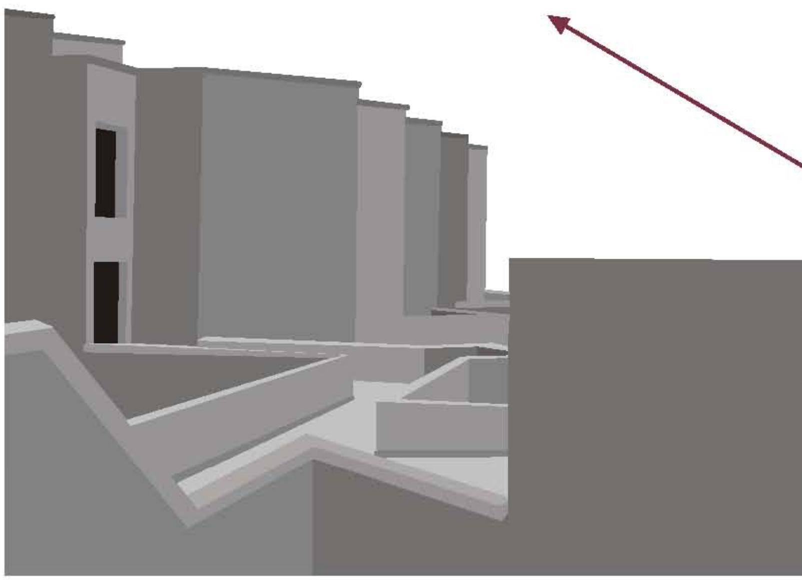

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A. E. Kyprianou, J. C. Pardo and K. van Schaik

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A Wiener-Hopf Monte Carlo simulation technique for Lévy processes.

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A. E. Kyprianou[1], J. C. Pardo[2], K. van Schaik[3]

Abstract

We develop a new method for simulating the joint law of the position and running maximum at a fixed time of a general Lévy process with a view to application in insurance and financial mathematics. Although different, our method takes lessons from Carr's so-called 'Canadization' technique as well as Doney's method of stochastic bounds for Lévy processes; see Carr [6] and Doney [8]. We rely fundamentally on the Wiener-Hopf decomposition for Lévy processes as well as taking advantage of recent developments in factorisation techniques of the latter theory due to Vigon [20] and Kuznetsov [11]. We illustrate our Wiener-Hopf Monte Carlo method on a number of different processes, including a new family of Lévy processes called hypergeometric Lévy processes.

Key words and phrases: Lévy processes, exotic option pricing, Wiener-Hopf factorisation.

MSC 2000 subject classifications: 65C05, 68U20.

[1]Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, U.K. E-mail: a.kyprianou@bath.ac.uk

[2]Centro de Investigación en Matemáticas A.C. Calle Jalisco s/n. 36240 Guanajuato, México. E-mail: jcpardo@cimat.mx

[3]Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, U.K. E-mail: k.van.schaik@bath.ac.uk

1 Introduction

Let us suppose that $X = \{X_t : t \geq 0\}$ is a general Lévy process with law $\mathbb{P}$ and Lévy measure Π. That is to say, X is a Markov process with paths that are right continuous with left limits such that the increments are stationary and independent and whose characteristic function at each time t is given by the Lévy-Khinchine representation

$$\mathbb{E}(e^{\mathrm{i}\theta X_t}) = e^{-t\Psi(\theta)}, \quad \theta \in \mathbb{R},$$

where

$$\Psi(\theta) = \mathrm{i}\theta a + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,+\infty)} (1 - e^{\mathrm{i}\theta x} + \mathrm{i}\theta x \mathbf{1}_{\{|x|<1\}})\Pi(\mathrm{d}x). \tag{1.1}$$

We have $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and Π is a measure supported on $\mathbb{R}$ with $\Pi(\{0\}) = 0$ and $\int_{\mathbb{R}}(x^2 \wedge 1)\Pi(\mathrm{d}x) < \infty$. Starting with the early work of Madan and Seneta [16], Lévy processes have played a central role in the theory of financial mathematics and statistics (see for example the books [4, 7, 17, 18]). More recently they have been extensively used in modern insurance risk theory (see for example Klüppelberg et al. [10], Song and Vondraček [19]). The basic idea in financial mathematics and statistics is that the log of a stock price or risky asset follows the dynamics of a Lévy process whilst in insurance mathematics, it is the Lévy process itself which models the surplus wealth of an insurance company until ruin. There are also extensive applications of Lévy processes in queuing theory, genetics and mathematical biology as well as through their appearance in the theory of stochastic differential equations.

In both financial and insurance settings, a key quantity of generic interest is the joint law of the current position and the running maximum of a Lévy process at a fixed time if not the individual marginals associated with the latter bivarite law. For example, if we define $\overline{X}_t = \sup_{s\leq t} X_s$ then the pricing of barrier options boil down to evaluating expectations of the form $\mathbb{E}(f(x + X_t)\mathbf{1}_{\{x+\overline{X}_t>b\}})$ for some appropriate function $f(x)$ and threshold $b > 0$. Indeed if $f(x) = (K - e^x)^+$ then the latter expectation is related to the value of an 'up-and-in' put. In credit risk one is predominantly interested in the quantity $\widehat{\mathbb{P}}(\overline{X}_t < x)$ as a function in x and t, where $\widehat{\mathbb{P}}$ is the law of the dual process $-X$. Indeed it is as a functional of the latter probabilities that the price of a credit default swap is computed; see for example the recent book of Schoutens and Cariboni [18]. One is similarly interested in $\widehat{\mathbb{P}}(\overline{X}_t \geq x)$ in ruin theory as these probabilities are also equivalent to the finite-time ruin probabilities.

One obvious way to do Monte Carlo simulation of expectations involving the joint law of $(X_t, \overline{X}_t)$ that takes advantage of the stationary and independent increments of Lévy processes is to take a random walk approximation to the Lévy process, simulate multiple paths, taking care to record the maximum for each run. Whilst one is able to set things up in this way so that one samples exactly from the distribution of X_t, the law of the maximum of the underlying random walk will not agree with the law of $\overline{X}_t$.

Taking account of the fact that all Lévy processes respect a fundamental path decomposition known as the Wiener-Hopf factorisation, it turns out there is another very straightforward way to perform Monte Carlo simulations for expectations involving the joint law of $(X_t, \overline{X}_t)$ which we introduce in this paper. Our method allows for exact sampling from the law of $(X_g, \overline{X}_g)$ where g is a random time whose distribution can be concentrated arbitrarily close around t.

There are several advantages of the technique we use which are discussed in detail in the subsequent sections of this paper. Firstly, when it is taken in context with very recent developments in Wiener-Hopf theory for Lévy processes, for example recent advances in the theory of scale functions for spectrally negative processes (see Kyprianou et al. [14]), new complex analytical techniques due to Kuznetsov [11] and Vigon's theory of philanthropy (see [20]), one may quickly progress the algorithm to quite straightforward numerical work. Secondly, our Wiener-Hopf method takes advantage of a similar feature found in the, now classical, 'Canadization' method of Carr [6] for numerical evaluation of optimal stopping problems. The latter is generally acknowledged as being more efficient than appealing to classical random walk approximation Monte Carlo methods. Indeed, later in this paper, we present our numerical findings with some indication of performance against the method of random walk approximation for the case of Brownian motion, one of the very few examples for which the joint law of $(X_t, \overline{X}_t)$ is known analytically. In this case, our Wiener-Hopf method appears to be extremely effective. Thirdly, in principle, our method handles better the phenomena of discontinuities which can occur with functionals of the form $\mathbb{E}(f(x + X_t)\mathbf{1}_{\{x+\overline{X}_t>b\}})$ at the boundary point $x = b$. It is now well understood that the issue of regularity of the upper and lower half line for the underlying Lévy process (see Chapter 6 of [12] for a definition) is responsible the appearance of a discontinuity at $x = b$ in such functions (cf. [1]). The nature of our Wiener-Hopf method naturally builds the distributional atom which is responsible for this discontinuity into the simulations.

2 Wiener-Hopf Monte Carlo simulation technique

The basis of the algorithm is the following simple observation which was pioneered by Carr [6] and subsequently used in several contexts within mathematical finance for producing approximate solutions to free boundary value problems that appear as a result of optimal stopping problems that characterise the value of an American-type option.

Suppose that $\mathbf{e}_1, \mathbf{e}_2, \cdots$ are a sequence of i.i.d. exponentially distributed random variables with unit mean. Suppose they are all defined on a common product space with product law $\mathbf{P}$ which is orthogonal to the probability space on which the Lévy process X is defined. For all $t > 0$, we know from the Strong Law of Large Numbers that

$$\sum_{i=1}^{n} \frac{t}{n}\mathbf{e}_i \to t \text{ as } n \uparrow \infty \tag{2.2}$$

$\mathbf{P}$-almost surely. The random variable on the left hand side above is equal in law to a Gamma random variable with parameters n and n/t. Henceforth we write it $\mathbf{g}(n, n/t)$. Recall that $(X, \mathbb{P})$ is our notation for a general Lévy process. Then writing $\overline{X}_t = \sup_{s\leq t} X_s$ we argue the case that, for sufficiently large n, a suitable approximation to $\mathbb{P}(X_t \in dx,\, \overline{X}_t \in dy)$ is $\mathbf{P} \times \mathbb{P}(X_{g(n,n/t)} \in dx,\, \overline{X}_{g(n,n/t)} \in dy)$.

This approximation gains practical value in the context of Monte Carlo simulation when we take advantage of the fundamental path decomposition that applies to all Lévy processes over exponential time periods known as the Wiener-Hopf factorisation.

Theorem 1. *For all $n \in \{1, 2, \cdots\}$ and $\lambda > 0$ define $\mathbf{g}(n,\lambda) := \sum_{i=1}^{n} \mathbf{e}_i/\lambda$. Then*

$$(X_{\mathbf{g}(n,\lambda)}, \overline{X}_{\mathbf{g}(n,\lambda)}) \stackrel{d}{=} (V(n,\lambda), J(n,\lambda))$$

where

$$V(n,\lambda) = \sum_{j=1}^{n} \{S_\lambda^{(j)} + I_\lambda^{(j)}\} \text{ and } J(n,\lambda) := \bigvee_{i=0}^{n-1} \left(\sum_{j=1}^{i} \{S_\lambda^{(j)} + I_\lambda^{(j)}\} + S_\lambda^{(i+1)} \right).$$

Here, $S_\lambda^{(0)} = I_\lambda^{(0)} = 0$, $\{S_\lambda^{(j)} : j \geq 1\}$ are an i.i.d. sequence of random variables with common distribution equal to that of $\overline{X}_{\mathbf{e}_1/\lambda}$ and $\{I_\lambda^{(j)} : j \geq 1\}$ are another i.i.d. sequence of random variable with common distribution equal to that of $\underline{X}_{\mathbf{e}_1/\lambda}$.

Proof. Suppose we define $\overline{X}_{s,t} = \sup_{s \leq u \leq t} X_u$. Then it is trivial to note that

$$\overline{X}_{\mathbf{g}(n,\lambda)} = \bigvee_{i=1}^{n} \overline{X}_{\mathbf{g}(i-1,\lambda),\mathbf{g}(i,\lambda)} \tag{2.3}$$

where $\mathbf{g}(0,\lambda) := 0$.

Next we prove by induction that for each $k = 0, 1, \cdots$

$$(X_{\mathbf{g}(n,\lambda)} : n \leq k) \stackrel{d}{=} \left(\sum_{j=1}^{n} \{S_\lambda^{(j)} + I_\lambda^{(j)}\} : n \leq k \right). \tag{2.4}$$

Note first that the above equality is trivially true when $k = 1$ on account of the Wiener-Hopf factorisation. Indeed the latter tells us that $\overline{X}_{\mathbf{e}_1/\lambda}$ and $X_{\mathbf{e}_1/\lambda} - \overline{X}_{\mathbf{e}_1/\lambda}$ are independent and the second of the pair is equal in distribution to $\underline{X}_{\mathbf{e}_1/\lambda}$. Now suppose that (2.4) is true for $k \geq 1$. Then stationary and independent increments of X together with the Wiener-Hopf factorisation imply that

$$X_{\mathbf{g}(k+1,\lambda)} \stackrel{d}{=} X_{\mathbf{g}(k,\lambda)} + X^{(k+1)}_{\mathbf{e}_{k+1}/\lambda} = X_{\mathbf{g}(k,\lambda)} + S_\lambda^{(k+1)} + I_\lambda^{(k+1)}.$$

where $X^{(k+1)}$ is an independent copy of X, $S_\lambda^{(k+1)} := \sup_{s \leq \mathbf{e}_{k+1}/\lambda} X_s^{(k+1)}$ and $I_\lambda^{(k+1)} := \inf_{s \leq \mathbf{e}_{k+1}/\lambda} X_s^{k+1}$. The induction hypothesis thus holds for $k+1$.

For $n = 0, 1, \cdots$, stationary and independent increments of X allows us to write

$$\overline{X}_{\mathbf{g}(n,\lambda),\mathbf{g}(n+1,\lambda)} \stackrel{d}{=} X_{\mathbf{g}(n,\lambda)} + \sup_{s \leq \mathbf{e}_{n+1}/\lambda} X_s^{(n+1)} = X_{\mathbf{g}(n,\lambda)} + S_\lambda^{(n+1)}.$$

Hence for $k = 0, 1, \cdots$

$$(\overline{X}_{\mathbf{g}(n,\lambda),\mathbf{g}(n+1,\lambda)} : n \le k) \stackrel{d}{=} \left(\sum_{j=1}^{n} \{S_\lambda^{(j)} + I_\lambda^{(j)}\} + S_\lambda^{(n+1)} : n \le k \right).$$

From (2.3) and (2.4) the result now follows. □

Note that the idea of embedding a random walk into the path of a Lévy process with two types of step distribution determined by the Wiener-Hopf factorisation has been used in a different, and more theoretical context, by Doney [8].

The previous theorem combined with the strong law of large numbers in (2.2), which occurs on a probability space orthogonal to that of the Lévy process X, gives us the following important corollary.

Corollary 1. *We have as $n \uparrow \infty$*

$$(V(n, n/t), J(n, n/t)) \to (X_t, \overline{X}_t)$$

where the convergence is understood in the distributional sense.

The above corollary now suggests that we need only to be able to simulate i.i.d. copies of the distributions of $S_{n/t} := S_{n/t}^{(1)}$ and $I_{n/t} := I_{n/t}^{(1)}$ and then by a simple functional transformation we may produce a realisation of the random variable $X_{\mathbf{g}(n,n/t)}$. Given a suitably nice function F, using standard Monte-Carlo methods one estimates for large k

$$\mathbb{E}(F(X_t, \overline{X}_t)) \simeq \frac{1}{k} \sum_{m=1}^{k} F(V^{(m)}(n, n/t), J^{(m)}(n, n/t))$$

where $(V^{(m)}(n, n/t), J^{(m)}(n, n/t))$ are i.i.d. copies of $(V(n, n/t), J(n, n/t))$. Indeed the strong law of large numbers and Corollary 1 imply that the right hand side above converges almost surely as $k \uparrow \infty$ to $\mathbf{E} \times \mathbb{E}(F(X_{\mathbf{g}(n,n/t)}, \overline{X}_{\mathbf{g}(n,n/t)}))$ which in turn converges as $n \uparrow \infty$ to $\mathbb{E}(F(X_t, \overline{X}_t))$.

3 Implementation

The algorithm described in the previous section only has practical value if one is able to sample from the distributions of $\overline{X}_{\mathbf{e}_1/\lambda}$ and $-\underline{X}_{\mathbf{e}_1/\lambda}$. It would seem that this, in itself, is not that much different from the problem that it purports to solve. However, it turns out that there are many tractable examples and in all cases this is due to the part tractability of their Wiener-Hopf factorisations.

Whilst several concrete cases can be handled from the class of spectrally one-sided Lévy processes thanks to recent development in the theory of scale functions which can be used to described the laws of $\overline{X}_{\mathbf{e}_1/\lambda}$ and $-\underline{X}_{\mathbf{e}_1/\lambda}$ (cf. [9, 15]), we give here two large families of two sided jumping Lévy processes that have pertinence to mathematical finance to show how the algorithm may be implemented.

3.1 Kuznetsov's β-class of Lévy processes

The β-class of Lévy processes, introduced by Kuznetsov [11], is a 10-parameter Lévy process which has characteristic exponent

$$\begin{aligned}\Psi(\theta) &= \mathrm{i}a\theta + \frac{1}{2}\sigma^2\theta^2 + \frac{c_1}{\beta_1}\left\{B(\alpha_1, 1-\lambda_1) - B(\alpha_1 - \frac{\mathrm{i}\theta}{\beta_1}, 1-\lambda_1)\right\} \\ &\quad + \frac{c_2}{\beta_2}\left\{B(\alpha_2, 1-\lambda_2) - B(\alpha_2 - \frac{\mathrm{i}\theta}{\beta_2}, 1-\lambda_2)\right\}\end{aligned}$$

with parameter range $a, \sigma \in \mathbb{R}$, $c_1, c_2, \alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and $\lambda_1, \lambda_2 \in (0,3)$ which is the result of an underlying Lévy measure Π with density π given by

$$\pi(x) = c_1 \frac{e^{-\alpha_1\beta_1 x}}{(1-e^{-\beta_1 x})^{\lambda_1}}\mathbf{1}_{\{x>0\}} + c_2\frac{e^{\alpha_2\beta_2 x}}{(1-e^{\beta_2 x})^{\lambda_2}}\mathbf{1}_{\{x<0\}}.$$

Although Ψ takes a seemingly complicated form, this particular family of Lévy processes has a number of very beneficial virtues from the point of view of mathematical finance which are discussed in [11]. Moreover, the large number of parameters also allows one to choose Lévy processes within the β-class that have paths that are both of unbounded variation (when at least one of the conditions $\sigma \neq 0$, $\lambda_1 \in (2,3)$ or $\lambda_2 \in (2,3)$ holds) and bounded variation (when all of the conditions $\sigma = 0$, $\lambda_1 \in (0,2)$ and $\lambda_2 \in (0,2)$ hold) as well as having infinite and finite activity in the jumps component (accordingly as both $\lambda_1, \lambda_2 \in (1,3)$ or not).

What is special about the β-class is that all the roots of the equation $\lambda + \Psi(\theta) = 0$ are completly identifiable which leads to semi-explicit identities for the laws of $\overline{X}_{\mathbf{e}_{1/\lambda}}$ and $-\underline{X}_{\mathbf{e}_{1/\lambda}}$ as the following result lifted from Kuznetsov [11] shows.

Theorem 2. *For $\lambda > 0$, all the roots of the equation*

$$\lambda + \Psi(\theta) = 0$$

are simple and occur on the imaginary axis. They can be enumerated by $\{\mathrm{i}\zeta_n^+ : n \geq 0\}$ on the positive imaginary axis and $\{\mathrm{i}\zeta_n^- : n \geq 0\}$ on the negative imaginary axis in order of increasing absolute magnitude where

$$\begin{aligned}&\zeta_0^+ \in (0, \beta_2\alpha_2),\ \zeta_0^- \in (-\beta_1\alpha_1, 0),\\ &\zeta_n^+ \in (\beta_2(\alpha_2+n-1), \beta_2(\alpha_2+n)) \text{ for } n \geq 1\\ &\zeta_n^- \in (\beta_1(-\alpha_1-n), \beta_1(-\alpha_1-n+1)) \text{ for } n \geq 1.\end{aligned}$$

Moreover, for $x > 0$,

$$\mathbb{P}(\overline{X}_{\mathbf{e}_{1/\lambda}} \in \mathrm{d}x) = -\left(\sum_{k\geq 0} c_k^- \zeta_k^- e^{\zeta_k^- x}\right)\mathrm{d}x \tag{3.5}$$

where

$$c_0^- = \prod_{n\geq 1} \frac{1+\frac{\zeta_0^-}{\beta_1(n-1+\alpha_1)}}{1-\frac{\zeta_0^-}{\zeta_n^-}} \quad and \quad c_k^- = \frac{1+\frac{\zeta_k^-}{\beta_1(k-1+\alpha_1)}}{1-\frac{\zeta_k^-}{\zeta_0^-}} \prod_{n\geq 1, n\neq k} \frac{1+\frac{\zeta_k^-}{\beta_1(n-1+\alpha_1)}}{1-\frac{\zeta_k^-}{\zeta_n^-}}.$$

A similar expression holds for $\mathbb{P}(-\underline{X}_{\mathbf{e}_{1/\lambda}} \in \mathrm{d}x)$ *with the role of* $\{\zeta_n^- : n \geq 0\}$ *being played by* $\{-\zeta_n^+ : n \geq 0\}$ *and* α_1, β_1 *replaced by* α_2, β_2.

Note that when 0 is irregular for $(0,\infty)$ the distribution of $\overline{X}_{\mathbf{e}_{1/\lambda}}$ will have an atom at 0 which can be computed from (3.5) and is equal to $1-\sum_{k\geq 0} c_k^-$. Alternatively, from Remark 6 in [11] this can equivalently be written as $\prod_{n\geq 0}(-1)^n \zeta_n^-/\beta_1(n+\alpha_1)$. A similar statement can be made concerning an atom at 0 for the distribution of $-\underline{X}_{\mathbf{e}_{1/\lambda}}$ when 0 is irregular for $(-\infty,0)$. Conditions for irregularity are easy to check thanks to Bertoin [3]; see also the summary in Kyprianou and Loeffen [13].

By making a suitable truncation of the series (3.5) one may easily perform independent sampling from the distributions $\overline{X}_{\mathbf{e}_{1/\lambda}}$ and $\underline{X}_{\mathbf{e}_{1/\lambda}}$ as required for our Monte Carlo methods.

3.2 Philanthropy and General Hypergeometric Lévy processes.

The forthcoming discussion will assume familiarity with classical excursion theory of Lévy processes for which the reader is referred to Chapter VI of [2] or Chapter 6 of [12].

According to Vigon's theory of philanthropy, a (killed) subordinator is called a *philanthropist* if its Lévy measure has a decreasing density on $\mathbb{R}_+$. Moreover, given any two subordinators H_1 and H_2 which are philanthropists, providing that at least one of them is not killed, there exist a Lévy process X such that H_1 and H_2 have the same law as the ascending and descending ladder height processes of X, respectively. Suppose we denote the killing rate, drift coefficient and Lévy measures of H_1 and H_2 by the respective triples (k,δ,Π_{H_1}) and $(\widehat{k},\widehat{\delta},\Pi_{H_2})$. Then [20] shows that the Lévy measure of X satisfies the following identity

$$\overline{\Pi}_X^+(x) = \int_0^\infty \Pi_{H_1}(x+du)\overline{\Pi}_{H_2}(u) + \widehat{\delta}\,\pi_{H_1}(x) + \widehat{k}\,\overline{\Pi}_{H_1}(x), \quad x>0, \tag{3.6}$$

where π_{H_1} is the density of Π_{H_1}. By symmetry, an obvious analogue of (3.6) holds for the negative tail $\overline{\Pi}_X^-(x) := \Pi_X(-\infty,x)$, $x<0$.

A particular family of subordinators which will be of interest to us is the class subordinators which is found within the definition of Kuznetsov's β-class of Lévy processes. These processes have characteristics (c,α,β,γ) where $\gamma\in(0,1)$, $\beta,c>0$ and $\alpha\in(-\infty,1]$. The Lévy measure of such subordinators is of the type

$$c\frac{e^{\alpha\beta x}}{(e^{\beta x}-1)^{1+\gamma}}1_{\{x>0\}}\,\mathrm{d}x. \tag{3.7}$$

From Proposition 9 in [11], the Laplace exponent of a β-class subordinator satisfies

$$\Phi(\theta) = \mathtt{k} + \delta\theta + c\frac{\Gamma(1-\gamma)}{\beta\gamma}\left(\frac{\Gamma(\theta/\beta+1-\alpha+\gamma)}{\Gamma(\theta/\beta+1-\alpha)} - \frac{\Gamma(1-\alpha+\gamma)}{\Gamma(1-\alpha)}\right) \tag{3.8}$$

for $\theta \geq 0$ where δ is the drift coefficient and $\mathtt{k}$ is the killing rate.

Let H_1 and H_2 be two independent subordinators from the β-class where for $i = 1, 2$, with respective drift coefficients $\delta_i \geq 0$, killing rates $\mathtt{k}_i \geq 0$ and Lévy measure parameters $(c_i, \alpha_i, \beta_i, \gamma_i)$. Their respective Laplace exponents are denoted by Φ_i, $i = 1, 2$. In Vigon's theory of philanthropy it is required that $\mathtt{k}_1\mathtt{k}_2 = 0$. Under this assumption, let us denote by X the Lévy process whose ascending and descending ladder height processes have the same law as H_1 and H_2, respectively. In other words, the Lévy process whose characteristic exponent is given by $\Phi_1(-i\theta)\Phi_2(i\theta)$, $\theta \in \mathbb{R}$. From (3.6), the Lévy measure of X is such that

$$\begin{aligned}\overline{\Pi}_X^+(x) = {} & c_1c_2\int_x^\infty \frac{e^{\beta_1\alpha_1 u}}{(e^{\beta_1 u}-1)^{\gamma_1+1}}\int_{u-x}^\infty \frac{e^{\alpha_2\beta_2 z}}{(e^{\beta_2 z}-1)^{\gamma_2+1}}dz\,du \\ & + \delta_2 c_1\frac{e^{\beta_1\alpha_1 x}}{(e^{\beta_1 x}-1)^{\gamma_1+1}} + \mathtt{k}_2 c_1\int_x^\infty \frac{e^{\beta_1\alpha_1 u}}{(e^{\beta_1 u}-1)^{\gamma_1+1}}\,\mathrm{d}x.\end{aligned}$$

Making use of the general binomial expansion, we develop the integrals above in series form,

$$\begin{aligned}\overline{\Pi}_X^+(x) = {} & \frac{c_1c_2}{\beta_2}\sum_{n,k=0}^\infty \frac{(\gamma_1+1)_k(\gamma_2+1)_n}{k!n!(\gamma_2+1-\alpha_2+n)}\frac{e^{-\beta_1(\gamma_1+1-\alpha_1+k)x}}{\beta_1(\gamma_1+1-\alpha_1+k)+\beta_2(\gamma_2+1-\alpha_2+n)} \\ & + \delta_2 c_1\frac{e^{\beta_1\alpha_1 x}}{(e^{\beta_1 x}-1)^{\gamma_1+1}} + \frac{\mathtt{k}_2 c_1}{\beta_1}\sum_{k=0}^\infty \frac{(\gamma_1+1)_k}{k!(\gamma_1+1-\alpha_1+k)}e^{-\beta_1(\gamma_1+1-\alpha_1+k)x}\end{aligned}$$

where $(z)_n = \Gamma(z+n)/\Gamma(z)$, $z \in \mathbb{C}$. An identical expression holds for $\overline{\Pi}_X^-(x)$ by exchanging the roles of the constants $(\mathtt{k}_1, \delta_1, c_1, \alpha_1, \beta_1, \gamma_1)$ and $(\mathtt{k}_2, \delta_2, c_2, \alpha_2, \beta_2, \gamma_2)$. It is important to note that the Gaussian component of the process X is given by $2\delta_1\delta_2$.

We define a General Hypergeometric process to be the 14 parameter Lévy process with characteristic exponent given in compact form

$$\Psi(\theta) = \mathtt{d}\mathrm{i}\theta + \frac{1}{2}\sigma^2\theta^2 + \Phi_1(-\mathrm{i}\theta)\Phi_2(\mathrm{i}\theta),\ \theta \in \mathbb{R} \tag{3.9}$$

where $\mathtt{d}, \sigma \in \mathbb{R}$. The inclusion of the two additional parameters $\mathtt{d}, \sigma$ is largely with applications in mathematical finance in view. Without these two additional parameters it is difficult to disentangle the Gaussian coefficient and the drift coefficients from parameters appearing in the jump measure. Note that the Gaussian coefficient in (3.9) is now $\sigma^2/2 + 2\delta_1\delta_2$. The definition of General Hypergeometric Lévy processes includes previously defined Hypergeometric Lévy processes in Kyprianou et al. [14] and Lamperti-stable Lévy processes in Caballero et al. [5]. Note that this is also the case of Kuznetsov's β-class of Lévy processes.

Just as with the case of the β-family of Lévy processes, because Ψ can be written as a linear combination of a quadratic form and beta functions, it turns out that one can identify all the roots of the equation $\Psi(\theta) + q = 0$ which is again sufficient to describe the laws of $\overline{X}_{\mathbf{e}_1/\lambda}$ and $-\underline{X}_{\mathbf{e}_1/\lambda}$. We get the following two results whose proofs are briefly outlined in the Appendix.

Theorem 3. *For $\lambda > 0$, all the roots of the equation*

$$\lambda + \Psi(\theta) = 0$$

are simple and occur on the imaginary axis. Moreover, they can be enumerated by $\{\mathrm{i}\xi_n^+ : n \geq 0\}$ on the positive imaginary axis and $\{\mathrm{i}\xi_n^- : n \geq 0\}$ on the negative imaginary axis in order of increasing absolute magnitude where

$$\begin{aligned} \xi_0^+ \in \big(0, \beta_2(1+\gamma_2-\alpha_2)\big),\ \xi_0^- &\in \big(-\beta_1(1+\gamma_1-\alpha_1), 0\big) \\ \xi_n^+ &\in \big(\beta_2(\gamma_2-\alpha_2+n), \beta_2(1+\gamma_2-\alpha_2+n)\big) \qquad \text{for } n \geq 1 \\ \xi_n^- &\in \big(-\beta_1(1+\gamma_1-\alpha_1+n), -\beta_1(\gamma_1-\alpha_1+n)\big) \qquad \text{for } n \geq 1. \end{aligned}$$

Moreover, for $x > 0$,

$$\mathbb{P}\big(\overline{X}_{\mathbf{e}_1/\lambda} \in \mathrm{d}x\big) = -\left(\sum_{k\geq 0} c_k^- \xi_k^- e^{\xi_k^- x}\right) \mathrm{d}x, \tag{3.10}$$

where

$$c_0^- = \prod_{n\geq 1} \frac{1 + \frac{\xi_0^-}{\beta_1(\gamma_1-\alpha_1+n)}}{1 - \frac{\xi_0^-}{\xi_n^-}} \quad \text{and} \quad c_k^- = \frac{1 + \frac{\xi_k^-}{\beta_1(\gamma_1-\alpha_1+k)}}{1 - \frac{\xi_k^-}{\xi_0^-}} \prod_{n\geq 1, n\neq k} \frac{1 + \frac{\xi_k^-}{\beta_1(\gamma_1-\alpha_1+n)}}{1 - \frac{\xi_k^-}{\xi_n^-}}.$$

Moreover, a similar expression holds for $\mathbb{P}(-\underline{X}_{\mathbf{e}_1/\lambda} \in \mathrm{d}x)$ with the role of $\{\xi_n^- : n \geq 0\}$ replaced by $\{-\xi_n^- : n \geq 0\}$ and $\alpha_1, \beta_1, \gamma_2$ replaced by $\alpha_2, \beta_2, \gamma_2$.

Similar remarks to those made after Theorem 2 regarding the existence of atoms in the distribution of $\overline{X}_{\mathbf{e}_1/\lambda}$ and $-\underline{X}_{\mathbf{e}_1/\lambda}$ also apply here.

Remark 1. It is important to note that the hypergeometric Lévy process is but one of many examples of Lévy processes which may be constructed using Vigon's theory of philanthropy. With the current Monte Carlo algorithm in mind, it should be possible to engineer other favourable Lévy processes in this way.

4 Simulations

Our starting point is to look at how our method compares against the obvious way of simulating the distribution of $(X_1, \overline{X}_1)$ using a random walk Monte Carlo method. We do this for the most basic of cases, namely Brownian motion, for the simple reason that we are able to compare simulations of the distribution of $\overline{X}_1$ and the joint distribution of $(X_1, \overline{X}_1)$ against an exact value. In this experiment we simulated $m = 10^6$ paths of the quantities $(V(n,n), J(n,n))$ for $n = 10, 10^2$ and 10^3 thereby producing numerical estimates of $\mathbb{P}(\overline{X}_1 \leq z)$ for different values of z and $\mathbb{P}(X_1 \leq z_1, \overline{X}_1 \geq z_2)$ for different pairs z_1, z_2 with $z_1 \leq z_2$. Note that the two Wiener-Hopf random variables we are required to sample from are both known in exact form as exponential distributions. For fair comparison, we also simulated numerical estimates of the aforementioned probabilities using the obvious Gaussian random walk approximation to a Brownian motion with $2n$ steps for the same values of n. (Recall that in our method, each of the n exponential periods that make up the $\mathbf{g}(n,n)$ time horizon has associated to it two steps coming from each of the Wiener-Hopf factors which explains the comparison against $2n$ steps in the random walk simulation). The results are illustrated in Tables 1 and 2, both of which appear to indicate clear benefits from the Wiener-Hopf method (indicated by w.h.) over the alternative random walk method (indicated by r.w.).

As mentioned before, a typical example of a financial contract whose payoff depends on the joint density of $(X_t, \overline{X}_t)$ is a barrier option. All of the forthcoming simulations take as a test case an up-and-out European call option with unit time horizon whose value function satisfies

$$V(x) = \mathbb{E}(e^{-r}(e^{x+X_1} - K)^+ \mathbf{1}_{\{\exp(x+\overline{X}_1)<b\}}). \tag{4.11}$$

We take the following parameters: strike price $K = 5$, barrier $b = 10$ and $r = 5/100$. Below we show some plots of V (as a function of $s = e^x$) obtained by simulating $m = 10^6$ paths of the quantities $(V(n,n), J(n,n))$ for $n = 100$. Each plot takes about 40 seconds to produce and the simulations were programmed using the open source computer algebra system SAGE (`www.sagemath.org`) on a standard 2009 laptop.

In all cases the linear drift is chosen such that $\Psi(-\mathrm{i}) = -r$, for no other reason that this is a risk neutral setting which makes the process $\{\exp(X_t - rt) : t \geq 0\}$ is a martingale. Figure 1 shows for an example of the classic Black-Scholes model, i.e. $X_t = \sigma B_t + \mu t$, where B is standard Brownian motion, the exact value of V, the simulated value of V and the difference between the two. Figure 2 shows how different the value function can be when X is a member of the β-family. The five diagrams cover the case that there is a Gaussian component with a jump structure which has finite activity, bounded and unbounded variation paths respectively as well as the case of no Gaussian component with a jump structure which has bounded and unbounded variation paths.

The fourth diagram in Figure 2 exhibits the interesting phenomenon of a discontinuity in $V(x)$ at the boundary. The discontinuity should be there and occurs due to the fact that, for those particular parameter choices, there is irregularity of the upper half line. Irregularity of the upper half line is equivalent to there being an atom at zero in the distribution of $\overline{X}_t$ for any $t > 0$ (also at independent and exponentially distributed random times). This means that the Wiener-Hopf Monte Carlo algorithm correctly builds in an atom at zero into the approximating distribution of $\overline{X}_1$ if and only if an atom is supposed to be present. By contrast the random walk Monte Carlo simulation method will always build in an atom at zero into the approximating distribution of $\overline{X}_1$ irrespective of whether it is supposed to be there or not.

Finally in Figure 3 we look at two simulations of $V(x)$ for a hypergeometric Lévy process with and without a Gaussian component. Both diagrams in Figure 3 come from processes with unbounded variation paths.

Appendix

Our objective is to give a brief overview of the proof of Theorem 3. Let us start by writing out the Lévy-Khintchine formula for the General Hypergeometric Lévy process in full detail. We shall use the usual notation for the beta function $B(x;y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. The next lemma is proved using straightforward algebra making use of the following identity which is obtained using integration by parts and Proposition 9 in [11]. Let $\theta \in \mathbb{R}$, then

$$
\begin{aligned}
\mathrm{i}\theta \int_0^\infty (1 - e^{\mathrm{i}\theta}) \frac{e^{\alpha\beta x}}{(e^{\beta x} - 1)^{1+\gamma}}\,\mathrm{d}x &= \alpha\Big(B(1+\gamma-\alpha-\mathrm{i}\theta/\beta; -\gamma) - B(1+\gamma-\alpha;-\gamma)\Big) \\
&\quad - (1+\gamma)\Big(B(1+\gamma-\alpha-\mathrm{i}\theta/\beta; -(1+\gamma)) - B(1+\gamma-\alpha; -(1+\gamma))\Big) \\
&\quad + i\frac{\theta\alpha}{\beta} B(1+\gamma-\alpha;-\gamma)\big(\psi(1+\gamma-\alpha) - \psi(1-\alpha)\big) \\
&\quad - i\frac{\theta(1+\gamma)}{\beta} B(1+\gamma-\alpha; -(1+\gamma))\big(\psi(1+\gamma-\alpha) - \psi(-\alpha)\big).
\end{aligned}
$$

Lemma 1. *Up to a multiplicative constant, the characteristic exponent of any hypergeometric Lévy process X can be written for $\theta \in \mathbb{R}$ in the form*

$$
\Psi(\theta) = \mathrm{i}\theta(\mathtt{d} + \mu) + \frac{1}{2}(\sigma^2 + 2\delta_1\delta_2)\theta^2 + \Psi_1(\theta) - \Psi_1(0),
$$

where

$$
\begin{aligned}
\Psi_1(\theta) &= -\delta_2 c_1(1+\gamma_1)B(-\mathrm{i}\theta/\beta_1 + 1 + \gamma_1 - \alpha_1; -(1+\gamma_1)) \\
&\quad - \delta_1 c_2 (1+\gamma_2) B(\mathrm{i}\theta/\beta_2 + 1 + \gamma_2 - \alpha_2; -(1+\gamma_2)) \\
&\quad - \frac{c_2}{\beta_2}\left(\mathtt{k}_1 - \delta_1\beta_2\alpha_2 + \frac{c_1}{\beta_1}B(1+\gamma_1-\alpha_1;-\gamma_1)\right) B(\mathrm{i}\theta/\beta_2 + 1 + \gamma_2 - \alpha_2; -\gamma_2) \\
&\quad - \frac{c_1}{\beta_1}\left(\mathtt{k}_2 - \delta_2\beta_1\alpha_1 + \frac{c_2}{\beta_2}B(1+\gamma_2-\alpha_2;-\gamma_2)\right) B(-\mathrm{i}\theta/\beta_1 + 1 + \gamma_1 - \alpha_1; -\gamma_1) \\
&\quad + \frac{c_1 c_2}{\beta_1\beta_2} B(-\mathrm{i}\theta/\beta_1 + 1 + \gamma_1 - \alpha_1; -\gamma_1) B(\mathrm{i}\theta/\beta_2 + 1 + \gamma_2 - \alpha_2; -\gamma_2)
\end{aligned}
$$

for some constant μ which, without loss of generality, can be ignored and absorbed into the quantity $\mathtt{d}$.

Proof of Theorem 3. As noted in the previous lemma, we may take without loss of generality the constant $\mu = 0$. Using the explicit expression of the characteristic exponent Ψ, we rewrite the equation $\lambda + \Psi(\mathrm{i}\theta) = 0$ as follows

$$\theta \mathtt{d} + \frac{1}{2}(\sigma^2 + 2\delta_1\delta_2)\theta^2 - \lambda = \Psi_1(\mathrm{i}\theta) - \Psi_1(0).$$

Let us denote the right hand side of the above equation by $R(\theta)$ and the left hand side by $L(\theta)$. We first verify that there exist a solution to the identity of above on $(-\beta_1(1+\gamma_1-\alpha_1), 0)$ and $\big(0, \beta_2(1+\gamma_2-\alpha_2)\big)$. We observe that $R(0) = 0$ and that $R(\theta) \searrow -\infty$ as $\theta \searrow -\beta_1(1+\gamma_1-\alpha_1)$ and as $\theta \nearrow \beta_2(1+\gamma_2-\alpha_2)$. On the other hand $L(\theta)$ is continuous and negative at $\theta = 0$, thus we have at least one solution $\xi_0^+ \in (0, \beta_2(1+\gamma_2-\alpha_2))$ and at least one solution $\xi_0^- \in (-\beta_1(1+\gamma_1-\alpha_1), 0)$. In fact, we have that we have one solution on each interval since the function R is concave on $(-\beta_1(1+\gamma_1-\alpha_1), \beta_2(1+\gamma_2-\alpha_2))$ while L is convex. The proof for $n \neq 0$ follows from similar arguments.

The agenda for the remainder of the proof is identical to those of Theorems 8 and 11 of Kuznetsov [11]. Two key issues are that we need to choose an entire function which has zeros at the poles of $\Psi(\theta)$ with the same multiplicity. The the obvious choice here is

$$\frac{1}{\Gamma(1+\gamma_1-\alpha_1-i\theta/\beta_1)\Gamma(1+\gamma_2-\alpha_2+i\theta/\beta_2)}.$$

Secondly, we need to establish certain asymptotics of the roots to the equation $\lambda + \Psi(\mathrm{i}\theta) = 0$. From this it follows that one may identify the space-time Wiener-Hopf factorisation,

$$\frac{\lambda}{\lambda + \Psi(\theta)} = \frac{1}{1+\frac{\mathrm{i}\theta}{\xi_0^-}} \prod_{n\geq 1} \frac{1 - \frac{\mathrm{i}\theta}{\beta_1(\gamma_1-\alpha_1+n)}}{1+\frac{\mathrm{i}\theta}{\xi_n^-}} \times \frac{1}{1+\frac{\mathrm{i}\theta}{\xi_0^+}} \prod_{n\geq 1} \frac{1 + \frac{\mathrm{i}\theta}{\beta_2(\gamma_2-\alpha_2+n)}}{1+\frac{\mathrm{i}\theta}{\xi_n^+}},$$

where the infinite product converges uniformly on the compact subsets of the complex pane excluding zeros/poles of $\lambda + \Psi(\theta)$. The distribution of $\overline{X}_{\mathbf{e}_{1/\lambda}}$ (and subsequently that of $-\underline{X}_{\mathbf{e}_{1/\lambda}}$) follows from a straightforward Fourier inversion using residues.

The whole proof in the case at hand thus rests on establishing the asymptotic behaviour of the roots of $\lambda + \Psi(\theta)$. Just as in [11], one may do this by making use of the asymptotics of $B(\alpha+\theta;\gamma)$ and $B(\alpha-\theta;\gamma)$ that are given in the proof of Theorem 10 in the same paper.

For example, if $\delta_1, \delta_2 > 0$ and $\theta \to \infty$, we use the explicit form of Ψ in Lemma 1 and rewrite equation $\lambda + \Psi(\mathrm{i}\theta) = 0$ as

$$\frac{\sin\big(\pi(\theta/\beta_2 + \alpha_2)\big)}{\sin\big(\pi(\theta/\beta_2 - 1 + \alpha_2 - \gamma_2)\big)} = \frac{\beta_2^{1+\gamma_2}}{2\delta_1 c_2 \Gamma(-\gamma_2)}(\sigma^2 + 2\delta_1\delta_2)\theta^{1-\gamma_2} + O(\theta^{\gamma_1-\gamma_2}) + O(\theta^{-\gamma_2}),$$

while if $\sigma = \delta_1 = \delta_2 = 0$ and $\theta \to \infty$, we have

$$\frac{\sin\big(\pi(\theta/\beta_2 - 1 + \alpha_2)\big)}{\sin\big(\pi(\theta/\beta_2 - 1 + \alpha_2 - \gamma_2)\big)} = \frac{\beta_2^{1+\gamma_2}}{c_2\Gamma(-\gamma_2)}\left(\mathtt{k}_2 + \frac{c_2}{\beta_2}B(1+\gamma_2-\alpha_2; -\gamma_2)\right)\theta^{-\gamma_2} + O(\theta^{-\gamma_1-\gamma_2}).$$

The case when $\theta \to -\infty$ can be obtained in a similar way.

There are several other cases to consider depending on the equality with zero of some of the parameters in the defintion of Ψ. We summarise all the possible asymptotics of the roots below:

$$\xi_n^- = -\beta_1(n - \alpha_1 + \omega_1) - A(n - \alpha_1 + \omega_1)^{\varrho_1} + O(n^{\varrho_1 - \epsilon})$$
$$\xi_n^+ = \beta_2(n - \alpha_2 + \omega_2) + C(n - \alpha_2 + \omega_2)^{\varrho_2} + O(n^{\varrho_2 - \epsilon}) \qquad \text{as} \quad n \to \infty.$$

where the coefficients $\omega_1, \omega_2, \varrho_1, \varrho_2, A$ and C are presented in Table 3 and Table 4. □

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		$z = 0.1$	$z = 0.2$	$z = 0.3$	$z = 0.4$	$z = 0.5$	$z = 1$	$z = 1.5$	$z = 2$
	exact	0.0797	0.1585	0.2358	0.3108	0.3829	0.6827	0.8664	0.9545
$n = 10$	w.h.	0.0828	0.1644	0.2447	0.3219	0.3955	0.6944	0.8700	0.9523
	error	3.88%	3.74%	3.75%	3.56%	3.28%	1.71%	0.41%	-0.23%
	r.w.	0.1886	0.2593	0.3315	0.4020	0.4689	0.7389	0.8951	0.9661
	error	136.76%	63.57%	40.56%	29.36%	22.44%	8.23%	3.32%	1.21%
$n = 100$	w.h.	0.0803	0.1592	0.2372	0.3125	0.3843	0.6852	0.8672	0.9546
	error	0.79%	0.41%	0.58%	0.52%	0.35%	0.36%	0.09%	0.01%
	r.w.	0.1122	0.1909	0.2675	0.3411	0.4116	0.7018	0.8764	0.9586
	error	40.90%	20.40%	13.45%	9.72%	7.48%	2.80%	1.16%	0.43%
$n = 1000$	w.h.	0.0792	0.1581	0.2357	0.3112	0.3837	0.6839	0.8665	0.9546
	error	-0.53%	-0.27%	-0.07%	0.12%	0.20%	0.17%	0.03%	0.00%
	r.w.	0.0899	0.1684	0.2456	0.3206	0.3925	0.6896	0.8699	0.9559
	error	12.91%	6.24%	4.16%	3.12%	2.50%	1.01%	0.41%	0.15%

Table 1: Computing $\mathbb{P}(\overline{X}_1 \leq z)$ for different values of z when X is a standard Brownian motion.

		$z_2 = 0.1$	$z_2 = 0.3$	$z_2 = 0.5$	$z_2 = 1$
$z_1 = -2$	exact	0.0139	0.0047	0.0014	0.00003
	w.h.	0.0138	0.0046	0.0013	0.00003
	error	-0.93%	-1.93%	-1.33%	-5.27%
	r.w.	0.0128	0.0043	0.0012	0.00002
	error	-7.92%	-8.22%	-10.51%	-24.22%
$z_1 = -1$	exact	0.1151	0.0548	0.0228	0.0014
	w.h.	0.1147	0.0544	0.0225	0.0013
	error	-0.28%	-0.65%	-0.91%	-5.77%
	r.w.	0.1095	0.0515	0.0210	0.0012
	error	-4.87%	-6.12%	-7.54%	-14.36%
$z_1 = 0$	exact	0.4207	0.2743	0.1587	0.0228
	w.h.	0.4205	0.2738	0.1576	0.0223
	error	-0.06%	-0.18%	-0.68%	-2.02%
	r.w.	0.4101	0.2653	0.1518	0.0211
	error	-2.54%	-3.26%	-4.34%	-7.18%
$z_1 = 1$	exact				0.1587
	w.h.				0.1583
	error				-0.24%
	r.w.				0.1519
	error				-4.23%

Table 2: Computing $\mathbb{P}(X_1 \leq z_1, \overline{X}_1 \geq z_2)$ for different values of z_1, z_2 when X is a standard Brownian motion.

Case	ω_1	A	ϱ_1
$\sigma^2, \delta_1, \delta_2 > 0$	$1+\gamma_1$	$\frac{2\delta_2 c_1}{\beta_1\Gamma(1+\gamma_1)(\sigma^2+2\delta_1\delta_2)}$	$\gamma_1 - 1$
$\sigma = 0, \delta_1, \delta_2 > 0$	$1+\gamma_1$	$\frac{c_1}{\beta_1\Gamma(1+\gamma_1)\delta_1}$	$\gamma_1 - 1$
$\sigma^2, \delta_2 > 0, \delta_1 = 0$	$1+\gamma_1$	$\frac{2\delta_2 c_1}{\beta_1\Gamma(1+\gamma_1)\sigma^2}$	$\gamma_1 - 1$
$\sigma^2, \delta_1 > 0, \delta_2 = 0$	$1+\gamma_1$	$\frac{2c_1c_2\Gamma(1-\gamma_2)}{\beta_2^{1+\gamma_2}\beta_1^{2-\gamma_1}\Gamma(1+\gamma_1)\gamma_2\sigma^2}$	$\gamma_1+\gamma_2-2$
$\delta_2 > 0, \sigma = \delta_1 = 0$	0	$\frac{\sin(\pi\gamma_1)}{\pi}\frac{\beta_1^2\gamma_1(\mu+\mathtt{d})}{\delta_2 c_1\Gamma(1-\gamma_1)}$	$-\gamma_1$
$\delta_1 > 0, \sigma = \delta_2 = 0$	$1+\gamma_1$	$\frac{c_1}{\beta_1\delta_1\Gamma(1+\gamma_1)}$	$\gamma_1 - 1$
$\sigma^2 > 0, \delta_1 = \delta_2 = 0$	$1+\gamma_1$	$\frac{2c_1c_2\Gamma(1-\gamma_2)}{\beta_2^{1+\gamma_2}\beta_1^{2-\gamma_1}\Gamma(1+\gamma_1)\gamma_2\sigma^2}$	$\gamma_1+\gamma_2-2$
$\sigma = \delta_1 = \delta_2 = 0$	1	$\frac{\beta_1^2\gamma_1}{c_1\Gamma(1-\gamma_1)}\frac{\sin(\pi\gamma_1)}{\pi}\left(\mathtt{k}_1+\frac{c_1}{\beta_1}B(1+\gamma_1-\alpha_1;-\gamma_1)\right)$	$-\gamma_1$

Table 3: Coefficients for the asymptotic expansion of ξ_n^-.

Case	ω_2	C	ϱ_2
$\sigma^2, \delta_1, \delta_2 > 0$	$1+\gamma_2$	$\frac{2\delta_1 c_2}{\beta_2\Gamma(1+\gamma_2)(\sigma^2+2\delta_1\delta_2)}$	$\gamma_2 - 1$
$\sigma = 0, \delta_1, \delta_2 > 0$	$1+\gamma_2$	$\frac{c_2}{\beta_2\Gamma(1+\gamma_2)\delta_2}$	$\gamma_2 - 1$
$\sigma^2, \delta_2 > 0, \delta_1 = 0$	$1+\gamma_2$	$\frac{2c_1c_2\Gamma(1-\gamma_1)}{\beta_1^{1+\gamma_1}\beta_2^{2-\gamma_2}\Gamma(1+\gamma_2)\gamma_1\sigma^2}$	$\gamma_1+\gamma_2-2$
$\sigma^2, \delta_1 > 0, \delta_2 = 0$	$1+\gamma_2$	$\frac{2\delta_1 c_2}{\beta_2\Gamma(1+\gamma_2)\sigma^2}$	$\gamma_2 - 1$
$\delta_2 > 0, \sigma = \delta_1 = 0$	$1+\gamma_2$	$\frac{c_2}{\beta_2\delta_2\Gamma(1+\gamma_2)}$	$\gamma_2 - 1$
$\delta_1 > 0, \sigma = \delta_2 = 0$	0	$\frac{\sin(\pi\gamma_2)}{\pi}\frac{\beta_2^2\gamma_2(\mu+\mathtt{d})}{\delta_1 c_2\Gamma(1-\gamma_2)}$	$-\gamma_2$
$\sigma^2 > 0, \delta_1 = \delta_2 = 0$	$1+\gamma_2$	$\frac{2c_1c_2\Gamma(1-\gamma_1)}{\beta_1^{1+\gamma_1}\beta_2^{2-\gamma_2}\Gamma(1+\gamma_2)\gamma_1\sigma^2}$	$\gamma_1+\gamma_2-2$
$\sigma = \delta_1 = \delta_2 = 0$	1	$\frac{\beta_2^2\gamma_2}{c_2\Gamma(1-\gamma_2)}\frac{\sin(\pi\gamma_2)}{\pi}\left(\mathtt{k}_2+\frac{c_2}{\beta_2}B(1+\gamma_2-\alpha_2;-\gamma_2)\right)$	$-\gamma_2$

Table 4: Coefficients for the asymptotic expansion of ξ_n^+.

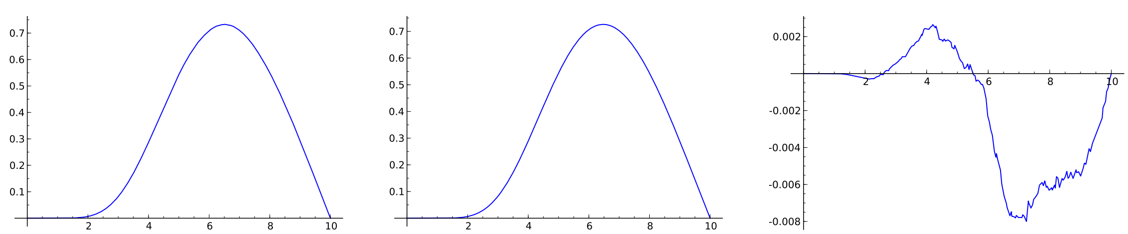

Figure 1: From left to right, a simulation of $V(x)$ using the Wiener-Hopf method, an exact plot of $V(x)$ and the difference between the two curves. Here the underlying Lévy process takes the form $X_t = 0.4B_t - 0.03t$

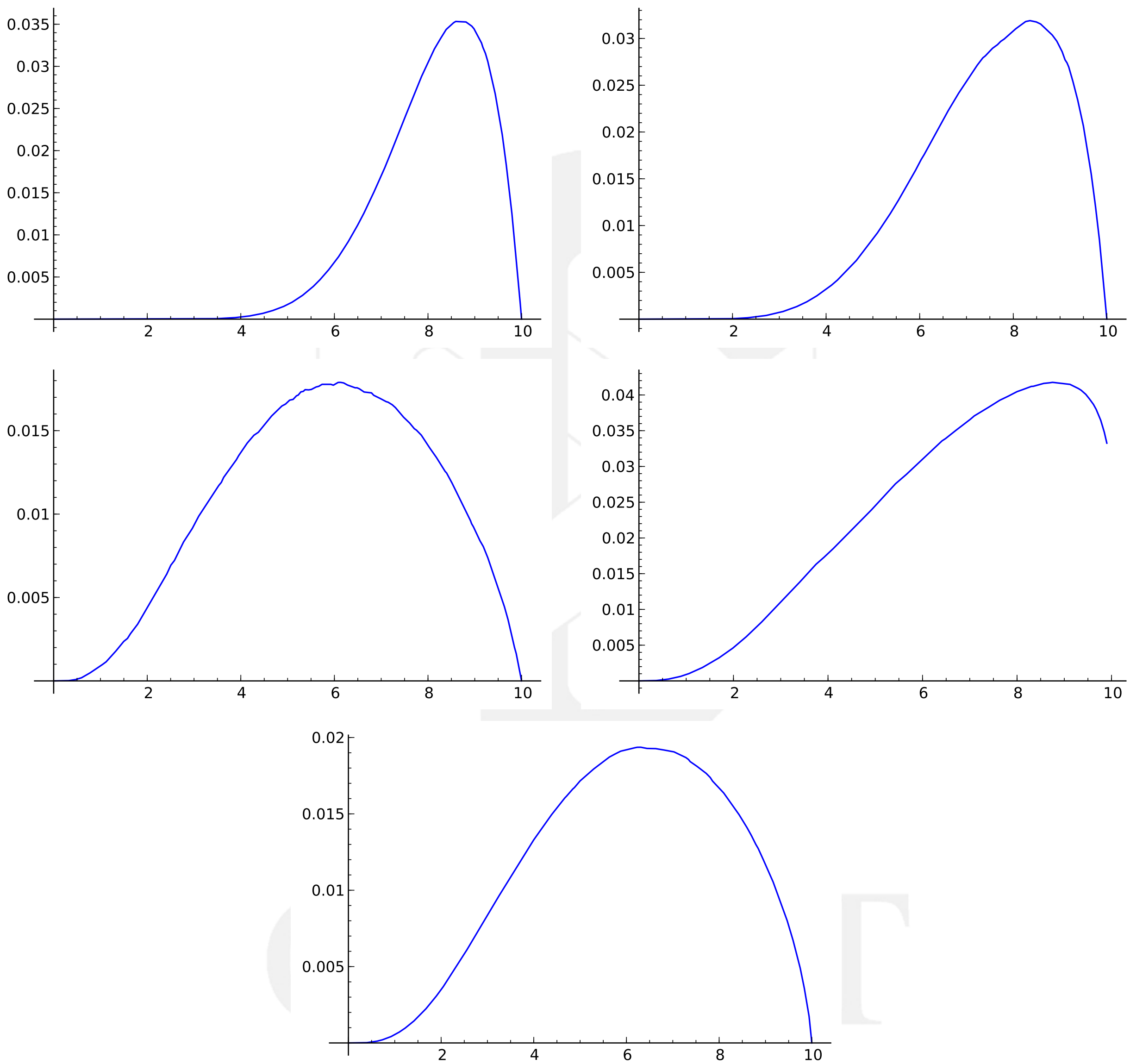

Figure 2: These four diagrams plot the simulated values of $V(x)$ for a β-process with parameters $(a, \sigma, \alpha_1, \beta_1, \lambda_1, c_1, \alpha_2, \beta_2, \lambda_2, c_2) = (a, 0.4, 1, 1.5, \lambda, 1, 1, 1.5, \lambda, 1)$ where a is chosen so that $-r = \Psi(-\mathrm{i})$ and, from left to right, top to bottom, $\sigma = 0.4$ and $\lambda = 0.5, 1.5, 2.5$ for the first three diagrams (corresponding to the cases of compound Poisson, bounded variation and unbounded variation jump component respectively) and $\sigma = 0$ with $\lambda = 1.5, 2.5$ (corresponding to bounded variation and unbounded variation jump component respectively) for the final two diagrams.

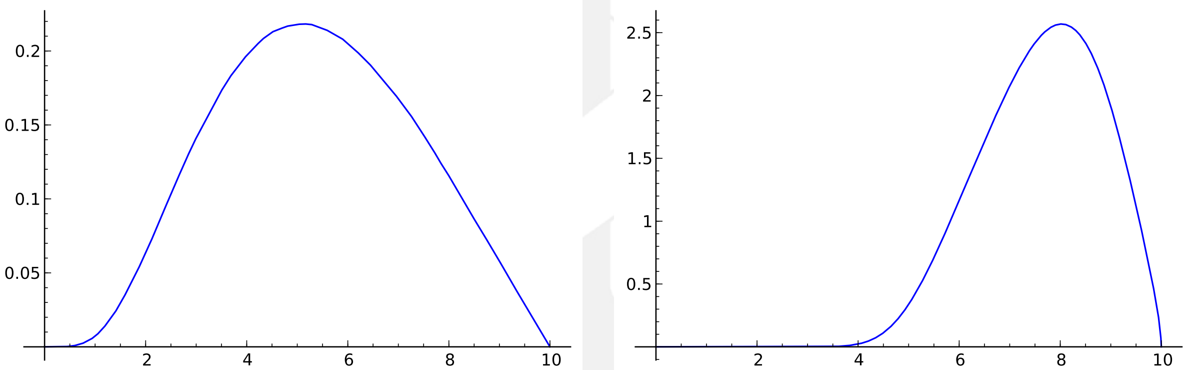

Figure 3: From left to right, these two diagrams plot the simulated values of $V(x)$ for a hypergeometric Lévy process with parameters $(\mathtt{d}, \sigma, \mathtt{k}_1, \delta_1, c_1, \alpha_1, \beta_1, \gamma_1, \mathtt{k}_2, \delta_2, c_2, \alpha_2, \beta_2, \gamma_2) = (\mathtt{d}, \sigma, \delta_1, 0.5, 1, 0.5, 10, 0.5, 0, 0.5, 1, 0.5, 10, 0.5)$ chosen such that $(\sigma, \delta_1) = (0.3, 1)$ and $(0, 0)$ respectively and $\mathtt{d}$ is chosen so that $-r = \Psi(-\mathrm{i})$.