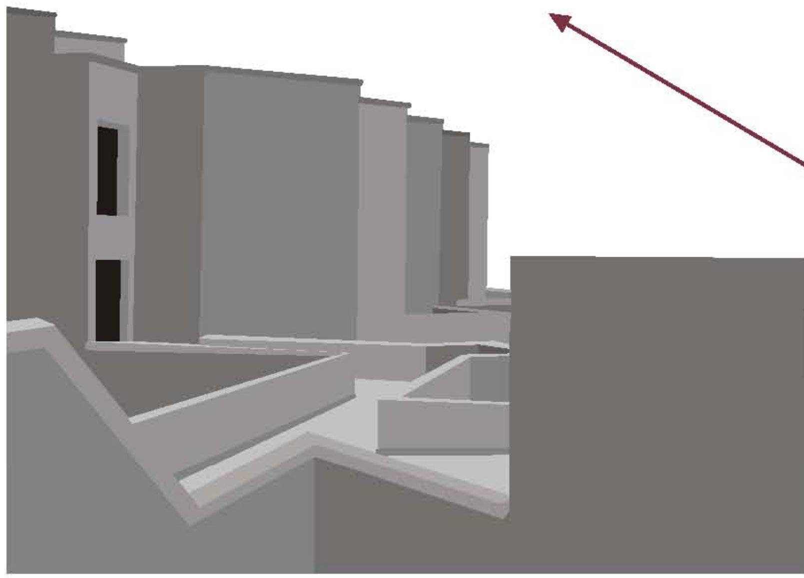

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Law of Large Numbers for the Occupation Time of an Age-dependent Branching System

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Abstract

The occupation time of an age-dependent branching particle system is considered, where the particles are subject to random migration, random lifetimes and branching. Under the usual independence assumptions in branching processes, we give a characterization of the occupation time of such branching system by means of its transition Laplace functional. In the case of finite variance branching (and large mobility of individuals), a large time limit theorem for the occupation time process is obtained.

1 Introduction and background

Consider a branching population in the d-dimensional Euclidean space $\mathbb{R}^d$, evolving in the following fashion. Any given individual develops a Markovian migration during its lifetime τ, where τ is a random variable having a continuous density f, and at the end of its life it branches, leaving behind a population consisting of $\zeta \in \{0, 1, \dots\}$ individuals, all appearing at their mother's death position and evolving independently in the same way as their progenitor.

When τ has an exponential distribution it is well known that, under the customary independence assumptions, the resulting stochastic process $\{X(t),\, t \geq 0\}$, is a Markov process, where, for each $t \geq 0$, $X(t)$ denotes the simple counting measure on $\mathbb{R}^d$ whose atoms are the positions of particles alive at time t. In the literature there is a lot of work about this Markovian model. Our main objective here is to investigate the case when τ is not necessarily an exponential random variable, in which case $\{X(t),\, t \geq 0\}$ is no longer a Markov process. In order to get a Markov process out of this model, Kaj and Sagitov [14] and Fleischmann, Vatutin and Wakolbinger [9] extended the population's phase space by attaching to each individual its remaining lifetime.

In this work we will Markovianize the process $\{X(t),\, t \geq 0\}$ by means of the ages of particles. This is done by considering a population in $\mathbb{R} \times \mathbb{R}^d$, where the first component t represents the age, and the second component z, the position of an individual $\delta_{(t,z)}$, $(t,z) \in \mathbb{R} \times \mathbb{R}^d$. With this in mind, we need to re-formulate our model, and this is done rigorously in section 2.

We need some notations. Recall that the age and position of an individual are described by a point $e = (u,x) \in E := \mathbb{R}_+ \times \mathbb{R}^d$. Let $e_i = (u_i, x_i)$, $i = 1,2$, and define

$$d_E(e_1, e_2) = 1 \wedge |u_1 - u_2| + |x_1 - x_2|,$$

which is a metric in E making (E, d_E) a Polish space. Recall that the (spherically symmetric) α-stable process is a Markov process $\{\xi_t, t \geq 0\}$ in $\mathbb{R}^d$ with infinitesimal generator

$$\Delta_\alpha := -(-\Delta)^{\alpha/2},$$

where $\alpha \in (0,2]$ and Δ is the d-dimensional Laplacian. We denote by $\{\mathcal{T}_t^\alpha, t \geq 0\}$ and $\{p_t^\alpha(.,.), t \geq 0\}$ the corresponding semigroup and the transition function, respectively. When there is no danger of confusion, we shall simply write $\mathcal{T}_t$ and p_t.

Let $p \in (d, d+\alpha]$ be fixed, where α is the index of the stable process. Define the reference function $\Phi_p(x) := (1 + |x|^2)^{-p/2}$, $x \in \mathbb{R}^d$, and let $\hat{C}_p := \hat{C}_p(E)$ be the space of continuous functions $\psi : E \to \mathbb{R}$ such that

$$\|\psi\| := \sup_{(u,x)\in E} \left| \frac{\psi(u,x)}{\Phi_p(x)} \right| < \infty;$$

the map

$$(u,x) \longmapsto \frac{\psi(u,x)}{\Phi_p(x)},$$

can be extended continuously to a function over $\dot{E} := \dot{\mathbb{R}}_+ \times \dot{\mathbb{R}}^d$, where $\dot{\mathbb{R}}_+$ and $\dot{\mathbb{R}}^d$ are the one-point compactification of $\mathbb{R}_+$ and $\mathbb{R}^d$, respectively. The vector space $(\hat{C}_p, \|.\|)$ is a separable Banach space.

We finish this section by recalling some facts about the so called *age-process*, as it is described by Joffe [13]. Let τ be a random variable whose distribution function F is supported on $[0,\infty)$ with $F(0) = 0$ and $F(u) < 1$ for all $u \in [0,\infty)$, and having a continuous density f. Let $\{\tau_i\}_{i=1}^\infty$ be a sequence of independent copies of τ.

Let $\{\eta_t, t \geq 0\}$ be the stochastic process defined as follows: assume that $\eta_0 = u$, where $u \in [0,\infty)$. Then η_t increasses linearly for $t \in (0, \tau_1 - u)$, and jumps to 0 at $\tau_1 - u$. Again η_t increases linearly between $(\tau_1 - u, \tau_1 - u + \tau_2)$ (now starting from 0), jumps to 0 at $\tau_1 - u + \tau_2$ and so on. As we can see, $\{\eta_t\ t \geq 0\}$ is a renewal proccess, and the random variable η_t represents the *age* since the last renewal. The process η is called *age-process*.

Now we are going to write down the semigroup $\mathcal{S}$ associated to the age-process. Let us define

$$F_u(v) := P(\tau \leq u + v \mid \tau > u),$$

for $u, v \in [0, \infty)$, and write $\bar{F}(u) := 1 - F(u)$. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be a bounded uniformly continuous function. Then, for $t, u \in [0, \infty)$

$$\begin{aligned} \mathcal{S}_t\varphi(u) \quad := \quad & \varphi(u+t)\frac{\bar{F}(u+t)}{\bar{F}(u)} + \int_0^t \Big[\varphi(t-s)\bar{F}(t-s) \\ & + \int_0^{t-s} \varphi(t-s-v)\bar{F}(t-s-v)dU(v)\Big]\, dF_u(s), \end{aligned} \tag{1}$$

where U is the renewal function $U(v) = \sum_{i=0}^{\infty} F^{*i}(v)$. If we take φ to be a differentiable function, then we can write down the infinitesimal generator $\mathcal{A}$ of the age-proccess by

$$\mathcal{A}\varphi(u) = \varphi'(u) + \lambda(u)[\varphi(0) - \varphi(u)], \tag{2}$$

where $\lambda(u) = f(u)/\bar{F}(u)$.

2 The Model

We consider the following branching model: assume that a particle living in $\mathbb{R}^d$ develops a Markovian migration ξ during its random lifetime τ. When it dies, it procreates a random number ζ of new particles, where ζ is an integer-valued random variable. This random number represents new (i.e. zero-aged) particles that are positioned at the same point where the progenitor died. All of these new particles behave independently in the fashion described above. The migration process of each particle is characterized by the stable process ξ, described in Section 1. In addition, we assume that the probability generating function (pgf) of ζ is given by

$$\Phi(s) := \mathbb{E}s^{\zeta} = s + c(1-s)^{1+\beta}, \tag{3}$$

where $|s| \leq 1$, $\beta \in (0, 1]$ and $c \in (0, \frac{1}{1+\beta}]$. In particular $\mathbb{E}\zeta = 1$, that is, the branching is critical. When $\beta = 1$ and $c = 1/2$ we get the critical binary branching.

Let $\mathcal{M} := \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^d)$ denote the space of Radon measures on the Borel sets of $\mathbb{R}_+ \times \mathbb{R}^d$, and let $\mathcal{N} := \mathcal{N}(\mathbb{R}_+ \times \mathbb{R}^d) \subset \mathcal{M}$ be the subspace of counting measures. We will denote by $\mathcal{M}_F$ and $\mathcal{N}_F$ the corresponding subsets of finite measures. Our branching population is properly described by a stochastic process $\{\bar{X}_t : t \geq 0\}$ taking values in $\mathcal{N}_F$. In particular, if $\bar{X}_t = \sum_{k=1}^{n} \delta_{(\eta_t^k, \xi_t^k)}$, we have at time t, n particles where the k-th particle has age $\eta_t^k \in \mathbb{R}_+$ and position $\xi_t^k \in \mathbb{R}^d$. More generally, $\bar{X}_t([a, b] \times B)$ is the number of particles at time t in the set $B \in \mathbb{R}^d$ with ages in the interval $[a, b]$. The process $\bar{X}$ is a Markov process.

3 The Basic Process

In this section we consider the average motion of individuals. Such displacement is described by a process $\tilde{\xi} := \{(\eta(t), \xi(t)) : t \geq 0\}$ which takes values on $\mathbb{R}_+ \times \mathbb{R}^d$, where the process η represents the age of the particle and ξ its position on $\mathbb{R}^d$; here ξ is the α-stable Lévy process with $\alpha \in (0, 2]$. We denote by $\tilde{T}$ the semigroup correponding to $\tilde{\xi}$.

Our first goal is to find the infinitesimal generator and an invariant measure for the process $\tilde{\xi}$.

First, we find the infinitesimal generator. The jumps of the process $\tilde{\xi}$ that we are interested in are when the particle dies, i.e., the jumps of the process η which ocurr at rate $\lambda(u) := f(u)/\bar{F}(u)$, where $u \in \mathbb{R}_+$ and $\bar{F}(u) := 1 - F(u)$ (See [18], p. 480). This means that, if R is the first jumping time of η, then

$$P\{R > t + u | R > u\} = e^{-\int_u^{t+u} \lambda(s) ds}, \quad u \geq 0.$$

The infinitesimal generator of the process $\tilde{\xi}$, which we will denote by $\mathcal{L}$, is given in the following proposition.

Proposition 3.1 *The process $\{\tilde{\xi}_t, t \geq 0\}$ is a homogeneous Markov process whose infinitesimal generator $\mathcal{L}$, is given by*

$$\mathcal{L}\psi(u, x) = \frac{\partial \psi(u, x)}{\partial u} + \Delta_\alpha \psi(u, x) - \lambda(u)[\psi(u, x) - \psi(0, x], \tag{4}$$

where $\psi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is any bounded function such that $\psi(., x) \in C_b^1(\mathbb{R}_+)$ for all $x \in \mathbb{R}^d$, and $\psi(u, .) \in Dom(\Delta_\alpha)$ for all $u \in \mathbb{R}_+$.

Proof: We need to find the limit

$$\mathcal{L}(t)\psi(u, x) = \lim_{h \downarrow 0} h^{-1} \mathbb{E}\left[\psi(\tilde{\xi}_{t+h}) - \psi(u, x) | \tilde{\xi}_t = (u, x)\right], \quad \text{for every } \ t \geq 0.$$

For each $t \geq 0$ we have that:

$$
\begin{aligned}
& h^{-1}\mathbb{E}\left[\psi(\tilde{\xi}_{t+h}) - \psi(u,x) | \tilde{\xi}_t = (u,x)\right] \\
= \; & h^{-1}\mathbb{E}\left[\psi(\tilde{\xi}_{t+h}) | \tilde{\xi}_t = (u,x)\right] - h^{-1}\psi(u,x) \\
= \; & h^{-1}\left\{ e^{-\int_u^{u+h}\lambda(s)ds}\mathcal{T}_h^\alpha \psi(u+h,.)(x) + \int_0^h \lambda(u+r)e^{-\int_u^{u+r}\lambda(s)ds} \right. \\
& \left. \times \int_{\mathbb{R}^d} p_r^\alpha(x,y)dydr\mathbb{E}[\psi(\tilde{\xi}_{t+h}) \mid \tilde{\xi}_{t+r} = (0,y)] \right\} - h^{-1}\psi(u,x) \\
= \; & \frac{1}{h}e^{-\int_u^{u+h}\lambda(s)ds}\left[\mathcal{T}_h^\alpha \psi(u+h,.)(x) - \mathcal{T}_h^\alpha \psi(u,.)(x)\right] \\
& + \frac{1}{h}e^{-\int_u^{u+h}\lambda(s)ds}\left[\mathcal{T}_h^\alpha \psi(u,.)(x) - \psi(u,.)(x)\right] + \frac{1}{h}\left(e^{-\int_u^{u+h}\lambda(s)ds} - 1\right)\psi(u,x) \\
& + \frac{1}{h}\int_0^h \lambda(u+r)e^{-\int_u^{u+r}\lambda(s)ds}\mathcal{T}_r^\alpha \mathbb{E}[\psi(\tilde{\xi}_{t+h}) \mid \tilde{\xi}_{t+r} = (0,.)](x)dr.
\end{aligned}
$$

Therefore, letting h tend to infinity and applying the strong continuity of $\mathcal{T}^\alpha$ in the first term on the last equality, we get (4).$\triangleleft$

Remark 3.2 *Taking $\psi(u,x) \equiv \psi(u)$ in the preceding proposition yields the expression (2) for the infinitesimal generator of the age-process.*

Now, we will focus on finding an invariant measure for $\mathcal{S}$ and with this at hand we obtain an invarinat measure for $\tilde{T}$.

Let ψ be a function from $\mathbb{R}_+ \times \mathbb{R}^d$ to $\mathbb{R}$ such that $\psi(u,x) = \psi_1(u)\psi_(x)$, for each $(u,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Then, for each $t \geq 0$, $\mathcal{S}_t\mathcal{T}_t\psi(u,x) = \mathcal{S}_t\psi_1(u)\mathcal{T}_t\psi_2(x)$.

In fact,

$$
\begin{aligned}
\lim_{t\downarrow 0}\frac{\mathcal{S}_t\mathcal{T}_t\psi(u,x) - f(u,x)}{t} &= \lim_{t\downarrow 0}\frac{\mathcal{S}_t\psi_1(u)\mathcal{T}_t\psi_2(x) - \psi_1(u)\psi_2(x)}{t} \\
&= \lim_{t\downarrow 0}\{\mathcal{S}_t\psi_1(u)[\mathcal{T}_t\psi_2(x) - \psi_2(x)] + \psi_2(x)[\mathcal{S}_t\psi_1(u) - \psi_1(u)]\} \\
&= \psi_1(u)\Delta_\alpha\psi_2(x) + \psi_2(x)\left(\frac{\partial\psi_1(u)}{\partial u} + \lambda(u)[\psi_1(0) - \psi_1(x)]\right) \\
&= \mathcal{L}\psi(u,x),
\end{aligned}
$$

where the last expression is obtained using the strong continuity of $\{\mathcal{S}_t : t \geq 0\}$ and $\{\mathcal{T}_t : t \geq 0\}$.

Similarly, we have that

$$
\lim_{t\downarrow 0}\frac{\mathcal{T}_t\mathcal{S}_t\psi(u,x) - \psi(u,x)}{t} = \mathcal{L}\psi(u,x).
$$

Therefore, by Stone-Weierstrass' Theorem, for each $t \geq 0$ $\tilde{T}_t = \mathcal{S}_t\mathcal{T}_t$ in $\hat{C}_p$.

Lemma 3.3 *The semigroup $\{\mathcal{S}_t, t \geq 0\}$ has an invariant measure G^* which, for any $\varphi \in C_b^1(\mathbb{R}_+)$, is given by*

$$\int_0^\infty \varphi(u) G^*(du) = \int_0^\infty \int_0^t \varphi(s) ds f(t) dt.$$

Proof: *First, we observe that it is enough to show that*

$$\int_0^\infty \mathcal{A}\varphi(u) G^*(du) = 0.$$

Applying Fubini's Theorem we have that

$$\begin{aligned}
\int_0^\infty \mathcal{A}\varphi(u) G^*(du) &= \int_0^\infty \int_0^t \left[\varphi'(u) + \lambda(u)[\varphi(0) - \varphi(u)]\right] du f(t) dt \\
&= \int_0^\infty \int_0^t \varphi'(u) du f(t) dt + \int_0^\infty \frac{f(u)}{1 - F(u)} [\varphi(0) - \varphi(u)] \int_u^\infty f(t) dt \\
&= \int_0^\infty [\varphi(0) - \varphi(u)] f(u) du + \int_0^\infty [\varphi(u) - \varphi(0)] f(u) du \\
&= 0,
\end{aligned}$$

where the last equality is because φ is bounded. Therefore, G^* is an invariant measure for $\mathcal{S}$.$\triangleleft$

Remark 3.4 *In the case that the lifetime has finite mean, $\{\mathcal{S}_t; t \geq 0\}$ has a unique invariant law ([5] pag. 130),which is denoted by $\tilde{G}$ and is given as follows:*

$$\int_0^\infty \varphi(u) \tilde{G}(du) = \frac{1}{\mu} \int_0^\infty \int_0^t \varphi(s) ds f(t) dt,$$

where μ is the mean of the lifetime.

Notation: If μ is a measure and ψ is a μ-intregrable function, $< \mu, \psi >:= \int f d\mu$. $*$ will denote integration with respect to the age component and $\bullet$ denotes integration in the spatial component.

The next proposition shows the existence of an invariant measure for $\tilde{T}$.

Proposition 3.5 *The product measure $G^* \times \Lambda$ is an invariant measure for $\{\tilde{T}_t, t \geq 0\}$.*

Proof: *We know that Λ is an invariant measure for the semigroup $\{\mathcal{T}_t : t \geq 0\}$. Then, for $\psi \in \hat{C}_p$ and $t \geq 0$ we have, by Lemma 3.3, that*

$$\begin{aligned}
< G^* \times \Lambda, \tilde{T}_t \psi_1(*) \psi_2(\bullet) > &= < G^* \times \Lambda, \left(\mathcal{S}_t \mathcal{T}_t \psi\right)(*, \bullet) > \\
&= < G^*, < \Lambda, \left(\mathcal{S}_t \mathcal{T}_t \psi\right)(*, \bullet) >> \\
&= < G^*, \mathcal{S}_t < \Lambda, (\mathcal{T}_t \psi)(*, \bullet) > \\
&= < G^*, \mathcal{S}_t (< \Lambda, \psi(*, \bullet) >) > \\
&= < G^*, < \Lambda, \psi(*, \bullet) >> \\
&= < G^* \times \Lambda, \psi(*, \bullet) > .
\end{aligned}$$

Therefore, $G^* \times \Lambda$ is an invariant measure for the semigroup $\{\tilde{T}_t : t \geq 0\}$.

4 Age-Dependent Branching Particle System

In this section we will derive the infinitesimal generator for the branching particle system on certain cylindrical functions. Namely, let ψ be a function from $\mathbb{R}_+ \times \mathbb{R}^d$ to $(0,1]$ with compact support and such that $\psi \in \mathrm{Dom}(\mathcal{L})$. Given $\nu \in \mathcal{M}$ define

$$G_\psi(\nu) := \exp < \nu, \log \psi > .$$

The infinitesimal genertor of $\{\bar{X}_t; t \geq 0\}$, evaluated at the function $G_\psi(\nu)$, is defined as follows:

$$\mathcal{G}G_\psi(\delta_{(u,x)}) := \lim_{t \to 0} t^{-1} \mathbb{E}\left[G_\psi(\bar{X}_t) - G_\psi(\delta_{(u,x)}) \mid \bar{X}_0 = \delta_{(u,x)}\right].$$

We note that, due to the independence assumptions between branching mechanism and diffusion, $\mathcal{L}$ can be expresed as

$$\mathcal{L}G_\psi(\nu) = \mathcal{B}G_\psi(\nu) + \mathcal{D}G_\psi(\nu),$$

where $\mathcal{B}$ corresponds to the branching part and $\mathcal{D}$ to the diffusion part.

Firstly, we evaluate the branching part, this is given by

$$\begin{aligned} \mathcal{B}G_\psi(\nu) &= < \nu, \lambda(*,.) \sum_{k=0}^{\infty} p_k \left[G_\psi(\nu - \delta_{(*,.)} + k\delta_{(0,.)}) - G_\psi(\nu)\right] > \\ &= < \nu, \lambda(*) \sum_{k=0}^{\infty} p_k \left[\frac{\psi^k(0,.)}{\psi(*,.)} - 1\right] > \\ &= G_\psi(\nu) < \nu, \lambda(*) \frac{\Phi(\psi(0,.)) - \psi(*,.)}{\psi(*,.)} > . \end{aligned} \tag{5}$$

Now, we want to see what the diffusion part is. Let τ_* be the time of the first branching and assuming that $\bar{X}_0 = \sum_{k=1}^n \delta_{(u_k,x_k)}$ we get that

$$\begin{aligned} & \mathbb{E}[G_\psi(\bar{X}_h) \mid \bar{X}_0, \tau_* > h] P(\tau_* > h \mid \tau_1 > u_1, \cdots, \tau_n > u_n) \\ = & \sum_{k=1}^n e^{-\int_{u_k}^{u_k+h} \lambda(r)dr} \mathcal{T}_h \phi(u_k + h, x_k) \prod_{j=1, j\neq k} e^{-\int_{u_j}^{u_j+h} \lambda(r)dr} \mathcal{T}_h \phi(u_j + h, x_j). \end{aligned}$$

Therefore, by independence between the particles we have that

$$\mathbb{E}[G_\psi(\bar{X}_h) - G_\psi(\bar{X}_0) \mid \bar{X}_0] = \prod_{k=1}^n e^{-\int_{u_k}^{u_k+h} \lambda(r)dr} \mathcal{T}_h \phi(u_k + h, x_k),$$

hence,

$$\begin{aligned}
\mathcal{D}G_\psi(\bar{X}_0) &:= \lim_{h\downarrow 0} h^{-1}\mathbb{E}[G_\psi(\bar{X}_h) - G_\psi(\bar{X}_0) \mid \bar{X}_0] \\
&= \lim_{h\downarrow 0} h^{-1}\left[\prod_{k=1}^{n} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\mathcal{T}_h\psi(u_k+h,x_k) - \prod_{k=1}^{n}\mathcal{T}_h\psi(u_k,x_k)\right] \\
&= \lim_{h\downarrow 0} h^{-1}\left[\prod_{k=1}^{n} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\mathcal{T}_h\psi(u_k+h,x_k) - \prod_{k=1}^{n} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\phi(u_k+h,x_k)\right. \\
&\quad + \prod_{k=1}^{n} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\psi(u_k+h,x_k) - \prod_{k=1}^{n} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\psi(u_k,x_k) \\
&\quad \left. + \prod_{k=1}^{n} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\psi(u_k,x_k) - \prod_{k=1}^{n}\psi(u_k,x_k)\right] \\
&= \sum_{k=1}^{n}\left[\Delta_\alpha\psi(u_k,x_k) + \frac{\partial}{\partial u}\psi(u_k,x_k) - \lambda(u_k)\psi(u_k,x_k)\right]\prod_{j=1,j\neq k}^{n}\psi(u_j,x_j). \qquad (6)
\end{aligned}$$

We will prove the last equality only for $n=2$, for general n the calculations are similar but more involved.

Take $n=2$ and denote by (I), (II) and (III) the first, second and third term, rescpectively, on the third equality.

Note that, by applaying the chain rule to the third term we get that

$$(III) = -\sum_{k=0}^{n}\lambda(u_k)\prod_{j=1,j\neq k}^{n}\psi(u_j,x_j).$$

For the second term,

$$\begin{aligned}
(II) &= \lim_{h\downarrow 0}\left[\prod_{k=1}^{2} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\psi(u_k+h,x_k) - \prod_{k=1}^{2} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\psi(u_k,x_k)\right] \\
&= \lim_{h\downarrow 0} h^{-1}\left\{e^{-\int_{u_1}^{u_1+h}\lambda(r)dr}[\psi(u_1+h,x_1) - \psi(u_1,x_1)]e^{-\int_{u_2}^{u_2+h}\lambda(r)dr}\psi(u_2,x_2)\right. \\
&\quad \left. + e^{-\int_{u_2}^{u_2+h}\lambda(r)dr}[\psi(u_2+h,x_2) - \psi(u_2,x_2)]e^{-\int_{u_1}^{u_1+h}\lambda(r)dr}\psi(u_1,x_1)\right\} \\
&= \psi(u_2,x_2)\frac{\partial}{\partial u}\psi(u_1,x_1) + \psi(u_1,x_1)\frac{\partial}{\partial u}\psi(u_2,x_2).
\end{aligned}$$

Finally, we evaluate the first term

$$
\begin{aligned}
(I) &= \lim_{h\downarrow 0} h^{-1}\left[\prod_{k=1}^{2} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\mathcal{T}_h\psi(u_k+h,x_k) - \prod_{k=1}^{2} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\phi(u_k+h,x_k)\right]\\
&= \lim_{h\downarrow 0} h^{-1}\Big\{ e^{-\int_{u_1}^{u_1+h}\lambda(r)dr}[\mathcal{T}_h\psi(u_1+h,x_1)-\psi(u_1,x_1)]e^{-\int_{u_2}^{u_2+h}\lambda(r)dr}\mathcal{T}_h\psi(u_2,x_2)\\
&\quad e^{-\int_{u_2}^{u_2+h}\lambda(r)dr}[\mathcal{T}_h\psi(u_2+h,x_2)-\psi(u_2,x_2)]e^{-\int_{u_1}^{u_1+h}\lambda(r)dr}\mathcal{T}_h\psi(u_1,x_1)\Big\}\\
&= \psi(u_2,x_2)\Delta_\alpha\psi(u_1,x_1)+\psi(u_1,x_1)\Delta_\alpha\psi(u_2,x_2).
\end{aligned}
$$

Therefore, by (I),(II) and (III)we have shown (6) for $n=2$.

Note that, (6) can be written as

$$
\mathcal{D}G_\psi(\nu) = G_\psi(\nu) < \nu, \frac{\Delta_\alpha\psi(*,.)+\frac{\partial}{\partial *}\psi(*,.)-\lambda(*)\psi(*,.)}{\psi(*,.)} >, \tag{7}
$$

hence, adding (5) and (7) we get that

$$
\mathcal{G}G_\psi(\nu) = G_\psi(\nu) < \nu, \frac{\mathcal{L}\psi(*,.)+\lambda(*)[\Phi(\psi(0,.))-\psi(0,.)]}{\psi(*,.)} > . \tag{8}
$$

Now, we summarize the above discussion in the following proposition.

Proposition 4.1 *Let $\{\bar{X}_t, t\geq 0\}$ be the branching particle system. Then for each $\psi \in Dom(\mathcal{L})$ such that $0<\|\psi\|\leq 1$ and $\psi>0$, the process*

$$
M_t := e^{<\bar{X}_t,\log\psi>} - \int_0^t e^{<\bar{X}_s,\log\psi>} < \bar{X}_s, \frac{\mathcal{L}\psi(*,\bullet)+\lambda(*)[\Phi(\psi(0,\bullet))-\psi(0,\bullet)]}{\psi(*,\bullet)} > ds, \tag{9}
$$

is a martingale.

5 Occupation Time

Given an $\mathcal{M}$-valued process $\{Y_t, t\geq 0\}$ and $\psi:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}_+$ a measurable compact supported function we define the *occupation time* process $\{\mathcal{J}_t, t\geq 0\}$ as follows

$$
\mathcal{J}_t^\psi := \int_0^t < Y_s, \psi > ds.
$$

In this section we will study the occupation time of the process $\bar{X}$. Our goal is to show that the occupation time satisfies an ergodic theorem (a kind of strong law of large numbers).

Occupation time has been extensively studied in the context of branching particle systems where the particles' lifetime are exponentialy distributed, (see [4],[15],[1],[2]). Also, [11] and [8] studied it in the context of Dawson-Watanabe supperprocess, i.e., measure-valued processes which are diffusion limits of branching particle systems with exponential life times.

Cox and Griffeath [4] proved the following central limit-type theromen for the occupation time of the critical binary branching Brownian motion, as $t \to \infty$

$$\frac{\mathcal{J}_t^{1_A} - < 1_A, \Lambda > t}{b_t} \overset{a.s.}{\Rightarrow} N(0, \sigma^2),$$

where b_t is a function that depends on the dimension and also σ^2 changes with the dimension. Méléard and Roelly [15] proved that the braching particle system with exponential lifetimes and α-stable migration on $\mathbb{R}^d$ satiafies an ergodic result. Namely, as t goes to infinity

$$t^{-1} < \mathcal{J}_t, \psi > \overset{a.s.}{\to} < \Lambda, \psi > .$$

Bojdecki et. al. ([1],[2]) studied the fluctuations of the reescaled occupation time proccess of this branching system. They have obtained different new process for example *fractional Brownian motion* and *sub-fractional Brownian motion.*

Iscoe [11] has several central limit-type therems for the occupation time in the context of supperprocesses. See also, Fleischmann and Gärtner [8] for more results on this direction.

6 An Ergodic Result

This Section is concerned with the extension of the ergodic result of [15] to the general case: particles' lifetime is any distribution function with continuous density. In fact, we shall show the following theorem.

Theorem 6.1 *Let $d > \alpha$, and let $f \in \hat{C}_p$ be such that $f(u, x) = f_1(u) f_2(x)$. Then, as $t \to \infty$*

$$\frac{< \mathcal{J}_t, f >}{t} \overset{c.s}{\to} < G^* \times \Lambda, f > .$$

To prove this theroem we will need some preliminaries result. The following proposition gives the joint Laplace functional of the branching system and its occupation time.

Proposition 6.2 *Let ϕ, ψ be a nonnegative, measurable compact supported functions from $[0, \infty) \times \mathbb{R}^d$ to $\mathbb{R}_+$. Then, the joint Laplace functional of the branching particle system and its occupation time is given by*

$$\mathbb{E}\left[e^{-<\bar{X}_t, \psi> - \int_0^t <\bar{X}_s, \phi> ds}\right] = e^{-<G^* \times \Lambda, V_t^{\psi} \phi>},$$

where $V_t^\psi\phi$ satisfies

$$\frac{\partial}{\partial t}V_t^\psi\phi(u,x)=\mathcal{L}V_t^\psi\phi(u,x)-\lambda(u)[\Phi(1-V_t^\psi\phi(0,x))-(1-V_t^\psi\phi(0,x))]+\phi(u,x)(1-V_t^\psi\phi(u,x)), \quad (10)$$

$V_0\psi\phi(u,x)=1-e^{-\psi(u,x)}$.

Proof: First we note that, we can extend Proposition (4.1) to a time-dependent function ψ. For example, $W_{T-t}^\psi\phi$ for $t\in[0,T]$ with $T>0$ fixed, where $W_t^\psi\phi$ is the solution of the following partial differential equation

$$\frac{\partial}{\partial t}W_t^\psi\phi(u,x)=\mathcal{L}W_t^\psi\phi(u,x)+\lambda(u)[\Phi(W_t^\psi\phi(0,x))-W_t^\psi\phi(0,x)]-f(u,x)W_t^f g(u,x),$$

with initial condition $W_0^\psi\phi(u,x)=e^{-\psi(u,x)}$.

We see that, for $T>0$ fixed, (9) can be written as

$$M'_t:=e^{<\bar{X}_t,\log W_{T-t}^\psi\phi>}-\int_0^t<\bar{X}_s,\phi>e^{<\bar{X}_s,\log W_{T-s}^\psi\phi>}ds,$$

hence M' is a martingale on $[0,T]$.

Now, applying Corollary 2.3.3 of [6] to M' we get that

$$\tilde{M}_t:=\exp\left(<\bar{X}_t,\log W_{T-t}^\psi\phi>-\int_0^t<\bar{X}_s,\phi>ds\right),$$

is a martingale on $[0,T]$.

Then, taking expectations and ussing the martingale property of $\tilde{M}$ we have that

$$\begin{aligned}\mathbb{E}\left[e^{-<\bar{X}_t,\psi>-\int_0^t<\bar{X}_t,\phi>ds}\right] &= \mathbb{E}\left[e^{<\bar{X}_0,\log W_t^\psi\phi>}\right]\\ &= e^{-<G^*\times\Lambda,1-W_t^\psi\phi>},\end{aligned}$$

in the last idnetity we are using the fact that the initial population is Poissonian with intensity measure $G^*\times\Lambda$. Finally, to finish the proof denife $V_t^\psi\phi:=1-W_t^\psi\phi$.$\triangleleft$

The next Lemma gives the mean of the occupation time.

Lemma 6.3 *Let $\phi:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}_+$ be a measurable function with compact support. Then, for each $t\geq 0$*

$$\mathbb{E}_{G^*\times\Lambda}\mathcal{J}_t^\phi=<\phi,G^*\times\Lambda>t.$$

Proof: Given $k\geq 0$ define

$$\begin{aligned}L_t(k\phi) &= \mathbb{E}\left[e^{-k\int_0^t<\bar{X}_s,\phi>ds}\right]\\ &= e^{-<G^*\times\Lambda,V_t(k\phi)>}, \qquad (11)\end{aligned}$$

where $V_t(k\phi)$ satisfies (10) with ϕ substituted by $k\phi$ and $\psi \equiv 0$.

Now, we note that

$$
\begin{aligned}
\mathbb{E}_{G^*\times\Lambda}\mathcal{J}_t^{\phi} &= -\frac{d}{dk}\mathbb{E}_{G^*\times\Lambda}\exp -\mathcal{J}_t^{k\phi}\,|_{k=0^+} \\
&= <\frac{d}{dk}V_t^0(k\phi), G^*\times\Lambda> \exp - <V_t^0(k\phi), G^*\times\Lambda>|_{k=0^+}\,.
\end{aligned}
$$

Then, defining $\dot{V}_t\phi := \frac{d}{d\theta}V_t^0(\theta\phi)\,|_{k=0^+}$ and recalling that $V_t^0(0\phi)=0$, we obtain

$$
\begin{aligned}
\tfrac{\partial}{\partial t}\dot{V}_t\phi(u,x) &= \mathcal{L}\dot{V}_t\phi(u,x)+\phi(u,x) \\
\dot{V}_0\phi(u,x) &= 0.
\end{aligned}
$$

Therefore,

$$
\dot{V}_t\phi(u,x) = \int_0^t \tilde{T}_{t-s}\phi(u,x)ds,
$$

consecuently

$$
\begin{aligned}
\mathbb{E}_{G^*\times\Lambda}\mathcal{J}_t^{\phi} &= <\dot{V}_t\phi, G^*\times\Lambda> \\
&= <\int_0^t \tilde{T}_{t-s}\phi ds, G^*\times\Lambda> \\
&= \int_0^t <\phi, G^*\times\Lambda> ds \\
&= <\phi, G^*\times\Lambda> t.\triangleleft
\end{aligned}
$$

The next Lemma gives a bound for the variance of the occupation time.

Lemma 6.4 *Let $\phi : \mathbb{R}_+\times\mathbb{R}^d \to \mathbb{R}_+$ be a measurable function with compact support such that $\phi(u,x)=\phi_1(u)\phi_2(x)$ for all $(u,x)\in\mathbb{R}_+\times\mathbb{R}^d$. Then, for each $t\geq 0$*

$$
\begin{aligned}
Var\mathcal{J}_t^{\phi} \leq\;& \Phi''(1)K_{\phi_1}^2 <G^*,\lambda> Cte(\phi_2)(t+t^{3-d/\alpha}) \\
&+2K_{\phi_1}<G^*,\phi_1> Cte(\phi_2)(t+t^{2-d/\alpha}).
\end{aligned}
\tag{12}
$$

Proof: Deriving $V_t(k\phi)$ with respect to k (under the assumption that $\beta=1$) and ussing equation (10), we have that

$$
\begin{aligned}
\frac{\partial^2}{\partial t\partial K}V_t(k\phi)(u,x) =\;& A\frac{\partial}{\partial K}V_t(k\phi)(u,x)+\phi(u,x)(1-V_t(k\phi)(0,x))-k\phi(u,x)\frac{\partial}{\partial K}V_t(k\phi)(u,x) \\
&-\lambda(u)\left[-\Phi'(1-V_t(k\phi)(0,x))\frac{\partial}{\partial K}V_t(k\phi)(0,x)+\frac{\partial}{\partial K}V_t(k\phi)(0,x)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^3}{\partial t \partial K^2} V_t(k\phi)(u,x) &= A\frac{\partial^2}{\partial K^2} V_t(k\phi)(u,x) - 2\phi(u,x)\frac{\partial}{\partial K} V_t(k\phi)(u,x) - k\phi(u,x)\frac{\partial^2}{\partial K^2} V_t(k\phi)(u,x) \\
&\quad -\lambda(u)\left[\Phi''\left(1 - V_t(k\phi)(0,x)\right)\left(\frac{\partial}{\partial K} V_t(k\phi)(u,x)\right)^2\right. \\
&\quad \left. -\Phi'(1 - V_t(k\phi)(0,x))\frac{\partial^2}{\partial K^2} V_t(k\phi)(0,x) + \frac{\partial^2}{\partial K^2} V_t(k\phi)(0,x)\right]
\end{aligned}
$$

Letting $\ddot{V}_t\phi = \frac{\partial^2}{\partial K^2} V_t(k\phi)|_{k=0^+}$ we get that

$$
\frac{\partial}{\partial t}\ddot{V}_t\phi(u,x) = A\ddot{V}_t\phi(u,x) - \lambda(u)\Phi''(1)\left(\dot{V}_t\phi(0,x)\right)^2 - 2\phi(u,x)\dot{V}_t\phi(u,x). \tag{13}
$$

From Lemmma (6.3) and (13) we obtain

$$
\ddot{V}_t\phi(u,x) = \int_0^t \tilde{T}_s\left[-\lambda(u)\Phi''(1)\left(\int_0^s \tilde{T}_r\phi(u,x)dr\right)^2 - 2\phi(u,x)\int_0^s \tilde{T}_r\phi(u,x)dr\right] ds.
$$

Note that, Var $< T_t, \phi >= - < G^* \times \Lambda, \ddot{V}_t\phi(*,\bullet) >$. Then,

$$
\begin{aligned}
\text{Var} < \mathcal{J}_t, \phi > &= < G^* \times \Lambda, \int_0^t \tilde{T}_s\left[\lambda(*)\Phi''(1)\left(\int_0^s \tilde{T}_r\phi(*,\bullet)dr\right)^2\right. \\
&\qquad \left. +2\phi(*,\bullet)\int_0^s \tilde{T}_r\phi(*,\bullet)dr\right] ds > \\
&= \int_0^t < G^* \times \Lambda, \lambda(*)\phi''(1)\left(\int_0^s \tilde{T}_r\phi(*,\bullet)dr\right)^2 > ds \\
&\quad +2\int_0^t < G^* \times \Lambda, \phi(*,\bullet)\int_0^s \tilde{T}_r\phi(*,\bullet)dr > ds
\end{aligned}
$$

Let (A) and (B) be the first and second term, respectively, in the last equation. Then for $\phi(u,x) = \phi_1(x)\phi_2(x)$ and $t \geq 0$, $\tilde{T}_t\phi(u,x) = \mathcal{S}_t\phi_1(u)\mathcal{T}_t\phi_2(x)$.

Note that $< G^*, \lambda >< \infty$. In fact, by Fubini's Theorem we have that

$$
\begin{aligned}
< G^*, \lambda > &= \int_0^\infty \left(\int_0^t \lambda(s)ds\right) f(t)dt \\
&= \int_0^\infty \frac{f(s)}{1-F(s)}\left(\int_s^\infty f(t)dt\right) ds \\
&= 1.
\end{aligned}
$$

Afterward,

$$\begin{aligned}
(A) &= \int_0^t < G^* \times \Lambda, \lambda(*)\Phi''(1)\left(\int_0^s \mathcal{S}_r\phi_1(*)\mathcal{T}_r\phi_2(\bullet)dr\right)^2 > ds \\
&\leq \int_0^t < G^* \times \Lambda, \lambda(*)\Phi''(1)K_{\phi_1}^2\left(\int_0^s \mathcal{T}_r\phi_2(\bullet)dr\right)^2 > ds \\
&= \int_0^t < G^*, \lambda(*)\Phi''(1)K_{\phi_1}^2 >< \Lambda, \left(\int_0^s \mathcal{T}_r\phi_2(\bullet)dr\right)^2 > ds \\
&= \Phi''(1)K_{\phi_1}^2 < G^*, \lambda(*) > \int_0^t < \Lambda, \left(\int_0^s \mathcal{T}_r\phi_2(\bullet)dr\right)^2 > ds,
\end{aligned}$$

where K_{ϕ_1} is such that for all $t \geq 0$, $\|\mathcal{S}_t\phi_1\| \leq K_{\phi_1}$. Also, it can be proven that

$$\int_0^t < \Lambda, \left(\int_0^s \mathcal{T}_r\phi_2(\bullet)dr\right)^2 > ds \leq \mathrm{Cte}(\phi_2)(t + t^{3-d/\alpha}),$$

and consecuently,

$$(A) \leq \Phi''(1)K_{\phi_1}^2 < G^*, \lambda > \mathrm{Cte}(\phi_2)(t + t^{3-d/\alpha}).$$

Similarly, for ϕ_1 such that $< G^*, \phi_1 >< \infty$ we obtain that

$$\begin{aligned}
(B) &= 2\int_0^t < G^* \times \Lambda, \phi_1(*)\phi_2(\bullet)\int_0^s \mathcal{S}_r\phi_1(*)\mathcal{T}_r\phi_2(\bullet)dr > ds \\
&\leq 2\int_0^t < G^* \times \Lambda, \phi_1(*)\phi_2(\bullet)K_{\phi_1}\int_0^s \mathcal{T}_r\phi_2(\bullet)dr > ds \\
&= 2K_{\phi_1} < G^*, \phi_1(*) > \int_0^t < \Lambda, \phi_2(\bullet)\int_0^s \mathcal{T}_r\phi_2(\bullet)dr > ds,
\end{aligned}$$

where

$$\int_0^t < \Lambda, \phi_2(\bullet)\int_0^s \mathcal{T}_r\phi_2(\bullet)dr > ds \leq \mathrm{Cte}(\phi_2)(t + t^{2-d/\alpha}),$$

hence,

$$(B) \leq 2K_{\phi_1} < G^*, \phi_1 > \mathrm{Cte}(\phi)(t + t^{2-d/\alpha}).$$

Finally, adding the bounds for (A) and (B) we get the result.$\triangleleft$

Proof of Theorem 6.1. Applying Chebyshev's inequality we have that for each $t \geq 0$ and $\epsilon > 0$

$$P\left\{\frac{|< \mathcal{J}_t, \phi > - < G^* \times \Lambda, \phi >|}{t} > \epsilon\right\} \leq \frac{1}{t^2\epsilon^2}Var < \mathcal{J}_t, \phi > .$$

Let $a \in (1,\infty)$ and $k_n = a^n$, for $n \in \mathbb{N}$. Then, by Lemma 6.4

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left\{\frac{|<\mathcal{J}_{k_n},\phi> - <G^*\times\Lambda,\phi>|}{k_n} > \epsilon\right\} \\
\leq\ & \frac{1}{\epsilon^2}\sum_{n=1}^{\infty} Var <\mathcal{J}_{k_n},\phi> \\
\leq\ & \frac{1}{\epsilon^2}\left\{\Phi''(1)K_{\phi_1}^2 <G^*,\lambda> \mathrm{Cte}(\phi)\sum_{n=1}^{\infty}(k_n + k_n^{3-d/\alpha})/k_n^2\right. \\
& \left. + 2K_{\phi_1} <G^*,\phi_1> \mathrm{Cte}(\phi)\sum_{n=1}^{\infty}(k_n + k_2^{2-d/\alpha})/k_n^2\right\} \\
\leq\ & \frac{1}{\epsilon^2}\left\{\Phi''(1)K_{\phi_1}^2 <G^*,\lambda> \mathrm{Cte}(\phi)\sum_{n=1}^{\infty}(a^{-n} + a^{(1-d/\alpha)n})\right. \\
& \left. + 2K_{\phi_1} <G^*,\phi_1> \mathrm{Cte}(\phi)\sum_{n=1}^{\infty}(a^{-n} + a^{(-d/\alpha)^n}\right\} \\
<\ & \infty.
\end{aligned}$$

Hence, by Borel-Cantelli's Lemma

$$\frac{1}{k_n} <\mathcal{J}_{k_n},\phi> \stackrel{c.s.}{\to} <G^*\times\Lambda,\phi>,$$

as $n \to \infty$.

Now, let $t \geq 1$ be fixed, then there exists $n \in \mathbb{N}$ such that $a^n \leq t \leq a^{n+1}$ and

$$\frac{<\mathcal{J}_{a^n},\phi>}{a^{n+1}} \leq \frac{<\mathcal{J}_t,\phi>}{t} \leq \frac{<\mathcal{J}_{a^{n+1}},\phi>}{a^n},$$

then,

$$\frac{<G^*\times\Lambda,\phi>}{a} \leq \liminf_{t\to\infty}\frac{<\mathcal{J}_t,\phi>}{t} \leq \limsup_{t\to\infty}\frac{<\mathcal{J}_t,\phi>}{t} \leq <G^*\times\Lambda,\phi> a,$$

the last inequalities are true for all $a > 1$. Letting $a \to 1$ we get the result.

Remark 6.5 *If the lifetime has an exponential distribution with mean λ^{-1}, $\phi_1 \equiv 1$ and G^* changes to $\tilde{G}$ then Theorem 6.1 reduces to Theorem 4 of [15].*

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