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Gerber-Shiu functionals for two-sided jumps risk processes perturbed by an α -stable motion

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Abstract

We study the Gerber-Shiu functional of two-sided jumps risk processes perturbed by an α -stable motion for a wide a class of penalty functions. We obtain a formula for the Laplace transform of such functional which extend previous work of Furrer (1998), Albrecher, Gerber and Yang (2010), Zhang, Yang and Li (2010), Labbé, Sendov and Sendova (2011).

Keywords and phrases: two-sided risk process, stable process, ruin probability, severity of ruin, surplus before ruin.

1 Introduction.

For a risk process $X = \{X(t), t \geq 0\}$ the discounted Gerber-Shiu penalty function of X , as a function of an initial capital $u \geq 0$, is defined by

$$\phi_X(u) = \mathbb{E}[e^{-\delta\tau_0}\omega(|X(\tau_0)|, X(\tau_0-))I_{\{\tau_0 < \infty\}} | X(0) = u], \quad (1)$$

where $\tau_0 = \inf\{t > 0 : X(t) < 0\}$ is the time of ruin of X , $\delta \geq 0$ is a discounted force of interest and $\omega(x, y) : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a given non-negative function called penalty function. This functional was introduced in 1998 by Gerber and Shiu in the context of the classical risk model, generalizing the concept of ruin probability, and represents the joint distribution of the time of ruin, the surplus immediately before ruin and the deficit at time of ruin. Let us define the two-sided jumps classical risk process $X_0 := \{X_0(t), t \geq 0\}$

by the expression

$$X_0(t) = u + ct + \sum_{i=1}^{N_1(t)} X_{i1} - \sum_{i=1}^{N_2(t)} X_{i2}, \quad (2)$$

where u and c are nonnegative constants representing, respectively, initial capital and prime per time unit, and for $j = 1, 2, \dots$, $\{X_{ij}\}$ are sequences of independent and identically distributed random variables with a common distribution function $F_j(x)$ such that $F_j(0) = 0, j = 1, 2$ and with densities given by f_j . In addition, $\{N_j(t), t \geq 0\}$ are independent homogeneous Poisson processes with parameters $\lambda_j > 0$, which means that $Z_j = \{Z_j(t), t \geq 0\}, j = 1, 2$ are two independent compound Poisson processes with rates $\lambda_j > 0$.

Albrecher et al. (2010) and Labbé et al. (2011) studied the process X_0 in many cases of the upward jumps distribution and obtained expressions for the Laplace transform of the corresponding Gerber-Shiu functional. In fact, in Labbé et al. (2011) the authors obtain a renewal equation for the Gerber-Shiu functional of X_0 for the case when f_1 is an Erlang density $\Gamma(n, \beta)$ with $n \in \mathbb{N}$ and $\beta > 0$, while in Albrecher et al. (2010) the authors present a method to obtain several expressions for the Gerber-Shiu functional of X_0 , assuming that the penalty function ω depends only on the severity of ruin, and f_1 is an Hyperexponential distribution (i.e. a sum of exponential distributions of the form $F(x) = 1 - \sum_{j=1}^m A_j e^{-\beta_j x}, x > 0$, where $A_j > 0$ for all j and $\sum_j A_j = 1$). They also consider the case of the perturbed two-sided jumps risk process

$$X_B^*(t) = u + ct + \sum_{i=1}^{N_1(t)} X_{i1} - \sum_{i=1}^{N_2(t)} X_{i2} + \sigma B(t), \quad (3)$$

where $\{B(t), t \geq 0\}$ is a standard Brownian motion. This process has also been studied in Zhang et. al (2010) in a more general case when f_1 has Laplace transform given by

$$\widehat{f}_1(r) = \frac{Q(r)}{\prod_{i=1}^N (q_i + r)^{m_i}} \quad (4)$$

where $N, m_i \in \mathbb{N}$ with $m_1 + m_2 + \dots + m_N = m, m_i > 0, i = 1, 2, \dots, N, q_i \neq q_j, \forall i \neq j, 0 < q_1 < q_2 < \dots < q_m$ and $Q(r)$ is a polynomial function of degree $m - 1$ or less, satisfying $Q(0) = \prod_{i=1}^N q_i^{m_i}$. This family of distributions includes the phase-type distributions, from which the Erlang, Hyper-exponential and Coxian distributions arise as particular cases. The authors present an expression for the corresponding Gerber-Shiu functional

and obtain asymptotic expressions for the ruin probability, under some restrictions on the claim size distribution F_2 .

In this communication we consider the risk process $V_\alpha = \{V_\alpha(t), t \geq 0\}$ defined as:

$$V_\alpha(t) = u + ct + \sum_{i=1}^{N_1(t)} X_{i1} - \sum_{i=1}^{N_2(t)} X_{i2} - \eta W_\alpha(t), \quad \eta > 0, t \geq 0, \quad (5)$$

where $u, c, \sum_{i=1}^{N_1(t)} X_{i1}$ and $\sum_{i=1}^{N_2(t)} X_{i2}$ are as in the definition of the process $X_0 = \{X_0(t), t \geq 0\}$ and $\{W_\alpha(t), t \geq 0\}$ is an independent standard α -stable process with index of stability $1 < \alpha < 2$ and skewness parameter $\beta = 1$. The assumption on α guarantees the existence of the first moment of $V_\alpha(t)$, while the assumption on $\beta = 1$ ensures that the only upward jumps in $V_\alpha(t)$ are given by $Z_1(t)$. The case when $\alpha = 2$ is obtained as a limit, and it reduces to the process $X_B^* = \{X_B^*(t), t \geq 0\}$ studied in Zhang et al. (2010). We denote by $\psi(u)$ the probability of ruin $\mathbb{P}[\tau_0 < \infty | V_\alpha(0) = u]$ and also assume that the following conditions hold:

Hypothesis 1

1. The upward density f_1 has a Laplace transform of the form (4).
2. For $c \geq 0$ we have $E[V_\alpha(1) - u] = c + \lambda_1 \mu_1 - \lambda_2 \mu_2 > 0$.
3. The penalty function ω is bounded and such that

$$\mathbb{P}[(|V(\tau_0)|, V(\tau_0-)) \in \mathcal{D}_\omega] = 0,$$

where \mathcal{D}_ω is the discontinuity set of ω . Note that condition 2 above is just the Net Profit Condition, which ensures we do not have the trivial case of ruin occurring with probability 1.

The results obtained here generalize our previous result for perturbed classical risk processes with only negative jumps; we consider also here a more general class of penalty functions.

The main difficulties in working with risk processes perturbed by a stable Levy processes consist in the lack of a closed expression for the stable density, and the fact that we can not use the standart tool of first step analysis, because of the infinite number of jumps of the α -stable process in each time interval. We avoid this difficulties by constructing a sequence of classical two-sided risk processes which converge weakly to V_α and prove

the convergence of the corresponding Gerber-Shiu functionals. In order to obtain this last convergence, we make some assumptions on the penalty function $\omega(x, y)$, which are not so restrictive (see Hypothesis 1). We obtain a formula for the Laplace transform of the Gerber-Shiu function of the approximating sequence of processes and then obtain an expression for the Laplace transform of the Gerber-Shiu function of V_α by taking limits. After that this formula is inverted in order to obtain an expression for the Gerber-Shiu function of V_α as an infinite series of convolutions of given functions. As a particular case we obtain the results given in Albrecher et al. (2010), Zhang et al. (2010) and Labbé et al. (2011).

2 Basic concepts and main results

In this section we give some definitions and preliminary results, and present the main result. The proofs and applications will be given in the forthcoming Ph.D. thesis of Ehyter M. González.

We recall that for any nonnegative function f the Laplace transform of f is defined by

$$\hat{f}(r) = \int_{-\infty}^{\infty} e^{-rx} f(x) dx, \quad r \geq 0. \quad (6)$$

for each r where the integral above exists and is finite. In general, we write $\hat{f}(r) = \int_{-\infty}^{\infty} e^{-rx} f(x) dx$, for any complex r such that the latter integral exists.

Let us denote by $S_\alpha(\sigma, \beta, \mu)$, the α -stable distribution with parameters $0 < \alpha \leq 2$ (index of stability), $\sigma > 0$ (scale), $-1 \leq \beta \leq 1$, (skewness) $-\infty < \mu < \infty$ (shift), and density $g_{\alpha, \beta, \sigma, \mu}(x)$. According to Theorem C 3 in Zolotarev (1986), this probability density has Laplace transform given by

$$\mathbb{E}[e^{i\theta X}] = \begin{cases} e^{\sigma^\alpha(i\mu\theta - |\theta|^\alpha \exp\{-i(\pi/2)\beta K(\alpha) \operatorname{sgn}(\theta)\})} & \text{for } \alpha \neq 1, \\ e^{\sigma[i\mu\theta - |\theta|(\pi/2 + i\beta \log|\theta|) \operatorname{sgn}(\theta)]} & \text{for } \alpha = 1, \end{cases} \quad (7)$$

where $K(\alpha) = \alpha - 1 + \operatorname{sgn}(1 - \alpha)$ and $\operatorname{sgn}(\theta) = 1_{\{\theta > 0\}} + \theta 1_{\{\theta = 0\}} - 1_{\{\theta < 0\}}$.

Now we introduce the α -stable motion, which is a particular case of the class of Levy processes. We say that a stochastic process X is a Levy process if it satisfies the following conditions:

- $X(0) = 0$ a.s.
- X has \mathbb{P} -a.s. right-continuous paths with left limits (càdlàg trajectories)
- For $0 \leq s \leq t$, $X(t) - X(s) \stackrel{d}{=} X(t - s)$ and $X(t) - X(s)$ is independent of $\{X(u), u \leq s\}$

If $W_\alpha = \{W_\alpha(t), t \geq 0\}$ is a Levy process such that $W_\alpha(t) - W_\alpha(s) \sim S_\alpha [(t - s)^{1/\alpha}, \beta, \mu]$, $0 \leq s < t < \infty$, then W_α is called **α -stable Levy motion**. It is called **standard α -stable motion** when $\sigma = 1, \mu = 0$. If $1 < \alpha < 2$, the moments of W_α with order at most 1, are finite, and when $\beta = 1$, only positive jumps of W_α are possible.

For $\alpha = 2$, we obtain the Brownian motion $\{\sqrt{2}B(t), t \geq 0\}$. We refer the reader to Kyprianou (2006) and Sato(1999) for other properties of Levy processes.

Some important functions satisfying the above Hypothesis are:

1. Any constant $a > 0$ which gives a times the Laplace transform of the time to ruin when $\delta > 0$ and $a\psi(u)$ when $\delta = 0$,
2. $1_{\{x>a, y>b\}}$ for $a, b > 0$, the joint tail of the severity of ruin and surplus prior to ruin (when $\delta = 0$),
3. e^{-sx-ty} for fixed $s, t \geq 0$, which gives the Laplace transform of the joint distribution of $\tau_0, |V(\tau_0)|$ and $V(\tau_0-)$ when $\delta > 0$,
4. $1_{\{x+y>a\}}$ for $a > 0$, the tail of the distribution of the claim that causes ruin, conditioned on $\{\tau_0 < \infty\}$, when $\delta = 0$, and
5. $\max\{K - e^{a-y}, 0\}$ for $K, a > 0$, which is a particular case of a payoff function in option pricing.

We also obtain expressions for $\phi(u)$ when the penalty function is given by $\omega_1(x, y) = x, \omega_2(x, y) = y$ or $\omega(x, y) = xy$, for $\delta > 0$, although these three cases do not satisfy the above conditions for ω .

Let us define the functions

$$L_\alpha(r) = \lambda_2 \widehat{f}_2(r) + \lambda_1 \sum_{i=1}^n \sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} + cr + (\eta r)^\alpha - (\lambda_1 + \lambda_2 + \delta), \quad r \neq q_i. \quad (8)$$

and $Q_1(r) = \prod_{k=1}^N (q_k - r)^{m_k}$. We also denote the sets $\mathbb{C}_+ := \{z \in \mathbb{C} : \text{Re}(z) \geq 0\}$, $\mathbb{C}_{++} := \{z \in \mathbb{C} : \text{Re}(z) > 0\}$. We prove that for $\delta \geq 0$, the function L_α has exactly $m+1$ roots in \mathbb{C}_+ , denoted by $\{\rho_{1,\delta}, \rho_{2,\delta}, \dots, \rho_{m+1,\delta}\}$, which we assume are different. If $\delta = 0$, $\rho_{1,\delta} = 0$ is the only root on the imaginary axis.

Let us define

$$\tilde{N}(r) = \lambda_2 \int_0^\infty \int_u^\infty e^{-ru} \omega(x-u, u) f_2(x) dx du, \quad (9)$$

and

$$M_\alpha(r) = \frac{\eta^\alpha \alpha (\alpha - 1)}{\Gamma(2 - \alpha)} \int_0^\infty \int_u^\infty e^{-ru} \omega(x-u, u) x^{-1-\alpha} dx du. \quad (10)$$

The following result holds.

Theorem 1 *The Laplace transform of the Gerber-Shiu penalty function of the perturbed risk process V_α is given by*

$$\hat{\phi}(r) = \frac{\sum_{j=1}^{m+1} \frac{Q_1(\rho_{j,\delta}) \prod_{i=1, i \neq j}^{m+1} (\rho_{i,\delta} - r)}{\prod_{i=1, i \neq j}^{m+1} (\rho_{i,\delta} - \rho_{j,\delta})} \left[\tilde{N}(\rho_{j,\delta}) - \tilde{N}(r) + M_\alpha(\rho_{j,\delta}) - M_\alpha(r) \right]}{L_\alpha(r) \left[\sum_{j=1}^{m+1} \frac{Q_1(\rho_{j,\delta}) \prod_{i=1, i \neq j}^{m+1} (\rho_{i,\delta} - r)}{\prod_{i=1, i \neq j}^{m+1} (\rho_{i,\delta} - \rho_{j,\delta})} \right]}, \quad (11)$$

or equivalently

$$\hat{\phi}(r) = \frac{\sum_{j=1}^{m+1} \frac{Q_1(\rho_{j,\delta})}{\prod_{i=1, i \neq j}^{m+1} (\rho_{i,\delta} - \rho_{j,\delta})} \left[\frac{\tilde{N}(\rho_{j,\delta}) - \tilde{N}(r)}{\rho_{j,\delta} - r} + \frac{M_\alpha(\rho_{j,\delta}) - M_\alpha(r)}{\rho_{j,\delta} - r} \right]}{L_\alpha(r) \left[\sum_{j=1}^{m+1} \frac{Q_1(\rho_{j,\delta})}{\prod_{i=1, i \neq j}^{m+1} (\rho_{i,\delta} - \rho_{j,\delta}) (\rho_{i,\delta} - r)} \right]}. \quad (12)$$

Remark 1 *In the particular case when f_1 is a Hyperexponential distribution, the roots to Lundberg function $L_\alpha(r)$ are all real and different (see Bowers et al. (1997) p. 422). In this case, the formulae for $\hat{\phi}(r)$ given in Main Theorem I reduce to that given in Albrecher et al. (2010) when $\eta = 0$. In the case when $\hat{f}_1(r) = \left(\frac{q}{q+r}\right)^m$ and $\eta = 0$, we obtain Corollary 6.2 in Labbé et al. (2011).*

In the case when $\alpha = 2$ our risk process has the form

$$V_B(t) = u + ct + \sum_{i=1}^{N_1(t)} X_{i1} - \sum_{i=1}^{N_2(t)} X_{i2} + \eta^2 \sqrt{2} B(t), \quad (13)$$

where $B = \{B(t), t \geq 0\}$ is an standard Brownian motion. In this case we obtain the result in Zhang et al. (2010) which is the following corollary.

Corollary 1 *Let us denote by $\phi_B(u)$ the Gerber-Shiu functional for the process V_B defined in (13).*

$$\widehat{\phi}_B(r) = \frac{\sum_{j=1}^{m+1} \frac{Q_1(\rho_j, \delta) \prod_{i=1, i \neq j}^{m+1} (\rho_i, \delta - r)}{\prod_{i=1, i \neq j}^{m+1} (\rho_i, \delta - \rho_j, \delta)} [\widetilde{N}(\rho_j, \delta) - \widetilde{N}(r)]}{L_2(r) \left[\sum_{j=1}^{m+1} \frac{Q_1(\rho_j, \delta) \prod_{i=1, i \neq j}^{m+1} (\rho_i, \delta - r)}{\prod_{i=1, i \neq j}^{m+1} (\rho_i, \delta - \rho_j, \delta)} \right]} + \frac{\omega(0, 0) \sum_{j=1}^{m+1} \frac{Q_1(\rho_j, \delta) \prod_{i=1, i \neq j}^{m+1} (\rho_i, \delta - r)}{\prod_{i=1, i \neq j}^{m+1} (\rho_i, \delta - \rho_j, \delta)} [r - \rho_j, \delta]}{L_2(r) \left[\sum_{j=1}^{m+1} \frac{Q_1(\rho_j, \delta) \prod_{i=1, i \neq j}^{m+1} (\rho_i, \delta - r)}{\prod_{i=1, i \neq j}^{m+1} (\rho_i, \delta - \rho_j, \delta)} \right]}, \quad (14)$$

where

$$L_2(r) = \lambda_2 \widehat{f}_2(r) + \lambda_1 \sum_{j=1}^N \sum_{l=1}^{m_j} \beta_{jl} \frac{q_j^l}{(q_j - r)^l} + cr + \sqrt{2}\eta^2 r^2 - (\lambda_1 + \lambda_2 + \delta).$$

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