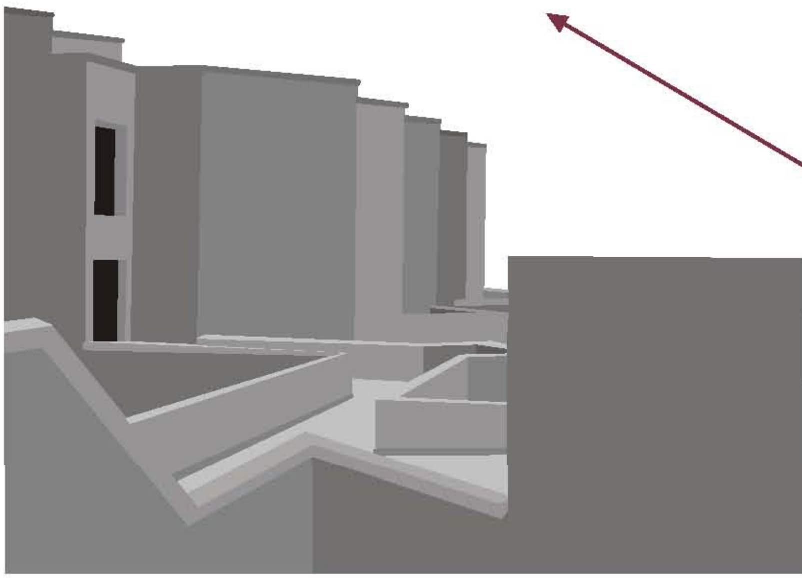

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The Infinitely Many Zeros of Stochastic Coupled Oscillators Driven by Random Forces

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The infinitely many zeros of stochastic coupled oscillators driven by random forces

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Abstract

This work deals with the oscillatory behavior around 0 of the stochastic coupled oscillators driven by random forces. We focus on three main aspects: 1) the analysis of this oscillatory behavior for the case of coupled harmonic oscillators, a property that has only been demostrated for simple harmonic oscillators; 2) the capability of some numerical integrators for reproducing this dynamical property; and 3) the identification of some classes of coupled nonlinear oscillators that can be shown to have this oscillatory dynamics by reducing their analysis to equivalent linear oscillators.

1 Introduction

Motivated by their capability to describe the time evolution of complex random phenomena, models of nonlinear oscillators driven by random forces have become a focus of intensive studies (see, e.g., [4],[14],[7],[8]). Naturally, the added noise modifies the dynamics of the deterministic oscillators and so new distinctive dynamical features arise in these random systems. Since the complexity of the random dynamics depends on the type of nonlinearity and the level of noise, many of the results on this matter have been achieved for specific classes of stochastic oscillators. In particular, a number properties have been studied for the simple harmonic oscillator such as the stationary probability distribution, the linear growth of energy along the paths, the oscillatory behavior around 0, and the symplectic structure of Hamiltonian oscillators, among other (see, e.g., [9]).

On the other hand, demanded by an increasing number of practical applications, the numerical simulations of stochastic oscillators has also a high interest. In particular, it is required specialized numerical integrators that preserve the dynamics of the oscillators since general multipurpose integrators fail to achieve this target. Consequently, specific oriented integrators for stochastic oscillators have also been proposed, for instance, in (see, e.g., [10],[2],[11],[13]). Distinctively, in [3], the family of the Locally Linearized methods have been proved to simultaneously reproduce various dynamical properties of the stochastic harmonic oscillators.

In this work, we are interested in the study of the oscillatory behavior around 0 of the stochastic coupled oscillators driven by random forces. We focus on three main aspects: 1) the analysis of

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this oscillatory behavior for the case of coupled harmonic oscillators, a property that has only been demonstrated for simple harmonic oscillators ([9],[8]); 2) the capability of the Locally Linearized methods for reproducing this dynamical property; and 3) the identification of some classes of coupled nonlinear oscillators that display this dynamics.

2 The infinitely many zeros of the coupled harmonic oscillators

Let us consider the undamped harmonic oscillator defined by the $2d$-dimensional Stochastic Differential Equation (SDE) with additive noise

$$d\mathbf{x}\left(t\right) = \mathbf{A}\mathbf{x}\left(t\right)dt + \mathbf{B}d\mathbf{w}_t, \tag{1}$$

for $t \geq 0$, with initial condition $\mathbf{x}(t_0) = (x_0, y_0) \in \mathbb{R}^d \times \mathbb{R}^d$ and $d > 1$. Here,

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\Lambda^2 & \mathbf{0} \end{bmatrix}, \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \Pi \end{bmatrix},$$

being $\Lambda \in \mathbb{R}^{d\times d}$ a nonsingular symmetric matrix, $\Pi \in \mathbb{R}^{d\times m}$ a matrix, $\mathbf{I}$ the $d-$dimensional identity matrix, and $\mathbf{w}_t$ an m-dimensional standard Wiener process on the filtered complete probability space $\left(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq t_0}, \mathbb{P}\right)$.

The following lemma will be useful.

Lemma 1 *Let $\eta_1, ..., \eta_n, ...$ be independent Gaussian random variables with zero mean and variance 1. Let $\{\sigma_{nr}\}$ be a bounded triangular array of real numbers. Set $S_n = \sum\limits_{r=1}^{n} \sigma_{nr}\eta_r$ and $s_n^2 = \sum\limits_{r=1}^{n} \sigma_{nr}^2$. If $\liminf\limits_{n\to\infty} \frac{s_n^2}{n} > 0$, then*

$$P\left(\limsup_{n\to\infty} \frac{S_n}{2s_n^2 \log\log s_n^2} \geq 1\right) = 1 \tag{2}$$

and

$$P\left(\liminf_{n\to\infty} \frac{S_n}{2s_n^2 \log\log s_n^2} \leq -1\right) = 1. \tag{3}$$

Proof. This is a direct consequence of Corollary 1 of Theorem 2 in [12]. ■

The following theorem shows the infinitely many oscillations of the paths of coupled harmonic oscillators (1), which extends the Theorem 4.1 in [8] (Section 8.4) that refers to the paths of simple harmonic oscillators (i.e., those defined by (1) with $d = 1$).

Theorem 2 *Consider the coupled harmonic oscillator (1). Then, almost surely, each component of the solution $\mathbf{x}(t)$ has infinitely many zeros on $[t_0\ \infty)$ for every $t_0 \geq 0$.*

Proof. Let us start considering the first component x^1 of the solution of (1). By the spectral theorem for the real nonsingular symmetric matrix Λ we have the factorization

$$\Lambda = P diag[\lambda_1, \ldots, \lambda_d] P^\intercal,$$

were $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of Λ, and P is a real orthogonal matrix with entries $[P_{k,j}]$ for all $k, j = 1, \ldots, d$. Then, for $f(\Lambda) = \sin(\Lambda)$ and for $f(\Lambda) = \cos(\Lambda)$, we have (see, e.g., [5])

$$f(\Lambda) = P diag[f(\lambda_1), \ldots, f(\lambda_d)]P^{\intercal}.$$

Since the solution of (1) satisfies (see, e.g., [8])

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos(\Lambda t) & \Lambda^{-1}\sin(\Lambda t) \\ -\Lambda\sin(\Lambda t) & \cos(\Lambda t) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \int\limits_{t_0}^{t} \begin{bmatrix} \Lambda^{-1}\sin(\Lambda(t-s)) \\ \cos(\Lambda(t-s)) \end{bmatrix} \Pi d\mathbf{w}_s,$$

$$x^1(t) = D(t) + V(t), \tag{4}$$

where

$$D(t) = \sum_{k=1}^{d} \left(P_{1k}\cos(\lambda_k t) < P_k, x_0 > + P_{1k}\lambda_k^{-1}\sin(\lambda_k t) < P_k, y_0 > \right)$$

and

$$V(t) = \sum_{l=1}^{m} \left(\int\limits_{t_0}^{t} \left(\sum_{k=1}^{d} c_k^l \sin(\lambda_k(t-s)) \right) d\mathbf{w}_s^l \right), \tag{5}$$

being $c_k^l = P_{1k}\lambda_k^{-1} < P_k, \Pi_l >$ and P_k, Π_l the column vectors of P and Π, respectively.

Without loss of generality, let us assume that $\lambda_k > 0$ and $\lambda_k \neq \lambda_j$ for all $k \neq j$. Indeed, when there are only $d^* < d$ different values λ_j^* of $|\lambda_k|$, the expression (5) can be rewritten as

$$V(t) = \sum_{l=1}^{m} \left(\int\limits_{t_0}^{t} \left(\sum_{j=1}^{d^*} e_j^l \sin\left(\lambda_j^*(t-s)\right) \right) d\mathbf{w}_s^l \right),$$

where $e_j^l = \sum\limits_{k=1}^{d} c_k^l \delta_{\lambda_j^*}^{|\lambda_k|}(1_{\lambda_k>0} - 1_{\lambda_k<0})$, and δ is the Kronecker delta. For this expression the analysis below would be the same.

Consider an arbitrary $\Delta > 0$ and the time instants $t_n = t_0 + n\Delta$, with $n = 1, 2 \ldots,$. In addition, for all n, define

$$S_n := V(t_n) = \sum_{r=1}^{n} V_{nr}, \tag{6}$$

where $V(t_n)$ is defined in (5), and

$$V_{nr} = \sum_{l=1}^{m} \left(\int\limits_{t_0+(r-1)\Delta}^{t_0+r\Delta} \left(\sum_{k=1}^{d} c_k^l \sin(\lambda_k(t_n - s)) \right) d\mathbf{w}_s^l \right)$$

for all $n, r = 1, 2 \ldots$. Because the independence of $\mathbf{w}_s^1, \ldots, \mathbf{w}_s^m$ and the independence of the increments of $\mathbf{w}_s^l$ on disjoint intervals, $\{V_{nr}\}_{r\geq 1}$ defines a sequence of independent Gaussian random variables with zero mean and variance

$$\begin{aligned} \sigma_{nr}^2 &= E(V_{nr}^2) \\ &= \int\limits_{t_0+(r-1)\Delta}^{t_0+r\Delta} \sum_{l=1}^{m} \left(\sum_{k=1}^{d} c_k^l \sin(\lambda_k(t_n - s)) \right)^2 ds. \end{aligned} \tag{7}$$

In this way, (6) can be written as

$$S_n = \sum_{r=1}^{n} \sigma_{nr} \eta_r,$$

where $\eta_1, ..., \eta_n$ are independent $\mathcal{N}(0,1)$ random variables. On the other hand,

$$s_n^2 = \sum_{r=1}^{n} \sigma_{nr}^2.$$

The expression (7) and the identity $sin(\theta) = (\exp(i\theta) - \exp(-i\theta))/(2i)$ (where $i = \sqrt{-1}$) imply that

$$s_n^2 = -\frac{1}{2}\sum_{l=1}^{m}\sum_{k,j=1}^{d} c_k^l c_j^l \operatorname{Re}\left\{\exp(i(\lambda_j+\lambda_k)t_n)\int_{t_0}^{t_n}\exp(i(\lambda_j+\lambda_k)s)ds - \exp(i(\lambda_j-\lambda_k)t_n)\int_{t_0}^{t_n}\exp(i(\lambda_j-\lambda_k)s)ds\right\},$$

where Re denotes the real part of a complex number. Since

$$\int_{t_0}^{t_n} \exp(i\theta s)ds = \begin{cases} n\Delta & \text{if } \theta = 0 \bmod 2\pi \\ (\exp(i\theta t_n) - \exp(-i\theta t_n))/(i\theta) & \text{otherwise} \end{cases},$$

$$s_n^2 = \frac{1}{2}\sum_{l=1}^{m}\sum_{k=1}^{d}\left(c_k^l\right)^2 n\Delta + C_n,$$

where C_n is uniformly bound for all n. Thus,

$$\lim_{n\to\infty}\frac{s_n^2}{n} = \frac{1}{2}\sum_{l=1}^{m}\sum_{k=1}^{d}\left(c_k^l\right)^2\Delta > 0.$$

In addition, since

$$\begin{aligned}\sigma_{nr}^2 &\leq 2^d \int_{t_0+(r-1)\Delta}^{t_0+r\Delta}\sum_{l=1}^{m}\sum_{k=1}^{d}\left(c_k^l\right)^2 ds \\ &\leq 2^d\Delta\sum_{l=1}^{m}\sum_{k=1}^{d}\left(c_k^l\right)^2\end{aligned}$$

for all n and r, the Law of the Iterated Logarithms of Lemma 1 holds for S_n. Thus, for $0 < \varepsilon < 1$, (2) implies that

$$S_n > (1-\varepsilon)\, 2s_n^2 \left(\log\log s_n^2\right) \quad \text{for infinitely many values of } n \text{ (a.s.)}.$$

In addition, since

$$|D(t_n)| \leq |P|^2 \left(|x_0| + |y_0| \max_k \left\{\lambda_k^{-1}\right\}\right)$$

for all n, for the fist component (4) of the solution of (1) we have that

$$x^1(t_n) > 0 \text{ infinitely often as } n \to \infty \text{ (a.s.)}.$$

Similarly, (3) implies that

$$S_n < (-1+\varepsilon)\, 2s_n^2 \left(\log\log s_n^2\right) \quad \text{for infinitely many values of } n \text{ (a.s.)}$$

for $0 < \varepsilon < 1$, and so

$$x^1(t_n) < 0 \text{ infinitely often as } n \to \infty \text{ (a.s.)}.$$

Thus, since the sample path of the solution to (1) is continuous, $x^1(t)$ must have, almost surely, infinitely many zeros on $[t_0\ \infty)$. For the rest of the components of the solution of (1) we can proceed in a similar manner. This concludes the proof. ■

3 The infinitely many zeros of the Local Linearized schemes for coupled harmonic oscillators

Let $(t)_h = \{t_n : n = 0, 1, \ldots, N\}$ be a partition of the time interval $[t_0, T]$ with maximum stepsize h defined as a sequence of times $0 = t_0 < t_1 < \ldots < t_N = T$ such that $h_n = t_{n+1} - t_n \leq h$ for $n = 0, \ldots, N-1$.

The Locally Linearized integrator for the equation (1) is defined by the recursive expression [3]

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \mathbf{u}_n + \mathbf{z}_{n+1} \tag{8}$$

for $n = 0, \ldots, N-1$, where $\mathbf{x}_n = (x_n, y_n)$, $\mathbf{u}_n = \mathbf{L}e^{\mathbf{C}_n h_n}\mathbf{r}$, and $\mathbf{z}_{n+1} = \mathbf{Q}\Delta\mathbf{w}_n$, with

$$\mathbf{C}_n = \begin{bmatrix} \mathbf{0}_d & \mathbf{I}_d & y_n \\ -\Lambda^2 & \mathbf{0}_d & -\Lambda^2 x_n \\ \mathbf{0}_{1\times d} & \mathbf{0}_{1\times d} & 0 \end{bmatrix} \in \mathbb{R}^{(2d+1)\times(2d+1)},$$

$\mathbf{L} = [\mathbf{I}_{2d} \;\; \mathbf{0}_{2d\times 1}]$, $\mathbf{r} = [\mathbf{0}_{1\times 2d} \;\; 1]^{\intercal}$, $\Delta\mathbf{w}_n = \mathbf{w}_{t_{n+1}} - \mathbf{w}_{t_n}$, and $\mathbf{Q} = \left[\begin{array}{c} Q_1 \\ Q_2 \end{array} \right]$ a $2d \times m$ matrix with Q_1, $Q_2 \in \mathbb{R}^{d\times m}$.

Theorem 3 *Let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of Λ, and $\lambda^*_{\max} = \max(|\lambda_1|, \ldots, |\lambda_d|)$. For the coupled harmonic oscillator (1), each components of the Locally Linearized integrator switches signs infinitely many times as $n \to \infty$, almost surely, for any integration stepsize $h < \pi/\lambda^*_{\max}$.*

Proof. Lemma 3.2 in [3] states that the Locally Linearized schemes (8) can be written as

$$\mathbf{x}_{n+1} = \mathbf{M}^{n+1}\mathbf{x}_0 + \sum_{r=0}^{n} \mathbf{M}^r \mathbf{Q}\Delta\mathbf{w}_{n-r},$$

where

$$\mathbf{M}^r = \begin{bmatrix} \cos(r\Lambda h) & \Lambda^{-1}\sin(r\Lambda h) \\ -\Lambda\sin(r\Lambda h) & \cos(r\Lambda h) \end{bmatrix}.$$

Likewise in the proof of Theorem 2, by using the Spectral Theorem, the first component x^1_{n+1} of $\mathbf{x}_{n+1}$ can be written

$$x^1_{n+1} = D_{n+1} + S_n, \tag{9}$$

where

$$D_{n+1} = \sum_{k=1}^{d} \left(P_{1k}\cos((n+1)h\lambda_k) < P_k, x_0 > + P_{1k}\lambda_k^{-1}\sin((n+1)h\lambda_k) < P_k, y_0 > \right)$$

and

$$S_n = \sum_{r=0}^{n} V_{nr}, \tag{10}$$

being

$$V_{nr}=\sum_{l=1}^{m}\sum_{j=1}^{d}\left(e_j^l\cos(r\lambda_j h)+(f_j^l\lambda_j^{-1})\sin(r\lambda_j h)\right)\Delta\mathbf{w}_{n-r}^l, \tag{11}$$

and $e_j^l,\ f_j^l\in\mathbb{R}$.

Without loss of generality, let us assume that $\lambda_k>0$ and $\lambda_k\neq\lambda_j$ for all $k\neq j$. Indeed, when there are only $d^*<d$ different values λ_j^* of $|\lambda_k|$, the expression (11) can be rewritten as

$$V_{nr}=\sum_{l=1}^{m}\sum_{j=1}^{d^*}\left(E_j^l\cos(r\lambda_j^* h)+(F_j^l(\lambda_j^*)^{-1})\sin(r\lambda_j h)\right)\Delta\mathbf{w}_{n-r}^l,$$

where $E_j^l=\sum_{k=1}^{d}e_k^l\delta_{\lambda_j^*}^{|\lambda_k|}(1_{\lambda_k>0}-1_{\lambda_k<0})$, $F_j^l=\sum_{k=1}^{d}f_k^l\delta_{\lambda_j^*}^{|\lambda_k|}(1_{\lambda_k>0}-1_{\lambda_k<0})$, and δ is the Kronecker delta. For this expression the analysis below would be the same.

Since $\Delta\mathbf{w}_{n-r}^l$ are independent Gaussian random variables with zero mean and variance h, $\{V_{nr}\}$ defines a sequence of independent Gaussian random variable with zero mean and variance

$$\sigma_{nr}^2=h\left(\sum_{l=1}^{m}\sum_{j=1}^{d}e_j^l\cos(r\lambda_j h)+(f_j^l\lambda_j^{-1})\sin(r\lambda_j h)\right)^2.$$

Thus, (10) can be rewritten as

$$S_n=\sum_{r=1}^{n}\sigma_{nr}\eta_r,$$

where $\eta_1,...,\eta_n$ are independent $\mathcal{N}(0,1)$ random variables.

On the other hand, let us compute

$$s_n^2=\sum_{r=0}^{n}\sigma_{nr}^2. \tag{12}$$

First note that, for all $j=1,...,d$,

$$\sum_{l=1}^{m}e_j^l\cos(r\lambda_j h)+\sum_{l=1}^{m}(f_j^l\lambda_j^{-1})\sin(r\lambda_j h)=c_j\cos\left(r\lambda_j h-\alpha_j\right),$$

where $c_j^2=\left(\sum_{l=1}^{m}e_j^l\right)^2+\left(\sum_{l=1}^{m}f_j^l\lambda_j^{-1}\right)^2$, $\alpha_j=\arctan\left(\sum_{l=1}^{m}f_j^l\lambda_j^{-1}/\sum_{l=1}^{m}e_j^l\right)$ for $\sum_{l=1}^{m}e_j^l\neq 0$, and $\alpha_j=\frac{\pi}{2}$ for $\sum_{l=1}^{m}e_j^l=0$. From this and by using the identity $\cos(\theta)=\left(\exp(i\theta)+\exp(-i\theta)\right)/2$, we obtain that

$$\begin{aligned}\sigma_{nr}^2&=h\left(\sum_{j=1}^{d}c_j\cos\left(r\lambda_j h-\alpha_j\right)\right)^2\\&=\frac{h}{2}\sum_{j,k=1}^{d}c_kc_j\operatorname{Re}\left\{\exp(ir(\lambda_k+\lambda_j)h)\exp(-i(\alpha_k+\alpha_j))+\exp(ir(\lambda_k-\lambda_j)h)\exp(i(\alpha_j-\alpha_k))\right\},\end{aligned} \tag{13}$$

where Re denotes the real part of a complex number. Under the assumption $h < \pi/\lambda^*_{\max}$, it holds that $\theta \neq 0 \bmod 2\pi$ for all $\theta = h(\lambda_j + \lambda_k)$ with $1 \leq j, k \leq d$, and for all $\theta = h(\lambda_k - \lambda_j)$ with $1 \leq j \neq k \leq d$. Therefore, from (13) and the known expression of the partial sum of the geometric series,

$$\sum_{r=1}^{n} \exp(ir\theta) = \begin{cases} n & \text{if } \theta = 0 \text{ mod } 2\pi \\ \frac{1-\exp(i\theta(n+1))}{1-\exp(i\theta)} - 1 & \text{otherwise} \end{cases},$$

it is obtained that

$$s_n^2 = n\frac{h}{2}\sum_{k=1}^{d} c_k^2 + C_n,$$

where C_n is uniformly bound for all n. Thus, the assumption $h < \pi/\lambda^*_{\max}$ implies that

$$\lim_{n\to\infty} \frac{s_n^2}{n} = \frac{h}{2}\sum_{k=1}^{d} c_k^2 > 0.$$

Since σ_{nr}^2 is bound for all n and r, the Law of the Iterated Logarithms stated in Lemma 1 holds for S_n. Thus, for $0 < \varepsilon < 1$, (2) implies that

$$S_n > (1-\varepsilon)\, 2s_n^2 \left(\log\log s_n^2\right) \quad \text{for infinitely many values of } n \text{ (a.s.)}.$$

In addition, since

$$|D_{n+1}| \leq |P|^2 \left(|x_0| + |y_0| \max_k \left\{\lambda_k^{-1}\right\}\right)$$

for all n, the fist component (4) of the solution of (1) satisfies

$$x_{n+1}^1 > 0 \text{ infinitely often as } n \to \infty \text{ (a.s.)}.$$

Similarly, (3) implies that

$$S_n < (-1+\varepsilon)\, 2s_n^2 \left(\log\log s_n^2\right) \text{ for infinitely many values of } n\text{(a.s.)}.$$

for $0 < \varepsilon < 1$, and so

$$x_{n+1}^1 < 0 \text{ infinitely often as } n \to \infty \text{ (a.s.)}.$$

Similarly we can proceed to prove that the other components of $\mathbf{x}_n$ also change sign infinitely often. This completes the proof. ■

It was shown in [3] that, likewise the exact solution of the simple harmonic oscillator (equation (1) with $d = 1$), the path of the Local Linearized integrator (8) switches signs infinitely many times as $n \to \infty$ almost surely for any integration stepsize h. However, according to Theorem 3, in the case of the coupled oscillator (1), this dynamics of the Local Linearized integrator (8) is only guaranteed for stepsizes $h < \pi/\max(|\lambda_1|, \ldots, |\lambda_d|)$, where $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of Λ.

Theorem 3 complements the results obtained in [3] that demonstrate the capability of the Local Linearized integrators for reproducing other essential dynamics of the coupled harmonic oscillators: the mean value, the linear growth of energy along the paths, and the symplectic structure of Hamiltonian oscillators.

In addition, it is worth to mention that, since the exponential and trigonometric integrators considered in [11] and [2] reduce to the expression (8) when they are applied to equation (1), the Theorem 3 can also be applied. In this way, these integrators with stepsize $h < \pi/\max(|\lambda_1|, \ldots, |\lambda_d|)$ also switch signs infinitely many times as $n \to \infty$ almost surely.

4 The infinitely many zeros of coupled nonlinear oscillators

In what follows, $|\cdot|$ denotes the Frobenious norm for vectors and matrices.

Lemma 4 *Let* $(x(t), y(t)) \in \mathbb{R}^{2d}$ *be the unique solution of the harmonic oscillator equation (1) on* $[0,T]$ *for any* $T > 0$. *Suppose that* $\Phi_t := \phi(x(t), y(t)) : \mathbb{R} \to \mathbb{R}^m$ *is a function satisfying the linear growth condition*

$$|\phi(x(t), y(t))| \leq K(1 + |x(t)| + |y(t)|). \tag{14}$$

Then, there is a probabilistic measure $\widetilde{\mathbb{P}}$ *on* $(\Omega, \mathfrak{F})$ *with the same null sets than* $\mathbb{P}$ *and an* m*-dimensional standard Wiener process* $\widetilde{\mathbf{w}}_t$ *on* $\left(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t \geq t_0}, \widetilde{\mathbb{P}}\right)$ *such that* $(x(t), y(t))$ *is also the unique solution of the nonlinear equation*

$$\begin{aligned} dx(t) &= y(t)dt \\ dy(t) &= \left(-\Lambda^2 x(t) + \Pi \Phi_t\right) dt + \Pi d\widetilde{\mathbf{w}}_t \end{aligned} \tag{15}$$

on $[0,T]$.

Proof. Let $\mathbf{x}_t = (x(t), y(t))$ be the solution of the equation (1) on $[0,T]$. From the condition (14) it follows that

$$|\Phi_t|^2 \leq C\left(1 + |\mathbf{x}_t|^2\right),$$

where $C = 4K^2$.

Since $\mathbf{x}_t$ is the solution of the linear SDE with additive noise (1), $\mathbf{x}_t \sim N_{2d}(\mu_t, \Sigma_t)$ for all $t \in [0,T]$, where the mean μ_t and the variance Σ_t of $\mathbf{x}_t$ are continuous functions on $[0,T]$ (see, e.g., **[1]**). Here, N_{2d} denotes $2d-$variate normal distribution. The random vector $\mathbf{x}_t$ can be written as $\mathbf{x}_t = \mu_t + \Sigma_t^{1/2} Z_t$, where $\Sigma_t^{1/2}$ is the symmetric squared root of Σ_t, and $Z_t \sim N_{2d}(\mathbf{0}, \mathbf{I})$. Therefore,

$$\begin{aligned} E\left(\exp\left(|\Phi_t|^2\right)\right) &\leq \exp(C)\, E\left(\exp\left(C\left|\mu_t + \Sigma_t^{1/2} Z_t\right|^2\right)\right) \\ &\leq \exp\left(C + 2C|\mu_t|^2\right) E\left(\exp\left(2C\left|\Sigma_t^{1/2}\right|^2 |Z_t|^2\right)\right). \end{aligned}$$

Since $|Z_t|^2$ is a random variable that has chi-squared distribution with $2d$ degrees of freedom, $E\left(\exp\left(\alpha |Z_t|^2\right)\right) \leq 1/(1-2\alpha)^{d/2}$ for $\alpha < 1/2$ ([6], pp. 420). Therefore, for all $a < 1/\left(8C \max_{t\in[0,T]} |\Sigma_t|\right)$, it holds that

$$\begin{aligned} E\left(\exp\left(a|\Phi_t|^2\right)\right) &\leq E\left(\exp\left(\frac{1}{4}|Z_t|^2\right)\right) \\ &\leq D 2^{d/2}, \end{aligned}$$

where $D = \exp\left(aC + 2aC \max_{t\in[0,T]} |\mu_t|^2\right)$. The proof is then completed as a direct consequence of the Cameron-Martin-Girsanov theorem (see, e.g., [8], pp. 274). ■

Let us consider the coupled nonlinear oscillator defined by the $2d$-dimensional ($d > 1$) SDE with additive noise

$$
\begin{aligned}
dx(t) &= y(t)dt \\
dy(t) &= -f(x(t), y(t))dt + \Pi d\widetilde{\mathbf{w}}_t,
\end{aligned} \tag{16}
$$

where $\Pi \in \mathbb{R}^{d\times m}$ is a matrix, $\widetilde{\mathbf{w}}_t$ is a m-dimensional standard Wiener process, and $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a smooth function satisfying the linear growth condition

$$
|f(x, y)| \leq K_1(1 + |x| + |y|) \tag{17}
$$

for some positive constant K_1.

Theorem 5 *Let $(x(t), y(t)) \in \mathbb{R}^{2d}$ be the unique solution of the harmonic oscillator equation (1) on $[0, T]$ for $T > 0$. Suppose that there is a function $\Phi_t := \phi(x(t), y(t)) : \mathbb{R} \to \mathbb{R}^m$ such that*

$$
\Pi\Phi_t = \Lambda^2 x(t) - f(x(t), y(t)), \tag{18}
$$

where the function f satisfies the linear growth condition (17). Then, there is a probabilistic measure $\widetilde{\mathbb{P}}$ on $(\Omega, \mathfrak{F})$ with the same null sets than $\mathbb{P}$ and an m-dimensional standard Wiener process $\widetilde{\mathbf{w}}_t$ on $\left(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq t_0}, \widetilde{\mathbb{P}}\right)$ such that $(x(t), y(t))$ is also the unique solution of the nonlinear oscillator equation (16) on $[0, T]$.

Proof. Since Φ_t solves the equation (18), $\Phi_t = \Pi^- \left(\Lambda^2 x(t) - f(x(t), y(t))\right)$, where the matrix Π^- is a generalized inverse of Π. This and condition (17) imply that Φ_t satisfies the linear growth condition (14). Then, the assumptions of Lemma 4 are fulfilled, which completes the proof. ■

Notice that the assumptions of Theorem 5 are directely fulfilled in the case that Π in (1) is a nonsingular $d \times d$ matrix.

The next theorem deals with the infinite oscillations of the paths of the coupled nonlinear oscillator (16).

Theorem 6 *Under condition of Lemma 4 (resp. Theorem 5), each component of the solution $(x(t), y(t))$ of the coupled nonlinear oscillator (15) (resp. (16)) has infinitely many zeros on $[t_0\ \infty)$ for every $t_0 \geq 0$ almost surely.*

Proof. Lemma 4 states that, for properties holding almost surely, the analysis of the nonlinear oscillator (15) with growth condition (14) reduces to that of the harmonic oscillator (1). In this way, since by Theorem 2 the harmonic oscillator (1) has infinitely many zeros on $[t_0\ \infty)$, the nonlinear oscillator (15) will also has infinitely many zeros on $[t_0\ \infty)$ for every $t_0 \geq 0$. Likewise, the proof for the nonlinear oscillators (16) can be derived. ■

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