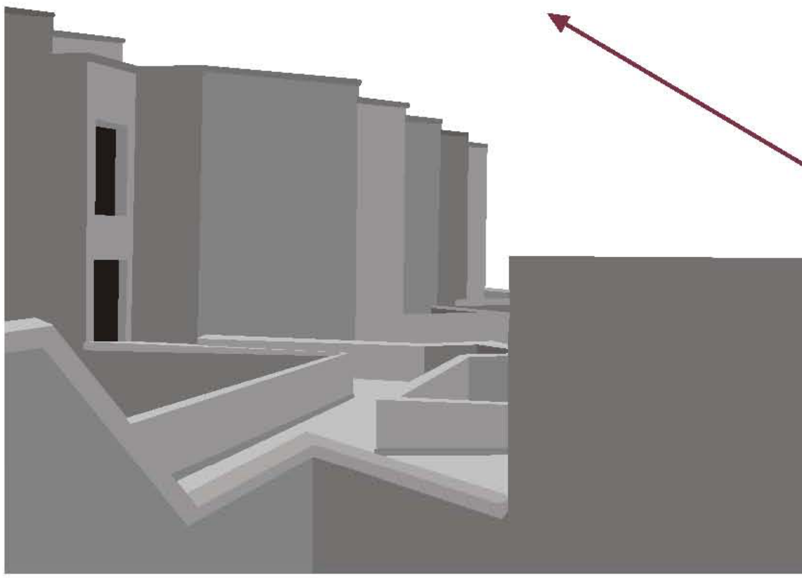

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Fluctuation limit theorems for age-dependent critical binary branching systems

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Abstract

We consider an age-dependent branching particle system in $\mathbb{R}^d$, where the particles are subject to α-stable migration, $0 < \alpha \leq 2$, critical binary branching, and general (non-arithmetic) lifetimes distribution. The population starts off from a Poisson random field in $\mathbb{R}^d$ with Lebesgue intensity. We prove functional central limit theorems and strong laws of large numbers under two rescalings: high density, and a space-time rescaling. Properties of the limit processes, such as Markov property, almost sure continuity of paths, Langevin equation and spectral measure, are also investigated.

1 Introduction

In this paper, we investigate high density and space-time scaling limits of a random population living in the d-dimensional Euclidean space $\mathbb{R}^d$. The evolution of the population is as follows. Any given individual independently develops a spherically symmetric α-stable process during its lifetime τ, where $0 < \alpha \leq 2$ and τ is a random variable having a non-arithmetic distribution function, and at the end of its life it either disappears, or is replaced at the site where it died by two newborns, each event occurring with probability 1/2. The population starts off from a Poisson random field having Lebesgue measure Λ as its intensity. We postulate the usual independence assumptions in branching systems.

Two regimes for the distribution of τ are considered: either τ has finite mean $\mu > 0$, or

τ possesses a distribution function F such that $F(0) = 0$, $F(x) < 1$ for all $x \in [0, \infty)$, and

$$\bar{F}(u) := 1 - F(u) \sim u^{-\gamma}\Gamma(1+\gamma)^{-1} \quad \text{as} \quad u \longrightarrow \infty, \tag{1}$$

where $\gamma \in (0,1)$ and Γ denotes the Gamma function, i.e. F belongs to the normal domain of attraction of a γ-stable law. In particular, this allows to consider lifetimes with infinite mean.

Let $X \equiv \{X(t),\, t \geq 0\}$, where $X(t)$ denotes the simple counting measure on $\mathbb{R}^d$ whose atoms are the positions of particles alive at time t. When τ has an exponential distribution it is well known that the measure-valued process X is Markov. In the literature there is a lot of work about the Markovian model. Our objective here is to investigate the case when τ is not necessarily an exponential random variable, in whose case $\{X(t),\, t \geq 0\}$ is no longer a Markov process. Another striking difference with respect to the case of exponential lifetimes arises when the particle lifetime distribution satisfies (1). When the distribution of τ possesses heavy tails, a kind of compensation occurs between longevity of individuals and clumping of the population: heavy-tailed lifetimes enhance the mobility of individuals, favouring in this way the spreading out of particles, and thus counteracting the clumping of the population. Since clumping goes along with local extinction (due to critical branching), a smaller exponent γ suits better for stability of the population. As a matter of fact, Vatutin and Wakolbinger [29] and Fleischmann, Vatutin and Wakolbinger [11] proved that X admits a nontrivial equilibrium distribution if and only if $d \geq \gamma\alpha$. This contrasts with the case of finite-mean (or exponentially distributed) lifetimes, where the necessary and sufficient condition for stability is $d > \alpha$. As we shall see, such qualitative departure from the Markovian model propagates also to other aspects of the branching particle system, such as the large-time behavior of the limit theorem mentioned at the beginning of this introduction.

As we mentioned above we investigate the so-called *high density* and *space-time scaling* limits of our age-dependent branching system. The high density limit consist in increasing the initial intensity by a factor K which will tend to infinity, see [22] for the physical motivation of this rescaling. We are interested in the fluctuations process, i.e, we center the process around its mean measure and normalize it by $K^{1/2}$; this entails to change the state-space of X and the underlaying notion of convergence. We show that the fluctuations process converges to an $S'(\mathbb{R}^d)$-valued centered Gaussian process whose covariance functional is calculated explicitly, where $S'(\mathbb{R}^d)$ s the space of tempered distributions, dual of the space $S(\mathbb{R}^d)$ of rapidly decreasing functions. Also we prove several properties of the limit process, namely, Markov property and almost sure continuity of paths in the norm $\|\cdot\|_{-p}$ for some $p \geq 1$, see the following section for this technical points. These results are valid for a general non-arithmetic lifetime distribution. When the lifetime distribution possesses a continuous density, we also show that the limit process satisfies a generalized Langevin equation. These results were known only in the case of exponentially distributed lifetimes; see [13] for the general mono-type branching case, and [19] for systems with multi-type branching.

For the space-time scaling limit we again assume that the lifetime distribution has a tail of the form (1). The coordinates in space and time are respectively Kx and $K^\alpha t$, again K being a parameter which will tend to infinity. In this case we need to assume that $d > \alpha\gamma$, i.e., we require supercritical dimension for persistence. The normalizing constant for the fluctuation process is $K^{d+\alpha\gamma}$, with $K \longrightarrow \infty$. (Recall that, for exponentially distributed lifetimes, the normalizing function is $K^{d+\alpha}$; see [13]). The limit process is again an $S'(\mathbb{R}^d)$-valued centered Gaussian process, it is a Markov process and possesses a version which is continuous in the norm $\|\cdot\|_{-p}$ for some $p \geq 1$. Also, it satisfies a generalized Langevin equation. Heavy-tailed lifetimes play a key role in the space-time scaling because the power γ of the tail decay figures explicitly in the limit theorems, the effect being similar to the one that it has in the diffusion limit approximation of [18]; see equation (5.1) there.

It is well known that, in order to prove weak convergence of a sequence $\{P_n\}_{n=1}^{\infty}$ of probability measures in the Skorokhood space, it is sufficient to show weak convergence of the finite-dimensional marginals, and tightness (or relative compactness) of $\{P_n\}$. In our proof of the fluctuation limit theorems mentioned above, convergence of finite-dimensional distributions is achieved by the usual method, showing convergence of characteristic functionals and using the Minlos-Sasonov's theorem. The proof of tightness can not be carried out as in the classical case of exponentially distributed life times because, as we mentioned above, $\{X_t, t \geq 0\}$ is not a Markov process, and many of the steps in the proof of tightness are based on this property. To overcome this difficulty, we consider the Markov process $\{X_t \times \bar{X}_t, t \geq 0\}$, where $\{\bar{X}_t, t \geq 0\}$ is a Markovianization of the branching system $\{X_t, t \geq 0\}$ obtained by enlarging the phase-state, including the "elapsed time" or "age" of each individual (see [27] for a more detailed discussion, and [18] and [11] for a related procedure based on the residual lifetime of each particle).

The occupation time process of a branching system is another object that has been extensively studied in the context of exponentially distributed lifetimes (see [7], [23], [4], [5]). [16] and [10] investigated the occupation time of Dawson-Watanabe superprocesses, i.e., measure-valued processes which are diffusion limits of branching particle systems with exponential lifetimes. The authors have investigated laws of large numbers for the occupation time of the critical binary age-dependent branching system, see [27].

2 Some notation and technical points

For each $p \geq 0$ we define the reference function $\phi_p(x) = (1 + |x|^2)^{-p}$, $x \in \mathbb{R}^d$, we denote by $\mathcal{M}_p(\mathbb{R}^d)$ the space of non-negative Radon measures μ on $\mathbb{R}^d$, such that $\int \phi_p d\mu < \infty$, and endow $\mathcal{M}_p(\mathbb{R}^d)$ with the p-vague topology, i.e., the minimal topology under which the maps $\mu \longmapsto \int \phi d\mu$ are continuous for $\phi \in K_p(\mathbb{R}^d)_+$, where $K_p(\mathbb{R}^d)_+ = C_c(\mathbb{R}^d)_+ \cup \{\phi_p\}$, where $C_c(\mathbb{R}^d)$ denotes the space of continuous functions with compact support. $\mathcal{M}_p(\mathbb{R}^d)$ is a complete, separable metric space, and the finite atomic measures are dense in it. The Lebesgue measure on $\mathbb{R}^d$ belongs to $\mathcal{M}_p(\mathbb{R}^d)$ for $p > d/2$. $D(\mathbb{R}_+, \mathcal{M}_p(\mathbb{R}^d))$ denotes the space of functions from $\mathbb{R}_+$ to $\mathcal{M}_p(\mathbb{R}^d)$ which are continuous from the right with limits from the left. It is well know that $D(\mathbb{R}_+, \mathcal{M}_p(\mathbb{R}^d))$ equipped with the Skorokhod topology ([9]) is a complete, separable metric space. The process X takes values on $D(\mathbb{R}_+, \mathcal{M}_p(\mathbb{R}^d))$.

Now we introduce the state-space which will be need when we Markovianize the process X. Let $E = \mathbb{R} \times \mathbb{R}^d$, and let $\hat{C}_p = \hat{C}_p(E)$ denote the space of all continuous functions $\psi : E \longrightarrow \mathbb{R}$ such that

$$\lfloor \psi \rfloor_p := \sup_{(u,x)\in E} \left| \frac{\psi(u,x)}{\phi_p(x)} \right|,$$

and such that the map

$$(u, x) \longmapsto \frac{\psi(u,x)}{\phi_p(x)}$$

on E can be extended continuously to a function on $\dot{\mathbb{R}}_+ \times \dot{\mathbb{R}}^d$, where $\dot{\mathbb{R}}_+$ and $\dot{\mathbb{R}}^d$ are the one-point compactifications of $\mathbb{R}_+^d$ and $\mathbb{R}^d$, respectively. Then $(\hat{C}_p, \lfloor\cdot\rfloor_p)$ is a separable Banach space.

Let $\hat{\mathcal{M}}_p = \hat{\mathcal{M}}_p(E)$ be the set of all p-tempered measures on E, that is, measures μ on E such that the integral

$$\langle \phi_p, \mu \rangle := \int_E \phi_p(x)\mu\,(d(u,x))$$

is finite. Introduce the weakest topology in $\hat{\mathcal{M}}_p$ such that for each $\psi \in \hat{C}_p$ the mapping $\mu \longmapsto \langle \psi, \mu \rangle$ is continuous. Note that $\nu \times \Lambda$ belongs to $\hat{M}_p$ for any finite measure ν (since $\langle \phi_p, \Lambda \rangle < \infty$). The process $\bar{X}$ takes values on $D(\mathbb{R}_+, \hat{\mathcal{M}}_p)$.

Regarding p, α and d above, we assume that $p > d/2$, and in addition $p < (d+\alpha)/2$ if $\alpha < 2$ (see [8] and [16] on this condition).

Let $S(\mathbb{R}^d)$ be the space of rapidly decreasing functions, i.e. functions $\phi : \mathbb{R}^d \to \mathbb{R}$ such that ϕ is infinitely differentiable, and for all $p = 0, 1, 2, \cdots$,

$$\|\phi\|_p = \left(\sum_{|k|=0}^{p} \int_{\mathbb{R}^d} \left(1+|x|^2\right)^p |D^k \phi(x)|^2 dx \right)^{1/2} < \infty, \tag{2}$$

where $x = (x_1, \cdots, x_d)$, $k = (k_1, \cdots, k_d)$, $|k| = k_1 + \cdots + k_d$ and $D^k = \partial^{|k|}/\partial x_1^{k_1} \cdots \partial x_d^{k_d}$. It can be shown that $S(\mathbb{R}^d) \subset C_p(\mathbb{R}^d)$, where $C_p(\mathbb{R}^d)$ denotes the space of real-valued continuous functions ϕ on $\mathbb{R}^d$ such that $|\phi|_p := \sup_{x\in\mathbb{R}^d} |\phi(x)/\phi_p(x)| < \infty$

The space $S(\mathbb{R}^d)$ endowed with the topology induced by the system of Hilbert's norms $\{\|\cdot\|_p, p \geq 0\}$ is a metric space which is separable, complete and nuclear. Let $S_p(\mathbb{R}^d)$ be completion of $S(\mathbb{R}^d)$ with respect to the norm $\|\cdot\|_p$. Then $S_m(\mathbb{R}^d) \subset S_n(\mathbb{R}^d)$ for $n \leq m$, $S(\mathbb{R}^d) = \cap_{p\geq 0} S_p(\mathbb{R}^d)$, and for each $p \geq 0$, $S_p(\mathbb{R}^d)$ is a Hilbert space. In particular, $S_0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. Let us denote by $S'_p(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$ the strong dual space of $S_p(\mathbb{R}^d)$ and $S(\mathbb{R}^d)$, respectively. $S'(\mathbb{R}^d)$ is nuclear and is called the *Schwartz's space of tempered distributions* on $\mathbb{R}^d$.

For each $p = 0, 1, 2, \cdots$, $S'_p(\mathbb{R}^d)$ is a Hilbert space with norm

$$\|F\|_{-p} := \sup_{\|\phi\|_p=1} |\langle F, \phi \rangle|, \quad F \in S'_p(\mathbb{R}^d), \quad \phi \in S_p(\mathbb{R}^d), \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denotes the canonic bilinear form in $S'(\mathbb{R}^d) \times S(\mathbb{R}^d)$ and $S'_p(\mathbb{R}^d) \times S_p(\mathbb{R}^d)$. We denote by $D(\mathbb{R}_+, S'(\mathbb{R}^d))$ the space of functions from $\mathbb{R}_+$ to $S'(\mathbb{R}^d)$ which are continuous from the right with limits from the left, endowed with the Skorokhod topology (see [26]). For more details on this topic see [12] and [28].

3 Results

1. High density. The initial intensity changes to $K\Lambda$. The resulting branching particle system is denoted by $X^{1,K} \equiv \{X_t^{1,K}, t \geq 0\}$.

2. Space-time re-scaling. Let us suppose that $d > \alpha\gamma$. We recall that $\gamma = 1$ can be regarded as finite mean life times case. The coordinates in space-time are Kx and $K^\alpha t$, respectively. The branching particle system is denoted by $X^{2,K} \equiv \{X_t^{2,K}, t \geq 0\}$.

The fluctuation processes corresponding to these re-scalings are respectively

$$M^{l,K} = K^l \left(X^{l,K} - \mathbb{E}X^{l,K}\right)$$

, where $K^1 = K^{-1/2}$ and $K^2 = K^{-(d+\alpha\gamma)/2}$, $l = 1, 2$.

Theorem 3.1 *(Functional central limit theorems) $M^{l,K} \Longrightarrow M^l$, $l = 1, 2$, in $D\left([0,\infty), S'(\mathbb{R}^d)\right)$, as $K \longrightarrow \infty$, where M^l, $l = 1, 2$, are centered Gaussian process whose covariance functionals*

$$\mathcal{K}^l(s,\varphi;t,\psi) \equiv \mathbb{E}\left(\langle\varphi, M_s^l\rangle\langle\psi, M_t^l\rangle\right), \quad l = 1, 2,$$

are given by

$$\mathcal{K}^1(s,\varphi;t,\psi) = \langle\varphi\mathcal{S}_{t-s}\psi, \Lambda\rangle + \int_0^s \langle(\mathcal{S}_{s-r}\varphi)(\mathcal{S}_{t-r}\psi), \Lambda\rangle dU(r), \tag{4}$$

where $U(r) = \sum_{k=0}^\infty F^{*k}(r)$; and

$$\mathcal{K}^2(s,\varphi;t,\psi) := \frac{1}{\Gamma(1+\gamma)} \int_0^s \langle(\mathcal{S}_{t-u}\psi)(\mathcal{S}_{s-u}\varphi), \Lambda\rangle\gamma u^{\gamma-1} du \tag{5}$$

for all $0 \leq s \leq t < \infty$ and $\varphi, \psi \in S(\mathbb{R}^d)$.

Theorem 3.2 *(Laws of large numbers) For each $t \geq 0$ and $\varphi \in S(\mathbb{R}^d)$:*

$$\frac{\langle\varphi, X_t^{1,K}\rangle}{K} \longrightarrow \langle\varphi, \Lambda\rangle,$$

and

$$\frac{\langle \varphi^K, X_t^{2,K}\rangle}{K^d} \longrightarrow \langle \varphi, \Lambda\rangle,$$

in $L^2(\mathbb{R}^d)$*, as* $K \longrightarrow \infty$*.*

Theorem 3.3 *(Properties of the fluctuation limits) (a) For* $l = 1, 2$, M^l *is a Markov process, and*

$$\langle \psi, M_t^l\rangle - \int_0^t \langle \psi, M_s^l\rangle ds, \ t \geq 0, \tag{6}$$

is a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{\langle \phi, M_r^l\rangle, \ r \leq t, \ \phi \in S(\mathbb{R}^d)\}, \ t \geq 0$*.*
(b) There exists $p \geq 1$ *such that* M^l, $l = 1, 2$*, has a continuous version in the norm* $\|\cdot\|_{-p}$*.*
(c) Assume that F *has a continuous density* f*. Then, the process* M^1 *satisfies the generalized Langevin equation*

$$\begin{aligned} dM_t^1 &= \Delta_\alpha M_t^1 + d\mathcal{W}_t, \\ M_0^1 &= W, \end{aligned} \tag{7}$$

where W *is a centered spatial white noise and the Wiener process* $\mathcal{W}$ *is associated to the family of operators* $\{Q_t^1, t \geq 0\}$ *such that for each* $\varphi, \psi \in S(\mathbb{R}^d)$*,*

$$\langle \varphi, Q_t \psi\rangle = \langle \varphi\psi, \Lambda\rangle u(t) - 2\langle \varphi \Delta_\alpha \psi, \Lambda\rangle, \tag{8}$$

where $u(t) = dU(t)/dt$*.*

Remark 3.4 *(a) The assumption in the theorem above that* F *has a continuous density cannot be dropped; without such assumption we cannot guarantee differentiability of the function* $t \longmapsto \mathcal{K}^1(t, \varphi; t, \varphi)$*.*
(b) Assuming that $F(t) = 1 - e^{-Vt}$, $t \geq 0$*, and* $\alpha = 2$ *we get that* $U(dt) \equiv V\,dt$*. Hence, (8) is equivalent to*

$$\langle \varphi, Q_t \psi\rangle = V\langle \varphi\psi, \Lambda\rangle + \langle \nabla\varphi \cdot \nabla\psi, \Lambda\rangle,$$

which recovers a result from [13] for critical binary branching.
(c) By Remark (a) of Theorem 3.6 in [2], without any regularity condition on F *we still have*

$$\langle \varphi, M_t\rangle = \langle \varphi, W\rangle + \int_0^t \langle \varphi, M_s\rangle ds + \langle \varphi, \mathcal{W}_t\rangle, \ \ t \geq 0,$$

where $\{\mathcal{W}_t, t \geq 0\}$ is a continuous $S'(\mathbb{R}^d)$-valued Gaussian process with covariance functional

$$\begin{aligned}\mathbb{E}\left[\langle \varphi, W_s\rangle \langle \varphi, W_t\rangle\right] &= \mathcal{K}(s\wedge t, \varphi; s\wedge t, \psi) \\ &\quad - \int_0^{s\wedge t} (\mathcal{K}(u, \Delta_\alpha \varphi; u, \psi) - \mathcal{K}(u, \varphi; u, \Delta_\alpha \psi)) du,\end{aligned}$$

for all $s, t \geq 0$ and $\varphi, \psi \in S(\mathbb{R}^d)$.

(d) When $\alpha < 2$, (7) has to be understood in a generalized sense, because of $\Delta_\alpha S(\mathbb{R}^d) \nsubseteq S(\mathbb{R}^d)$, see [8].

4 Some moment calculations

Let $Z_t(A)$ denote the number of individuals living in $A \in \mathcal{B}(\mathbb{R}^d)$ at time t, in a population starting with one particle at time $t = 0$. Following [18] we define

$$Q_t\varphi(x) := \mathbb{E}_x\left[1 - e^{-\langle \varphi, Z_t\rangle}\right], \quad x \in \mathbb{R}^d,\ t \geq 0, \tag{9}$$

where $\mathbb{E}_x$ means that the initial particle is located at $x \in \mathbb{R}^d$ and $\varphi \in C_c^+(\mathbb{R}^d)$. Since the initial population X_0 is Poissonian, we have

$$\begin{aligned}\mathbb{E}e^{-\langle \varphi, X_t\rangle} &= \exp\left(-\int \mathbb{E}_x\left[1 - e^{-\langle \varphi, Z_t\rangle}\right] dx\right) \\ &= \exp\left(-\int Q_t\varphi(x) dx\right), \quad \varphi \in C_c(\mathbb{R}^d).\end{aligned} \tag{10}$$

Let $\{\tau_k, k \geq 1\}$ be a sequence of i.i.d. random variables with common distribution function F, and let

$$N_t = \sum_{k=1}^{\infty} 1_{\{S_k \leq t\}}, \quad t \geq 0,$$

where the random sequence $\{S_k, k \geq 0\}$ is recursively defined by

$$S_0 = 0, \quad S_{k+1} = S_k + \tau_k, \quad k \geq 0.$$

For any $p = 1, 2, \ldots$, $0 < t_p \leq t_{p-1}, \ldots, t_1 < \infty$, $\varphi_1, \varphi_2, \ldots, \varphi_p \in C_c(\mathbb{R}^d)$ and $\theta_1, \ldots, \theta_p \in \mathbb{R}$, we define $\bar{t} = (t_1, t_2, \ldots, t_p)$, $\bar{t} - s = (t_1 - s, t_2 - s, \ldots, t_p - s)$, $\theta_{(p)} = (\theta_1, \ldots, \theta_p)'$ and

$$Q_{\bar{t}}^p \theta_{(p)}(x) = \mathbb{E}_x\left[1 - e^{-\sum_{j=1}^p \theta_j \langle \varphi_j, Z_{t_j}\rangle}\right].$$

Let $\{B_s,\ s \geq 0\}$ denote the spherically symmetric α-stable process in $\mathbb{R}^d$, with transition density functions $\{p_t(x,y) := p_t(x-y),\ t>0,\ x,y \in \mathbb{R}^d\}$, and semigroup $\{\mathcal{S}_t,\ t \geq 0\}$, we will use the following upper-bound

$$p_t(x) \leq ct|x|^{-d-\alpha},\ t>0,\ x \in \mathbb{R}^d, \tag{11}$$

for some positive constant c.

Proposition 4.1 *([18]) The function $Q^p_{\bar{t}}\theta_{(p)}$ satisfies*

$$\begin{aligned} Q^p_{\bar{t}}\theta_{(p)}(x) &= \mathbb{E}_x\Bigg[1-e^{-\sum_{j=1}^p \theta_j\varphi_j(B_{t_j})} - \int_0^{t_p} \frac{1}{2}\left(Q^p_{\bar{t}-s}\theta_{(p)}(B_s)\right)^2 dN_s \\ &\quad - \sum_{i=1}^{p-1}\left(1-e^{-\sum_{j=i+1}^p \theta_j\varphi_j(B_{t_j})}\right)\int_{t_{i+1}}^{t_i} \frac{1}{2}\left(Q^i_{\bar{t}-s}\theta_{(i)}(B_s)\right)^2 dN_s\Bigg]. \end{aligned}$$

As in (10), since the initial population is Poissonian we have

$$\begin{aligned} \mathbb{E}\left[e^{-\sum_{j=1}^p \theta_j\langle\varphi_j, X_{t_j}\rangle}\right] &= \exp\left(-\int \mathbb{E}_x\left[1-e^{-\sum_{j=1}^p \theta_j\langle\varphi, Z_{t_j}\rangle}\right]dx\right) \\ &= \exp\left(-\int Q^p_{\bar{t}}\theta_{(p)}(x)dx\right). \end{aligned} \tag{12}$$

Using criticality of the branching, and that Lebesgue measure is invariant for the semigroup of the symmetric α-stable process, it is easy to see that

$$m(t,\varphi) := \mathbb{E}[\langle\varphi, X_t\rangle] = \langle\varphi, \Lambda\rangle, \quad t \geq 0, \quad \varphi \in C_c(\mathbb{R}^d). \tag{13}$$

Lemma 4.2 *Let $0 < s \leq t < \infty$ and $\psi, \varphi \in C_c(\mathbb{R}^d)$. Then,*

$$\begin{aligned} C_x(s,\varphi;t,\psi) &:= \mathbb{E}_x\left[\langle\varphi, Z_s\rangle\langle\psi, Z_t\rangle\right] \\ &= \mathbb{E}_x\left[\varphi(B_s)\psi(B_t) + \int_0^s m_{B_r}(t-r,\psi)m_{B_r}(s-r,\varphi)dN_r\right]. \end{aligned} \tag{14}$$

Proof: In order to preserve the notation in Proposition 4.1, we put $p=2$, $t_1=t$, $t_2=s$, $\varphi_1=\psi$ and $\varphi_2=\varphi$. Then we have

$$C_x(t_1,\varphi_1;t_2,\varphi_2) = -\frac{\partial^2}{\partial\theta_1\partial\theta_2}Q^2_{\bar{t}}\theta_{(2)}(x)\bigg|_{\theta_1=\theta_2=0^+},$$

where

$$\begin{aligned}
\frac{\partial^2}{\partial\theta_1\partial\theta_2}Q^2_{\bar{t}}\theta_{(2)}(x) &= \mathbb{E}_x\Big[-\varphi_1(B_{t_1})\varphi_2(B_{t_2})e^{-\theta_1\varphi(B_{t_1})-\theta_2\varphi_2(B_{t_2})} \\
&\quad -\int_0^{t_2}\frac{\partial}{\partial\theta_2}Q^2_{\bar{t}-r}\theta_{(2)}(B_r)\frac{\partial}{\partial\theta_1}Q^2_{\bar{t}-r}\theta_{(2)}(B_r)dN_r \\
&\quad -\int_0^{t_2}\left(Q^2_{\bar{t}-r}\theta_{(2)}(B_r)\right)\frac{\partial^2}{\partial\theta_2\partial\theta_1}Q^2_{\bar{t}-r}\theta_{(2)}(B_r)dN_r \\
&\quad -\varphi_2(B_{t_2})e^{-\theta\varphi_2(B_{t_2})}\int_{t_1}^{t_2}\left(Q^1_{t_2-r}\theta_1(B_r)\right)\frac{\partial}{\partial\theta_1}Q^1_{t_2-r}\theta_1(B_r)dN_r\Big].
\end{aligned}$$

Evaluating at $\theta_1=\theta_2=0$ we finish the proof. □

Proposition 4.3 *Let $0<s\le t<\infty$ and $\psi,\varphi\in C_c(\mathbb{R}^d)$. Then,*

$$C(s,\varphi;t,\psi):=\mathrm{Cov}\left(\langle\varphi,X_s\rangle,\langle\psi,X_t\rangle\right)=\langle\varphi\mathcal{S}_{t-s}\psi,\Lambda\rangle+\int_0^s\langle\left(\mathcal{S}_{s-r}\varphi\right)\left(\mathcal{S}_{t-r}\psi\right),\Lambda\rangle dU(r), \quad (15)$$

where $U(r)=\sum_{k=0}^{\infty}F^{*k}(r)$.

Proof: We put $p=2$ in (12) and use the same notations as in the proof of Lemma 4.2. Then,

$$\begin{aligned}
\mathbb{E}\left[\langle\varphi_1,X_{t_1}\rangle\langle\varphi_2,X_{t_2}\rangle\right] &= \frac{\partial^2}{\partial\theta_1\partial\theta_2}\exp\left(-\int Q^2_{\bar{t}}\theta_{(2)}(x)\,dx\right)\Big|_{\theta_1=\theta_2=0^+} \\
&= \Big[-\frac{\partial^2}{\partial\theta_1\partial\theta_2}\int Q^2_{\bar{t}}\theta_{(2)}(x)dx \\
&\quad +\int\frac{\partial}{\partial\theta_1}\int Q^2_{\bar{t}}\theta_{(2)}(x)dx\int\frac{\partial}{\partial\theta_1}\int Q^2_{\bar{t}}\theta_{(2)}(x)dx\Big]\Big|_{\theta_1=\theta_2=0^+} \\
&= \int C_x(t_1,\varphi_1;t_2,\varphi_2)dx+\int m_x(t_1,\varphi_1)dx\int m_x(t_2,\varphi_2)dx,
\end{aligned}$$

and from Lemma 4.2 we obtain

$$C(s,\varphi;t,\psi)=\int_{\mathbb{R}^d}\mathbb{E}_x\left[\varphi(B_s)\psi(B_t)+\int_0^s m_{B_r}(t-r,\psi)m_{B_r}(s-r,\varphi)dN_r\right]dx, \quad (16)$$

which completes the proof. □

5 Markovianizing an age-dependent branching system

In this Section we discuss a Markovianized version of the the critical binary age-dependent branching system, which will be needed to prove weak convergence in the Skorokhod space

$D(\mathbb{R}_+, S'(\mathbb{R}^d))$. We recall that, by a well known result of [26], to show tightness of the sequence $\{M_t^{l,K}, t \geq 0\}$, $K = 1, 2, \cdots$, is enough to prove tightness of the sequence of real-values processes $\{\langle \varphi, M_t^{l,K} \rangle, t \geq 0\}$, $K = 1, 2, \cdots$, for each $\varphi \in S'(\mathbb{R}^d)$.

Let $X \equiv \{X_t,\, t \geq 0\}$ be the branching system defined in Section 1. For any $t \geq 0$, let $\bar{X}_t$ denote the population in $\mathbb{R} \times \mathbb{R}^d$ obtained by attaching to each individual $\delta_x \in X_t$ its age. Namely, for each $t \geq 0$,

$$\bar{X}_t = \sum_i \delta_{(\eta_t^i, \xi_t^i)}, \tag{17}$$

where η_t^i and ξ_t^i denotes respectively, the age and position of the i^{th} particle at time t, and the summation is over all particles alive at time t. Let us assume that $\bar{X}_0$ is a Poisson random field on $\mathbb{R}_+ \times \mathbb{R}^d$ with intensity measure $F \times \Lambda$. Here, F also means the Lebesgue-Stieltjes measure corresponding to F. The probability generating function of the branching law is denoted by Φ. Thus, for critical binary branching, $\Phi(s) \equiv \frac{1}{2}(1 + s^2)$, $-1 \leq s \leq 1$.

Given a counting measure ν on $\mathbb{R}_+ \times \mathbb{R}^d$, and a measurable function $\phi : \mathbb{R} \times \mathbb{R}^d \longrightarrow (0, 1]$, we define

$$G_\phi(\nu) := \exp\left(\langle \log \phi, \nu \rangle\right).$$

It can be shown that the infinitesimal generator of $\{\bar{X}_t,\ t \geq 0\}$ evaluated at the function $G_\phi(\nu)$ is given by

$$\mathcal{G}G_\phi(\nu) = G_\phi(\nu) \left\langle \frac{\mathcal{L}\phi(*, \cdot) + \lambda(*)[\Phi(\phi(0, \cdot)) - \phi(0, \cdot)]}{\phi(*, \cdot)}, \nu \right\rangle, \tag{18}$$

where

$$\lambda(u) = \frac{f(u)}{1 - F(u)}, \quad u \geq 0, \tag{19}$$

and

$$\mathcal{L}\phi(u, x) = \frac{\partial \phi(u, x)}{\partial u} + \Delta_\alpha \phi(u, x) - \lambda(u)\left[\phi(u, x) - \phi(0, x)\right], \tag{20}$$

where the function ϕ is such that $\phi(\cdot, x) \in C_b^1(\mathbb{R}_+)$ for any $x \in \mathbb{R}^d$, and $\phi(u, \cdot) \in C_c^\infty(\mathbb{R}^d)$ for any $u \in \mathbb{R}_+$. Here $C_b^1(\mathbb{R}_+)$ denotes the set of all bounded functions with continuous first derivative, and $C_c^\infty(\mathbb{R}^d)$ denotes the space of infinitely differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}$, having compact support. The operator $\mathcal{L}$ is the infinitesimal generator of a Markov process on $\mathbb{R} \times \mathbb{R}^d$ whose semigroup is denoted by $\{\tilde{T}_t, t \geq 0\}$, see [27] for details.

Now consider the process $\hat{X} := \{X_t \times \bar{X}_t,\, t \geq 0\}$, which is a Markov process taking values in the Skorokhod space $D(\mathbb{R}_+, \mathcal{M}_p(\mathbb{R}^d) \times \hat{\mathcal{M}}_p)$. Next we give the infinitesimal generator of the process $\hat{X}$ for certain cylindrical functions. Define

$$g(\mu_1, \mu_2) := G(\langle \varphi, \mu_1 \rangle) \text{ for } \varphi \in S(\mathbb{R}^d),\ \mu_1 \in \mathcal{M}(\mathbb{R}^d),\ \mu_2 \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^d),$$

where $G \in C^3(\mathbb{R})$ is such that $G''' \equiv 0$. It can be seen ([27]) that the infinitesimal generator $\mathcal{G}$, is given by

$$\begin{aligned}\mathcal{G}g(\mu_1, \mu_2) &= \langle \Delta_\alpha \varphi, \mu_1 \rangle G'(\langle \varphi, \mu_1 \rangle) + \frac{1}{2} \langle \Delta_\alpha \varphi^2 - 2\varphi \Delta_\alpha \varphi, \mu_1 \rangle G''(\langle \varphi, \mu_1 \rangle) \\ &\quad + \langle \lambda(*) \sum_{k=0}^{\infty} p_k [G(\langle \varphi, \mu_1 + (k-1)\delta_{\cdot} \rangle) - G(\langle \varphi, \mu_1 \rangle)], \mu_2 \rangle. \end{aligned} \tag{21}$$

Putting $G(y) = y$ for all $y \in \mathbb{R}$, from (21) we get that

$$\begin{aligned}\mathcal{G}g(\mu_1, \mu_2) &= \langle \Delta_\alpha \varphi, \mu_1 \rangle + \langle \lambda(*) \sum_{k=0}^{\infty} p_k (k-1) \varphi(\cdot), \mu_2 \rangle \\ &= \langle \Delta_\alpha \varphi, \mu_1 \rangle,\end{aligned}$$

where the second equality follows from criticality of the branching. Then, from the Markov property we have that

$$Y_t(\varphi) := \langle \varphi, X_t \rangle - \int_0^t \langle \Delta_\alpha \varphi, X_s \rangle ds, \quad t \geq 0 \text{ and } \varphi \in S(\mathbb{R}^d), \tag{22}$$

is a martingale (with respect to the filtration generated by the process $\hat{X}$).

Proposition 5.1 *Let $\bar{X} \equiv \{\bar{X}_t,\, t \geq 0\}$ as before and let $\bar{X}_0$ be a Poisson random field on $\mathbb{R}_+ \times \mathbb{R}^d$ with intensity measure $F \times \Lambda$. The joint Laplace functional of the branching particle system $\bar{X}$ and its occupation time is given by*

$$\mathbb{E}\left[e^{-\langle \psi, \bar{X}_t \rangle - \int_0^t \langle \phi, \bar{X}_s \rangle ds} \right] = e^{-\langle V_t^\psi \phi, F \times \Lambda \rangle}, \quad t \geq 0,$$

for all measurable functions $\psi, \phi : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}_+$ with compact support, where $V_t^\psi \phi$ satisfies, in the mild sense, the non-linear evolution equation

$$\begin{aligned}\frac{\partial}{\partial t} V_t^\psi \phi(u, x) &= \mathcal{L} V_t^\psi \phi(u, x) - \lambda(u)[\Phi(1 - V_t^\psi \phi(0, x)) - (1 - V_t^\psi \phi(0, x))] \\ &\quad + \phi(u, x)(1 - V_t^\psi \phi(u, x)), \\ V_0 \psi \phi(u, x) &= 1 - e^{-\psi(u, x)}.\end{aligned} \tag{23}$$

Proof: The proof is carried out using the martingale problem for $\{\bar{X}_t,\, t \geq 0\}$, and Itô's formula. We omit the details. □

6 Proofs

6.1 Proof of Theorem 3.3

Proof of (a): First, we show that $C(s,\varphi;s,\mathcal{S}_{t-s}\psi) = C(s,\varphi;t,\psi)$ for all $s \leq t$ and $\varphi,\psi \in \mathcal{S}$. In fact,

$$\begin{aligned} C(s,\varphi;s,\mathcal{S}_{t-s}\psi) &= \langle \varphi \mathcal{S}_{s-s}\mathcal{S}_{t-s}\psi, \Lambda\rangle + \int_0^s \langle (\mathcal{S}_{s-r}(\mathcal{S}_{t-r}\psi)(\mathcal{S}_{s-r}\varphi), \Lambda\rangle dU(r) \\ &= \langle \varphi \mathcal{S}_{t-s}\psi, \Lambda\rangle + \int_0^s \langle (\mathcal{S}_{t-r}\psi)(\mathcal{S}_{s-r}\varphi), \Lambda\rangle dU(r) \\ &= C(s,\varphi;t,\psi). \end{aligned} \tag{24}$$

Hence, the Markov property follows from Theorem 6 in [22]. □

Proof of (b): We will show that there exists $p \geq 1$ such that M is almost surely continuous in the norm $\|\cdot\|_{-p}$. To this end, we will use that

$$\sup_{T\in\mathbb{R}_+} \frac{V_T(\phi)}{g(T)} < \infty, \tag{25}$$

where g is a positive locally bounded function on $[0,\infty)$ and

$$V_T(\phi) := \mathbb{E}\left[\sup_{0\leq t\leq T} \langle \phi, M_t\rangle^2\right],$$

with $\phi \in$. Taking for granted (25), the result follows from a theorem in [24].

The proof of (25) follows along the same lines as in [13]. Namely, by applying Doob's inequality to the martingale (6). We omit the details. □

Proof of (c): We will show that M^1 satisfies all the conditions of Theorem 3.6 in [2]. Condition 1 follows from part (b) and condition 4 follows from part (a) of this theorem;

condition 3 holds by hypothesis. It remains to show Condition 2. We have that, for each $t \geq 0$ and $\varphi \in S(\mathbb{R}^d)$,

$$\begin{aligned} \mathcal{K}^1(t,\varphi;t,\varphi) &= \langle \varphi^2, \Lambda\rangle + \int_0^t \langle (\mathcal{S}_{t-r}\varphi)^2, \Lambda\rangle dU(r) \\ &= \langle \varphi^2, \Lambda\rangle + \int_0^t \langle (\mathcal{S}_{t-r}\varphi)^2, \Lambda\rangle u(r)dr, \end{aligned}$$

the second inequality is a consequence of lifetimes distribution with continuous density. Hence, the function $t \longmapsto \mathcal{K}^1(t,\varphi;t,\varphi)$ is continuously differentiable. Then, M^1 satisfies all the conditions in Theorem 3.6 from [2].

It remains to show equation (8). Notice that for $s = t$, (4) can be written as follows

$$\mathcal{K}^1(t,\varphi;t,\psi) = \langle \varphi\psi, \Lambda\rangle + \int_0^t \langle \varphi(\mathcal{S}_{2(t-r)}\psi), \Lambda\rangle u(r)dr, \ \ 0 \leq t, \ \ \varphi, \phi \in S'(\mathbb{R}^d). \tag{26}$$

Therefore,

$$\begin{aligned} \langle \varphi, Q_t\varphi\rangle &\equiv \frac{d}{dt}\mathcal{K}^1(t,\varphi;t,\varphi) - 2\mathcal{K}^1(t,\Delta_\alpha\varphi;t,\varphi) \\ &= \langle \varphi^2, \Lambda\rangle - 2\langle (\Delta_\alpha\varphi)\varphi, \Lambda\rangle. \end{aligned}$$

Notice that, (8) can be deduced from

$$\langle \varphi, Q_t\psi\rangle = \frac{1}{2}\left[\langle (\varphi+\psi), Q_t(\varphi+\psi)\rangle - \langle \varphi, Q_t\varphi\rangle - \langle \psi, Q_t\psi\rangle\right].$$

□

6.2 Proof of the limit theorems

We give the proofs only for the case $l = 1$, since the case $l = 2$ are similar. Notice that, from Proposition 4.3, for $0 \leq t_1 \leq t_2 < \infty$ and $\varphi_1, \varphi_2 \in S(\mathbb{R}^d)$,

$$\mathrm{Cov}(\langle \varphi_1, X^{1,K}_{t_1}\rangle, \langle \varphi_2, X^{1,K}_{t_2}\rangle) = K\mathcal{K}^1(t_1,\varphi_1;t_2,\varphi_2). \tag{27}$$

The next lemma gives convergence of finite-dimensional distributions of $M^{1,K}$ to those of M.

Lemma 6.1 *$M^{1,K} \Longrightarrow_f M$ as $K \longrightarrow \infty$, i.e., for each $p \geq 1$, $0 < t_p \leq t_{p-1} \leq \cdots \leq t_1 < \infty$, $\varphi_1, \cdots, \varphi_p \in S(\mathbb{R}^d)$ and $\theta_1, \cdots, \theta_p \in \mathbb{R}$,*

$$\mathbb{E}\left[e^{i\sum_{j=1}^p \theta_j \langle \varphi_j, M^{1,K}_{t_j}\rangle}\right] \longrightarrow \exp\left(-\frac{1}{2}\sum_{j=1}^p \sum_{k=1}^p \theta_j\theta_k \mathcal{K}^1(t_j,\varphi_j;t_k,\varphi_k)\right),$$

as $K \to \infty$.

Proof: The proof of this lemma uses Minlos-Sasonv's theorem ([17]). First we note that

$$\begin{aligned}
\mathbb{E}\left[e^{i\sum_{j=1}^p \theta_j \langle \varphi_j, M^{1,K}_{t_j}\rangle}\right] &= \mathbb{E}\left[\exp\left(i\sum_{j=1}^p \theta_j \frac{\langle \varphi_j, X^{1,K}_{t_j}\rangle - K\langle \varphi_j, \Lambda\rangle}{K^{-1/2}}\right)\right] \\
&= \exp\left(-K\int_{\mathbb{R}^d} \mathbb{E}_x\left[1 - e^{i\sum_{j=1}^p \theta_j K^{-1/2}\langle \varphi_j, Z_{t_j}\rangle}\right]dx\right) \\
&\quad \times \exp\left(-iK^{1/2}\sum_{j=1}^p \theta_j \langle \varphi_j, \Lambda\rangle\right) \\
&= \exp\left(-\frac{1}{2}\int_{\mathbb{R}^d} \mathbb{E}_x\left(\sum_{j=1}^p \theta_j \langle \varphi_j, Z_{t_j}\rangle\right)^2 dx\right) \\
&\quad \times \exp\left(\int K\left[\mathbb{E}_x e^{i\sum_{j=1}^p K^{-1/2}\theta_j\langle \varphi_j, Z_{t_j}\rangle} - 1\right.\right. \\
&\quad \left.\left. -iK^{-1/2}\sum_{j=1}^p \theta_j \mathbb{E}_x\langle \varphi_j, Z_{t_j}\rangle + \frac{1}{2}K^{-1}E_x\left(\sum_{j=1}^p \theta_j\langle \varphi_j, Z_{t_j}\rangle\right)^2\right]dx\right),
\end{aligned}$$

where the integrand converges to 0, as $K \longrightarrow \infty$, and is bounded by $c\sum_{j=1}^p \theta_j^2 \mathbb{E}_x\langle \varphi_j, Z_{t_j}\rangle^2$ for some constant $c > 0$ (see [6] Proposition 8.44). Hence,

$$\lim_{K\to\infty} \mathbb{E}\left[e^{i\sum_{j=1}^p \theta_j \langle \varphi_j, M^{1,K}_{t_j}\rangle}\right] = \exp\left(-\frac{1}{2}\int_{\mathbb{R}^d} \mathbb{E}_x\left(\sum_{j=1}^p \theta_j\langle \varphi_j, Z_{t_j}\rangle\right)^2 dx\right).$$

□

Proof of Theorem 3.2. Note that, from (27), for all $\varphi \in S(\mathbb{R}^d)$

$$\mathbb{E}\left(\frac{\langle \varphi, X^{1,K}_t\rangle}{K} - \langle \varphi, \Lambda\rangle\right)^2 = \frac{1}{K^2}\mathrm{Var}\left(\langle \varphi, X^{1,K}_t\rangle\right) = \frac{1}{K}\mathcal{K}^1(t,\varphi;t,\varphi).$$

Letting $K \longrightarrow \infty$ yields the result. □

Proof of Theorem 3.1. We will show that the sequence $\{M^{1,K}, K = 1, 2, \cdots\}$ satisfies all the conditions in Theorem 2.1 from [14]. First we note that, by Theorem 3.3 (b), the process M possesses continuous paths. Condition (b) is proved in Lemma 6.1. To prove conditions (c) and (d) we show that

$$\sup_{K\geq 1}\sup_{0\leq t\leq T} \mathbb{E}\langle \varphi, M^{1,K}\rangle^2 < \infty, \tag{28}$$

for each $T > 0$ and $\varphi \in S(\mathbb{R}^d)$, see Remark (1) after Theorem 2.1 in [14]. In fact, from (27) we have that

$$\mathbb{E}\langle \varphi, M_t^{1,K}\rangle^2 = \langle \varphi^2, \Lambda\rangle + \int_0^t \langle (\mathcal{S}_{t-r}\varphi)^2, \Lambda\rangle dU(r), \tag{29}$$

for each $t \geq 0$ and $\varphi \in S(\mathbb{R}^d)$. Note that, (29) can be bounded from above as follows

$$\mathbb{E}\langle \varphi, M_t^{1,K}\rangle^2 \leq \langle \varphi^2, \Lambda\rangle + \int_0^t \langle (\mathcal{S}_{t-r}|\varphi|)^2, \Lambda\rangle dU(r).$$

Hence, without loss of generality we can assume that $\varphi > 0$. Now, we observe that

$$\langle (\mathcal{S}_{t-r}\varphi)^2, \Lambda\rangle = \langle \frac{\mathcal{S}_{t-r}\varphi}{\phi_p}\phi_p \mathcal{S}_{t-r}\varphi, \Lambda\rangle \leq |\varphi|_p\langle \varphi, \Lambda\rangle. \tag{30}$$

Hence

$$\begin{aligned}\sup_{0\leq t\leq T} \mathbb{E}\langle \varphi, M^{1,K}\rangle^2 &\leq \sup_{0\leq t\leq T}\left(\langle \varphi^2, \Lambda\rangle + \int_0^t |\varphi|_p\langle \varphi, \Lambda\rangle dU(r)\right)\\ &\leq \langle \varphi^2, \Lambda\rangle + |\varphi|_p\langle \varphi, \Lambda\rangle U(T),\end{aligned}$$

which implies (28).

It remains to verify Condition (a), for this we use the Markovianized process discussed in Section 5. From (22) and the fact that $\langle \Delta_\alpha\varphi, \Lambda\rangle = 0$, we can deduce that for all $\varphi \in S(\mathbb{R}^d)$,

$$\langle \varphi, M_t^{(K)}\rangle - \int_0^t \langle \Delta_\alpha\varphi, M_s^{(K)}\rangle ds, \;\; t \geq 0, \tag{31}$$

is a martingale, seen as a process in $D\left([0,\infty), S'(\mathbb{R}^d)\times S'(\mathbb{R}^{d+1})\right)$.

Finally, notice that Lemma 6.1 gives $M^{1,K} \Longrightarrow_f M$, from this follows that $(M^{1,K}, 0) \Longrightarrow_f (M, 0)$. Hence, we have shown that $(M^{1,K}, 0) \Longrightarrow (M, 0)$, this convergence holds in the space $D\left([0,\infty), S'(\mathbb{R}^d)\times S'(\mathbb{R}^{d+1})\right)$. The proof can be completed by using Continuous Mapping Theorem. □

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