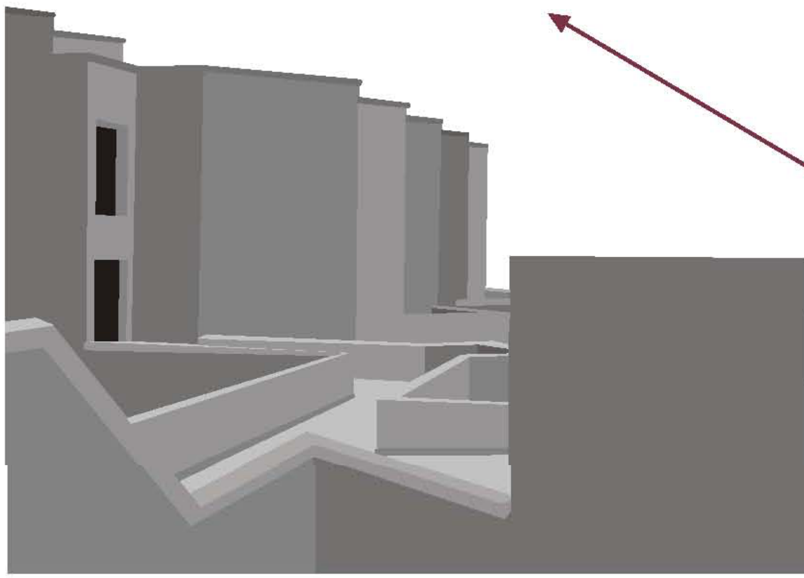

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Perpetual integral functionals of Brownian motion and blowup of a nonlinear system of SPDEs

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Abstract

We investigate existence of non global positive solutions of a system of nonlinear heat equations perturbed by a multiplicative Gaussian noise, where the noise factor is a Wiener process W. We obtain a lower bound for the probability of explosion in finite time of such solutions, which is given in terms of a perpetual integral functional of W.

Key words: Perpetual integral functionals; Blowup of systems of semi-linear equations; Systems of stochastic partial differential equations

Mathematics Subject Classification: 35R60; 60H15; 74H35.

1 Introduction

In this paper we investigate existence of non global positive weak solutions of the nonlinear system of stochastic partial differential equations

$$\begin{aligned} \mathrm{d}u(t,x) &= \left(\Delta u^{1+\nu}(t,x) + v^p(t,x)\right)\mathrm{d}t + \kappa u(t,x)\,\mathrm{d}W_t, \quad t>0 \\ \mathrm{d}v(t,x) &= \left(\Delta v^{1+\mu}(t,x) + u^q(t,x)\right)\mathrm{d}t + \theta v(t,x)\,\mathrm{d}W_t, \quad t>0 \qquad (1) \\ u(0,x) &= u_0(x), \quad v(0,x) = v_0(x), \qquad x \in D, \\ u(t,x) &= v(t,x) = 0, \quad t \geq 0, \quad x \in \partial D, \end{aligned}$$

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where $\mu, \nu > 0$, $p, q > 1$ and $\kappa, \theta \in \mathbb{R}$ are constants, u_0, v_0 are nonnegative bounded continuous functions which not identically vanish, $W \equiv \{W_t,\ t \geq 0\}$ is a one-dimensional standard Brownian motion and $D \subset \mathbb{R}^d$ is a bounded domain with smooth boundary ∂D.

The system above arises as a simplified model of the process of diffusion of heat and combustion in a two-component continuous media, subject to random perturbations proportional to the temperatures u and v of the reactant substances.

Using the transformation (5) we convert (1) into a parabolic system of random non-linear equations; see section 3.1 below. Then we produce a suitable subsolution for the random parabolic system which allows us to obtain bounds for the explosion times of system (1), as well as estimates for the probability of finite-time blowup of (1). The bounds for the blowup times are stopping times given in terms of functionals of the form $\int_0^t f(W_s)\,ds$, where $t \in [0, \infty]$ and f is a measurable function, which are termed perpetual integral functionals of W. We also calculate explicitly the probability distribution of a perpetual integral functional of Brownian motion which is relevant in our approach and which we think is of independent interest, see section 2. We deal mainly with the cases in which $\kappa = \theta$, $p = 1 + \mu$, $q = 1 + \nu$ and either $\mu = \nu$ or $\mu > \nu$. Other parameter configurations can be dealed with using the same techniques.

We remark that in this manuscript we report only our main results, without any proofs. Detailed proofs of our main theorems and other auxiliary results will appear in the PhD thesis of the second-named author.

2 An exponential functional of Brownian motion

Let $\{W_t,\ t \geq 0\}$ be a one-dimensional standard Brownian motion defined on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Let σ and μ be positive constants. It is well known (see, for instance, [6]) that Dufresne's functional $\int_0^\infty e^{\sigma W_s-\mu s}\mathrm{d}s$ has the distribution

$$\mathbb{P}\left(\int_0^\infty e^{\sigma W_s-\mu s}\mathrm{d}s > c\right) = \frac{\gamma\left(\frac{2\mu}{\sigma^2}, \frac{2}{\sigma^2 c}\right)}{\Gamma\left(\frac{2\mu}{\sigma^2}\right)}, \tag{2}$$

for all $c>0$, where $\gamma(a,x)=\int_0^x e^{-s}s^{a-1}\mathrm{d}s$ and $\Gamma(a)=\gamma(a,\infty)$, for all $a>0, x\geq 0$.

Let $\mathcal{X}_t = \sigma W_t - \mu t$, $t\geq 0$. The motivation of this section is to study, from an analytical point of view, some distributional properties of the exponential functional

$$\int_0^\infty e^{-(\mathcal{X}_t+x)}\mathbb{1}_{\{\mathcal{X}_t+x\geq 0\}}\mathrm{d}t, \quad x\geq 0.$$

This kind of functionals, also known as one-sided variants of Dufresne's functional, emerges for instance in the problem of explosion in finite time of systems of SPDEs. In particular we calculate explicitly its Laplace transform and its distribution at $x=0$.

Theorem 1. *The relation*

$$\mathbb{E}\left[\exp\left(-z\int_0^\infty e^{-\mathcal{X}_t}\mathbb{1}_{\{\mathcal{X}_t\geq 0\}}dt\right)\right] = \frac{4\mu I_{\frac{2\mu}{\sigma^2}}\left(\frac{\sqrt{8z}}{\sigma}\right)}{\sigma\sqrt{8z}I_{\frac{2\mu}{\sigma^2}-1}\left(\frac{\sqrt{8z}}{\sigma}\right)}, \tag{3}$$

holds for every $z>0$, where

$$I_\nu(x) := \sum_{k\geq 0}\frac{(x/2)^{2k+\nu}}{k!\Gamma(k+1+\nu)}, \quad \nu\in\mathbb{R},\ x\in\mathbb{R},$$

is the modified Bessel function of the first kind of order ν.

Let σ be the distribution function of $\int_0^\infty e^{-\mathcal{X}_t}\mathbb{1}_{\{\mathcal{X}_t\geq 0\}}dt$. Let $\left\{j_{\frac{2\mu}{\sigma^2}-1,n}\right\}_{n\geq 1}$ be the increasing sequence of all positive zeros of the Bessel function of the first kind of order $\frac{2\mu}{\sigma^2}-1>-1$ and

$$J_{\frac{2\mu}{\sigma^2}-1}(z) := \sum_{m\geq 0}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m+\frac{2\mu}{\sigma^2}-1}}{m!\Gamma\left(m+\frac{2\mu}{\sigma^2}\right)}, \quad z\in\mathbb{C}.$$

Theorem 2. *σ is absolutely continuous with respect to the Lebesgue measure. Furthermore, if*

$$h(y) := \mu \sum_{n\geq 1} \exp\left\{-\left(\frac{\sigma^2}{8} j^2_{\frac{2\mu}{\sigma^2}-1,n}\right) y\right\}, \quad y \geq 0,$$

then

$$\sigma(dy) = h(y)\, dy. \tag{4}$$

3 Conditions for existence and non existence of global solutions

3.1 A related system of RPDEs

In this section we obtain an upper bound for the explosion time of system (1). For this we first construct a suitable subsolution of (1) by means of the change of variables

$$\widetilde{u}(t,x) := \exp\{-\kappa W_t\} u(t,x), \qquad \widetilde{v}(t,x) := \exp\{-\theta W_t\} v(t,x), \tag{5}$$

where (u,v) is a weak solution of (1) to get a related system of random partial differential equations (RPDEs). Similarly as in [5] one can show that the vector function $\left(\widetilde{u},\widetilde{v}\right)$ is a weak solution of the system of RPDEs

$$\begin{aligned}
\frac{\partial}{\partial t}\widetilde{u}(t,x) &= e^{\kappa\nu W_t}\Delta\widetilde{u}^{1+\nu}(t,x) - \frac{\kappa^2}{2}\widetilde{u}(t,x) + e^{(\theta p-\kappa)W_t}\widetilde{v}^{p}(t,x), \quad t>0\\
\frac{\partial}{\partial t}\widetilde{v}(t,x) &= e^{\theta\mu W_t}\Delta\widetilde{v}^{1+\mu}(t,x) - \frac{\theta^2}{2}\widetilde{v}(t,x) + e^{(\kappa q-\theta)W_t}\widetilde{u}^{q}(t,x), \quad t>0\\
u(0,x) &= u_0(x) \geq 0, \quad x \in D,\\
v(0,x) &= v_0(x) \geq 0, \quad x \in D,\\
u(t,x) &= v(t,x) = 0, \quad t \geq 0, \quad x \in \partial D,
\end{aligned} \tag{6}$$

with the same assumptions as in (1).

3.2 Condition for global solvability

Let $\lambda > 0$ and ψ be, respectively, the first eigenvalue and eigenfunction of $-\Delta$ in D, with $\psi > 0$ in D, $\psi \in \mathrm{H}_0^1(D)$ and normalized so that $\int_D \psi(x)\,\mathrm{d}x = 1$.

Theorem 3. *Assume that $\kappa = \theta$, $p = 1+\mu$, $q = 1+\nu$, $\mu = \nu$ and $\lambda > 1$. If $\left(\widetilde{u}, \widetilde{v}\right)$ is a weak solution of (6) and $\widetilde{u}_0, \widetilde{v}_0 \in L^{1+\nu}(D)$, then for all $T \in (0,\infty)$, $\widetilde{u}, \widetilde{v} \in L^{\infty}([0,T]; L^{2+\nu}(D)) \cap L^{2+\nu}([0,T]; L^{2+\nu}(D))$ and $\widetilde{u}^{1+\nu}, \widetilde{v}^{1+\nu} \in L^2([0,T]; \mathrm{H}_0^1(D))$, $\mathbb{P}$-a.s.*

3.3 Conditions for non existence of global solutions

Let

$$E(0) = \int_D u(0,x)\varphi(x)\,dx + \int_D v(0,x)\varphi(x)\,dx,$$

and let $\tau \in [0,\infty]$ be the explosion time of (1).

Theorem 4. *Let suppose $\kappa = \theta$, $p = 1+\mu$, $q = 1+\nu$ and $\lambda < 1$. Suppose (u,v) is a weak solution of (1).*

(a) If $\nu = \mu > 0$ then $\tau \leq \tau^*$, where

$$\tau^* := \inf\left\{ t \geq 0 : \int_0^t e^{\kappa\nu W_s - \frac{\kappa^2}{2}\nu s} ds > E^{-\nu}(0)\, 2^{\nu} \nu^{-1} (1-\lambda)^{-1} \right\}. \tag{7}$$

(b) Let $\mu > \nu$. Let $\epsilon_0 := 1 \wedge \left(\frac{\widetilde{R}^{1+\nu}(0)}{A_0} \right)^{\frac{\mu-\nu}{1+\nu}} \wedge \left(\frac{2^{-1-\nu} E^{1+\nu}(0)}{A_0} \right)^{\frac{\mu-\nu}{1+\nu}}$ and $M_s := e^{\kappa\nu W_s} \wedge e^{\kappa\mu W_s}$, $s \geq 0$. Then $\tau \leq \tau^{**}$, where

$$\tau^{**} := \inf\left\{ t \geq 0 : \int_0^t e^{-\frac{\kappa^2}{2}\mu s} M_s ds > \frac{E^{-\nu}(0)\, \nu^{-1} (1-\lambda)^{-1}}{2^{-\nu}\epsilon_0 - \epsilon_0^{\frac{1+\mu}{\mu-\nu}} A_0 E^{-1-\nu}(0)} \right\}. \tag{8}$$

4 Lower bound for the probability of explosion in finite time

Theorem 5. *Suppose that κ, θ, p and q are as in Theorem 4.*

(a) If $\nu = \mu$, then

$$\mathbb{P}(\tau < \infty) \geq \frac{\gamma\left(\nu^{-1}, E^{\nu}(0)\,\kappa^{-2}2^{1-\nu}\nu^{-1}(1-\lambda)\right)}{\Gamma(\nu^{-1})}. \tag{9}$$

(b) If $\mu > \nu$, then

$$\mathbb{P}(\tau < \infty) \geq \frac{4\mu}{\nu^2}\sum_{n\geq 1}\frac{e^{-\frac{\kappa^2\nu^2}{8}C_0 j^2_{\frac{\mu}{\nu^2}-1,n}}}{j^2_{\frac{\mu}{\nu^2}-1,n}}, \tag{10}$$

where $C_0 = \frac{E^{-\nu}(0)\nu^{-1}(1-\lambda)^{-1}}{2^{-\nu}\epsilon_0 - \epsilon_0^{\frac{1+\mu}{\mu-\nu}}A_0E^{-1-\nu}(0)}$.

Remark 6. Notice that C_0 is small for sufficiently large $E(0)$. Therefore the relationship

$$\frac{4\mu}{\nu^2}\sum_{n\geq 1}\frac{e^{-\frac{\kappa^2\nu^2}{8}C_0 j^2_{\frac{\mu}{\nu^2}-1,n}}}{j^2_{\frac{\mu}{\nu^2}-1,n}} \sim 1 - \frac{\sqrt{2}\kappa\mu}{\sqrt{\pi}\nu}C_0^{1/2}$$

(which follows from formula (39) in [3]) implies that $\mathbb{P}(\tau < \infty) \sim 1$. This means that for sufficiently large initial conditions, system (1) explodes in finite time with high probability.

Remark 7. Another lower bound for $\mathbb{P}(\tau < \infty)$ can e obtain as follows: the random variable τ^{****} defined by

$$\tau^{****} := \inf\left\{t \geq 0 : \int_0^t e^{\kappa\nu W_s - \frac{\kappa^2}{2}\mu s}\mathbb{1}_{\left\{\kappa\nu W_s - \frac{\kappa^2}{2}\mu s \geq 0\right\}}\mathrm{d}s \geq C_0\right\} \tag{11}$$

satisfies $\tau \leq \tau^{****}$. Hence using Markov's inequality

$$\begin{aligned}
\mathbb{P}\left(\tau=\infty\right) \leq \mathbb{P}\left(\tau^{****}=\infty\right) &= \mathbb{P}\left(\int_0^t e^{\kappa\nu W_s-\frac{\kappa^2}{2}\mu s}\mathbb{1}_{\left\{\kappa\nu W_s-\frac{\kappa^2}{2}\mu s\geq 0\right\}}\mathrm{d}s \geq C_0\right) \\
&\leq e^{C_0}\mathbb{E}\left[\exp\left(\int_0^t e^{\kappa\nu W_s-\frac{\kappa^2}{2}\mu s}\mathbb{1}_{\left\{\kappa\nu W_s-\frac{\kappa^2}{2}\mu s\geq 0\right\}}\mathrm{d}s\right)\right] \\
&= e^{C_0}\frac{2\left(\frac{\mu}{\nu^2}\right)K_{\frac{\mu}{\nu^2}}\left(\frac{\sqrt{8}}{\kappa\nu}\right)}{\frac{2\Gamma\left(1+\frac{\mu}{\nu^2}\right)}{\Gamma\left(\frac{\mu}{\nu^2}\right)}K_{\frac{\mu}{\nu^2}}\left(\frac{\sqrt{8}}{\kappa\nu}\right)+\frac{\sqrt{8}}{\kappa\nu}K_{\frac{\mu}{\nu^2}-1}\left(\frac{\sqrt{8}}{\kappa\nu}\right)},
\end{aligned}$$

where we have used Theorem 2.2 in [11].

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