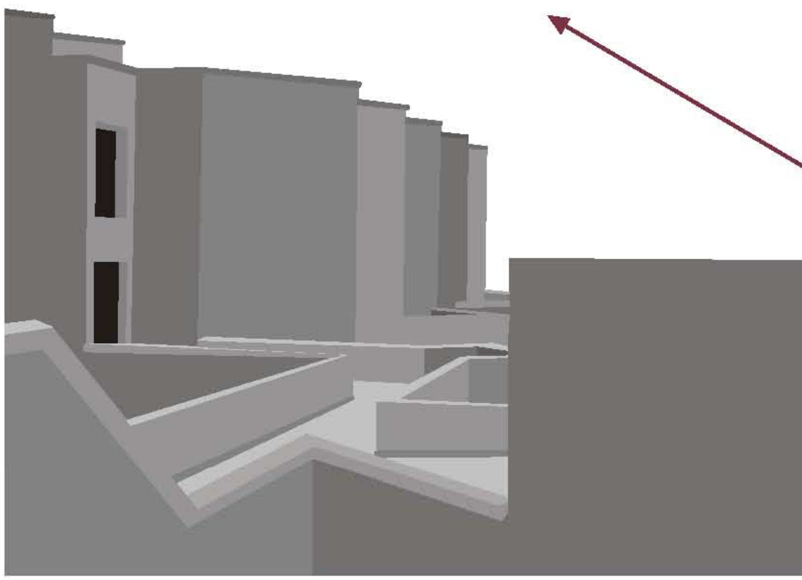

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ON THE NOTION OF DIGROUP

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On the notion of digroup

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Abstract. We show that the study of digroups is equivalent to the study of right G-spaces with a fixed point, and that this implies that digroups can be viewed as trivial associated bundles. We consider a generalization of this construction, that relates the "coquecigrue" problem of Loday to the study of associated bundles.

1. Introduction.

Digroups were independently introduced, as a generalization of groups, almost simultaneously by Liu [L], Felipe [F] and Kinyon [K]. Their definition was largely motivated by Loday´s work on Leibniz algebras and dialgebras and, more specifically, by the question (also raised by Loday) of the existence of integral manifolds to a Leibniz algebra, the so-called "coquecigrue" problem (see [Lo]), which is also the motivation for this article.

Formally, a digroup D is a set with *two* associative binary operations, often denoted by $\vdash, \dashv$, satisfying the associativity relations

$$\begin{aligned} (x \vdash y) \vdash z &= (x \dashv y) \vdash z \\ x \dashv (y \dashv z) &= x \dashv (y \vdash z) \\ (x \vdash y) \dashv z &= x \vdash (y \dashv z)\,; \end{aligned}$$

a distinguished element, e, satisfying the relations (defining a so-called *bar unit*)

$$e \vdash x = x \dashv e = x \quad \forall\, x \in D;$$

and such that, for each $x \in D$ a unique element exists, x^{-1}, so that

$$x \vdash x^{-1} = x^{-1} \dashv x = e.$$

I refer the reader to the works cited above for a more complete discussion of this notion, but I wish to recall two well known facts:

First, that to any digroup we can associate two distinguished subsets: The set of *inverses*

$$G = \left\{ y \in D \mid \exists x \text{ such that } y = x^{-1} \right\}$$

and the set of bar units

$$J = \{y \in D \mid \forall x \quad y \vdash x = x \dashv y = x\}.$$

And second, that the operations restricted to G not only are closed, but actually coincide, which in particular implies that G is a group.

Perhaps the deepest result in this context is a theorem of Kinyon (Theorem 4.8 of [K]) to the effect that D is isomorphic to $G \times J$, not only as a set but as a digroup. Kinyon based his result on semigroup theory, and more concretely on the existence of a very natural epimorphism of digroups $D \to G$, given by $x \mapsto \left(x^{-1}\right)^{-1}$, whose kernel is precisely J.

Now, Kinyon´s paper is elegant and readable, so there is no point in trying to reproduce his arguments; rather, my twofold aim is: to give a somewhat different proof of his result —arguing directly from very basic properties of digroups, but in a way that, as I hope to show, gives a still better understanding of their structure, bringing to the fore the group action implicit in this construction; and then to give a mostly self-contained discussion of some implications of this result, mostly from a differential geometric point of view. In doing so, I will also give a unified approach that considerably expands some results I obtained in previous works ([O1], [O2]), where I analyzed the geometric structure of some very special types of digroups, and their relation to the coquecigrue problem. (In honesty, however, I should add right away that the full coquecigrue problem still remains a challenge.)

2. The algebraic structure of a digroup.

2.1. Digroups and group actions. Let us start with a couple of very simple lemmas, that allow us to easily identify the result of multiplying by inverses and the bar unit in the "reverse" order:

LEMMA 1. *Let D be a digroup and $x \in D$. Then, $x \dashv x^{-1}$ and $x^{-1} \vdash x$ belong to J.*

PROOF. If y is any element in D, then, using the properties of the products we get

$$\left(x \dashv x^{-1}\right) \vdash y = \left(x \vdash x^{-1}\right) \vdash y = e \vdash y = y;$$

the remaining identities are entirely similar. □

LEMMA 2. *Let D be a digroup and $x \in D$. Then, $\left(x^{-1}\right)^{-1} = x \vdash e = e \dashv x$.*

PROOF. Since inverses are unique, this also follows immediately from the properties of the products:

$$x^{-1} \vdash (x \vdash e) = \left(x^{-1} \vdash x\right) \vdash e = \left(x^{-1} \dashv x\right) \vdash e = e \vdash e = e,$$

and again the other assertion is similar. □

The key point here is that the lemmas show that to any $x \in D$ we can associate, in a simple way, a bar unit and an inverse, that moreover have very nice expressions: $x^{-1} \vdash x$ and $x \vdash e$, respectively. Of course, strictly speaking there are actually two bar units to choose from, and there is also an even simpler choice of inverse, namely x^{-1}; but the arguments that follow do not depend on the choice of the bar unit in a significant way, while choosing x^{-1} instead of $\left(x^{-1}\right)^{-1}$ would introduce some mildly unpleasant, but also not essential, changes of order in what follows, and so from now on I will work with the choices stated above.

At any rate, using this we can restate (a part of) Kinyon´s result as follows:

Proposition 1. *Let D be a digroup, G its subgroup of inverses and J its set of bar units. Then, the map*

$$D \to J \times G : x \mapsto (x^{-1} \vdash x, x \vdash e)$$

is a bijection with inverse

$$(\alpha, a) \mapsto a \vdash \alpha;\ \ \alpha \in J, a \in G$$

Proof. The proof is again quite simple, so let us just verify that the second formula gives indeed the inverse of the first map. Thus, assume $\alpha = x^{-1} \vdash x$ and $a = x \vdash e$, then,

$$\begin{aligned} a \vdash \alpha &= (x \vdash e) \vdash \left(x^{-1} \vdash x\right) = (x \dashv e) \vdash \left(x^{-1} \vdash x\right) \\ &= x \vdash \left(x^{-1} \vdash x\right) = \left(x \vdash x^{-1}\right) \vdash x = e \vdash x = x, \end{aligned}$$

as desired. □

Remark 1. *Up to this point the setting has been entirely algebraic; but for future reference, observe that, if the digroup is a topological or a differentiable manifold, and the operations are accordingly continuous or differentiable, then the identifications above are homeomorphisms or diffeomorphisms, respectively. Thus, from a topologico-geometrical point of view, digroups can be regarded as trivial fiber bundles; either over G with fiber J, or over J with fiber G (see section 3.1). Later on I shall try to argue which one is more convenient; but for the time being, observe moreover that, given the choices mentioned above, the operations in the digroup already provide a natural way of writing the projections onto the factors of D.*

Let us now look at the digroup operations using the above identification. First, $x \vdash y$ goes to:

$$\begin{aligned} \left((x \vdash y)^{-1} \vdash (x \vdash y), x \vdash y \vdash e\right) &= \left(\left(y^{-1} \vdash x^{-1}\right) \vdash (x \vdash y), x \vdash y \vdash e\right) \\ &= \left(y^{-1} \vdash \left(x^{-1} \dashv x\right) \vdash y, x \vdash y \vdash e\right) \\ &= \left(y^{-1} \vdash e \vdash y, x \vdash y \vdash e\right) \\ &= \left(y^{-1} \vdash y, x \vdash y \vdash e\right). \end{aligned} \tag{2.1}$$

Then, $x \dashv y$ corresponds to:

$$\begin{aligned} \left((x \dashv y)^{-1} \vdash (x \dashv y), (x \dashv y) \vdash e\right) &= \left(\left(y^{-1} \dashv x^{-1}\right) \vdash (x \dashv y), x \vdash y \vdash e\right) \\ &= \left(\left(y^{-1} \vdash x^{-1}\right) \vdash (x \dashv y), x \vdash y \vdash e\right) \\ &= \left(y^{-1} \vdash \left(x^{-1} \vdash x\right) \dashv y, x \vdash y \vdash e\right) \end{aligned} \tag{2.2}$$

and the last expression cannot be further simplified. Moreover, since $x \vdash y \vdash e = (x \vdash e) \vdash (y \vdash e)$, the second element in both pairs is nothing but the product $\left(x^{-1}\right)^{-1} \vdash \left(y^{-1}\right)^{-1}$ (and since both elements lie in G, the choice of product is irrelevant here).

Also, observe that the first component of expression (2) is (up to a change of order) the "conjugation" $x^{-1} \vdash y \dashv x$, giving the "rack" structure of D considered by Kinyon in his paper. Let us now have a closer look at this "action." So, let $x, y \in D$, then

$$x^{-1} \vdash y \dashv x = x^{-1} \vdash y \dashv (x \dashv e) = x^{-1} \vdash y \dashv (x \vdash e) = x^{-1} \vdash y \dashv \left((x)^{-1}\right)^{-1}.$$

Hence, the action does not require that x be an arbitrary element of D, since it only depends on its projection onto G. Since on G both operations coincide, it follows immediately that this defines in fact a *true* action of the group G on D; given our choices it turns out to be a *right* action.

Furthermore, if y is any bar unit, then, for any $z \in D$ we have

$$\left(x^{-1} \vdash y \dashv x\right) \vdash z = \left(x^{-1} \vdash y \vdash x\right) \vdash z = \left(x^{-1} \vdash x\right) \vdash z = z,$$

so J is invariant under the action, and therefore is itself a right G-space.

Remark 2. *For completeness, it is perhaps worthwhile to recall here that a right action of a group G on a space X is just a map $\rho : X \times G \to X$, satisfying $\rho(x, g_1 g_2) = \rho(\rho(x, g_1), g_2)$ (left actions are similarly defined); the space X is then said to be a right G-space, or sometimes a "right homogeneous space" for G, although the latter is often reserved for a more specific type of action. For the sake of simplicity, actions are frequently denoted by the less explicit, but more suggestive, notation $\rho(x, g) = x \cdot g$ (a practice we shall follow here), so the action condition becomes $x \cdot (g_1 g_2) = (x \cdot g_1) \cdot g_2$.*

Observe, finally, that since $\{e\} = G \cap J$, it follows that e is a *fixed point* for the action, that is $e \cdot a = e$, $\forall a \in G$ (however, we shall see that there is no reason —nor need— for it to be unique).

Gathering the above, we can strengthen proposition 4 in the following form:

Proposition 2. *Let D be a digroup, with bar unit e, subgroup of inverses G, and set of bar units J. Then, J is a right G-space for the action $(\alpha, a) \mapsto a^{-1} \vdash \alpha \dashv a$, with fixed point e, and D is isomorphic as a digroup to (the digroup given by) the product $J \times G$ with operations*

$$(\alpha, a) \vdash (\beta, b) = (\beta, ab) \ ; \ (\alpha, a) \dashv (\beta, b) = \left(b^{-1} \vdash \alpha \dashv b, ab\right)$$

(here, and henceforth, we have suppressed the symbol for the product when dealing exclusively with elements of G).

Thus, given the choice of projections, a digroup determines in a very natural way a right G-space structure on its set of bar units. Now, let us show that this is indeed all that is needed: a right G-space J with a fixed point.

Proposition 3. *Let G be a group, with unit element e, and J a right G-space; denote the action $J \times G \to J : (a,b) \mapsto a \cdot b$, and assume it has a fixed point ε. Then $D = J \times G$ has the structure of a digroup with operations*

$$(\alpha, a) \vdash (\beta, b) = (\beta, ab) \;\; ; \;\; (\alpha, a) \dashv (\beta, b) = (\alpha \cdot b, ab)$$

and the point (ε, e) as the preferred bar unit. Inverses with respect to this bar unit are given by $(\alpha, a)^{-1} = \left(\varepsilon, a^{-1}\right)$.

Proof. Once again, the computations are quite straightforward, so let us just verify the identity $(x \vdash y) \dashv z = x \vdash (y \dashv z)$. Let $x = (\alpha, a)$, $y = (\beta, b)$ and $z = (\gamma, c)$, then

$$\begin{aligned}
(x \vdash y) \dashv z &= ((\alpha, a) \vdash (\beta, b)) \dashv (\gamma, c) \\
&= (\beta, ab) \dashv (\gamma, c) \\
&= (\beta \cdot c, abc) \\
&= (\alpha, a) \vdash (\beta \cdot c, bc) \\
&= (\alpha, a) \vdash ((\beta, b) \dashv (\gamma, c)) \\
&= x \vdash (y \dashv z)
\end{aligned}$$

□

With the last proposition we have finally recovered all of Kinyon´s result; but what I would like to point out here is that, since the specification of the G-space J already involves the information of the group G, in a very definite sense this is the only datum required to construct the digroup $J \times G$. In fact, more can be said, because proposition 8 actually implies a kind of functorial converse to proposition 7, in the following sense:

Proposition 4. *Let $D = J \times G$ be a digroup constructed as in proposition 8. Identify J and G as subsets of D in the natural way:*

$$\alpha \leftrightarrow (\alpha, e) \;\; , \;\; a \leftrightarrow (\varepsilon, a)$$

(observe that ε and e get both identified with (ε, e)). Then the element $\alpha \cdot a$ gets identified with $a^{-1} \vdash \alpha \dashv a$.

Proof. Simply observe that

$$\begin{aligned}
a^{-1} \vdash \alpha \dashv a \leftrightarrow \left(\varepsilon, a^{-1}\right) \vdash (\alpha, e) \dashv (\varepsilon, a) &= \left(\alpha, a^{-1}\right) \dashv (\varepsilon, a) \\
&= \left(\alpha \cdot a, a^{-1} a\right) = (\alpha \cdot a, e) \leftrightarrow \alpha \cdot a
\end{aligned}$$

□

Example 1. *A simple class of digroups can be constructed as follows: Let V be a vector space (for the sake of definiteness, say it is a k-dimensional real vector space), and consider a non-identically null linear functional φ, and a fixed element $e \in V$ such that $\varphi(e) = 1$. Endow V with the operations $x \vdash y = \varphi(x) y$, $x \dashv y = x \varphi(y)$ (so that V becomes a* dialgebra*).*

Obviously e is a bar unit for these operations, and if $x \notin \ker \varphi$, we can declare its inverse with respect to e to be $\varphi(x)^{-1}$, then, the open subset $D = V \setminus \ker \varphi$ becomes a digroup, which I called a φ-digroup.

From our point of view, $G=\mathbb{R}\diagdown\{0\}$ (to be more precise, but somewhat pedantic, $G=(\mathbb{R}\diagdown\{0\})\otimes\{e\}$), while J is the affine space $N=e+\ker\varphi$, and the identification $D=J\times G$ takes on the form $x\leftrightarrow\left(\varphi(x)^{-1}x,\varphi(x)\right)$; the right action is given by $(\alpha,a)\mapsto a^{-1}\alpha a$.

Moreover, the relatively straightforward generalization to a matrix case, obtained by considering instead of V a space of matrices with entries in V, $Mat_n(V)=Mat_n(\mathbb{R})\otimes V$, and extending φ to a map $\varphi: Mat_n(V)\to Mat_n(\mathbb{R})$ entry-wise, is significantly more interesting for the coquecigrue problem, since the Leibniz algebra related to this matrix case is not abelian (I refer the reader to [O1] for details).

For future reference, let us close this section stressing the following: If $x=(\alpha,a)$, $y=(\beta,b)\in D$, with the identification of proposition 9, the products on the digroup can be written in the following form:

$$x\vdash y=(\beta,ab)\ ; \tag{2.3}$$

$$x\dashv y=(\alpha\cdot b,ab)\,. \tag{2.4}$$

In particular, the product $\vdash$ is the same for any digroup supported by the manifold $J\times G$. The moral here is therefore that the construction of a digroup requires a right G-space J, that is, a right action of G on J, but in fact *only one* non-trivial product , together with a simple agreement on the way we associate bar units to elements of the digroup; this, in turn, corrsponds to a choice of one of the two possible natural projections of the product space $J\times G$ on J given by the algebraic structure of the digroup. With our choices, the non-trivial product is the one denoted $\dashv$, and this non-trivial product is in fact equivalent to the given right action.

2.2. Morphisms of digroups. The digroup approach raises some natural questions; chief amongst them is the one regarding morphisms between digroups, which we now consider. The appropriate definition is as follows:

DEFINITION 1. *Let D and D' be two digroups, with bar units e and e' respectively. A morphism between them is a map $\varphi: D\to D'$ such that for all $x,y\in D$,*

$\varphi(x\vdash y)=\varphi(x)\vdash\varphi(y)$, $\varphi(x\dashv y)=\varphi(x)\dashv\varphi(y)$, and $\varphi(e)=e'$.

(Here, to avoid cluttering the notations, we have denoted the operations on both digroups with the same symbols.)

Now, since our point of view here is to regard D as the product $J\times G$, we want to see how the above definition translates to this setting. Let us first state a preliminary lemma:

LEMMA 3. *Let $D=J\times G$, and $D'=J'\times G'$ be two digroups, and $\Phi: D\to D'$ a morphism. Then $\Phi(G)\subset G'$ and $\Phi(J)\subset J'$.*

PROOF. For the first assertion, it suffices to show that both products restricted to $\Phi(G)$ coincide, which characterizes the subgroup of inverses in a digroup; this is true because Φ, being a morphism of digroups, for arbitrary $x, y \in G$ we have

$$\Phi(x) \vdash \Phi(y) = \Phi(x \vdash y) = \Phi(x \dashv y) = \Phi(x) \dashv \Phi(y).$$

For the second assertion, it suffices to prove that for any bar unit $\alpha \in J$ the inverse associated to $\Phi(\alpha)$ is e', or what is the same, that $\pi_2'(\alpha) = e'$ (where π_i' denotes the projections onto the i-th factor of D'). But

$$\pi_2'(\Phi(\alpha)) = \Phi(\alpha) \vdash e' = \Phi(\alpha \vdash e) = \Phi(e) = e'$$

as desired. □

The consequence here is that a digroup morphism splits into two maps, $\varphi : G \to G'$ and $\psi : J \to J'$, so that we can write

$$\Phi(x) \leftrightarrow (\psi(\alpha), \varphi(a)), \tag{2.5}$$

where as before we are identifying x with (α, a).

PROPOSITION 5. *Let $\Phi : D \to D'$ be a morphism of digroups, and identify it with the pair (ψ, φ) as above. Then $\varphi : G \to G'$ is a homomorphism of groups, and ψ is a φ-equivariant map from the right G-space J to the right G'-space J', mapping the preferred fixed point $\varepsilon \in J$ to $\varepsilon' \in J'$.*

REMARK 3. *Before giving the proof, it is perhaps useful to remind the reader that if J and J' are as in the proposition, and $\varphi : G \to G'$ is a group homomorphism, a map $\psi : J \to J'$ is said to be φ-equivariant if for $\alpha \in J$, $a \in G$ we have $\psi(\alpha \cdot a) = \psi(\alpha) \cdot \varphi(a)$. This is in fact mostly terminology, since $\psi(\alpha) \cdot a = \psi(\alpha) \cdot \varphi(a)$ defines an action of G on $\psi(J)$, which can then be extended to an action of G on J declaring that G acts trivially on the elements in $J' \diagdown \psi(J)$, so both spaces, J and J', can be seen as G-spaces; if we denote the action of G on J by, say ρ, and the one by G' on J' by ρ', what we have defined is the so-called pullback action, usually denoted $\varphi^*\rho'$, and the map ψ is then equivariant in the usual sense (for ρ and $\varphi^*\rho'$). The terminology is nevertheless convenient in our setting, because when we deal with two digroups there are naturally involved two groups, which of course need not be the same.*

PROOF. (Of Proposition 10) That φ is a group homomorphism is immediate from the lemma, since the digroup operations coincide on the G factor. Equivariance is likewise clear from formula (4) of the previous section.

Finally, to see that the fixed points are preserved, since equivariant maps send fixed points to fixed points, we just invoke the fact that the bar units in D and D' are given by (ε, e), and (ε', e'), respectively, together with the fact that under the natural injections $G, J \hookrightarrow D$, $G', J' \hookrightarrow D'$, these are the only points in the intersections $G \cap J$, and $G' \cap J'$. □

Since the φ-equivariance of the map ψ implies knowledge of the G-action on J', and hence of the group homomorphism φ, again we can say that knowledge of the morphism Φ is equivalent to knowledge of the φ-equivariant map ψ. Moreover, since the composition of homomorphisms and equivariant maps is of course again a homomorphism or an equivariant map respectively, we see that if Φ_1 and Φ_2 are two digroup morphisms, with the obvious conventions the morphism $\Phi_1 \circ \Phi_2$

is associated to the pair $(\psi_1 \circ \psi_2, \varphi_1 \circ \varphi_2)$; thus, the identification of the digroups $D \leftrightarrow J \times G$ preserves the composition of morphisms, and is therefore categorical.

Remark 4. *And incidentally, we see now that had we chosen $x \dashv x^{-1}$ as the bar unit to associate to x, the only noticeable change would have been the interchange of the role of the products, so that the "trivial" product would have been the one we have denoted $\dashv$ instead of $\vdash$. On the other hand, if instead of $\left(x^{-1}\right)^{-1}$ had we chosen x^{-1}, the action we would recover in Proposition 9 would be the* ***left*** *action $a \vdash \alpha \dashv a^{-1}$, and not the original action; there seems to be no way around this (in a sense this is akin to a change of orientation).*

So, to be technically more precise, we might say that what the above results show is that the digroup operations give two functorial, but naturally equivalent, ways of doing the identification $D \leftrightarrow J \times G$. Thus, and to summarize our statements up to this point: *the algebraic study of digroups is formally equivalent to the study of spaces with a right group action with a fixed point, and their corresponding equivariant maps.*

3. The geometric structure of a digroup.

3.1. Bundles and digroups. In spite of the comment made at the end of the last section, I believe the notion of digroup is interesting in its own right, precisely because of its relationship to the coquecigrue problem mentioned at the introduction. This is because the homogeneous space J defining a digroup cannot be viewed, in a natural way by itself, as an integral manifold of a given Leibniz algebra (that is, as a coquecigrue); we need to consider the whole product $J \times G$.

On the other hand, as Kinyon already pointed out, digroups cannot be the full solution to the coquecigrue problem, and what I intend to do next will also address, at least partially, this question.

Now, the coquecigrue problem is not only an algebraic question, but also a differential geometrical one. So, we will now assume that our objects are differentiable manifolds, although topological manifolds —or even orbifolds— could probably suffice —or perhaps even be better suited— to discuss some aspects of what follows; this surely warrants some investigation, but it would certainly involve some technical complications that I wish to avoid here, mostly because, as I said, the approach here still leaves an important part of the coquecigrue problem unanswered. So, hereafter G is a Lie group, and J a smooth manifold (and smooth means C^∞).

As mentioned, we can regard $D = J \times G$ as a fiber bundle in two different ways, by considering the projections $\pi_1 : D \to J$ and $\pi_2 : D \to G$. Both bundles are of course trivial, but let us dig a bit deeper into the relationship of these bundle structures on D to the algebraic structure of D.

Consider first the bundle structure given by $\pi_1 : D \to J$. Then, from formula (3) of section 1.1, that shows that for fixed y the product $\vdash$ only depends on the projection of y onto its first factor β, this gives a left action of G on D preserving the fibers of π_1. Indeed, what the product $\vdash$ does is simply transfer the group structure of G to the fiber $\pi^{-1}(\beta)$, and therefore this endows D with the structure of a (trivial and principal, see below) bundle of groups over J. Since from our point of view the "trivial" product $\vdash$ is the same for any digroup supported by the manifold $J \times G$, this gives little, if any, information about the specific digroup... but then again, we are dealing here with the "trivial" product.

Now, for the projection $\pi_2 : D \to G$, the bundle structure of D can no longer —in general— be that of a principal bundle; however, D will be an associated bundle. For the sake of completeness, but also to fix some conventions, let us first recall a few basic facts about principal and associated bundles.

Definition 2. *Let G be a Lie group, $\pi : P \to M$ a manifold submersion. Then P is a principal bundle over M with structure group G, or simply a G-principal bundle, if G acts on P on the right in such a way that: The action of G preserves each fiber $\pi^{-1}(x)$, $x \in M$, and is transitive on it; and for each $x \in M$ there exists a neighborhood U, such that $\pi^{-1}(U)$ is diffeomorphic to $G \times U$, and such that the diffeomorphism can be written in the form*

$$\Psi : \pi^{-1}(U) \to G \times U : p \mapsto (\psi(p), \pi(p))$$

where ψ is equivariant with respect to the action of G on P and to the action of G on $G \times U$ given by right multiplication in G, that is, $\psi(p \cdot a) = \psi(p)\, a$, for $p \in P$ and $a \in G$.

(It follows in particular that the fibers of P are diffeomorphic to the structure group G, and that the base manifold M is diffeomorphic to the quotient space P/G.)

The last condition, called *local triviality*, says that P locally looks like the product space $G \times U$, and any neighborhood U where $\pi^{-1}(U)$ is diffeomorphic to $G \times U$ is called a *trivializing neighborhood* (principal bundles are therefore a special type of what are called *locally trivial fiber bundles*). The key point is that they are in general not globally diffeomorphic to a products pace. To make this statement precise, recall that we say that two bundles P and P' are isomorphic if there exists a diffeomorphism, $\Phi : P \to P'$, such that Φ preserves fibers and is equivariant; and a bundle is trivial if it is isomorphic to the trivial bundle $G \times M$. A necessary and sufficient condition for global triviality of a principal bundle is the existence of a *global section* to π; that, is a map $\sigma : M \to P$ such that $\pi \circ \sigma$ is the identity on M.

If $\{U_i\}$ is a covering of M by trivializing neighborhoods, with corresponding diffeomorphisms $\Psi_i = (\psi_i, \pi)$, then whenever $U_i \cap U_j \neq \varnothing$, by equivariance, the map $\psi_j \circ \psi_i^{-1} : G \times (U_i \cap U_j) \to G$ depends only on $\pi(p)$, and therefore defines a map $g_{ij} : U_i \cap U_j \to G$, called a *transition function* for the bundle. These transition functions satisfy the *2-cocycle condition*: if $U_i \cap U_j \cap U_k \neq \varnothing$ then $g_{ij} g_{jk} = g_{ik}$ (in particular, $g_{ii} : U_i \to G$ is the constant map with value e, and $(g_{ij}(p))^{-1} = g_{ji}(p)$), so one usually calls the collection $\{g_{ij}\}$ a cocycle for the bundle. The knowledge of such a cocycle actually determines the bundle, since it can be recovered by considering the disjoint union $\bigcup_i G \times U_i$ modulo the equivalence relation $(g_1, x_1) \in G \times U_1 \sim (g_2, x_2) \in G \times U_2$ iff $x_1 = x_2$ and $g_2 = g_1 g_{21}(x_1)$ (well defined due to the cocycle condition), which turns out to be isomorphic to the original bundle.

Example 2. *Plainly, from the above definition, a digroup $D = J \times G$ is a trivial principal bundle with respect to π_1, with base manifold J, and action of G on D given by right multiplication on the second factor.*

.

Definition 3. *If $\pi : P \to M$ is a principal bundle, and J is a right G-space, with action $\rho : J \times G \to J$ (whose effect we will denote as usual $(\alpha, a) \mapsto \alpha \cdot a$), then the bundle associated to P via ρ is constructed as follows: On the Cartesian product $J \times P$ define the equivalence relation: $(\alpha, p) \sim (\beta, q)$ if there exists $g \in G$ such that $(\beta, q) = (\alpha \cdot g, q \cdot g)$, and endow the quotient space $E = J \times P / \sim$ with the quotient topology and the projection $[\alpha, p] \mapsto \pi(p)$.*

The manifold E is then a fiber bundle, and with the same trivializing neighborhoods as P, but with fiber diffeomorphic to J; the group G is then called the *structure group* of the bundle. In fact, the associated bundle can also be recovered from the knowledge of a cocycle for the corresponding principal bundle, by considering this time the disjoint union $\bigcup_i J \times U_i$, modulo the equivalence relation $(\alpha_1, x_1) \in J \times U_1 \sim (\alpha_2, x_2) \in J \times U_2$ iff $x_1 = x_2$ and $\alpha_2 = \alpha_1 \cdot g_{21}(x_1)$.

Remark 5. *The reader should perhaps be warned that in most of the literature the action on the fiber is taken to be a left action; the equivalence relation is then $(g_1, x_1) \in G \times U_1 \sim (g_2, x_2) \in G \times U_2$ iff $x_1 = x_2$ and $g_2 = g_{21}(x_1)\, g_1$. This is due to the fact that the most important class of associated bundles are the vector bundles, where the fiber is a vector space V, and the group $G = GL(V)$ acts by linear transformations which are usually written on the left; the principal bundle associated in this case is the so-called* bundle of frames. *I will nevertheless stick to the convention stated above, since it seems to me more natural to relate bundles to digroups.*

With these preliminaries, let us turn back to digroups, and their non-trivial product.

So let $D = J \times G$, be a digroup, with non-trivial product $\dashv$ determined by the right action $\rho : J \times G \to J$. Notice that if we consider only elements of the form $y = (\varepsilon, b) \in G$, formula (2) becomes $(\alpha, a) \dashv (\varepsilon, b) = (\alpha \cdot b, b)$, which, from what has been said, is a fiber preserving right action of G on D. This action is trivially equivariant, and thus, with respect to the projection $\pi_2 : D \to G$, D becomes an associated bundle (to the trivial principal bundle $G \times G$, via the action ρ).

Although this may seem almost as superficial as the description of D as a principal bundle over J, it certainly is not, because this involves the non-trivial product, which does indeed distinguish the different digroups supported by $J \times G$. To see our gain, let us consider again the matrix φ-digroups mentioned in example 9:

Example 3. *Let V be a k-dimensional real vector space, $G \approx GL(k, \mathbb{R})$, φ a linear functional and the G-space J essentially be the vector space $N = \ker \varphi$, where $\varphi : Mat_n(\mathbb{R}) \otimes V \to Mat_n(\mathbb{R})$. The associated bundle here is therefore a vector bundle, but the principal bundle associated by our construction is not the bundle of frames, since G is in general a strict subgroup of $GL(N)$ (technically, we could say that viewing the digroup as an associated bundle over G corresponds to a reduction of the structure group).*

But more interestingly, two particular features of this example are that the operations are linear, and that the vector bundle can be considered as a principal bundle; not with respect to the original action, but with respect to the right action given by addition in the fibers (which of course, are themselves groups). This is not *entirely trivial because, by linearity of the operations, for fixed* y *the product* $y \vdash x \dashv x^{-1}$ *(which from our point of view corresponds to* $\left(\alpha \cdot a^{-1}, b\right)$*), describes a subspace of* $T_y D$*, and thereby we get a distribution (i.e., a subbundle of the tangent bundle* TD*), which is equivariant for the latter action, and therefore gives a connection in this bundle.*

Remark 6. *However, the fact that the associated bundle we get here is also principal somehow obscures the deeper meaning of looking at* D *as a bundle over* G*. Moreover, in addition to the bundle being trivial, which as we have seen holds for any digroup, the connection here turns out to be flat, so these digroups are indeed of a very special type. (Nonetheless, for a time I conjectured that the solution to the coquecigrue problem could come from the analysis of principal bundles with flat connections; I refer the reader to [O1] and [O2] for more details.)*

3.2. A generalization of digroups. Now, as mentioned in the introduction, the coquecigrue problem calls for the integration of Leibniz algebras. To state it more precisely, recall the following definition

Definition 4. *A (left) Leibniz algebra is a vector space, endowed with a bilinear operation* $[\cdot,\cdot]$*, satisfying the Leibniz identity*

$$[[x,y],z] = [[x,z],y] + [x,[y,z]].$$

Leibniz algebras clearly generalize Lie algebras, in the sense that a Lie bracket satisfies the previous equation; but it also satisfies the additional requirement that it be anti-symmetric. This allows some rewording of the Leibniz identity, particularly the so-called *Jacoby identity* $[[x,y],z]+[[y,z],x]+[[z,x],y]=0$, which is the usual way of presenting the requirements for a Lie bracket; this is not available for Leibniz algebras.

In any case, just as for Lie groups, to recover the Leibniz algebra from the digroup, we can consider the rack structure given by conjugation, $x \vdash y \dashv x^{-1}$, and differentiate (see *e.g.*, [K] or [O1]): More precisely, if X, Y are two tangent vectors at $T_e D$, we can consider two curves $x(s)$, $y(t)$, such that $x(0) = e = y(0)$, and $x'(0) = X$, $y'(0) = Y$; then we can compute the Leibniz bracket $[X,Y]$ as

$$[X,Y] = \frac{\partial^2}{\partial s \partial t}\, |_{s=t=0} \left(x(s) \vdash y(t) \dashv x^{-1}(s)\right). \tag{3.1}$$

Upon identifying $x=(\alpha,a)$, $y=(\beta,b)$, and using what we have seen, this can be rewritten as

$$[X,Y] = \left(\frac{\partial^2}{\partial s \partial t}\, |_{s=t=0} \left(\beta(t) \cdot a(s)\right), [A,B]\right), \tag{3.2}$$

where $A = a'(0)$, $B = b'(0)$,and their bracket is computed in the Lie algebra $\mathfrak{g}$ of G.

The last expression makes two things clear: First, that the Leibniz algebra $T_e D$ of a digroup is *split*, meaning that it decomposes as a direct sum $\mathfrak{j} \oplus \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G; and second, that the essential new ingredient in the Leibniz bracket comes again from the action of G on J. (The first observation is in fact a somewhat more precise statement of the fact mentioned above, that digroups cannot solve the general coquecigrue problem, because not every Leibniz algebra is split.)

As for the second, we see that we need to linearize the non-trivial product, which in turn is related to the bundle structure $\pi_2 : D \to G$. The important remark here, however, is that computation of formulas (6) and (7) does not require a knowledge of the global structure of D, and in particular, of its decomposition as a product $J \times G$; it only requires a knowledge of its structure in a neighborhood of some special point ξ. Therefore, one way to generalize the construction above of integral manifolds for Leibniz algebras would be to replace digroups by associated bundles that are not necessarily trivial, with a distinguished point. Let us now discuss how this can be done.

So, we let G be a Lie group, P a G-principal bundle over G, J a right G-space, and E a bundle associated to P via some action, with projection $\pi : E \to G$. Our aim is to define a binary operation $E \times E \to E : (x, y) \mapsto p \odot q$, that upon differentiation gives a Leibniz algebra structure to $T_\xi E$, where ξ is some distinguished point. Since for trivial bundles this should recover the case of digroups, there are some natural requirements that we might impose to this operation (although perhaps not necessary conditions, in a strict sense).

The first would be the existence of the distinguished point ξ. In contrast to the digroup case, we do not have *a priori* the natural choice $\xi = (\varepsilon, e)$ at our disposal here. However, from what has been said before, the existence of a distinguished point should require the existence of a fixed point $\varepsilon \in J$ for the action. This leads to the following:

Proposition 6. *Let G be a Lie group, P a G-principal bundle over a manifold M, J a right G-space, and E a bundle associated to P, with projection $\pi : E \to M$, via an action with a fixed point $\varepsilon \in J$. Then E admits a natural global section σ_ε associated to ε.*

Proof. Let $\Psi = (\psi, \pi)$ be any trivialization over an open set $\pi^{-1}(U)$; then $\sigma_{\varepsilon,U} : U \to E$, given by $\sigma_{\varepsilon,U}(a) = \Psi^{-1}(\varepsilon, a)$, clearly defines a section over U. To see that this defines a global section, if $\Psi_1 = (\psi_1, \pi)$, $\Psi_2 = (\psi_2, \pi)$ are two trivializations defined on $\pi^{-1}(U_i)$, where U_i are two overlapping trivializing neighborhoods for the bundle, and the relation between them on the intersection $U_1 \cap U_2$:

$$\psi_1(x) = \psi_2(x) \cdot g_{12}(a),$$

where g_{ij} is a cocycle for the bundle. Then for $a \in U_1 \cap U_2$ we have to show that $\sigma_{\varepsilon,U_1}(a) = \sigma_{\varepsilon,U_2}(a)$, that is, that $\Psi_1^{-1}(\varepsilon, a) = \Psi_2^{-1}(\varepsilon, a)$. Since Ψ_i are diffeomorphisms, this in turn is equivalent to $\Psi_1 \circ \Psi_2^{-1}(\varepsilon, a) = (\varepsilon, a)$. But

$$\begin{aligned}
\Psi_1 \circ \Psi_2^{-1}(\varepsilon, a) &= \left(\psi_1 \circ \left(\Psi_2^{-1}(\varepsilon, a)\right), a\right) \\
&= \left(\psi_2 \circ \left(\Psi_2^{-1}(\varepsilon, a)\right) \cdot g_{12}(a), a\right) \\
&= \left(\varepsilon \cdot g_{12}(a), a\right) = (\varepsilon, a).
\end{aligned}$$

The same computations show that, indeed, σ_ε is independent of the trivialization. □

In particular, we see that the choice of a fixed point ε indeed determines the distinguished point $\xi = \sigma_\varepsilon(e)$ (but we stress the fact that such a special point need not be unique).

Remark 7. *It is also perhaps worthwhile to recall here that, while for a principal bundle existence of a global section implies triviality, associated bundles can* *certainly have global sections without being trivial. The standard example, which is in fact a special case of the proposition above, is that of vector bundles, which always possess the zero section.*

The second requrement would be that, upon projection, the operation $\odot$ should correspond to the product in the group G. If we let $\pi(x) = a$, $\pi(y) = b$, this condition is simply $\pi(x \odot y) = ab$,

Finally, we would like $\odot$ to be described by equation (4), at least in a neighborhood of ξ, so we are actually generalizing the notion of digroup. In other words, if we let $\Psi = (\psi, \pi)$ be a trivialization of the bundle, defined on an open neighborhood U, the operation should be expressed as

$$\Psi((x \odot y)) = (\psi(x) \cdot b, ab). \tag{3.3}$$

Now,since E can be described as equivalence classes of pairs, (z, a), $z \in J$, $a \in U_i$, this suggests the following definition of the product $\odot$: Let $x = [(\alpha, a)]$, $y = [(\beta, b)] \in E$, then define

$$x \odot y = [(\alpha \cdot b, ab)]. \tag{3.4}$$

If we do so, first we need to see under what conditions the operation is well defined. So, consider two arbitrary representatives for the classes, $(\alpha_1, a_1), (\alpha_2, a_2) \in x, (\beta_1, b_1), (\beta_2, b_2) \in y$; we need to see when the elements $(\alpha_1 \cdot b_1, a_1 b_1)$ and $(\alpha_2 \cdot b_2, a_2 b_2)$ are in the same class. By hypothesis, however, $a_1 = a_2 = a$, and $b_1 = b_2 = b$, so we only need to see when $\alpha_1 \cdot b \sim \alpha_2 \cdot b$. Now, both points $\alpha_1 \cdot b \sim \alpha_2 \cdot b$, are to be regarded as lying on the fiber over ab, possibly on two distinct trivializing open sets U_1, U_2; so this amounts to show that there exists a transition function g_{12} defined on $U_1 \cap U_2$ such that $\alpha_1 \cdot b = \alpha_2 \cdot b g_{12}(ab)$. Now by hypothesis, there exists a transition function $\tilde{g}_{12}(a)$ such that $\alpha_1 = \alpha_2 \cdot \tilde{g}_{12}(a)$; acting on this equality by b, we get therefore the relation

$$\alpha_2 \cdot b g_{12}(ab) = \alpha_2 \cdot \tilde{g}_{12}(a) b,$$

as a condition on the associated bundle that would allow a global product, defined by relation (9), and this will satisfy all three requirements stated above.

The previous observations prove the following:

Theorem 1. *Let G be a Lie group, J a right G-space, with a fixed point $\varepsilon \in J$. Let P a G-principal bundle over G,and E a bundle associated to P via the action, with projection $\pi : E \to G$. Let ξ be the fixed point in the fiber over the identity element of the group, e, given by the canonical section determined by ε. Assume that P admits a cocycle satisfying the following condition: For all $a, b \in G$ and $z \in J$, there exist transition functions g_{ij}, $\tilde{g}_{ij}$ such that*

$$\alpha \cdot b g_{12}(ab) = \alpha \cdot \tilde{g}_{12}(a) b,$$

for all $\alpha \in J$.

Then, if we define a binary operation $E \times E \to E : (x, y) \mapsto x \odot y$, by

$$x \odot y = [(\alpha \cdot b, ab)],$$

upon differentiation this gives a Leibniz algebra structure to $T_\xi E$, and is therefore a solution to the coquecigrue problem.

The condition stated in the theorem seems rather hard to analyze in general, because it involves both the topology of the bundle and the action. On the other hand, it will be clearly satisfied if we demand the simpler condition that the cocycle g_{ij} satisfy the relation

$$g_{ij}(ab) = b^{-1}\tilde{g}_{ij}(a)\,b; \tag{3.5}$$

Moreover, since for the coquecigrue problem we are actually only interested in the tangent space at the special point ξ in the fiber $\pi^{-1}(e)$, and thus we need to compute some derivatives, we could restrict ourselves to the requirement that formula (9) holds only for a small neighbourhood V about e, contained in the intersection of two trivializing neighborhoods of e, and such that if $a, b \in V$, then $ab \in U_i \cap U_j$, which is certainly possible by continuity of the product, for then we could take $g_{ij} = \tilde{g}_{ij}$ and setting $a = e$ in the last condition, this would be reduced to

$$g_{ij}(b) = b^{-1}g_{ij}(e)\,b. \tag{3.6}$$

for all terms of the cocycle defined in a neighbourhood of e.

Remark 8. *The last conditions certainly seem cohomological in nature, but not exactly. The condition for two cocycles $\{g_{ij}\}$ and $\{\tilde{g}_{ij}\}$ (which can be assumed to be associated to the same family of trivializng neighbourhoods $\{U_i\}$) to be cohomologous is that there exists a family of maps $\{g_i : U_i \to G\}$ (a 1-cocycle) such that $g_{ij}(a) = g_i^{-1}(a)\,\tilde{g}_{ij}(a)\,g_j(a)$; and in contrast to the previous formulas, all maps are evaluated at the same point. This certainly requires further investigation.*

Now, let us see that the above constructions are meaningful, by considering in some detail the following simple but non-trivial example:

Example 4. *Let $G = U(1)$ be the unit circle group in the complex plane, Consider the $U(1)$-principal bundle P over $U(1)$ defined by the following cocycle: Let $U_1 = G\diagdown\{i\}$, $U_2 = G\diagdown\{-i\}$; $\{U_1, U_2\}$ define an open covering of the circle, then we can define a $U(1)$-principal bundle via the cocycle $g_{12} : U_1 \cap U_2 \to U(1) : e^{i\theta} \mapsto -1$. This bundle P, which on the intersection $U_1 \cap U_2$ identifies a point in the fiber with its antipode, is sometimes called a* twisted torus, *and topologically is a* Klein bottle, *so it is certainly not trivial.*

Thus, if we let $J = D = \{\mid z \mid < 1\}$ be the open unit disk, where G acts in the standard way by rotations, defining E to be the associated bundle via these data, then E is also not trivial, so it cannot be a digroup.

Nevertheless, following the above constructions we can provide a (local) product that will solve the coquecigrue problem. Indeed, over U_1 we can consider the explicit trivialization of J, (z, a), while over U_2 we can take $(-z, a)$. Then, if $V = \{e^{i\theta}; \theta \in (-\pi/2, \pi/2)\}$, clearly $VV \subset U_1 \cap U_2$, and so we if take $x = (z, a)$, $y = (w, b) \in V$, we can define their product as

$$x \odot y = (zb, ab)\,. \tag{3.7}$$

Since the action has only one fixed point, namely 0, the distinguished point is necessarily $\xi = (0, 1)$; differentiation of the action at ξ gives the following Leibniz algebra structure on $T_\xi J$: If $X = (Z, A) = (z'(0), a'(0))$ and $Y = (W, B) = (w'(0), b'(0))$ are two tangent vectors, then, from formula (7),

$$[X, Y] = (z'(0), [a'(0), b'(0)]) = (Z, 0)\,.$$

.

One might rightfully say that there is some cheating here, in asmuch as we can not define the product globally this way, because condition (10) is not satisfied for all members of the cocycle at all points. To understand what difficulties result from this, notice that formula (9) can be used to define a product on a set larger than V; indeed, the product would not be well defined using formula (9) only if $ab = \pm i$, therefore, we could define define the product on the complement of the set $W = \{ab = \pm i\}$, which is an open and dense subset of $G \times G$. The problem is that, roughly speaking, crossing W brings a jump discontinuity, the product passing from (zb, ab) to $(-zb, ab)$, so we can not define a global product in a continuous way.

Remark 9. *On the other hand, the same considerations show that, at least in this case, we can amend the product by, for instance, considering a modified product* $x \odot' y = (\eta(b)zb, ab)$*, where* η *is a smooth "bump" function, with value* 1 *in a small closed neighborhood* $V_0 \subset V$ *of* $b = 1$*, and* 0 *outside* V*, thus getting a still reasonable solution to the coquecigrue problem that is not a digroup.*

To conclude, our results here suggest that a good way to attack the coquecigrue problem is via associated bundles. In other words, a general coquecigrue can perhaps be constructed by a suitable deformation of a digroup, somehow like the way an associated bundle is a deformation of a product space. As we have seen, this poses a number of non-trivial technical problems, among them some of a cohomological nature, which seems appropriate, since this was the original motivation for Leibniz algebras. I will try to address some of these questions in a forthcoming paper.

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