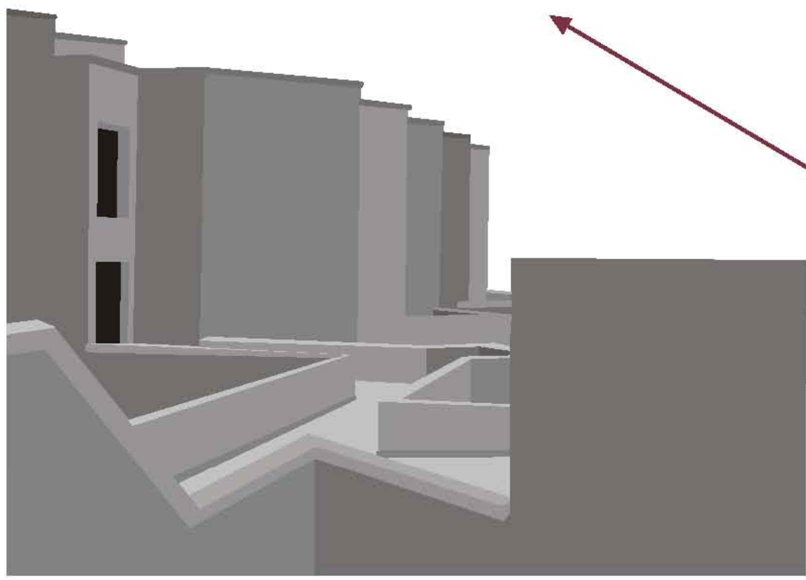

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GLOBAL AND NONGLOBAL SOLUTIONS OF A SYSTEM OF NONAUTONOMOUS SEMILINEAR EQUATIONS WITH ULTRACONTRACTIVE LEVY GENERATORS

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Global and nonglobal solutions of a system of nonautonomous semilinear equations with ultracontractive Lévy generators

José Alfredo López-Mimbela* Aroldo Pérez †

Abstract

We consider a semilinear system of the form $\partial u_i(t,x)/\partial t = k(t)\mathcal{A}u_i(t,x) + u_{i'}^{\beta_i}(t,x)$, with Dirichlet boundary conditions on a bounded open set $D \subset \mathbb{R}^d$, where $k : [0,\infty) \to [0,\infty)$ is continuous, $\mathcal{A}$ is the infinitesimal generator of a symmetric Lévy process $Z \equiv \{Z(t)\}_{t\geq 0}$, $\beta_i > 1$, $i \in \{1,2\}$ and $i' = 3-i$. We give conditions on D and on the Lévy measure of Z under which our system possesses global positive solutions, or exhibits blow up in finite time. Our approach is based on the intrinsic ultracontractivity property of the semigroup generated by the process Z killed on leaving D.

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1 Introduction and background

Consider a semilinear problem of the form

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= k(t)\Delta_\alpha u(t,x) + u^\beta(t,x), \\ u(0,x) &= f(x), \quad x \in D, \end{aligned} \tag{1}$$

where $k : [0,\infty) \to [0,\infty)$ is continuous and not identically zero, Δ_α is the fractional power $-(-\Delta)^{\alpha/2}$ of the Laplacian, $0 < \alpha \leq 2$, $D \subset \mathbb{R}^d$ is an open set, $\beta > 1$ is a constant and $f \geq 0$ is a bounded measurable function. It is known that the factor $k(t)$ has a strong effect on the asymptotic behavior of positive solutions of (1): when $D = \mathbb{R}^d$ we proved in [14] that integrability of k already excludes existence of global solutions, and if $\int_0^t k(s)\,ds \sim t^\rho$ as $t \to \infty$ for some $\rho \geq 1$, then the blow up behavior of (1) (i.e. finite-time blow up vs. existence of global solutions) parallels that of the equation $\partial u(t)/\partial t = \Delta_{\alpha/\rho} u + u^\beta$. If D is a bounded smooth domain, H. Fujita [7] proved, in the case of $k \equiv 1$ and $\alpha = 2$, that for any nontrivial, nonnegative initial value $f \in L^2(D)$ such that

$$\int_D f(x)\varphi_0(x)\,dx > \lambda_0^{1/(\beta-1)}, \tag{2}$$

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the solution of equation (1) with Dirichlet boundary condition blows up in finite time. Here $\lambda_0 > 0$ is the first eigenvalue of the Laplacian on D, and φ_0 the corresponding eigenfunction normalized so that $\|\varphi_0\|_{L^1} = 1$.

In this paper we investigate the dichotomy: finite-time blow up versus existence, globally in time, of positive solutions of the nonautonomous semilinear system

$$\begin{aligned} \frac{\partial u_1(t,x)}{\partial t} &= k(t)\,\mathcal{A}u_1(t,x) + u_2^{\beta_1}(t,x), \quad t>0,\ x\in D, \\ \frac{\partial u_2(t,x)}{\partial t} &= k(t)\,\mathcal{A}u_2(t,x) + u_1^{\beta_2}(t,x), \quad t>0,\ x\in D, \\ u_i(0,x) &= f_i(x), \quad x\in D, \quad u_i|_{D^c}\equiv 0,\ i=1,2, \end{aligned} \tag{3}$$

where $\mathcal{A}$ is the infinitesimal generator of a symmetric Lévy process $\{Z(t)\}_{t\geq 0}$, $\beta_i > 1$ are constants, $D \subset \mathbb{R}^d$ is a bounded open set with $d \geq 1$, and the initial values f_i, $i = 1, 2$, are nonnegative functions in the space $C_0(D)$ of continuous functions on D vanishing on D^c. As before, the function $k : [0,\infty) \to [0,\infty)$ is assumed to be continuous and not identically zero.

System (3) provides a simplified model of the process of diffusion of heat and burning in a two-component continuous media with temporary-inhomogeneous thermal conductivity. In this model u_1 and u_2 represent the temperatures of the two reactant components, the thermal conductivity is supposed equal for both substances but it might be discontinuous, and even evolve solely by jumps. See [8], [9] and [5] for similar models with nonlinear conductivity; see [13] and [6] for other related results.

Recall [16] that the Lévy process $Z \equiv \{Z(t)\}_{t\geq 0}$ is called symmetric when $Z(t)$ and $-Z(t)$ have the same distribution for all $t \geq 0$. The probability law of Z is uniquely determined by the probability measure $\mu(B) := \Pr\{Z(1) \in B\}$, $B \in \mathcal{B}(\mathbb{R}^d)$ (here $\mathcal{B}(\mathbb{R}^d)$ denotes the system of Borel sets in $\mathbb{R}^d$), which is infinitely divisible and therefore, by the Lévy-Khintchine formula, its normalized Fourier transform $\widehat{\mu}$ admits the representation

$$\widehat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + \mathrm{i}\langle \gamma, z\rangle + \int_{\mathbb{R}^d}\left(e^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z,x\rangle 1_{\{x:\,|x|\leq 1\}}(x)\right)\nu(dx)\right], \quad z\in\mathbb{R}^d, \mathrm{i}=\sqrt{-1},$$

where $A = (a_{jk})$ is a symmetric nonnegative-definite matrix, $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{R}^d$ and ν is a measure on $\mathbb{R}^d$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\,\nu(dx) < \infty$, which is termed Lévy measure. The operator $\mathcal{A}$ arises as the generator of the strongly continuous semigroup of contractions $\{S(t)\}_{t\geq 0}$ defined by $S(t)f(x) = \mathbb{E}\,[f(x + Z(t))]$, $f \in C_0(\mathbb{R}^d)$, and is given by

$$\mathcal{A}f(x) = \frac{1}{2}\sum_{j,k=1}^{d}\frac{a_{jk}\partial^2 f}{\partial x_j\partial x_k}(x) + \sum_{j=1}^{d}\frac{\gamma_j\partial f}{\partial x_j}(x) + \int_{\mathbb{R}^d}\left(f(x+y) - f(x) - \sum_{j=1}^{d}\frac{y_j\partial f}{\partial x_j}(x)1_{\{x:\,|x|\leq 1\}}(y)\right)\nu(dy)$$

for any twice continuously differentiable $f \in C_0(\mathbb{R}^d)$. Special instances of $\mathcal{A}$ include the Laplacian Δ and its fractional powers Δ_α with $0 < \alpha \leq 2$.

Dirichlet boundary value problems of the above type in the Gaussian case $\nu \equiv 0$ have been studied by many authors. In the present paper we focus on the purely non-Gaussian symmetric case in which $A = 0$ and ν is a nontrivial Lévy measure, hence Z is a pure-jump process which leaves D only when it hits D^c. This gives rise to the condition $u_i|_{D^c} = 0$ in (3), which is the form that the Dirichlet boundary condition takes in our setting; see [1] and [2].

We are going to assume that D is an open bounded set, and that the semigroup $\{S_D(t)\}_{t\geq 0}$ of the process Z killed on exiting D is intrinsically ultracontractive. Our main result, Theorem 6, gives a criterion in terms of the system parameters which is useful to determine for which initial values our semilinear system explodes in finite time. In particular, System (3) exhibits finite-time blow up provided that

$$\min_{i\in\{1,2\}} \int_D f_i(x)\varphi_0(x)\,dx > \text{Const.} \max_{i\in\{1,2\}} \left[\int_0^\infty \min_{i\in\{1,2\}} \left(e^{-\lambda_0 K(r,0)}\right)^{\beta_i - 1} dr\right]^{(\beta_i+1)/(1-\beta_1\beta_2)};$$

here

$$K(t,s) = \int_s^t k(r)\,dr,\ 0 \leq s \leq t, \tag{4}$$

and $\lambda_0 > 0$ and φ_0 are, respectively, the first eigenvalue and corresponding eigenfunction of the infinitesimal generator of the semigroup $\{S_D(t)\}_{t\geq 0}$ (see Section 2 below). Hence, when k is a continuous integrable function, positive solutions corresponding to initial values satisfying the above inequality cannot be global.

The approach we use to prove Theorem 6 uses in an essential way that the semigroup $\{S_D(t)\}_{t\geq 0}$ is intrinsically ultracontractive. The notion of intrinsic ultracontractivity was introduced by Davies and Simon in [4], and has been investigated by many authors since then, specially for diffusions (both symmetric and nonsymmetric). The cases in which $\mathcal{A} = -(-\Delta)^{\alpha/2}$, $0 < \alpha \leq 2$, were explored in [3], [4] and [11]. For symmetric Lévy processes, this notion was studied by T. Grzywny [10]. Several of the results and hypothesis from [10] are going to be used in our arguments, specially in Section 2 where we introduce the additive process generated by the family of generators $\{k(t)\,\mathcal{A}\}_{t\geq 0}$, and the corresponding killed process. In Section 3 we prove existence of local solutions of (3) using the classical fixed-point argument, adapted to our needs. Conditions ensuring existence of a global positive solution of (3) are given in Section 4. Theorem 6 is proved in Section 5.

2 Killed additive process

Let $Z \equiv \{Z(t)\}_{t\geq 0}$ be a symmetric Lévy process in $\mathbb{R}^d$ with generator $\mathcal{A}$, and whose Lévy measure ν is not identically zero. We assume that Z posesess a family of transition densities $p(t,x,y) \equiv p(t,x-y)$ which are continuous for every $t > 0$, and that for any $\delta > 0$ there exists a constant $c = c(\delta)$ such that $p(t,x) \leq c$ for $t > 0$ and $|x| \geq \delta$. In [12] (Lemma 2.5) and [1] (Lemma 1.1), sufficient conditions are given for continuity on $\mathbb{R}^d\backslash\{0\}$ and boundedness for every $t > 0$ of the transition densities of isotropic unimodal pure-jump Lévy processes.

Notice that $\{\nu(t,\cdot) := K(t,0)\,\nu(\cdot)\}_{t\geq 0}$ is a family of Lévy measures satisfying

$$\begin{aligned}
\nu(0,\cdot) &= 0,\\
\nu(s,B) &\leq \nu(t,B) \text{ for every } B \in \mathcal{B}(\mathbb{R}^d) \text{ whenever } 0 \leq s \leq t,\\
\nu(s,B) &\to \nu(t,B) \text{ for every } B \in \mathcal{B}(\mathbb{R}^d) \text{ as } s \to t.
\end{aligned}$$

Hence (see [16], Chapter 2.9) there exists an additive process in law $\{W(t)\}_{t\geq 0}$, uniquely determined up to identity in law, such that the infinitely divisible probability measure $\Pr[W(t)\in\cdot\,]$ has Lévy measure $\nu(t,\cdot)$, $t\geq 0$. Moreover, $\{W(t)\}_{t\geq 0}$ is a Markov process with transition probability function $P(s,x,t,B) := \Pr[W(t)-W(s)\in B-x]$, $0\leq s\leq t$, $x\in\mathbb{R}^d$, $B\in\mathcal{B}(\mathbb{R}^d)$. Since $\mu(\cdot)=\Pr[Z(1)\in\cdot\,]$ has Lévy measure $\nu(\cdot)$, the probability $\mu^{K(t,0)}(\cdot)$ has the Lévy measure $\nu(t,\cdot)$ defined above. Therefore

$$\Pr[W(t)\in\cdot\,]=\mu^{K(t,0)}(\cdot)=\Pr[Z(K(t,0))\in\cdot\,] \tag{5}$$

an thus

$$\begin{aligned} P(s,x,t,B) &= \Pr[Z(K(t,0))-Z(K(s,0))\in B-x] \\ &= \Pr[Z(K(t,0)-K(s,0))\in B-x] \\ &= \Pr[Z(K(t,s))\in B-x] \\ &= (S(K(t,s))\,1_B)(x), \end{aligned}$$

where $\{S(t)\}_{t\geq 0}$ denotes the semigroup with generator $\mathcal{A}$ and 1_B is the indicator function of B. Since the function $(t,x)\mapsto S(K(t,s))\,f(x)$, $(t,x)\in[s,\infty)\times\mathbb{R}^d$, is the unique solution of

$$\begin{aligned} \frac{\partial w(t,x)}{\partial t} &= k(t)\mathcal{A}w(t,x), \quad t>s, \quad x\in\mathbb{R}^d, \\ w(s,x) &= f(x), \quad f\in C_0(\mathbb{R}^d), \end{aligned}$$

we call $\{W(t)\}_{t\geq 0}$ the time-inhomogeneous Markov process corresponding to the family of generators $\{k(t)\,\mathcal{A}\}_{t\geq 0}$. Letting

$$p(s,x,t,y)\equiv p(K(t,s),x,y)\,, \quad 0\leq s\leq t, \quad x,y\in\mathbb{R}^d,$$

we see that $p(s,x,t,y)$ is a continuous transition density function for the process $\{W(t)\}_{t\geq 0}$. We define

$$\tau_D=\inf\{t>0: W(t)\notin D\} \quad \text{and} \quad \widehat{\tau}_D=\inf\{t>0: Z(t)\notin D\}\,.$$

Using (5) we obtain that

$$\widehat{\tau}_D=K(\tau_D,0)\,. \tag{6}$$

Let $\{S_D(t)\}_{t\geq 0}$ be the semigroup associated to the process $\{Z(t)\}_{t\geq 0}$ killed on exiting D, and let $p_D(t,x,y)$ be the transition density function of $\{S_D(t)\}_{t\geq 0}$, i.e.

$$S_D(t)\,f(x) := E^x[f(Z(t))\,;t<\widehat{\tau}_D]=\int_D f(y)\,p_D(t,x,y)\,dy, \quad x\in D,\ t>0,\ f\in\mathbb{B}^+(\mathbb{R}^d),$$

where $\mathbb{B}^+(\mathbb{R}^d)$ is the space of nonnegative bounded measurable functions on $\mathbb{R}^d$. Here and in the sequel P^x and E^x denote, respectively, the distribution and expectation with respect to the process $\{x+Z(t)\}_{t\geq 0}$ starting in $x\in\mathbb{R}^d$, but we use the same symbol $\{Z(t)\}_{t\geq 0}$ for the resulting process. It is known [10] that $p_D(t,x,y)=p_D(t,y,x)$ and $p_D(t,x,y)\leq p(t,x,y)$ for all $t>0$

and $x, y \in D$, and that $\{S_D(t)\}_{t\geq 0}$ is a strongly continuous semigroup of contractions on the space $L^2(D)$. Moreover, the linear operators $S_D(t)$, $t \geq 0$, are compact, and there exists an orthonormal basis of eigenfunctions $\{\varphi_n\}_{n=0}^{\infty}$ with corresponding eigenvalues $\left\{e^{-\lambda_n t}\right\}_{n=0}^{\infty}$ satisfying $0 < \lambda_0 < \lambda_1 \leq \lambda_2 \ldots$, and $\lim_{n\to\infty} \lambda_n = \infty$. All eigenfunctions φ_n are continuous and real-valued (see [10]). Let $B_r(x) = \left\{y \in \mathbb{R}^d : |y - x| < 1\right\}$ be the open ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$. If, in addition to the above assumptions

(H1) D is a connected open bounded set or,

(H2) D is a bounded open set and for every $x \in \mathbb{R}^d$ and $r > 0$, $\nu(B_r(x)) > 0$,

then the transition density $p_D(t, \cdot, \cdot)$, $t > 0$ is strictly positive on $D \times D$ and the eigenfunction $\varphi_0(x) > 0$ for every $x \in D$ (see [10], Proposition 2.2).

Let $\{W_D(t)\}_{t\geq 0}$ be the additive process $\{W(t)\}_{t\geq 0}$ killed on exiting D, namely

$$W_D(t) = \begin{cases} W(t) & \text{on } \{t < \tau_D\}, \\ \partial & \text{on } \{t \geq \tau_D\}, \end{cases}$$

where ∂ is a cemetery point. The state space of $\{W_D(t)\}_{t\geq 0}$ is the set $D_\partial = D \cup \{\partial\}$, and from (6) it follows that its transition function is given by

$$P_D(s, x, t, \Gamma) \;=\; P^x\left[Z(K(t,s)) \in \Gamma; K(t,s) < \hat{\tau}_D\right], \; 0 \leq s < t, \; x \in D, \; \Gamma \in \mathcal{B}(D),$$

where $\mathcal{B}(D)$ denotes the Borel σ-field on D. Hence the transition density function of $\{W_D(t)\}_{t\geq 0}$ is given by $p_D(s, x, t, y) = p_D(K(t,s), x, y)$ and thus, for every $f \in L^2(D)$,

$$U_D(t,s) f(x) \equiv \int_D f(y) p_D(s, x, t, y)\, dy = S_D(K(t,s)) f(x), \qquad 0 \leq s < t, \quad x \in D. \tag{7}$$

Proposition 1 *If* (H1) *or* (H2) *holds, then the function $p_D(s, x, t, y)$ is a density of $P_D(s, x, t, \Gamma)$, which is strictly positive, symmetric and continuous on $D \times D$.*

Proof. This follows easily from the fact that $p_D(t, x, y)$ is a density of $P_D(t, x, \Gamma)$, which is strictly positive, symmetric and continuous on $D \times D$ (see [10], Proposition 2.2). ∎

Using (7) and the fact that $\{S_D(t)\}_{t\geq 0}$ is a strongly continuous semigroup of contractions on $L^2(D)$, we obtain that $\{U_D(t,s)\}_{t\geq s\geq 0}$ is an evolution family of contractions on $L^2(D)$. In [10] (Theorem 3.1) it is proved that either condition (H2) or

(H3) D is an open bounded connected Lipschitz set, and for every $x \in S$, $\gamma \in (0, \pi/2]$ and $r > 0$,

$$\nu(\Gamma_\gamma(x) \cap B_r(0)) > 0,$$

where S denotes the unit sphere in $\mathbb{R}^d$ and $\Gamma_\gamma(x) = \{y \in \mathbb{R}^d : \langle x, y\rangle > |y| \cos\gamma\}$, imply that $\{S_D(t)\}_{t\geq 0}$ is an intrinsically ultracontractive semigroup, i.e., for all $t > 0$ there exists a positive constant $c = c(t, D)$ such that, for all $f \in L^2(D)$,

$$|S_D(t) f(x)| < c\varphi_0(x) \|f\|_{L^2(D)}, \quad x \in D; \tag{8}$$

see [4], Theorem 3.2.

3 Local existence of a mild solution

A solution of the integral system

$$u_i(t,x) = U_D(t,0) f_i(x) + \int_0^t U_D(t,r)\, u_{i'}^{\beta_i}(r,x)\, dr, \quad t \geq 0,\ x \in D, \tag{9}$$

is called a mild solution of (3); here and in the sequel, $i \in \{1,2\}$ and $i' = 3 - i$.

We are going to assume that f_1 and f_2 are nonnegative functions in $L^\infty(D)$, where $L^\infty(D)$ is the space of real-valued essentially bounded functions defined on D.

Our proof of the existence of local solutions is an adaptation, to our case, of the proof given in [17]. For any constant $\tau > 0$ let

$$E_\tau \equiv \{(u_1,u_2) : [0,\tau] \to L^\infty(D) \times L^\infty(D), |||(u_1,u_2)||| < \infty\},$$

where

$$|||(u_1,u_2)||| \equiv \sup_{0 \leq t \leq \tau} \{||u_1(t,\cdot)||_\infty + ||u_2(t,\cdot)||_\infty\}.$$

The couple $(E_\tau, |||\ |||)$ is a Banach space and $P_\tau \equiv \{(u_1,u_2) \in E_\tau : u_1 \geq 0, u_2 \geq 0\}$ and $C_R \equiv \{(u_1,u_2) \in E_\tau : |||(u_1,u_2)||| \leq R\}$, $R > 0$, are closed subsets of E_τ.

Theorem 2 *Let $f_i : D \to [0,\infty)$ be in $L^\infty(D)$, $i = 1,2$. There exists a constant $\tau = \tau(f_1,f_2) > 0$ such that the integral system (9) posesses a unique nonnegative local solution in $L^\infty([0,\tau] \times D) \times L^\infty([0,\tau] \times D)$.*

Proof. Define the operator Ψ on $C_R \cap P_\tau$ by

$$\begin{aligned} \Psi(u_1,u_2)(t,x) &= (U_D(t,0) f_1(x), U_D(t,0) f_2(x)) \\ &\quad + \left(\int_0^t U_D(t,r)\, u_2^{\beta_1}(r,x)\, dr, \int_0^t U_D(t,r)\, u_1^{\beta_2}(r,x)\, dr \right). \end{aligned} \tag{10}$$

We are going to show that Ψ is a contraction on $C_R \cap P_\tau$ for suitably chosen $R > 0$ and $\tau > 0$. In fact, if $(u_1,u_2), (\tilde{u}_1,\tilde{u}_2) \in C_R \cap P_\tau$, then

$$|||\Psi(u_1,u_2) - \Psi(\tilde{u}_1,\tilde{u}_2)||| \leq \sup_{0 \leq t \leq \tau} \int_0^t \|u_2^{\beta_1}(r,\cdot) - \tilde{u}_2^{\beta_1}(r,\cdot))\|_\infty dr + \sup_{0 \leq t \leq \tau} \int_0^t \|u_1^{\beta_2}(r,\cdot) - \tilde{u}_1^{\beta_2}(r,\cdot))\|_\infty dr.$$

From the elementary inequality $|a^p - b^p| \leq p(a \vee b)^{p-1}|a-b|$, which holds for all $a,b > 0$ and $p \geq 1$, we get

$$\begin{aligned} |||\Psi(u_1,u_2) - \Psi(\tilde{u}_1,\tilde{u}_2)||| &\leq \beta_1 R^{\beta_1 - 1} \int_0^\tau \|u_2(r,\cdot) - \tilde{u}_2(r,\cdot)\|_\infty dr \\ &\quad + \beta_2 R^{\beta_2 - 1} \int_0^\tau \|u_1(r,\cdot) - \tilde{u}_1(r,\cdot)\|_\infty dr \\ &\leq (\beta_1 R^{\beta_1 - 1} \vee \beta_2 R^{\beta_2 - 1}) |||(u_1,u_2) - (\tilde{u}_1,\tilde{u}_2)||| \tau. \end{aligned} \tag{11}$$

Noticing that

$$|||\Psi(u_1, u_2)||| \leq ||f_1||_\infty + ||f_2||_\infty + \tau \left(R^{\beta_1} + R^{\beta_2} \right),$$

by taking $R > 0$ big enough and $\tau > 0$ sufficiently small it follows from (11) that Ψ is a contraction mapping on $C_R \cap P_\tau$. Thus, the Banach fixed-point theorem implies that (9) posesses a unique solution (u_1, u_2) such that $u_i \geq 0$, $i = 1, 2$. ■

4 Global existence of the mild solution

Here we suppose again that $f_i \in L^\infty(D)$. Our proof of the next theorem follows closely the proof of Theorem 2.2 in [14].

Theorem 3 *Let f_i be nonnegative, and let $g \equiv f_1 \vee f_2$. If*

$$(\beta_i - 1) \int_0^\infty \|U_D(t, 0)g\|_\infty^{\beta_i - 1} dt < 1, \quad i = 1, 2,$$

then the solution of the integral system (9) *is global.*

Proof. If g is identically zero, the nonnegative solution of (9) is clearly $(u_1, u_2) \equiv (0, 0)$, which is global. Now, if g is not identically zero, putting

$$B_i(t) = \left[1 - (\beta_i - 1) \int_0^t \|U_D(r, 0)g\|_\infty^{\beta_i - 1} dr \right]^{-\frac{1}{\beta_i - 1}}$$

we get $B_i(0) = 1$ and

$$\begin{aligned} \frac{d}{dt} B_i(t) &= -\frac{1}{\beta_i - 1} \left[1 - (\beta_i - 1) \int_0^t \|U_D(r, 0)g\|_\infty^{\beta_i - 1} dr \right]^{-\frac{1}{\beta_i - 1} - 1} [-(\beta_i - 1) \|U_D(t, 0)g\|_\infty^{\beta_i - 1}] \\ &= \|U_D(t, 0)g\|_\infty^{\beta_i - 1} B_i^{\beta_i}(t), \end{aligned}$$

which gives

$$B_i(t) = 1 + \int_0^t \|U_D(r, 0)g\|_\infty^{\beta_i - 1} B_i^{\beta_i}(r) dr, \quad i = 1, 2. \tag{12}$$

Since the evolution system $\{U_D(t, s)\}_{t \geq s \geq 0}$ is positivity-preserving (due to (7)), we can choose two continuous functions $v_i : [0, \infty) \times D \to [0, \infty)$, $i = 1, 2$, such that $v_i(t, \cdot) \in C_b(D)$ for all $t \geq 0$ and

$$0 \leq v_i(t, x) \leq (B_1(t) \wedge B_2(t)) U_D(t, 0) g(x), \quad t \geq 0, \quad i = 1, 2.$$

Let us define

$$\mathcal{F}_i v_i(t, x) := U_D(t, 0) f_i(x) + \int_0^t U_D(t, r) v_{i'}^{\beta_i}(r, x)\, dr.$$

Because of $g \geq f_i$ and $B_i(r) \geq B_1(r) \wedge B_2(r),\ i = 1, 2,$

$$\begin{aligned}
\mathcal{F}_i v_i(t,x) &\leq U_D(t,0)g(x) + \int_0^t B_i^{\beta_i}(r) U_D(t,r) \left(U_D(r,0)g(x)\right)^{\beta_i} dr \\
&\leq U_D(t,0)g(x) + \int_0^t B_i^{\beta_i}(r) U_D(t,r) U_D(r,0)g(x) \|U_D(r,0)g\|_\infty^{\beta_i - 1} dr \\
&= U_D(t,0)g(x) \left[1 + \int_0^t \|U_D(r,0)g\|_\infty^{\beta_i-1} B_i^{\beta_i}(r) dr\right] \\
&= B_i(t) U_D(t,0)g(x),
\end{aligned}$$

where we used (12) in the last equality. Therefore,

$$0 \leq \mathcal{F}_i v_i(t,x) \leq \left(B_1(t) \vee B_2(t)\right) U_D(t,0)g(x), \quad t \geq 0,\ x \in D.$$

We now define

$$u_{i,0}(t,x) = U_D(t,0) f_i(x) \quad \text{and} \quad u_{i,n+1}(t,x) = \mathcal{F}_i u_{i,n}(t,x), \quad n = 0, 1, \ldots.$$

Using that $u_{i,0}(t,x) \leq u_{i,1}(t,x)$ for all $t \geq 0,\ x \in D$, and again that $U_D(t,s)$ preserves positivity, it follows by induction that $u_{i,n}(t,x) \leq u_{i,n+1}(t,x)$, $n \geq 0$. Hence

$$u_i(t,x) \equiv \limsup_{n \to \infty} u_{i,n}(t,x) \leq \left(B_1(t) \vee B_2(t)\right) U_D(t,0)g(x) < \infty$$

for all $t \geq 0$ and $x \in D$. From the monotone convergence theorem we conclude that $u_i(t,x)$ satisfies

$$u_i(t,x) = U_D(t,0) f_i(x) + \int_0^t U_D(t,r) u_{i'}^{\beta_i}(r,x) dr, \quad t \geq 0,\ x \in D.$$

Therefore, (u_1, u_2) is a global mild solution of (3). ∎

5 Blow up in finite time of the positive mild solution

In the sequel we assume that (H2) or (H3) holds.

Recall that φ_0 is the eigenfunction corresponding to the first eigenvalue λ_0 of the infinitesimal generator of the semigroup $\{S_D(t)\}_{t \geq 0}$. Arguing as in the case of Brownian motion in a bounded domain (see [15], p. 287), it can be shown that $\varphi_0^2(x)\, dx$ is the unique invariant measure of the semigroup $\{Q(t)\}_{t \geq 0}$ given by

$$Q(t)\, g(x) = \frac{e^{\lambda_0 t}}{\varphi_0(x)} S_D(t) \left(g \varphi_0\right)(x), \ x \in D,\ g \in C_b(D),\ t \geq 0.$$

Thus, defining

$$E[h] := \int h(x)\, \varphi_0^2(x)\, dx, \quad h \in C_b(D),$$

and

$$T(t,s)\, g(x) = \frac{e^{\lambda_0 K(t,s)}}{\varphi_0(x)} S_D\left(K(t,s)\right) \left(g \varphi_0\right)(x), \quad x \in D, \quad g \in C_b(D), \quad t \geq s \geq 0,$$

we have that for any $t \geq s \geq 0$ and $g \in C_b(D)$,

$$E\left[Q(t)g\right] = E\left[g\right] \quad \text{and} \quad T(t,s)g = Q(K(t,s))g. \tag{13}$$

Lemma 4 *For any $t \geq s \geq 0$ and $g \in C_b(D)$,*

$$E\left[T(t,s)g\right] = E\left[g\right].$$

Proof. This is a direct consequence of (13). ■

Proposition 5 *Let $f_i = g_i\varphi_0$, where $g_i \in C_b(D)$ is nonnegative and not identically zero, $i = 1,2$. If*

$$\min_{i\in\{1,2\}} \langle f_i, \varphi_0\rangle > \max_{i\in\{1,2\}} \left[\left(\frac{\beta_1\beta_2 - 1}{\beta_i + 1}\right)\left(\frac{\beta_i + 1}{\beta_{i'} + 1}\right)^{\frac{\beta_i}{\beta_i+1}} \int_0^\infty \min_{i\in\{1,2\}} \left(\frac{e^{-\lambda_0 K(r,0)}}{\|\varphi_0\|_1}\right)^{\beta_i - 1} dr\right]^{\frac{\beta_i+1}{1-\beta_1\beta_2}}, \tag{14}$$

then the mild solution of (3) *blows up in finite time.*

Proof. Notice that $\langle f_i, \varphi_0\rangle = E\left[g_i\right] > 0$, $i = 1,2$. We define

$$w_i(t,x) = \frac{e^{\lambda_0 K(t,0)} u_i(t,x)}{\varphi_0(x)} \quad \text{and} \quad z(t,x) = e^{-\lambda_0 K(t,0)}\varphi_0(x), \ x \in D, \ t \geq 0,$$

where (u_1, u_2) is the mild solution of (3), i.e., (u_1, u_2) solves the integral system (9). Multiplying both sides of (9) by $\varphi_0(x)^{-1}\exp(\lambda_0 K(t,0))$ we get

$$\begin{aligned}
&w_i(t,x)\\
&= T(t,0)g_i(x) + \int_0^t \frac{\exp(\lambda_0 K(t,0))}{\varphi_0(x)} U_D(t,r)\, u_{i'}^{\beta_i}(r,x)\,dr\\
&= T(t,0)g_i(x) + \int_0^t \frac{\exp(\lambda_0 K(t,0))}{\varphi_0(x)} U_D(t,r)\left(\frac{u_{i'}^{\beta_i}(r,x)}{\varphi_0^{\beta_i-1}(x)}\varphi_0^{\beta_i-1}(x)\right)dr\\
&= T(t,0)g_i(x) + \int_0^t \exp(\lambda_0 K(r,0))\frac{\exp(\lambda_0 K(t,r))}{\varphi_0(x)} U_D(t,r)\left(\frac{u_{i'}^{\beta_i}(r,x)}{\varphi_0^{\beta_i-1}(x)}\varphi_0^{\beta_i-1}(x)\right)dr\\
&= T(t,0)g_i(x) + \int_0^t \exp(\lambda_0 K(r,0))\, T(t,r)\left(\frac{u_{i'}^{\beta_i}(r,x)}{\varphi_0^{\beta_i}(x)}\varphi_0^{\beta_i-1}(x)\right)dr\\
&= T(t,0)g_i(x) + \int_0^t T(t,r)\Big(\frac{\exp(\lambda_0 K(r,0)\beta_i)\, u_{i'}^{\beta_i}(r,x)}{\varphi_0^{\beta_i}(x)}\\
&\quad \cdot \exp(-\lambda_0 K(r,0)(\beta_i - 1))\,\varphi_0^{\beta_i-1}(x)\Big)dr\\
&= T(t,0)g_i(x) + \int_0^t T(t,r)\, w_{i'}^{\beta_i}(r,x)\, z^{\beta_i-1}(r,x)\,dr.
\end{aligned}$$

The last equality renders

$$E\left[w_i(t,\cdot)\right] = E\left[T(t,0)\,g_i\right] + \int_0^t E\left[T(t,r)\left(w_{i'}^{\beta_i}(r,\cdot)\,z^{\beta_i-1}(r,\cdot)\right)\right]dr,$$

and due to Lemma 4,

$$E\left[w_i(t,\cdot)\right] = E\left[g_i\right] + \int_0^t E\left[w_{i'}^{\beta_i}(r,\cdot)\,z^{\beta_i-1}(r,\cdot)\right]dr.$$

It follows that for any $\epsilon > 0$,

$$E\left[w_i(t+\epsilon,\cdot)\right] - E\left[w_i(t,\cdot)\right] = \int_t^{t+\epsilon} E\left[w_{i'}^{\beta_i}(r,\cdot)\,z^{\beta_i-1}(r,\cdot)\right]dr, \tag{15}$$

with

$$\begin{aligned} E\left[w_{i'}^{\beta_i}(r,\cdot)\,z^{\beta_i-1}(r,\cdot)\right] &= e^{-\lambda_0 K(r,0)(\beta_i-1)} \int \left[w_{i'}(r,x)\,\varphi_0(x)\right]^{\beta_i}\varphi_0(x)\,dx \\ &\geq e^{-\lambda_0 K(r,0)(\beta_i-1)}\,\|\varphi_0\|_1 \left(\int w_{i'}(r,x)\,\frac{\varphi_0^2(x)}{\|\varphi_0\|_1}dx\right)^{\beta_i} \\ &= \left(\frac{\exp\left(-\lambda_0 K(r,0)\right)}{\|\varphi_0\|_1}\right)^{\beta_i-1} E\left[w_{i'}(r,\cdot)\right]^{\beta_i}, \end{aligned} \tag{16}$$

where we have used Jensen's inequality with respect to the probability measure $\|\varphi_0\|_1^{-1}\varphi_0(x)\,dx$. Let $h_i(t) := E\left[w_i(t,\cdot)\right]$. Plugging (16) into (15), and afterward multiplying the resulting inequality by ϵ^{-1} with $\epsilon \to 0$, we obtain that

$$h_i'(t) \geq \left(\|\varphi_0\|_1^{-1}\exp\left(-\lambda_0 K(t,0)\right)\right)^{\beta_i-1} h_{i'}^{\beta_i}(t), \qquad h_i(0) = \langle f_i, \varphi_0\rangle. \tag{17}$$

Let

$$c(t) = \min_{i\in\{1,2\}}\left\{\left(\|\varphi_0\|_1^{-1}\exp(-\lambda_0 K(t,0))\right)^{\beta_i-1}\right\}, \quad N = \min_{i\in\{1,2\}}\left\{\langle f_i,\varphi_0\rangle\right\} > 0,$$

and consider the ordinary differential system

$$p_1'(t) = c(t)p_2^{\beta_1}(t), \quad p_2'(t) = c(t)p_1^{\beta_2}(t), \quad p_i(0) = N,\ i = 1,2. \tag{18}$$

It follows that $\int_0^t p_1^{\beta_2}(r)p_1'(r)\,dr = \int_0^t p_2^{\beta_1}(r)p_2'(r)\,dr$, and

$$\frac{1}{\beta_2+1}\left[p_1^{\beta_2+1}(t) - N^{\beta_2+1}\right] = \frac{1}{\beta_1+1}\left[p_2^{\beta_1+1}(t) - N^{\beta_1+1}\right].$$

Notice that if $N \leq \left(\frac{\beta_2+1}{\beta_1+1}\right)^{\frac{1}{\beta_2+1}} N^{\frac{\beta_1+1}{\beta_2+1}}$, then

$$\frac{1}{\beta_2+1}p_1^{\beta_2+1}(t) \leq \frac{1}{\beta_1+1}p_2^{\beta_1+1}(t) \tag{19}$$

and, if $N \geq \left(\frac{\beta_2+1}{\beta_1+1}\right)^{\frac{1}{\beta_2+1}} N^{\frac{\beta_1+1}{\beta_2+1}}$, then

$$\frac{1}{\beta_2+1} p_1^{\beta_2+1}(t) \geq \frac{1}{\beta_1+1} p_2^{\beta_1+1}(t). \tag{20}$$

If (19) holds, then

$$p_2(t) \geq \left(\frac{\beta_1+1}{\beta_2+1}\right)^{\frac{1}{\beta_1+1}} p_1^{\frac{\beta_2+1}{\beta_1+1}}(t).$$

Substituting this into the first equation of (18), we get

$$p_1'(t) \geq c(t) \left(\frac{\beta_1+1}{\beta_2+1}\right)^{\frac{\beta_1}{\beta_1+1}} p_1^{\frac{\beta_1(\beta_2+1)}{\beta_1+1}}(t),$$

which is the same as

$$p_1^{-\frac{\beta_1(\beta_2+1)}{\beta_1+1}}(t) p_1'(t) \geq c(t) \left(\frac{\beta_1+1}{\beta_2+1}\right)^{\frac{\beta_1}{\beta_1+1}}.$$

Integrating both sides of the above inequality from 0 to t yields

$$\frac{\beta_1+1}{1-\beta_1\beta_2}\left[p_1^{\frac{1-\beta_1\beta_2}{\beta_1+1}}(t) - N^{\frac{1-\beta_1\beta_2}{\beta_1+1}}\right] \geq \left(\frac{\beta_1+1}{\beta_2+1}\right)^{\frac{\beta_1}{\beta_1+1}} \int_0^t c(r)dr.$$

Thus, in view of $\beta_1, \beta_2 > 1$,

$$p_1(t) \geq \left[N^{\frac{1-\beta_1\beta_2}{\beta_1+1}} - \left(\frac{\beta_1\beta_2-1}{\beta_1+1}\right)\left(\frac{\beta_1+1}{\beta_2+1}\right)^{\frac{\beta_1}{\beta_1+1}} \int_0^t c(r)dr\right]^{\frac{\beta_1+1}{1-\beta_1\beta_2}}. \tag{21}$$

Similarly, if (20) holds, we can show that

$$p_2(t) \geq \left[N^{\frac{1-\beta_1\beta_2}{\beta_2+1}} - \left(\frac{\beta_1\beta_2-1}{\beta_2+1}\right)\left(\frac{\beta_2+1}{\beta_1+1}\right)^{\frac{\beta_2}{\beta_2+1}} \int_0^t c(r)dr\right]^{\frac{\beta_2+1}{1-\beta_1\beta_2}}. \tag{22}$$

Since the function $\int_0^t c(r)\,dr$ is continuous and increases to $\int_0^\infty c(r)\,dr$, (21) and (22) implies finite-time blow up of (3) provided that

$$\min_{i\in\{1,2\}} \langle f_i, \varphi_0\rangle > \max_{i\in\{1,2\}} \left[\left(\frac{\beta_1\beta_2-1}{\beta_i+1}\right)\left(\frac{\beta_i+1}{\beta_{i'}+1}\right)^{\frac{\beta_i}{\beta_i+1}} \int_0^\infty \min_{i\in\{1,2\}} \left(\frac{e^{-\lambda_0 K(r,0)}}{\|\varphi_0\|_1}\right)^{\beta_i-1} dr\right]^{\frac{\beta_i+1}{1-\beta_1\beta_2}}.$$

$\blacksquare$

Theorem 6 *Let $f_1, f_2 \in C_0(D)$ be two nonnegative functions which are not identically zero. If Condition* (14) *holds, then the mild solution of (3) blows up in finite time.*

Proof. Let $i \in \{1,2\}$. Since by assumption f_i is not identically zero, there exists $x_i \in D$ such that $f_i(x_i) > 0$. Using the continuity of f_i we get $r_i > 0$ such that $f_i(x) > 0$ on $B_{r_i}(x_i) \subset D$. By Urysohn's lemma there exists a continuous function $q_i : \mathbb{R}^d \to [0,1]$ such that $q_i = 1$ on the closed ball $\overline{B_{r_i/3}(x_i)}$, and $q_i = 0$ on $\left(B_{2r_i/3}(x_i)\right)^c$. Hence the support of q_i is contained in $B_{r_i}(x_i)$. Putting $h_i = \frac{1}{2}\left(f_i \wedge q_i\right)$ we get a continuous function which is not identically zero, and whose support $\widetilde{C}_i$ is compact, has positive Lebesgue measure and is contained in D. Moreover, $0 \le h_i < f_i$, on $\widetilde{C}_i$.

Let $\{t_n\}$ be any given sequence of positive numbers with $t_n \downarrow 0$. It follows from the strong continuity of $\{U_D(t,s)\}_{t \ge s \ge 0}$, that

$$U_D\left(t_n, 0\right) h_i \to h_i \text{ in } L^2\left(D\right), \quad i = 1, 2,$$

and therefore

$$U_D\left(t_n, 0\right) h_1 \to h_1 \text{ in } L^2\left(\widetilde{C}_1\right).$$

Using Egoroff's theorem, there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$, and a set $C_1 \subset \widetilde{C}_1$ of positive Lebesgue measure such that

$$U_D\left(t_{n_k}, 0\right) h_1 \to h_1 \text{ uniformly in } C_1. \tag{23}$$

Writing $s_k \equiv t_{n_k}$ and arguing as above, we find a subsequence $\{s_{k_n}\}$ of $\{s_k\}$, and a set $C_2 \subset \widetilde{C}_2$ of positive Lebesgue measure such that

$$U_D\left(s_{k_n}, 0\right) h_2 \to h_2 \text{ uniformly in } C_2. \tag{24}$$

From (23) and (24) it follows that there exists $t_0 > 0$ such that

$$U_D\left(t_0, 0\right) h_1\left(x\right) < f_1\left(x\right) \text{ for all } x \in C_1$$

and

$$U_D\left(t_0, 0\right) h_2\left(x\right) < f_2\left(x\right) \text{ for all } x \in C_2.$$

Defining

$$\widetilde{f}_i(x) = 1_{C_i}(x) U_D\left(t_0, 0\right) h_i\left(x\right), \quad x \in D, \ i = 1, 2,$$

where 1_{C_i} is the indicator function of C_i, we obtain

$$u_i(t,x) = U_D(t,0) f_i(x) + \int_0^t U_D\left(t,r\right) u_{i'}^{\beta_i}\left(r,x\right)\, dr \ge U_D(t,0)\widetilde{f}_i(x) + \int_0^t U_D\left(t,r\right) u_{i'}^{\beta_i}\left(r,x\right)\, dr.$$

Let (v_1, v_2) be the mild solution of (3) with initial value $\left(\widetilde{f}_1, \widetilde{f}_2\right)$, which is given by

$$v_i(t,x) = U_D(t,0)\widetilde{f}_i(x) + \int_0^t U_D\left(t,r\right) v_{i'}^{\beta_i}\left(r,x\right) dr, \quad i = 1, 2.$$

We define the operator $\widetilde{\Psi}$ by

$$\begin{aligned} \widetilde{\Psi}\left(v_1, v_2\right)\left(t,x\right) &= \left(U_D(t,0)\widetilde{f}_1(x), U_D(t,0)\widetilde{f}_2(x)\right) \\ &\quad + \left(\int_0^t U_D\left(t,r\right) v_2^{\beta_1}\left(r,x\right)\, dr, \int_0^t U_D\left(t,r\right) v_1^{\beta_2}\left(r,x\right)\, dr\right). \end{aligned}$$

For any real numbers a, b, c, d, let us write $(a, b) \leq (c, d)$ when $a \leq c$ and $b \leq d$. Thus

$$\widetilde{\Psi}\left(u_1, u_2\right)(t, x) \leq \Psi\left(u_1, u_2\right)(t, x),$$

where Ψ is defined in (10). Similarly as in [17], we define the sequences $\{(v_{1,n}, v_{2,n})\}_{n=0}^{\infty}$ and $\{(u_{1,n}, u_{2,n})\}_{n=0}^{\infty}$ by

$$\left(v_{1,n+1}, v_{2,n+1}\right) = \widetilde{\Psi}\left(v_{1,n}, v_{2,n}\right) \quad \text{and} \quad \left(u_{1,n+1}, u_{2,n+1}\right) = \Psi\left(u_{1,n}, u_{2,n}\right),$$

respectively. If $(v_{1,n}(t, x), v_{2,n}(t, x)) \leq (u_{1,n}(t, x), u_{2,n}(t, x))$, then

$$\left(v_{1,n+1}(t, x), v_{2,n+1}(t, x)\right) \leq \widetilde{\Psi}\left(u_{1,n}, u_{2,n}\right)(t, x) \leq \Psi\left(u_{1,n}, u_{2,n}\right)(t, x) = \left(u_{1,n+1}(t, x), u_{2,n+1}(t, x)\right).$$

The contraction mapping property in a Banach space implies that the sequence $\{(v_{1,n}, v_{2,n})\}_{n=0}^{\infty}$, with $(v_{1,0}, v_{2,0}) = (0, 0)$, converges in the norm $|||\cdot|||$ to the unique fixed point (v_1, v_2) of $\widetilde{\Psi}$, namely

$$\left(v_{1,n}, v_{2,n}\right) \to \left(v_1, v_2\right) \quad \text{and} \quad \widetilde{\Psi}\left(v_{1,n}, v_{2,n}\right) \to \left(v_1, v_2\right)$$

in the norm $|||\cdot|||$ as $n \to \infty$. Similarly, the sequence $\{(u_{1,n}, u_{2,n})\}_{n=0}^{\infty}$ with $(u_{1,0}, u_{2,0}) = (0, 0)$ converges to the unique fixed point (u_1, u_2) of Ψ, that is

$$\left(u_{1,n}, u_{2,n}\right) \to \left(u_1, u_2\right) \quad \text{and} \quad \Psi\left(u_{1,n}, u_{2,n}\right) \to \left(u_1, u_2\right)$$

in the norm $|||\cdot|||$ as $n \to \infty$. Choosing $(v_{1,0}, v_{2,0}) = (u_{1,0}, u_{2,0}) = (0, 0)$, we obtain

$$\left(v_1(t, x), v_2(t, x)\right) \leq \left(u_1(t, x), u_2(t, x)\right).$$

Therefore, it suffices to prove that, under the hypothesis of Theorem 6, the mild solution of (3) blows up in finite time for all initial conditions of the form

$$u_i(0, x) = 1_{C_i}(x) U_D(t_0, 0) h_i(x), \quad x \in D, \quad i = 1, 2.$$

On the other hand, intrinsic ultracontractivity (8) implies that

$$\frac{U_D(t_0, 0) h_i}{\varphi_0} = \frac{S_D(K(t_0, 0)) h_i}{\varphi_0} \in C_b(D).$$

Thus, we can assume that the initial conditions in (3) are of the form $u_i(0, x) = 1_{C_i} p_i(x) \varphi_0(x)$, with $0 < p_i \in C_b(D)$. The assertion of the theorem now follows from Proposition 5 by taking continuous function g_i, with support contained in C_i, such that $0 \leq g_i \leq 1_{C_i} p_i$, $i = 1, 2$. ■

Remark 7 Notice that the above theorem is consistent with the corresponding result for the case of a single equation obtained in [15], which establishes that for a single Dirichlet boundary problem with $k \equiv 1$, $\mathcal{A} = \Delta$, $\beta > 1$ and a nonnegative initial condition $f \in C_0(D)$, where D is a bounded regular domain, the nonnegative mild solution blows up in finite time if $\langle f, \varphi_0 \rangle > \lambda_0^{\frac{1}{\beta-1}} \|\varphi_0\|_1$.

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