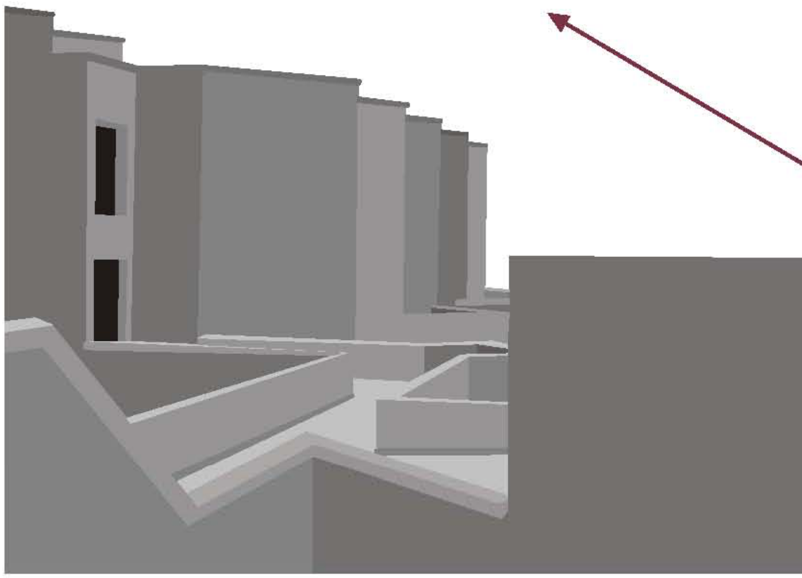

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Lower and upper bounds of the explosion time of a reaction-diffusion system perturbed by Brownian motion

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Abstract

We investigate lower and upper bounds for the blow-up time of a system of semilinear stochastic partial differential equations (SPDEs). From these bounds we obtain lower and upper bounds for the probability of explosion in finite time of the system. The lower bound is obtained from a related system of random partial differential equations, and is given in terms of the Laplace transform of a perpetual integral functional of a standard Brownian motion. The upper bound is given in terms of the expected value of a similar perpetual integral functional. We also extend the approach introduced by Chow (2011) to our system of SPDEs, and get an explosion result in L^p-norm, for any $1 \leq p < \infty$.

Key words: Semilinear PDEs, finite-time blow-up, blow-up time, integral functional of Brownian motion.

Mathematics Subject Classification: 60H30, 35K57, 35B35, 60J57, 60E07, 60J75.

1 Introduction

In this paper we investigate upper and lower bounds for the explosion time, and for the probability of explosion in finite time of the system of semilinear SPDEs

$$\begin{aligned} du_1(t,x) &= \left(\Delta_\alpha u_1(t,x) + u_2^{1+\beta_1}(t,x)\right) dt + \kappa_1 u_1(t,x)\, dW_t, \\ du_2(t,x) &= \left(\Delta_\alpha u_2(t,x) + u_1^{1+\beta_2}(t,x)\right) dt + \kappa_2 u_2(t,x)\, dW_t, \\ u_i(0,x) &= f_i(x), \quad x \in D, \\ u_i(t,x) &= 0, \quad t \geq 0, \quad x \in \mathbb{R}^d \setminus D, \quad i \in \{1,2\}, \end{aligned} \tag{1.1}$$

where Δ_α is the fractional power $-(-\Delta)^{\alpha/2}$ of the Laplacian, $0 < \alpha \leq 2$, β_i and κ_i are strictly positive constants, $\{W_t,\ t \geq 0\}$ is a one-dimensional standard Brownian motion defined on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, and $D \subset \mathbb{R}^d$ is a bounded and smooth domain. The initial values $f_i \in C_0^2\left(\bar{D}\right)$ are nonnegative and not identically zero.

When $\kappa_1 = \kappa_2 = 0$ system (1.1) provides a simplified model of the process of diffusion of heat and burning in a two-component continuous media. In such model u_1 and u_2 represent the temperatures of the two reactant components, the thermal conductivity is supposed to be the same for both substances but it might be discontinuous, and even evolve solely by jumps.

By adapting the approach of Dozzi et al. (2013) to our system, we lower-bound the probability of explosion in finite time of weak nontrivial positive solutions of (1.1). This entails to transform (1.1) into a parabolic system of random semilinear equations by means of the transformation $v_i(t,x) = e^{-\kappa_i W_t} u_i(t,x)$, $i \in \{1,2\}$, and then to produce a suitable subsolution for the weak solutions of the random parabolic system. We distinguish two cases: one in which $\beta_1 = \beta_2 > 0$, where we show that the probability of finite-time blowup of (1.1) is bigger than $1 - e^K H$. Here $K > 0$ is a constant whose value is explicitly given, and H is the Laplace transform of a related perpetual integral functional of $\{W_t : t \geq 0\}$. The other case corresponds to the parameter configuration $\beta_1 > \beta_2 > 0$. In this setting the probability of explosion in finite time of (1.1) is lower bounded by $1 - e^C H$ with a constant C (different from K) which again is explicitly computed. Then we proceed to derive upper bounds for the probability of explosion in finite time of non-trivial positive solutions of (1.1). The bounds we get are obtained following closely the approach of Dozzi and López-Mimbela (2010), which is based on the classical Picard's approximation scheme. At the end of the paper we also explore the concept of explosion in the $L^p(D)$-norm of (1.1). This notion of explosion was introduced by Chow (2011). Under suitable assumptions on the system parameters, we prove that any nontrivial positive solution of (1.1) explodes in finite time in the $L^p(D)$-norm sense for any $1 \leq p < \infty$.

We remark that in this manuscript we report our main results only (without proofs). Detailed proofs of our main theorems and other auxiliary result will appear in the PhD thesis of the second-named author.

Let A and c be positive constants. Let $H(x,z)$ be the unique solution of the integral equation

$$H(x,z) = 1 - c^{-1} z \mathrm{e}^{\frac{2c}{A^2}x} \int_x^\infty \mathrm{e}^{-\left(1+\frac{2c}{A^2}\right)u} H(u,z)\, \mathrm{d}u - c^{-1} z \int_0^x \mathrm{e}^{-u} H(u,z)\, \mathrm{d}u, \tag{1.2}$$

which is strictly positive and exists under suitable assumptions on $x \geq 0$ and $z \geq 0$.

2 A lower bound for the probability of explosion in finite time

In this section we obtain a lower bound for the probability of explosion in finite time of the system (1.1). For this we first construct a suitable subsolution of (1.1) by means of the change of variable $v_i(t,x) := \exp\{-\kappa_i W_t u_i(t,x)\}$ to get a related system of random partial differential equations (RPDEs), with u_i a weak solution of (1.1).

Consider the system of RPDEs

$$\begin{aligned}
\frac{\partial}{\partial t} v_1(t,x) &= \left(\Delta_\alpha v_1(t,x) - \frac{\kappa_1^2}{2} v_1(t,x)\right) + \mathrm{e}^{((1+\beta_1)\kappa_2-\kappa_1)W_t} v_2^{1+\beta_1}(t,x), \\
\frac{\partial}{\partial t} v_2(t,x) &= \left(\Delta_\alpha v_2(t,x) - \frac{\kappa_2^2}{2} v_2(t,x)\right) + \mathrm{e}^{((1+\beta_2)\kappa_1-\kappa_2)W_t} v_1^{1+\beta_2}(t,x), \\
v_i(0,x) &= f_i(x) \geq 0,\ x \times D, \\
v_i(t,x) &= 0,\ (t,x) \in \mathbb{R}_0^+ \times \mathbb{R}^d \setminus D,\ i \in \{1,2\}
\end{aligned} \tag{1.3}$$

with the same assumptions as in (1.1). As is shown in Dozzi et al. (2013), the function $v_i(t,x)$ is a weak solution (1.3).

2.1 Case $\boldsymbol{\beta_1 = \beta_2 > 0}$

Denote by λ and ψ the first eigenvalue and eigenfunction of Δ_α in D, respectively, with ψ normalized so that $\int \psi(x)\,\mathrm{d}x = 1$. Using the fact that $\Delta_\alpha \psi(x) = -\lambda\psi(x)$, it is easy to show that the vector function $(h_1.h_2)$ given by

$$\begin{aligned}
\frac{\partial}{\partial t} h_i(t) &= -\left(\lambda + \frac{\kappa_i^2}{2}\right) h_i(t) + \mathrm{e}^{((1+\beta_i)\kappa_j-\kappa_i)W_t} h_j^{1+\beta_i}(t), t \in \mathbb{R}^+, \\
h_i(0) &= \int_D f_i(x)\,\psi(x)\,\mathrm{d}x, \quad i = 1,2,
\end{aligned}$$

is, componentwise, a subsolution of (v_1, v_2). Let

$$E(t) := h_1(t) + h_2(t), \quad t \geq 0, \tag{1.4}$$

$$m := \lambda + \frac{\max\{\kappa_1^2, \kappa_2^2\}}{2}$$

and

$$M_t := \mathrm{e}^{((1+\beta_1)\kappa_2 - \kappa_1)W_t} \wedge \mathrm{e}^{((1+\beta_2)\kappa_1 - \kappa_2)W_t}.$$

Let I be given by

$$I(t) = \mathrm{e}^{-mt}\left[I^{-\beta_1}(0) - 2^{-(1+\beta_1)}\beta_1 \int_0^t \mathrm{e}^{-m\beta_1 s} M_s \mathrm{d}s\right]^{-\frac{1}{\beta_1}}, \quad t \in [0, \tau^*),$$

where

$$\tau^* := \inf\left\{t \geq 0 : \int_0^t \mathrm{e}^{-m\beta_1 s} M_s \mathrm{d}s \geq 2^{1+\beta_1}\beta_1^{-1} I^{-\beta_1}(0)\right\}.$$

Let

$$A := \min\{(1+\beta_1)\kappa_2 - \kappa_1, (1+\beta_1)\kappa_1 - \kappa_2\},$$

and assume that $A > 0$. Writing τ for the blow up time of system (1.1), we have that $\tau \leq \tau^*$.

Theorem 1. *If $\beta_1 = \beta_2 > 0$, then*

$$\mathbb{P}(\tau < \infty) \geq 1 - \exp\left(2^{1+\beta_1}\beta_1^{-1} I^{-\beta_1}(0)\right) H(0,1), \tag{1.5}$$

where $H(x,z)$ is the solution of (1.2) with $c = m\beta_1$.

2.2 Case $\beta_1 > \beta_2 > 0$

Let $A_0 = \left(\frac{1+\beta_1}{1+\beta_2}\right)^{-\frac{1+\beta_2}{\beta_1-\beta_2}} \frac{\beta_1-\beta_2}{1+\beta_1}$. Using Young's inequality one can deduce, as in Dozzi et al. (2013), that $E(t) \geq I(t)$ for $t \geq 0$, where E is defined in (1.4) and

I is the solution of

$$\begin{aligned} \frac{\partial}{\partial t} I(t) &= -mI(t) + \frac{2^{-(1+\beta_2)}\epsilon_0 E^{1+\beta_2}(0) - \epsilon_0^{\frac{1+\beta_2}{\beta_1-\beta_2}} A_0}{E^{1+\beta_2}(0)} M_t I^{1+\beta_2}(t), \quad t \in \mathbb{R}_0^+, \\ I(0) &= E(0), \end{aligned}$$

which is given by

$$I(t) = \mathrm{e}^{-mt}\left[I^{-\beta_2}(0) - \frac{2^{-(1+\beta_2)}\epsilon_0 E^{1+\beta_2}(0) - \epsilon_0^{\frac{1+\beta_2}{\beta_1-\beta_2}} A_0}{E^{1+\beta_2}(0)}\beta_2 \int_0^t \mathrm{e}^{-m\beta_2 s} M_s \mathrm{d}s\right]^{-\frac{1}{\beta_2}},$$

for all $t \in [0, \tau^{***})$, where

$$\tau^{***} := \inf\left\{t \geq 0 : \int_0^t \mathrm{e}^{-m\beta_2 s} M_s \mathrm{d}s \geq \left(\frac{2^{-(1+\beta_2)}\epsilon_0 E^{1+\beta_2}(0) - \epsilon_0^{\frac{1+\beta_2}{\beta_1-\beta_2}} A_0}{E^{1+\beta_2}(0)}\beta_2\right)^{-1} I^{-\beta_2}(0)\right\}.$$

Putting $C := \left(\left(\frac{2^{-(1+\beta_2)}\epsilon_0 E^{1+\beta_2}(0) - \epsilon_0^{\frac{1+\beta_2}{\beta_1-\beta_2}} A_0}{E^{1+\beta_2}(0)}\beta_2\right)^{-1} I^{-\beta_2}(0)\,\beta_2\right)^{-1} I^{-\beta_2}(0)$, and defining

$$\tau^{****} := \inf\left\{t \geq 0 : \int_0^t \mathrm{e}^{-(AW_s - m\beta_2 s)} \mathbb{1}_{\{AW_s - m\beta_2 s \geq 0\}} \mathrm{d}s \geq C\right\},$$

then $\tau \leq \tau^{***} \leq \tau^{****}$, and we get (as in case $\beta_1 = \beta_2$) that $\mathbb{P}(\tau = \infty) \leq \mathrm{e}^C H(0,1)$. Thus we obtain:

Theorem 2. *If $\beta_1 > \beta_2 > 0$, then*

$$\mathbb{P}(\tau < \infty) \geq 1 - e^C H(0,1).$$

3 An upper bound for the probability of explosion in finite time

Suppose that $\{Y_t : t \geq 0\}$ is an isotropic α-stable Lévy process with infinitesimal generator Δ_α. Let $\tau^D := \inf\{t > 0 : Y_t \notin D\}$ and consider the killed process $\left\{Y_t^D : t \geq 0\right\}$ given by

$$Y_t^D = \begin{cases} Y_t & t < \tau^D \\ \partial & t \geq \tau^D, \end{cases}$$

where ∂ is a cemetery point. Recall that a pair of $\mathcal{F}_t$-adapted random fields $\{v_i(t,x) : x \in D, t \geq 0\}$, $i \in \{1,2\}$ is a mild solution of (1.3) in the interval $[0,\tau)$ if

$$v_i(t,x) = \mathrm{e}^{-\frac{\kappa_i^2}{2}t} P_t^D f_i(x) + \int_0^t \mathrm{e}^{-\frac{\kappa_i^2}{2}(t-r)} \mathrm{e}^{((1+\beta_i)\kappa_j - \kappa_i)W_r} P_{t-r}^D \left[v_j^{1+\beta_i}(r,x)\right] \mathrm{d}r, \tag{1.6}$$

$\mathbb{P}$-a.s., where $\left(P_t^D\right)$ is the semigroup of the process $\left(Y_t^D\right)_{t\geq 0}$.

Theorem 3. *For all $x \in D$ we have*

$$v_i(t,x) \leq \frac{e^{-\frac{\kappa_i^2}{2}t} P_t^D f_i(x)}{\left(1 - \beta_i \int_0^t e^{((1+\beta_i)\kappa_j - \kappa_i)W_r} \left\| e^{-\frac{\kappa_i^2}{2}r} P_r^D f_i \right\|_\infty^{\beta_i} dr\right)^{\frac{1}{\beta_i}}},$$

for all $t \in [0, \tau_*)$, where

$$\begin{aligned} \tau_* \quad = \quad \inf\Bigg\{ t \geq 0 : \int_0^t e^{((1+\beta_1)\kappa_2 - \kappa_1)W_r} \left\| e^{-\frac{\kappa_1^2}{2}r} P_r^D f_1 \right\|_\infty^{\beta_1} dr \geq \beta_1^{-1} \ \textit{or} \\ \int_0^t e^{((1+\beta_2)\kappa_1 - \kappa_2)W_r} \left\| e^{-\frac{\kappa_2^2}{2}r} P_r^D f_2 \right\|_\infty^{\beta_2} dr \geq \beta_2^{-1} \Bigg\}. \end{aligned}$$

In the sequel we consider $f_i = L_i\psi$ in D, where $L_i > 0$ is a constant, $i \in \{1,2\}$. In this case

$$\tau_* = \inf\left\{t \geq 0 : \int_0^t \max_{i\in\{1,2\}}\left\{\mathrm{e}^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r - \beta_i\left(\lambda+\frac{\kappa_i^2}{2}\right)r}\right\}\mathrm{d}r \geq \min_{i\in\{1,2\}}\left\{\frac{1}{\beta_i L_i^{\beta_i}\left\|\psi\right\|_\infty^{\beta_i}}\right\}\right\}$$

From the definition of τ_*, it is clear that the relation $\tau_* \leq \tau$ holds.

Consider the random variable τ_{**} defined by

$$\tau_{**} :=$$

$$\inf\left\{t \geq 0 : \int_0^t \mathrm{e}^{-\min\limits_{i\in\{1,2\}}\left\{\beta_i\left(\lambda+\frac{\kappa_i^2}{2}\right)\right\}r} \max_{i\in\{1,2\}}\left\{\mathrm{e}^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r}\right\}\mathrm{d}r \geq \min_{i\in\{1,2\}}\left\{\frac{1}{\beta_i L_i^{\beta_i}\left\|\psi\right\|_\infty^{\beta_i}}\right\}\right\}.$$

It is easy to see that $\tau_{**} \leq \tau_*$. For simplicity we denote by

$$\begin{aligned}
\Lambda &:= \min_{i\in\{1,2\}}\left\{\beta_i\left(\lambda + \frac{\kappa_i^2}{2}\right)\right\} \\
a &:= \min_{i\in\{1,2\}}\left\{(1+\beta_i)\,\kappa_{3-i} - \kappa_i\right\} \\
b &:= \max_{i\in\{1,2\}}\left\{(1+\beta_i)\,\kappa_{3-i} - \kappa_i\right\} \\
M &:= \min_{i\in\{1,2\}}\left\{\frac{1}{\beta_i L_i^{\beta_i}\left\|\psi\right\|_\infty^{\beta_i}}\right\}
\end{aligned}$$

Theorem 4. *If* $\alpha := 1 - M^{-1}\mathbb{E}\left[\int_0^\infty e^{-\Lambda s}\max\left\{e^{aW_s}, e^{bW_s}\right\}ds\right] \in (0,1)$, *then*

$$\mathbb{P}\left(\tau < \infty\right) \leq 1 - \alpha. \tag{1.7}$$

A condition under which $\alpha \in (0,1)$ is given in the next theorem.

Theorem 5. *If* $2\Lambda > \max\left\{a^2, b^2\right\}$ *and* $M > \frac{\sqrt{2}}{2\sqrt{\Lambda}\left(\sqrt{2\Lambda}+a\right)} + \frac{\sqrt{2}}{2\sqrt{\Lambda}\left(\sqrt{2\Lambda}-b\right)}$, *then* $\alpha \in (0,1)$.

4 Explosion in $L^p(D)$-norm

In this section we analyze another kind of explosion which was defined by Chow (2011). We say that a solution u explodes in $L^p(D)$-norm if there exists $T_p \in \mathbb{R}_0^+ \cup \{\infty\}$ such that

$$\lim_{t \to T_p^-} \mathbb{E}\left[\|u(t,\cdot)\|_{L^p(D)}\right] = \infty. \tag{1.8}$$

When $T_p < \infty$, we say that u explodes in finite time in $L^p(D)$-norm and T_p is called the explosion time. For (1.1) we say that the system explodes in $L^p(D)$-norm if either u_1 or u_2 explodes in the $L^p(D)$-norm. For (1.1) we say that the system explodes in finite time in $L^p(D)$-norm if u_1 or u_2 explode in finite time in $L^p(D)$-norm. In this case, $\min\left\{T_p^1, T_p^2\right\}$ is called the explosion time of the system (1.1), where T_p^i is the explosion time of u_i, $i \in \{1,2\}$. The notion of explosion in $L^p(D)$-norm was investigated by Chow for a single SPDE. In order to prove explosion in $L^p(D)$-norm for all $1 \leq p < \infty$ of a system of SPDEs, we follow the methodology of Dozzi et al. (2013).

Let $u_i(t,\psi) := \int_D u_i(t,x)\,\psi(x)\,\mathrm{d}x$, $i \in \{1,2\}$. Then

$$u_i(t,\psi) = u_i(0,\psi) + \int_0^t u_i(s,\Delta_\alpha\psi)\,\mathrm{d}s + \int_0^t u_j^{1+\beta_i}(s,\psi)\,\mathrm{d}s + \int_0^t \kappa_i u_i(s,\psi)\,\mathrm{d}W_s. \tag{1.9}$$

Let us write $\mu_i(t) := \mathbb{E}[u_i(t,\psi)]$. Using Fubini's theorem and Jensen's inequality we get from (1.9) that

$$\mu_i(t) \geq \mu_i(0) - \lambda\int_0^t \mu_i(s)\,\mathrm{d}s + \int_0^t \mu_j^{1+\beta_i}(s)\,\mathrm{d}s$$

Now consider the system of ODEs:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}h_i(t) &= -\lambda h_i(t) + h_j^{1+\beta_i}(t), \quad t \in \mathbb{R}^+, \\ h_i(0) &= \mu_i(0), \end{aligned}$$

for $i \in \{1,2\}$, $j = 3-i$, and consider as before $E(t) = h_1(t) + h_2(t)$, $t \geq 0$. We have the following theorem.

Theorem 6. *1. Assume that $\beta_1 = \beta_2 > 0$ and $E(0) > \left(2^{1+\beta_1}\lambda\right)^{1/\beta_1}$. Then there exists $T^* \in \mathbb{R}^+$ such that $\min\left\{T_p^1, T_p^2\right\} \leq T^*$ for all $p \in [1,\infty)$.*

2. Let $\beta_1 > \beta_2 > 0$, and let $A_0 = \left(\frac{1+\beta_1}{1+\beta_2}\right)^{-\frac{1+\beta_2}{\beta_1-\beta_2}} \frac{\beta_1-\beta_2}{1+\beta_1}$, and suppose that there exists $\epsilon_0 \in (0,1]$ such that

$$C := 2^{-(1+\beta_2)}\epsilon_0 E^{1+\beta_2}(0) - \epsilon_0^{\frac{1+\beta_2}{\beta_1-\beta_2}} A_0 > \lambda E(0).$$

Then there exists $T^* \in \mathbb{R}^+$ such that $\min\left\{T_p^1, T_p^2\right\} \leq T^*$ for all $p \in [1,\infty)$.

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