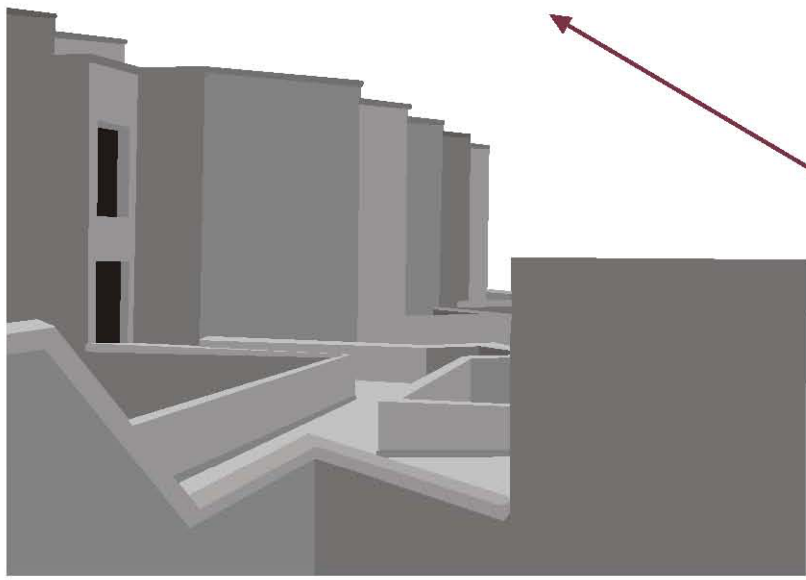

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CORRECTION TO "A NOTE ON THE RATIO OF NORMAL AND LAPLACE RANDOM VARIABLES", BY NADARAJAH, S. AND KOTZ, S. in Statistical Methods and Applications, 2006, 15, 151-158

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Comunicación del CIMAT No I-08-08/21-02-2008
(PE/CIMAT)

CIMAT

Correction to "A note on the ratio of normal and Laplace random variables", by Nadarajah, S. and Kotz, S., *Statistical Methods and Applications, 2006*, **15**, p. 151-158

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Abstract

Some errors were found in the expressions given for the distribution and the density functions of the absolute value of a ratio of a Normal to a Laplace random variables in the paper mentioned in the title. As a consequence, negative values can be obtained for this distribution. The corrected expressions for these formulae are presented here. Complementary error function distribution and density functions of the ratio of Normal and Laplace variables.

1 Introduction

Nadarajah and Kotz (2006) provide expressions for the exact distribution of the absolute value of the ratio $Z = |X/Y|$ where X and Y are independent random variables distributed as Normal and Laplace, respectively. The corresponding density functions are

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad x \in \mathbb{R}, \tag{1}$$

and

$$f_Y(y) = \frac{\lambda}{2} \exp\left(-\lambda |y|\right), \ y \in \mathbb{R}, \tag{2}$$

where σ and λ are scale parameters, therefore positive, and μ is a location parameter, thus $\mu \in \mathbb{R}$. To derive their Theorem 1 where the expressions for the cdf $F(z)$ and pdf $f(z)$ are given, Nadarajah and Kotz use a formula, presented in Prudnikov *et al.* (1986, 2.8.9.1, p.110) which is also given in their Lemma 1, for calculating integrals of the form

$$\int_0^\infty \exp(-px)\operatorname{erfc}(cx+b)dx, \tag{3}$$

where $p>0$, and erfc is the complementary error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp\left(-t^2\right) dt.$$

The central point to note is that the formula provided in Prudnikov *et al.* is only valid for $c>0$. When c is negative, some limits of integration must be modified in a change of variable required in the derivation of this formula that lead to different expressions both for $F(z)$ and $f(z)$.

Furthermore, if the corrections presented here are not considered, the original expression of Theorem 1 in Nadarajah and Kotz (2006) can provide negative results for $F(z)$. As an example of this situation, consider the case of $\mu=-2$, $\sigma=\lambda=1$ that yields a negative value of $F(z=1)=-11.978$, while the correct value is 0.204.

2 Correction

The correct expression for the integral (3) is given in the following lemma.

Lemma 1 *The following integral can be expressed as:*

$$\int_0^\infty \exp(-px)\operatorname{erfc}(cx+b)dx = \frac{1}{p}\operatorname{erfc}(b) + \begin{cases} -\frac{1}{p}\exp\left(\frac{bp}{c}+\frac{p^2}{4c^2}\right)\operatorname{erfc}\left(b+\frac{p}{2c}\right), & \text{if } c>0, \\ \frac{1}{p}\exp\left(\frac{bp}{c}+\frac{p^2}{4c^2}\right)\operatorname{erfc}\left(-b-\frac{p}{2c}\right), & \text{if } c<0. \end{cases} \tag{4}$$

Proof. The detailed proof is given in the Appendix. ∎

The first row of this expression is a special case of Prudnikov's formula for $c>0$. The second row of (4) is the correction that is presented here. Both rows are derived in the Appendix to this note.

This correction must be considered when applying the formulas given in Lemma 1 to Nadarajah and Kotz's two integrals given in their formula (6). This formula can be derived from Proposition

6.1.12 in Laha and Rohatgi (1979) and was rewritten here for convenience as

$$F(z) = \begin{cases} \dfrac{\lambda}{2}\left[\displaystyle\int_0^\infty \exp(-\lambda y)\,\mathrm{erfc}\left(\frac{\mu - zy}{\sqrt{2}\sigma}\right) dy \right. \\ \left. -\displaystyle\int_0^\infty \exp(-\lambda y)\,\mathrm{erfc}\left(\frac{\mu + zy}{\sqrt{2}\sigma}\right) dy\right], & \text{if } z > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

For the first integral in (5), let

$$p = \lambda > 0, \quad b = \frac{\mu}{\sqrt{2}\sigma}, \text{ and } c_1 = -\frac{z}{\sqrt{2}\sigma}.$$

Since Z takes non negative values, the corresponding values of c_1 are non positive so that the second row of Lemma 1 must be used to express the first integral in (5), using (p, b, c_1).

For the second integral in (5), let

$$p = \lambda > 0, \quad b = \frac{\mu}{\sqrt{2}\sigma}, \text{ and } c_2 = \frac{z}{\sqrt{2}\sigma}.$$

Note that c_2 is always non negative. Therefore the first row of Lemma 1 must be used to express this second integral in (5), using (p, b, c_2).

When the appropriate values of (b, c, p) in (4) are applied to the integrals given in (5) , the following corrected version of Nadarajah and Kotz 's Theorem 1 is obtained.

Theorem 2 *Suppose X and Y are distributed according to* (1) *and* (2). *Then, the cdf of $Z = |X/Y|$ can be expressed as follows,*

$$F(z) = \begin{cases} \dfrac{1}{2}\left[\exp\left(-\dfrac{\mu\lambda}{z} + \dfrac{\lambda^2\sigma^2}{2z^2}\right)\mathrm{erfc}\left(-\dfrac{\mu}{\sqrt{2}\sigma} + \dfrac{\lambda\sigma}{\sqrt{2}z}\right)\right. \\ \left. +\exp\left(\dfrac{\mu\lambda}{z} + \dfrac{\lambda^2\sigma^2}{2z^2}\right)\mathrm{erfc}\left(\dfrac{\mu}{\sqrt{2}\sigma} + \dfrac{\lambda\sigma}{\sqrt{2}z}\right)\right], & \textit{if } z > 0, \\ 0, & \textit{otherwise.} \end{cases} \tag{6}$$

Proof. The proof is a direct application of (4) to the integrals in (5) . ■

The probability density function of Z is provided in the following theorem.

Theorem 3 *Suppose X and Y are distributed according to* (1) *and* (2). *Then, the pdf of $Z = |X/Y|$ can be expressed as follows,*

$$f(z) = \begin{cases} \dfrac{\sqrt{2}\lambda\sigma}{\sqrt{\pi}z^2}\exp\left(-\dfrac{\mu^2}{2\sigma^2}\right) \\ +\dfrac{1}{2}\left(\dfrac{\mu\lambda}{z^2} - \dfrac{\lambda^2\sigma^2}{z^3}\right)\exp\left(-\dfrac{\mu\lambda}{z} + \dfrac{\lambda^2\sigma^2}{2z^2}\right)\mathrm{erfc}\left(-\dfrac{\mu}{\sqrt{2}\sigma} + \dfrac{\lambda\sigma}{\sqrt{2}z}\right) \\ -\dfrac{1}{2}\left(\dfrac{\mu\lambda}{z^2} + \dfrac{\lambda^2\sigma^2}{z^3}\right)\exp\left(\dfrac{\mu\lambda}{z} + \dfrac{\lambda^2\sigma^2}{2z^2}\right)\mathrm{erfc}\left(\dfrac{\mu}{\sqrt{2}\sigma} + \dfrac{\lambda\sigma}{\sqrt{2}z}\right), & \textit{if } z > 0, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. The pdf is obtained by differentiating with respect to z the expression given in (6), taking into account that

$$\frac{d}{dz}\,\mathrm{erfc}\left(\frac{a}{z}+b\right)=\frac{2a}{\sqrt{\pi}z^2}\exp\left[-\left(\frac{a}{z}+b\right)^2\right].\ \blacksquare$$

3 Appendix

The integral (3) can be solved by integration by parts and noting that

$$\frac{d}{dx}\,\mathrm{erfc}\,(ax+d)=-\frac{2a}{\sqrt{\pi}}\exp\left[-\left(ax+d\right)^2\right].$$

Therefore,

$$\begin{aligned}\int_0^\infty \exp(-px)\,\mathrm{erfc}(cx+b)dx &= \lim_{x\to\infty}\left[-\frac{1}{p}\exp(-px)\,\mathrm{erfc}\,(cx+b)\right]+\frac{1}{p}\,\mathrm{erfc}\,(b)\\ &\quad -\frac{2c}{p\sqrt{\pi}}\int_0^\infty \exp\left[-\left(cx+b\right)^2-px\right]dx. \qquad (7)\end{aligned}$$

The limit in the first term in the right side above, is zero since $p>0$ and

$$\lim_{x\to\infty}\mathrm{erfc}\,(cx)=\begin{cases}0, & \text{if } c>0,\\ 2, & \text{if } c<0.\end{cases}$$

The integral in the last term in the right side of (7) yields different results depending on the sign of c as will be shown below. This integral can be expressed as

$$\int_0^\infty \exp\left[-\left(cx+b\right)^2-px\right]dx=\exp\left(\frac{bp}{c}+\frac{p^2}{4c^2}\right)\int_0^\infty \exp\left[-\left(cx+b+\frac{p}{2c}\right)^2\right]dx.$$

Making the following change of variable in this last integral, $w=cx+b+p/(2c)$, yields different limits of integration depending on the sign of c :

$$w\in\begin{cases}\left(b+\dfrac{p}{2c},\infty\right), & \text{if } c>0,\\ \left(b+\dfrac{p}{2c},-\infty\right), & \text{if } c<0.\end{cases} \qquad (8)$$

The more general formula (2.8.9.1) of Prudnikov *et al.* (1986, p.110) provides the correct result for the integral (3) but only for positive values of c. Nadarajah and Kotz (2006) apparently overlooked the fact that for $c<0$ a different result had to be used, noting the different limits of integration given in (8) that must be considered when $c < 0$. Therefore (7) can be expressed as

$$\int_0^\infty \exp(-px)\,\mathrm{erfc}(cx+b)dx = \frac{1}{p}\,\mathrm{erfc}\,(b) - \frac{2c}{p\sqrt{\pi}}\int_0^\infty \exp\left[-\left(cx+b\right)^2 - px\right]dx$$

$$= \frac{1}{p}\,\mathrm{erfc}\,(b) + \begin{cases} -\dfrac{1}{p}\exp\left(\dfrac{bp}{c}+\dfrac{p^2}{4c^2}\right)\dfrac{2}{\sqrt{\pi}}\int_{b+p/(2c)}^{\infty}\exp\left(-w^2\right)dw, & \text{if } c>0, \\ \dfrac{1}{p}\exp\left(\dfrac{bp}{c}+\dfrac{p^2}{4c^2}\right)\dfrac{2}{\sqrt{\pi}}\int_{-\infty}^{b+p/(2c)}\exp\left(-w^2\right)dw, & \text{if } c<0, \end{cases}$$

$$= \frac{1}{p}\,\mathrm{erfc}\,(b) + \begin{cases} -\dfrac{1}{p}\exp\left(\dfrac{bp}{c}+\dfrac{p^2}{4c^2}\right)\mathrm{erfc}\left(b+\dfrac{p}{2c}\right), & \text{if } c>0, \\ \dfrac{1}{p}\exp\left(\dfrac{bp}{c}+\dfrac{p^2}{4c^2}\right)\mathrm{erfc}\left(-b-\dfrac{p}{2c}\right), & \text{if } c<0, \end{cases} \tag{9}$$

noting that

$$\frac{2}{\sqrt{\pi}}\int_{-\infty}^{b+p/(2c)}\exp\left(-w^2\right)dw = 2-\mathrm{erfc}\left(b+\frac{p}{2c}\right) = \mathrm{erfc}\left(-b-\frac{p}{2c}\right).$$

The expression (9) was given here in (4) and is the one that should be used to obtain the correct version of Theorem 1 provided in this note.

References

[1] Laha RG, Rohatgi VK (1979) Probability Theory. John Wiley & Sons, New York

[2] Nadarajah S, Kotz S (2006) A note on the ratio of normal and Laplace random variables. Statistical Methods and Applications 15:151-158

[3] Prudnikov AP, Brychkov YA, Marichev OI (1986) Integrals and Series, vol 1-3. Gordon and Breach, New York