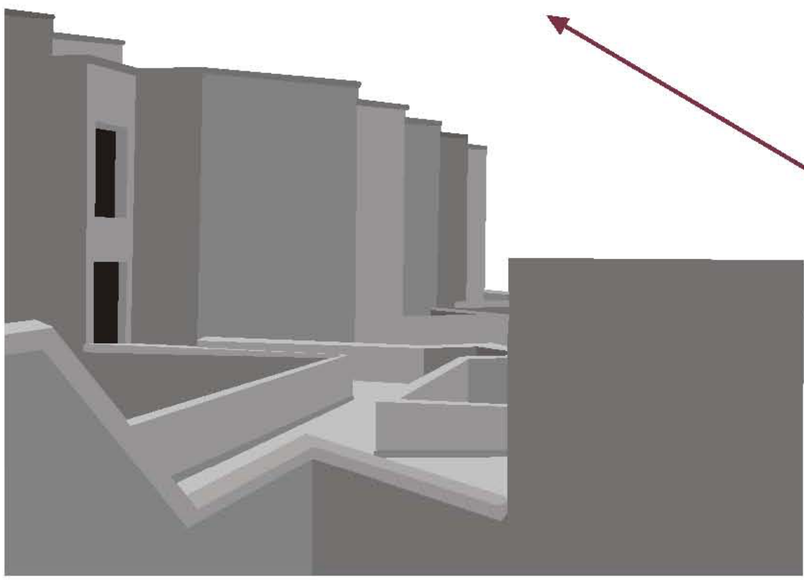

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On K-B quasi-Jordan algebras and their relation with Leibniz algebras

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Abstract

In this paper we show some basic properties of quasi-Jordan algebras and we study the definition of Leibniz-Jordan algebra introduced by P. Kolesnikov and restrictive quasi-Jordan algebras introduced by M. R. Bremner. We show that these definitions are equivalent and we define K-B quasi-Jordan algebras. We present a characterization of K-B quasi-Jordan algebras by Jordan bimodules and construct right units over a K-B quasi-Jordan algebras.

On the other hand, we prove that there are inner derivations (classical and left derivations) in K-B quasi-Jordan algebras, we find the relationship between K-B quasi-Jordan algebras and Leibniz algebras and we construct Leibniz algebras from K-B quasi-Jordan algebras.

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Introduction

There are three strongly related algebras: Associative, Jordan and Lie algebras. It is known that any associative algebra A becomes a Lie algebra under the skew-symmetric product (Lie bracket) $[x, y] := xy - yx$ and at the same time it becomes a Jordan algebra with respect to the product $x \bullet y := \frac{1}{2}(xy + yx)$. On the other hand, it is known that the universal enveloping algebra of a Lie algebra has the structure of an associative algebra. Finally, we recall that from the works of J. Tits, I. Kantor and M. Koecher follows that any Jordan algebra can be embedded into a Lie algebra.

In 1993, J. L. Loday introduced the notion of Leibniz algebras (see [15]), which is a generalization of the Lie algebras where the skew-symmetry of the bracket is dropped and the Jacobi identity is changed by the Leibniz identity. Loday also showed that the relationship between Lie algebras and associative algebras translate into an analogous relationship between Leibniz algebras and the so-called Dialgebras (see [15]) which are a generalization of associative algebras possessing two operations. In particular Loday showed that any dialgebra $(D, \dashv, \vdash)$ becomes a Leibniz algebra D_{Leib} under the Leibniz bracket $[x, y] := x \dashv y - y \vdash x$ and the universal enveloping algebra of a Leibniz algebra has the structure of a Dialgebra (see [15] or [16]).

On the other hand, the authors have introduced the notion of quasi-Jordan algebras over fields of characteristic other than two. These algebras satisfy the identities

$$x(yz) = x(zy)$$

and

$$(yx)x^2 = (yx^2)x,$$

The quasi-Jordan algebras will have a relationship the Leibniz algebras similar to the one existing among the Jordan algebras and the Lie algebras. In fact in [18] we attached a quasi-Jordan algebra L_x to any ad-nilpotent element x with index of nilpotence 3 (Q-Jordan element) in a Leibniz algebra L. Thus the quasi-Jordan algebras are a generalization of Jordan algebras where the commutative law is changed by a quasi-commutative identity and a special form of the Jordan identity is retained. In [18] we showed a few results about the relationship between Jordan algebras and quasi-Jordan algebras. Also, we compared quasi-Jordan algebras with some known structures.

In Addition, Velásquez and Felipe introduced the concept of split quasi-Jordan algebras and add right units to split quasi-Jordan algebras. In [19] the authors studied the relationship between quasi-Jordan algebras and split quasi-Jordan algebras. In particular, they showed that every quasi-Jordan algebra is isomorphic to a subalgebra of a split quasi-Jordan algebra.

Independently, K. Liu introduced the notion of generalized Jordan algebra from associative $\mathbb{Z}_2$-algebras over fields of characteristic other than two and three (see [12]). The generalized Jordan algebra satisfies three identities, the first two identities are the same identities in the definition of quasi-Jordan algebras, and the third identity, called the *Hu-Liu identity*, is

$$(x, y, x^2) = 2(x^2, y, x),$$

where $(\cdot, \cdot, \cdot)$ denotes the associator.

More recently, P. Kolesnikov introduced the notion of Leibniz-Jordan algebras over fields of characteristic other than two and three (see [11]).

The Leibniz-Jordan algebras were obtained from the relationship between conformal algebras and dialgebras and satisfy three identities, the first two identities are the same identities in definition of quasi-Jordan algebras. The third identity is

$$(z, y, x^2) = 2(zx, y, x).$$

In 2008, M. R. Bremner, using computer algebra, introduced the definition of restrictive quasi-Jordan algebras over fields of characteristic other than two and three (see [1]). The definition of restrictive quasi-Jordan algebras are obtained from the Jordan product

$$xy = \frac{1}{2}(x \dashv y + y \vdash x),$$

where x and y are elements in a dialgebra D.

These algebras satisfy three identities, the first two identities are the same identities in the definition of quasi-Jordan algebras. The third identity is

$$(y, x^2, z) = 2(y, x, z)x.$$

M. R. Bremner and L. A. Peresi showed in [2] that there exist exceptional (non-special) restrictive quasi-Jordan algebras, i.e. there exist restrictive quasi-Jordan algebras that are not generated by dialgebras.

Finally, we show that the definitions of Leibniz-Jordan algebras and restrictive quasi-Jordan algebras are equivalent, and we called this algebras *K-B quasi-Jordan algebras*. Moreover, this definition implies the Liu's definition of generalaized Jordan algebra.

In this paper we will work with K-B Quasi-Jordan algebras.

1 Leibniz algebras and dialgebras

Around 1990, J. L. Loday introduced the notions of Leibniz algebras and dialgebras (see [15] and [16]). Leibniz algebras are a generalization of Lie algebras where the skew-symmetry of the bracket is suppressed and the Jacobi identity is changed by the Leibniz identity.

Definition 1 *A* **Leibniz algebra** *over a field* $\mathbb{K}$ *is a* $\mathbb{K}$*-vector space* L *equipped with a binary operation, called a Leibniz bracket,* $[\cdot,\cdot] : L \times L \to L$ *which satisfies the Leibniz identity*

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]\,, \qquad \textit{for all} \;\; x, y, z \in L \qquad \text{(L)}$$

Example 2 *Let* (A, d) *be a differential associative algebra. Therefore* $d(ab) = da\, b + a\, db$ *and* $d^2 = 0$*. Define the bracket on* A *by the formula*

$$[a, b] := a\, db - db\, a$$

The vector space A *equipped with this bracket is a Leibniz algebra.*

It is known that the universal enveloping algebra of a Lie algebra has the structure of an associative algebra. Loday showed that the relationship between Lie algebras and associative algebras can be translated into an analogous relationship between Leibniz algebras and dialgebras.

Dialgebras are a generalization of associative algebras with two associative products.

Definition 3 *A* **dialgebra** *over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space D equipped with two bilinear associative products*

$$\dashv : D \times D \to D$$

$$\vdash : D \times D \to D$$

satisfying the identities:

$$x \dashv (y \dashv z) = x \dashv (y \vdash z) \tag{D1}$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z) \tag{D2}$$

$$(x \vdash y) \vdash z = (x \dashv y) \vdash z \tag{D3}$$

This example of dialgebra was studied by R. Felipe, F. Ongay, N. López and R. Velásquez (see [4]).

Example 4 *Let V be a vector space and fix $\varphi \in V'$, where V' denotes the dual space of V. Then one can define a dialgebra structure on V by setting $x \dashv y = \varphi(y)x$ and $x \vdash y = \varphi(x)y$, denoted by V_φ. If $\varphi \neq 0$, then V_φ is a dialgebra with non-trivial bar-units ($e \in D$ such that $x \dashv e = x = e \vdash x$ for all $x \in D$). Moreover, its halo (set of bar units) is an affine space modelled after the subspace $Ker\varphi$.*

If D is a dialgebra and we define the bracket $[\cdot,\cdot] : D \times D \to D$ by

$$[x, y] := x \dashv y - y \vdash x\,, \quad \text{for all } x, y \in D,$$

then $(D, [\cdot,\cdot])$ is a Leibniz algebra.

Moreover, Loday showed that the following diagram is commutative

$$\begin{array}{ccc} Dias & \xrightarrow{-} & Leib \\ \uparrow & & \uparrow \\ As & \xrightarrow{-} & Lie \end{array}$$

where *Dias*, *As*, *Lie* and *Leib* denote, respectively, the categories of dialgebras, associative, Lie and Leibniz algebras.

2 Quasi-Jordan algebras

In 2008, R. Velásquez and R. Felipe introduced the notion of quasi-Jordan algebras (see [18]. Quasi-Jordan algebras are a generalization of the Jordan algebras for which the commutative law is changed by a quasi-commutative identity and a special form of the Jordan identity is retained.

Definition 5 *A* **quasi-Jordan algebra** *is a vector space $\Im$ over a field $\mathbb{K}$ of characteristic other than 2 equipped with a bilinear product $\triangleleft : \Im \times \Im \to \Im$ that satisfies*

$$x \triangleleft (y \triangleleft z) = x \triangleleft (z \triangleleft y) \qquad \textit{(right commutativity)} \tag{QJ1}$$

$$(y \triangleleft x) \triangleleft x^2 = (y \triangleleft x^2) \triangleleft x \qquad \textit{(right Jordan identity),} \tag{QJ2}$$

for all $x, y, z \in \Im$, where $x^2 = x \triangleleft x$.

If we translate the quasi-multiplication (Jordan product) to the dialgebra framework, we obtain a quasi-Jordan algebra.

If D is a dialgebra over a field $\mathbb{K}$ of characteristic other than 2 and we define the product $\triangleleft : D \times D \to D$ by

$$x \triangleleft y := \frac{1}{2}(x \dashv y + y \vdash x), \tag{$\triangleleft$}$$

for all $x, y \in D$, then $(D; \triangleleft)$ is a quasi-Jordan algebra.

If D is a unital dialgebra, with a specific bar-unit e, we have that $x \triangleleft e = x$, for all x in D. This implies that e is a right unit for the algebra $(D, \triangleleft)$.

Example 6 *Let V be a vector space over a field $\mathbb{K}$ with characteristic other than 2 and let $gl^+(V)$ be a Jordan algebra of linear transformations over V with product defined by*

$$A \bullet B = \frac{1}{2}(AB + BA).$$

We consider the vector space $gl^+(V) \times V$ and we define the product $\triangleleft : (gl^+(V) \times V) \times (gl^+(V) \times V) \to gl^+(V) \times V$ by

$$(A, u) \triangleleft (B, v) = (A \bullet B, Bu)$$

for all $A, B \in gl(V)$ and $u, v \in V$. Then $(gl^+(V) \times V, \triangleleft)$ is a quasi-Jordan algebra.

Moreover, (Id, v) is a right unit for all $v \in V$, but (Id, v) is not a left unit. If the field $\mathbb{K}$ has characteristic zero, then $gl^+(V) \times V$ is a power-associative algebra.

Definition 7 *For a quasi-Jordan algebra $\Im$ we define*

$$Z^r(\Im) = \{z \in \Im | x \triangleleft z = 0, \ \forall x \in \Im\},$$

and we denote by $\Im^{ann}$ the subspace of $\Im$ spanned by elements of the form $x \triangleleft y - y \triangleleft x$, with $x, y \in \Im$.

We have that $\Im$ is a Jordan algebra if and only if $\Im^{ann} = \{0\}$. Besides, $\Im^{ann} \subset Z^r(\Im)$.

We have the following properties of $\Im^{ann}$ and $Z^r(\Im)$

Lemma 8 *Let $\Im$ be a quasi-Jordan algebra. Then $\Im^{ann}$ and $Z^r(\Im)$ are two-sided ideals of $\Im$. Moreover,*

$$(Z^r(\Im) \triangleleft \Im) \subset \Im^{ann}.$$

Let $(\Im, \triangleleft)$ be a quasi-Jordan algebra. If we consider the quotient algebra $\Im_{Jor} := \Im / \Im^{ann}$, then we see that $\Im_{Jor}$ is a Jordan algebra.

Besides, the ideal $\Im^{ann}$ is the smallest two-sided ideal in $\Im$ such that $\Im / \Im^{ann}$ is a Jordan algebra.

The quotient map $\pi : \Im \to \ \Im_{Jor}$ is a homomorphism of quasi-Jordan algebras. Moreover, π is universal with respect to all homomorphisms from $\Im$ to another Jordan algebra J, this is equivalent to say that the following diagram commutes

$$\begin{tikzcd} \Im \arrow[r, "\pi"] \arrow[dr] & \Im_{Jor} \arrow[d] \\ & J \end{tikzcd} .$$

A **right unit** in a quasi-Jordan algebra $\Im$ is an element e in $\Im$ such that $x \triangleleft e = x$, for all $x \in \Im$.

Let $\Im$ be a quasi-Jordan algebra, if there is an element ϵ in $\Im$ such that $\epsilon \triangleleft x = x$ then $\Im$ is a classical Jordan algebra and ϵ is a unit. For this reason we only consider right units over quasi-Jordan algebras.

We denote by $U_r(\Im)$ the set of all right units of a quasi-Jordan algebra $\Im$. A **right unital quasi-Jordan algebra** is a quasi-Jordan algebra with a specified right unit e.

Example 9 *Let V be a vector space and fix $\varphi \in V'$ with $\varphi \neq 0$. We define the product $\triangleleft : V \times V \to V$ by $x \triangleleft y = \varphi(y)x$, for all $x, y \in V$. Then (V, φ) is a quasi-Jordan algebra and all elements x in V such that $\varphi(x) \neq 0$ define a right unit $x/\varphi(x)$. Moreover, $U_r(V)$ is an affine space modelled after the $Ker\varphi$.*

We will show the following characterization of the ideal $\Im^{ann}$ and the set $U_r(\Im)$ of all right units.

Lemma 10 *Let $\Im$ be a right unital quasi-Jordan algebra, with a specific right unit e. Then*

$$\Im^{ann} = Z^r(\Im),$$

$$\Im^{ann} = \{x \in \Im | e \triangleleft x = 0\}$$

and

$$U_r(\Im) = \{x + e | x \in \Im^{ann}\}$$

3 Kolesnikov's and Bremner's definitions

In 2008 Pavel Kolesnikov (see [11]), using techniques of conformal algebras, showed that the product $(\triangleleft)$ satisfies the identities

$$x(yz) = x(zy), \tag{K1}$$

$$(y(xz))u + (y(zu))x + (y(ux))z = (yx)(zu) + (yz)(ux) + (yu)(xz) \tag{K2}$$

$$x(y(zu)) + ((xu)y)z + ((xz)y)u = (xy)(zu) + (xz)(yu) + (xu)(yz) \tag{K3}$$

The last identities over a field $\mathbb{K}$ with characteristic other than 2 and 3 are equivalent to

$$x(yz) = x(zy), \tag{K1'}$$

$$(yx^2)x = (yx)x^2 \tag{K2'}$$

$$(z, y, x^2) = 2(zx, y, x), \tag{K3'}$$

where $(\cdot, \cdot, \cdot)$ denotes the associator.

Then, P. Kolesnikov proposed the following more restrictive definition of quasi-Jordan algebras, called Leibniz-Jordan algebras.

Definition 11 *An algebra J over a field $\mathbb{K}$ with characteristic different from 2 and 3 is said to be a* ***Leibniz-Jordan algebra*** *if it satisfies the identities (K1'), (K2') and (K3').*

More recently, in 2009, Murray Bremner (see [1]), using computer algebra, showed that the product ($\triangleleft$) in a Dialgebra D over a field $\mathbb{K}$ with characteristic other than 2 and 3 satisfies the identities

$$x(yz) = x(zy), \tag{B1}$$

$$(y(xz))u + (y(zu))x + (y(ux))z = (yx)(zu) + (yz)(ux) + (yu)(xz) \tag{B2}$$

$$((xy)u)z + ((xz)u)y + x((yz)u) = (x(yz))u + (x(yu))z + (x(zu))y \tag{B3}$$

These equations are equivalent to the identities

$$x[y,z] = 0, \tag{B1'}$$

$$(yx^2)x = (yx)x^2 \tag{B2'}$$

$$(y, x^2, z) = 2(y, x, z)x \tag{B3'}$$

Then M. Bremner proposed the following definition.

Definition 12 *A quasi-Jordan algebra over a field $\mathbb{K}$ with characteristic different from 2 and 3 is a nonassociative algebra satisfying the polynomial identities (B1'), (B2') and (B3').*

It is simple to see that Kolesnikov and Bremner linear identities are equivalent, since the identities (K1) and (K2) are equal to the identities (B1) and (B2), respectively. If we change x with y in (B3), we obtain the identity

$$((yx)u)z + ((yz)u)x + y((xz)u) = (y(xz))u + (y(xu))z + (y(zu))x.$$

Since the right side of the last identity is equal to left side of the identity (B2), then we obtain the identity (K3). In a similar form we obtain (B3) from (K2) and (K3).

Remark 13 *The definition of Leibniz-Jordan algebra due to Kolesnikov and the definition of quasi-Jordan algebra due to Bremner are equivalent. We call these algebras* ***K-B quasi-Jordan algebras****.*

With respect to generalization of Jordan algebras in this context, K. Liu introduced in 2006 (see [12]) the notion of generalized Jordan algebra. The definition is a follows.

Definition 14 *A* ***generalized Jordan algebra*** *over a field $\mathbb{K}$ with characteristic other than 2 and 3 is a nonassociative algebra satisfying the polynomial identities*

$$x[y,z] = 0, \tag{L1}$$

$$(yx^2)x = (yx)x^2 \tag{L2}$$

$$(x, y, x^2) = 2(x^2, y, x), \tag{HL}$$

where $(\cdot,\cdot,\cdot)$ denotes the associator.

In 2007, K. Liu proved a generalization of Cohns Theorem on Jordan algebras by generalized Jordan algebras (see [13]).

It is easy to see that the Liu's definition of generalized Jordan algebras is a particular case of the Kolesnikov's definition of Leibniz-Jordan algebras. In this context, we are going to work only with the K-B quasi-Jordan algebras in the rest of this paper.

The first step is to give a characterization of K-B quasi-Jordan algebras in terms of Jordan algebras and Jordan bimodules.

We introduce the following definition to construct a characterization of K-B quasi-Jordan algebras.

Definition 15 *Let J be a Jordan algebra and let M be a vector space over the same field as J. Then M is a* **Jordan bimodule** *for J, if there are two bilinear maps $(m, a) \mapsto ma$ and $(m, a) \mapsto am$, for all $m \in M$ and $a \in J$, satisfying*

$$ma = am,$$
$$a^2(ma) = (a^2m)a$$

and

$$(a^2, b, m) = 2(a, b, am),$$

for all $m \in M$ and $a, b \in J$.

Example 16 *Let J be a Jordan algebra over a field $\mathbb{K}$ of characteristic $\neq 2, 3$ and let M be a Jordan bimodule. A linear map $f : M \to J$ is called* **J-equivariant** *over M if $f(am) = a \bullet f(m)$, for all $m \in M$ and $a \in J$. If f is a J-equivariant map over M, and we define a product from $M \times M$ to M by*

$$mn = f(n)m\,, \quad \text{for all } m, n \in M.$$

Then M is a K-B quasi-Jordan algebra.

Indeed, for all $m, n, s \in M$ we have that

$$m(ns) = m(f(s)n) = f(f(s)n)m = (f(s)f(n))m$$

and

$$m(sn) = m(f(n)s) = f(f(n)s)m = (f(n)f(s))m = (f(s)f(n))m,$$

then $m(ns) = m(sn)$.

On the other hand,

$$(nm^2)m = f(m)(f(m)^2n) = (f(m)^2n)f(m)$$

and

$$\begin{aligned}(nm)m^2 &= (f(m)n)(f(m)m) = f(f(m)m)(f(m)n) = f(m)^2(f(m)n)\\ &= f(m)^2(nf(m)),\end{aligned}$$

since $am = ma$, for all $a \in J$ and $m \in M$. Therefore,

$$(nm^2)n = (nm)m^2.$$

Finally, since

$$\begin{aligned}(s,n,m^2) &= (sn)m^2 - s(nm^2) = f(m)^2(f(n)s) - (f(m)^2 f(n))s \\ &= -(f(m)^2, f(n), s)\end{aligned}$$

and

$$\begin{aligned}(sm,n,m) &= ((sm)n)m - (sm)(nm) \\ &= f(m)(f(n)(f(m)s)) - (f(m)f(n))(f(m)s) \\ &= -(f(m), f(n), f(m)s),\end{aligned}$$

then $(s,n,m^2) = 2(sm,n,m)$.

Hence by Kolesnikov's definition we have that M is a K-B quasi-Jordan algebra.

Proposition 17 *Let $\Im$ be a K-B quasi-Jordan algebra and $J := \Im/\Im^{ann}$ its canonical Jordan algebra. Then there exists a J-bimodule structure on $\Im$ and exists a J-equivariant map f over $\Im$ such that the K-B quasi-Jordan structure on $\Im$ is recovered by $xy = f(y)x$.*

Proof. We have that $\Im$ is a Jordan bimudule for the Jordan algebra $J := \Im/\Im^{ann}$, with actions $\overline{y}x = x\overline{y} = xy$, for all $\overline{y} \in J$ and $x \in \Im$.

The map $f : \Im \to J$, defined by $f(x) = \overline{x}$, is J-invariant over $\Im$, since

$$f(\overline{y}x) = f(xy) = \overline{xy} = \overline{yx} = \overline{y}\,\overline{x} = \overline{y}f(x),$$

for all $\overline{y} \in J$ and $x \in \Im$.

The axioms for Jordan bimodule imply that the K-B quasi-Jordan structure on $\Im$ is recovered by $xy = f(y)x$. ■

The main idea in this part is to construct right units over K-B quasi-Jordan algebras.

It is well known that if J is a Jordan algebra which is not unital, then the algebra $\widehat{J} := \mathbb{K} \oplus J$, where $\mathbb{K}$ is the field over J, with product defined by

$$(\alpha + x)\hat{\bullet}(\beta + y) = (\alpha\beta) + (\alpha y + \beta x + xy),$$

for all $\alpha, \beta \in \mathbb{K}$ and $x, y \in J$, is a unital Jordan algebra, with unit $1 + 0$. Additionally, $\{0\} \oplus J \cong J$ and J is embedded in $\widehat{J}$ as a subalgebra.

We are going to use this construction and the characterization of K-B quasi-Jordan algebras by Jordan bimodules to construct right units over K-B quasi-Jordan algebras.

We suppose that $\Im$ is a K-B quasi-Jordan algebra over the field $\mathbb{K}$ of characteristic other than 2 and 3, and $J := \Im/\Im^{ann}$ its canonical Jordan algebra. We consider the vector space $\widehat{\Im} := \mathbb{K} \oplus J \oplus \Im$ and we define the product on $\widehat{\Im}$ by

$$(\alpha \oplus \overline{a} \oplus x)(\beta \oplus \overline{b} \oplus y) = \alpha\beta \oplus \alpha\overline{b} + \beta\overline{a} + \overline{a}\overline{b} \oplus \beta x + xb, \qquad (*)$$

for any $\alpha, \beta \in \mathbb{K}$, $\overline{a}, \overline{b} \in J$ and $x, y \in \Im$.

Remark 18 *The product (*) is well defined, since for* $\overline{b} = \overline{b'}$ *we have* $b - b' \in \Im^{ann}$*, and therefore* $0 = x(b-b') = xb - xb'$*, which is equivalent to* $xb = xb'$*.*

In Addition, if $\Im$ *is a Jordan algebra, then* $J = \Im$ *and we obtain the classical construction.*

If $\Im$ is not a Jordan algebra, then the product (*) is not commutative. Besides, this product satisfies the following identities for all $\alpha, \beta, \gamma \in \mathbb{K}$, $\overline{a}, \overline{b}, \overline{c} \in J$ and $x, y, z \in \Im$.

1. $(0 \oplus \overline{a} \oplus x)(0 \oplus \overline{b} \oplus y) = (0 \oplus \overline{a}\overline{b} \oplus xb)$
2. $\widehat{\Im}$ is a right unital algebra, since $(\alpha \oplus \overline{a} \oplus x)(1 \oplus \overline{0} \oplus y) = \alpha \oplus \overline{a} \oplus x$

The space $\widehat{\Im}$ can be written in the form $\widehat{J} \oplus \Im$ and the product (*) can be written, over $\widehat{J} \oplus \Im$, in the form

$$(\alpha \oplus \overline{a} \oplus x)(\beta \oplus \overline{b} \oplus y) = (\alpha \oplus \overline{a}) \hat{\bullet} (\beta \oplus \overline{b}) \oplus (\beta x + xb). \qquad (**)$$

If we write $(\alpha \oplus \overline{a} \oplus x)(\beta \oplus \overline{b} \oplus y) = (\alpha \oplus \overline{a}) \hat{\bullet} (\beta \oplus \overline{b}) \oplus x(\beta \oplus \overline{b})$, we have the following result.

Lemma 19 *Let* $\Im$ *be a K-B quasi-Jordan algebra over the field* $\mathbb{K}$ *of characteristic other than 2 and 3, and* $J := \Im/\Im^{ann}$ *its canonical Jordan algebra. Then* $\widehat{J} \oplus \Im$ *is a right unital K-B quasi-Jordan algebra with respect to the product defined by (**).*

Proof. Because $\widehat{J}$ is a Jordan algebra with the product $\hat{\bullet}$, we only need to show that the K-B quasi-Jordan algebra axioms are satisfied by the third component.

First, we have that

$$(\beta \oplus \overline{b} \oplus y)(\gamma \oplus \overline{c} \oplus z) = (\beta \oplus \overline{b}) \hat{\bullet} (\gamma \oplus \overline{c}) \oplus (y(\gamma \oplus \overline{c}))$$

and

$$(\gamma \oplus \overline{c} \oplus z)(\beta \oplus \overline{b} \oplus y = (\gamma \oplus \overline{c}) \hat{\bullet} (\beta \oplus \overline{b}) \oplus (y(\gamma \oplus \overline{c})).$$

Since the product $\hat{\bullet}$ is commutative, then these two expressions are different only in the last terms.

From

$$\begin{aligned}
&(\alpha \oplus \overline{a} \oplus x)\left((\beta \oplus \overline{b} \oplus y)(\gamma \oplus \overline{c} \oplus z)\right) \\
&\qquad = (\alpha \oplus \overline{a} \oplus x)\left((\beta \oplus \overline{b}) \hat{\bullet} (\gamma \oplus \overline{c}) \oplus y(\gamma \oplus \overline{c})\right) \\
&\qquad = (\alpha \oplus \overline{a}) \hat{\bullet} \left((\beta \oplus \overline{b}) \hat{\bullet} (\gamma \oplus \overline{c})\right) \oplus x\left((\beta \oplus \overline{b}) \hat{\bullet} (\gamma \oplus \overline{c})\right)
\end{aligned}$$

and

$$\begin{aligned}
&(\alpha \oplus \overline{a} \oplus x)\left((\gamma \oplus \overline{c} \oplus z)(\beta \oplus \overline{b} \oplus y)\right) \\
&\qquad = (\alpha \oplus \overline{a} \oplus x)\left((\gamma \oplus \overline{c}) \hat{\bullet} (\beta \oplus \overline{b}) \oplus z(\beta \oplus \overline{b})\right) \\
&\qquad = (\alpha \oplus \overline{a}) \hat{\bullet} \left((\beta \oplus \overline{b}) \hat{\bullet} (\gamma \oplus \overline{c})\right) \oplus x\left((\beta \oplus \overline{b}) \hat{\bullet} (\gamma \oplus \overline{c})\right),
\end{aligned}$$

we have that $\widehat{\Im}$ satisfies the identity (B1).

Also, from $(\alpha \oplus \overline{a} \oplus x)^2 = (\alpha \oplus \overline{a})^2 \oplus x(\alpha \oplus \overline{a})$, we have that

$$
\begin{aligned}
(\beta \oplus \overline{b} \oplus y)(\alpha \oplus \overline{a} \oplus x)^2 &= (\beta \oplus \overline{b} \oplus y)\left((\alpha \oplus \overline{a})^2 \oplus x(\alpha \oplus \overline{a})\right) \\
&= (\beta \oplus \overline{b})\hat{\bullet}(\alpha \oplus \overline{a})^2 \oplus y(\alpha \oplus \overline{a})^2 \\
&= (\beta \oplus \overline{b})\hat{\bullet}(\alpha \oplus \overline{a})^2 \oplus y(\alpha^2 \oplus 2\alpha\overline{a} + \overline{a}^2).
\end{aligned}
$$

Therefore, since

$$
\begin{aligned}
&((\beta \oplus \overline{b} \oplus y)(\alpha \oplus \overline{a} \oplus x)^2)(\alpha \oplus \overline{a} \oplus x) \\
&= \left((\beta \oplus \overline{b})\hat{\bullet}(\alpha \oplus \overline{a})^2 \oplus y(\alpha^2 \oplus 2\alpha\overline{a} + \overline{a}^2)\right)(\alpha \oplus \overline{a} \oplus x) \\
&= \left((\beta \oplus \overline{b})\hat{\bullet}(\alpha \oplus \overline{a})^2\right)(\alpha \oplus \overline{a}) \oplus \left(y(\alpha^2 \oplus 2\alpha\overline{a} + \overline{a}^2\right)(\alpha \oplus \overline{a}) \\
&= \left((\beta \oplus \overline{b})\hat{\bullet}(\alpha \oplus \overline{a})^2\right)(\alpha \oplus \overline{a}) \oplus (\alpha^2 y + 2\alpha y a + y a^2)(\alpha \oplus \overline{a}) \\
&= \left((\beta \oplus \overline{b})\hat{\bullet}(\alpha \oplus \overline{a})^2\right)(\alpha \oplus \overline{a}) \oplus \left(\alpha^3 y + 3\alpha^2 y a + \alpha y a^2 + 2\alpha(ya)a + (ya^2)a\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&((\beta \oplus \overline{b} \oplus y)(\alpha \oplus \overline{a} \oplus x))(\alpha \oplus \overline{a} \oplus x)^2 \\
&= \left((\beta \oplus \overline{b})\hat{\bullet}(\alpha \oplus \overline{a}) \oplus y(\alpha \oplus \overline{a})\right)\left((\alpha \oplus \overline{a})^2 \oplus x(\alpha \oplus \overline{a})\right) \\
&= \left((\beta \oplus \overline{b})\hat{\bullet}(\alpha \oplus \overline{a})\right)\hat{\bullet}(\alpha \oplus \overline{a})^2 \oplus (y(\alpha \oplus \overline{a}))(\alpha \oplus \overline{a})^2 \\
&= \left((\beta \oplus \overline{b})\hat{\bullet}(\alpha \oplus \overline{a})\right)\hat{\bullet}(\alpha \oplus \overline{a})^2 \oplus (\alpha y + ya)(\alpha^2 \oplus 2\alpha\overline{a} + \overline{a}^2) \\
&= \left((\beta \oplus \overline{b})\hat{\bullet}(\alpha \oplus \overline{a})\right)\hat{\bullet}(\alpha \oplus \overline{a})^2 \oplus \left(\alpha^3 y + 3\alpha^2 y a + \alpha y a^2 + 2\alpha(ya)a + (ya)a^2\right),
\end{aligned}
$$

then $\widehat{\Im}$ satisfies the identity (B2).

Finally, since

$$
\begin{aligned}
&((\beta \oplus \overline{b} \oplus y)(\alpha \oplus \overline{a} \oplus x)^2)(\gamma \oplus \overline{c} \oplus z) \\
&= \left((\beta \oplus \overline{b})\hat{\bullet}(\alpha \oplus \overline{a})^2 \oplus y(\alpha^2 \oplus 2\alpha\overline{a} + \overline{a}^2)\right)(\gamma \oplus \overline{c} \oplus z) \\
&= \left((\beta \oplus \overline{b})\hat{\bullet}(\alpha \oplus \overline{a})^2\right)\hat{\bullet}(\gamma \oplus \overline{c}) \oplus \left(\alpha^2 y + 2\alpha y a + y a^2\right)(\gamma \oplus \overline{c}) \\
&= \left((\beta \oplus \overline{b})\hat{\bullet}(\alpha \oplus \overline{a})^2\right)\hat{\bullet}(\gamma \oplus \overline{c}) \oplus \left(\alpha^2\gamma y + 2\alpha\gamma y a + \gamma y a^2 + \alpha^2 y c + 2\alpha(ya)c + (ya^2)c\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&(\beta \oplus \overline{b} \oplus y)\left((\alpha \oplus \overline{a} \oplus x)^2(\gamma \oplus \overline{c} \oplus z)\right) \\
&= (\beta \oplus \overline{b} \oplus y)\left(\left((\alpha \oplus \overline{a})^2 \oplus x(\alpha \oplus \overline{a})\right)(\gamma \oplus \overline{c} \oplus z)\right) \\
&= (\beta \oplus \overline{b} \oplus y)\left((\alpha \oplus \overline{a})^2\hat{\bullet}(\gamma \oplus \overline{c}) \oplus (x(\alpha \oplus \overline{a}))(\gamma \oplus \overline{c})\right) \\
&= (\beta \oplus \overline{b})\hat{\bullet}\left((\alpha \oplus \overline{a})^2\hat{\bullet}(\gamma \oplus \overline{c})\right) \oplus y(\alpha^2\gamma \oplus \alpha^2\overline{c} + 2\alpha\gamma\overline{a} + \gamma\overline{a}^2 + 2\alpha\overline{a}\overline{c} + \overline{a}^2\overline{c}) \\
&= (\beta \oplus \overline{b})\hat{\bullet}\left((\alpha \oplus \overline{a})^2\hat{\bullet}(\gamma \oplus \overline{c})\right) \oplus \left(\alpha^2\gamma\, y + \alpha^2 y c + 2\alpha\gamma y a + \gamma y a^2 + 2\alpha y(ac) + y(a^2 c)\right),
\end{aligned}
$$

then

$$
\begin{aligned}
&(\beta \oplus \overline{b} \oplus y, (\alpha \oplus \overline{a} \oplus x)^2, \gamma \oplus \overline{c} \oplus z) \\
&= ((\beta \oplus \overline{b} \oplus y)(\alpha \oplus \overline{a} \oplus x)^2)(\gamma \oplus \overline{c} \oplus z) - (\beta \oplus \overline{b} \oplus y)\left((\alpha \oplus \overline{a} \oplus x)^2(\gamma \oplus \overline{c} \oplus z)\right) \\
&= (\beta \oplus \overline{b}, (\alpha \oplus \overline{a})^2, \gamma \oplus \overline{c})_{\hat{\bullet}} \oplus \left(2\alpha(y, a, c)_{\Im} + (y, a^2, c)_{\Im}\right),
\end{aligned}
$$

where $(\cdot,\cdot,\cdot)_{\hat{\bullet}}$ denotes the associator in $\widehat{J}$ and $(\cdot,\cdot,\cdot)_{\Im}$ denotes the associator in $\Im$.

In a similar way, we can show that

$$
\begin{aligned}
(\beta \oplus \overline{b} \oplus y, &\alpha \oplus \overline{a} \oplus x, \gamma \oplus \overline{c} \oplus z)(\alpha \oplus \overline{a} \oplus x) \\
&= \left((\beta \oplus \overline{b}, \alpha \oplus \overline{a}, \gamma \oplus \overline{c})_{\hat{\bullet}} \hat{\bullet} (\alpha \oplus \overline{a})\right) \oplus \left(\alpha(y,a,c)_{\Im} + (y,a,c)_{\Im} a\right),
\end{aligned}
$$

hence $\widehat{\Im}$ satisfies the identity (B3), and therefore $\widehat{J} \oplus \Im$ is a right unital K-B quasi-Jordan algebra. ■

If we consider the vector space $\widehat{\Im}_0$, generated by the elements $0 \oplus \overline{x} \oplus x$, for all $x \in \Im$, we have

$$(0 \oplus \overline{x} \oplus x)(0 \oplus \overline{y} \oplus y) = 0 \oplus \overline{xy} \oplus xy,$$

for all $0 \oplus \overline{x} \oplus x$, $0 \oplus \overline{y} \oplus y$ in $\widehat{\Im}_0$. Therefore $\widehat{\Im}_0$ is a subalgebra of $\widehat{\Im}$.

Moreover, if we consider the application $\Im \to \widehat{\Im}_0$ define by $x \mapsto 0 \oplus \overline{x} \oplus x$, then we have that $\Im$ is isomorphic to $\widehat{\Im}_0$ and we have the following result.

Lemma 20 *Any K-B quasi-Jordan algebra $\Im$ is embedded in the unital K-B quasi-Jordan algebra $\widehat{\Im}$. In addition, we have*

1. $\widehat{\Im}^{ann} = Z^r(\widehat{\Im}) = \{0\} \oplus \{\overline{\mathbf{0}}\} \oplus \Im$
2. $\widehat{\Im}_0^{ann} = \{0\} \oplus \{\overline{\mathbf{0}}\} \oplus \Im^{ann}$
3. $U_r(\widehat{\Im}) = \{1\} \oplus \{\overline{\mathbf{0}}\} \oplus \Im$.

In 2009, Velásquez and Felipe introduced the definition of split quasi-Jordan algebra and showed some properties.

Definition 21 *Let $\Im$ be a quasi-Jordan algebra and let I be an ideal in $\Im$ such that $\Im^{ann} \subset I \subset Z^r(\Im)$. We said that $\Im$ is* **split** *over I if there is a subalgebra J of $\Im$ such that $\Im = I \oplus J$, as direct sum of subspaces.*

It is clear from the previous definition that if $\Im$ is split over an ideal I with complement J, then J is a Jordan algebra with respect to restricted product $\triangleleft$ over J. This is equivalent to saying that $(J, \triangleleft|_J)$ is a Jordan algebra.

In addition, for $u, v \in I$ and $x, y \in J$ we have

$$(u + x) \triangleleft (v + y) = u \triangleleft y + x \triangleleft y,$$

since $I \subset Z^r(\Im)$.

From the definition of split quasi-Jordan algebra and Lemma 20, we have

Lemma 22 *Let $\Im$ be a K-B quasi-Jordan algebra. Then $\widehat{\Im} = \widehat{J} \oplus \Im$ and therefore $\widehat{\Im}$ is a split K-B quasi-Jordan algebra.*

4 Relationship between K-B quasi-Jordan algebras and Leibniz algebras

In this section we will work with K-B quasi-Jordan algebras and Leibniz algebras L over a field $\mathbb{K}$ of characteristic other than 2 and 3 . In particular, we have that L is 2 and 3-torsion free.

In 2008, Velásquez and Felipe showed that it is possible to attach a quasi-Jordan algebra to some elements of a Leibniz algebra L. For this construction we need to introduce the following definitions.

Definition 23 *Let L be a Leibniz algebra. For all $x \in L$, we define the* **adjoint** *map $ad_x : L \to L$ by $ad_x y = [y, x]$, for all $y \in L$.*

Remark 24 *The map $ad : L \to gl(L)$, $x \mapsto ad_x$, where $gl(L)$ is the Lie algebra of linear maps over L with a Lie bracket $[T, S] = TS - ST$, it is an antihomomorphism of Leibniz algebras, this is*

$$ad_{[x,y]} = [ad_y, ad_x] \,, \quad \text{for all } x, y \in L$$

The set $ad(L) = \{ad_x | x \in L\}$ with the bracket defined by $[ad_x, ad_y] := ad_x ad_y - ad_y ad_x$ turns out to be a Lie algebra, in particular it is a Lie subalgebra of $gl(L)$.

Notation 25 *We will use capital letters to denote the adjoint maps (the elements of $ad(L)$): $X = ad_x$, $Y = ad_y$, etcetera. In this notation the last identity has the form*

$$ad_{[x,y]} = [Y, X]$$

Definition 26 *Let L be a Leibniz algebra and let x be an element in L. We say that x is an* **ad-nilpotent element** *if there is a positive integer m such that $ad_x^m = 0$.*

We say that an element x in a Leibniz algebra L is a **Q-Jordan element** *if x is an ad-nilpotent element of index at most 3.*

In the rest of this section we will work with Leibniz algebras over a field $\mathbb{K}$ containing $1/6$. In particular, we have that L is 2 and 3-torsion free.

First, we have the following lemma.

Lemma 27 *Let x be a Jordan element of a Leibniz algebra L. For any $a, b \in L$ and $\alpha \in \mathbb{K}$, we have*

1. $X^2 AX = XAX^2$
2. $X^2 AX^2 = 0$
3. $X^2 A^2 XAX^2 = X^2 AXA^2 X^2$
4. $[X^2(a), X(b)] = -[X(a), X^2(b)]$
5. $ad_x^2([a, [b, x]]) = [X(a), X^2(b)]$
6. $X^2 ad_{[a,X^2(b)]} = ad_{[X^2(a),b]} X^2$
7. $ad^2_{X^2(a)} = X^2 A^2 X^2$
8. *αx, $ad_x^2(a)$ are Jordan elements in L,*

where $A = ad_a$

Theorem 28 *Let L be a Leibniz algebra and let x be a Q-Jordan element of L. Then L with the product defined by*

$$ab := \frac{1}{2}[a, [b, x]]$$

is a nonassociative algebra, denoted by $L^{(x)}$, such that

$$Ker_L(x) := \{a \in L | [[a, x], x] = 0\}$$

is an ideal of $L^{(x)}$.

We attach a quasi-Jordan algebra to any Q-Jordan element x of a Leibniz algebra L in the following form.

Theorem 29 *Let L be a Leibniz algebra and let x be a Q-Jordan element of L. Then $L_x := L^{(x)}/Ker_L(x)$ is a quasi-Jordan algebra. Moreover, L_x is a noncommutative algebra in general.*

Definition 30 *For any Q-Jordan element x of a Leibniz algebra L, the quasi-Jordan algebra L_x we have just introduced will be called the* **quasi-Jordan algebra of L at** *x.*

The next example of Leibniz algebras appeared in the work of K. Liu [14]. This Leibniz algebra generates a noncommutative quasi-Jordan algebra and it follows that in general the quasi-Jordan algebras obtained from Leibniz algebras and Q-Jordan elements are noncommutative algebras.

Example 31 *(K. Liu) Let L be a Leibniz algebra over a field $\mathbb{K}$ (of characteristic zero) with basis $\{h, e, f, u, v, w\}$ defined by*

$$[h, e] = 2e + 2\alpha u \quad [h, f] = -2f + \beta w \quad [e, h] = -2e \quad [e, f] = h + \alpha v$$

$$[f, h] = 2f \quad [f, e] = -h - \beta v \quad [u, h] = -2u \quad [u, f] = -v$$

$$[v, e] = -2u \quad [v, f] = -w \quad [w, h] = 2w \quad [w, e] = -2v,$$

where the omitted products are equal to zero and α, β are fixed elements of the field $\mathbb{K}$.

We have that e is a Q-Jordan element in L and $Ker_L(e)$ is the subspace generated by $\{e, h, u, v\}$.

Since $\overline{u}\,\overline{f} = \overline{f}$ and $\overline{f}\,\overline{u} = \overline{0}$, then L_e is not commutative.

The relationship between Leibniz algebras with K-B quasi-Jordan algebras is given by the following lemma.

Lemma 32 *Let L be a Leibniz algebra and let x be a Q-Jordan element of L. Then the quasi-Jordan algebra of L at x, L_x, satisfies the identity*

$$(\overline{b}, \overline{a}^2, \overline{c}) = 2(\overline{b}, \overline{a}, \overline{c}) \triangleleft \overline{a},$$

and therefore L_x is a K-B quasi-Jordan algebra.

Proof. First, for all $a, b, c \in L$ we have that

$$\begin{aligned}(ba^2)c &= [ba^2, [c,x]] = [[b,[a^2,x]],[c,x]] = [[b,[[a,[a,x]],x]],[c,x]] \\ &= [X,C][X,[[X,A],A]](b),\end{aligned}$$

$$\begin{aligned}b(a^2c) &= [b,[a^2c,x]] = [b,[[a^2,[c,x]],x]] = [b,[[[a,[a,x]],[c,x]],x]] \\ &= [X,[[X,C],[[X,A],A]]](b),\end{aligned}$$

$$\begin{aligned}((ba)c)a &= [(ba)c,[a,x]] = [[ba,[c,x],[a,x]] = [[[b,[a,x]],[c,x]],[a,x]] \\ &= [X,A][X,C][X,A](b)\end{aligned}$$

and

$$\begin{aligned}(b(ac))a &= [b(ac),[a,x]] = [[b,[ac,x]],[a,x]] = [[b,[[a,[c,x]],x],[a,x]] \\ &= [X,A][X,[[X,C],A]](b).\end{aligned}$$

Now we consider b as a variable and we will work with the maps in the right hand side of the last equations. The identities in the Lemma 27 imply

$$\begin{aligned}[X,C][X,[[X,A],A]] = {} & XCX^2A^2 - 2XCXAXA + 2XCAXAX \\ & - XCA^2X^2 + 2CX^2AXA - 2CXAXAX \\ & + CXA^2X^2,\end{aligned} \tag{1}$$

$$\begin{aligned}[X,[[X,C],[[X,A],A]]] = {} & -2X^2CAXA + 2XCXAXA + X^2CA^2X \\ & - 2XCXA^2X + X^2A^2CX + 2XAXAXC \\ & - 2XAXACX + XA^2XCX + 2CX^2A^2X \\ & - 2X^2A^2XC + 2XCAXAX - 2CXAXAX \\ & - XCA^2X^2 + XA^2XCX - XA^2CX^2 \\ & - 2AXAXCX + 2AXACX^2,\end{aligned} \tag{2}$$

$$\begin{aligned}[X,A][X,C][X,A] = {} & XAXCXA - XAXCAX - XACX^2A \\ & + XACXAX - AX^2CXA + AX^2CAX \\ & + AXCX^2A - AXCXAX\end{aligned} \tag{3}$$

and

$$\begin{aligned}[X,A][X,[[X,C],A]] = {} & XAX^2CA - XAXCXA + AX^2XCA \\ & - XAXAXC + AX^2AXC + XAXACX \\ & - AX^2ACX - XAXCAX + AX^2CAX \\ & + XACXAX - AXCXAX + XA^2XCX \\ & - AXAXCX - XA^2CX^2 + AXACX^2.\end{aligned} \tag{4}$$

Now, if we apply the map X^2 on the left hand side of (1), (2), (3) and (4), we obtain

$$X^2([X,A][X,[[X,C],A]]) = -2X^2CXAXAX, \tag{5}$$

$$\begin{aligned} X^2([X,[[X,C],[[X,A],A]]]) = &-2X^2CXAXAX - 2X^2AXAXCX \\ &+ 2X^2AXACX^2, \end{aligned} \tag{6}$$

$$X^2([X,A][X,C][X,A]) = -X^2AXCXAX \tag{7}$$

and

$$\begin{aligned} X^2([X,A][X,[[X,C],A]]) = &-X^2AXCXAX - X^2AXAXCX \\ &+ X^2AXACX^2. \end{aligned} \tag{8}$$

The identities (5), (6), (7) and (8) imply

$$\begin{aligned} X^2([X,A][X,[[X,C],A]] &- [X,[[X,C],[[X,A],A]]] \\ &- 2[X,A][X,C][X,A] + 2[X,A][X,[[X,C],A]]) = \mathbf{0} \end{aligned}$$

and therefore

$$\begin{aligned} ([X,A][X,[[X,C],A]] &- [X,[[X,C],[[X,A],A]]] \\ &- 2[X,A][X,C][X,A] + 2[X,A][X,[[X,C],A]])(b) \in Ker_L(x), \end{aligned} \tag{9}$$

for all $b \in L$.

The identity 9 is equivalent to

$$(ba^2)c - b(a^2c) - 2((ba)c)a + 2(b(ac))a \in Ker_L(x),$$

for all $a, b, c \in L$, and this implies that

$$(\overline{b}\overline{a}^2)\overline{c} - \overline{b}(\overline{a}^2\overline{c}) - 2((\overline{b}\overline{a})\overline{c})\overline{a} + 2(\overline{b}(\overline{a}\overline{c}))\overline{a} = \overline{0},$$

for all $\overline{a}, \overline{b}, \overline{c} \in L_x$. Therefore

$$(\overline{b}, \overline{a}^2, \overline{c}) = 2(\overline{b}, \overline{a}, \overline{c})\overline{a},$$

for all $\overline{a}, \overline{b}, \overline{c} \in L_x$. ■

5 Leibniz algebras generated by K-B quasi-Jordan algebras

In the late 1960s I. Kantor and M. Koecher independently showed how build a Lie algebra $TKK(J) := L_1 \oplus L_0 \oplus L_{-1}$ with short 3-grading $[L_i, L_j] \subseteq Li + j$ by taking $L_{\pm 1}$ two copies of any Jordan algebra J glued together by the structure Lie algebra $L_0 = Strl(J) := R(J) \oplus Der(J)$ and by the inner Lie algebra $L_0 = Inn(J) := R(J) \oplus [R(J), R(J)]$, where $R(J)$ is spanned by all right multiplication operators R_x, $Der(J)$ is the Lie algebra spanned by all derivations and $[R(J), R(J)]$ is spanned by all inner derivations $[R_x, R_y] = R_x R_y - R_y R_x$, where J is a Jordan algebra and $x, y \in J$ (see [10] and [17]).

In this section we are going to show that there are inner derivations (classical and left derivations) in K-B quasi-Jordan algebras and we will define the structure and inner Leibniz algebra by K-B quasi-Jordan algebras. These results were obtained by R. Velásquez in [20] and some other results were obtained by R. Felipe in [7].

We start with the notions of derivation and left derivation over algebras.

Definition 33 *Let A be an algebra over a field $\mathbb{K}$ and let D and ∂ be a linear transformations over A. Then D is called a* ***derivation*** *of A if*

$$D(ab) = Da\, b + a\, Db, \quad \forall a, b \in A.$$

The linear transformation ∂ is called a ***left derivation*** *of A if*

$$\partial(ab) = \partial a\, b + \partial b\, a, \quad \forall a, b \in \Im.$$

If the algebra A is commutative, then derivations and left derivations agree.

It is well known that a linear transformation D of A is a derivation if and only if $L_{Da} = [D, L_a]$ and if and only if $R_{Da} = [D, R_a]$, for all $a \in A$, where L_a and R_a denotes the left and right multiplication operators, respectively, and $[B, C] = BC - CB$ denotes the Lie bracket of linear transformations.

Moreover, if D_1 and D_2 are a derivations of A, then the bracket $[D_1, D_2]$ is a derivation of A. Therefore the vector space generated by derivations over A, denoted by $Der(A)$, is a Lie algebra over $\mathbb{K}$, namely a subalgebra of the Lie algebra of linear transformations.

From the definition of left derivations, we obtain the following characterization.

Corollary 34 *Let ∂ be a linear transformation of A. Then ∂ is a left derivation of A if and only if $L_{\partial a} = [\partial, R_a]$, for all $a \in A$.*

The Lie bracket of left derivations is not in general a left derivation, but the Lie bracket of a derivation and a left derivation is a left derivation. Indeed, if D is a derivation and ∂ is a left derivation of A, then for all $a \in A$

$$\begin{aligned} L_{[\partial,D](a)} &= L_{\partial(Da)-D(\partial Da)} = L_{\partial(Da)} - L_{D(\partial Da)} \\ &= [\partial, R_{Da}] - [D, L_{\partial a}] = [\partial, [D, R_a]] - [D, [\partial, R_a]] \\ &= -[[D, R_a], \partial] - [[R_a, \partial], D] = [[\partial, D], R_a], \end{aligned}$$

hence $[\partial, D]$ is a left derivation of A.

The vector space generated by left derivations of A it is denoted by $LDer(A)$.

If we consider the direct sum of the vector spaces

$$LD(A) := LDer(A) \oplus Der(A)$$

and we define product $\langle\cdot,\cdot\rangle : LD(A)\times LD(A)\to LD(A)$ by

$$\left\langle \partial\oplus D,\partial'\oplus D'\right\rangle = [\partial,D']\oplus[D,D'],$$

for all $\partial\oplus D,\partial'\oplus D'\in LD(A)$, we obtain the following result.

Lemma 35 *Let A be an algebra, then $LD(A)$ is a Leibniz algebra with the product $\langle\cdot,\cdot\rangle$.*

Proof. Let $\partial_1,\partial_2,\partial_3$ be a left derivations and let D_1,D_2,D_3 be a derivations. Then

1. $\langle\langle\partial_1\oplus D_1,\partial_2\oplus D_2\rangle,\partial_3\oplus D_3\rangle = [[\partial_1,D_2],D_3]\oplus[[D_1,D_2],D_3]$
2. $\langle\langle\partial_1\oplus D_1,\partial_3\oplus D_3\rangle,\partial_2\oplus D_2\rangle = [[\partial_1,D_3],D_2]\oplus[[D_1,D_3],D_2]$
3. $\langle\partial_1\oplus D_1,\langle\partial_2\oplus D_2,\partial_3\oplus D_3\rangle\rangle = [\partial_1,[D_2,D_3]]\oplus[D_1,[D_2,D_3]]$

By the skew-symmetry and the Jacobi identity of the Lie bracket, we have that

$$[[\partial_1,D_2],D_3] = [[\partial_1,D_3],D_2]+[\partial_1,[D_2,D_3]]$$

and

$$[[D_1,D_2],D_3] = [[D_1,D_3],D_2]+[D_1,[D_2,D_3]].$$

Replacing the last identities in the items 1, 2 and 3, then

$$\begin{aligned}&\langle\langle\partial_1\oplus D_1,\partial_2\oplus D_2\rangle,\partial_3\oplus D_3\rangle =\\ &\qquad\langle\langle\partial_1\oplus D_1,\partial_3\oplus D_3\rangle,\partial_2\oplus D_2\rangle+\langle\partial_1\oplus D_1,\langle\partial_2\oplus D_2,\partial_3\oplus D_3\rangle\rangle,\end{aligned}$$

for all $\partial_1\oplus D_1,\partial_2\oplus D_2,\partial_3\oplus D_3\in LD(A)$. Therefore $(LD(A),\langle\cdot,\cdot\rangle)$ is a Leibniz algebra. ■

In this section we are going to construct Leibniz algebras defined by K-B quasi-Jordan algebras. In particular, we introduce two structure Leibniz algebras and two inner Leibniz algebras.

Let $\Im$ be a K-B quasi-Jordan algebra, then we have two equivalent sets of linear identities, Kolesnikov's identities

$$x(yz) = x(zy), \tag{K1}$$
$$(y(xz))u+(y(zu))x+(y(ux))z = (yx)(zu)+(yz)(ux)+(yu)(xz), \tag{K2}$$
$$x(y(zu))+((xu)y)z+((xz)y)u = (xy)(zu)+(xz)(yu)+(xu)(yz) \tag{K3}$$

and Bremner's identities

$$x(yz) = x(zy), \tag{B1}$$
$$(y(xz))u+(y(zu))x+(y(ux))z = (yx)(zu)+(yz)(ux)+(yu)(xz), \tag{B2}$$
$$((xy)u)z+((xz)u)y+x((yz)u) = (x(yz))u+(x(yu))z+(x(zu))y. \tag{B3}$$

Because of the consequences of definition 12, we will work only with Bremner's identities.

Rewriting Bremner's identities using the left and right multiplication operators, we obtain from (B1) the identities

$$L_xL_y = L_xR_y, \tag{B1'}$$

$$R_{xy} = R_{yx}. \tag{B1”}$$

From (B2) we obtain

$$R_xR_{yz} + R_yR_{zx} + R_zR_{xy} = R_{yz}R_x + R_{zx}R_y + R_{xy}R_z, \tag{B2’}$$

or equivalently

$$[R_x, R_{yz}] + [R_y, R_{zx}] + [R_z, R_{xy}] = 0$$

and

$$L_{x(yz)} + R_yL_xR_z + R_zL_xR_y = R_{yz}L_x + L_{xz}L_y + L_{xy}L_z. \tag{B2”}$$

The bracket form of the (B2’) identity was obtained by R. Felipe in relation with split quasi-Jordan algebras in [7].

From (B3) we obtain

$$R_xR_yR_z + R_zR_xR_y + R_{zy}x = R_xR_{yz} + R_yR_{zx} + R_zR_{xy} \tag{B3’}$$

and

$$L_xR_{yz} + R_yL_{xz} + R_zL_{xy} = L_{xy}R_z + L_{xz}R_y + R_{yz}L_x. \tag{B3”}$$

On the other hand (B3”) is equivalent to

$$[L_x, R_{yz}] + [R_y, L_{xz}] + [R_z, L_{xy}] = 0$$

and

$$L_{(xz)y} + R_zR_yL_x + L_xR_yR_z = R_yL_xR_z + L_{x(zy)} + R_zL_xR_y. \tag{B3*}$$

The last identity (B3*) implies that

$$L_{[L_x,R_y](z)} = [[L_x, R_y], R_z] \tag{LD}$$

and we conclude from this identity that $[L_x, R_y] \in LDer(\Im)$.

Moreover, the left hand side of (B2’) is equal to the right hand side of (B3’) and hence we obtain the identity

$$R_{yz}R_x + R_{zx}R_y + R_{xy}R_z = L_xR_{yz} + R_yL_{xz} + R_zL_{xy}.$$

Interchanging x and y in this identity and subtracting this from the resulting identity we get

$$R_{[R_x,R_y](z)} = [[R_x, R_y], R_z] \tag{D}$$

and we conclude that $[R_x, R_y] \in Der(\Im)$. The identity (D) was obtained by R. Felipe in relation with quasi-Jordan algebras (see [7]).

We are now going to construct two structure and inner algebras of a K-B quasi-Jordan algebra.

5.1 The structure and inner Leibniz algebras

In this subsection we will assume that $\Im$ is a K-B quasi-Jordan algebra with a specific right unit e.

We recall (see [18]) that if $\Im$ is a unital K-B quasi-Jordan algebra, with a specific right unit e, then

$$\Im^{\mathrm{ann}} = Z^r(\Im),$$

$$\Im^{\mathrm{ann}} = \{x \in \Im | e \triangleleft x = 0\}$$

and

$$U_r(\Im) = \{x + e | x \in \Im^{\mathrm{ann}}\}.$$

First, we are going to show the action of left derivations and derivations over right units. Let $\partial \in LDer(\Im)$ and $D \in Der(\Im)$, then

$$\partial e = \partial(ee) = \partial e\, e + \partial e\, e,$$

this is equivalent to $\partial e = 0$, and

$$De = D(ee) = De\, e + e\, e = De + e\, De.$$

Since $x \in \Im^{\mathrm{ann}}$ if and only if $ex = 0$, then $De \in \Im^{\mathrm{ann}}$.

Since $L_x(e) = x$, for all $x \in \Im$, we have that $L(\Im) \cap LDer(\Im) = \{\mathbf{0}\}$, where $L(\Im)$ denotes the vector space generated by left multiplication operators.

Then we consider the direct sum

$$SL(\Im) := L(\Im) \oplus LDer(\Im).$$

Next, we consider the vector space generated by the right multiplication operators, denoted by $R(\Im)$. For any $x \in \Im$, we have

1. If $x \in \Im^{\mathrm{ann}}$, then $R_x = \mathbf{0}$.
2. If $R_x = \mathbf{0}$, then $R_x(e) = ex = 0$ and therefore $x \in \Im^{\mathrm{ann}}$.

These results imply that $R_x = \mathbf{0}$ if and only if $x \in \Im^{\mathrm{ann}}$.

Let $x \notin \Im^{\mathrm{ann}}$ and we suppose $R_x e \in \Im^{\mathrm{ann}}$. Then $ex \in \Im^{\mathrm{ann}}$ and

$$ex = e(xe) = e(ex) = 0.$$

This is equivalent to $x \in \Im^{\mathrm{ann}}$, but this is a contradiction. Hence we conclude that $R_x e \notin \Im^{\mathrm{ann}}$.

Since $R_x e \notin \Im^{\mathrm{ann}}$, for any $x \notin \Im^{\mathrm{ann}}$, $R_x = \mathbf{0}$ if and only if $x \in \Im^{\mathrm{ann}}$ and $De \in \Im^{\mathrm{ann}}$, we have $R(\Im) \cap Der(\Im) = \{\mathbf{0}\}$, and therefore we can consider the direct sum

$$SR(\Im) = R(\Im) \oplus Der(\Im).$$

Remark 36 *If $\Im$ is a Jordan algebra, then $SL(\Im)$ and $SR(\Im)$ are the same vector space. This vector space is the classical structure algebra (Lie algebra) of the Jordan algebra $\Im$.*

For our purpose, we take the formal sum of vector spaces

$$S(\Im) := LS(\Im)\bar{\oplus}SR(\Im),$$

and we define the product

$$[(L_x \oplus \partial\bar{\oplus}R_y \oplus D), (L_z \oplus \partial'\bar{\oplus}R_w \oplus D')] :=$$

$$([L_x, D']+[\partial, R_w])\oplus([L_x, R_w]+[\partial, D'])\bar{\oplus}([R_y, D']+[D, R_w])\oplus([R_y, R_w]+[D, D']),$$

for any $a, b, c, d \in \Im$, $\partial, \partial' \in LDer(\Im)$ and $D, D' \in Der(\Im)$, where $[\cdot,\cdot]$ denoted the Lie bracket in $End(\Im)$.

Theorem 37 *Let $\Im$ be a unital K-B quasi-Jordan algebra, then $(S(\Im), [\cdot,\cdot])$ is a Leibniz algebra.*

Proof. From the identity

$$[[A, B], C] = [[A, C], B] + [A, [B, C]],$$

for all A, B, C in the Lie algebra $End(\Im)$, the result follows from a straightforward calculations. ■

Definition 38 *The Leibniz algebra $S(\Im)$ is called the **structure Leibniz algebra** of the unital K-B quasi-Jordan algebra $\Im$.*

Since the identities (LD) and (D) imply that $[L_x, R_y] \in LDer(\Im)$ and $[R_z, R_w] \in Der(\Im)$, for all $x, y, z, w \in \Im$, respectively, we can consider the vector space $[L(\Im), R(\Im)]$ spanned by the products $[L_x, R_y]$ and the vector sapce $[R(\Im), R(\Im)]$ panned by the products $[R_z, R_w]$.

The direct sum $[L(\Im), R(\Im)] \oplus [R(\Im), R(\Im)]$ is a subalgebra of the Leibniz algebra $(LD(\Im), \langle\cdot,\cdot\rangle)$.

Now, if we consider the direct sums $L(\Im) \oplus [L(\Im), R(\Im)]$ and $R(\Im) \oplus [R(\Im), R(\Im)]$, we can define the Leibniz algebra

$$L(\Im) \oplus [L(\Im), R(\Im)]\overline{\oplus}R(\Im) \oplus [R(\Im), R(\Im)],$$

with the bracket defined in Theorem 37. This Leibniz algebra is called the **inner Leibniz algebra** of $\Im$.

Remark 39 *If $\Im$ is a Jordan algebra, then $[L(\Im), R(\Im)]$ and $[R(\Im), R(\Im)]$ are the same vector space. This vector space is the classical inner algebra (Lie algebra) of the Jordan algebra $\Im$, and it is a subalgebra of the structure algebra.*

5.2 The quasi-structure and quasi-inner Leibniz algebras

The constructions of the structure Leibniz algebra, $S(\Im)$, and the inner Leibniz algebra, $L(\Im) \oplus [L(\Im), R(\Im)]\overline{\oplus}R(\Im) \oplus [R(\Im), R(\Im)]$, of a unital K-B quasi-Jordan algebra $\Im$, give two copies of the classical structure and inner algebras of a Jordan algebra, respectively, if $\Im$ is a Jordan algebra. Hence it is necessary to consider other type of construction of structure and inner Leibniz algebras for K-B quasi-Jordan algebras.

To make this other construction. We consider of vector subspaces of $End(\Im)$:

1. The vector space $M(\Im)$ spanned by the multiplication operators L_x and R_y, for all $x, y \in \Im$.
2. The vector space $D(\Im)$ spanned by the left derivations and derivations of $\Im$,

where $\Im$ is a K-B quasi-Jordan algebra.

We define the bracket $[\cdot,\cdot]$ over the direct sum $M(\Im) \oplus D(\Im)$ by

$$[L_x + R_y, L_z + R_w] = [L_x, R_w] + [R_y, R_w] \in D(\Im), \tag{10}$$

$$[L_x + R_y, \partial' + D'] = [L_x, D'] + [R_y, D'] \in M(\Im), \tag{11}$$

$$[\partial + D, L_z + R_w] = [\partial, R_w] + [D, R_w] \in M(\Im) \tag{12}$$

and

$$[\partial + D, \partial' + D'] = [\partial, D'] + [D, D'] \in D(\Im). \tag{13}$$

From this definition and some straightforward calculations we obtain the following.

Theorem 40 *Let $\Im$ be a K-B quasi-Jordan algebra. Then $M(\Im) \oplus D(\Im)$ equipped with the bracket defined by (10), (11), (12) and (13) is a Leibniz algebra.*

Considering the vector space spanned by $[L_x, R_y]$ and $[R_z, R_w]$, for all $x, y, z, w \in \Im$, denoted by $[M(\Im), M(\Im)]$, we get the following.

Lemma 41 *Let $\Im$ be a K-B quasi-Jordan algebra. Then $M(\Im) \oplus [M(\Im), M(\Im)]$ is a Leibniz subalgebra of $M(\Im) \oplus D(\Im)$.*

Since $L_x = R_x$, for all $x \in J$ and the left derivations of J are a derivations of J, where J is a Jordan algebra, hence the Leibniz algebras $M(\Im) \oplus D(\Im)$ and $M(\Im) \oplus [M(\Im), M(\Im)]$ are the classical structure and inner algebras of $\Im$ (Lie algebras), if $\Im$ is a Jordan algebra.

Definition 42 *The Leibniz algebras $M(\Im) \oplus D(\Im)$ and $M(\Im) \oplus [M(\Im), M(\Im)]$ are called the **quasi-structure Leibniz algebra** and the **quasi-inner Leibniz algebra** of the K-B quasi-Jordan algebra $\Im$, respectively.*

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