# Inference in elliptical configuration distributions 

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#### Abstract

The inference procedure for any elliptical configuration density is set in this work in terms of published efficient algorithms involving infinite confluent hypergeometric type series of zonal polynomials. The finite configuration density study is proposed and it is applied in a finite Kotz configuration density subfamily, including normal; then inference procedure is addressed exact for this subfamily, and it is applied in a sort of experiments available in other shape literature contexts.


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## 1 Introduction

Statistical shape theory based on Euclidean transformation has been studied extensively in literature, see Dryden and Mardia (1998) and the references there in. By another hand, Goodall and Mardia (1993) (corrected by DíazGarcía et al. (2003) and revised again by Caro-Lopera et al (2008)) proposed a normal shape (called configuration) density based on affine transformations; it opened as usual, a distributional problem for elliptical generalizations and inference based on exact distributions.

Recently, Caro-Lopera et al (2008), derived the noncentral configuration density under an elliptical model and by using partition theory, a number of explicit configuration densities were obtained; i.e. configuration densities associated with the matrix variate symmetric Kotz type distributions (it includes normal), the matrix variate Pearson type VII distributions (it includes $t$ and Cauchy distributions), the matrix variate symmetric Bessel distribution (it includes Laplace distribution) and the matrix variate symmetric logistic distribution. The configuration density of any elliptical model was set in terms of zonal polynomials which now can be efficiently computed by Koev and Edelman (2006), and in consequence, the inference problem can be studied and solved with the exact densities instead of usual constraints and asymptotic distributions, and approximations of the statistical shape theory works (see Goodall and Mardia (1993), Dryden and Mardia (1998) and the references there in). The general procedure becomes very clear now and the underlying problem, the programming problem, is simply time consuming.

Thus two perspectives can be explored, first, the inference based on exact distributions and second, their applications in shape theory.

The general procedure for performing inference of any elliptical model is proposed and it is set in such manner that the published efficient numerical algorithms for confluent infinite series type involving zonal polynomials, can be used; this is outlined in section 2.

More over, a further simplification of the closed computational problem is also proposed, the study of finite configuration densities (section 3); a subfamily of them is derived and as a simple example of their use, exact inference for testing configuration location differences in some applied problems are provided in section 4. The applications involve Biology (mouse vertebra, gorilla skulls, girl and boy craniofacial studies), Medicine (brain MR scans of schizophrenic patients) and image analysis (postcode recognition).

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## 2 Inference for elliptical configuration models

First we recall the basic definitions of elliptical distributions and configurations (see Gupta and Varga (1993) and Goodall and Mardia (1993), respectively).

We say that $\mathbf{X}: N \times K$ has a matrix variate elliptically contoured distribution if its density respect to the Lebesgue measure is given by:

$$
f_{\mathbf{X}}(\mathbf{X})=\frac{1}{|\boldsymbol{\Sigma}|^{K / 2}|\boldsymbol{\Theta}|^{N / 2}} h\left(\operatorname{tr}\left((\mathbf{X}-\mu)^{\prime} \boldsymbol{\Sigma}^{-1}(\mathbf{X}-\mu) \boldsymbol{\Theta}^{-1}\right)\right)
$$

where $\mu: N \times K, \boldsymbol{\Sigma}: N \times N, \boldsymbol{\Theta}: K \times K, \boldsymbol{\Sigma}$ positive definite $(\boldsymbol{\Sigma}>\mathbf{0}), \boldsymbol{\Theta}>\mathbf{0}$. Such a distribution is denoted by $\mathbf{X} \sim \mathcal{E}_{N \times K}(\mu, \boldsymbol{\Sigma}, \boldsymbol{\Theta}, h)$.

Definition 1 Two figures $\mathbf{X}: N \times K$ and $\mathbf{X}_{1}: N \times K$ have the same configuration, or affine shape, if $\mathbf{X}_{1}=\mathbf{X E}+\mathbf{1}_{N} e^{\prime}$, for some translation $e: K \times 1$ and a nonsingular $\mathbf{E}: K \times K$.

The configuration coordinates are constructed in two steps summarized in the expression

$$
\begin{equation*}
\mathbf{L X}=\mathbf{Y}=\mathbf{U E} \tag{1}
\end{equation*}
$$

The matrix $\mathbf{U}: N-1 \times K$ contains configuration coordinates of $\mathbf{X}$. Let $\mathbf{Y}_{1}: K \times K$ be nonsingular and $\mathbf{Y}_{2}: q=N-K-1 \geq 1 \times K$, such that $\mathbf{Y}=\left(\mathbf{Y}_{1}^{\prime} \mid \mathbf{Y}_{2}^{\prime}\right)^{\prime}$. Define also $\mathbf{U}=\left(\mathbf{I} \mid \mathbf{V}^{\prime}\right)^{\prime}$, then $\mathbf{V}=\mathbf{Y}_{2} \mathbf{Y}_{1}^{-1}$ and $\mathbf{E}=\mathbf{Y}_{1}$. Where $\mathbf{L}$ is an $N-1 \times N$ Helmert sub-matrix.

Now, from Caro-Lopera (2008), Caro-Lopera et al (2008) the configuration density under a non-isotropic noncentral elliptical model, is given by

Theorem 2 If $\mathbf{Y} \sim \mathcal{E}_{N-1 \times K}\left(\mu_{N-1 \times K}, \boldsymbol{\Sigma}_{N-1 \times N-1} \otimes \mathbf{I}_{K}, h\right)$, for $\boldsymbol{\Sigma}>\mathbf{0}, \mu \neq$ $\mathbf{0}_{N-1 \times K}$, then the configuration density is given by

$$
\begin{array}{r}
\frac{\pi^{K^{2} / 2} \Gamma_{K}\left(\frac{N-1}{2}\right)}{|\boldsymbol{\Sigma}|^{\frac{K}{2}}\left|\mathbf{U}^{\prime} \boldsymbol{\Sigma}^{-1} \mathbf{U}\right|^{\frac{N-1}{2}} \Gamma_{K}\left(\frac{K}{2}\right)} \sum_{t=0}^{\infty} \frac{1}{t!\Gamma\left(\frac{K(N-1)}{2}+t\right)} \sum_{r=0}^{\infty} \frac{1}{r!}\left[\operatorname{tr}\left(\mu^{\prime} \boldsymbol{\Sigma}^{-1} \mu\right)\right]^{r} \\
\times \sum_{\tau} \frac{\left(\frac{N-1}{2}\right)_{\tau}}{\left(\frac{K}{2}\right)_{\tau}} C_{\tau}\left(\mathbf{U}^{\prime} \boldsymbol{\Sigma}^{-1} \mu \mu^{\prime} \boldsymbol{\Sigma}^{-1} \mathbf{U}\left(\mathbf{U}^{\prime} \boldsymbol{\Sigma}^{-1} \mathbf{U}\right)^{-1}\right) S, \tag{2}
\end{array}
$$

where

$$
S=\int_{0}^{\infty} h^{(2 t+r)}(y) y^{\frac{K(N-1)}{2}+t-1} d y<\infty
$$

Our proposal is to use the elliptically contoured distribution to model population configurations (2) for some particular cases. For this, we consider a random sample of $n$ independent and identically distributed observations $\mathbf{U}_{1}, \ldots, \mathbf{U}_{n}$ obtained from

$$
\mathbf{Y}_{i} \sim E_{N-1 \times K}\left(\mu_{N-1 \times K}, \sigma^{2} \mathbf{I}_{N-1} \otimes \mathbf{I}_{K}, h\right), \quad i=1, \ldots, n
$$

by mean of (1).
Now we define the configuration population parameters. Let $C D\left(\mathbf{U} ; \mathcal{U}, \sigma^{2}\right)$ be the exact configuration density, where $\mathcal{U}$ is the location parameter matrix of the configuration population (we just say configuration location) and $\sigma^{2}$ is population scale parameter. Both $\mathcal{U}$ and $\sigma^{2}$ are the parameters to estimate. More exactly, let $\mu \neq \mathbf{0}_{N-1 \times K}$ be the parameter matrix of the elliptical density Y considered in theorem 2; if we write it as $\mu=\left(\mu_{1}^{\prime} \mid \mu_{2}^{\prime}\right)^{\prime}$, where $\mu_{1}: K \times K$ (nonsingular) and $\mu_{2}: q=N-K-1 \geq 1 \times K$, then, according to (1), we can define the configuration location parameter matrix $\mathcal{U}: N-1 \times K$ as follows: $\mathcal{U}=\left(\mathbf{I}_{K} \mid \mathcal{V}^{\prime}\right)^{\prime}$ where $\mathcal{V}=\mu_{2} \mu_{1}^{-1} ;$ and $\mathcal{V}: q=N-K-1 \geq 1 \times K$ contains $q \times K$ configuration location parameters to estimate. Then, taking into account this remark and using the same notation of Dryden and Mardia (1998), p. 144-145. we have:

$$
\log L\left(\mathbf{U}_{1}, \ldots, \mathbf{U}_{n} ; \mathcal{V}, \sigma^{2}\right)=\sum_{i=1}^{n} \log C D\left(\mathbf{U}_{i} ; \mathcal{V}, \sigma^{2}\right)
$$

Finally, the maximum likelihood estimators for location and scale parameters are

$$
\begin{equation*}
\left(\tilde{\mathcal{V}}, \widetilde{\sigma^{2}}\right)=\underset{\mathcal{V}, \sigma^{2}}{\arg \sup \log L\left(\mathbf{U}_{1}, \ldots, \mathbf{U}_{n} ; \mathcal{V}, \sigma^{2}\right) . . . . . . . . .} \tag{3}
\end{equation*}
$$

Now, for the numerical optimization we can use a number of routines, which, clearly, are based on the initial point for estimation. In our case, consider the Helmertized landmark data $\mathbf{Y}_{i} \sim E_{N-1 \times K}\left(\mu_{N-1 \times K}, \sigma^{2} \mathbf{I}_{N-1} \otimes \mathbf{I}_{K}, h\right) i=$ $1, \ldots, n$ (see (1)) and let $\widetilde{\mu}=\left(\widetilde{\mu}_{1}^{\prime} \mid \widetilde{\mu}_{2}^{\prime}\right)^{\prime}$ and $\tilde{\sigma}^{2}$ be the maximum likelihood estimators of the location parameter matrix $\mu_{N-1 \times K}$ and the scale parameter $\sigma^{2}$ of the elliptical distribution under consideration, so, given that

$$
\mathbf{U}_{i}^{\prime} \boldsymbol{\Sigma}^{-1} \mu \mu^{\prime} \boldsymbol{\Sigma}^{-1} \mathbf{U}_{i}\left(\mathbf{U}_{i}^{\prime} \boldsymbol{\Sigma}^{-1} \mathbf{U}_{i}\right)^{-1}=\mathbf{Y}_{i}^{\prime} \boldsymbol{\Sigma}^{-1} \mu \mu^{\prime} \boldsymbol{\Sigma}^{-1} \mathbf{Y}_{i}\left(\mathbf{Y}_{i}^{\prime} \boldsymbol{\Sigma}^{-1} \mathbf{Y}_{i}\right)^{-1}
$$

then an initial point can be $x_{0}=\left(\operatorname{vec}^{\prime}\left(\mathcal{V}_{0}^{\prime}\right), \sigma_{0}^{2}\right)$, where $\mathcal{V}_{0}=\widetilde{\mu}_{2} \widetilde{\mu}_{1}^{-1}$ and $\sigma_{0}^{2}=$ $\tilde{\sigma}^{2}$.

So, the exact inference procedure can be outlined in the next few steps.

### 2.1 Step I. Available distributions: Families of isotropic elliptical configuration densities

A first step considers a list of configuration densities, they are full derived in Caro-Lopera et al (2008). The classical elliptical configuration densities included are the Kotz, Pearson type VII, Bessel, Logistic. But we must note that any elliptical function $h(\cdot)$ which satisfies theorem 2 is appropriated.

Most of the applications in statistical theory of shape reside on the isotropic model (see Dryden and Mardia (1998)), so in the case of the noncentral elliptical configuration density if we take $\boldsymbol{\Sigma}=\sigma^{2} \mathbf{I}_{N-1}$ in the general configuration density, we get a list of suitable distributions for inference; which, are expanded in terms of zonal polynomials and they can be computed efficiently by Koev and Edelman (2006). Note, that we can consider a more enriched estructure, for example $\boldsymbol{\Sigma}=\operatorname{diag}\left(\sigma_{1}^{2}, \sigma_{2}^{2}, \ldots, \sigma_{N-1}^{2}\right)$ (which suppose a different scale parameter in each landmark component), and similar diagonal structures.

We will not write down such densities here, only we will performed inference with a special Kotz subfamily; but we must highlight that the four step inference procedure can be studied with the densities provided in Caro-Lopera et al (2008).

The particular Kotz density which will be studied in applications is the following

Corollary 3 If $\mathbf{Y} \sim E_{N-1 \times K}\left(\mu_{N-1 \times K}, \sigma^{2} \mathbf{I}_{N-1} \otimes \mathbf{I}_{K}, h\right)$ and $T=1$, then the Kotz type isotropic noncentral configuration density simplifies to

$$
\begin{align*}
& \frac{\Gamma_{K}\left(\frac{N-1}{2}\right) \operatorname{etr}\left(\frac{R}{\sigma^{2}} \mu^{\prime} \mathbf{U}\left(\mathbf{U}^{\prime} \mathbf{U}\right)^{-1} \mathbf{U}^{\prime} \mu-\frac{R}{\sigma^{2}} \mu^{\prime} \mu\right)}{\pi^{K q / 2}\left|\mathbf{I}_{K}+\mathbf{V}^{\prime} \mathbf{V}\right|^{\frac{N-1}{2}} \Gamma_{K}\left(\frac{K}{2}\right)} \\
& \quad \quad_{1} F_{1}\left(-\frac{q}{2} ; \frac{K}{2} ;-\frac{R}{\sigma^{2}} \mu^{\prime} \mathbf{U}\left(\mathbf{U}^{\prime} \mathbf{U}\right)^{-1} \mathbf{U}^{\prime} \mu\right), \tag{4}
\end{align*}
$$

and where $R=\frac{1}{2}$, we get the normal configuration density.

### 2.2 Step II. Choosing the elliptical configuration density

Here, the main advantage of working with elliptical models appears, we have the possibility of choosing a distribution for the landmark data. Recall that the main assumptions for inference in this works are supported by independent
and identically elliptically contoured distributed observations

$$
\mathbf{Y}_{i} \sim E_{N-1 \times K}\left(\mu_{N-1 \times K}, \sigma^{2} \mathbf{I}_{N-1} \otimes \mathbf{I}_{K}, h\right), i=1, \ldots, n
$$

According to our assumptions we can consider Schwarz (1978) as an appropriate technique for choosing the elliptical model. Explicitly, the procedure is as follows: consider $k$ elliptical models, then perform the maximization of the likelihood function separately for each model $j=1, \ldots, k$, obtaining say, $M_{j}\left(\mathbf{Y}_{1}, \ldots, \mathbf{Y}_{n}\right)$, then Schwarz's criterion for a large-sample is given by

Choose the model for which $\log M_{j}\left(\mathbf{Y}_{1}, \ldots, \mathbf{Y}_{n}\right)-\frac{1}{2} k_{j} \log n$ is largest,
where $k_{j}$ is the dimension (number of parameters) of the model $j$.
Remark 4 The preceding result can be implemented for choosing a shape model, i.e. given an independent and identically distributed random sample of landmark data and a list of shape distributions: pre-shape, size and shape, shape, reflection shape, reflection size and shape, cone, disk, (all of them supported by Euclidean transformations), configuration (supported by affine transformations), and projection, etc. we can select the best shape-transformationmodel. However, it is constrained by the computation of the densities, and as we can check in the statistical shape literature, the Euclidean based shape densities have important difficulties for computations even in the gaussian case (most of them have not an elliptical version yet), see Dryden and Mardia (1998) and the references there in, but it is not the case with our configuration densities. We will let these comparisons for a subsequent work.

### 2.3 Step III. Configuration Location

Once the elliptical model is selected, we find the estimators of location and scale parameters of configuration by mean of (3). The crucial point here is the computation of the configuration density; if the selected model is the Gaussian one, then the matlab algorithms for confluent hypergeometric functions of matrix argument by Koev and Edelman (2006) gives the solution very efficiently, this solves in fact the inference problem proposed by Goodall and Mardia (1993). We highlight that the cited computation of the ${ }_{1} F_{1}(a ; c ; \mathbf{X})$ series is restricted to the truncation and it is an open problem addressed in the last section of Koev and Edelman (2006), however the fast algorithms let a sort of numerical experiments until a given precision is reached, so the optimization problem remains in terms of the truncation and the set precision, but this occurs, clearly, since it is an intrinsic problem of any numerical optimization problem.

But, if the selected model is not Gaussian, we could think that the problems remains open, but fortunately, the configuration densities can be computed efficiently by using the same work of Koev and Edelman (2006).

First let us denote the elliptical configuration density of theorem 2 by

$$
\begin{equation*}
A_{1} P_{1}(f(t): a ; c ; \mathbf{X}), \tag{5}
\end{equation*}
$$

where

$$
\begin{gathered}
A=\frac{\pi^{K^{2} / 2} \Gamma_{K}\left(\frac{N-1}{2}\right)}{|\boldsymbol{\Sigma}|^{\frac{K}{2}}\left|\mathbf{U}^{\prime} \boldsymbol{\Sigma}^{-1} \mathbf{U}\right|^{\frac{N-1}{2}} \Gamma_{K}\left(\frac{K}{2}\right)},{ }_{1} P_{1}(f(t): a ; c ; \mathbf{X})=\sum_{t=0}^{\infty} \frac{f(t)}{t!} \sum_{\tau} \frac{(a)_{\tau}}{(c)_{\tau}} C_{\tau}(\mathbf{X}), \\
f(t)=\sum_{r=0}^{\infty} \frac{\left[\operatorname{tr}\left(\mu^{\prime} \boldsymbol{\Sigma}^{-1} \mu\right)\right]^{r}}{r!\Gamma\left(\frac{K(N-1)}{2}+t\right)} \int_{0}^{\infty} h^{(2 t+r)}(y) y^{\frac{K(N-1)}{2}+t-1} d y \\
\mathbf{X}=\mathbf{U}^{\prime} \boldsymbol{\Sigma}^{-1} \mu \mu^{\prime} \boldsymbol{\Sigma}^{-1} \mathbf{U}\left(\mathbf{U}^{\prime} \boldsymbol{\Sigma}^{-1} \mathbf{U}\right)^{-1}, a=\frac{N-1}{2}, c=\frac{K}{2} .
\end{gathered}
$$

Unfortunately, the configuration density $A_{1} P_{1}(f(t): a ; c ; \mathbf{X})$ is an infinite series, given that $a$ and $c$ are positive, (recall that $N$ is the number of landmarks, $K$ is de dimension and $N-K-1 \geq 1)$. So a truncation is needed if we want to use it directly by using computation of zonal polynomials.

Expression (5) belongs to the class of eq. (1.6), p. 3 from Koev and Edelman (2006), and as they affirm "With minimal changes our algorithms (for hypergeometric functions of one matrix arguments) can approximate the hypergeometric function of two matrix arguments..., and more generally functions of the form $G(x)=\sum_{k=0}^{\infty} \sum_{\kappa} a_{\kappa} C_{\kappa}^{\alpha}(\mathbf{X})$, for arbitrary coefficients $a_{\kappa}$ at a similar computational cost," see, eq. (6.5) of Koev and Edelman (2006), and they add "Although the expression (6.5) is not a hypergeometric function of a matrix argument, its truncation for $|\kappa| \leq m$ has the form (1.6), and is computed analogously."

Then, in principle, the configuration densities can be evaluated efficiently with the fast algorithms of Koev and Edelman (2006) and the corresponding inference problem can be solved numerically. And at this stage, by using for example the compatible matlab routine fminsearch (unconstrained nonlinear optimization) with the modified matlab files of Koev and Edelman (2006), we have the estimators for the configuration location and the scale parameter of the "best" elliptical model chosen with Schwarz's criterion. We arrive then, to the final step.

Finally, given that the likelihood can be evaluated and optimized, then a sort of likelihood ratio tests can be performed for testing a particular configuration for a population, or testing for differences in configuration between two populations, or testing one-dimensional uniform shear of two populations, etc.

In the statistical shape analysis, the large sample standard likelihood ratio tests are the most frequently used, see for example Dryden and Mardia (1998), by mean of Wilk's theorem. Explicitly, for testing whether $H_{0}: \mathcal{U} \in \Omega_{0}$ versus $H_{a}: \mathcal{U} \in \Omega_{1}$, where $\Omega_{0} \subset \Omega_{1} \subseteq \mathbb{R}^{K q}$, with $\operatorname{dim}\left(\Omega_{0}\right)=p<K q$ and $\operatorname{dim}\left(\Omega_{a}\right)=r \leq K q$. Thus, the $-2 \log$-likelihood ratio is given by

$$
-2 \log \Lambda=2 \sup _{H_{a}} \log L\left(\mathcal{U}, \sigma^{2}\right)-2 \sup _{H_{0}} \log L\left(\mathcal{U}, \sigma^{2}\right),
$$

and by the Wilk's theorem for large samples, the distribution of the null hypothesis $H_{0}$ obeys (see Dryden and Mardia (1998))

$$
-2 \log \Lambda \approx \chi_{r-p}^{2}
$$

In a similar way we can test differences in configuration between two populations, etc. Suppose that the last hypothesis is rejected, then an interesting test can be performed one-dimensional uniform shear of two populations which determines the amount of deformation axes by axes. Note that the classical statistical shape analysis (pre-shape, size and shape, shape, reflection shape, reflection size and shape, cone, disk,) which is based on Euclidean transformations assume that any shape is uniform deformed in any dimension, which certainly is very idealistic, but the configuration density accept different uniform shearing among the axes.

Explicitly, if we want to test uniform shear in the $i$ coordinate of two populations, then the testing procedure lies on $H_{0}: \mu_{1} B=\mu_{1} B$ versus $H_{0}$ : $\mu_{1} B \neq \mu_{1} B$, where $B=(0, \ldots, i, \ldots, 0)^{\prime}$ and the configuration density $\mathbf{U}$ goes to UB. Note that the new configuration density is simpler, since it is just a vector density and it is easier of computing.

Thus, the whole inference procedure of the above four steps can be carried out for a particular landmark data (for example from Dryden and Mardia (1998), Bookstein (1991)), and up here we can consider the inference problem numerically solvable.

## 3 Further simplifications: finite configuration densities

Even the whole elliptical configuration problem is cleared. There are interesting simplifications which open a promissory future work. We explore a little the problem in this section, before ending the work with some applications.

However, the zonal polynomials are computable very fast the problem now resides in the convergence and the truncation of the above series for performing the numerical optimization, in fact in the same reference of Koev and Edelman (2006) we read:
"Several problems remain open, among them automatic detection of convergence .... and it is unclear how to tell when convergence sets in. Another open problem is to determine the best way to truncate the series. "

Thus the implicit numerical difficulties for truncation of any configuration density series of type (5) motivates two areas of investigation: one, continue the numerical approach started by (Koev and Edelman (2006)) with the confluent hypergeometric functions and extend it to the case of some configuration series type Kotz, Pearson, Bessel, Logistic, for example; or second, propose a theoretical approach for solving analytically the problem.

In the next few lines we establish the second question and leave their implications for a future works.

First represent the configuration density as it was done in (5).
The above series can be finite if we use the following basic principle.
Lemma 5 Let $N-K-1 \geq 1$ as usual, and consider the infinite configuration density

$$
C D_{1}=A_{1} P_{1}\left(f(t): \frac{N-1}{2} ; \frac{K}{2} ; \mathbf{X}\right) .
$$

If the dimension $K$ is even (odd) and the number of landmarks $N$ is odd (even), respectively, then the equivalent configuration density

$$
C D_{2}=A_{1} P_{1}\left(g(t):-\left(\frac{N-1}{2}-\frac{K}{2}\right) ; \frac{K}{2} ; h(\mathbf{X})\right),
$$

is a polynomial of degree $K\left(\frac{N-1}{2}-\frac{K}{2}\right)$ in the latent roots of the matrix $\mathbf{X}$ (otherwise the series is infinite), for suitable $f(t), g(t)$ and $h(\mathbf{X})$.

Proof. Recall that $\tau=\left(t_{1}, \ldots, t_{K}\right), t_{1} \geq t_{2} \geq \cdots t_{K} \geq 0$, is a partition of $t$ and

$$
(\alpha)_{\tau}=\prod_{i=1}^{K}\left(\alpha-\frac{1}{2}(i-1)\right)_{t_{i}},
$$

where

$$
(\alpha)_{t}=\alpha(\alpha+1) \cdots(\alpha+t-1), \quad(\alpha)_{0}=1 .
$$

Now, If $K$ is even (odd) and $N$ is odd (even) then $-\left(\frac{N-1}{2}-\frac{K}{2}\right)=-\frac{q}{2}$ is a negative integer and clearly $\left(-\frac{q}{2}\right)_{\tau}=0$ for every $t \geq \frac{K q}{2}+1$, then $C D_{2}$ is a polynomial of degree $\frac{K q}{2}$ in the latent roots of $\mathbf{X}$.

So, the addressed truncation problem of an infinite configuration density can be solved by finding an equivalent finite configuration density according to the preceding lemma and selecting an appropriate number of landmarks in the figure.

Given an elliptical configuration density $C D_{1}$ indexed by function $f(t)$, $a=$ $\frac{N-1}{2}>0, c=\frac{K}{2}>0$, the crucial point consist of finding an integral representation valid for $c-a=-\frac{q}{2}<0$ leading an equivalent elliptical configuration density $C D_{2}$ indexed by some function $g(t)$. Then the finiteness of $C D_{2}$ follows from $K$ even (odd) and $N$ odd (even), respectively.

We already met a kind of this relations, when $f(t)$ is a constant, i.e. in corollary 3 ; in this case lemma 5 is reduced to the Kummer relations; and the corresponding configuration densities (which includes Gaussian) are finite by selecting an odd (even) number of landmarks $N$ according to an even (odd) dimension $K$, respectively. The implications of the finiteness for applications will avoid the addressed open problem for truncation proposed in Koev and Edelman (2006).

The above discussion it is important for a generalization of Kummer type relations; for example, equalities for non constant $f(t)$, i.e. expressions of $g(t)$ and $h(\mathbf{X})$ for non R -normal models (4). Some advances in this direction are available from the authors, for example, the generalized Kummer relations for a Kotz type ( $T$ positive integer), and a Pearson type VII based con a Beta type integral representation has ratified that ${ }_{1} P_{1}(f(t): a ; c ; \mathbf{X})={ }_{1}$ $P_{1}(g(t): c-a ; c ;-\mathbf{X})$, for the corresponding $f, g$, but in the case of $c-a>0$. The next step is to prove the relations for $c-a<0$, by a Laplace representation type, then lemma 5 can be applied to Kotz type and Pearson type VII configuration densities and the respective series become finite.

Meanwhile, fortunately, we can performed inference with the finite series of corollary 3 , specially with the Gaussian case $R=\frac{1}{2}$.

Corollary 6 If $\mathbf{Y} \sim \mathcal{N}_{N-1 \times K}\left(\mu_{N-1 \times K}, \sigma^{2} \mathbf{I}_{N-1} \otimes \mathbf{I}_{K}\right)$, $K$ is even (odd) and $N$ is odd (even), respectively, then the finite isotropic noncentral normal configuration density is given by

$$
\begin{aligned}
\frac{\Gamma_{K}\left(\frac{N-1}{2}\right)}{\pi^{K q / 2}\left|\mathbf{I}_{K}+\mathbf{V}^{\prime} \mathbf{V}\right|^{\frac{N-1}{2}} \Gamma_{K}\left(\frac{K}{2}\right)} \operatorname{etr} & \left(\frac{1}{2 \sigma^{2}} \mu^{\prime} \mathbf{U}\left(\mathbf{U}^{\prime} \mathbf{U}\right)^{-1} \mathbf{U}^{\prime} \mu-\frac{1}{2 \sigma^{2}} \mu^{\prime} \mu\right) \\
& \times{ }_{1} F_{1}\left(-\frac{q}{2} ; \frac{K}{2} ;-\frac{1}{2 \sigma^{2}} \mu^{\prime} \mathbf{U}\left(\mathbf{U}^{\prime} \mathbf{U}\right)^{-1} \mathbf{U}^{\prime} \mu\right),
\end{aligned}
$$

and it is a polynomial of degree $K\left(\frac{N-1}{2}-\frac{K}{2}\right)$ in the latent roots of

$$
\frac{1}{2 \sigma^{2}} \mu^{\prime} \mathbf{U}\left(\mathbf{U}^{\prime} \mathbf{U}\right)^{-1} \mathbf{U}^{\prime} \mu
$$

## 4 Applications

In this section we consider planar classical application in the statistical shape analysis. The following situations are sufficiently studied by shape based on euclidian transformations and asymptotic formulae. We will use here exact inference in the sense that we will use the exact densities and compute the likelihood exactly by using zonal polynomials theory.

We will test configuration differences under the exact gaussian configuration density, and the applications include Biology (mouse vertebra, gorilla skulls, girl and boy craniofacial studies), Medicine (brain MR scans of schizophrenic patients) and image analysis (postcode recognition).

First we start with the two dimensional case, then corollary 6 turns:
Corollary 7 If $\mathbf{Y} \sim \mathcal{N}_{N-1 \times 2}\left(\mu_{N-1 \times 2}, \sigma^{2} \mathbf{I}_{N-1} \otimes \mathbf{I}_{2}\right)$, and $N$ is odd, then the finite two dimensional isotropic noncentral normal configuration density is given by

$$
\begin{aligned}
\frac{\Gamma_{2}\left(\frac{N-1}{2}\right)}{\pi^{N-3}\left|\mathbf{I}_{2}+\mathbf{V}^{\prime} \mathbf{V}\right|^{\frac{N-1}{2}} \Gamma_{K}(1)} & \operatorname{etr}\left(\frac{1}{2 \sigma^{2}} \mu^{\prime} \mathbf{U}\left(\mathbf{U}^{\prime} \mathbf{U}\right)^{-1} \mathbf{U}^{\prime} \mu-\frac{1}{2 \sigma^{2}} \mu^{\prime} \mu\right) \\
& \times{ }_{1} F_{1}\left(-\frac{N-3}{2} ; 1 ;-\frac{1}{2 \sigma^{2}} \mu^{\prime} \mathbf{U}\left(\mathbf{U}^{\prime} \mathbf{U}\right)^{-1} \mathbf{U}^{\prime} \mu\right)
\end{aligned}
$$

and it is a polynomial of degree $N-3$ in the two latent roots of

$$
\frac{1}{2 \sigma^{2}} \mu^{\prime} \mathbf{U}\left(\mathbf{U}^{\prime} \mathbf{U}\right)^{-1} \mathbf{U}^{\prime} \mu
$$

Then, we apply the confluent hypergeometric's given in the appendix in a sort of problems and as motivations of future works with other elliptical models and situations.

### 4.1 Biology: mouse vertebra

This problem has been studied deeply by Dryden and Mardia (1998). The data come from an investigation into the effects of selection for body weight on the shape of mouse vertebra and the experiments consider the second thoracic vertebra T2 of 30 control (C), 23 large (L) and 23 small (S) bones. The control group contains unselected mice, the large group contains mice selected at each generation according to large body weight and the small group was selected for small body weight. In order to apply the finite densities we do not consider the third landmark of the total 6, they proposed (see Dryden and Mardia (1998), p. 10 and the data given in p. 313-316).

Inference is based on (A.1), a confluent hypergeometric polynomial of two degree in the two eigenvalues of the zonal polynomial argument, then after a very simple computation we have the following configuration locations of the three groups.

| Group | $\widetilde{\mathcal{V}_{11}}$ | $\widetilde{\mathcal{V}_{12}}$ | $\widetilde{\mathcal{V}_{21}}$ | $\widetilde{\mathcal{V}_{22}}$ | $\widetilde{\sigma^{2}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Control | -0.10789 | 0.15594 | 0.0005299 | -0.97056 | 0.00165 |
| Large | -0.084983 | 0.12436 | 0.00049203 | -1.0787 | 0.0021303 |
| Small | -0.092342 | 0.21291 | 0.00052577 | -1.018 | 0.0019863 |

A test for scala parameters reveals significantly differences between C-L and LS , but equality in the C-S case. Then the likelihood ratios (based on $-2 \log \Lambda \approx$ $\left.\chi_{4}^{2}\right)$ for the paired tests $H_{0}: \mathcal{U}_{1}=\mathcal{U}_{2}$ vs $H_{a}: \mathcal{U}_{1} \neq \mathcal{U}_{2}$, give the p-values: $2.8 E-9$ for C-L; $177.769 E-7$ for L-S and $3 E-10$ for C-S. So, we can say that there are strong evidence for different configuration changes, and the most important is given between small and large, as we expected.

### 4.2 Biology: gorilla skulls

In this application Dryden and Mardia (1998) investigate the cranial differences between the 29 male and 30 female apes by studying 8 anatomical landmarks. For the finiteness of the configuration density we remove landmark $o$ (see Dryden and Mardia (1998) p.11, and the data in p. 317-318) and the corresponding confluent hypergeometric is a polynomials of fourth degree, see (A.2).

The estimators of the configuration location and scale parameters are given below

| Group | $\widetilde{\mathcal{V}_{11}}$ | $\widetilde{\mathcal{V}_{12}}$ | $\widetilde{\mathcal{V}_{21}}$ | $\widetilde{\mathcal{V}_{22}}$ |
| :---: | :---: | :---: | :---: | :---: |
| Female | -0.28033 | 0.31315 | -0.42269 | -0.59672 |
| Male | -0.33313 | 0.42484 | -0.43594 | -0.5734 |


| $\cdots$ | $\widetilde{\mathcal{V}_{31}}$ | $\widetilde{\mathcal{V}_{32}}$ | $\widetilde{\mathcal{V}_{41}}$ | $\widetilde{\mathcal{V}_{42}}$ | $\widetilde{\sigma^{2}}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $\cdots$ | 0.27398 | -1.4695 | 0.7363 | -1.2665 | 0.0042665 |
| $\cdots$ | 0.30563 | -1.306 | 0.73169 | -1.0594 | 0.010057 |

A test for scale parameters reveals significant difference between the two populations, so the likelihood ratio (based on $-2 \log \Lambda \approx \chi_{8}^{2}$ ) for $H_{0}: \mathcal{U}_{1}=\mathcal{U}_{2}$ vs $H_{a}: \mathcal{U}_{1} \neq \mathcal{U}_{2}$ of configuration location cranial difference between the sexes of the apes, gives a p-value of $12.74 E-13$. Which clearly ratifies strong evidence for differences between the female and male configuration locations.

### 4.3 Biology: The university school study subsample

In this experiment Bookstein (1991) studies sexes shapes differences between 8 craniofacial landmarks for 36 normal Ann Arbor boys and 26 girls near the ages of 8 years. In order to get a finite configuration density we discard the landmark Sella (see Bookstein (1991), p.401-405), then the hypergeometric functions is a polynomial of fourth degree, see (A.2).

Then, the estimators of the configuration location and scale parameters are the following

| Group | $\widetilde{\mathcal{V}_{11}}$ | $\widetilde{\mathcal{V}_{12}}$ | $\widetilde{\mathcal{V}_{21}}$ | $\widetilde{\mathcal{V}_{22}}$ |
| :---: | :---: | :---: | :---: | :---: |
| Male | -1.2425 | 2.1948 | 0.46435 | -1.3752 |
| Female | -1.2483 | 2.2331 | 0.43685 | -1.3845 |


| $\cdots$ | $\widetilde{\mathcal{V}_{31}}$ | $\widetilde{\mathcal{V}_{32}}$ | $\widetilde{\mathcal{V}_{41}}$ | $\widetilde{\mathcal{V}_{42}}$ | $\widetilde{\sigma^{2}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\cdots$ | -0.91487 | 0.66127 | 0.15775 | -0.069042 | 0.0032908 |
| $\cdots$ | -0.92903 | 0.70439 | 0.1616 | -0.077236 | 0.0059142 |

In this case, strong evidence for differences in the scale parameter is reveled
by a test between the two populations, thus the likelihood ratio (based on $-2 \log \Lambda \approx \chi_{8}^{2}$ ) for $H_{0}: \mathcal{U}_{1}=\mathcal{U}_{2}$ vs $H_{a}: \mathcal{U}_{1} \neq \mathcal{U}_{2}$ of configuration location cranialfacial difference between the boys and girls, gives a p-value of 0.7053 . And the difference between these two configuration locations is insignificant. A similar global conclusion gives Bookstein (1991), however a more detailed study of landmark subset is required, then possible differences can be detected, as Bookstein (1991) ratifies in a different shape context.

### 4.4 Medicine: brain MR scans of schizophrenic patients

We return to the applications in Dryden and Mardia (1998), in this case, they study 13 landmarks on a near midsagittal two dimensional slices from magnetic resonance (MR) brain scans of 14 schizophrenic patients and 14 normal patients. Given that the number of two dimensional landmarks is odd we preserve them leading a 10 degree confluent hypergeometric polynomial, easily to compute, see (A.5).

Thus, the estimators of the configuration location and scale parameters are given by

| Group | $\widetilde{\mathcal{V}_{11}}$ | $\widetilde{\mathcal{V}_{12}}$ | $\widetilde{\mathcal{V}_{21}}$ | $\widetilde{\mathcal{V}_{22}}$ | $\widetilde{\mathcal{V}_{31}}$ | $\widetilde{\mathcal{V}_{32}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Normal | -0.64099 | 2.6942 | -1.2744 | -2.8323 | -0.42155 | -1.003 |
| Squizo. | -0.68623 | 2.393 | -1.145 | -2.8484 | -0.37349 | -1.0744 |


| $\widetilde{\mathcal{V}_{41}}$ | $\widetilde{\mathcal{V}_{42}}$ | $\widetilde{\mathcal{V}_{51}}$ | $\widetilde{\mathcal{V}_{52}}$ | $\widetilde{\mathcal{V}_{61}}$ | $\widetilde{\mathcal{V}_{62}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| -0.31011 | -2.3094 | -0.30236 | -3.5261 | 0.36 | -0.90135 |
| -0.23173 | -2.1929 | -0.20173 | -3.3226 | 0.38123 | -0.84316 |


| $\widetilde{\mathcal{V}_{71}}$ | $\widetilde{\mathcal{V}_{72}}$ | $\widetilde{\mathcal{V}_{81}}$ | $\widetilde{\mathcal{V}_{82}}$ | $\widetilde{\mathcal{V}_{91}}$ | $\widetilde{\mathcal{V}_{92}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.1597 | -2.2205 | 0.8518 | -0.7578 | 1.8686 | 0.86501 |
| 0.20429 | -2.109 | 0.84683 | -0.56588 | 1.7948 | 0.88466 |


| $\widetilde{\mathcal{V}_{10,1}}$ | $\widetilde{\mathcal{V}_{10,2}}$ | $\widetilde{\sigma^{2}}$ |
| :---: | :---: | :---: |
| -0.14205 | 0.20718 | 0.010843 |
| -0.079005 | 0.1378 | 0.054064 |

A test for scale parameters in the two populations revels significant differences in this topic. Dryden and Mardia (1998) advert about the small sample size of this experiment and obviously it can be explain the opposite result based on $-2 \log \Lambda \approx \chi_{20}^{2}$, which means a p-value of 0.9174 . Mean shape differences is concluded in Dryden and Mardia (1998), but configuration difference is definitely insignificant. The controversial configuration location results could suggest some a deep study for small sample likelihood and perhaps it can ratify important different conclusions of a sort of studies about schizophrenia classification based only on MR scans. But the most important fact here is the geometrical meaning of the data, because it certainly differs of the preceding ones, which have a explicit geometrical explanation.

In general the study of scale parameter, certainly is complicated and it deserves a deeper study.

### 4.5 Image analysis: postcode recognition

Again, a 13 landmark problem, which supposes a 10 degree confluent hypergeometric appears, in this case Dryden and Mardia (1998) studies a 30 random sample of handwritten digit 3 for postcode recognition. The data is available in Dryden and Mardia (1998), p. 318-320.

The next table shows, the configuration location and scale parameter estimates, joint the configuration coordinates of a template number 3 digit, with two equal sized arcs, and 13 landmarks (two coincident) lying on two regular octagons see Dryden and Mardia (1998), p. 153.

| Group | $\widetilde{\mathcal{V}_{11}}$ | $\widetilde{\mathcal{V}_{12}}$ | $\widetilde{\mathcal{V}_{21}}$ | $\widetilde{\mathcal{V}_{22}}$ | $\widetilde{\mathcal{V}_{31}}$ | $\widetilde{\mathcal{V}_{32}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Digit 3 | -0.79087 | 1.9432 | -2.1073 | 1.5875 | -2.713 | 0.81862 |
| Template | -2.0908 | 2.2071 | -4.0409 | 2.8051 | -4.5904 | 2.2904 |


| $\widetilde{\mathcal{V}_{41}}$ | $\widetilde{\mathcal{V}_{42}}$ | $\widetilde{\mathcal{V}_{51}}$ | $\widetilde{\mathcal{V}_{52}}$ | $\widetilde{\mathcal{V}_{61}}$ | $\widetilde{\mathcal{V}_{62}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| -2.8084 | -0.066901 | -2.5712 | 0.71315 | -2.6934 | 1.2955 |
| -4.2069 | 1.3688 | -3.3126 | 1.7582 | -3.5881 | 2.7053 |


| $\widetilde{\mathcal{V}_{71}}$ | $\widetilde{\mathcal{V}_{72}}$ | $\widetilde{\mathcal{V}_{81}}$ | $\widetilde{\mathcal{V}_{82}}$ | $\widetilde{\mathcal{V}_{91}}$ | $\widetilde{\mathcal{V}_{92}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| -3.1548 | 1.6802 | -3.8004 | 1.34 | -4.0517 | 0.33141 |
| -5.4996 | 4.0629 | -7.5557 | 4.8428 | -8.2514 | 4.4208 |


| $\widetilde{\mathcal{V}_{10,1}}$ | $\widetilde{\mathcal{V}_{10,2}}$ | $\widetilde{\sigma^{2}}$ |
| :---: | :---: | :---: |
| -3.7659 | -0.6583 | 0.22904 |
| -6.9108 | 2.8899 |  |

And clearly, this enormous difference most be reveled in the corresponding test based on $-2 \log \Lambda \approx \chi_{20}^{2}$, with approximately zero p-value. This result was corroborated with probability $\approx 0.0002$ by Dryden and Mardia (1998), p. 153. under a shape model. In any case an strong evidence that the configuration location does not have the configuration of the ideal template for digit 3 .

Finally we must note that the remaining planar applications in Dryden and Mardia (1998), and Bookstein (1991), etc. can be studied with the finite configuration densities and exact formulae for zonal polynomials; in fact the three dimensional applications available in the literature (see Goodall and Mardia (1993)) and others in genetics for 3D DNA part, etc, can be studied in an exact form with the help of corollary 3 via lemma 5 and exact formulae for zonal polynomials of third degree in James (1964), avoiding the open truncation problems implicit in Koev and Edelman (2006). Even more, some comparisons among the shape models can be performed via remark 4 and other tests for uniform deformations (see step IV) can be performed. This will be considered in a subsequent work. Of course the study of finite configuration densities associated to Pearson, Bessel, logistic and the general Kotz, will facilitate exact inference and will avoid the addressed truncation problem, but it will depends on some developments in integration and series representation. In fact, the distribution of the likelihood ratio could be potentially studied by using the low degree finite configuration densities. These topics are been investigated.

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## A Finite series for planar applications of maximum 21 landmarks

Given that most of the applications in shape theory comes from two dimensional images (see Dryden and Mardia (1998)), then it is important to give explicit expressions for the finite series ${ }_{1} F_{1}\left(-\frac{N-3}{2} ; 1 ;-\frac{1}{2 \sigma^{2}} \mu^{\prime} \mathbf{U}\left(\mathbf{U}^{\prime} \mathbf{U}\right)^{-1} \mathbf{U}^{\prime} \mu\right)$ involved in corollary 7 when $N=5,7,9, \ldots$ is small. Let $x, y$ be the eigenvalues of $\Omega=\frac{1}{2 \sigma^{2}} \mu^{\prime} \mathbf{U}\left(\mathbf{U}^{\prime} \mathbf{U}\right)^{-1} \mathbf{U}^{\prime} \mu$, then we have for $N=5,7, \ldots, 21$ the following list of polynomials of degree $K q / 2=N-3$; these expressions are useful for exact inference of the corresponding configuration densities. We use in this case exact formulae for zonal polynomials given by James (1968) see also Caro-Lopera et al (2007). In fact all the applications studied in Dryden and Mardia (1998) have maximum 21 landmarks (which supposes a polynomial of 18 degree in the two eigenvalues of corresponding matrix), so the following confluent hypergeometric expressions are sufficient for their corresponding configuration analysis. Note that the cited applications demand formulae for zonal polynomials of second order up maximum 18 degree, and this expressions are available since 60 's, so the numerical algorithms of Koev and Edelman (2006), very useful for infinite series, but with the addressed problem of truncations, are not needed here and the exact inference on con-
figuration densities, historically could be studied since the configurations were proposed by Goodall and Mardia (1993).

Observe that the selection of an odd number of landmarks for planar applications suggest deleting one of them of the available tables usually studied for approximations methods, clearly it is also possible to reduce in one, any group of preset even landmarks, however we leave the decision to an expert. According to the number of odd landmarks, we suggest some problems studied by Dryden and Mardia (1998) but in the context of finite gaussian configuration densities (we put in parenthesis the original number of landmarks studied by Dryden and Mardia (1998)).

The involved series up 15 landmarks are easily computed as (parenthesis indicates the number of landmarks in the original source of Dryden and Mardia (1998)(DM) and Bookstein (1991)(B), respectively):

- $N=5$ : Mouse vertebra (6)(DM),

$$
\begin{equation*}
1+y+x+2 y x \tag{A.1}
\end{equation*}
$$

- $N=7$ : Gorilla skulls (8)(DM), the university school study subsample (8)(B),

$$
\begin{equation*}
1+2 y+2 x+\frac{1}{2} y^{2}+7 y x+\frac{1}{2} x^{2}+2 y^{2} x+2 y x^{2}+\frac{2}{3} y^{2} x^{2} \tag{A.2}
\end{equation*}
$$

- $N=9$ :

$$
\begin{align*}
& 1+3 y+3 x+\frac{3}{2} y^{2}+15 y x+\frac{3}{2} x^{2}+\frac{1}{6} y^{3}+\frac{17}{2} y^{2} x \\
& +\frac{17}{2} y x^{2}+\frac{1}{6} x^{3}+y^{3} x+\frac{16}{3} y^{2} x^{2}+y x^{3}+\frac{2}{3} y^{3} x^{2} \\
& +\frac{2}{3} y^{2} x^{3}+\frac{4}{45} y^{3} x^{3} \tag{A.3}
\end{align*}
$$

- $N=11$ : Sooty mangabeys (12)(DM).

$$
\begin{align*}
& 1+4 x+22 y^{2} x+4 y+\frac{81}{4} y^{2} x^{2}+5 y^{2} x^{3}+5 y^{3} x^{2}+22 y x^{2} \\
& +\frac{31}{6} y^{3} x+\frac{31}{6} y x^{3}+26 y x+\frac{1}{24} x^{4}+\frac{1}{3} y x^{4}+\frac{1}{3} y^{4} x \\
& +\frac{2}{315} y^{4} x^{4}+\frac{4}{45} y^{4} x^{3}+\frac{4}{45} y^{3} x^{4}+\frac{1}{3} y^{4} x^{2}+\frac{58}{45} y^{3} x^{3} \\
& +\frac{1}{3} y^{2} x^{4}+3 y^{2}+3 x^{2}+\frac{2}{3} y^{3}+\frac{2}{3} x^{3}+\frac{1}{24} y^{4} \tag{A.4}
\end{align*}
$$

- $N=13$ : Brain MR scans of schizophrenic patients (13)(DM), postcode recognition (13)(DM)

$$
\begin{align*}
& 1+5 x+45 y^{2} x+5 y+\frac{655}{12} y^{2} x^{2}+\frac{241}{12} y^{2} x^{3}+\frac{241}{12} y^{3} x^{2}+45 y x^{2} \\
& +\frac{95}{6} y^{3} x+\frac{95}{6} y x^{3}+40 y x+\frac{5}{24} x^{4}+\frac{49}{24} y x^{4}+\frac{49}{24} y^{4} x+\frac{1}{12} y x^{5} \\
& +\frac{4}{14175} y^{5} x^{5}+\frac{2}{315} y^{5} x^{4}+\frac{2}{315} y^{4} x^{5}+\frac{2}{45} y^{5} x^{3}+\frac{46}{315} y^{4} x^{4} \\
& +\frac{2}{45} y^{3} x^{5}+\frac{1}{9} y^{5} x^{2}+\frac{47}{45} y^{4} x^{3}+\frac{47}{45} y^{3} x^{4}+\frac{1}{9} y^{2} x^{5} \\
& +\frac{1}{12} y^{5} x+\frac{8}{3} y^{4} x^{2}+\frac{689}{90} y^{3} x^{3}+\frac{8}{3} y^{2} x^{4}+5 y^{2} \\
& +5 x^{2}+\frac{5}{3} y^{3}+\frac{5}{3} x^{3}+\frac{5}{24} y^{4}+\frac{1}{120} y^{5}+\frac{1}{120} x^{5} \tag{A.5}
\end{align*}
$$

- $N=15$ :

$$
\begin{align*}
& 1+6 x+80 y^{2} x+6 y+\frac{1445}{12} y^{2} x^{2}+\frac{353}{6} y^{2} x^{3}+\frac{353}{6} y^{3} x^{2}+80 y x^{2} \\
& +\frac{75}{2} y^{3} x+\frac{75}{2} y x^{3}+57 y x+\frac{5}{8} x^{4}+\frac{29}{4} y x^{4}+\frac{29}{4} y^{4} x+\frac{71}{120} y x^{5} \\
& +\frac{134}{14175} y^{5} x^{5}+\frac{34}{315} y^{5} x^{4}+\frac{34}{315} y^{4} x^{5}+\frac{1}{36} y^{6} x^{2}+\frac{23}{45} y^{5} x^{3} \\
& +\frac{263}{210} y^{4} x^{4}+\frac{23}{45} y^{3} x^{5}+\frac{1}{36} y^{2} x^{6}+\frac{1}{60} y^{6} x+\frac{35}{36} y^{5} x^{2}+\frac{181}{30} y^{4} x^{3} \\
& +\frac{181}{30} y^{3} x^{4}+\frac{35}{36} y^{2} x^{5}+\frac{1}{60} y x^{6}+\frac{2}{135} y^{6} x^{3}+\frac{2}{135} y^{3} x^{6}+\frac{1}{315} y^{6} x^{4} \\
& +\frac{1}{315} y^{4} x^{6}+\frac{4}{14175} y^{6} x^{5}+\frac{4}{467775} y^{6} x^{6}+\frac{4}{14175} y^{5} x^{6}+\frac{1}{720} x^{6} \\
& +\frac{1}{720} y^{6}+\frac{71}{120} y^{5} x+\frac{187}{16} y^{4} x^{2}+\frac{5339}{180} y^{3} x^{3}+\frac{187}{16} y^{2} x^{4}+\frac{15}{2} y^{2} \\
& +\frac{15}{2} x^{2}+\frac{10}{3} y^{3}+\frac{10}{3} x^{3}+\frac{5}{8} y^{4}+\frac{1}{20} y^{5}+\frac{1}{20} x^{5} \tag{A.6}
\end{align*}
$$

- $N=17$ :

$$
\begin{align*}
& 1+7 x+7 y+\frac{1}{945} y^{4} x^{7}+77 y x+\frac{21}{2} y^{2}+\frac{21}{2} x^{2}+\frac{259}{2} y^{2} x \\
& +\frac{259}{2} y x^{2}+\frac{93}{4} y^{2} x^{2}+\frac{35}{6} y^{3}+\frac{35}{6} x^{3}+\frac{455}{6} y^{3} x+\frac{455}{6} y x^{3} \\
& +\frac{567}{4} y^{3} x^{2}+\frac{567}{4} y^{2} x^{3}+\frac{3199}{36} y^{3} x^{3}+\frac{8}{42567525} y^{7} x^{7}+\frac{35}{24} y^{4} \\
& +\frac{35}{24} x^{4}+\frac{469}{24} y^{4} x+\frac{469}{24} y x^{4}+\frac{1799}{48} y^{4} x^{2}+\frac{1799}{48} y^{2} x^{4} \\
& +\frac{1}{945} y^{7} x^{4}+\frac{3457}{144} y^{4} x^{3}+\frac{3457}{144} y^{3} x^{4}+\frac{416}{63} y^{4} x^{4}+\frac{2}{14175} y^{5} x^{7} \\
& +\frac{7}{40} y^{5}+\frac{7}{40} x^{5}+\frac{287}{120} y^{5} x+\frac{287}{120} y x^{5}+\frac{1121}{240} y^{5} x^{2}+\frac{1121}{240} y^{2} x^{5} \\
& +\frac{73}{24} y^{5} x^{3}+\frac{73}{24} y^{3} x^{5}+\frac{2}{14175} y^{7} x^{5}+\frac{107}{126} y^{5} x^{4}+\frac{107}{126} y^{4} x^{5} \\
& +\frac{523}{4725} y^{5} x^{5}+\frac{4}{467775} y^{7} x^{6}+\frac{4}{467775} y^{6} x^{7}+\frac{7}{720} y^{6}+\frac{7}{720} x^{6} \\
& +\frac{97}{720} y^{6} x+\frac{97}{720} y x^{6}+\frac{4}{15} y^{6} x^{2}+\frac{4}{15} y^{2} x^{6}+\frac{19}{108} y^{6} x^{3}+\frac{19}{108} y^{3} x^{6} \\
& +\frac{47}{945} y^{6} x^{4}+\frac{47}{945} y^{4} x^{6}+\frac{31}{4725} y^{6} x^{5}+\frac{31}{4725} y^{5} x^{6}+\frac{184}{467775} y^{6} x^{6} \\
& +\frac{1}{5040} y^{7}+\frac{1}{5040} x^{7}+\frac{1}{360} y^{7} x+\frac{1}{360} y x^{7}+\frac{1}{180} y^{7} x^{2} \\
& +\frac{1}{180} y^{2} x^{7}+\frac{1}{270} y^{7} x^{3}+\frac{1}{270} y^{3} x^{7} \tag{A.7}
\end{align*}
$$

- $N=19$ :

$$
\begin{align*}
& 1+8 x+8 y+\frac{31}{1890} y^{4} x^{7}+100 y x+14 y^{2}+14 x^{2}+196 y^{2} x \\
& +196 y x^{2}+\frac{819}{2} y^{2} x^{2}+\frac{28}{3} y^{3}+\frac{28}{3} x^{3}+\frac{413}{3} y^{3} x+\frac{413}{3} y x^{3} \\
& +\frac{896}{3} y^{3} x^{2}+\frac{896}{3} y^{2} x^{3}+\frac{10073}{45} y^{3} x^{3}+\frac{44}{3869775} y^{7} x^{7}+\frac{35}{12} y^{4} \\
& +\frac{35}{12} x^{4}+\frac{133}{3} y^{4} x+\frac{133}{3} y x^{4}+\frac{1183}{12} y^{4} x^{2}+\frac{1183}{12} y^{2} x^{4}+\frac{31}{1890} y^{7} x^{4} \\
& +\frac{1357}{18} y^{4} x^{3}+\frac{1357}{18} y^{3} x^{4}+\frac{104071}{4032} y^{4} x^{4}+\frac{41}{14175} y^{5} x^{7}+\frac{7}{15} y^{5} \\
& +\frac{7}{15} x^{5}+\frac{217}{30} y^{5} x+\frac{217}{30} y x^{5}+\frac{1}{40320} x^{8}+\frac{491}{30} y^{5} x^{2}+\frac{491}{30} y^{2} x^{5} \\
& +\frac{1}{2520} y^{8} x+\frac{9151}{720} y^{5} x^{3}+\frac{9151}{720} y^{3} x^{5}+\frac{41}{14175} y^{7} x^{5}+\frac{247}{56} y^{5} x^{4} \\
& +\frac{247}{56} y^{4} x^{5}+\frac{3089}{4050} y^{5} x^{5}+\frac{122}{467775} y^{7} x^{6}+\frac{122}{467775} y^{6} x^{7}+\frac{1}{1080} y^{2} x^{8} \\
& +\frac{7}{180} y^{6}+\frac{7}{180} x^{6}+\frac{11}{18} y^{6} x+\frac{11}{18} y x^{6}+\frac{2017}{1440} y^{6} x^{2}+\frac{2017}{1440} y^{2} x^{6} \\
& +\frac{1189}{1080} y^{6} x^{3}+\frac{1189}{1080} y^{3} x^{6}+\frac{1}{2520} y x^{8}+\frac{2921}{7560} y^{6} x^{4}+\frac{2921}{7560} y^{4} x^{6} \\
& +\frac{319}{4725} y^{6} x^{5}+\frac{319}{4725} y^{5} x^{6}+\frac{1129}{187110} y^{6} x^{6}+\frac{1}{40320} y^{8}+\frac{1}{630} y^{7} \\
& +\frac{1}{630} x^{7}+\frac{127}{5040} y^{7} x+\frac{1}{1080} y^{8} x^{2}+\frac{127}{5040} y x^{7}+\frac{7}{120} y^{7} x^{2}+\frac{7}{120} y^{2} x^{7} \\
& +\frac{5}{108} y^{7} x^{3}+\frac{5}{108} y^{3} x^{7}+\frac{1}{1350} y^{8} x^{3}+\frac{1}{1350} y^{3} x^{8}+\frac{1}{3780} y^{8} x^{4} \\
& +\frac{1}{3780} y^{4} x^{8}+\frac{2}{42525} y^{8} x^{5}+\frac{2}{42525} y^{5} x^{8}+\frac{2}{467775} y^{8} x^{6}+\frac{2}{467775} y^{6} x^{8} \\
& +\frac{8}{42567525} y^{8} x^{7}+\frac{8}{42567525} y^{7} x^{8}+\frac{2}{638512875} y^{8} x^{8} \tag{A.8}
\end{align*}
$$

- $N=21:$ Microfossils (21)(DM).

$$
\begin{align*}
& 1+9 x+9 y+\frac{109}{840} y^{4} x^{7}+\frac{1}{8100} y^{3} x^{9}+126 y x+18 y^{2}+18 x^{2} \\
& +282 y^{2} x+282 y x^{2}+\frac{1343}{2} y^{2} x^{2}+14 y^{3}+14 x^{3}+\frac{1}{7560} y^{9} x^{2}+231 y^{3} x \\
& +231 y x^{3}+\frac{2}{1403325} y^{6} x^{9}+\frac{1141}{2} y^{3} x^{2}+\frac{1141}{2} y^{2} x^{3}+\frac{1}{85050} y^{9} x^{5} \\
& +\frac{7462}{15} y^{3} x^{3}+\frac{349}{1576575} y^{7} x^{7}+\frac{1}{18900} y^{9} x^{4}+\frac{21}{4} y^{4}+\frac{21}{4} x^{4}+\frac{357}{4} y^{4} x \\
& +\frac{357}{4} y x^{4}+\frac{1}{18900} y^{4} x^{9}+\frac{903}{4} y^{4} x^{2}+\frac{903}{4} y^{2} x^{4}+\frac{109}{840} y^{7} x^{4}+\frac{4013}{20} y^{4} x^{3} \\
& +\frac{4013}{20} y^{3} x^{4}+\frac{184099}{2240} y^{4} x^{4}+\frac{2}{638512875} y^{8} x^{9}+\frac{1613}{56700} y^{5} x^{7}+\frac{21}{20} y^{5} \\
& +\frac{21}{20} x^{5}+\frac{91}{5} y^{5} x+\frac{91}{5} y x^{5}+\frac{1}{4480} x^{8}+\frac{2809}{60} y^{5} x^{2}+\frac{2809}{60} y^{2} x^{5} \\
& +\frac{23}{5760} y^{8} x+\frac{10133}{240} y^{5} x^{3}+\frac{10133}{240} y^{3} x^{5}+\frac{1613}{56700} y^{7} x^{5}+\frac{353047}{20160} y^{5} x^{4} \\
& +\frac{353047}{20160} y^{4} x^{5}+\frac{1711063}{453600} y^{5} x^{5}+\frac{1061}{311850} y^{7} x^{6}+\frac{1061}{311850} y^{6} x^{7}+\frac{1}{8100} y^{9} x^{3} \\
& +\frac{2}{189} y^{2} x^{8}+\frac{7}{60} y^{6}+\frac{7}{60} x^{6}+\frac{41}{20} y^{6} x+\frac{41}{20} y x^{6}+\frac{2563}{480} y^{6} x^{2}+\frac{2563}{480} y^{2} x^{6} \\
& +\frac{21041}{4320} y^{6} x^{3}+\frac{21041}{4320} y^{3} x^{6}+\frac{23}{5760} y x^{8}+\frac{2}{1403325} y^{9} x^{6}+\frac{643}{315} y^{6} x^{4} \\
& +\frac{643}{315} y^{4} x^{6}+\frac{50333}{113400} y^{6} x^{5}+\frac{50333}{113400} y^{5} x^{6}+\frac{2}{638512875} y^{9} x^{8} \\
& +\frac{49279}{935550} y^{6} x^{6}+\frac{1}{7560} y^{2} x^{9}+\frac{1}{4480} y^{8}+\frac{1}{140} y^{7}+\frac{1}{140} x^{7}+\frac{71}{560} y^{7} x \\
& +\frac{2}{189} y^{8} x^{2}+\frac{71}{560} y x^{7}+\frac{4}{42567525} y^{7} x^{9}+\frac{3361}{10080} y^{7} x^{2}+\frac{3361}{10080} y^{2} x^{7} \\
& +\frac{221}{720} y^{7} x^{3}+\frac{221}{720} y^{3} x^{7}+\frac{1}{20160} y x^{9}+\frac{1}{20160} y^{9} x+\frac{53}{5400} y^{8} x^{3} \\
& +\frac{53}{5400} y^{3} x^{8}+\frac{79}{18900} y^{8} x^{4}+\frac{79}{18900} y^{4} x^{8}+\frac{4}{97692469875} y^{9} x^{9} \\
& +\frac{157}{170100} y^{8} x^{5}+\frac{157}{170100} y^{5} x^{8}+\frac{1}{85050} y^{5} x^{9}+\frac{52}{467775} y^{8} x^{6} \\
& +\frac{52}{467775} y^{6} x^{8}+\frac{1}{362880} x^{9}+\frac{62}{8513505} y^{8} x^{7}+\frac{62}{8513505} y^{7} x^{8} \\
& +\frac{2}{8292375} y^{8} x^{8}+\frac{1}{362880} y^{9}+\frac{4}{42567525} y^{9} x^{7}  \tag{A.9}\\
& +\mathrm{A} .9 \\
& +5
\end{align*}
$$


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