} = \frac{1}{2^k}\sum_{m=0}^{k}\binom{k}{m}K_{q-k+2m}(z).$$

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Then, the k-th derivative of (15) follows after some simplification as

$$
\begin{aligned}
h^{(k)} = & \frac{1}{2^{q+K(N-1)-1}\pi^{\frac{K(N-1)}{2}} r^{K(N-1)+q}\Gamma\left(q+\frac{K(N-1)}{2}\right)} \\
& \times \sum_{m=0}^{k}\binom{k}{m}\left[\prod_{j=0}^{m-1}\left(\frac{q}{2}-j\right)\right]\sum_{\kappa\in P_{k-m}}\frac{(k-m)!\prod_{j=0}^{k-m-1}\left(\frac{1}{2}-j\right)^{\sum_{i=j+1}^{k-m}\nu_i}}{2^{\sum_{i=1}^{k-m}\nu_i} r^{\sum_{i=1}^{k-m}\nu_i}\prod_{i=1}^{k-m}\nu_i!(i!)^{\nu_i}} \\
& \times \sum_{n=0}^{\sum_{i=1}^{k-m}\nu_i}\binom{\sum_{i=1}^{k-m}\nu_i}{n} K_{q-\sum_{i=1}^{k-m}\nu_i+2n}\left(\frac{1}{r}y^{\frac{1}{2}}\right) y^{\sum_{i=1}^{k-m}\left(\frac{1}{2}-i\right)\nu_i+\frac{q}{2}-m}.
\end{aligned}
$$

Thus the integral S in (10) can be now computed:

$$
\begin{aligned}
S = & \int_0^{\infty} h^{(2t+r)}(y) y^{\frac{K(N-1)}{2}+t-1}dy \\
= & \frac{1}{\pi^{\frac{K(N-1)}{2}}\Gamma\left(q+\frac{K(N-1)}{2}\right)}\sum_{m=0}^{2t+r}\binom{2t+r}{m}\left[\prod_{j=0}^{m-1}\left(\frac{q}{2}-j\right)\right] \\
& \times \sum_{\kappa\in P_{2t+r-m}}\frac{(2t+r-m)!\prod_{j=0}^{2t+r-m-1}\left(\frac{1}{2}-j\right)^{\sum_{i=j+1}^{2t+r-m}\nu_i}}{(2r)^{2\sum_{i=1}^{2t+r-m}i\nu_i+2m-2t}\prod_{i=1}^{2t+r-m}\nu_i!(i!)^{\nu_i}} \\
& \times \sum_{n=0}^{\sum_{i=1}^{2t+r-m}\nu_i}\binom{\sum_{i=1}^{2t+r-m}\nu_i}{n}\Gamma\left(\sum_{i=1}^{2t+r-m}(1-i)\nu_i - m + \frac{K(N-1)}{2}+t-n\right) \\
& \times \Gamma\left(-\sum_{i=1}^{2t+r-m} i\nu_i + q - m + \frac{K(N-1)}{2} + t + n\right),
\end{aligned}
$$

The conditions for the existence of S are the same indicated for the Bessel distribution plus the conditions demanded by the arguments of the gamma's which must be positive.

Thus the configuration density follows from (9)

Corollary 19 *If* $\mathbf{Y} \sim \mathcal{E}_{N-1\times K}(\mu_{N-1\times K}, \mathbf{\Sigma}_{N-1\times N-1}\otimes \mathbf{I}_K, h)$, *with* $\mathbf{\Sigma} > \mathbf{0}$, *then the Bessel non-isotropic noncentral configuration density is given by*

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$$
\begin{aligned}
&\frac{\Gamma_K\left(\frac{N-1}{2}\right)}{\pi^{\frac{Kq}{2}}|\mathbf{\Sigma}|^{\frac{K}{2}}|\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}|^{\frac{N-1}{2}}\Gamma_K\left(\frac{K}{2}\right)}\sum_{t=0}^{\infty}\frac{1}{t!\Gamma\left(\frac{K(N-1)}{2}+t\right)\Gamma\left(q+\frac{K(N-1)}{2}\right)} \\
&\times\sum_{r=0}^{\infty}\frac{1}{r!}\left[\operatorname{tr}\left(\mu'\mathbf{\Sigma}^{-1}\mu\right)\right]^r\sum_{\tau}\frac{\left(\frac{N-1}{2}\right)_\tau}{\left(\frac{K}{2}\right)_\tau}C_\tau\left(\mathbf{U}'\mathbf{\Sigma}^{-1}\mu\mu'\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U})^{-1}\right) \\
&\times\sum_{m=0}^{2t+r}\binom{2t+r}{m}\left[\prod_{j=0}^{m-1}\left(\frac{q}{2}-j\right)\right] \\
&\times\sum_{\kappa\in P_{2t+r-m}}\frac{(2t+r-m)!\prod_{j=0}^{2t+r-m-1}\left(\frac{1}{2}-j\right)^{\sum_{i=j+1}^{2t+r-m}\nu_i}}{(2r)^{2\sum_{i=1}^{2t+r-m}i\nu_i+2m-2t}\prod_{i=1}^{2t+r-m}\nu_i!(i!)^{\nu_i}} \\
&\times\sum_{n=0}^{\sum_{i=1}^{2t+r-m}\nu_i}\binom{\sum_{i=1}^{2t+r-m}\nu_i}{n}\Gamma\left(\sum_{i=1}^{2t+r-m}(1-i)\,\nu_i-m+\frac{K(N-1)}{2}+t-n\right) \\
&\times\Gamma\left(-\sum_{i=1}^{2t+r-m}i\nu_i+q-m+\frac{K(N-1)}{2}+t+n)\right).
\end{aligned}
$$

Finally, if $q=0$ and $r=\frac{\delta}{\sqrt{2}}$, $\delta>0$ in corollary 19, then we have the Laplace non-isotropic noncentral configuration density.

Again it is important to note that for doing inference the above series must be truncated and the Bessel parameters chosen in order that the gamma's exists.

We end this work with a distribution which has no closed form for its expression, because the integrals involved can not be expressed in terms of classical functions.

4.4 Logistic configuration density

For this case we put h as (see Gupta and Varga (1993))

$$
h(y)=c\exp(-y)\left[1+\exp(-y)\right]^{-2},
$$

then by partition theory the k-th derivative can be computed after some simplification as

$$
h^{(k)}=c\sum_{m=0}^{k}\binom{k}{m}\sum_{\kappa\in P_{k-m}}\frac{(k-m)!\left(\sum_{i=1}^{k-m}\nu_i+1\right)!\exp(-(1+\sum_{i=1}^{k-m}\nu_i)y)}{(-1)^{m+\sum_{i=1}^{k-m}(1+i)\nu_i}\prod_{i=1}^{k-m}\nu_i!(i!)^{\nu_i}\left[1+\exp(-y)\right]^{2+\sum_{i=1}^{k-m}\nu_i}}.
$$

So (10) becomes

$$
\begin{aligned}
S &= \int_0^\infty h^{(2t+r)}(y) y^{\frac{K(N-1)}{2}+t-1} dy \\
&= c \sum_{m=0}^{2t+r} \binom{2t+r}{m} \sum_{\kappa \in P_{2t+r-m}} \frac{(2t+r-m)! \left(\sum_{i=1}^{2t+r-m} \nu_i + 1\right)!}{(-1)^{m+\sum_{i=1}^{2t+r-m}(1+i)\nu_i} \prod_{i=1}^{2t+r-m} \nu_i!(i!)^{\nu_i}} \\
&\quad \times \int_0^\infty \frac{\mathrm{e}^{-(1+\sum_{i=1}^{2t+r-m}\nu_i)y}}{(1+\mathrm{e}^{-y})^{2+\sum_{i=1}^{2t+r-m}\nu_i}} y^{\frac{K(N-1)}{2}+t-1} dy,
\end{aligned}
$$

And finally we get the configuration from (9)

Corollary 20 *If* $\mathbf{Y} \sim \mathcal{E}_{N-1\times K}(\mu_{N-1\times K}, \mathbf{\Sigma}_{N-1\times N-1} \otimes \mathbf{I}_K, h)$, *with* $\mathbf{\Sigma} > \mathbf{0}$, *then the logistic non-isotropic noncentral configuration density is given by*

$$
\frac{\pi^{K^2/2}\Gamma_K\left(\frac{N-1}{2}\right)}{|\mathbf{\Sigma}|^{\frac{K}{2}}|\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U}|^{\frac{N-1}{2}}\Gamma_K\left(\frac{K}{2}\right)} \sum_{t=0}^{\infty} \frac{1}{t!\Gamma\left(\frac{K(N-1)}{2}+t\right)} \sum_{r=0}^{\infty} \frac{1}{r!}\left[\mathrm{tr}\left(\mu'\mathbf{\Sigma}^{-1}\mu\right)\right]^r
$$

$$
\times \sum_{\tau} \frac{\left(\frac{N-1}{2}\right)_\tau}{\left(\frac{K}{2}\right)_\tau} C_\tau\left(\mathbf{U}'\mathbf{\Sigma}^{-1}\mu\mu'\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}'\mathbf{\Sigma}^{-1}\mathbf{U})^{-1}\right)
$$

$$
\times c \sum_{m=0}^{2t+r} \binom{2t+r}{m} \sum_{\kappa \in P_{2t+r-m}} \frac{(2t+r-m)! \left(\sum_{i=1}^{2t+r-m} \nu_i + 1\right)!}{(-1)^{m+\sum_{i=1}^{2t+r-m}(1+i)\nu_i} \prod_{i=1}^{2t+r-m} \nu_i!(i!)^{\nu_i}}
$$

$$
\times \int_0^\infty \frac{\exp(-(1+\sum_{i=1}^{2t+r-m}\nu_i)y)}{[1+\exp(-y)]^{2+\sum_{i=1}^{2t+r-m}\nu_i}} y^{\frac{K(N-1)}{2}+t-1} dy.
$$

And by definition of ν_i and t, providing that the integral in c exists, we can see that the above integral also exists, however, for inference and for a meaningful sample of configuration, the above series must be truncated enough and them much of the terms in the derivatives vanish.

The inference problem is studied in Caro-Lopera (2008) and it will be consider in the second part of this paper, however we must note that all the preceding densities can be computed by slight modifications of the algorithms by Koev and Edelman (2006). And this is clearly the most important contribution of the paper, all the distributions here derived can be computed and then exact inference can be carried out, without the classical approximations in the shape theory works.

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