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OF LÉVY PROCESSES

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On the density of exponential functionals of Lévy processes

J. C. Pardo*, V. Rivero† and K. van Schaik‡

Abstract

In this paper, we study the existence of the density associated to the exponential functional of the Lévy process ξ ,

$$I_{\mathbf{e}_q} := \int_0^{\mathbf{e}_q} e^{\xi_s} ds,$$

where \mathbf{e}_q is an independent exponential r.v. with parameter $q \geq 0$. In the case when ξ is the negative of a subordinator, we prove that the density of $I_{\mathbf{e}_q}$, here denoted by k , satisfies an integral equation that generalizes the one found by Carmona et al. [7]. Finally when $q = 0$, we describe explicitly the asymptotic behaviour at 0 of the density k when ξ is the negative of a subordinator and at ∞ when ξ is a spectrally positive Lévy process that drifts to $+\infty$.

Keywords: Lévy processes, exponential functional, subordinators, self-similar Markov processes.

AMS 2000 subject classifications: 60G51

1 Introduction

A real-valued Lévy process is a stochastic process issued from the origin with stationary and independent increments and almost sure right continuous paths with left-limits. We write $\xi = (\xi_t, t \geq 0)$ for its trajectory and \mathbb{P} for its law. The law \mathbb{P} of a Lévy processes is characterized by its one-time transition probabilities. In particular there always exists

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a triple (a, σ^2, Π) where $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and Π is a measure on $\mathbb{R} \setminus \{0\}$ satisfying the integrability condition $\int_{\mathbb{R}} (1 \wedge x^2) \Pi(dx) < \infty$, such that, for $t \geq 0$ and $z \in \mathbb{R}$

$$\mathbb{E}[e^{iz\xi t}] = \exp \{-\Psi(z)t\}, \quad (1.1)$$

where

$$\Psi(z) = iaz + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \left(1 - e^{izx} + izx \mathbf{1}_{\{|x|<1\}}\right) \Pi(dx).$$

In the case when ξ is a subordinator, the Lévy measure Π has support on $[0, \infty)$ and fulfils the extra condition $\int_{(0, \infty)} (1 \wedge x) \Pi(dx) < \infty$. Hence, the characteristic exponent Ψ can be expressed as

$$\Psi(z) = -icz + \int_{(0, \infty)} \left(1 - e^{izx}\right) \Pi(dx),$$

where $c \geq 0$ and is known as the drift coefficient. It is well-known that the function Ψ can be extended analytically on the complex upper half-plane, so the Laplace exponent of ξ is given by

$$\phi(\lambda) := -\log \mathbb{E}[e^{-\lambda\xi_1}] = \Psi(i\lambda) = c\lambda + \int_{(0, \infty)} \left(1 - e^{-\lambda x}\right) \Pi(dx).$$

Similarly, in the case when ξ is a spectrally negative Lévy process (i.e. has no positive jumps), the Lévy measure Π has support on $(-\infty, 0)$ and the characteristic exponent Ψ can be written as

$$\Psi(z) = iaz + \frac{1}{2}\sigma^2 z^2 + \int_{(-\infty, 0)} \left(1 - e^{izx} + izx \mathbf{1}_{\{x>-1\}}\right) \Pi(dx).$$

It is also well-known that the function Ψ can be extended analytically on the complex lower half-plane, so its Laplace exponent satisfies

$$\psi(\lambda) := \log \mathbb{E}[e^{\lambda\xi_1}] = -\Psi(-i\lambda) = a\lambda + \frac{1}{2}\sigma^2 \lambda^2 + \int_{(-\infty, 0)} \left(e^{\lambda x} - 1 + -\lambda x \mathbf{1}_{\{x>-1\}}\right) \Pi(dx).$$

In this article, one of our aims is to study the existence of the density associated to the exponential functional

$$I_{\mathbf{e}_q} := \int_0^{\mathbf{e}_q} e^{\xi_s} ds,$$

where \mathbf{e}_q is an exponential r.v. independent of the Lévy process ξ with parameter $q \geq 0$. If $q = 0$, then \mathbf{e}_q is understood as ∞ . In this case, we assume that the process ξ drifts towards $-\infty$ since it is a necessary and sufficient condition for the almost sure finiteness of $I := I_\infty$, see for instance Theorem 1 in Bertoin and Yor [4].

Up to our knowledge nothing is known about the existence of the density of $I_{\mathbf{e}_q}$ when $q > 0$. In the case when $q = 0$, the existence of the density of I has been proved by Carmona et al. [7] for Lévy processes whose jump structure is of finite variation and recently by Bertoin et al. [2] (see Theorem 3.9) for any real-valued Lévy process. In

particular when ξ is the negative of a subordinator such that $\mathbb{E}[|\xi_1|] < \infty$, Carmona et al. [7] (see Proposition 2.1) proved that the r.v. I has a density, k , that is the unique (up to a multiplicative constant) L^1 positive solution to the equation

$$(1 - cx)k(x) = \int_x^\infty \bar{\Pi}(\log(y/x))k(y) dy, \quad x \in (0, 1/c), \quad (1.2)$$

where $\bar{\Pi}(x) := \Pi(x, \infty)$.

Here, we generalize the above equation. Indeed, we establish an integral equation for the density of I_{e_q} , $q \geq 0$, when ξ is the negative of a subordinator and we note that when $q = 0$, the condition $\mathbb{E}[|\xi_1|] < \infty$ is not essential for the existence of its density and the validity of (1.2).

Another interesting problem is determining the behaviour of the density of the exponential functional I at 0 and at ∞ . This problem has been recently studied by Kuznetsov [14] for Lévy processes with rational Laplace exponent (at 0 and at ∞), by Kuznetsov and Pardo [16] for hypergeometric Lévy processes (at 0 and at ∞) and by Patie [19] for spectrally negative Lévy processes (at ∞). In most of the applications, it is enough to have estimates of the tail behaviour $\mathbb{P}(I \leq t)$ when t goes to 0 and/or $\mathbb{P}(I \geq t)$ when t goes to ∞ . The tail behaviour $\mathbb{P}(I \leq t)$ was studied by Pardo [20] in the case where the underlying Lévy process is spectrally positive and its Laplace exponent is regularly varying at infinity with index $\gamma \in (1, 2)$, and by Caballero and Rivero [6] in the case when ξ is the negative of a subordinator whose Laplace exponent is regularly varying at 0. Furthermore, the tail behaviour $\mathbb{P}(I \geq t)$ has been studied in a general setting, see [8, 18, 21, 22]. The second main result of this paper is related to this problem. Namely, we describe explicitly the asymptotic behaviour at 0 of the density of I when ξ is a subordinator which in particular implies the behaviour of $\mathbb{P}(I < t)$ near 0.

The paper is organized as follows: in Section 2 we state our main results, in particular we study the density of I_{e_q} and the asymptotic behaviour at 0 of the density of the exponential functional associated to the negative of a subordinator. Section 3 is devoted to the proof of the main results and in Section 4, we give some examples and some numerical results for the density of I_{e_q} when the driving process is the negative of a subordinator.

2 Main results

Our first main result states that I_{e_q} has a density, for $q > 0$. Before we establish our first Theorem, we need to introduce some notation and recall some facts about positive self-similar Markov processes (pssMp) which will be our main tool in this first part.

Let $(\xi_t^\dagger, t \geq 0)$ be the process obtained by killing ξ at an independent exponential time of parameter $q > 0$, here denoted by e_q . The law and the lifetime of ξ^\dagger are denoted by \mathbb{P}^\dagger and β , respectively.

We first note that

$$(I, \mathbb{P}^\dagger) = \left(\int_0^\beta \exp \{ \xi_t^\dagger \} dt, \mathbb{P}^\dagger \right) \stackrel{d}{=} \left(\int_0^{e_q} e^{\xi_t} dt, \mathbb{P} \right).$$

For $x \geq 0$ let \mathbb{Q}_x be the law of $X^{(x)}$, the positive self-similar Markov process with self-similarity index 1 issued from x , associated to ξ^\dagger via its Lamperti's representation (see [17] for more details on this representation), that is for $x > 0$

$$X_t^{(x)} = \begin{cases} x \exp \{ \xi_{\tau(t/x)}^\dagger \}, & \text{if } \tau(t/x) < \infty \\ 0, & \text{if } \tau(t/x) = \infty, \end{cases} \quad t \geq 0;$$

where

$$\tau(s) = \inf \left\{ r > 0 : \int_0^r e^{\xi_t^\dagger} dt > s \right\}, \quad \inf \{ \emptyset \} = \infty,$$

and 0 is understood as a cemetery state. The process $X^{(x)}$ is a strong Markov process and it fulfills the scaling property, i.e. for $k > 0$,

$$(kX_{t/k}^{(x)}, t \geq 0) \stackrel{d}{=} (X_t^{(kx)}, t \geq 0).$$

We denote by $T_0^{(x)} := \inf \{ t > 0 : X_t^{(x)} = 0 \}$, the first hitting time of $X^{(x)}$ at 0. Observe that for $s > 0$, we have the following equivalences,

$$\tau(s) < \infty \quad \text{iff} \quad \tau(s) \leq \beta \quad \text{iff} \quad s \leq \int_0^\beta e^{\xi_t^\dagger} dt.$$

Hence, it follows from the construction of X that the following equality in law follows

$$(T_0, \mathbb{Q}_1) \stackrel{d}{=} \left(\int_0^{e_q} e^{\xi_t} dt, \mathbb{P} \right).$$

Now, we have all the elements to establish our first main result. It concerns the existence of the density of I_{e_q} .

Theorem 2.1. *Let $q > 0$, then the function*

$$h(t) := q \mathbb{Q}_1 \left[\frac{1}{X_t} \mathbf{1}_{\{t < T_0\}} \right], \quad t \geq 0,$$

is a density for the law of I_{e_q} .

Corollary 2.2. *Assume $q > 0$ and that ξ is a subordinator. Then the law of the r.v. I_{e_q} is a mixture of exponential, that is its law has a density h on $(0, \infty)$ which is completely monotone. Furthermore, $\lim_{t \downarrow 0} h(t) = q$.*

In the sequel, we assume that $\xi = -\zeta$ where ζ is a subordinator and we denote by $U_q(dx)$ the renewal measure of the killed subordinator $(\zeta_t, t \leq \mathbf{e}_q)$, i.e.

$$\mathbb{E} \left[\int_0^{\mathbf{e}_q} f(\zeta_t) dt \right] = \int_{[0, \infty)} f(x) U_q(dx), \quad (2.3)$$

where f is a positive measurable function. If the renewal measure is absolutely continuous with respect to the Lebesgue measure, the function $u_q(x) = U_q(dx)/dx$, is usually called the renewal density. If $q = 0$, we denote U_0 and u_0 by U and u .

Before stating our first main result, which is a generalization of the integral equation (1.2) of Carmona et al. for subordinators, in the next proposition we establish that the density of $I_{\mathbf{e}_q}$ solves an integral equation in terms of the renewal measure U_q .

Proposition 2.3. *Let $q \geq 0$. The random variable $I_{\mathbf{e}_q}$ has a density that we denote by k , and it solves the equation*

$$\int_y^\infty k(x) dx = \int_0^\infty k(ye^x) U_q(dx), \quad \text{almost everywhere.} \quad (2.4)$$

The next result generalizes (1.2).

Theorem 2.4. *Let $q \geq 0$. The random variable $I_{\mathbf{e}_q}$ has a density that we denote by k , and it solves*

$$(1 - cx)k(x) = \int_x^\infty \bar{\Pi}(\log(y/x))k(y)dy + q \int_x^\infty k(y)dy \quad x \in (0, 1/c). \quad (2.5)$$

Conversely, if a density on $(0, 1/c)$ satisfies this equation then it is the density of $I_{\mathbf{e}_q}$.

The importance of the above result will be illustrated in Theorem 2.5 where we study the asymptotic behaviour at 0 of the density k , and in Section 4 where we provide some examples where k can be computed explicitly. Further applications have been provided in Haas [11] and Haas and Rivero [12] where this equation has been used to estimate the right tail behavior of the law of I and to study the maximum domain of attraction of I .

The following corollary is another important application of equation (2.5). In particular, it says that if we know the density of the exponential functional of the negative of a subordinator, say k , then for $\rho \geq 0$, $x^\rho k(x)$ adequately normalized is the density of the exponential functional associated to the negative of a new subordinator. The proof of this fact follows easily by multiplying in both sides of equation (2.5) by x^ρ . Such result also appears in Chazal et al. [9] but in terms of the distribution of $I_{\mathbf{e}_q}$ not in terms of its density.

Corollary 2.5. *Let $q \geq 0, \rho > 0, c_\rho$ a positive constant satisfying*

$$c_\rho = \int_{(0, \infty)} x^\rho k(x) dx,$$

and suppose that when $q > 0$ the renewal measure U_q has a density. Then the function $h(x) := c_\rho^{-1}x^\rho k(x)$ is the density of the exponential functional of the negative of a subordinator whose Laplace exponent is given by

$$\phi_\rho(\lambda) = \frac{\lambda}{\lambda + \rho} (\phi(\lambda + \rho) + q). \quad (2.6)$$

Moreover, the density h solves the equation

$$(1 - cx)h(x) = \int_x^\infty \bar{\Pi}_\rho(\log y/x)h(y)dy \quad x \in (0, 1/c), \quad (2.7)$$

where $\bar{\Pi}_\rho(z) = \bar{\Pi}(z)e^{-\rho z} + qe^{-\rho z}$.

We remark that the transformation studied in Chazal et al. [9] is more general than the one presented in (2.6) and that they applied such transformation to Lévy processes with one-sided jumps. We also remark that the subordinator whose Laplace exponent is given by ϕ_ρ has an infinite lifetime in any case.

Our next goal is to study the behavior of the density of I_{e_q} near 0. When $q = 0$, we work with the following assumption:

(A) The Lévy measure Π belongs to the class \mathcal{L}_α for some $\alpha \geq 0$, that is to say that the tail Lévy measure $\bar{\Pi}$ satisfies

$$\lim_{x \rightarrow \infty} \frac{\bar{\Pi}(x+y)}{\bar{\Pi}(x)} = e^{-\alpha y}, \quad \text{for all } y \in \mathbb{R}. \quad (2.8)$$

Observe that regularly varying and subexponential tail Lévy measures satisfy this assumption with $\alpha = 0$ and that convolution equivalent Lévy measures are examples of Lévy measures satisfying (2.8) for some index $\alpha > 0$.

Theorem 2.6. *Let $q \geq 0$ and $\xi = -\zeta$, where ζ is a subordinator such that when $q = 0$ the Lévy measure Π satisfies assumption (A). The following asymptotic behaviour holds for the density function k of the exponential functional I_{e_q} .*

i) If $q > 0$, then

$$k(x) \rightarrow q \quad \text{as } x \downarrow 0.$$

ii) If $q = 0$, then $\mathbb{E}[I^{-\alpha}] < \infty$ and

$$k(x) \sim \mathbb{E}[I^{-\alpha}] \bar{\Pi}(\log 1/x) \quad \text{as } x \downarrow 0.$$

In the sequel we will assume that $q = 0$. The above result will help us to describe the behaviour at ∞ of the density of the exponential functional of a particular spectrally negative Lévy processes associated to the subordinator ζ . In order to explain such relation, we

need the following assumptions. Assume that U , the renewal measure of the subordinator ζ , is absolutely continuous with respect to the Lebesgue measure with density u which is non-increasing and convex. We also suppose that $\mathbb{E}[\zeta_1] < \infty$. According to Theorem 2 in Kyprianou and Rivero [13] there exists a spectrally negative Lévy process $Y = (Y_t, t \geq 0)$ that drifts to $+\infty$, whose Laplace exponent is described by

$$\psi(\lambda) = \lambda\phi^*(\lambda) = \frac{\lambda^2}{\phi(\lambda)}, \quad \text{for } \lambda \geq 0,$$

where ϕ^* is the Laplace exponent of another subordinator and satisfies

$$\phi^*(\lambda) := q^* + c^*\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x}) \Pi^*(dx),$$

where

$$q^* = \left(c + \int_{(0,\infty)} x\Pi(dx) \right)^{-1}, \quad c^* = \begin{cases} 0 & c > 0, \text{ or } \Pi(0, \infty) = \infty, \\ 1/\Pi(0, \infty), & c = 0 \text{ and } \Pi(0, \infty) < \infty, \end{cases}$$

and the Lévy measure Π^* satisfies

$$u(x) = c^* \mathbf{1}_{\{x=0\}} + q^* + \bar{\Pi}^*(x), \quad \text{for } x \geq 0.$$

Let I_ψ be the exponential functional associated to $-Y$, i.e.

$$I_\psi = \int_0^\infty e^{-Y_s} ds,$$

and denote its density by k_ψ . From the proof of Proposition 4 in Rivero [21] the density k_ψ satisfies

$$k_\psi(x) = q^* \frac{1}{x} k\left(\frac{1}{x}\right), \quad \text{for } x > 0. \quad (2.9)$$

The following corollary give us the asymptotic behaviour at ∞ of the density of the exponential functional of $-Y$.

Corollary 2.7. *Suppose that ζ is a subordinator satisfying assumption (A) and such that its renewal measure has a density which is non-increasing and convex and let Y be its associated spectrally negative Lévy process defined as above. Then the following asymptotic behaviour holds for the density function k_ψ ,*

$$k_\psi(x) \sim q^* \mathbb{E}[I^{-\alpha}] \frac{1}{x} \bar{\Pi}(\log x) \quad \text{as } x \rightarrow \infty.$$

3 Proofs

Proof of Theorem 2.1. We start the proof by showing that the function

$$h(t, x) := q\mathbb{Q}_x \left[\frac{1}{X_t} \mathbf{1}_{\{t < T_0\}} \right], \quad t \geq 0, x > 0$$

is such that

$$\int_0^\infty h(t, x) dt = 1, \quad \text{for } x > 0. \quad (3.1)$$

Then the result follows from the identity (3.1) and the fact that

$$h(t + s) = q\mathbb{Q}_1 \left[h(s, X_t) \mathbf{1}_{\{t < T_0\}} \right], \quad \text{for } s, t \geq 0,$$

which is a straightforward consequence of the Markov property.

Let us prove (3.1). From the definition of X and the change of variables $u = \tau(t/x)$, which implies that $du = x^{-1} \exp\{-\xi_{\tau(t/x)}^\dagger\} dt$, we get

$$\begin{aligned} \int_0^\infty h(t, x) dt &= q \int_0^\infty dt \mathbb{E} \left[x^{-1} \exp \left\{ -\xi_{\tau(t/x)}^\dagger \right\} \mathbf{1}_{\{\tau(t/x) < \infty\}} \right] \\ &= q\mathbb{E} \left[\int_0^\infty x^{-1} \exp \left\{ -\xi_{\tau(t/x)}^\dagger \right\} \mathbf{1}_{\{t \leq x \int_0^\beta e^{\xi_s^\dagger} ds\}} dt \right] \\ &= q\mathbb{E} \left[\int_0^\infty \mathbf{1}_{\{u \leq \beta\}} du \right] = q\mathbb{E}(\beta) = 1. \end{aligned}$$

We now prove that

$$\int_t^\infty h(s) ds = \mathbb{P} \left(I_{\mathbf{e}_q} > t \right), \quad t > 0.$$

Indeed, let $t > 0$ making a change of variables, using the semi-group property, and Fubini's theorem we have

$$\int_t^\infty h(s) ds = \int_0^\infty h(s + t, 1) ds = \mathbb{Q}_1 \left[\left(\int_0^\infty h(s, X_t) ds \right) \mathbf{1}_{\{t < T_0\}} \right] = \mathbb{Q}_1(t < T_0).$$

The result follows from the identity $\mathbb{Q}_1(t < T_0) = \mathbb{P} \left(I_{\mathbf{e}_q} > t \right)$. \square

Proof of Corollary 2.2. Here, we use the same notation as above and we follow similar arguments as in the proofs of Lemma 5 and Proposition 1 in [3]. We first prove that for every $0 \leq t < T_0$ and $p > 0$, the variable

$$X_t^p \int_t^{T_0} \frac{1}{X_s^{p+1}} ds$$

is independent from $\sigma\{X_s, 0 \leq s \leq t\}$ and is distributed as

$$\int_0^{\mathbf{e}_q} e^{-p\xi_s} ds.$$

As a consequence of the Markov property at time t , we only need to show that under \mathbb{Q}_x , the variable

$$x^p \int_0^{T_0} \frac{1}{X_s^{p+1}} ds$$

is distributed as $\int_0^{e_q} e^{-p\xi_s} ds$. Then the change of variables $t = \tau(s/x)$, $s = x \int_0^t e^{\xi_u} du$, yields

$$\begin{aligned} x^p \int_0^{T_0} \frac{1}{X_s^{p+1}} ds &= x^{-1} \int_0^{T_0} e^{-(p+1)\xi_{\tau(s/x)}^\dagger} ds \\ &= \int_0^\beta e^{-(p+1)\xi_t^\dagger} e^{\xi_t^\dagger} dt \\ &= \int_0^\beta e^{-p\xi_t^\dagger} dt, \end{aligned}$$

which implies the desired identity in law since $(\xi_t^\dagger, 0 \leq t \leq \beta)$ and $(\xi_t, 0 \leq t \leq e_q)$ have the same law. Hence, we have

$$\mathbb{Q}_1 \left[\int_t^{T_0} \frac{1}{X_s^{p+1}} ds \right] = \frac{\mathbb{Q}_1 [X_t^{-p}; t < T_0]}{\phi(p) + q},$$

which implies

$$\frac{\partial \mathbb{Q}_1 [X_t^{-p}; t < T_0]}{\partial t} = -(\phi(p) + q) \mathbb{Q}_1 [X_t^{-(p+1)}; t < T_0].$$

By iteration, we get that the function $t \mapsto \mathbb{Q}_1 [X_t^{-p}; t < T_0]$ is completely monotone and takes value 1 for $t = 0$. Thus taking $p = 1$, we deduce that $h(t)$ is completely monotone on $(0, \infty)$ and that $\lim_{t \downarrow 0} h(t) = q$. Finally from Theorem 51.6 and Proposition 51.8 in [23], we have that the law of I_{e_q} is a mixture of exponentials. \square

Proof of Proposition 2.3. The proof follows from the identity

$$\mathbb{E} [I_{e_q}^n] = \frac{n}{\phi(n) + q} \mathbb{E} [I_{e_q}^{n-1}], \quad n > 0$$

Indeed, on the one hand it is clear that

$$\mathbb{E} [I_{e_q}^n] = \int_0^\infty x^n k(x) dx = n \int_0^\infty dy y^{n-1} \int_y^\infty k(x) dx.$$

On the other hand from the identity (2.3) with $f(x) = e^{-nx}$ and a change of variables, we

get

$$\begin{aligned}
\frac{n}{\phi(n) + q} \mathbb{E} \left[I_{\mathbf{e}_q}^{n-1} \right] &= n \int_0^\infty U_q(\mathrm{d}x) e^{-nx} \int_0^\infty y^{n-1} k(y) \mathrm{d}y \\
&= n \int_0^\infty U_q(\mathrm{d}x) \int_0^\infty y^{n-1} e^{-nx} k(y) \mathrm{d}y \\
&= n \int_0^\infty U_q(\mathrm{d}x) \int_0^\infty z^{n-1} k(ze^{-x}) \mathrm{d}z \\
&= n \int_0^\infty \mathrm{d}z z^{n-1} \int_0^\infty k(ze^{-x}) U_q(\mathrm{d}x).
\end{aligned}$$

Then putting the pieces together, we have

$$\int_0^\infty \mathrm{d}y y^{n-1} \int_y^\infty k(x) \mathrm{d}x = \int_0^\infty \mathrm{d}y y^{n-1} \int_0^\infty k(ye^{-x}) U_q(\mathrm{d}x), \quad \text{for } n > 0,$$

which implies the desired result because the density

$$y \mapsto \frac{1}{\mathbb{E}(I_{\mathbf{e}_q})} \int_y^\infty k(x) \mathrm{d}x,$$

is determined by its entire moments, which in turn is an easy consequence of the fact that k is so. \square

Proof of Theorem 2.4. By Theorem 2.1 (when $q > 0$) and Theorem 3.9 in [2] (when $q = 0$), we know that there exists a density of $I_{\mathbf{e}_q}$, for $q \geq 0$, that we denote by h . Moreover, in [7] it has been proved that the moments of $I_{\mathbf{e}_q}$ are given by

$$\mathbb{E} \left[I_{\mathbf{e}_q}^n \right] = \frac{n!}{\prod_{i=1}^n (q + \phi(i))}, \quad n \in \mathbb{N} \tag{3.2}$$

where the product is understood as 1 when $n = 0$.

We first prove that the function $\tilde{h} : (0, \infty) \rightarrow (0, \infty)$ defined via

$$\tilde{h}(x) = \begin{cases} cxh(x) + \int_x^\infty \bar{\Pi}(\log(y/x))h(y) \mathrm{d}y + q \int_x^\infty h(y) \mathrm{d}y, & \text{if } x \in (0, 1/c), \\ 0, & \text{elsewhere,} \end{cases}$$

is a density for the law of $I_{\mathbf{e}_q}$ and hence that $h = \tilde{h}$ a.e. Then we prove that the equality (2.5) holds. In order to do so, it is enough to verify that

$$\int_0^\infty x^n \tilde{h}(x) \mathrm{d}x = \frac{n!}{\prod_{i=1}^n (q + \phi(i))}, \quad n \in \mathbb{N},$$

since the law of $I_{\mathbf{e}_q}$ is determined by its entire moments.

Indeed, elementary computations, identity (2.4) and the fact that

$$\int_0^\infty e^{-\theta y} U_q(dy) = \frac{1}{\phi(\theta) + q}, \quad \theta \geq 0,$$

give that for any integer $n \geq 0$,

$$\begin{aligned} \int_0^\infty x^n \tilde{h}(x) dx &= c \int_0^\infty dx x^{n+1} h(x) + \int_0^\infty dx x^n \int_x^\infty dy \bar{\Pi}(\log(y/x)) h(y) \\ &\quad + q \int_0^\infty dx x^n \int_0^\infty h(xe^y) U_q(dy) \\ &= \frac{n!(n+1)c}{\prod_{i=1}^{n+1} (q + \phi(i))} + \int_0^\infty dy h(y) \int_0^y dx x^n \bar{\Pi}(\log(y/x)) \\ &\quad + q \int_0^\infty U_q(dy) \int_0^\infty dx x^n h(xe^y) \\ &= \frac{n!(n+1)c}{\prod_{i=1}^{n+1} (q + \phi(i))} + \int_0^\infty dy h(y) y^{n+1} \int_0^\infty dz e^{-(n+1)z} \bar{\Pi}(z) \\ &\quad + q \int_0^\infty U_q(dy) e^{-(n+1)y} \int_0^\infty dz z^n h(z) \\ &= \frac{n!(n+1)c}{\prod_{i=1}^{n+1} (q + \phi(i))} + \frac{(n+1)!}{\prod_{i=1}^{n+1} (q + \phi(i))} \frac{\int_0^\infty (1 - e^{-(n+1)z}) \bar{\Pi}(dz)}{n+1} \\ &\quad + q \frac{n!}{\prod_{i=1}^n (q + \phi(i))} \int_0^\infty U_q(dy) e^{-(n+1)y} \\ &= \frac{n!}{\prod_{i=1}^n (q + \phi(i))} \frac{(n+1)c + \int_0^\infty (1 - e^{-(n+1)z}) \bar{\Pi}(dz) + q}{q + \phi(n+1)} \\ &= \frac{n!}{\prod_{i=1}^n (q + \phi(i))}. \end{aligned}$$

Now, let $\mathcal{N} = \{x \in \mathbb{R} : h(x) \neq \tilde{h}(x)\}$. By the above arguments, we know that the Lebesgue measure of \mathcal{N} is zero. Let $k : (0, \infty) \rightarrow (0, \infty)$ be the function defined by

$$k(x) = \begin{cases} h(x), & \text{if } x \in \mathcal{N}^c, \\ \frac{1}{1-cx} \left(\int_x^\infty \bar{\Pi}(\log(y/x)) h(y) dy + q \int_x^\infty h(y) dy \right), & \text{if } x \in \mathcal{N}. \end{cases}$$

We now prove that $k(x)$ satisfies equation (2.5) everywhere. If $x \in \mathcal{N}^c$ then we have that $k(x) = h(x) = \tilde{h}(x)$, and hence equation (2.5) is verified. Finally, if $x \in \mathcal{N}$, we have the

following equalities

$$\begin{aligned}
cxk(x) + \int_x^\infty \bar{\Pi}(\log(y/x))k(y)dy + q \int_x^\infty k(y)dy \\
&= cxk(x) + \int_x^\infty \bar{\Pi}(\log(y/x))k(y)\mathbf{1}_{\{y \in \mathcal{N}^c\}}dy + q \int_x^\infty k(y)\mathbf{1}_{\{y \in \mathcal{N}^c\}}dy \\
&= cxk(x) + \int_x^\infty \bar{\Pi}(\log(y/x))h(y)\mathbf{1}_{\{y \in \mathcal{N}^c\}}dy + q \int_x^\infty h(y)\mathbf{1}_{\{y \in \mathcal{N}^c\}}dy \\
&= \frac{cx}{1-cx} \left(\int_x^\infty \bar{\Pi}(\log(y/x))h(y)dy + q \int_x^\infty h(y)dy \right) \\
&\quad + \int_x^\infty \bar{\Pi}(\log(y/x))h(y)dy + q \int_x^\infty h(y)dy \\
&= k(x).
\end{aligned}$$

Conversely if k is a density on $(0, 1/c)$ satisfying equation (2.5), from the above computations it is clear that k and I_{e_q} have the same entire moments. This implies that the k is a density of the exponential functional I_{e_q} . \square

Proof of Theorem 2.6. The proof consists of three steps. First we show that when $q = 0$, $\mathbb{E}[I^{-\alpha}] < \infty$, then for $q \geq 0$ we obtain a technical estimate on the maximal growth of $k(x)$ as $x \downarrow 0$, and finally the statement of the theorem.

Step 1. Here we assume that $q = 0$ and prove that $\mathbb{E}[I^{-\alpha}] < \infty$. The case $\alpha = 0$ is obvious. For $\alpha \in (0, 1)$, we have from Theorem 2 in [4] that there exists a random variable R , independent of ξ , such that $IR \stackrel{d}{=} \mathbf{e}$, where \mathbf{e} follows a unit mean exponential distribution. Since $\mathbb{E}[\mathbf{e}^{-\alpha}] < \infty$, the result follows.

Finally let $\alpha \geq 1$. With (2.5) and some standard computations, we find

$$\begin{aligned}
\int_0^\infty x^{-\beta-1}k(x) dx &= c \int_0^\infty dx x^{-\beta}k(x) + \int_0^\infty dx x^{-\beta-1} \int_x^\infty dy \bar{\Pi}(\log(y/x))k(y) \\
&= c\mathbb{E}[I^{-\beta}] + \int_0^\infty dy k(y) \int_0^y dx x^{-\beta-1} \bar{\Pi}(\log(y/x)) \\
&= c\mathbb{E}[I^{-\beta}] + \int_0^\infty dy y^{-\beta}k(y) \int_0^\infty du e^{\beta u} \bar{\Pi}(u) \\
&= -\frac{1}{\beta} \mathbb{E}[I^{-\beta}] \left(-c\beta + \int_0^\infty (1 - e^{\beta z}) \Pi(dz) \right),
\end{aligned}$$

that is to say

$$\mathbb{E}[I^{-\beta-1}] = \mathbb{E}[I^{-\beta}] \frac{\phi(-\beta)}{-\beta}, \tag{3.3}$$

where ϕ is the Laplace exponent of ξ , which can be extended to $(-\alpha, \infty)$ since for $\beta < \alpha$

$$\int_0^\infty (e^{\beta u} - 1) \Pi(du) = \beta \int_1^\infty \bar{\Pi}(\log(z))z^{\beta-1} dz < \infty. \tag{3.4}$$

To see that (3.4) holds, note that $\bar{\Pi}(\log(z))$ is regularly varying with index $-\alpha$ by (2.8). Hence $\bar{\Pi}(\log z) = z^{-\alpha}\ell(z)$ for a slowly varying function ℓ and we can apply Proposition 1.5.10 from Bingham et al. [5].

Now, by iteratively using (3.3) we see that for $\mathbb{E}[I^{-\alpha}] < \infty$ it is enough to have $\mathbb{E}[I^{-\alpha'}] < \infty$ for some $\alpha' \in [0, 1)$. But this obviously holds if $\alpha' = 0$, while if $\alpha' \in (0, 1)$ it holds by the same argument as we used above for the case $\alpha \in (0, 1)$.

Step 2. We assume that $q \geq 0$. For $q = 0$, let p be any function such that $p(0) = 0$ and $\min\{\alpha - 1, 0\} < p(\alpha) < \alpha$ for all $\alpha > 0$. When $q > 0$ the function p will be taken as zero and hence the symbol $p(\alpha)$ will be taken as 0. The goal of this step is to show

$$\frac{k(x)}{x^{p(\alpha)}} \quad \text{stays bounded as} \quad x \downarrow 0. \quad (3.5)$$

Observe that when $q > 0$ it follows from (2.5) that $\liminf_{x \rightarrow 0} k(x) \geq q$. Set $h(x) := k(x)/x^{p(\alpha)}$. We can write (2.5) as

$$1 - cx = x \int_1^\infty \bar{\Pi}(\log(z)) z^{p(\alpha)} \frac{h(xz)}{h(x)} dz + \frac{qx^{p(\alpha)}\mathbb{P}(I_{\mathbf{e}_q} > x)}{h(x)}. \quad (3.6)$$

We argue by contradiction. Take some $\hat{x} \in (0, 1/c)$. If h were not bounded at $0+$, then $\mathbf{1}_{\{x \leq \hat{x}\}}h(x)$ would keep on attaining new maxima as $x \downarrow 0$. (Note that \hat{x} is present just to make sure this statement also holds if k is not bounded at $1/c-$.) In particular this means that a sequence of points $(x_n)_{n \geq 0}$ exists with $x_n \downarrow 0$ as $n \rightarrow \infty$ and such that $h(x_n) \geq \sup_{x \in [x_n, \hat{x}]} h(x)$. We will show that this implies

$$x_n \int_1^\infty \bar{\Pi}(\log(z)) z^{p(\alpha)} \frac{h(x_n z)}{h(x_n)} dz + \frac{qx_n^{p(\alpha)}\mathbb{P}(I_{\mathbf{e}_q} > x_n)}{h(x_n)} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,$$

which indeed contradicts (3.6) since $1 - cx_n \rightarrow 1$ as $n \rightarrow \infty$. Observe that if $q > 0$ and h is not bounded at $0+$ then the second term in the latter equation tends to 0, because $p(\alpha) = 0$, by construction. So we just have to prove that the first term in the latter equation tends to 0. For this, we have

$$\begin{aligned} x_n \int_1^\infty \bar{\Pi}(\log(z)) z^{p(\alpha)} \frac{h(x_n z)}{h(x_n)} dz &= x_n \int_1^{\hat{x}/x_n} \bar{\Pi}(\log(z)) z^{p(\alpha)} \frac{h(x_n z)}{h(x_n)} dz \\ &\quad + x_n \int_{\hat{x}/x_n}^\infty \bar{\Pi}(\log(z)) z^{p(\alpha)} \frac{h(x_n z)}{h(x_n)} dz. \end{aligned} \quad (3.7)$$

We first deal with the first integral on the right hand side of (3.7). By construction of the sequence $(x_n)_{n \geq 0}$, we have $h(x_n z) \leq h(x_n)$ for any $z \in [1, \hat{x}/x_n]$, hence

$$x_n \int_1^{\hat{x}/x_n} \bar{\Pi}(\log(z)) z^{p(\alpha)} \frac{h(x_n z)}{h(x_n)} dz \leq x_n \int_1^{\hat{x}/x_n} \bar{\Pi}(\log(z)) z^{p(\alpha)} dz. \quad (3.8)$$

If $q > 0$ or $\alpha = 0$ (recall $p(0) = 0$), we can take any $1 < z_0$ and write

$$\begin{aligned} x_n \int_1^{\hat{x}/x_n} \bar{\Pi}(\log(z)) dz &= x_n \int_1^{z_0} \bar{\Pi}(\log(z)) dz + x_n \int_{z_0}^{\hat{x}/x_n} \bar{\Pi}(\log(z)) dz \\ &\leq x_n \int_1^{z_0} \bar{\Pi}(\log(z)) dz + x_n \left(\frac{\hat{x}}{x_n} - z_0 \right) \bar{\Pi}(\log(z_0)), \end{aligned}$$

where the inequality uses that $\bar{\Pi}$ is decreasing. Letting $n \rightarrow \infty$, recalling that $x_n \downarrow 0$, we see that the first integral on the right hand side vanishes while the second term tends to $\hat{x}\bar{\Pi}(\log z_0)$. As we can make this term arbitrarily small by choosing z_0 large enough, since $\bar{\Pi}(\log z) \rightarrow 0$ as $z \rightarrow \infty$, it follows indeed that (3.8) vanishes.

Next, let $\alpha > 0$. Since $\alpha - 1 < p(\alpha) < \alpha$, we can choose some $\beta \in (0, \alpha)$ such that $p(\alpha) - \beta + 1 \in (0, 1)$. Using this we find

$$\begin{aligned} x_n \int_1^{\hat{x}/x_n} \bar{\Pi}(\log(z)) z^{p(\alpha)} dz &= x_n \int_1^{\hat{x}/x_n} \bar{\Pi}(\log(z)) z^{\beta-1} z^{p(\alpha)-\beta+1} dz \\ &\leq x_n \left(\frac{\hat{x}}{x_n} \right)^{p(\alpha)-\beta+1} \int_1^{\hat{x}/x_n} \bar{\Pi}(\log(z)) z^{\beta-1} dz, \end{aligned}$$

and the right hand side indeed vanishes as $n \rightarrow \infty$, again since $x_n \downarrow 0$ and by (3.4).

It remains to show that the second integral on the right hand side of (3.7) vanishes as $n \rightarrow \infty$. We have

$$\begin{aligned} x_n \int_{\hat{x}/x_n}^{\infty} \bar{\Pi}(\log(z)) z^{p(\alpha)} \frac{h(x_n z)}{h(x_n)} dz &\leq x_n \bar{\Pi}(\log(\hat{x}/x_n)) \frac{1}{h(x_n)} \int_{\hat{x}/x_n}^{\infty} z^{p(\alpha)} h(x_n z) dz \\ &= \frac{\bar{\Pi}(\log(\hat{x}/x_n))}{x_n^{p(\alpha)}} \frac{1}{h(x_n)} \int_{\hat{x}}^{\infty} k(u) du, \end{aligned}$$

where the inequality uses that $\bar{\Pi}$ is decreasing and the equality the definition of h together with the substitution $u = x_n z$. Since k is a density and by assumption $h(x_n) \rightarrow \infty$ as n goes to ∞ , for the right hand side to vanish it remains to show that $\bar{\Pi}(\log \hat{x}/x_n)/x_n^{p(\alpha)}$ stays bounded as n increases. If $q > 0$ or $\alpha = 0$ (recall $p(0) = 0$) it is immediate since $\bar{\Pi}$ is decreasing. If $\alpha > 0$, for any $1 < z_0 < z$ integration by parts yields

$$\bar{\Pi}(\log(z)) z^{p(\alpha)} = p(\alpha) \int_{z_0}^z \bar{\Pi}(\log(u)) u^{p(\alpha)-1} du + \int_{z_0}^z u^{p(\alpha)} d\bar{\Pi}(\log(u)) + \bar{\Pi}(\log(z_0)) z_0^{p(\alpha)}.$$

Now, if we let z goes to ∞ , then since $p(\alpha) < \alpha$ we see from (3.4) that the first integral in the right hand side stays bounded while the second integral is negative on account of the fact that $\bar{\Pi}$ is decreasing. Consequently the left hand side has to stay bounded and we are done.

Step 3, case $q = 0$. Denote $C_\alpha = \mathbb{E}[I^{-\alpha}]$, which is finite by Step 1. From (2.5) we obtain for all $x > 0$,

$$(1 - cx) \frac{k(x)}{\bar{\Pi}(\log(1/x))} = \int_x^\infty \frac{\bar{\Pi}(\log(y/x))}{\bar{\Pi}(\log(1/x))} k(y) dy. \quad (3.9)$$

Using this equation together with $k \geq 0$, Fatou's lemma and the fact that Π has an exponential tail (cf. (2.8)) yields

$$\begin{aligned} \liminf_{x \downarrow 0} \frac{k(x)}{\overline{\Pi}(\log(1/x))} &= \liminf_{x \downarrow 0} \frac{cxk(x)}{\overline{\Pi}(\log(1/x))} + \liminf_{x \downarrow 0} \int_x^\infty \frac{\overline{\Pi}(\log(y/x))}{\overline{\Pi}(\log(1/x))} k(y) dy \\ &\geq \int_0^\infty y^{-\alpha} k(y) dy = C_\alpha. \end{aligned}$$

On the other hand, for any $\varepsilon > 0$ we have as $x \downarrow 0$

$$\int_\varepsilon^\infty \frac{\overline{\Pi}(\log(y/x))}{\overline{\Pi}(\log(1/x))} k(y) dy \rightarrow \int_\varepsilon^\infty y^{-\alpha} k(y) dy \leq C_\alpha.$$

If $\alpha > 0$, this follows from the fact that the convergence (2.8) is uniform over $y \in [\varepsilon, \infty)$, see e.g. Theorem 1.5.2 in [5]. If $\alpha = 0$ this uniformity holds only over intervals of the form $[\varepsilon, x_0]$, in which case we can write the left hand side as the sum of integrals over $[\varepsilon, x_0]$ and $[x_0, \infty)$, the former in the limit again is bounded above by C_α , while for the latter we can use that $\overline{\Pi}$ is decreasing to see

$$\int_{x_0}^\infty \frac{\overline{\Pi}(\log(y/x))}{\overline{\Pi}(\log(1/x))} k(y) dy \leq \frac{\overline{\Pi}(\log(x_0/x))}{\overline{\Pi}(\log(1/x))} \int_{x_0}^\infty k(y) dy,$$

then letting first $x \rightarrow \infty$, thereby using (2.8), and then $x_0 \rightarrow \infty$ it follows that this term vanishes.

So it remains to show that

$$\limsup_{x \downarrow 0} \int_x^\varepsilon \frac{\overline{\Pi}(\log(y/x))}{\overline{\Pi}(\log(1/x))} k(y) dy \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

For this, we get for ε small enough and $x < \varepsilon$:

$$\begin{aligned} \frac{1}{\overline{\Pi}(\log(1/x))} \int_x^\varepsilon \overline{\Pi}(\log(y/x)) k(y) dy &= \frac{x}{\overline{\Pi}(\log(1/x))} \int_1^{\varepsilon/x} \overline{\Pi}(\log(z)) k(xz) dz \\ &\leq \frac{Cx}{\overline{\Pi}(\log(1/x))} \int_1^{\varepsilon/x} \overline{\Pi}(\log(z)) (xz)^{p(\alpha)} dz \\ &= \frac{Cx^{1+p(\alpha)}}{\overline{\Pi}(\log(1/x))} \int_1^{\varepsilon/x} \overline{\Pi}(\log(z)) z^{p(\alpha)} dz \\ &\sim \frac{C'x^{1+p(\alpha)}}{\overline{\Pi}(\log(1/x))} \left(\frac{\varepsilon}{x}\right)^{p(\alpha)+1} \overline{\Pi}(\log(\varepsilon/x)) \quad \text{as } x \downarrow 0, \end{aligned}$$

where C and C' are constants, the inequality holds by Step 2 (cf. (3.5)) and the asymptotics follows from Karamata's theorem (see e.g. Theorem 1.5.11 in [5]), which indeed applies here since $\overline{\Pi}(\log(z))$ is regularly varying with index $-\alpha$ (cf. (2.8)) and by construction (see Step 2) $p(\alpha) \geq \alpha - 1$. Now, using (2.8) we see that the ultimate right hand side goes to $C'\varepsilon^{p(\alpha)+1-\alpha}$ as $x \downarrow 0$ and this indeed vanishes as $\varepsilon \rightarrow 0$ since by construction $p(\alpha) + 1 - \alpha > 0$ for all $\alpha \geq 0$.

Step 3, case $q > 0$. We will prove that

$$\int_x^\infty \bar{\Pi}(\log(y/x))k(y)dy \xrightarrow{x \rightarrow 0} 0.$$

By Step 2, we can assume that k is bounded by $K \geq q$, in a neighborhood of $0 +$. Let $\delta > 1$ fixed, for x small enough we have that

$$\begin{aligned} \int_x^{x\delta} \bar{\Pi}(\log(y/x))k(y)dy &\leq K \int_x^{x\delta} \bar{\Pi}(\log(y/x))dy \\ &= K \int_0^{\log \delta} \bar{\Pi}(u)xe^u du \\ &\leq Kx\delta \int_0^{\log \delta} \bar{\Pi}(u)du \xrightarrow{x \rightarrow 0} 0. \end{aligned}$$

Also, we have that

$$\int_{x\delta}^\infty \bar{\Pi}(\log(y/x))k(y)dy \leq \bar{\Pi}(\log \delta) \int_{x\delta}^\infty k(y)dy \xrightarrow{x \rightarrow 0} \bar{\Pi}(\log \delta),$$

We conclude by making $\delta \rightarrow \infty$. Indeed, using equation (2.5) and the above arguments we conclude that

$$(1 - cx)k(x) - q\mathbb{P}(I_{e_q} > x) \xrightarrow{x \rightarrow 0} 0,$$

from where the result follows. □

4 Examples and some numerics

In this section, we illustrate Theorem 2.3 , Corollary 2.4 and equation (2.9) with some examples and we also provide some applications of Theorem 2.5.

Example 1. Let $q > 0$ and consider the case when the subordinator is just a linear drift with $c > 0$. By a simple Laplace inversion, we deduce $u_q(x) = c^{-1}e^{-\frac{q}{c}x}$. Thus, from identities (2.5) and (2.4) we get

$$(1 - cx)k(x) = \frac{q}{c} \int_{[0, \infty)} k(xe^y)e^{-\frac{q}{c}y}dy, \quad x \in (0, 1/c).$$

After straightforward computations, we deduce that the density of I_{e_q} is of the form

$$k(x) = q(1 - cx)^{\frac{q}{c}-1}, \quad x \in (0, 1/c).$$

Let $\rho > 0$ and note that

$$\phi_\rho(\theta) = c\theta + q\frac{\theta}{\theta + \rho} \quad \text{and} \quad c_\rho = \frac{q}{c^{\rho+1}} \frac{\rho(\rho + 1)\Gamma(q/c)}{\Gamma(\rho + q/c + 1)}.$$

According to Corollary 2.4, the density of the exponential functional of the subordinator whose Laplace exponent is given by ϕ_ρ , satisfies

$$h(x) = c^{\rho+1} \frac{\Gamma(\rho + q/c + 1)}{\Gamma(\rho + 1)\Gamma(q/c)} x^\rho (1 - cx)^{q/c-1}, \quad \text{for } x \in (0, 1/c),$$

in other words the exponential functional has the same law as $c^{-1}B(\rho + 1, q/c)$, where $B(\rho + 1, q/c)$ is a beta r. v. with parameters $(\rho + 1, q/c)$.

Now, let us consider the associated spectrally negative Levy process Y whose Laplace exponent is written as follows

$$\psi(\lambda) = \frac{\lambda^2}{\phi_\rho(\lambda)} = \frac{\lambda(\lambda + \rho)}{c(\lambda + \rho) + q},$$

From (2.9), we deduce that the density of the exponential functional I_ψ associated to Y satisfies

$$k_\psi(x) = \frac{\rho c^{\rho+1}}{c\rho + q} \frac{\Gamma(\rho + q/c + 1)}{\Gamma(\rho + 1)\Gamma(q/c)} x^{-(\rho+q/c)} (x - c)^{q/c-1} \quad \text{for } x > c.$$

Hence I_ψ has the same law as $c(B(\rho, q/c))^{-1}$.

Example 2. Let $q = c = 0$, $\beta > 0$ and

$$\bar{\Pi}(z) = \frac{\beta}{\Gamma(a + 1)} e^{-\frac{(s-1)}{a}z} \left(e^{\frac{z}{a}} - 1 \right)^{a-1},$$

where $a \in (0, 1]$ and $s \geq a$. Thus, the Laplace exponent ϕ has the form

$$\phi(\theta) = \beta \frac{\theta \Gamma(a(\theta - 1) + s)}{\Gamma(a\theta + s)}.$$

In this case, the equation (2.5) can be written as follows

$$\begin{aligned} k(x) &= \frac{\beta}{\Gamma(a + 1)} \int_x^\infty (y/x)^{-\frac{s-1}{a}} \left((y/x)^{\frac{1}{a}} - 1 \right)^{a-1} k(y) dy \\ &= \frac{\beta x}{\Gamma(a)} \int_0^\infty (z + 1)^{a-s} z^{a-1} k(x(z + 1)^a) dz, \end{aligned}$$

where we are using the change of variable $z = (y/x)^{\frac{1}{a}} - 1$. After some computations we deduce that

$$k(z) = \frac{\beta^{s/a}}{a\Gamma(s)} z^{\frac{s-a}{a}} e^{-(\beta z)^{\frac{1}{a}}}, \quad \text{for } z \geq 0. \quad (4.10)$$

In other words I has the same law as $\beta^{-1}\gamma_s^a$, where γ_s is a gamma r.v. with parameter s .

If $a = 1$, the process ξ is a compound Poisson process of parameter $\beta > 0$ with exponential jumps of mean $(s - 1)^{-1} > 0$. From (4.10), it is clear that the law of its

associated exponential functional has the same law as $\gamma_{(s,\beta)}$, a gamma r.v. with parameters (s, β) .

We now consider the associated spectrally negative Levy process Y whose Laplace exponent satisfies

$$\psi(\lambda) = \frac{\lambda^2}{\phi(\lambda)} = \frac{\lambda\Gamma(a\lambda + s)}{\beta\Gamma(a(\lambda - 1) + s)}.$$

The density of the exponential functional I_ψ associated to Y is given by

$$k_\psi(x) = \frac{\beta^{\frac{s-a}{a}}}{a\Gamma(s-a)} x^{-s/a} e^{-(\beta/x)^{1/a}}, \quad x > 0.$$

We remark that when $a = 1$, the process ξ is a Brownian motion with drift and that the exponential functional I_ψ has the same law as $\gamma_{(s-1,\beta)}^{-1}$. This identity in law has been established by Dufresne [10].

Next, let $\rho > 0$ and note that

$$\phi_\rho(\theta) = \beta \frac{\theta\Gamma(a(\theta + \rho - 1) + s)}{\Gamma(a(\theta + \rho) + s)} \quad \text{and} \quad c_\rho = \frac{\Gamma(a\rho + s)}{\beta^\rho\Gamma(s)}.$$

According to Corollary 2.4, the density of the exponential functional of the subordinator whose Laplace exponent is given by ϕ_ρ , satisfies

$$h(x) = \frac{\beta^{(s+a\rho)/a}}{a\Gamma(a\rho + s)} x^{(a\rho+s-a)/a} e^{-(\beta x)^{1/a}} \quad \text{for } x > 0,$$

i.e. it has the same law as $\beta^{-1}\gamma_{a\rho+s}^a$. In particular, the density of the exponential functional of its associated spectrally negative Levy process satisfies

$$k_\psi(x) = \frac{\beta^{(s+a\rho-a)/a}}{a\Gamma(a(\rho - 1) + s)} x^{-(a\rho+s)/a} e^{-(\beta/x)^{1/a}}, \quad x > 0.$$

Example 3. Finally, let $a \in (0, 1)$, $\beta \geq a$, $c = 0$, $q = \Gamma(\beta)/\Gamma(\beta - a)$,

$$\bar{\Pi}(z) = \frac{1}{\Gamma(1-a)} \int_z^\infty \frac{e^{(1+a-\beta)x/a}}{(e^{x/a} - 1)^{1+a}} dx \quad \text{and} \quad u_q(z) = \frac{1}{\Gamma(a+1)} e^{-(\beta-1)z/a} \left(e^{\frac{z}{a}} - 1 \right)^{a-1}.$$

The process ξ with such characteristics is a killed Lamperti subordinator with parameters $(1/\Gamma(1-a), 1+a-\beta, 1/a, a)$, see Section 3.2 in Kuznetsov et al. [15] for a proper definition. From Theorem 1.3 the density of I_{e_q} satisfies the equation

$$k(x) = \int_0^\infty \left(\frac{x e^y}{\Gamma(1-a)} \int_z^\infty \frac{e^{(1+a-\beta)x/a}}{(e^{x/a} - 1)^{1+a}} dx + \frac{\Gamma(\beta) e^{-(\beta-1)z/a}}{\Gamma(\beta-a)\Gamma(a+1)} \left(e^{\frac{y}{a}} - 1 \right)^{a-1} \right) k(x e^y) dy.$$

Since the above equation seems difficult to solve, we use the method of moments in order to determine the law of I_{e_q} . We first note that

$$\mathbb{E} \left[I_{e_q}^n \right] = \frac{n! \Gamma(\beta)}{\Gamma(an + \beta)},$$

and that in the case $\beta = 1$, the exponential functional I_{e_q} has the same distribution as X_a^{-a} , where X_a is a stable random variable, i.e.

$$\mathbb{E}\left[e^{-\lambda X_a}\right] = \exp\{-\lambda^a\}, \quad \lambda \geq 0,$$

see Section 3 in [3]. Recall that the negative moments of X_a are given by

$$\mathbb{E}\left[X_a^{-n}\right] = \frac{\Gamma(1 + n/a)}{\Gamma(1 + n)}, \quad n \geq 0.$$

Now we introduce $L_{(a,\beta)}$ and A , two independent r.v. whose laws are described as follows,

$$\mathbb{P}(L_{(a,\beta)} \in dy) = \mathbb{E}\left[\frac{a\Gamma(\beta)}{\Gamma(\beta/a)X_a^\beta}; \frac{1}{X_a^a} \in dy\right],$$

and

$$\mathbb{P}(A \in dy) = (\beta/a - 1)(1 - x)^{\beta/a - 2} \mathbf{1}_{[0,1]}(x) dx.$$

It is important to note from example 1, that A has the same law as the exponential functional associated to the subordinator σ which is defined as follows

$$\sigma_t = t + \beta/a - 1, \quad t \geq 0.$$

On the one hand, it is clear that

$$\mathbb{E}\left[L_{(a,\beta)}^n\right] = \frac{a\Gamma(\beta)}{\Gamma(\beta/a)} \mathbb{E}\left[X_a^{-(an+\beta)}\right] = \frac{\Gamma(\beta)}{\Gamma(\beta/a)} \frac{\Gamma(n + \beta/a)}{\Gamma(an + \beta)},$$

and on the other hand, we have

$$\mathbb{E}\left[A^n\right] = \frac{\Gamma(n + 1)\Gamma(\beta/a)}{\Gamma(n + \beta/a)},$$

which implies the I_{e_q} has the same law as $L_{a,\beta}A$.

Finally we numerically illustrate the density k and its asymptotic behaviour at 0 for some particular subordinators ζ . Let us first shortly discuss the method we used. Clearly the equation (1.2) motivates the following straightforward discretisation procedure: approximate k by a step function \tilde{k} , i.e.

$$\tilde{k}(x) = \sum_{i=0}^{N-1} \mathbf{1}_{\{x \in [x_i, x_{i+1})\}} y_i,$$

where $0 = x_0 < x_1 < \dots < x_N = 1/c$ forms a grid on the x -axis. The heights y_i can then be found by iterating over $i = N - 1, \dots, 0$, thereby at each step using (1.2) with $x = x_i$ and k replaced by \tilde{k} . Two remarks are in place here.

Firstly, as (1.2) is linear in k the condition that k is a density is required to uniquely determine the solution. This translates to the fact that the numerical procedure discussed above requires a starting point, i.e. the value $y_{N-1} > 0$ should be known. (Of course, starting with $y_N = 0$ yields $\tilde{k} \equiv 0$.) We proceed by first leaving y_{N-1} undetermined, run the iteration so that every y_i in fact becomes a linear function of y_{N-1} , and then find y_{N-1} by requiring that \tilde{k} integrates to 1.

The second remark is that even though any choice of grid would in principle work, we found one in particular to be useful. Indeed, if we set $x_n = (1/c)\Delta^{N-n}$ for some Δ less than (but typically very close to) 1, equation (2.5) yields the following relation:

$$\begin{aligned} (1 - cx_n)y_n &= \int_{x_n}^{\infty} \bar{\Pi}(\log(y/x_n))\tilde{k}(y) dy = x_n \int_1^{\infty} \bar{\Pi}(\log(z))\tilde{k}(x_n z) dz \\ &= x_n \sum_{i=n}^{N-1} y_i \int_1^{\infty} \bar{\Pi}(\log(z))\mathbf{1}_{\{x_n z \in [x_i, x_{i+1}]\}} dz = x_n \sum_{i=n}^{N-1} y_i \int_{\Delta^{n-i}}^{\Delta^{n-i-1}} \bar{\Pi}(\log(z)) dz. \end{aligned}$$

The approximation this setup yields is very efficient in comparison with e.g. the approximation using a standard equidistant grid, due to the fact that in this case we need to evaluate only N different integrals numerically*.

First we consider two examples for which the density k of I is explicitly known. The first one is taken from Example 2 with $a = 1$, $\beta = 2$ and $s = 3/2$. In this case from (4.10), we have

$$k(x) = \frac{2^{5/2}}{\sqrt{\pi}} x^{1/2} e^{-2x} \quad \text{for } x > 0.$$

See Figures 1-4 for plots of the density k , the difference $\tilde{k} - k$ (where \tilde{k} is obtained by the above method with $\Delta = 0.998$, yielding a grid of ≈ 4500 points and a few minutes computation time on an average laptop), the ratio $k(x)/\bar{\Pi}(\log(1/x))$ and the ratio $\tilde{k}(x)/\bar{\Pi}(\log(1/x))$ respectively.

The second explicit example is taken again from Example 2 with $\beta = 1$ and $s = 1$ and $a = 1/2$. In this case from (4.10), we have

$$k(x) = 2xe^{-x^2} \quad \text{for } x > 0.$$

It is important to note that $\bar{\Pi}$ satisfies **(A)** with $\alpha = 1$. In this case, Figures 5 -8 show plots plots of the density k , the difference $\tilde{k} - k$, the ratio $k(x)/\bar{\Pi}(\log(1/x))$ and the ratio $\tilde{k}(x)/\bar{\Pi}(\log(1/x))$ respectively.

Next we look at two examples where no formula for k is available. The first one is when ξ is a stable subordinator with drift, i.e. $c = 1$ and $\Pi(dx) = x^{-1-a} dx$, where we take $a = 1/4$. See Figures 9 & 10 for a plot of \tilde{k} and the ratio $\tilde{k}(x)/\bar{\Pi}(\log(1/x))$ respectively. Note that this is an example of a Lévy measure satisfying (2.8) with parameter 0.

Finally, the second example is a subordinator ξ with zero drift and Lévy measure of the form $\Pi(dx) = x^{-1/4} \exp(-x^n) dx$. Figure 11 shows \tilde{k} for $n = 1$ (blue), $n = 2$ (purple)

*All computations were done in the open source computer algebra system SAGE: www.sagemath.org

and $n = 3$ (green) respectively. Figure 12 shows the ratio $\tilde{k}(x)/\bar{\Pi}(\log 1/x)$ for the case $n = 1$, since then (A) is satisfied with $\alpha = 1$.

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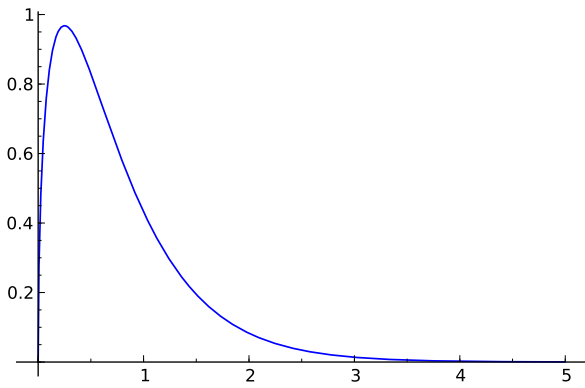


Figure 1: The density function k

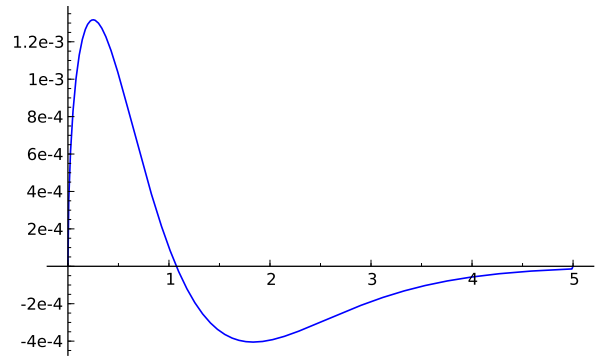


Figure 2: The difference $\tilde{k} - k$

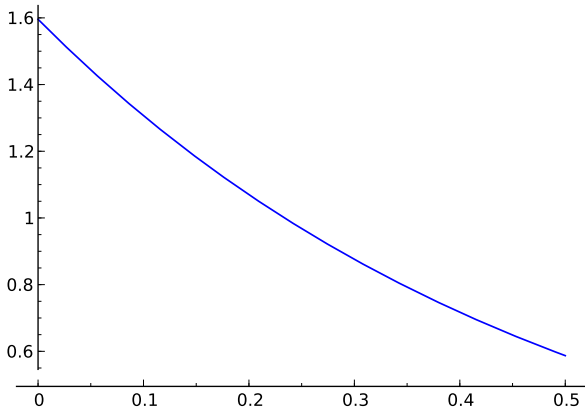


Figure 3: The ratio $k(x)/\bar{\Pi}(\log 1/x)$

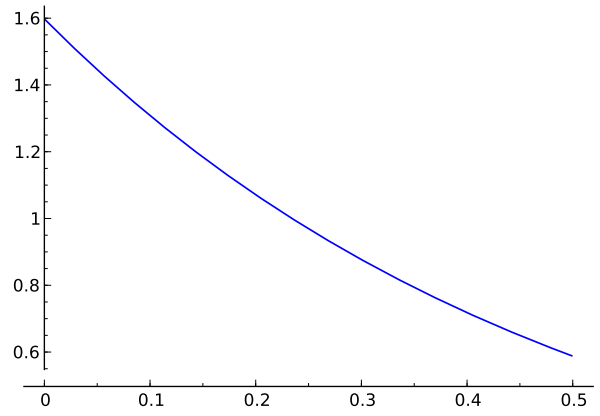


Figure 4: The ratio $\tilde{k}(x)/\bar{\Pi}(\log 1/x)$

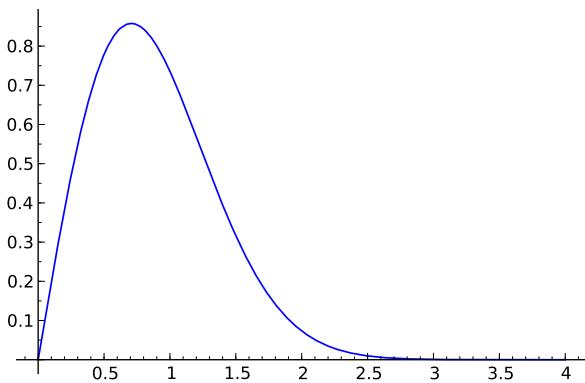


Figure 5: The density function k

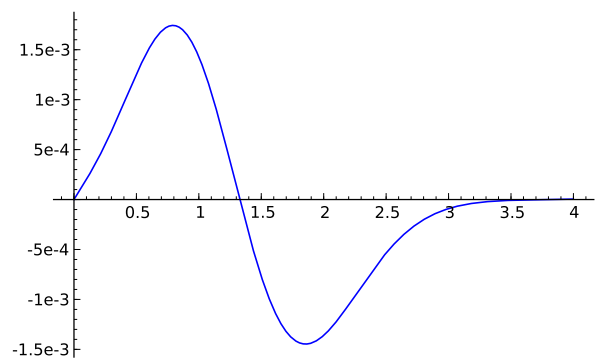


Figure 6: The difference $\tilde{k} - k$

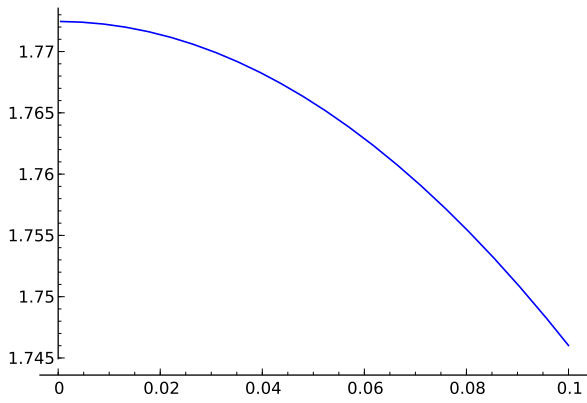


Figure 7: The ratio $k(x)/\bar{\Pi}(\log 1/x)$

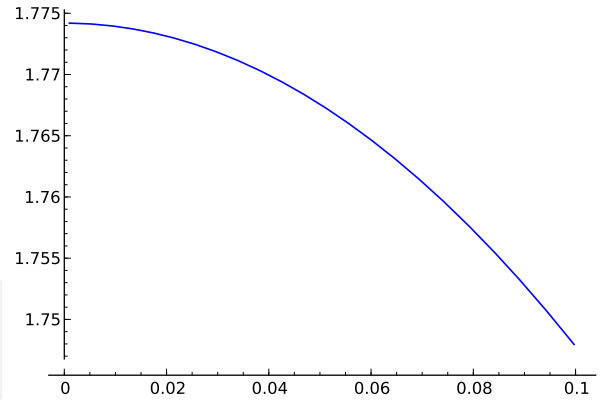


Figure 8: The ratio $\tilde{k}(x)/\bar{\Pi}(\log 1/x)$

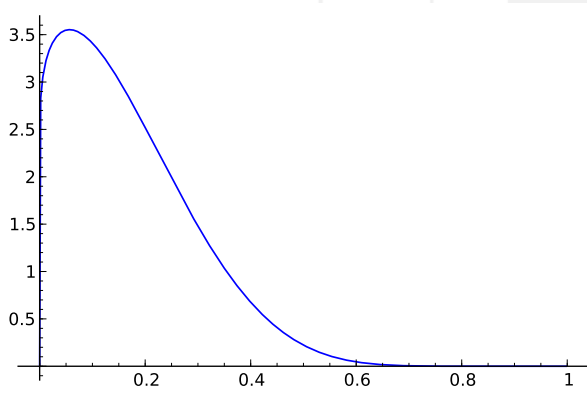


Figure 9: The density function \tilde{k}

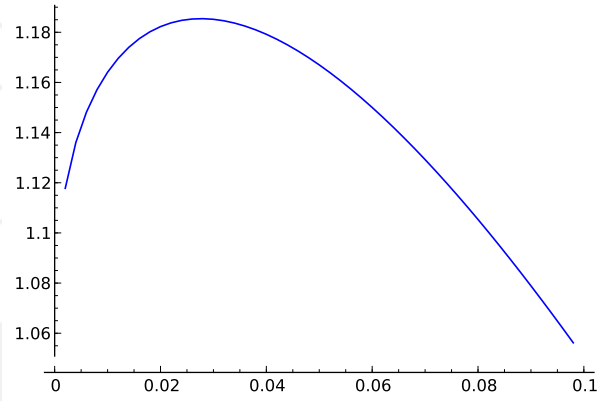


Figure 10: The ratio $\tilde{k}(x)/\bar{\Pi}(\log 1/x)$

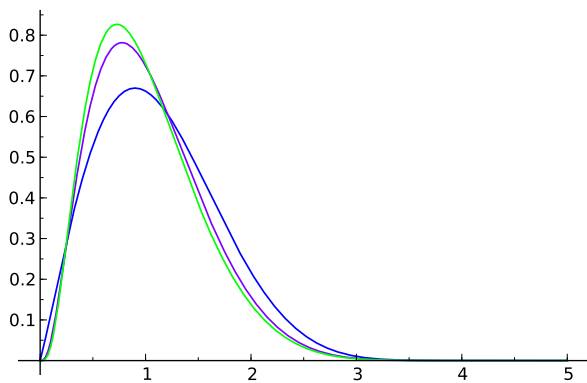


Figure 11: Density functions \tilde{k}

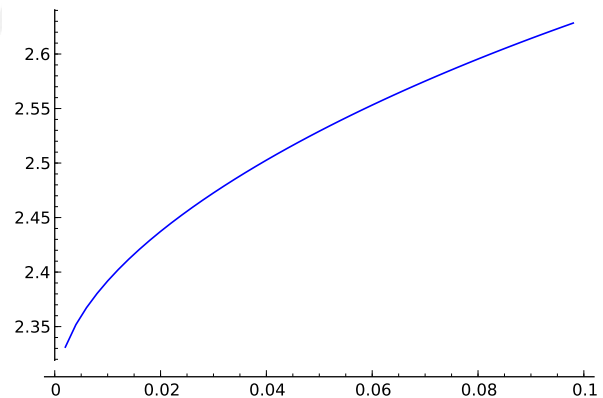


Figure 12: The ratio $\tilde{k}(x)/\bar{\Pi}(\log 1/x)$