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ON THE POSSIBILITY TO CREATE A
COMPATIBLE-COMPLETE UNARY COMPARISON
METHOD FOR EVOLUTIONARY
MULTIOBJECTIVE ALGORITMS

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Abstract

There are several studies on the desirable properties that a comparison method for evolutionary multiobjective algorithms must have. One of these properties is called *compatibility and completeness*. There is a theorem that states that, in the general case (infinite size sets), it is not possible to create a unary comparison method with the property mentioned before. As a consequence, unary performance measures are considered to be less reliable than binary performance measures. In this paper we provide a further analysis for practical conditions (finite size sets). We prove, for these conditions, that the impossibility to create a compatible and complete unary comparison method, is no longer valid. Our result opens the door for future research to establish whether or not is possible to create such a comparison method.

1 Introduction

The evaluation of the performance of multiobjective evolutionary algorithms is an open problem. The output of these algorithms are sets of non–dominated points (also known as non–dominated sets, approximation sets, NS). So, the evaluation of multiobjective algorithms is usually reduced to evaluate the quality of their outputs. Many methods to evaluate the quality of non–dominated set (popularly know as metrics or performance measures) have been proposed. Some

examples are the S–metric [7], the C–metric [7], etc. All of them have their advantages and disadvantages as is shown in [5] and [8].

Performance measures (PM) are usually classified based on the number of non–dominated sets they take as an input. For example, a PM is unary if it takes only one non–dominated set as an argument and returns a real number as an output. This number is a evaluation for the NS. So, we can compare different NSs based on this number. A binary PM uses two NS as input and evaluates which one is better than the other. The evaluation of a binary PM is valid only for those two NSs it takes as input and cannot be used to compare them with others.

Unary metrics are easier to use than binary metrics. If we compare m NSs (for $m > 2$), with a unary metric we only need m evaluations, while with a binary metric we need $O(m^2)$ evaluations. Also, binary metrics can induce cycles, for example for NSs A, B and C it is possible for a binary metric to consider A better than B, B better than C and C better than A. These are some of the reasons why unary metrics were more popular than binary metrics.

All this changed when an extensive study in performance measures were published [8]. In this study, the authors defined the property of *compatibility and completeness* (CC). According to that study, this is a very important property for a PM to have, because it ensures that the PM is able to decide whether a NS is better than another. They also introduced the concept of Comparison Method, that is a more formal definition of a methodology to compare non–dominated sets. One of the most important results in the study is that, in the general case, it is not possible to construct a unary comparison method that is compatible and complete. To prove this, they assumed that there exists a unary metric that is compatible and complete, then they showed that this leads to an absurd result.

In this work, we make a revision and corroborate some affirmations made in [8]. Also, we obtain a different theorem that states that the absurd result mentioned before is not present if we consider other conditions usually met in practice. The rest of the work is organized as follows: in Section 2, we make an introduction to the basic terminology from multiobjective optimization and set theory that are needed in the rest of the sections. Some important demonstrations about the cardinality of the sets of non–dominated sets are introduced in Section 3. In Section 4, we present a demonstration of an important theorem that restricts the utility of unary comparison methods. In Section 5, we review some special conditions in which it is possible to construct a compatible and complete unary comparison method and give an explanation why this is possible. In Section 6, we discuss the practical conditions under which the absurd result mentioned before does not occur. In Section 7, we state our conclusions.

2 Basic Concepts

In this section we introduce some concepts and terminologies about both multiobjective optimization and set theory. This information is useful to simplify

the following explanations.

2.1 Multiobjective Optimization

Multiobjective optimization (MOO) consist of maximizing/minimizing a vector of objective functions $F(x) = \{f_1(x), f_2(x), \ldots, f_n(x)\}$ subject to constraints. The objective functions and constraints depend on a vector of variables $x \in \mathbf{R}^m$, these vectors x are possible solutions for the problem, so they are also known as "solutions". We also refer to these vectors as "points", because they are usually visualized as points in R^m. We define the set $\Psi \subseteq \mathbf{R}^m$ as all vectors x that do not violate the constraints, and Z as the projection of Ψ in objective functions space. Without loss of generality, we consider hereafter that we are minimizing the objective functions.

A popular way to deal with MOO problems is to use the Pareto Optimality Criteria (POC). POC is defined through the relation between two vectors $x, y \in \mathbf{R}^m$ known as Pareto dominance or dominance. We have that x dominates y ($x \succeq y$) if $\forall i \in \{1, 2, \ldots, n\}$, $f_i(x) \leq f_i(y) \land \exists j \in \{1, 2, \ldots, n\} \mid f_j(x) < f_j(y)$. The goal is to find a set of vectors known as the Pareto Set (PS) defined as $PS = \{x \in \Psi \mid \forall y \in \Psi, y \nsucceq x\}$. According to POC, all elements of PS are optimal, because they represent different tradeoffs between objective functions where it is not possible to improve one objective without degrading another. The projection of PS in objective function space is called the Pareto Front (PF) and is usually described as a surface that represents the best tradeoff possible between the objective functions.

In recent years, many evolutionary algorithms based on POC have been developed [1] [9]. These algorithms are based on populations and use the principles of evolution to solve a problem. Instead of generating a single solution, these algorithms generate a finite set X of explicit vector solutions x that approximate the PS. These approximation sets have the characteristic that $\forall x, y \in X, x \nsucceq y \land y \nsucceq x$ and are usually called non–dominated sets. We assume that for $x_1 \neq x_2 \in X, F(x_1) \neq F(x_2)$. The set of all non–dominated sets for a multiobjective problem is represented by Ω.

It is valid to refer to a non–dominated set by their elements in the space of variables Ψ or by its projection in the space of objective functions Z. Performance measures are usually evaluated in the space of objective functions, so in the rest of the article we locate points, sets, solutions and vectors in this space.

As more evolutionary multiobjective algorithms were published, new performance measures to evaluate these algorithms were proposed, as mentioned in Section 1. These measures were created based more on intuition than on a formal theory. A reason for that was that practically there were no studies about performance measures for multiobjective algorithms.

In order to establish a minimum of what we expect from a quality indicator, Hansen and Jaszkiewicz [3], defined the three following relationships for $A, B \in \Omega$.

Weak outperformance: A weakly outperforms B ($A \ O_W \ B$), if for every point $b \in B$ there exists a point $a \in A$ so that $a \succeq b$ or $a = b$ and there exists

at least a point $c \in A$ so that $c \notin B$.

Strong outperformance: A strongly outperforms B ($A \ O_S \ B$), if for every point $b \in B$ there exists a point $a \in A$ so that $a \succeq b$ or $a = b$ and there exists at least a pair of points $r \in A$ and $s \in B$ such that $r \succeq s$.

Complete outperformance: A completly outperforms B ($A \ O_C \ B$), if for every point $b \in B$ there exists at least one point $a \in A$ so that $a \succeq b$.

These outperformance relations are used to establish a minimum of what we expect from a comparison method. It is easy to understand that $A \ O_C \ B$ implies that A is better than B, because for every vector in B there is a better one in A. So, if we have a comparison method R, and it evaluates B as better than A, then R is not reliable. We expect the same with respect to O_S and $A \ O_W \ B$. Hansen and Jaszkiewicz [3] also define the property of *compatibility with an outperformance relation O*, where O can be O_W, O_S or O_C, as follows:

Compatibility. A performance measure R is compatible with O if $A \ O \ B$ implies that R will evaluate A as better than B ($R(A > B)$). In other words, $A \ O \ B \implies R(A > B)$.

These concepts established a base to evaluate the effectiveness of quality indicators, considering only the concept of Pareto dominance. The work of Hansen and Jaszkiewicz has had an important influence and is a point of reference in the area.

Zitzler et al [8] went further and made a more formal characterization of what a performance measure is, through the following definition.

Definition 1. *A comparison method $C_{I,E}$, is a combination of k performance measures $\boldsymbol{I} = (I_1, I_2, \ldots, I_k)$ and a function $E{:}\boldsymbol{R}^k \times \boldsymbol{R}^k \to \{false, true\}$ that somehow interprets two vectors $\boldsymbol{I}$. If $\boldsymbol{I}$ consists of unary metrics only, we have a unary comparison method $C_{I,E} = E(\boldsymbol{I}(A), \boldsymbol{I}(B))$, where $A, B \in \Omega$ and $\boldsymbol{I}(Y) = (I_1(Y), I_2(Y), \ldots, I_k(Y))$ for $Y \in \Omega$.*

The later authors also defined the property of compatible and completeness (CC), that essentially transforms the implication in the definition of compatibility, into a double implication.

Compatibility and completeness. A comparison method C is compatible and complete with an outperformance relation O, when A outperforms B if and only if C evaluates A as better than B. In other words, $A \ O \ B \iff C(A > B)$.

Compatibility and completeness established a stricter criteria of what properties are desirable for a performance measure. According to Zitzler et al. [8] it is important because a compatible and complete comparison method is able to decide whether a NS is better than another. In the same study [8], it is demonstrated that in general, unary comparison methods can not be compatible and complete.

2.2 Set Theory

The concepts in this subsection were taken from [2] and [4]. The demonstrations of the results presented here can also be consulted in [2] and [4].

Two sets, A and B, are equivalent ($A \sim B$) if it is possible to make a correspondence between the elements of both sets in such a way that to every element of A correspond one and only one element of B; and to every element of B correspond one and only one element of A. This kind of correspondence is called a one to one correspondence or a mapping. The equivalence property is reflexive, transitive and symmetric. An injection from A to B is a mapping from A to a subset of B.

One of the most important concepts of set theory is that of cardinal numbers, or cardinality. Cardinal numbers are related to the size of a set. A cardinal number refers to an arbitrary member of a family of mutually equivalent sets. For example, the cardinal number 4 represents any set equivalent to {1,2,3,4}, like {a,b,c,d}, {"dog", "rat", "cat", "mouse"}, etc. We represent the cardinal number of a set A by $|A|$, for example $|\{a, b\}| = 2$. Infinite sets also have cardinal numbers. For the set of positive integers $\mathbf{N} = \{1, 2, 3, \ldots\}$, we represent its cardinal number by a. For real numbers $\mathbf{R}$ we represent $|\mathbf{R}|$ by c. We represent the cardinal number of the set of functions defined in a continuous interval by f.

It is not possible to make an injection from A to B if $|A| > |B|$ because there are not enough different elements in B to be associated with the elements in A. If $A \subset B$ then $|A| \leq |B|$. An interesting result from set theory is that it is impossible to make an injection from $\mathbf{R}$ to $\mathbf{N}$, the set of natural numbers is somehow "smaller" than the set of real numbers. For two infinite sets A and B, $|A| < |B|$ if and only if there is an injection from A to B but there is no injection from B to A. If we can make an injection from A to B, than $|A| \leq |B|$. It is proved that $a < c < f$.

A set A with cardinal number a is called countable and it is equivalent to the set of natural numbers. When listing its elements, a countable set is usually represented using "$\ldots$", for example $A = \{a_1, a_2, \ldots\}$.

An interesting property of the infinite sets, like $\mathbf{N}$ and $\mathbf{R}$, is that it is possible to make a one–to–one correspondence between an infinite set and some of its subsets.

For two sets A and B, their union is represented by $A + B$. Their Cartesian product is represented by $A \times B$. The cartesian product of a set with itself can be represented by an exponent. For example, $A \times A \times A = A^3$.

The power set of A ($P(A)$) is the set whose elements are all possible subsets of A and it is proved that $|A| < |P(A)|$. If $|A| = |B|$ then $|P(A)| = |P(B)|$.

The cardinal numbers of infinite sets are called transfinite numbers. The smallest transfinite number is a and all finite numbers are smaller than any transfinite number. We present a list of results of set theory, where $k > 0$ is a finite cardinal number and $m, n > 0$ are finite numbers.

a: $A \sim B \iff |A| = |B|$. Two sets are equivalent if and only if they have the same cardinal number.

b: $c + k = c + a = c + c = c$. For a set A of cardinality c, if we add a finite number of elements to A, the resulting set has cardinality c. The same

occurs if we add a–many elements to A or if we add c–many elements to A. For example $|\mathbf{R} - \{a, b, c\}| = |\mathbf{R}| = c$, $|(0, 1) + \mathbf{N}| = c$, $|[0, 1) + [1, 2)| = |[0, 1)| = c$. Similarly, we can subtract a finite number of elements from a set of cardinality c and the resulting set has cardinality c.

c: $c \cdot k = c \cdot a = c \cdot c = c$. The cartesian product of a set A with cardinality c with a finite set results in a set with cardinality c. The same result is obtained if the cartesian product is evaluated with a set of cardinality a or c. For example $|\mathbf{R} \times \{1, 2, 3\}| = c$, $|\mathbf{R} \times \mathbf{N}| = c$, $|\mathbf{R} \times \mathbf{R}| = c$.

d: $c^m = c^a = c$. A set A with cardinality c elevated to a finite exponent results in an equivalent set. For example, $|\mathbf{R}^5| = |\mathbf{R}| = c$. The same result is obtained if A is elevated to a.

e: Let $|A| = c$. $|A^c| = |P(A)|$. If we elevate a set A of cardinality c to the exponent c, the result is a set with a bigger cardinality. The same cardinality of the power set of A. For example, $\mathbf{R}^c \sim P(\mathbf{R})$.

f: $c^m \times c^n = c^{(m+n)} = c$. For example, $\mathbf{R}^5 \times \mathbf{R}^2 \sim \mathbf{R}^7 \sim \mathbf{R}$.

g: $c + c + \ldots = c$. The sum of a–many sets, each of them with cardinality c, results in a set of cardinality c. For example $[0, 1) + [1, 2) + \ldots + [k, k + 1) + \ldots \sim \mathbf{R}$.

h: $|A| \sim |B| \implies |P(A)| = |P(B)|$. If two sets have the same cardinality, then their power sets are equivalent.

i: $A \subset B \implies |A| \leq |B|$. If A is a subset of B, then the cardinal number of A is less or equal to that of B.

j: Let $|A| = c$ and $|B| = a$: $f = |P(A)| > c = |P(B)| > a$.

k: If $C \subset B$ then: $A \sim C \implies |A| \leq |B|$.

These results are used in the following Sections.

3 Some demonstrations about the cardinality of non–dominated sets

We introduced some demonstrations about the cardinality of the sets of non–dominated sets. These demonstrations are interesting by themselves from the theoretical point of view but they are also useful to prove the theorems we present later. We use extensively the Theorems (a)–(k) from Section 2.2.

In Section 2.1 we called Z the space of objective functions. Depending on the multiobjective problem, Z can have many topologies, for example, it can be discrete or continuous. We consider in the following demonstrations the more general case, where Z is equal to R^n where n is the number of objective functions. We also defined the set Ω of all non–dominated sets we can create
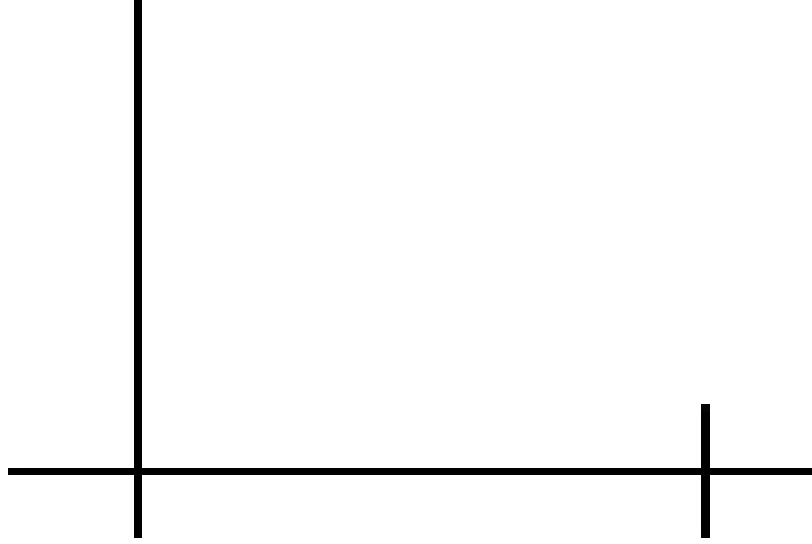
Figure 1: A non–dominated set $S \subset \mathbf{R}^2$

from Z. Ω is a very interesting set with respect to performance measures for multiobjective optimization. As we see later, the cardinality of this set restricts the utility of unary performance measures for the general case.

We are interested in the following question. How many non–dominated sets can we create from Z? In other words, what is the cardinality of Ω? The answer is shown in the following Lemma.

Lemma 1. *The cardinal number of Ω is f.*

In order to demonstrate Lemma 1 first we enunciate the following definition:

Definition 2. *Choose $a, b \in \mathbf{R}$ with $a < b$. The line $S \subset \mathbf{R}^n$ is defined as $S = \{(z^1, z^2, \ldots, z^n) \in Z \mid z^i = (a+b)/2 \text{ for } 3 < i < n, \ z^1 \in (a, b) \text{ and } z^2 = b + a - z^1\}$. Ω_S is the set of all non–dominated sets we can generate from S.*

Due to its construction, S is a non–dominated set. An example of the line S is shown in Figure 1. This line is a special construction that we borrowed from [8] and we use it in the demonstration of some of the lemmas in this Section. Ω_S is equivalent to the power set of S, because any subset of non–dominated set is also a non–dominated set with the exception of the empty set[1].

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[1] In order to make the demonstrations shorter, we consider that the empty set is not a non–dominated set. This makes no difference because our demonstrations holds; even if we consider that the empty set is a non–dominated set.

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Proof of Lemma 1. Consider the line $S \subset Z$ described before. The argumentation is the following:

(1): $\Omega_S \subset \Omega \subset P(\mathbf{R}^n)$, because of the definition of S, Ω and Ω_S.

(2): $|\Omega_S| \leq |\Omega| \leq |P(\mathbf{R}^n)|$, because of (1) and (i).

(3): $\Omega_S = P(S) - \{\emptyset\}$, because the subset of a NS is also a NS.

(4): $|\Omega_S| = f$, because the power set of a set with cardinality c has cardinality f (j), S has cardinality c and (3).

(5): $P(\mathbf{R}^n) \sim P(\mathbf{R})$, because of (d) and (h).

(6): $|P(\mathbf{R}^n)| = f$, because of (5), (a) and (j).

(7): $f \leq |\Omega| \leq f$, because of (2), (4) and (6).

(8): $|\Omega| = f$, this is a direct consequence of (7). $\square$

Other demonstrations that will be useful in the rest of this work are about the cardinality of the sets of non–dominated sets of a fixed size. For example, define Ω_k as the set of all non–dominated sets in R^n of size k, for $k > 0$. What is the cardinality of Ω_1, Ω_2 and in general Ω_k?

The cardinality of Ω_1 is c, because by definition, Ω_1 is a set in the form $\{\{z\}|z \in Z\}$, so we can make a one to one correspondence between the elements $\{z\} \in \Omega_1$ with the corresponding $z \in Z$. The cardinality of Z is c, so recalling Theorem (a) from Section 2.2 we can conclude that $|\Omega_1| = c$.

The cardinality of Ω_2 is c. This demonstration is more extensive than that for Ω_1, and is presented below:

Proof. Consider the line S described before. Choose a point $s_1 \in S$ and define $S' = S - \{s_1\}$. Define Ω_2' as the set of all sets in the form $\{s_1, s\}$ for $s \in S'$. We have that all elements of Ω_2' are non–dominated sets of size two.

(1): $|S'| = c$, because if we take from a set of cardinality c a finite number of elements, the resulting set has cardinality c (b). We define S' as $S - \{s_1\}$ and the cardinality of S is c, so the cardinal number of S' is c.

(2): $|\Omega_2'| = c$, because we can make a one to one correspondence between Ω_2' and S'. For this, associate each element $\{s_1, s\} \in \Omega_2'$ with the corresponding element $s \in S'$. But the cardinality of S' is c (1), so the cardinal number of Ω_2' is c (a).

(3): $\Omega_2' \subset \Omega_2$, because the elements of Ω_2' are non–dominated sets of size 2 and Ω_2 is the set of all non–dominated sets of size 2.

(4): $|\Omega_2'| \leq |\Omega_2|$, because Ω_2' is a subset of Ω_2 (3), its cardinality must be less or equal to that of Ω_2 (i).

(5): $c \leq |\Omega_2|$, because the cardinal number of Ω_2' is less or equal to the cardinal number of Ω_2 (4), and the cardinal number of Ω_2' is c (2).

(6): $|\Omega_2| \leq |R^{2n}|$, because of (k) and because we can make an injection from Ω_2 to R^{2n}. In order to make the injection mentioned before, sort the vectors $v = <v^1, v^2, ...v^n> \in A$ for $A \in \Omega_2$ using the following rule: v precedes u, for $v, u \in A$, if $v^1 < u^1$ or if $v^r < u^r$ when $v^i = u^i$ for $1 \leq i \leq r-1$. This way, every non–dominated set $A \in \Omega_2$ is associated with a unique pair of ordered vectors $v_1 = <v_1^1, v_1^2, \ldots, v_1^n>$, $v_2 = <v_2^1, v_2^2, \ldots, v_2^n>$. Associate each non–dominated

set $A \in \Omega_2$ with the point $(v_1^1, v_1^2, \ldots, v_1^n, v_2^1, v_2^2, \ldots, v_2^n) \in R^{2n}$ and we have the desired injection.

(7): $|\Omega_2| \leq c$, because the cardinal number of Ω_2 is less or equal to the cardinal number of R^{2n} (6) and the cardinal number of R^{2n} is c (d).

(8): $c \leq |\Omega_2| \leq c$, because of (5) and (7).

(9): $|\Omega_2| = c$. This is a direct consequence of (8).

$\square$

The demonstration presented above can be extended for Ω_k for any value of $k > 2$, as is shown in the following demonstration.

Proof. Consider the line S described before. Choose $k - 1$ different points $s_1, s_2, \ldots, s_{k-1} \in S$ and define $S' = S - \{s_1, s_2, \ldots, s_{k-1}\}$. Define Ω_k' as the set of all sets in the form $\{s_1, s_2, \ldots, s_{k-1}, s\}$ for $s \in S'$. We have that all elements of Ω_k' are non–dominated sets of size k.

(1): $|S'| = c$, because if we take from a set of cardinality c a finite number of elements, the resulting set has cardinality c (b). We define S' as $S - \{s_1, s_2, \ldots, s_{k-1}\}$ and the cardinality of S is c, so the cardinal number of S' is c.

(2): $|\Omega_k'| = c$, because we can make a one to one correspondence between Ω_k' and S'. For this, associate each element $\{s_1, s_2, \ldots, s_{k-1}, s\} \in \Omega_k'$ with the corresponding element $s \in S'$. But the cardinality of S' is c (1), so the cardinal number of Ω_k' is c (a).

(3): $\Omega_k' \subset \Omega_k$, because the elements of Ω_k' are non–dominated sets of size k and Ω_k is the set of all non–dominated sets of size k.

(4): $|\Omega_k'| \leq |\Omega_k|$, because Ω_k' is a subset of Ω_k (3), its cardinality must be less or equal to that of Ω_k (i).

(5): $c \leq |\Omega_k|$, because the cardinal number of Ω_k' is less or equal to the cardinal number of Ω_k (4), and the cardinal number of Ω_k' is c (1).

(6): $|\Omega_k| \leq |R^{kn}|$, because of (k) and because we can make an injection from Ω_k to R^{kn}. In order to make the injection mentioned before, sort the vectors $v = <v^1, v^2, \ldots v^n> \in A$ for $A \in \Omega_k$ using the following rule: v precedes u, for $v, u \in A$, if $v^1 < u^1$ or if $v^r < u^r$ when $v^i = u^i$ for $1 \leq i \leq r - 1$. This way, every non–dominated set $A \in \Omega_k$ is associated with a unique list of ordered vectors $v_1 = <v_1^1, v_1^2, \ldots, v_1^n>, v_2 = <v_2^1, v_2^2, \ldots, v_2^n>, \ldots, v_k = <v_k^1, v_k^2, \ldots, v_k^n>$. Associate each non–dominated set $A \in \Omega_k$ with the point $(v_1^1, v_1^2, \ldots, v_1^n, v_2^1, v_2^2, \ldots, v_2^n, \ldots, v_k^1, v_k^2, \ldots, v_k^n) \in R^{kn}$ and we have the desired injection.

(7): $|\Omega_k| \leq c$, because the cardinal number of Ω_k is less or equal to the cardinal number of R^{kn} (6) and the cardinal number of R^{kn} is c (f).

(8): $c \leq |\Omega_k| \leq c$, because of (5) and (7).

(9): $|\Omega_k| = c$. This is a direct consequence of (8).

$\square$

So, based on the demonstrations presented before, we introduce the following lemma:

Lemma 2. *The cardinal number of Ω_k, where k is a positive integer, is c.*

Lemma 2 will be useful in the following sections to clarify the demonstrations of some important theorems.

4   The impossibility to create a CC unary comparison method

In this section we present a demonstration of an important theorem presented in [8]. Our demonstration follows the same methodology used in the one presented [8], we assume that it is possible to construct a compatible and complete comparison method with a finite number of unary performance measures and then we show that this assumption leads to an absurd result.

Theorem 1. *For multiobjective problems with 2 or more objectives, there exits no unary comparison method with a finite number k of performance measures in $\boldsymbol{I}$, that is compatible and complete with any of the outperformance relations.*

Theorem 1 is one of the most important results obtained by Zitzler et al [8]. It has very important implications. It means that, in the more general case, unary metrics have a limited capacity to evaluate whether a non–dominated set is better than another. In order to demonstrate this theorem, the Ziztler et al first proved Lemma 3.

Lemma 3. *Let $Z = \{(z^1, z^2, \ldots, z^n) \in \boldsymbol{R}^n \mid a < z^i < b, 1 < i < n\}$, be an open hypercube in $\boldsymbol{R}^n$ with $n \geq 2$, $a, b \in \boldsymbol{R}$. If there exist a compatible and complete unary comparison method with $\boldsymbol{I} = (I_1, I_2, ..., I_k)$, and an interpretation function E, then for all $A, B \in \Omega$ with $A \neq B$ there is at least one I_j in $\boldsymbol{I}$ such that $I_j(A) \neq I_j(B)$.*

In other words, for a CC unary comparison method if $A \neq B$ then $\mathbf{I}(A) \neq \mathbf{I}(B)$. Define Υ as the set of all different vectors $\mathbf{I}$ we can generate with k quality indicators. The cardinal number of Υ is c, because $\Upsilon = \mathbf{R}^k \sim \mathbf{R}$, (see (d) in Section 2.2). So, there must be an injection from Ω to $\mathbf{R}$. The demonstration of Lemma 3 can be found in [8]. Next, we present a demonstration of Theorem 1 that is equivalent to the one presented in [8] using Lemma 1 from Section 3.

Proof of Theorem 1. Let $Z = R^n$.
    (1) $|\Omega| = f$, Lemma 1.
    (2) $|\Upsilon| = c$, because of the definition of Υ and (d).
    (3) $c < f$, because of (j).
    (4) We need to make an injection from Ω to Υ, because of Lemma 3.
    (5) It is impossible to make an injection from Ω to Υ, because of (1), (2) and (3).

<div align="right">□</div>

The conclusion is that no comparison method based on a finite number of unary performance measures can be compatible and complete, because it leads to an absurd result. The central part of the demonstration is that we can not make an injection from Ω to Υ. This part of the demonstration is important and for easy reference we called it the *cardinality contradiction*.

In the next section we review some conditions under which the cardinality contradiction does not occur.

5   Some special conditions

According to Theorem 1, in general it is not possible to construct a compatible and complete comparison method with a finite number of unary performance measures (CCUPM). This is because we cannot make an injection from Ω to Υ, because the cardinality of Ω is bigger than the cardinality of Υ. It is valid to ask if it is possible to construct a CCUPM under some special conditions. One strategy for this is to increment the size of Υ in such a way that its cardinality is equal to that of Ω. We state this in the following theorem.

Theorem 2. *If we use c–many unary performance measures, the cardinality contradiction vanishes.*

Proof. Redefine I as a combination of c–many unary performance measures.
   (1) $|\Omega| = f$, because of Lemma 1.
   (2) $|\Upsilon| = f$, because the number of different combinations of unary performance measures we can generate from I is $\mathbf{R}^c = f$ (e).
   (3) We need to make an injection from Ω to Υ, because of Lemma 3.
   (4) It is possible to make an injection from Ω to Υ, because of (1) and (2).
$\square$

So, the cardinality contradiction vanishes if we use c–many unary performance measures. Note that this is not true if we redefine $\mathbf{I}$ to contain a–many quality indicators, because in this case $|\Upsilon| = |\mathbf{R}^a| = c < f = |\Omega|$, so the cardinality contradiction holds.

In [8] it was already stated that it is possible to create a CCUCM with an infinite number of unary metrics. In Theorem 2 we explained how this condition makes it possible.

Another strategy to make possible the construction of a CCUCM is to reduce the size of Ω. An example of this strategy was also given in [8] in the form of the following theorem.

Theorem 3. *There exists a CCUPM if we restrict the size of the non–dominated sets we are going to compare, to be smaller than or equal to a fixed value l.*

The description of this CCUPM can be found in [8]. The existence of such a CCUPM generates another question. Why is it possible to construct such a comparison method when in general it is not possible? The reason is that in Theorem 3, new conditions are added, and these conditions lead us to a new result. We show this in the following theorem:

Theorem 4. *If we restrict the size of the non–dominated sets we are going to compare, to be smaller than or equal to a fixed value l the cardinality contradiction vanishes.*

In order to prove Theorem 4 we first demonstrate the following lemma.

Lemma 4. *Let $Z = R^n$. Define $\Omega_{\leq l}$ as the set of all non–dominated sets of size equal or less than l. The cardinality of $\Omega_{\leq l}$ is c.*

Proof. We can represent $\Omega_{\leq l}$ with the following sum:

$$\Omega_{\leq l} = \Omega_1 + \Omega_2 + \ldots + \Omega_l \qquad (\alpha)$$

(1) $|\Omega_{\leq l}| = |\Omega_1 + \Omega_2 + \ldots + \Omega_l|$, because of (α)

(2) $|\Omega_{\leq l}| = |\Omega_1| + |\Omega_2| + \ldots + |\Omega_l|$, because the different Ω_k are mutually disjoint.

(3) $|\Omega_{\leq l}| = l \cdot c$, because for Lemma 2, $|\Omega_k| = c$.

(4) $|\Omega_{\leq l}| = c$, because the product of c with a natural number $l > 0$ is equal to c (c) and (3). $\qquad\square$

Once Lemma 4 is proved, it is easy to prove Theorem 4.

Proof of Theorem 4. Redefine Ω so $\Omega = \Omega_{\leq l}$.

(1) $|\Omega| = c$, because of Lemma 4.

(2) $|\Upsilon| = c$, because the number of different combinations of unary performance measures we can generate from I with k unary metrics is $|\mathbf{R}^k| = c$ (d).

(3) We need to make an injection from Ω to Υ, because of Lemma 3.

(4) It is possible to make an injection from Ω to Υ, because of (1) and (2). $\qquad\square$

When we reduce the size of Ω, as described in Lemma 4, there is no reason to consider the construction of a CCUPM as impossible. Theorem 3 shows that it is possible through an example.

6   Another Special Condition

In the demonstration of Theorem 1, both the one presented here and the one in [8], it was considered that there is not restriction on the size of the sets we wish to evaluate, a very valid assumption in theory. However, under practical conditions, the size of the sets is *finite* because a MOEA can only generate sets of finite size. When we consider this new premise we arrive to a new result with important consequences. To show this, we first present the following definition.

Definition 3. $\Omega_{<a}$ *is the set of all possible non–dominated sets from Z of finite cardinality.*

As mentioned before, all MOEAs in literature only generate non–dominated sets in $\Omega_{<a}$. Based on this, we formulate the following theorem.

Theorem 5. *If we only compare non–dominated sets $A \in \Omega_{<a}$, the cardinality contradiction vanishes.*

In order to prove Theorem 5, we first demonstrate Lemma 5.

Lemma 5. *The cardinal number of $\Omega_{<a}$ is c.*

Proof. We can represent $\Omega_{<a}$ with the following sum:

$$\Omega_{<a} = \Omega_1 + \Omega_2 + \Omega_3 + \ldots \qquad (\beta)$$

Note that this sum has a–many elements, because we can make a mapping between the positive integers and the elements of the sum. For this we only need to associate each positive integer k with the corresponding Ω_k. The values of k increase without a limit, but they are always finite.

(1): $|\Omega_k| = c$, because of Lemma 2.

(2): $|\Omega_{\leq a}| = |\Omega_1 + \Omega_2 + \Omega_3 + \ldots|$, because of (β).

(3): $|\Omega_{\leq a}| = |\Omega_1| + |\Omega_2| + |\Omega_3| + \ldots$, because the different Ω_k are mutually disjoint and (2).

(4): $|\Omega_{\leq a}| = c + c + c + \ldots$, because of (1) and (3).

(5): $|\Omega_{\leq a}| = c$, because of (4) and (g). $\qquad\qquad\square$

With Lemma 5 demonstrated, we prove Theorem 5.

Proof of Theorem 5. Redefine Ω as $\Omega = \Omega_{\leq a}$

(1) $|\Omega| = c$, because of Lemma 5.

(2) $|\Upsilon| = c$, because of the definition of Υ and (d).

(3) $\Omega \sim \Upsilon$, because of (1), (2) and (a).

(4) We need to make an injection from Ω to Υ, because of Lemma 3.

(5) It is possible to make an injection from Ω to Υ, because of (3). $\qquad\square$

The consequence is that under the practical conditions of Theorem 5 we can not longer affirm that a compatible and complete comparison method based on a finite number of unary metrics is impossible to construct.

7 Conclusions

In this work we present a study based on the work of Zitzler et al [8]. In this study we make an analysis on the cardinality of the sets of non–dominated sets. The value of these cardinalities were used to explain why under some conditions it is possible to construct a compatible and complete unary comparison method. The most important result in this study is that under practical conditions there is not (known) restriction to the construction of a CCUCM. This result is obtained adding the premise that we are to evaluate non–dominated sets of arbitrary finite size. Unfortunately, we can neither affirm that such comparison method is possible, this is unknown at the moment. Further research

is necessary to determine whether it is possible to construct a compatible and complete comparison method under practical conditions.

8 Corrections

In previous versions of this Technical Report and in [6] we made some affirmations like "Theorem 1 (or its demonstration) is flawed" or "Theorem 1 does not apply". After a review of the work, is our obligation to report that these affirmations are incorrect, Theorem 1 is right and not flawed at all. Both Theorem 1 and Theorem 5 are correct, they consider different premises and that is why different results are obtained. We apologize for this error from our part.

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