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FREE INFINITE DIVISIBILITY AND FREE MULTIPLICATIVE
MIXTURES OF THE WIGNER DISTRIBUTION

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Free Infinite Divisibility of Free Multiplicative Mixtures of the Wigner Distribution

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Abstract

Let I^* and $I^{\boxplus}$ be the classes of all classical infinitely divisible distributions and free infinitely divisible distributions, respectively, and let Λ be the Bercovici-Pata bijection between I^* and $I^{\boxplus}$. The class type W of symmetric distributions in $I^{\boxplus}$ that can be represented as free multiplicative convolutions of the Wigner distribution is studied. A characterization of this class under the condition that the mixing distribution is 2-divisible with respect to free multiplicative convolution is given. A correspondence between symmetric distributions in $I^{\boxplus}$ and the free counterpart under Λ of the positive distributions in I^* is established. It is shown that the class type W does not include all symmetric distributions in $I^{\boxplus}$ and that it does not coincide with the image under Λ of the mixtures of the Gaussian distribution in I^*. Similar results for free multiplicative convolutions with the symmetric arcsine measure are obtained. Several well-known and new concrete examples are presented.

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Contents

1. Introduction

Let $\mathcal{P}$ denote the set of all Borel probability measures on $\mathbb{R}$ and let $\mathcal{P}_+$ and $\mathcal{P}_s$ be the sets of all Borel probability measures with support in $\mathbb{R}_+ = [0, \infty)$ and of all symmetric Borel probability measures (*i.e.* $\mu(B) = \mu(-B)$ for all Borel set B on $\mathbb{R}$), respectively.

The free additive convolution $\mu_1 \boxplus \mu_2$ is a binary operation from $\mathcal{P} \times \mathcal{P}$ to $\mathcal{P}$ that describes the spectral distribution of the sum $X + Y$ of two freely independent non-commutative random variables X and Y with spectral distributions μ_1 and μ_2, respectively, see [9], [14], [20], [28].

Free infinite divisibility of probability measures with respect to the free additive convolution $\boxplus$ has received increasing interest during the last years, see for example [7], [8], [9], [13], [15], [18], [25] and references therein. A key role in free infinite divisibility, similar to the Gaussian distribution in classical probability, is played by the Wigner or semicircle distribution w on $(-2, 2)$ defined as

$$\mathrm{w}(\mathrm{d}x) = \frac{1}{2\pi}\sqrt{4 - x^2}\,1_{(-2,2)}\mathrm{d}x. \tag{1}$$

On the other hand, if μ_1 and μ_2 are in $\mathcal{P}_+$, the free multiplicative convolution $\mu_1 \boxtimes \mu_2$ is the spectral distribution of $X^{1/2}YX^{1/2}$, where X and Y are freely independent positive non-commutative random variables with spectral distributions μ_1 and μ_2, respectively; see [9]. Recently, free multiplicative convolutions for μ_1 in $\mathcal{P}_+$ and μ_2 in $\mathcal{P}_s$ where considered in [2]; see also [23] when μ_1, μ_2 have bounded support. It was also shown in [2] that any free symmetric α-stable law μ_α, $0 < \alpha < 2$, can be written as $\mu_\alpha = \sigma_\beta \boxtimes \mathrm{w}$, where σ_β is a free positive β-stable law with $\beta = 2\alpha/(2 + \alpha)$.

The main purpose of the present paper is to study free multiplicative convolutions $\lambda \boxtimes \mathrm{w}$ of $\lambda \in \mathcal{P}_+$ with the Wigner distribution w. In particular, we are interested in a characterization

of the class of free type W distributions consisting of the measures $\lambda \boxtimes \mathrm{w}$ that are infinitely divisible with respect to the free additive convolution $\boxplus$.

The analogue problem in classical probability is the study of variance mixtures of the Gaussian distribution VZ, where $V > 0$ and Z are independent (classical) random variables, with Z normally distributed. Their product is distributed according to the classical multiplicative convolution of the laws of V and Z. It is well-known that VZ is infinitely divisible in the classical sense if V^2 is infinitely divisible and the law of VZ is called of *type G*, see [16], [24]. However, there are variance mixtures of the Gaussian distribution which are infinitely divisible but V^2 is not [16]. The relevance of classical type G distributions is that they are the distributions (at fixed time) of Lévy processes obtained by subordination of the classical one-dimensional Brownian motion.

A well-known analytic characterization of type G distributions is as follows. Recall that a probability measure μ on $\mathbb{R}$ belongs to the class I^* of all infinitely divisible distributions if and only if the logarithm of its Fourier transform, the so called the *classical cumulant transform* has the Lévy-Khintchine representation

$$\mathcal{C}_\mu^*(t) = ib_\mu t - \frac{1}{2}a_\mu t^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{itx} - 1 - itx1_{|x|\leq 1}\right)\nu_\mu(\mathrm{d}x), \quad t \in \mathbb{R}, \tag{2}$$

where $b_\mu \in \mathbb{R}, a_\mu \geq 0$, and ν_μ is a measure on $\mathbb{R}$ (called the *Lévy measure*) satisfying $\nu_\mu(\{0\}) = 0$ and $\int_{\mathbb{R}} \min(1, x^2)\nu_\mu(\mathrm{d}x) < \infty$. The triplet (a_μ, ν_μ, b_μ) is called the *classical generating triplet* of $\mu \in I^*$. We refer to the book by Sato [26] for the study of classical infinitely divisible distributions on $\mathbb{R}^d$. A distribution $\sigma \in I^*$ is concentrated on $\mathbb{R}_+$ if and only if it admits the *regular Lévy-Khintchine* representation

$$\mathcal{C}_\sigma^*(t) = ib_\sigma t + \int_{\mathbb{R}_+} \left(\mathrm{e}^{itx} - 1\right)\nu_\sigma(\mathrm{d}x), \quad t \in \mathbb{R}, \tag{3}$$

where $b_\sigma \geq 0$, $\nu_\sigma(-\infty, 0) = 0$ and $\int_{\mathbb{R}_+}(1 \wedge x)\nu(dx) < \infty$. Hence, if μ is the distribution of a type G random variable VZ, where V^2 has an infinitely divisible distribution σ, it holds that

$$\mathcal{C}_\mu^*(t) = \mathcal{C}_\sigma^*(it^2/2), \quad t \in \mathbb{R}. \tag{4}$$

The organization and the main results of the paper are as follows. In Section 2 we present preliminaries and notation on free additive $\boxplus$ and free multiplicative $\boxtimes$ convolutions, and the class $I^\boxplus$ of all $\boxplus$-infinitely divisible distributions. In particular we present the class of positive regular distributions in $I^\boxplus$. Section 3 introduces the concept of $\boxtimes$–2 divisible distributions in $\mathcal{P}_+$ and several examples and counter-examples of $\boxtimes$–2 divisible distributions are presented. It is shown that the free Poisson distributions $\mathrm{m}_c, c \geq 1$, are $\boxtimes$–2 divisible, but m_c is not for $c < 1$ small enough.

In Section 4 we consider symmetric distribution in $I^\boxplus$ and their corresponding positive regular distribution in $I^\boxplus$ for which an interesting relation between the corresponding free cumulant transforms similar to (4) is proved. This is surprisingly contrary to the classical case where (4) does not hold for all symmetric distributions in I^*. The class of free type G distributions considered in [3] is studied in the framework of this new relation.

In Section 5 we elaborate about free multiplicative mixtures with the Wigner distribution. We characterize the class of free type W distributions $\overline{\sigma} \boxtimes \mathrm{w}$, $\overline{\sigma} \in \mathcal{P}_+$, as those symmetric distributions in $I^\boxplus$ such that $\sigma = \overline{\sigma} \boxtimes \overline{\sigma}$ is positive regular in $I^\boxplus$ and $\boxtimes$–2 divisible. It is also

shown that the distribution in $\mathcal{P}_+$ induced by the the transformation $x \to x^2$ under $\overline{\sigma} \boxtimes \mathrm{w}$, $\overline{\sigma} \in \mathcal{P}_+$, is always $\boxplus$-infinitely divisible and moreover it is a free compound Poisson distribution. It is shown that there are free type W distributions that are not free type G distribution and that the Wigner measure is the free multiplicative convolution of the symmetric arcsine distribution but that the converse does not hold. Moreover, we prove that the class of free type W distributions is not the class of all symmetric free infinitely divisible distributions.

Finally, in Section 6 we consider symmetric distributions in $I^{\boxplus}$ which are free multiplicative convolutions with the symmetric arcsine distribution. We show that this class contains the free type W distributions and the free type G distributions but does not coincide with the full class of symmetric $\boxplus$-infinitely divisible distributions.

2. Preliminaries on Free Convolutions and Notation

In this section we collect some preliminary results and examples on free infinitely divisible distributions that are used in the remaining of this work.

2.1. Free additive convolution $\boxplus$ and infinite divisibility

Let $\mathbb{C}^+$ and $\mathbb{C}^-$ be the sets of all complex numbers satisfying $\mathrm{Im}(z) > 0$ and $\mathrm{Im}(z) < 0$, respectively. First, for any probability measure μ on $\mathbb{R}$, **the Cauchy transform** $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ is defined by

$$G_\mu(z) = \int_\mathbb{R} \frac{1}{z-x} \mu(\mathrm{d}x), \quad z \in \mathbb{C}^+.$$

The reciprocal of the Cauchy transform $F_\mu : \mathbb{C}^+ \to \mathbb{C}^+$ of $\mu \in \mathcal{P}$ is defined as $F_\mu(z) = \frac{1}{G_\mu(z)}$. It was shown in [9] that the right inverse function $F_\mu^{-1}(z)$ of $F_\mu(z)$ (i.e. $F_\mu(F_\mu^{-1}(z)) = z$) is defined on a region $\Gamma_{\eta,M} := \{z \in \mathbb{C}; |\mathrm{Re}(z)| < \eta \mathrm{Im}(z), \quad \mathrm{Im}(z) > M\}$. This allows us to define **free cumulant transform** (or R-transform) of a probability measure μ on $\mathbb{R}$ as $\mathcal{C}_\mu^{\boxplus}(z) = zF_\mu^{-1}(z^{-1}) - 1$, $z^{-1} \in \Gamma_{\eta,M}$.

From the analytic point of view, the **free additive convolution** of two probability measures μ_1, μ_2 on $\mathbb{R}$ is defined as the probability measure $\mu_1 \boxplus \mu_2$ on $\mathbb{R}$ such that

$$\mathcal{C}_{\mu_1 \boxplus \mu_2}^{\boxplus}(z) = \mathcal{C}_{\mu_1}^{\boxplus}(z) + \mathcal{C}_{\mu_2}^{\boxplus}(z) \quad z^{-1} \in \Gamma_{\eta,M} \tag{5}$$

for z in the common domain where $\mathcal{C}_{\mu_1}^{\boxplus}$ and $\mathcal{C}_{\mu_2}^{\boxplus}$ are defined.

A probability measure μ on $\mathbb{R}$ is **free infinitely divisible** (in short $\boxplus$–ID) if for any $n \in \mathbb{N}$ there exists a probability measure $\mu_{1/n}$ on $\mathbb{R}$ such that $\mu = \mu_{1/n} \boxplus \cdots \boxplus \mu_{1/n}$ (n times). We denote the class of all $\boxplus$–ID distributions by $I^{\boxplus}$. As in the classical case, there is a **free Lévy-Khintchine formula** due to Bercovici and Voiculescu [10]. In terms of (5), $\mu \in I^{\boxplus}$ if and only if

$$\mathcal{C}_\mu^{\boxplus}(z) = b_\mu z + a_\mu z^2 + \int_\mathbb{R} \left(\frac{1}{1-zx} - 1 - zx 1_{[-1,1]}(x) \right) \nu_\mu(\mathrm{d}x), \quad z \in \mathbb{C}^-, \tag{6}$$

where $b_\mu \in \mathbb{R}, a_\mu \geq 0$ and ν_μ, the Lévy measure, is such that $\nu_\mu(\{0\}) = 0$ and $\int_\mathbb{R} (1 \wedge |x|^2) \nu_\mu(\mathrm{d}x) < \infty$. The triplet (a_μ, ν_μ, b_μ) is unique. When we consider $\boxplus$–ID distributions, the free cumulant transform can be defined in the lower half plane $\mathbb{C}^-$. A probability measure μ is symmetric if and only if the Lévy measure ν_μ is symmetric, $b_\mu = 0$ and

$$\mathcal{C}_\mu^{\boxplus}(z) = a_\mu z^2 + \int_\mathbb{R} \left(\frac{1}{1-zx} - 1 \right) \nu_\mu(\mathrm{d}x), \quad z \in \mathbb{C}^-. \tag{7}$$

The **Bercovici-Pata bijection** $\Lambda : I^* \to I^{\boxplus}$ between classical and free infinitely divisible distributions was introduced in [8]. It is such that if $\mu \in I^*$ has classical triplet (a_μ, ν_μ, b_μ) then $\Lambda(\mu) \in I^{\boxplus}$ has free triplet (a_μ, ν_μ, b_μ).

In this work we consider free multiplicative convolutions with the following key examples. The **Wigner or semicircle distribution** $w_{b,a}$, with parameters $-\infty < b < \infty, a > 0$ is defined as

$$w_{b,a}(dx) = \frac{1}{2\pi a}\sqrt{4a - (x-b)^2}1_{[b-\sqrt{4a}, b+\sqrt{4a}]}(x)dx$$

and has the free cumulant transform $\mathcal{C}^{\boxplus}_{w_{b,a}}(z) = az^2 + bz$. The parameters b and a are the mean and the variance of this distribution, respectively. It is such that $w_{b,a} = \Lambda(\gamma_{b,a})$, where $\gamma_{b,a}$ is the classical Gaussian distribution with mean b and variance a. For this reason $w_{b,a}$ is also called the **free Gaussian distribution.** Especially, we simply write $w = w_{0,1}$, which is corresponding to the standard Gaussian distribution.

Another important example of a free infinitely divisible distribution is the **Marchenko-Pastur distribution** m_c with parameter $c > 0$, given by

$$m_c(dx) = \max(0, (1-c))\delta_0(dx) + \frac{1}{2\pi x}\sqrt{4c - (x-1-c)^2}1_{[(1-\sqrt{c})^2, (1+\sqrt{c})^2]}(x)dx. \quad (8)$$

It holds that $m_c = \Lambda(p_c)$ where p_c is the classical Poisson distribution of mean $c > 0$ and it has free triplet $(c, c\delta_1, 0)$, where δ_1 is the (probability) measure concentrated at one. For this reason m_c is also called the **free Poisson distribution.** In the case $c = 1$, we simply write $m = m_1$.

We now consider the case of free infinitely divisible distributions with nonnegative support, for which we have a situation different than for the classical case (3), where the drift is nonnegative, the Lévy measure is concentrated on $\mathbb{R}_+$ and there is not Gaussian part. In the free case we consider two situations. First, following a similar terminology as in [21], we propose to call a distribution $\sigma \in I^{\boxplus}_+$ **free regular** or simply regular, if its Lévy Khintchine representation is given by

$$\mathcal{C}^{\boxplus}_\sigma(z) = b_\sigma z + \int_{\mathbb{R}_+}\left(\frac{1}{1-zx} - 1\right)\nu_\sigma(dx), \quad z \in \mathbb{C}^-, \quad (9)$$

where $b_\sigma \geq 0$, $\nu_\sigma((-\infty, 0]) = 0$ and $\int_0^\infty \min(1, x)\nu_\sigma(dx) < \infty$. Not all nonnegative free infinitely divisible distributions are regular. For example, the standard Wigner distribution translated by 2, that is $w_{2,1}(dx) = \Lambda(\gamma_{2,1})(dx) = \frac{1}{2\pi}\sqrt{4 - (x-2)^2}1_{[0,4]}(x)dx$ has support on $(0, 4)$, its free triplet is $(1, 0, 2)$ and hence there is a non-zero Wigner part $a_{w_{2,1}} = 1$. Let $I^{\boxplus}_{r+}$ be the class of all regular distribution in $I^{\boxplus}$. It holds that $\Lambda(I^*_+) = I^{\boxplus}_{r+}$, but $I^{\boxplus}_{r+}$ is not equal to $\{\mu \in I^{\boxplus} \mid \text{the support of } \mu \text{ is in } \mathbb{R}_+\}$. The free Poisson distribution for all $c > 0$ is free regular infinitely divisible.

When a distribution μ has all moments, the **free cumulants** κ_n are the coefficients of the formal expansion

$$\mathcal{C}^{\boxplus}_\mu(z) = \sum_{n=1}^\infty \kappa_n z^n.$$

The following is a easy necessary criteria for free infinite divisibility which is used repeatedly in this work. It is based on the first four cumulants of a distribution. It is the analogue to a criterion in classical infinite divisibility in [27, pp 181]. It was pointed out to us by O.

Arizmendi. Its proof follows from the fact that the classical cumulant sequence (c_n) of $\mu \in I^*$ (i.e. the coefficients (c_n) of the series expansion

$$\mathcal{C}_\mu^*(t) = \sum_{n=1}^\infty \frac{c_n}{n!} t^n, \quad t \in \mathbb{R},$$

of the classical cumulant transform) coincide with the free cumulant sequence (κ_n) of $\Lambda(\mu)$. For other criteria using free cumulants see the recent paper [17]. The book [20] elaborates in general conditions related to the moment problem.

Lemma 1. *Let μ be an $\boxplus$–ID measure having four finite moments. Then $\kappa_2\kappa_4 \geq \kappa_3^2$.*

An example of a distribution that is not free infinitely divisible but plays an important role in this work is the **symmetric arcsine distribution** with parameter s

$$\mathrm{a}_s(\mathrm{d}x) = \frac{1}{\pi} \frac{1}{\sqrt{s - x^2}} 1_{(-\sqrt{s}, \sqrt{s})}(x)\mathrm{d}x. \tag{10}$$

If the parameter $s = 1$, we use the notation a for a_1. It is interesting that the distribution arises as the additive convolution $\mathrm{a} = \mathrm{d} \boxplus \mathrm{d}$, where d is the symmetric Bernoulli (atomic) measure $\mathrm{d} = \frac{1}{2}(\delta_{-1} + \delta_1)$.

2.2. Free multiplicative convolution $\boxtimes$ and the S-transform

For a probability measure μ on $\mathbb{R}$, the p–th push–forward measure of $\mu^{(p)}$ of μ is defined as

$$\mu^{(p)}(B) = \int_{\mathbb{R}} 1_B(|x|^p)\mu(\mathrm{d}x), \quad B \in \mathcal{B}((0,\infty)).$$

It is trivial to see that $\mathrm{w}^{(2)} = \mathrm{m}$ and $\mathrm{a}^{(2)} = \mathrm{a}^+$, where a^+ is is the positive arcsine law on $(0,1)$ given by (21) below.

To study the free "product" $\boxtimes$ of probability measures, it is useful to consider another analytic tool called the **S-transform,** which is defined as follows. For $\mu \in \mathcal{P}_+$ define

$$\Psi_\mu(z) = \int_{\mathbb{R}} \frac{zx}{1 - zx}\mu(\mathrm{d}x) = z^{-1}G_\mu(z^{-1}) - 1, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{11}$$

It was proved in [9] that for probability measures with support on $\mathbb{R}_+$ and such that $\mu(\{0\}) < 1$, the function $\Psi_\mu(z)$ has a unique inverse $\chi_\mu(z)$ in the left-half plane $i\mathbb{C}^+$ and $\Psi_\mu(i\mathbb{C}^+)$ is a region contained in the circle with diameter $(\mu(\{0\}) - 1, 0)$. In this case the S-transform of μ is defined as $S_\mu(z) = \chi_\mu(z)\frac{1+z}{z}$ It satisfies $z = \mathcal{C}_\mu^\boxplus(zS_\mu(z))$ for sufficiently small $z \in \Psi_\mu(i\mathbb{C}^+)$.

Following [9], the **free multiplicative convolution** of a probability measure μ_1, μ_2 supported on $\mathbb{R}_+$ is defined as the positive probability measure $\mu_1 \boxtimes \mu_2$ on $\mathbb{R}$ such that

$$S_{\mu_1\boxtimes\mu_2}(z) = S_{\mu_1}(z)S_{\mu_2}(z) \tag{12}$$

for z in a common region of $\Psi_{\mu_1}(i\mathbb{C}^+) \cup \Psi_{\mu_2}(i\mathbb{C}^+)$.

The definition of S-transform was extended for symmetric probability measures $\mu \in \mathcal{P}_s$ in [2]. Let $H = \{z \in \mathbb{C}^-; \ |\mathrm{Re}(z)| < |\mathrm{Im}(z)|\}$, $\widetilde{H} = \{z \in \mathbb{C}^+; \ |\mathrm{Re}(z)| < \mathrm{Im}(z)\}$. It was

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proved in [2] that when $\mu \in \mathcal{P}_s$ with $\mu(\{0\}) < 1$, the transform Ψ_μ has a unique inverse on H, $\chi_\mu : \Psi_\mu(H) \to H$ and a unique inverse on $\widetilde{H}$, $\widetilde{\chi}_\mu : \Psi_\mu(\widetilde{H}) \to \widetilde{H}$. In this case there are two S-transforms for μ given by

$$S_\mu(z) = \chi_\mu(z)\frac{1+z}{z} \text{ and } \widetilde{S}_\mu(z) = \widetilde{\chi}_\mu(z)\frac{1+z}{z} \tag{13}$$

and these are such that

$$S_\mu^2(z) = \frac{1+z}{z}S_{\mu^{(2)}}(z) \text{ and } \widetilde{S}_\mu^2(z) = \frac{1+z}{z}S_{\mu^{(2)}}(z) \tag{14}$$

for z in $\Psi_\mu(H)$ and $\Psi_\mu(\widetilde{H})$, respectively. Moreover the following result holds.

Lemma 2. *Assume that $\mu \in \mathcal{P}_s \cup \mathcal{P}_+$. For some sufficiently small $\varepsilon > 0$, we have a region D_ε that includes $\{-it; 0 < t < \varepsilon\}$ such that*

$$z = \mathcal{C}_\mu^{\boxplus}(zS_\mu(z)) \tag{15}$$

for $z \in D_\varepsilon$.

Proof For $\mu \in \mathcal{P}_+$ see [19]. Let $\mu \in \mathcal{P}_s$. We take some $\alpha > 0$ and $\beta > 0$ such that there exists $F_\mu^{-1}(z)$ for $z \in \Gamma_{\alpha,\beta}$. Then we have the inverse in $\{z \in \mathbb{C}^-; |z| > \alpha\Im z, |z| < 1/\beta\}$. From $\Psi_\mu(z) = \Psi_{\mu^{(2)}}(z^2)$, and [9, Proposition 6.1], $\lim_{it \to 0, t<0} \Psi_\mu(it) = 0$. Then $\lim_{t \to 0, t<0} \tilde{\chi}_\mu(t) = 0$. If we take sufficiently small $\varepsilon > 0$, $z\tilde{S}_\mu(z) = (z+1)\tilde{\chi}_\mu(z)$ maps from sufficiently small domain that contains $\{-it; 0 < t < \varepsilon' < \varepsilon\}$ to $\{z \in \mathbb{C}^-; |z| > \alpha\Im z, |z| < 1/\beta\}$ for some ε' smaller than ε. Then we always have some region where we have (15) from [2]. ∎

Following [2], the **free multiplicative convolution** of a probability measure μ_1 supported on $\mathbb{R}_+$ with a symmetric probability measure μ_2 on $\mathbb{R}$ is defined as the symmetric probability measure $\mu_1 \boxtimes \mu_2$ on $\mathbb{R}$ such that

$$S_{\mu_1 \boxtimes \mu_2}(z) = S_{\mu_1}(z)S_{\mu_2}(z). \tag{16}$$

It was also shown in [2] that

$$\mu_1 \boxtimes \mu_2^{(2)} \boxtimes \mu_1 = (\mu_1 \boxtimes \mu_2)^{(2)}. \tag{17}$$

For the convenience of the reader, we include other examples of the S-transform of important examples, which appear repeatedly in this paper. For the Wigner measure $\mathrm{w}_{0,a}$ with zero mean and variance a

$$S_{\mathrm{w}_{0,a}}(z) = \sqrt{\frac{1}{az}} \tag{18}$$

For the Marchenko-Pastur measure m_c with parameter $c > 0$

$$S_{\mathrm{m}_c}(z) = \frac{1}{z+c}. \tag{19}$$

For the arcsine distribution (10)

$$S_{\mathrm{a}_s}(z) = \sqrt{\frac{z+2}{sz}}. \tag{20}$$

For the **positive arcsine distribution** on $(0, s)$

$$\mathrm{a}_s^+(\mathrm{d}x) = \frac{1}{\pi} \frac{1}{\sqrt{x(s-x)}} 1_{(0,s)}(x)\mathrm{d}x \tag{21}$$

its S-transform is

$$S_{\mathrm{a}_s^+}(z) = \frac{z+2}{s(z+1)}. \tag{22}$$

When $s = 1$, we use the notation a^+ for a_1^+.

2.3. $\boxplus$-compound Poisson distributions

Using the Bercovici-Pata bijection we define **free compound Poisson distributions**. For a combinatorial treatment of the free compound Poisson distribution see the book [20].

Definition 3. *Let σ be a probability measure with $\sigma(\{0\}) = 0$ and let c be positive number. a) μ is compound Poisson distribution (c, σ) if its cumulant transform can be represented as*

$$\mathcal{C}_\mu^*(t) = c(\hat{\sigma}(t) - 1) = c \int_{\mathbb{R}} (\exp(itx) - 1)\sigma(\mathrm{d}x), \quad t \in \mathbb{R}.$$

b) μ is free compound Poisson distribution (c, σ) on $\mathbb{R}$ if $\Lambda^{-1}(\mu)$ is a classical compound Poisson distribution (c, σ). In this case

$$\mathcal{C}_\mu^\boxplus(z) = c \int_{\mathbb{R}_+} \left(\frac{1}{1-zx} - 1 \right) \sigma(\mathrm{d}x) \quad z \in \mathbb{C}^-. \tag{23}$$

We denote by $\mu^{\boxplus c}$ the c times free additive convolution of $\mu \in \mathcal{P}$.

Proposition 4. *a) If a probability measure μ in $I_{r+}^\boxplus$ or $I_s^\boxplus$ is free compound Poisson distribution (c, σ), then $\mu^{\boxplus 1/c} = m \boxtimes \sigma$.*

b) If $\mu = m \boxtimes \sigma$ for some σ in $\mathcal{P}_+$ or $\mathcal{P}_s$ respectively, then μ is the free compound Poisson distribution.

c) If $\sigma \in \mathcal{P}_+$, then $\mu = m \boxtimes \sigma \in I_{r+}^\boxplus$.

Proof a) Since μ is free infinitely divisible, for any $c > 0$ the measure $\mu^{\boxplus 1/c}$ exists. Therefore we have that $\mathcal{C}_{\mu^{\boxplus 1/c}}^\boxplus(z) = \int_{\mathbb{R}_+} \left(\frac{1}{1-zx} - 1 \right) \sigma(\mathrm{d}x)$ for $z \in \mathbb{C}^-$. Now, using (11) we have $\int_{\mathbb{R}} (\frac{zx}{1-zx})\sigma(\mathrm{d}x) = \int_{\mathbb{R}} (\frac{1}{1-zx} - 1)\sigma(\mathrm{d}x) = \Psi_\sigma(z)$. Then

$$\mathcal{C}_{\mu^{\boxplus 1/c}}^\boxplus(zS_{\mu^{\boxplus 1/c}}(z)) = z = \Psi_\sigma(zS_{\mu^{\boxplus 1/c}}(z)), \quad \text{for sufficiently small } z \in \Psi_{\mu^{\boxplus 1/c}}(i\mathbb{C}^+),$$

and therefore $\chi_\sigma(z) = zS_{\mu^{\boxplus 1/c}}(z)$. Thus from (13) and (19), $S_{\mu^{\boxplus 1/c}}(z) = S_m(z)S_\sigma(z)$ and we conclude that $\mu^{\boxplus 1/c} = m \boxtimes \sigma$.

b) Assume that $\mu = m \boxtimes \nu$. The S-transform of μ is $S_\mu(z) = \frac{1}{1+z}S_\nu(z) = \frac{1}{z}\chi_\nu(z)$. Therefore $\Psi_\nu(zS_\mu(z)) = z = \mathcal{C}_\mu^\boxplus(zS_\mu(z))$. We have $\mathcal{C}_\mu^\boxplus(z) = \Psi_\nu(z) = \int_{\mathbb{R}} (\frac{zx}{1-zx})\nu(\mathrm{d}x) = \int_{\mathbb{R}} (\frac{1}{1-zx} - 1)\nu(\mathrm{d}x)$. We then conclude that $\mu = \sigma \boxtimes \nu$ is free compound Poisson distribution.

(c) From (b), the Lévy measure of μ is σ. So, $\int_{\mathbb{R}_+} \min(1, x)\sigma(\mathrm{d}x) < \infty$ and σ is concentrated on $\mathbb{R}_+$. Therefore μ is $I_{r+}^\boxplus$. $\blacksquare$

3. ⊠–2 Divisibility of Probability Measures

In this section we consider the concept of ⊠–2 divisibility that it is used in Section 5 to characterize free type W distributions.

Definition 5. *A probability measure $\sigma \in \mathcal{P}_+$ is called ⊠-**2 divisible** if there exists a probability measure $\overline{\sigma} \in \mathcal{P}_+$ such that $\sigma = \overline{\sigma} \boxtimes \overline{\sigma}$.*

Bercovici and Voiculescu [9] consider the more general concept of infinite divisibility of probability measures on $\mathbb{R}_+$ with respect to the free multiplicative convolution ⊠. Namely, a probability measure μ on $\mathbb{R}_+$ is ⊠–infinitely divisible if for any $n \in \mathbb{N}$, there exists $\mu_n \in \mathcal{P}_+$ such that

$$\mu = \underbrace{\mu_n \boxtimes \mu_n \boxtimes \cdots \boxtimes \mu_n}_{n \text{ times}}.$$

Of course, any ⊠-infinitely divisible distribution is ⊠–2 divisible. For results on indecomposable measures we refer to the recent paper [11].

Examples of free multiplicative infinitely divisible distributions are the following.

Example 6. *(1) δ_0 is ⊠–infinitely divisible.*
(2) The free Poisson distribution m_c is ⊠–infinitely divisible if and only if $c \geq 1$. This is because if

$$\Sigma_{\mathrm{m}_c}(z) = S_{\mathrm{m}_c}\left(\frac{z}{1-z}\right) = \exp\left(-\log\frac{1-z}{(1-c)z+c}\right),$$

the function $v(z) := -\log\frac{1-z}{(1-c)z+c}$ is analytic on $\mathbb{C}\backslash\mathbb{R}_+$, $v(\overline{z}) = \overline{v(z)}$ and $v(\mathbb{C}^+) \subset \mathbb{C}^-$ if $c \geq 1$, and not analytic on $\mathbb{C}\backslash\mathbb{R}_+$ if $c < 1$. Hence Theorem 6.13. in [9] implies that m_c is ⊠-infinitely divisible if and only if $c \geq 1$ (see also Theorem 1.2. in [5]).
(3) The positive $\boxplus$–stable laws, with index $0 < \alpha < 1$, are also ⊠–infinitely divisible. This follows from [8, Proposition A 4.4].

We are able to obtain a class of examples of ⊠–2 divisible distributions, from distributions using Fuss-Catalan numbers $A_m(p,r)$ recently constructed in Młotkowski [18]. The Fuss-Catalan numbers are defined for, $p \in \mathbb{R}$ and $r \in \mathbb{R}$, as follows: $A_0(p,r) = 1$ and

$$A_m(p,r) := \frac{r}{m!}\prod_{i=1}^{m-1}(mp+r-i) \quad \text{if} \quad m \geq 1.$$

It was proved in [18] that when $p \geq 1$ and $0 \leq r \leq p$, the Fuss-Catalan numbers $\{A_m(p,r)\}_{m=1}^{\infty}$ are the moments of a probability measure concentrated on $\mathbb{R}_+$, denoted by $\mu_{(p,r)}$. We call $\mu_{(p,r)}$ the Młotkowski distribution with parameter p,r and the set of all such measures the Młotkowski class. Furthermore the following was shown in [18].

Lemma 7. *(a) The free cumulant sequence of $\mu_{(p,r)}$ is $\{A_m(p-r,r)\}_{m=1}^{\infty}$.*
(b) If $0 \leq 2r \leq p$ and $r + 1 \leq p$ then $\mu_{(p,r)}$ is $\boxplus$-infinitely divisible.
(c) $\mu_{(p_1,r)} \boxtimes \mu_{(1+p_2,1)} = \mu_{(p_1+rp_2,r)}$ for $r \neq 0$.

With the above lemma we can construct examples of ⊠–2 divisible distributions and consider their $\boxplus$-infinitely divisibility.

Example 8. *(1) Since all the cumulants of the Marchenko-Pastur distribution* $\mathrm{m} = \mathrm{m}_1$ *are equal to one, from (a) and (c) in the above example we have that* $\mathrm{m} = \overline{\mathrm{m}} \boxtimes \overline{\mathrm{m}}$ *is* $\boxtimes$*–2 divisible with* $\overline{\mathrm{m}} = \mu_{(3/2,1)}$.

(2) It is easy to see that $\mu_{(3/2,1)}$ *does not satisfy the condition in Lemma 1. Then* $\mu_{(3/2,1)}$ *is not* $\boxplus$*–infinitely divisible.*

(3) From Lemma 1 we have that the measure $\mu_{(5/4,1)}$ *is not* $\boxplus$*–infinitely divisible. This is because the first free cumulants of* $\mu_{(5/4,1)}$ *are* $\kappa_2 = 1/2$, $\kappa_3 = 3/24$ *and* $k_4 = 0$ *and they do not satisfy the condition in Lemma 1. However, the distribution* $\mu_{(3/2,1)} = \mu_{(5/4,1)} \boxtimes \mu_{(5/4,1)}$ *is* $\boxtimes$*–2 divisible but not* $\boxtimes$*–infinitely divisible.*

From Example 6 we have the following result.

Proposition 9. *For* $c \geq 1$ *the free Poisson distribution* m_c *is* $\boxtimes$*–2 divisible.*

On the other hand, for $0 < c < 1$ sufficiently small we can prove that m_c is not $\boxtimes$-2 divisible.

Proposition 10. *For* $0 < c < 1$ *sufficiently small the function* $S_{m_c}(z) = \frac{1}{\sqrt{z+2c}}$ *is not the S-transform of a probability measure on* $\mathbb{R}_+$.

Proof Assume there exists a probability measure σ with the S–transform $S_\sigma(z) = \frac{1}{z+c}$. From definition of the S–transform (13) we have

$$\Psi_\sigma\left(\frac{z}{1+z}S_\sigma(z)\right) = \Psi_\sigma\left(\frac{z}{(1+z)\sqrt{z+c}}\right) = z.$$

Consider the expansion

$$\frac{z}{(1+z)\sqrt{z+c}} = \frac{z}{\sqrt{c}} + \frac{(-2c-1)z^2}{2c^{3/2}} + \frac{\left(8c^2+4c+3\right)z^3}{8c^{5/2}} + \frac{\left(-16c^3-8c^2-6c-5\right)z^4}{16c^{7/2}} + O\left(z^5\right),$$

the inverse-series satisfies

$$\Psi_\sigma(z) = \sqrt{c}z + \frac{1}{2}(2c+1)z^2 + \frac{\left(8c^2+12c+1\right)z^3}{8\sqrt{c}} + \left(c^2+3c+1\right)z^4 + O\left(z^5\right).$$

So the first three moments of σ are $m_1(\sigma) = \sqrt{c}$, $m_2(\sigma) = \frac{1}{2}(2c+1)$, $m_3(\sigma) = \frac{8c^2+12c+1}{8\sqrt{c}}$. Then $\det(m_{i+j}(\sigma))_{0\leq i\leq 2, 0\leq j\leq 2} = \frac{8c-1}{64t}$. If we take $c < 1/8$, $\det(m_{i+j}(\sigma))_{0\leq i\leq 2, 0\leq j\leq 2} < 0$, which is a contradiction to the Stieltjes moment problem. ∎

From the above examples we have the following summary which is a useful result for the study of free type W distributions in Section 5.

Proposition 11. *(1) Let* $\overline{\sigma} = \mu_{(3/2,1)}$. *Then* $\overline{\sigma} \notin I_{r+}^{\boxplus}$ *but* $\sigma = \overline{\sigma} \boxtimes \overline{\sigma} \in I_{r+}^{\boxplus}$.
(2) Let $\overline{\sigma} = \mu_{(5/4,1)}$. *Then* $\overline{\sigma} \notin I_{r+}^{\boxplus}$ *and* $\sigma = \overline{\sigma} \boxtimes \overline{\sigma} \notin I_{r+}^{\boxplus}$.
(3) $\{\overline{\sigma} \in \mathcal{P}_+; \overline{\sigma} \boxtimes \overline{\sigma} \in I_{r+}^{\boxplus}\} \neq \mathcal{P}_+$.

It is an open problem to find a free regular probability measure $\overline{\sigma} \in I_{r+}^{\boxplus}$ such that $\sigma = \overline{\sigma} \boxtimes \overline{\sigma}$ is not $\boxplus$–infinitely divisible. We have the following table

Table 1: $\boxtimes$–2 divisibility and $\boxplus$–regular ID

σ \ $\overline{\sigma}$	$I_{r+}^{\boxplus}$	not $I_{r+}^{\boxplus}$
$I_{r+}^{\boxplus}$	$\mathrm{m}^{\boxtimes 2} = \mathrm{m} \boxtimes \mathrm{m}$ and $\boxplus$–stable case	$\mathrm{m} = \mu_{(3/2,1)} \boxtimes \mu_{(3/2,1)}$
not $I_{r+}^{\boxplus}$	No example	$\mu_{(3/2,1)} = \mu_{(5/4,1)} \boxtimes \mu_{(5/4,1)}$

4. Symmetric $\boxplus$–Infinitely Divisible Distributions

In this section we prove an interesting relation between the free cumulant transform of a symmetric free infinitely divisible distributions and that of an associated regular positive free infinitely divisible distributions. As to our knowledge, there is not such a relation in the classical case. We consider the particular case of the free type G distribution studied in [3] as the image of the classical type G distributions under the Bercovici-Pata bijection Λ. We prove that free type G distributions can be represented as mixtures of free multiplicative convolutions.

Using the push-forward notation of a measure, for a probability measure μ on $\mathbb{R}_+$, let $\mu^{(1/2)+}$ and $\mu^{(1/2)-}$ be the induced measures by μ on $(0,\infty)$ and $(-\infty,0)$ under the mappings $\sqrt{x} \to x$ and $-\sqrt{x} \to x$, respectively. We observe that $\mu^{(1/2)+}$ and $\mu^{(1/2)-}$ are the Lévy measures if μ is Lévy measure of a free regular infinitely divisible distribution.

4.1. A characterization

Theorem 12. $\mu \in I_s^{\boxplus}$ if and only if there is $\sigma \in I_{r+}^{\boxplus}$ such that

$$\mathcal{C}_\mu^{\boxplus}(z) = \mathcal{C}_\sigma^{\boxplus}(z^2). \quad z \in \mathbb{C} \backslash \mathbb{R}. \tag{24}$$

Moreover, the Lévy measures of μ and σ are related by a symmetrization

$$\nu_\mu = \frac{1}{2}\left(\nu_\sigma^{(1/2)+} + \nu_\sigma^{(1/2)-}\right) \tag{25}$$

and

$$\nu_\sigma = 2\nu_\mu^{(2)}. \tag{26}$$

Remark 13. a) Since $\mathcal{C}_{\mathrm{w}}^{\boxplus}(z) = z^2$ for the Wigner distribution w, from (24) we have that for any $\mu \in I_s^{\boxplus}$, it holds that there is $\sigma \in I_{r+}^{\boxplus}$ such that $\mathcal{C}_\mu^{\boxplus}(z) = \mathcal{C}_\sigma^{\boxplus}(\mathcal{C}_{\mathrm{w}}^{\boxplus}(z))$, $z \in \mathbb{C} \backslash \mathbb{R}$. In the classical case, we have from (4) that if γ_1 is the standard classical Gaussian distribution, it holds that only for for some $\mu' \in I_s^*$, $\mathcal{C}_{\mu'}^*(t) = \mathcal{C}_{\sigma'}^*(i\mathcal{C}_{\gamma_1}^*(t)), t \in \mathbb{R}$, with $\sigma' \in I_+^*$.

b) However, the classical corresponding of (24) is that for any $\Lambda^{-1}(\mu) \in I_s^*$ there is $\Lambda^{-1}(\sigma) \in I_+^*$ such that

$$\mathcal{C}_{\Lambda^{-1}(\mu)}^*(t) = \int_{\mathbb{R}_+} \left(\cos(t\sqrt{x} - 1)\, \nu_{\Lambda^{-1}(\sigma)}(\mathrm{d}x), \quad t \in \mathbb{R}.\right.$$

c) For illustration of this theorem, consider the standard Wigner distribution $\mu = \mathrm{w}$ for which $\mathcal{C}_{\mathrm{w}}^{\boxplus}(z) = z^2$. In this case $\sigma = \delta_1$ and $\mathcal{C}_\sigma^{\boxplus}(z) = z$. For the standard Cauchy

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distribution μ we have $\mathcal{C}_\mu^{\boxplus}(z) = -iz$ and $\nu_\mu(dx) = \frac{1}{x^2} 1_{\mathbb{R}}(x)dx$. In this case $\mathcal{C}_\sigma^{\boxplus}(z) = -i\sqrt{z}$ is the free cumulant transform of the one-side $1/2$-free stable distribution with Lévy measure $\nu_\sigma(dx) = \frac{1}{x^{3/2}} 1_{\mathbb{R}_+}(x)dx$.

Proof If we have $\sigma \in I_+^{\boxplus}$, $\mathcal{C}_\sigma^{\boxplus}(z)$ can be define on $\mathbb{C}\backslash\mathbb{R}_+$. Suppose $\mu \in I_s^{\boxplus}$ and let $z \in \mathbb{C}\backslash\mathbb{R}$. Then,

$$
\begin{aligned}
\mathcal{C}_\mu^{\boxplus}(z) &= a_\mu z^2 + \int_{\mathbb{R}} \left(\frac{1}{1-zx} - 1 \right) \nu_\mu(dx) \\
&= a_\mu z^2 + \int_{\mathbb{R}} \left(\frac{1}{1-zx} - 1 - zx 1_{|x|\leq 1}(x) \right) \nu_\mu(dx) \\
&= a_\mu z^2 + \int_{\mathbb{R}_+} \left(\frac{1}{1-zx} - 1 - zx 1_{|x|\leq 1}(x) \right) \nu_\mu(dx) \\
&\quad + \int_{\mathbb{R}\backslash\mathbb{R}_+} \left(\frac{1}{1-zx} - 1 - zx 1_{|x|\leq 1}(x) \right) \nu_\mu(dx).
\end{aligned}
$$

Since the Lévy measure ν_μ is symmetric, we have

$$
\begin{aligned}
\mathcal{C}_\mu^{\boxplus}(z) &= a_\mu z^2 + \int_{\mathbb{R}_+} \left(\frac{1}{1-zx} - 1 - zx 1_{|x|\leq 1}(x) \right) \nu_\mu(dx) \\
&\quad + \int_{\mathbb{R}_+} \left(\frac{1}{1+zx} - 1 + zx 1_{|x|\leq 1}(x) \right) \nu_\mu(dx) \\
&= a_\mu z^2 + \int_{\mathbb{R}_+} \left(\frac{1}{1-zx} + \frac{1}{1+zx} - 2 \right) \nu_\mu(dx) \\
&= a_\mu z^2 + 2 \int_{\mathbb{R}_+} \left(\frac{1}{1-z^2x^2} - 1 \right) \nu_\mu(dx) \\
&= a_\mu z^2 + 2 \int_{\mathbb{R}_+} \left(\frac{1}{1-z^2x} - 1 \right) \nu_\mu^{(2)}(dx).
\end{aligned}
\tag{27}
$$

Let $\nu_\sigma = 2\nu_\mu^{(2)}$. Then $\nu_\sigma((-\infty, 0]) = 0$ and using again the symmetry of ν_μ we have

$$
\begin{aligned}
\int_{\mathbb{R}_+} \min(1,x)\nu_\sigma(dx) &= \int_{\mathbb{R}_+} \min(1,x) 2\nu_\mu^{(2)}(dx) \\
&= 2 \int_{\mathbb{R}} \min(1, x^2)\nu_\mu(dx) < \infty,
\end{aligned}
$$

since ν_μ is a Lévy measure. Then ν_σ is the Lévy measure of a free regular infinitely divisible distribution σ and using (27) and the uniqueness of the Lévy-Khintchine representation we have $\mathcal{C}_\mu^{\boxplus}(z) = \mathcal{C}_\sigma^{\boxplus}(z^2)$. Here $\sigma \in I_{r+}^{\boxplus}$ has triplet $(0, \nu_\sigma, c_\sigma)$ with $c_\sigma = a_\mu$.

Conversely, if a $\sigma \in I_{r+}^{\boxplus}$ with triplet $(0, \nu_\sigma, c_\sigma)$, let μ be the symmetric free infinitely divisible with triplet $(a_\sigma, \nu_\sigma, 0)$ where $a_\mu = c_\mu$ and $\nu_\mu = \frac{1}{2}\left(\nu_\sigma^{(1/2)+} + \nu_\sigma^{(1/2)-} \right)$. Then ν_μ is a

symmetric Lévy measure and

$$
\begin{aligned}
\mathcal{C}_\mu^\boxplus(z) &= a_\mu z^2 + \int_\mathbb{R} \left(\frac{1}{1-zx} - 1 \right) \nu_\mu(\mathrm{d}x) \\
&= a_\mu z^2 + \frac{1}{2} \int_\mathbb{R} \left(\frac{1}{1-zx} - 1 \right) \left(\nu_\sigma^{(1/2)+}(\mathrm{d}x) + \nu_\sigma^{(1/2)-}(\mathrm{d}x) \right) \\
&= a_\mu z^2 + \frac{1}{2} \int_{\mathbb{R}_+} \left(\frac{1}{1-z\sqrt{x}} + \frac{1}{1+z\sqrt{x}} - 2 \right) \nu_\sigma(\mathrm{d}x) \\
&= a_\mu z^2 + \int_{\mathbb{R}_+} \left(\frac{1}{1-z^2 x} - 1 \right) \nu_\sigma(\mathrm{d}x) = \mathcal{C}_\sigma^\boxplus(z^2),
\end{aligned}
$$

which proves the result. ∎

The relations between the Lévy measures (25) and (26) hold in the free and classical infinitely divisible cases. The importance of the above theorem is in the relation (24) between the cumulant transforms.

When σ is absolutely continuous, we have the following formula.

Lemma 14. *If the Lévy measure of $\sigma \in I_+^*$ has density h_σ, then the $\nu_\sigma^{(1/2)+}$ is also absolutely continuous and has density $h_\sigma^{(1/2)+}(x)$ of $\nu_\sigma^{(1/2)+}$*

$$
h_\sigma^{(1/2)+}(x) = 2x h_\sigma(x^2) \quad (x > 0).
$$

4.2. Classical and free type G distributions

Following [3], we say that a probability distribution υ is in the class of **free type G** distributions, if there exists a classical type G distribution μ such that $\upsilon = \Lambda(\mu)$. That is, $\Lambda^{-1}(\upsilon)$ is the distribution of VZ where V and Z are independent, with Z having the standard Gaussian distribution and V^2 has a classical infinitely divisible distribution in I_+^* with the Lévy measure ρ_V.

The following result is a characterization of the Lévy measure of classical and free type G distributions in terms of the measure σ of Theorem 12. We also consider the relation between σ and the distribution of V^2 which is given in terms of mixtures of free multiplicative convolutions.

Theorem 15. *A non Gaussian symmetric probability measure μ on $\mathbb{R}$ is a type G distribution iff the Lévy measure of σ can be represented as*

$$
\nu_\sigma(\mathrm{d}x) = \frac{g(x)}{\sqrt{x}} \mathrm{d}x
$$

where $g(x)$ is a completely monotone function on $(0, \infty)$.

Proof If μ is type G distribution then its Lévy measure can be represented $\nu_\mu(\mathrm{d}x) = g(x^2)$ where $g(x)$ is a completely monotone function on $(0, \infty)$.

$$
\nu_\mu(\mathrm{d}x) = g(x^2)\mathrm{d}x = \frac{1}{2} \left(2g(x^2) 1_{(0,\infty)}(x) + 2g(x^2) 1_{(-\infty,0)}(x) \right) \mathrm{d}x
$$

So, $\nu_\sigma^{(1/2)+} = 2g(x^2) 1_{(0,\infty)}(x)\mathrm{d}x$ and $\nu_\sigma^{(1/2)-} = 2g(x^2) 1_{(-\infty,0)}(x)\mathrm{d}x$. By Lemma 14, $\nu_\sigma(\mathrm{d}x) = h_\sigma(x)\mathrm{d}x = \frac{g(x)}{\sqrt{x}}\mathrm{d}x$. The converse is trivial. ∎

It is well known that the Lévy measure of a type G distribution μ is of the form $\nu_\mu(dx) = v_\mu(x)dx$ where

$$v_\mu(x) = \int_{\mathbb{R}_+} \phi(x,s)\rho_V(ds) \tag{28}$$

for some Lévy measure ρ_V of a distribution in I_+^* and $\phi(x,s)$ is the Gaussian density of mean zero and variance s.

We have seen that a free type G distribution μ is related to two Lévy measures on $\mathbb{R}_+$; ν_σ given by Theorem 12 and ρ_V as in (28). The following result gives the relation between these two Lévy measures in terms of mixtures of cumulants of free multiplicative convolutions of the Marchenko-Pastur distribution with $\gamma_s^{(2)}$, the Gamma distribution with shape parameter $1/2$ and scale parameter s.

Theorem 16. *Let μ be a free type G distribution with the Lévy measure ν_μ given by (28) with the mixing Lévy measure ρ_V. Let $\sigma \in I_{r+}^\boxplus$ with the Lévy measure $\nu_\sigma = 2\nu_\mu^{(2)}$. Then*

$$\mathcal{C}_\sigma^\boxplus(z) = \int_{\mathbb{R}_+} \mathcal{C}_{m\boxtimes\gamma_s^{(2)}}^\boxplus(z)\rho_V(ds). \tag{29}$$

Moreover,

$$\nu_\sigma(dx) = \int_{\mathbb{R}} \gamma_s^{(2)}(dx)\rho_V(ds). \tag{30}$$

Proof Using the free regular representation (9), the fact that $\nu_\sigma = 2\nu_\mu^{(2)}$ and Proposition 4(b) with the free cumulant transform of the compound Poisson distribution $m \boxtimes \gamma_s^{(2)}$ in (23) we have

$$\begin{aligned}
\mathcal{C}_\sigma^\boxplus(z) &= a_\mu z + \int_{\mathbb{R}_+} \left(\frac{1}{1-zx}-1\right)\nu_\sigma(dx) \\
&= a_\mu z + 2\int_{\mathbb{R}_+} \left(\frac{1}{1-zx}-1\right)\nu_\mu^{(2)}(dx) \\
&= a_\mu z + \int_{\mathbb{R}} \left(\frac{1}{1-zx^2}-1\right)\nu_\mu(dx) \\
&= a_\mu z + \int_{\mathbb{R}} \left(\frac{1}{1-zx^2}-1\right)\int_{\mathbb{R}_+}\phi(x,s)\rho_V(ds)dx \quad \text{(use (28))} \\
&= a_\mu z + \int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(\frac{1}{1-zx^2}-1\right)\phi(x,s)dx\rho_V(ds) \\
&= a_\mu z + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \left(\frac{1}{1-zx}-1\right)\gamma_s^{(2)}(dx)\rho_V(ds) \\
&= a_\mu z + \int_{\mathbb{R}_+}\mathcal{C}_{m\boxtimes\gamma_s^{(2)}}^\boxplus(z)\rho_V(ds).
\end{aligned}$$

The second statement follows also from the above calculations. ∎

If in the above theorem we consider the variance mixture of the Normal distribution with the classical Poisson distribution, we obtain $\Lambda(\mathcal{L}(VZ)) = m \boxtimes \gamma_1$.

5. Free Type W Distributions

In this section we consider multiplicative convolutions of the Wigner measure with probability measures on $\mathbb{R}_+$ and their free additive infinite divisibility. We introduce the class of **free type W distributions** and show the role played by $\boxtimes$-2 divisible distributions.

5.1. Multiplicative mixtures of the Wigner distribution

Definition 17. *For a probability measure λ in $\mathcal{P}_+$, a symmetric probability measure $\mu = \lambda \boxtimes \mathrm{w}$ is called a free multiplicative mixture of the Wigner distribution. When μ is free infinitely divisible, it is called a **free type W distribution**.*

From [2, Theorem 12] we have that any free symmetric α-stable law is a free type W distribution. However, not all all symmetric distribution can be represented as free multiplicative mixture of the Wigner distribution. This is the case of the arcsine symmetric distribution a on $(-1, 1)$, as shown by the following result.

Proposition 18. *Let a be arcsine distribution on $(-1, 1)$. There does not exist $\lambda \in \mathcal{P}_+$ such that $\mathrm{a} = \lambda \boxtimes \mathrm{w}$.*

Proof If there is such a $\lambda \in \mathcal{P}_+$, from (16) we have $S_{\mathrm{a}}(z) = S_\lambda(z) S_{\mathrm{w}}(z)$. So, using (18) and (20) we obtain $S_\lambda(z) = \sqrt{z+2}$. But $S'_\lambda(t) = \frac{1}{2\sqrt{t+2}}$ is positive for all $t \in (-1, 0)$. Then, from Proposition 6.8 in [9], $S_\lambda(z) = \sqrt{z+2}$ cannot be the S-transform of a probability measure on $\mathbb{R}_+$. $\blacksquare$

A remarkable property of an arbitrary free multiplicative mixture of the Wigner measure is the fact that "its square" is always free infinitely divisible and moreover a free compound Poisson distribution.

Proposition 19. *Let $\overline{\sigma} \in \mathcal{P}_+$, $\sigma = \overline{\sigma} \boxtimes \overline{\sigma}$, w be Wigner measure and $\mu = \overline{\sigma} \boxtimes \mathrm{w}$. Then $\mu^{(2)} = \sigma \boxtimes \mathrm{m}$ is free infinitely divisible. Moreover, $\mu^{(2)}$ is a free compound distribution in $I_{r+}^{\boxplus}$.*

Remark 20. *In the classical case we have a similar result than in the above case. Namely, any square of a variance mixture of Gaussian (whether $*$-infinitely divisible or not) is always $*$-infinitely divisible. To see this, let $X = VZ$ for a positive random variable V independent of Z with the standard Gaussian distribution. Recall the well known fact that Z has the distribution of $E^{1/2}A$, where E and A are independent random variables, E with exponential distribution and A with a symmetric arcsine distribution on $(-1, 1)$. Then X^2 has the distribution of EV^2A^2 which is always $*$-infinitely divisible, since any mixture of the exponential distribution is $*$-infinitely divisible. Moreover, X^2 is in the Bondesson class of distributions characterized by $*$-infinitely divisible distributions with completely monotone Lévy density, see for example [4].*

Proof Since $\sigma = \overline{\sigma} \boxtimes \overline{\sigma}$ and $\mu = \overline{\sigma} \boxtimes \mathrm{w}$, using (12), (16) and the fact that $S_{\mathrm{w}}(z) = \frac{1}{\sqrt{z}}$ we have $S_\sigma(z) = S_{\overline{\sigma}}(z)^2$ and $S_\mu(z) = \frac{1}{\sqrt{z}} S_{\overline{\sigma}}(z)$. On the other hand, from (14) $S_\mu^2(z) = \frac{z+1}{z} S_{\mu^{(2)}}(z) = \frac{1}{z} S_\sigma(z)$. Hence, $S_{\mu^{(2)}}(z) = \frac{1}{z+1} S_\sigma(z)$ and since for the Marchenko-Pastur distribution m $S_{\mathrm{m}}(z) = \frac{1}{z+1}$, again using (12) we obtain $\mu^{(2)} = \sigma \boxtimes \mathrm{m}$. $\blacksquare$

In particular, if we consider the square of symmetric free stable law, the free compound Poisson distribution with one-side stable law appears.

Corollary 21. *Let υ_α be a symmetric free α-stable law, $0 < \alpha < 2$. Then $\upsilon_\alpha^{(2)} = (\sigma_\beta \boxtimes \sigma_\beta) \boxtimes$ m, where σ_β is a free positive β-stable law with $\beta = 2\alpha/(2 + \alpha)$.*

Proof The result follows from the last proposition, relation (17) and Proposition 4 (b). ∎

5.2. Type W and $\boxtimes$-2 divisibility

We now present a characterization of type W distributions where the concept of 2–$\boxtimes$ divisibility appears in connection with free regular infinitely distributions on $\mathbb{R}_+$. Specifically, a free multiplicative mixture of the Wigner distribution μ is free infinitely divisible if and only if the mixing measure σ is free regular and 2–$\boxtimes$ divisible. Moreover, σ is the distribution appearing in Theorem (12).

Theorem 22. *Let $\overline{\sigma} \in \mathcal{P}_+$ and w be the Wigner measure. Then $\sigma = \overline{\sigma} \boxtimes \overline{\sigma} \in I_{r+}^{\boxplus}$ if and only if $\mu = \overline{\sigma} \boxtimes$ w $\in I_s^{\boxplus}$ in which case*

$$\mathcal{C}_\mu^{\boxplus}(z) = \mathcal{C}_\sigma^{\boxplus}(z^2), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

Proof Assume first that $\overline{\sigma} \boxtimes \overline{\sigma} \in I_{r+}^{\boxplus}$. From theorem 12 there exists a symmetric $\boxplus$–infinitely divisible distribution μ such that $\mathcal{C}_\mu^{\boxplus}(z) = \mathcal{C}_{\overline{\sigma}\boxtimes\overline{\sigma}}^{\boxplus}(z^2)$. Then

$$\begin{aligned}
\mathcal{C}_{\overline{\sigma}\boxtimes\overline{\sigma}}^{\boxplus}(zS_{\overline{\sigma}}^2(z)) &= \mathcal{C}_\mu^{\boxplus}(\sqrt{z}S_{\overline{\sigma}}(z)) \\
&= \mathcal{C}_\mu^{\boxplus}(zS_{\overline{\sigma}\boxtimes\mathrm{w}}(z)) \\
&= \mathcal{C}_\mu^{\boxplus}(zS_\mu(z)).
\end{aligned}$$

Therefore, $S_\mu(z) = S_{\overline{\sigma}\boxtimes\mathrm{w}}(z)$ and by the uniqueness of the multiplicative convolution we obtain $\mu = \overline{\sigma} \boxtimes$ w.

Next, for a probability measure $\overline{\sigma}$ on $\mathbb{R}_+$ let $\mu = \overline{\sigma}\boxtimes$w, that is $S_\mu(z) = S_{\overline{\sigma}\boxtimes\mathrm{w}}(z) = \frac{1}{\sqrt{z}}S_{\overline{\sigma}}(z)$. Again, using theorem 12 there exists a regular $\boxplus$–infinitely divisible distribution σ (on $\mathbb{R}_+$) such that $\mathcal{C}_\mu^{\boxplus}(z) = \mathcal{C}_\sigma^{\boxplus}(z^2)$. Then

$$\begin{aligned}
\mathcal{C}_\mu^{\boxplus}(zS_\mu(z)) &= \mathcal{C}_\mu^{\boxplus}(\sqrt{z}S_{\overline{\sigma}}(z)) \\
&= \mathcal{C}_\mu^{\boxplus}(zS_{\overline{\sigma}}^2(z)) \\
&= \mathcal{C}_\sigma^{\boxplus}(zS_{\overline{\sigma}\boxtimes\overline{\sigma}}(z)).
\end{aligned}$$

Then, from (15) we have that

$$z = \mathcal{C}_\mu^{\boxplus}(zS_\mu(z)) = \mathcal{C}_\sigma^{\boxplus}(zS_{\overline{\sigma}\boxtimes\overline{\sigma}}(z)) = \mathcal{C}_\sigma^{\boxplus}(zS_\sigma(z))$$

and therefore $S_\sigma(z) = S_{\overline{\sigma}\boxtimes\overline{\sigma}}(z)$ and hence $\sigma = \overline{\sigma} \boxtimes \overline{\sigma} \in I_{r+}^{\boxplus}$. ∎

From Example 8(3) we have that if $\overline{\sigma} = \mu_{(5/4,1)}$, then $\sigma = \overline{\sigma} \boxtimes \overline{\sigma} = \mu_{(3/2,1)}$ is $\boxtimes$-2 divisible but it is not $\boxplus$–infinitely divisible. Then, from the above theorem we have that $\mu_{(5/4,1)} \boxtimes$ w is a free multiplicative mixture of the Wigner distribution but not a type W distribution. So we have the following result.

Proposition 23. *The class of all type W distributions is a proper subset of the class of all free multiplicative mixtures of the Wigner distribution.*

5.3. Relation to free type G distributions

We now give an example of a type W distribution which is not a free type G distribution and therefore type W distribution is not the image of the class of classical type G distributions under the Bercovici-Pata bijection Λ.

Let b_s be the symmetric Beta distribution $(1/2, 3/2)$ on $(-2\sqrt{s}, 2\sqrt{s})$ given by

$$\mu_{s,\alpha,\beta}(\mathrm{d}x) = \frac{1}{2B(\alpha,\beta)\sqrt{s}}|x|^{\alpha-1}(s-|x|)^{\beta-1}1_{(-2\sqrt{s},2\sqrt{s})}(x)\mathrm{d}x.$$

for $\alpha = 1/2, \beta = 3/2$. It was shown in [3] that b_s is $\boxplus$-infinitely divisible distribution which is not a free type G distribution. Moreover $b_s = a_s \boxtimes m$. Then $S_{b_1}(z) = S_a(z)S_m(z)$ and using (19) and (20) we obtain that

$$S_{b_1}(z) = \frac{1}{\sqrt{z}}\frac{\sqrt{z+2}}{z+1}. \tag{31}$$

With this we can prove that b_1 is a type W distribution.

Lemma 24. *The symmetric Beta distribution b_1 on $(-2,2)$ is a type W distribution. Moreover $b_1 = w \boxtimes a^+ \boxtimes \overline{m_2}$ where a^+ is the positive arcsine distribution on (0,1) and m_2 is the free Poisson distribution of parameter $c = 2$.*

Proof Using (31) we have

$$S_{b_1}(z) = \frac{1}{\sqrt{z}}\frac{\sqrt{z+2}}{z+1} = \frac{1}{\sqrt{z}}\frac{z+2}{z+1}\frac{1}{\sqrt{z+2}}.$$

The result follows, since from (19) and Proposition 9 $S_{\overline{m_2}}(z) = 1/\sqrt{z+2}$, and from (22) we obtain $S_{a^+}(z) = (z+2)/(z+1)$. $\blacksquare$

Finally, we show that type W distributions is a proper subclass of the class $I_s^{\boxplus}$ of all symmetric free infinitely divisible distributions. The example below is constructed from the family of the free Poisson distributions.

Example 25. *Let $\widetilde{m}_c$ be dual of m_c, that is $\widetilde{m}_c(B) = m_c(-B)$ for any Borel set B. Then for $c > 0$*

$$\widetilde{m}_c(\mathrm{d}x) = \max(0, (1-c))\delta_0(\mathrm{d}x) + \frac{1}{2\pi(-x)}\sqrt{4c - ((-x)-1-c)^2}1_{[-(1+\sqrt{c})^2,-(1-\sqrt{c})^2]}(x)\mathrm{d}x.$$

Let $\mu_c = m_c \boxplus \widetilde{m}_c$. Then μ_c is a symmetric $\boxplus$–infinitely divisible distribution with the free cumulant transform

$$\mathcal{C}_{\mu_c}^{\boxplus}(z) = \frac{2cz^2}{1-z^2},$$

and the S–transform

$$S_{\mu_c}(z) = \frac{1}{\sqrt{z}}\frac{1}{\sqrt{z+2c}}. \tag{32}$$

However, for small c, μ_c is not a free multiplicative convolution of the Wigner measure. Indeed, it was shown in Proposition 10 that for c small enough (for example $c < 1/16$), the function $1/\sqrt{z+2c}$ is not the S-transform of a probability measure on $\mathbb{R}_+$. Then, since $1/\sqrt{z}$ is the S-transform of the Wigner measure, from (32) and (16) we have that μ_c is not the S-transform of a multiplicative convolution of the Wigner measure. However, for $c \geq 1/2$, $m_c \boxplus \widetilde{m}_c = \overline{m}_{2c} \boxtimes w$.

We summarize some results of this section as follows.

Theorem 26. *(1) The class of all type W distributions is a proper subset of $I_s^{\boxplus}$.*
(2) The intersection of the class of all type W distributions and the class of all free type G distributions is not empty.
(3) The class of all type W distributions does not coincide with the class of all free type G distributions.

6. $\boxplus$–ID of Free Multiplicative Convolutions with the Arcsine Measure

We now define a new subclass of $\boxplus$–infinitely divisible distributions, the **type AS** distributions. We say that a distribution $\mu \in I_s^{\boxplus}$, belongs to the class type AS, if it is the free multiplicative convolution of the arcsine measure. That is, there exists a distribution $\lambda \in \mathcal{P}_+$ such that $\mu = \lambda \boxtimes a$.

It was already seen that $w = \overline{m_2} \boxtimes a$. Then the class type AS contains the class of type W distributions. We characterize the class of type AS distributions in a similar way as the class type W.

Theorem 27. *A symmetric distribution $\mu = \lambda \boxtimes a$, $\lambda \in \mathcal{P}_+$, is a type AS distribution if and only if there exists $\sigma \in I_{r+}^{\boxplus}$ such that $\lambda \boxtimes \lambda = m_2 \boxtimes \sigma$.*

Proof Assume $\mu = \lambda \boxtimes a$ is the type AS distribution. Then $S_\mu(z) = S_\lambda(z)S_a(z)$ and from Theorem 12, there exists $\sigma \in I_{r+}^{\boxplus}$ such that

$$\mathcal{C}_\mu^{\boxplus}(zS_\mu(z)) = \mathcal{C}_\sigma^{\boxplus}(z^2 S_\mu^2(z)).$$

Since $z = \mathcal{C}_\sigma^{\boxplus}(zS_\sigma(z))$, we have that $S_\sigma(z) = zS_\mu^2(z) = zS_\lambda^2(z)S_a^2(z)$. From (20) $S_a^2(z) = (z+2)/z$ and since $S_{m_2}(z) = 1/(z+2)$ we have $S_{m_2}(z)S_\sigma(z) = S_\lambda^2(z)$. Then we conclude $\lambda \boxtimes \lambda = m_2 \boxtimes \sigma$.

Next assume that $\mu = \lambda \boxtimes a$ and that there exist $\sigma \in I_{r+}^{\boxplus}$ such that $\lambda \boxtimes \lambda = m_2 \boxtimes \sigma$. From Theorem 12, we can find $\tilde{\mu} \in I_s^{\boxplus}$ satisfying

$$\mathcal{C}_{\tilde{\mu}}^{\boxplus}(z) = \mathcal{C}_\sigma^{\boxplus}(z^2).$$

Then

$$\mathcal{C}_{\tilde{\mu}}^{\boxplus}(zS_{\tilde{\mu}}(z)) = \mathcal{C}_\sigma^{\boxplus}(z^2 S_{\tilde{\mu}}^2(z)) = \mathcal{C}_\sigma^{\boxplus}(zS_\sigma(z))$$

from which we conclude that $zS_{\tilde{\mu}}^2(z) = S_\sigma(z) = (z+2)S_\lambda^2(z)$. Then $S_{\tilde{\mu}}^2(z) = S_a^2(z)S_\lambda^2(z)$ and therefore $\tilde{\mu} = \lambda \boxtimes a = \mu$. $\blacksquare$

Remark 28. *In the above theorem, if $\sigma \in I_{r+}^{\boxplus}$ is $\boxtimes$–2 divisible, $\lambda = \overline{m_2} \boxtimes \overline{\sigma}$.*

Example 29. *(1) If b_s is the symmetric beta $(1/2, 3/2)$ distribution, b_s is the free type AS distribution since $b_s = m \boxtimes a$. We recall that b_s is the free type W distribution but not the free type G distribution. Then the class of free type W distributions is a proper subclass of the class free type AS.*
(2) The Wigner-free Poisson distribution (i.e. $\mu = w \boxtimes m$) is the free type AS distribution. To see this observe that

$$S_\mu(z) = \frac{1}{\sqrt{z}(z+1)} = \sqrt{\frac{z+2}{z}}\frac{1}{z+1}\frac{1}{\sqrt{z+2}} = S_a(z)S_m(z)\frac{1}{\sqrt{z+2}},$$

where $\frac{1}{\sqrt{z+2}}$ is the S-transform of the probability measure $\overline{\mathrm{m}_2}$. Then $\mu = \mathrm{a} \boxtimes \mathrm{m} \boxtimes \overline{\mathrm{m}_2}$ and

$$(\mathrm{m} \boxtimes \overline{\mathrm{m}_2}) \boxtimes (\mathrm{m} \boxtimes \overline{\mathrm{m}_2}) = \mathrm{m}^{\boxtimes 2} \boxtimes \mathrm{m}_2$$

and therefore μ is $\boxplus$–infinitely divisible and thus the free type AS distribution.

We finally show that the class type AS distributions does not coincide with the class $I_s^{\boxplus}$ of all symmetric free infinitely distributions.

Example 30. *Let μ_c be as in example 25. Then*

$$S_{\mu_c}(z) = \sqrt{\frac{z+2}{z}} \frac{1}{\sqrt{(z+2)(z+2c)}}. \tag{33}$$

If we take $c = 1/15$, there is not a probability measure whose S–transform is $\frac{1}{\sqrt{(z+2)(z+2c)}}$ by a similar argument as in Proposition 10. Since $\sqrt{(z+2)/z}$ is the S-transform of the arcsine distribution a on $(-1, 1)$, from (33) and (16) we have that $S_{\mu_c}(z)$ cannot be the S-transform of a multiplicative convolution with a.

Acknowledgments

Symbols

$\mathcal{P}$: the set of all probability measures on $\mathbb{R}$.

$\mathcal{P}_+$: the set of all probability measures on $\mathbb{R}_+$.

$\mathcal{P}_s$: the set of all symmetric probability measures on $\mathbb{R}$.

$\mathcal{C}_\mu^*$: the classical cumulant transform of the probability measure μ.

G_μ : the Cauchy transform of the probability measure μ.

F_μ : the reciprocal of the Cauchy transform of the probability measure μ.

$\mathcal{C}_\mu^{\boxplus}$: the free cumulant (or R−)transform of the probability measure μ.

I^* : the set of all (classical) infinitely divisible distributions on $\mathbb{R}$.

I_+^* : the set of all positive (classical) infinitely divisible distributions on $\mathbb{R}_+$.

I_s^* : the set of all symmetric (classical) infinitely divisible distributions on $\mathbb{R}$.

$I^{\boxplus}$: the set of all free infinitely divisible distributions on $\mathbb{R}$.

$I_s^{\boxplus}$: the set of all free symmetric infinitely divisible distributions on $\mathbb{R}$.

$I_{r+}^{\boxplus}$: the set of all free regular infinitely divisible distributions on $\mathbb{R}_+$.

$\mathcal{L}(X)$: the law of a real random variable X.

ν_μ : the Lévy measure of $\mu \in I^*$ or $I^{\boxplus}$.

S_μ : the S-transform of the probability measure $\mu \in \mathcal{P}_+$.

$\mathrm{w}_{b,a}$: the Wigner (or semicircle) distribution with mean b and variance a.

w : the Wigner (or semicircle) distribution with mean 0 and variance 1.

m_c : the Marchenko-Pastur (or free Poisson) distribution with parameter $c > 0$.

m : the Marchenko-Pastur (or free Poisson) distribution with parameter 1.

a_s : the symmetric arcsine distribution on $(-s, s)$.

a : the symmetric arcsine distribution on $(-1, 1)$.

a_s^+ : the positive arcsine distribution on $(0, s)$.

a^+ : the positive arcsine distribution on $(0, 1)$.

$\gamma_{b,a}$: the Gaussian distribution with mean b and variance a.

γ_s : the Gaussian distribution with mean 0 and variance s.

p_c : the classical Poisson distribution with mean $c > 0$.

$\gamma_s^{(2)}$: the gamma distribution with shape parameter $1/2$ and scale parameter s.

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