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YAMABE PROBLEM ON RICCI SOLITONS, AND POISSON STRUCTURES ON SINGULAR FIBRATIONS

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To Morfeo

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Abstract

We study the Yamabe Problem in a special class of Riemannian manifolds, the Ricci solitons. We also explore contact and symplectic manifolds admitting a compatible Ricci soliton, where we obtain some observations. With respect to Poissons geometry, we provide local expressions for Poisson bivectors and the corresponding symplectic forms with broken Lefschetz and wrinkled singularities in dimensions 4, and 6, and discuss the higher dimension case.

With respect to the Yamabe Problem on compact Ricci solitons the main result that we obtained is:

Theorem *There exists a unique $U(2)$-invariant solution to the Yamabe equation on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ with the Koiso-Cao metric.*

We also explore the Ricci soliton equation. We use Hamiltonian and Liouville vector fields to derive some results concerning a Ricci soliton with a compatible symplectic structure. The main result obtained:

Theorem *Let (M,ω) be a symplectic manifold of dimension $n \geq 4$. If ω has compatible Ricci soliton g determined by a Hamiltonian or a Liouville holomorphic vector field, then g is Einstein.*

We also constructed singular Poisson structures on manifolds of dimension 4 and 6, where the singularties are given by a broken Lefschetz fibration or a wrinkled fibration. The main results are:

Theorem *A closed, orientable, smooth $4-$manifold equipped with a wrinkled fibration admits a complete Poisson structure. The fibres of the fibration are leaves of the symplectic foliation and both structures share the same singularities.*

Theorem *A generalized broken Lefschetz fibration admits a Poisson structure compatible with the fibration structure. Also, generalized wrinkled fibrations in dimension 6 admit compatible Poisson structures.*

Contents

Chapter 1

Introduction

A fundamental problem in geometry is the study of special distinguished structures on geometric manifolds. In particular to search for interesting metrics on a fixed manifold. There are different points of view to look at this problem.

The topology of a space creates restrictions on the geometry of metrics on it. For instance, in dimension 2, Gauss-Bonnet Theorem establishes a connection between the Euler characterisc and the Gaussian curvature of metrics on it: the integral of the curvature is independent of the metric. Also in dimension 2, the Uniformization Theorem states that in a given conformal class one can find a metric of constant curvature. For a proof see [51]. In higher dimiensions one would consider the scalar curvature, which is the average curvature of the metric at a point. In general, the integral of the scalar curvature is not constant, and it is known as the Hilbert-Einstein functional. It plays a fundamental role in studying the existence of constant scalar curvature metrics in a conformal class. This is known as the Yamabe Problem.

On the other hand, given a topological space we are interested in some special structure on it, then look for the most prevalent geometric structure in each dimension. For closed, orientable manifolds the existence of a symplectic form on a given manifold follows easily only in dimension 2, such structure always exists. In dimension 4 there are some known cases, but in higher dimension is more complicated, it is essentially unknown. A more general structure can be considered such as a Poisson structure. It can be regarded as a generalization of a symplectic structure. Every oriented 2-manifold admits a symplectic structure, and hence a Poisson structure. Every closed oriented 3-manifold admits a regular Poisson structure of rank 2 [33]. In dimension 4 only symplectic manifolds can have Poisson structures of rank 4. We will study singular Poisson manifolds in dimension 4 and 6 in Chapters 6 and 7.

This thesis is about these two topics: Yamabe Problem and Poisson structures. We study the Yamabe Problem in a special class of Riemannian manifolds, the Ricci solitons. We also explore contact and symplectic manifolds admitting a compatible Ricci soliton, where we obtain some observations. With respect to Poissons geometry, we provide local expressions for Poisson bivectors and the corresponding symplectic forms with broken Lefschetz and wrinkled singularities in dimensions 4 and 6, and discuss the higher dimensional case.

We now give a few more details on these subjects.

The Yamabe Problem

The Yamabe Problem consists in finding constant scalar curvature metrics in the conformal class of a given Riemannian manifold (M, g) of dimension n, $n \geq 3$. Consider the Hilbert–Einstein functional $\mathcal{S}$ defined in the space of Riemannian metrics on a smooth manifold M:

$$\mathcal{S}(g) = \frac{\int_M S_g dv_g}{\mathrm{Vol}(M, g)^{\frac{n-2}{n}}}$$

where dv_g is the volume element induced by g, $\mathrm{Vol}(M, g) = \int_M \cdot dv_g$ is the volume of (M, g) and S_g is the scalar curvature of (M, g). The power $\frac{n-2}{n}$ is chosen so that S_g is invariant by homothecies.

The Yamabe invariant is defined as:

$$Y(M) = \sup_{\{[g]\}} \inf_{h \in [g]} \mathcal{S}(h)$$

where $[g]$ is the conformal class of the Riemannian metric g.

Existence of constant scalar curvature metrics in a conformal class is equivalent to the existence of a smooth positive function f on M, and a constant λ such that the Yamabe equation holds:

$$-a_n \triangle_g f + S_g f = \lambda f^{p-1}$$

where $a_n = \frac{4(n-1)}{n-2}$, and $p = \frac{2n}{n-2}$.

If f satisfies the Yamabe equation, then $f^{p-2} \cdot g$ has constant scalar curvature with value $\lambda \in \mathbb{R}$.

On the other hand, if we write $\mathcal{S}(f^{p-2} \cdot g)$, we obtain:

$$\mathcal{Y}_g(f) := \mathcal{S}(f^{p-2} \cdot g) = \int_M \frac{(a_n |\nabla f|^2 + S_g f^2) dv_g}{||f||_p^2}.$$

The Euler-Lagrange equation of $\mathcal{Y}_g(f)$ is exactly the Yamabe equation, and therefore critical points of $\mathcal{S}|_{[g]}$ are metrics of constant scalar curvature in $[g]$.

The **Yamabe constant** of the conformal class $[g]$ is defined as:

$$Y(M, [g]) = \inf_{h \in [g]} \mathcal{Y}_g(h)$$

The Yamabe constant is finite and a fundamental theorem of Yamabe-Trudinger-Aubin-Schoen ([63, 58, 1, 50]) asserts that the infimum is always achieved. Then, the Yamabe equation has at least one solution: there exists at least one constant scalar curvature metric in every conformal class.

When $Y(M, [g]) \leq 0$, it is elementary to see that there is only one unit volume metric of constant scalar curvature in $[g]$. It is therefore a minimizer for $\mathcal{Y}_g$. For Einstein metrics different from the round metric on S^n, Obata's Theorem [43] gives uniqueness if the given Riemannian manifold is not the sphere with the standard metric. Hence, in this sense it is interesting to study uniqueness of the Yamabe problem in other more general situations. For this purpose we will focus on Ricci solitons. We would like to understand Yamabe metrics on the conformal class of non-Einstein Ricci solitons of positive scalar curvature.

Ricci solitons

A Ricci soliton is a n-dimensional Riemannian manifold (M, g) whose Ricci curvature satisfies the differential equation

$$-2Ric(g) = \mathcal{L}_X g + 2\mu g$$

for some complete vector field X and a scalar μ. Here $\mathcal{L}_X g$ is the Lie derivative of the metric in the direction of X.

The study of Ricci solitons has increased considerably in order to obtain a complete classification, and a better understanding of higher dimension geometry. There are few examples of non-trivial Ricci solitons. The only known compact non-trivial Ricci solitons are rotationally symmetric Kähler metrics. For real dimension 4 the first one was constructed by Koiso [36], and independently by Cao [10]. It is a non-Einstein shrinking soliton on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ with symmetry $U(2)$ and positive Ricci curvature. In fact, Cao give a family of homothetically gradient Kähler-Ricci solitons on $\mathbb{C}^n$ which includes the Hamilton soliton in case $n = 1$. The other example in dimension 4 was found by Wang-Zhu [61]. They proved the existence of a gradient Kähler-Ricci soliton on $\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$ with $U(1) \times U(1)$ symmetry.

Compact homogeneous Ricci solitons are Einstein. Thus, next case to study are cohomogeneity one Ricci solitons, studied for instance by Dancer, Hall and Wang [19], [20]. The only known $4-$dimensional non-trivial metric of cohomogeneity one is the example, mentioned above, constructed by Koiso and by Cao on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$.

With respect to the Yamabe Problem on compact Ricci solitons the main result that we obtained is [57]:

Theorem *There exists a unique $U(2)$-invariant solution to the Yamabe equation on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ with the Koiso-Cao metric.*

Contact and Symplectic solitons

One may plug into the Ricci soliton equation different vector fields. For closed manifolds, if the associated scalar is positive or zero, a soliton must be Einstein. But in general, there is no special behavior of the soliton for a fixed vector field. In order to explore other directions we discuss an idea about contact manifolds admitting a soliton [12]. Cho defines a contact soliton as a contact manifold with a metric structure given by a Ricci soliton. Cho studied Ricci solitons determined by the Reeb vector field. In dimension 3 Cho proved that any contact soliton has constant curvature, and that any contact Ricci soliton is a shrinking soliton. He also introduced transversal solitons. Given a contact soliton, those solitons are defined by orthogonal vector fields to the Reeb field, where he proved a rigidity result.

These ideas are explored here for a symplectic structure, and carried to extend Cho's results to higher dimensions. In the case of symplectic manifolds we use Hamiltonian and Liouville vector fields. On the other hand, Bowden-Crowley-Stipsicz proved that if M is a contact manifold, then $M \times S^2$ also admits a contact structure. Then we have obtained some results regarding M a contact soliton and S^2 as a shrinking soliton. Our contributions here are resumed in the following theorems.

Theorem *Let (M, ω) be a symplectic manifold of dimension $n \geq 4$. Any Ricci soliton g on M compatible with ω, determined by a Hamiltonian or a Liouville vector field is an Einstein metric.*

In the statement we consider the vector fields to be holomorphic.

There is an easy way to construct a symplectic manifold from a contact manifold, it is known as *symplectization*. In this case we obtained:

Theorem *The symplectization of a contact soliton manifold admits a shrinking Ricci soliton.*

Poisson structures

The study of smooth manifolds of dimension 4 has led to various interesting types of fibrations. Donaldson established a correspondence between Lefschetz pencils and symplectic 4-manifolds [22]. Since then, Lefschetz fibrations and their generalizations have been vital in symplectic geometry. These are maps to the 2-sphere with a finite number of isolated singular points where the rank of the derivative is zero.

In 2005, Auroux, Donaldson, and Katzarkov generalized this approach, introducing what is now known as a *broken Lefschetz fibration* [3]. There is an additional component in the singularity set of broken Lefschetz fibrations, a 1-submanifold of indefinite folds. Recently, near-symplectic structures and generalized broken Lefschetz fibrations have been studied in higher dimensions [60].

By a stable map it is understood one such that any nearby map in the space of smooth mappings can be perturbed to the original map after a change of coordinates in the domain and codomain. Broken Lefschetz fibrations are not stable. Lekili showed that the unstable Lefschetz singularities of a broken Lefschetz fibration can be substituted by cusps, then a stable map is obtained, with only folds and cusps as elements of its critical set [39]. These mappings are known as *wrinkled fibrations*.

A Poisson manifold is a pair $(M, \{\cdot, \cdot\})$, where M is a smooth manifold and $\{\cdot, \cdot\}$ is a bracket that defines a Lie algebra on $C^\infty(M)$ and it is a derivation in each factor:

$$\{gh, k\} = g\{h, k\} + h\{g, k\}$$

for any $g, h, k \in C^\infty(M)$.

The Symplectic Stratification Theorem states that a Poisson manifold can be decomposed into a disjoint union of symplectic manifolds. The symplectic form is the restriction of the Poisson bracket to each leaf [23].

Poisson geometry has been important, in particular in dimension 4. In [25] Suárez-Serrato, García-Naranjo and Vera exhibited a Poisson structure whose symplectic leaves coincide with the fibres of a broken Lefschetz fibration, and the singular sets of both structures coincide. The proof also implies that on any homotopy class of maps from a 4-manifold to S^2 there is such a singular Poisson structure. In a collaboration work with Suárez-Serrato [54] we showed:

Theorem *A closed, orientable, smooth* $4-$*manifold equipped with a wrinkled fibration admits a complete Poisson structure. The fibres of the fibration are leaves of the symplectic foliation and both structures share the same singularities.*

On the other hand, in collaboration with Suárez-Serrato and Vera [55] we studied higher dimensionnal broken Lefschetz fibrations, we define a generalized wrinkled fibration where we obtained a similar result.

Theorem *A generalized broken Lefschetz fibration admits a Poisson structure compatible with the fibration structure. Also, generalized wrinkled fibrations in dimension* 6 *admit compatible Poisson structures.*

This thesis is organized as follows. In Chapter 2 we introduce the background needed to follow each part of the thesis. Section 2.1 contains notation and results concerning the curvature of a Riemannian manifold. In Section 2.2 we show that the critical points of the Hilbert-Einstein functional are the Einstein metrics. The Yamabe constant is introduced in Section 2.3, where we discuss the uniqueness of solutions to the Yamabe equation. Obata's Theorem is presented here. We dedicate Section 2.5 to the basics facts about Poisson geometry, since one part of the work is about of the construction of Poisson manifolds. Following the definition and the Symplectic Stratification Theorem, we give the most known criterion used to decide when a Poisson manifold is linearizable or integrable. At the end of the chapter we present Broken Lefschetz and Wrinkled fibrations. These fibrations are the setting where singular Poisson structures are constructed in Chapters 6 and 7.

We give the definition of a Ricci soliton in Chapter 3, and review some properties and examples. In Chapter 4 we present some preliminary results on symplectic and contact manifolds with a compatible Ricci soliton. We prove that solitons generated by Hamiltonian or Liouville vector fields are Einstein metrics. Also we extend some results of Cho

([12, 13]) of contact solitons to certain Riemannian products.

The Chapter 5 is dedicated to prove the uniqueness of invariant solutions to the Yamabe equation on the Koiso-Cao soliton. The proof is given in detail in Section 5.3. Sections 5.1, 5.2 contain a different approach of the construction of the Koiso-Cao soliton, and we prove that the scalar curvature is a decreasing function on the orbit space. Also we were able to prove that it has positive Ricci curvature with our coordinates.

In Chapter 6 we provide a local formulæ for Poisson bivectors and symplectic forms on the leaves of Poisson structures associated to wrinkled fibrations on smooth 4–manifolds [54]. Following the construction given in [25] we use the Flaschka-Ratiu formula to produce a Poisson bracket with prescribed Casimirs. We use the coordinates of the local expression of each singularity as a Casimir for the Poisson structure that we want to construct. The singularities coincide precisely with those of a wrinkled fibration. The Poisson structure approaches zero near the singularities, so the corresponding symplectic forms approach infinity.

In Chapter 7 we show that generalized broken Lefschetz fibrations in arbitrary dimensions admit rank-2 Poisson structures. After extending the notion of wrinkled fibration to dimension 6 we prove that these wrinkled fibrations also admit compatible Poisson structures. We also discuss the case for other dimensions, where we describe a general procedure to construct similar Poisson structures, and their corresponding symplectic forms. Then we provide local expressions for each generalized fibration and their corresponding Lekili move. We discuss the linearization and integrability of the structures constructed.

Chapter 2

Preliminaries

2.1 Curvature

Let (M, g) be a Riemannian manifold of dimension n. We will denote by $\chi(M)$ the space of smooth vector fields on M. Let ∇ be the Levi-Civita connection of (M, g), the unique torsion-free connection, that is:

$$[Y, Z] = \nabla_Y Z - \nabla_Z Y,$$

which is compatible with g:

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(\nabla_Z Y, X).$$

It is is determined by the Koszul formula:

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y) \tag{2.1.1}$$

for every $X, Y, Z \in \chi(M)$.

***Definition* 2.1.1** *A* tensor T *of order* k *is a* $k-$*multilinear mapping over* $C^\infty(M)$*:*

$$T : \underbrace{\chi(M) \times \ldots \chi(M)}_{k\text{-}times} \to C^\infty(M).$$

The Levi-Civita connection gives a covariant derivative for tensors. Let T be a tensor of order k. The covariant derivative ∇T of T is a tensor of order $(k+1)$ given by the formula:

$$\begin{aligned}\nabla T(X_1, \ldots, X_k, X_{k+1}) =& X_{k+1}(T(X_1, \ldots, X_k)) - T(\nabla_{X_{k+1}} X_1, \ldots, X_k) - \ldots \\ & - T(X_1, \ldots, \nabla_{X_{k+1}} X_k)\end{aligned}$$

***Definition* 2.1.2** *Let* $f \in C^\infty(M)$*, the* Hessian *of* f *is defined by:*

$$\nabla(\nabla f) = Hess(f).$$

That is:

$$\begin{aligned} Hess(f)(X, Y) &= Y(\nabla f(X)) - \nabla f(\nabla_Y X) \\ &= Y(X(f)) - (\nabla_Y X)(f). \end{aligned}$$

Note that the Hessian is symmetric, since ∇ is a free-torsion connection.

***Definition* 2.1.3** *The* Laplacian *of a smooth function f associated to g is:*

$$\triangle_g(f) = -trace(Hess(f))$$

If $\{e_i\}_{i=1}^n$ is an orthonormal basis of T_pM, we extend it to a orthonormal frame $\{E_i\}_{i=1}^n$, and we obtain:

$$\triangle_g f(p) = -\left(\sum_{i=1}^n E_i(E_i(f)) - (\nabla_{E_i} E_i)f\right)(p)$$

***Definition* 2.1.4** *The* curvature tensor

$$R : \chi(M) \times \chi(M) \times \chi(M) \to \chi(M)$$

is defined by the formula:

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

for every X, Y, Z in $\chi(M)$.

If T is another vector field on M, the Riemann curvature tensor $\mathbf{R}$ is defined by:

$$\mathbf{R}(X,Y,Z,T) := g(R(X,Y)Z,T)$$

Let X, Y, Z, T, W be vector fields. The Riemann tensor satisfies the following properties:

1. $\mathbf{R}(X,Y,Z,T) = -\mathbf{R}(Y,X,Z,T) = -\mathbf{R}(X,Y,T,Z)$
2. $\mathbf{R}(X,Y,Z,T) = \mathbf{R}(Z,T,X,Y)$
3. First Bianchi Identity:
$$\mathbf{R}(X,Y,Z,T) + \mathbf{R}(Y,Z,X,T) + \mathbf{R}(Z,X,Y,T) = 0$$
4. Second Bianchi Identity:
$$\nabla\mathbf{R}(X,Y,Z,W,T) + \nabla\mathbf{R}(X,Y,W,T,Z) + \nabla\mathbf{R}(X,Y,T,Z,W) = 0$$

The **Ricci curvature**, Ric, is defined as the trace of the curvature tensor:

$$Ric(X,Y) = \sum_{i=1}^n g(R(X,e_i)Y,e_i).$$

The **scalar curvature**, S_g, is the trace of the Ricci tensor:

$$S_g = \sum_{j=1}^n Ric(e_j,e_j).$$

***Definition* 2.1.5** *A Riemannian manifold (M, g) is* Einstein *if it has constant Ricci curvature:*

$$Ric(g) = \lambda g$$

for some scalar $\lambda \in \mathbb{R}$.

If we take take traces we see that an Einstein metric has constanct scalar curvature

$$S_g = n\lambda.$$

***Definition* 2.1.6** *The* trace-free Ricci tensor *is the symmetric tensor of order two of zero trace*

$$T := Ric(g) - \frac{1}{n} S \cdot g.$$

If (U, φ) is a chart, we have a coordinate system $\{x_1, ..., x_n\}$ in $p \in M$. Denote by g_{ij} the components of the metric g, and g^{ij} the components of the inverse matrix g^{-1}. The vectors $\{\frac{\partial}{\partial x_j}|_p\}_{j=1}^n$ form a basis of the tangent space T_pM at the point $p \in M$. Then we have the local expression of the Christoffel symbols of the Levi-Civita connection:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

Thus the curvature tensor in coordinates is described by:

$$R_{ijk}^l = \frac{\partial}{\partial x_j} \Gamma_{ik}^l - \frac{\partial}{\partial x_i} \Gamma_{jk}^l - \Gamma_{ik}^s \Gamma_{js}^l - \Gamma_{jk}^s \Gamma_{is}^l.$$

The Riemann curvature tensor is expressed:

$$R_{ijkl} = R_{ijk}^m g_{lm}.$$

Contracting in the last expression, we obtain the Ricci tensor:

$$R_{ik} = R_{ijkl} g^{jl} = R_{ijk}^j.$$

Hence, the scalar curvature is:

$$S_g = R_{ij} g^{ij}.$$

Note that for a $2-$dimensional Riemannian manifold $T = 0$, but S_g is not necessarily constant. In this case g is Einstein if and only if S_g is constant.

We point out that Einstein summation convention is used to simplify notation of local expressions. We also denote: $\nabla_k := \nabla_{\frac{\partial}{\partial x_k}}(\cdot)$.

***Proposition* 2.1.7** *Let (M, g) be a Riemannian manifold of dimension $n \geq 3$ such that $T = 0$. Then g is Einstein (i.e., S_g is constant)*

Proof: The Second Bianchi identity in local coordinates reads as follows:

$$\nabla_m R_{ijkl} + \nabla_k R_{ijlm} + \nabla_l R_{ijmk} = 0.$$

Contracting in the indices j and m, i.e. multiplying by g^{jm}:

$$g^{jm}(\nabla_m R_{ijkl} + \nabla_k R_{ijlm} + \nabla_l R_{ijmk}) = 0.$$

Since covariant derivatives and contractions commute:

$$g^{jm}\nabla_m R_{ijkl} + \nabla_k g^{jm} R_{ijlm} + \nabla_l g^{jm} R_{ijmk} = 0.$$

From the symmetries of the Riemann tensor, the expression becomes:

$$\begin{aligned} g^{jm}\nabla_m R_{ijkl} + \nabla_k g^{jm} R_{ijlm} - \nabla_l g^{jm} R_{ijkm} &= 0 \\ g^{jm}\nabla_m R_{ijkl} + \nabla_k R_{il} - \nabla_l R_{ik} &= 0 \end{aligned}$$

Contracting in the indices i, l we obtain:

$$\begin{aligned} g^{jm}\nabla_m g^{il} R_{ijkl} + \nabla_k g^{il} R_{il} - \nabla_l g^{il} R_{ik} &= 0 \\ -g^{jm}\nabla_m g^{il} R_{jikl} + \nabla_k g^{il} R_{il} - \nabla_l g^{il} R_{ik} &= 0 \\ -g^{jm}\nabla_m R_{jk} + \nabla_k S_g - g^{il}\nabla_l R_{ik} &= 0. \end{aligned}$$

Since the first and third term are equal we have that:

$$\nabla_k S_g = 2g^{ik}\nabla_k R_{il}. \tag{2.1.2}$$

In local coordinates the fact that T vanishes is given by:

$$nR_{il} = S_g g_{il}.$$

We take covariant derivative and use the compatibility of ∇, that is $\nabla g = 0$:

$$n\nabla_k R_{il} = \nabla_k S_g g_{il}.$$

Contracting:

$$ng^{ik}\nabla R_{il} = \nabla_k S_g.$$

Hence, if $T = 0$, replacing it in the last equation and using again that $\nabla g = 0$:

$$\nabla_k S_g = \frac{2}{n} g^{ik} g_{il} \nabla_k S_g.$$

In consequence:

$$\nabla_k S_g = \frac{2}{n}\nabla_k S.$$

Thus, for $n \neq 2$, S is constant. Therefore g has constant Ricci curvature, and it is Einstein. □

The converse is also true, it follows from definition of the tensor T.

2.2 The Hilbert-Einstein functional

***Definition* 2.2.1** *Let M be a smooth compact oriented manifold of dimension n. The* Hilbert-Einstein functional *in the space of Riemannian metrics is defined as:*

$$g \to \mathcal{S}(g) = \frac{\int_M S_g dv_g}{\mathrm{Vol}(M,g)^{\frac{n-2}{n}}}. \tag{2.2.1}$$

The term $\mathrm{Vol}(M,g)^{\frac{n-2}{n}}$ is used to normalize.

We will assume $n \geq 3$. Note that when $n = 2$, $\mathcal{S}(g)$ is constant, by Gauss-Bonnet Theorem.

The critical points of $\mathcal{S}$ are the Einstein metrics on M. To see this, let (M,g) be a fixed Riemannian metric. Normalize it so that $\mathrm{Vol}(M,g) = 1$. Then we consider a variation of metrics g_t. If we write $g_t := g + th$, for any bilinear symmetric form h, we will compute $\partial_t \mathcal{S} := \frac{\partial}{\partial t}\mathcal{S}(g_t)|_{t=0}$.

Recall that the first variation of the volume is:

$$\partial_t \mathrm{Vol}(M, g_t) = \int_M \frac{1}{2} tr_g(h) d\omega_g.$$

Since we restricted to the class of unitary volume metrics, we regard volume preserving variations, that is, $\partial_t \mathrm{Vol}(M,g_t)|_{t=0} = 0$. Then the volume is preserved if and only if $\int_M tr_g(h) = 0$.

On the other hand, if we have a normal coordinate system around a point $p \in M$, the Christoffel symbols vanish at p and their first variation is:

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma^k_{ij}|_{t=0} &= \frac{\partial}{\partial t}|_{t=0}\left(\frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})\right) \\
&= \frac{1}{2}g^{kl}\frac{\partial}{\partial t}|_{t=0}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) + \frac{1}{2}\left(\frac{\partial}{\partial t}|_{t=0}g^{kl}\right)\Gamma^k_{ij} \\
&= \frac{1}{2}g^{kl}(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}).
\end{aligned}$$

For variation of the inverse of the metric g^{-1} we compute:

$$\begin{aligned}
0 = \partial_t(\delta_{ij}) &= \partial_t(g_t^{ij}(g_t)_{ij}) \\
&= g^{ij}h_{ij} + \frac{\partial}{\partial t}(g_t^{ij})g_{ij}.
\end{aligned}$$

Thus, we obtain:

$$\partial_t(g_t^{ij}) = -g^{ij}h_{ij}g^{ij}.$$

The curvature tensor is $R^l_{ijk} = \partial_j \Gamma^l_{ik} - \partial_i \Gamma^l_{jk}$, then its first variation is:

$$\begin{aligned}
\partial_t R^l_{ijk} = \frac{1}{2}g^{lm}\,&(\nabla_j\nabla_k h_{im} - \nabla_j\nabla_m h_{ik} - \nabla_i\nabla_k h_{jm} \\
&+ \nabla_i\nabla_m h_{jk} + \nabla_j\nabla_i h_{km} - \nabla_i\nabla_j h_{km}).
\end{aligned}$$

The Ricci tensor in local coordinates is given by:

$$R_{ij} = \partial_l \Gamma^l_{ij} - \partial_i \Gamma^l_{lj} = R^l_{ilj}.$$

Hence, from the curvature tensor variation we have:

$$\begin{aligned} \partial_t R_{ij} &= \frac{1}{2} g^{jm} \left(\nabla_j \nabla_k h_{im} - \nabla_j \nabla_m h_{ik} - \nabla_i \nabla_k h_{jm} \right. \\ &\qquad \left. + \nabla_i \nabla_m h_{jk} + \nabla_j \nabla_i h_{km} - \nabla_i \nabla_j h_{km} \right) \\ &= \frac{1}{2} \left(\nabla_j \nabla_k g^{jm} h_{im} - \triangle_g h_{ik} - \nabla_i \nabla_k g^{jm} h_{jm} \right. \\ &\qquad \left. + \nabla_i \nabla_m g^{jm} h_{jk} + \nabla_j \nabla_i g^{jm} h_{km} - \nabla_i \nabla_j g^{jm} h_{km} \right). \end{aligned}$$

Since $S_g = g^{ij} R_{ij}$, we obtain that its first variation is

$$\partial_t S|_{t=0} = -g^{ij} h_{ij} g^{ij} + g^{ij} \partial_t R_{ij},$$

and note that $g^{ij}\partial_t R_{ij}$ vanishes after integration on M, since all terms involved are of divergence type.

Therefore, all previous calculations give the first variation of the Hilbert-Einstein functional:

$$\begin{aligned} \partial_t \mathcal{S} = \frac{\partial}{\partial t} \left(\int_M S_{g_t} d\omega_{g_t} \right) \Big|_{t=0} &= \left(- \int_M g_t(h, Ric(g_t)) d\omega_{g_t} + \frac{1}{2} \int_M S_{g_t} tr_{g_t}(h) d\omega_g \right) \Big|_{t=0} \\ &= \left(- \int_M g_t(h, Ric(g_t) - \frac{1}{2} S_{g_t} g_t) d\omega_{g_t} \right) \Big|_{t=0} \\ &= - \int_M g \left(h, Ric(g) - \frac{1}{2} S_g \right) d\omega_{g_t}. \end{aligned}$$

Hence, a Riemannian metric g is a critical point of $\mathcal{S}$ if

$$\int_M g \left(h, Ric(g) - \frac{1}{2} S_g \right) d\omega_g = 0$$

for every h such that $\displaystyle\int_M tr_g(h) = 0$.

For a Riemannian metric g, set $V(t) := \mathrm{Vol}(M, g + th)$, then the normalized family $\widetilde{g_t} = V(t)^{-\frac{2}{n}} g_t$ satisfies $\mathrm{Vol}(M, \widetilde{g_t}) = 1$. The scalar curvature of $\widetilde{g_t}$ is $S_{\widetilde{g_t}} = V(t)^{\frac{n}{2}} S_{g_t}$, then:

$$\begin{aligned} \mathcal{S}(\widetilde{g_t}) &= \int_M S_{\widetilde{g_t}} d\omega_{\widetilde{g_t}} \\ &= V(t)^{\frac{2-n}{n}} \int_M S_{g_t} d\omega_{g_t}. \end{aligned}$$

As before, we obtain:

$$\begin{aligned} \frac{\partial}{\partial t} V(t) &= \frac{1}{2} \int_M tr_{g_t}(h) d\omega_{g_t} \\ &= \frac{1}{2} \int_M g_t(h, g_t) d\omega_{g_t}. \end{aligned}$$

It follows that:

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{S}(\widetilde{g}_t) &= V(t)^{\frac{2-n}{n}}\frac{\partial}{\partial t}\int_M S_{g_t}d\omega_{g_t} + \frac{2-n}{2n}V(t)^{\frac{2-2n}{n}}\left(\int_M g_t(h,g_t)d\omega_{g_t}\right)\mathcal{S}(g_t) \\
&= -V(t)^{\frac{2-n}{n}}\int_M g_t(h, Ric(g_t) - \frac{1}{2}S_{g_t}g_t)d\omega_{g_t} \\
&\quad +\frac{2-n}{2n}V(t)^{\frac{2-2n}{n}}\left(\int_M g_t(h,g_t)d\omega_{g_t}\right)\mathcal{S}(g_t) \\
&= -V(t)^{\frac{2-n}{n}}\int_M g_t(h, Ric(g_t) - \frac{1}{2}S_{g_t}g_t + \frac{n-2}{2n}\mathcal{S}(g_t)g_t)d\omega_{g_t}.
\end{aligned}$$

Therefore, a metric g with $\mathrm{Vol}(M,g)=1$ is a critical point of $\mathcal{S}$ if and only if:

$$Ric(g) - \frac{1}{2}S_g g + \frac{n-2}{2n}\mathcal{S}(g)g = 0$$

If we take traces in both sides,

$$S_g - \frac{n}{2}S_g + \frac{n-2}{2}\mathcal{S}(g) = 0,$$

and then:

$$S_g = \mathcal{S}(g).$$

Then, g with $\mathrm{Vol}(M,g)=1$, is a critical point of $\mathcal{S}$ if and only if:

$$\begin{aligned}
Ric(g) - \frac{1}{2}S_g g + \frac{n-2}{2n}S_g &= Ric(g) - \frac{1}{n}S_g g \\
&= T(g) = 0.
\end{aligned}$$

***Proposition* 2.2.2** *Let M be a closed oriented Riemannian manifold. A metric g is a critical point of $\mathcal{S}$, restricted to metrics with same volume as g, if and only if g is Einstein.*

2.3 Yamabe Problem

Two metrics g, h on a smooth manifold M are said to be conformal if there exists a smooth positive function $f : M \to \mathbb{R}_{>0}$ such that

$$h = f \cdot g$$

If (M,g) is a Riemannian manifold, the conformal class $[g]$ of the metric g is the family of metrics conformal to g:

$$[g] = \{f \cdot g | f \in C^\infty(M), f > 0\} = \{e^{2u}g | u \in C^\infty(M)\}.$$

A well-known problem in Riemannian geometry is to find metrics $\widetilde{g}$ of constant scalar curvature in a given conformal class $[g]$. Existence of such a metric $\widetilde{g}$ is known as the **Yamabe Problem**. It was first considered by Yamabe in [63]. Actually Yamabe claims to have solved the problem but his proof contained a mistake. The mistake was fixed in a sequel of articles by Trudinger [58], Aubin [1] and Schoen [50].

Let (M, g) be a Riemannian manifold of dimension $n \geq 3$. If we write $\widetilde{g} = f^{p-2} \cdot g$, with $p = \frac{2n}{n-2}$. Let S_g and $S_{\widetilde{g}}$ be the scalar curvatures of g and $\widetilde{g}$ respectively, then they satisfy the following relation:

$$-\frac{4(n-1)}{n-2}\triangle_g f + S_g f = S_{\widetilde{g}} f^{p-1}.$$

Hence $\widetilde{g}$ has constant scalar curvature λ if and only if f satisfies the ***Yamabe equation:***

$$-a_n \triangle_g f + S_g f = \lambda f^{p-1} \tag{2.3.1}$$

where $a_n = \frac{4(n-1)}{n-2}$.

This says that we have an equivalent PDE formulation for the Yamabe Problem, which leads to study existence and uniqueness of solutions to the Yamabe equation.

2.3.1 Yamabe constant

Let (M, g) be a closed Riemannian manifold of dimension n, with $n \geq 3$, and $p = \frac{2n}{n-2}$. Then $dv_{f^{p-2} \cdot g} = f^{\frac{n}{2}(p-2)}\, dv_g = f^p\, dv_g$.

If we restrict the Hilbert-Einstein functional 2.2.1 to a conformal class we obtain:

$$\begin{aligned}
\mathcal{S}(f^{p-2} \cdot g) &= \frac{\int_M S_{\widetilde{g}}\, dv_{\widetilde{g}}}{(\mathrm{Vol}(M, \widetilde{g}))^{\frac{n-2}{n}}} \\
&= \frac{\int_M f^{1-p}(-a_n \triangle_g f + S_g f) f^p\, dv_g}{(\int_M f^p\, dv_g)^{\frac{2}{p}}} \\
&= \frac{\int_M f(-a_n \triangle_g f + S_g f)\, dv_g}{||f||_p^2} \\
&= \frac{(\int_M a_n |\nabla f|^2 + S_g f^2)\, dv_g}{||f||_p^2}.
\end{aligned}$$

Definition 2.3.1 *The* Yamabe functional *of a Riemannian manifold (M, g) of dimension $n \geq 3$ is defined as:*

$$\mathcal{Y}_g(f) = \frac{(\int_M a_n |\nabla f|^2 + S_g f^2)\, dv_g}{||f||_p^2}. \tag{2.3.2}$$

If f is a smooth positive function and $\varphi \in C^\infty(M)$,

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{Y}_g(f + t\varphi)\Big|_{t=0} &= \frac{\partial}{\partial t} \frac{\int_M (a_n |\nabla(f + t\varphi)|^2 + S_g \cdot (f + t\varphi)^2)\, dv_g}{(\int_M (f + t\varphi)^p dv_g)^{\frac{2}{p}}}\Bigg|_{t=0} \\
&= \int_M (2a_n g(\nabla f, \nabla \varphi) + 2S_g f \varphi) dv_g \\
&\quad - \int_M (a_n |\nabla f|^2 + S_g f^2) dv_g \left(\int_M f^p\right)^{\frac{2}{p}-1} \cdot \int_M f^{p-1}\varphi.
\end{aligned}$$

It follows that a function $f \in C_+^\infty(M)$ is a critical point of $\mathcal{Y}_g$ if and only if:

$$-a_n \triangle_g f + S_g f = \frac{\mathcal{Y}_g(f)}{||f||_p^{p-2}} f^{p-1}.$$

Therefore, a function $f \in C_+^\infty(M)$ is a critical point of $\mathcal{Y}_g$ if and only if f satisfies the Yamabe equation (2.3.1). That is, if and only if $\widetilde{g} = f^{p-2} \cdot g$ has constant scalar curvature λ, with $\lambda = \frac{\mathcal{Y}_g(f)}{||f||_p^{p-2}}$. That means, the critical points of $\mathcal{S}|_{[g]}$ are the metrics of constant scalar curvature.

***Proposition* 2.3.2** *The Yamabe functional is bounded from below.*

Proof: We have that:

$$\begin{aligned} \mathcal{Y}_g(f) &\geq \frac{\int_M S_g f^2 dv_g}{||f||_p^2} \\ &\geq (\inf_M S_g) \frac{||f||_2^2}{||f||_p^2}. \end{aligned}$$

If $S_g \geq 0$, then $\mathcal{Y}_g(f) \geq 0$. If $\inf\limits_M S_g < 0$, we write $f^2 = f^2 \cdot 1$, and then $\frac{2}{p} + \frac{2}{n} = 1$. Thus, by the Hölder inequality:

$$||f||_2^2 \leq ||f^2||_{\frac{p}{2}} ||1||_{\frac{n}{2}} = ||f||_p^2 \ \ (\mathrm{Vol}(M,g))^{\frac{2}{n}}. \tag{2.3.3}$$

Thus, we obtain

$$\mathcal{Y}_g(f) \geq (\inf_M S_g)(\mathrm{Vol}(M,g))^{\frac{2}{n}}.$$

Therefore $\mathcal{Y}_g(f)$ is bounded from below. □

The last proposition allows to define:

***Definition* 2.3.3** *The* Yamabe constant *of a conformal class* $[g]$ *is:*

$$Y(M,[g]) = \inf_{h \in [g]} \mathcal{S}(h).$$

*If $h \in [g]$ satisfies $\mathcal{S}(h) = Y(M,[g])$, i.e. h realizes the infimum, then h has constant scalar curvature. These metrics are called **Yamabe metrics**.*

Note that the infimum can also be computed over all positive smooth functions on M,

$$Y(M,[g]) = \inf_{f \in C_+^\infty(M)} \mathcal{Y}_g(f).$$

T. Aubin [1] proved that the Yamabe constant of the conformal class of the round metric of the sphere is an upper bound for the Yamabe constant. Formally:

***Theorem* 2.3.4** *Let (M,g) be a n-dimensional closed Riemannian manifold of dimension $n \geq 3$. Then:*

$$Y(M,[g]) \leq Y_n.$$

Where Y_n is the Yamabe constant of (S^n, g_o^n), the sphere with the standard metric, $Y_n := Y(S^n,[g_o^n]) = n(n-1)\mathrm{Vol}(S^n, g_o^n)$.

Proof: Let $\pi : S^n \to \mathbb{R}^n$ be the stereographic projection. We consider the functions:

$$h_\lambda(x) = \left(\frac{2\lambda}{\lambda^2||x||^2+1}\right)^{\frac{n-2}{2}}.$$

Let $V_n := \mathrm{Vol}(S^n, g_o^n)$. Since $(h_\lambda)^{\frac{4}{n-2}}\langle,\rangle$ is isometric to g_o^n, it follows that:

$$\int_{\mathbb{R}^n}\left((h_\lambda)^{\frac{4}{n-2}}\right)^{\frac{n}{2}} dx = ||h_\lambda||_p^p = V_n.$$

Consider in $\mathbb{R}^n$ a ball of radius $\varepsilon > 0$, B_ε, and a dilation by λ, for λ big enough such that the complement of its image under π^{-1} in S^n is small enough. Hence,

$$\int_{B_\varepsilon}(h_\lambda)^p = V_n - \delta$$

for a small δ.

For λ fixed, we note that:

$$\int_{\mathbb{R}^n} h_\lambda^2 dx \sim \int_{||x||>>1}(\frac{1}{||x||^2})^{n-2}dx \sim \int_{||x||>>1}||x||^{4-2n}.$$

If $n = 3, 4$ the integral does not exist. Then, for $n > 4$:

$$\begin{aligned}\int_{\mathbb{R}^n} h_\lambda^2 dx &\sim \int_{b>>1}^{\infty} r^{n-1}r^{4-2n}dr\\ &= \int_{b>>1}^{\infty} r^{3-n}dr\\ &= \left.\frac{r^{4-n}}{4-n}\right|_{b>>1}^{\infty}\end{aligned}$$

and the limit goes to zero as $r \to \infty$.

Therefore, we are interested in computing $\int_{B_\varepsilon} h_\lambda^2$ for a small ε. This is the same as a big dilation.

Then for λ big enough:

$$\begin{aligned}\int_{B_{\frac{1}{\lambda}}}\left(\frac{2/\lambda}{||x||^2+1/\lambda^2}\right)^{n-2}dx &\leq \left(\frac{2}{\lambda}\right)^{n-2}\int_{B_{\frac{1}{\lambda}}}\left(\frac{1}{1/\lambda^2}\right)^{n-2}dx\\ &= 2^{n-2}\lambda^{n-2}\left(\frac{1}{\lambda}\right)^{n}\mathrm{Vol}(B^{n-1},\langle,\rangle) \sim \frac{1}{\lambda^2}\mathrm{Vol}(B^{n-1},\langle,\rangle).\end{aligned}$$

The limit goes to zero, as $\lambda \to \infty$.

On the other hand we have,

$$\begin{aligned}
\int_{B_\varepsilon - B_{\frac{1}{\lambda}}} \left(\frac{2/\lambda}{||x||^2+1/\lambda^2}\right)^{n-2} &\leq (\frac{2}{\lambda})^{n-2}\mathrm{Vol}(B^{n-1},\langle,\rangle)\int_{B_\varepsilon - B_{\frac{1}{\lambda}}}(\frac{1}{r^2})^{n-2}r^{n-1}dr \\
&= 2^{n-2}\lambda^{2-n}\left.\frac{r^{4-n}}{4-n}\right|_{\frac{1}{\lambda}}^{\varepsilon} \\
&= 2^{n-2}\lambda^{2-n}\left(\frac{\varepsilon^{4-n}}{4-n}-\frac{\lambda^{4-n}}{4-n}\right).
\end{aligned}$$

its limit goes to zero as $\lambda \to \infty$.

For $n = 4$ we obtain:

$$2^{n-2}\lambda^{2-n}\log r|_{\frac{1}{\lambda}}^{\varepsilon} = 4\lambda^{-2}(\log\varepsilon + \log\lambda).$$

It tends to zero as $\lambda \to \infty$. Therefore $\displaystyle\int_{\mathbb{R}^n} h_\lambda^2 \to 0$ as $\lambda \to \infty$.

Now, we choose at $p \in M$ a normal neighborhood and $\varepsilon > 0$ small enough such that $dv_g \sim dx$.

We can choose λ big enough such that $h_\lambda(r)$ be small enough. Furthermore, we can deform it in a such way that within an interval $I \subset (\varepsilon/2, \varepsilon)$ the function has slope 1. Hence, we are building a function $\widetilde{h}_\lambda$ with compact support contained inside B_ε. Then,

$$||\widetilde{h}_\lambda||_p \sim ||h_\lambda||_p.$$

Also:

$$||\nabla\widetilde{h}_\lambda||_2^2 \leq ||\nabla h_\lambda||_2^2.$$

and $||\widetilde{h}_\lambda||_2 \to 0$ as $\lambda \to \infty$.

Now, we use $\widetilde{h}_\lambda$ to build a function on M with support inside a normal neighborhood of p. Thus, we compute on M, since the volume elements are almost equal. In consequence we obtain:

$$\begin{aligned}
\mathcal{Y}_g(\widetilde{h_\lambda}) = \frac{a_n\int_M|\nabla\widetilde{h}_\lambda|^2+\int_M S_g\widetilde{h}_\lambda^2}{||\widetilde{h}_\lambda||_p^2} &\xrightarrow[t\to 0]{} a_n\frac{||\nabla h||_2^2}{||h_\lambda||_p^2} \\
&= \frac{a_n}{a_n/Y_n} = Y_n.
\end{aligned}$$

It means that at the limit, the left term is less than the right one, and it does not depend on λ. $\square$

Following arguments by Trudinger [58], Aubin proved that if $Y(M,[g]) < Y_n$ the infimum in the Yamabe constant is achieved. He also proved that this is the case if M has dimension $n \geq 6$ and is not locally conformally flat [1]. Finally, Schoen proved that the strict inequality also holds in the remaining cases [50]. So the infimum is achieved if and only if $Y(M,[g]) < Y_n$.

2.3.2 Uniqueness of solutions to the Yamabe equation

Let $h \in [g]$ be a metric such that $S_h > 0$, then for any $\widetilde{g} \in [g]$ there exists a smooth positive function $f : M \to \mathbb{R}_{>0}$ such that $\widetilde{g} = f^{p-2} \cdot h$, and $Y(M,[g]) > 0$. The converse is also true, that is, if $Y(M,[g]) > 0$, then there exists a metric $h \in [g]$ with $S_h > 0$. In fact, we have that $Y(M,[g]) < 0$ (or $Y(M,[g])$=0) if and only if there exists a metric $h \in [g]$ with $S_h < 0$ (or $S_h = 0$).

If $Y(M,[g]) > 0$ or $Y(M,[g]) < 0$, and h has constant scalar curvature λ, then $\lambda > 0$ or $\lambda < 0$, respectively. But if we have $Y(M,[g]) = 0$, then $\lambda \geq 0$.

Uniqueness to the Yamabe problem holds when $Y(M,[g]) \leq 0$.

Suppose that $Y(M,[g]) < 0$, and that there exists $h \in [g]$ a constant scalar cuvature metric, $S_h = \lambda \in \mathbb{R}$, $S_h < 0$. Then as in the proof of Proposition 2.3.2 we have that:

$$\mathcal{Y}_g(f) \geq \lambda \frac{||f||_2^2}{||f||_p^2}$$

and, again by the Hölder inequality, we have the inequality 2.3.3:

$$||f||_2^2 \leq ||f||_p^2 \, (\mathrm{Vol}(M,g))^{\frac{2}{n}} .$$

It follows that:

$$\begin{aligned} \mathcal{Y}_g(f) &\geq \lambda \, (\mathrm{Vol}(M,g))^{\frac{2}{n}} \\ &= \mathcal{S}(h). \end{aligned}$$

Since h is a minimizer, $\mathcal{Y}_g(f) = \mathcal{S}(h)$. This implies that the latter inequality is equality. Equality holds if and only if f is linearly independent to the constant function with value 1, that is, if f is constant.

If $Y(M,[g]) = 0$, and there exists $h \in [g]$ with $S_h = 0$, then

$$\int_M a_n |\nabla f|^2 = 0$$

and it follows that f has to be constant.

However, in general uniqueness does not hold in the positive case. Here we have a simple example in case of positive scalar curvature. Take $S^2 \times S^2$ with the standard metric on each factor $g_o^2 + g_o^2$. It has positive scalar curvature. We change the radius of both spheres. For $a, b > 0$:

$$ag_o^2 + bg_o^2 = b(\frac{a}{b} g_o^2 + g_o^2)$$

Then $ag_o^2 + bg_o^2$ is conformal to the metric $g_{ab} := \frac{a}{b}g_o^2 + g_o^2$. We compute:

$$\begin{aligned}
\mathcal{S}(g_{ab}) &= \frac{\int_{S^2\times S^2} \frac{2b+2a}{a} dv_{g_{ab}}}{(\mathrm{Vol}(S^2\times S^2, g_{ab})^{1/2}} \\
&= \left(\frac{2b+2a}{a}\right) \mathrm{Vol}(S^2\times S^2, g_{ab})^{1/2} \\
&= a\frac{(2b+2a)}{b}\mathrm{Vol}(S^2, g_o^2).
\end{aligned}$$

Therefore $\mathcal{S}(g_{ab}) > Y_n$, for b enough small, and so there exists more than one solution to Yamabe equation.

Multiplicity of solutions has been obtained on more general cases. Brendle constructed examples when the family of solutions to the Yamabe equation is not compact [9]. Petean proved multiplicity on Riemannian products $(N\times S^n, g_o^n + g)$, where g is a constant scalar curvature metric on N, and g_o^n is the round metric on S^n [47]. In [32] Henry and Petean obtained multiplicity of solutions for spheres products by studying the isoparametric hypersufaces. De Lima, Piccione and Zedda [40] proved multiplicity on arbitrary products with constant scalar curvature.

One also has uniqueness of solutions to the Yamabe Problem if g is Einstein and it is not the round metric on the sphere, since the Yamabe equation on the round sphere has infinitely many solutions.

***Theorem* 2.3.5 (Obata [43])** *Let (M,h) be a n-dimensional compact Einstein manifold, $n\geq 3$. If $g\in[h]$ is a constant scalar curvature metric, then g is Einstein. Such metric is unique unless (M,h) be isometric to the round sphere.*

Secondly, there exists a unique constant scalar curvature metric in the sphere up to conformal equivalences with its canonical metric g_o. In this case all constant curvature metrics have constant sectional curvature.

2.4 Contact and symplectic manifolds

***Definition* 2.4.1** *A* contact manifold (M,η) *is a differentiable manifold M of dimension $2n+1$ with a 1-form η such that $\eta\wedge(d\eta)^n \neq 0$ in M.*

The $2-$form $d\eta$ has rank $2n$ in the alternating algebra $\bigwedge T_p^*M$ for each $p\in M$. Thus, given $p\in M$ there exists a $1-$dimensional subspace:

$$\{v\in T_pM | d\eta(v, T_pM) = 0\}$$

where η does not vanishes, and which is complementary to the subspace defined by $\eta = 0$. If we choose ξ_p in this subspace normalized by $\eta(\xi_p) = 1$, we obtain a global vector field which satisfies:

$$d\eta(\xi, X) = 0 \qquad \eta(\xi) = 1$$

for any vector field X on M. The vector field ξ is called the ***Reeb vector field*** of the contact form η.

Let D be the distribution determined by:

$$D_p = \{v \in T_pM | \eta(v) = 0\}.$$

The requirement of $\eta \wedge (d\eta)^n \neq 0$ means that D is as far as possible from being integrable, in the sense of Frobenius.

***Definition* 2.4.2** *A* symplectic manifold (M, Ω) *is a smooth manifold of dimension* $2n$ *together with a closed* 2*-form* Ω *such that* $d\Omega = 0$ *and* $\Omega^n \neq 0$.

***Definition* 2.4.3** *Let* $H : M \to \mathbb{R}$ *be a function in a symplectic manifold* (M, Ω). *We define the* Hamiltonian vector field *as the unique vector field* X_H *given by*

$$\Omega(X_H, Y) = Y(H)$$

for every vector field Y *on* M.

Each function $H : M \to \mathbb{R}$ gives a Hamiltonian field. Note also that, since Ω is closed, the Lie derivative of Ω with respect to X_H vanishes:

$$\begin{aligned}\mathcal{L}_{X_H}\Omega &= di_{X_H}\Omega + i_{X_H}d\Omega \\ &= d(i_{X_H}\Omega) \\ &= d(dH) = 0\end{aligned} \tag{2.4.1}$$

where $i_{X_H}(\cdot)$ is the inner product of Ω with X_H.

A symplectic manifold has an associated metric, together with an almost-complex structure, as in the previous case. For a proof of the statement see [6].

***Theorem* 2.4.4** *Let* (M, Ω) *be a symplectic manifold. Then there exist a Riemannian metric* g *and an almost complex structure* J *such that*

$$g(X, JY) = \Omega(X, Y) \tag{2.4.2}$$

for all X, Y *vector fields. Such a metric is said to be compatible with the symplectic form* Ω.

***Definition* 2.4.5** *Let* (M, Ω) *be a symplectic manifold with a complex structure* J. *A* vector field X *is* holomorphic *if it preserves* J*:*

$$\mathcal{L}_X J = 0$$

2.5 Poisson geometry

In this section we include basic facts about Poisson geometry that we will use in the construction of Poisson structures associated to singular fibrations. For further details we refer [23, 37, 59].

***Definition* 2.5.1** *A* Poisson bracket *(or a* Poisson structure*) on a smooth manifold M is a bilinear operation $\{\cdot,\cdot\}$ on the set $C^\infty(M)$ that satisfies:*

(i) $(C^\infty(M), \{\cdot,\cdot\})$ *is a Lie algebra.*

(ii) $\{gh, k\} = g\{h, k\} + h\{g, k\}$ *for any* $g, h, k \in C^\infty(M)$.

A manifold M with such a Poisson bracket is called a *Poisson manifold.* Sometimes it will be denoted by $(M, \{\cdot,\cdot\})$.

Symplectic manifolds (M, ω) provide examples of Poisson manifolds. The bracket of M is defined by

$$\{g, h\} = \omega(X_g, X_h).$$

Property (ii) in Definition 2.5.1 allows us to define Hamiltonian vector fields for Poisson manifolds. For $h \in C^\infty(M)$ we assign it the *Hamiltonian vector field* X_h, defined via

$$X_h(\cdot) = \{\cdot, h\}.$$

It follows from (ii) that a Poisson bracket $\{g, h\}$ depends solely on the first derivatives of g and h. Hence we may think of the bracket as defining a bivector field π defined by

$$\{g, h\} = \pi(dg, dh). \tag{2.5.1}$$

A Poisson bivector π can be described locally, for coordinates $(x^1, \ldots, x^n)$, by

$$\pi(x) = \frac{1}{2}\sum_{i,j=1}^{n} \pi^{ij}(x)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.$$

Here $\pi^{ij}(x) = \{x^i, x^j\} = -\{x^j, x^i\}$.

By (i) Poisson brackets satisfy the Jacobi identity, it implies that π satisfies an integrability condition which in local coordinates is a system of first order semilinear partial differential equations for $\pi^{ij}(x)$ [37]. That is, the Jacobi identity is a local condition on π.

Given a bivector π on M, a point $q \in M$, and $\alpha_q \in T_q^*M$ it is possible to define a bundle map:

$$\mathcal{B} : T^*M \to TM \tag{2.5.2}$$

$$\mathcal{B}_q(\alpha_q)(\cdot) \;=\; \pi_q(\cdot, \alpha_q) \tag{2.5.3}$$

If π is a Poisson bivector, we have that $X_h = \mathcal{B}(dh)$.

We then define the *rank* of π at $q \in M$ to be equal to the rank of $\mathcal{B}_q : T_q^*M \to T_qM$. This is also the rank of the matrix $\pi^{ij}(x)$.

For $x_o \in M$ define the linear subspace:

$$\Sigma_{x_o}(M) = \{v \in T_{x_o}(M) | \exists \in C^\infty(M), X_f(x_o) = v\}.$$

Note that, $\Sigma_{x_o}(M) = Im(\mathcal{B}_{x_o})$.

The set $\Sigma(M) = \{\Sigma_{x_o}(M)\}$ is a differentiable distribution called the *characteristic distribution* of the Poisson structure. If the rank of $\Sigma(M)$ is constant, we call it a *regular distribution*; else, it is called a *singular distribution*.

***Theorem* 2.5.2 (*Symplectic Stratification Theorem*)** *The characteristic distribution $\Sigma(M)$ of the Poisson manifold (M, π) is completely integrable, and the Poisson structure induces symplectic structures on the leaves Σ_{x_o}. This foliation is integrable in the sense of Stefan-Sussman [23].*

As a set Σ_q, the symplectic leaf of M through the point q, is also the collection of points that may be joined via piecewise smooth integral curves of Hamiltonian vector fields. Write ω_{Σ_q} for the symplectic form on Σ_q. Observe that $T_q\Sigma_q$ is exactly the characteristic distribution of π through p. Therefore, given $u_q, v_q \in T_q\Sigma_q$ there exist $\alpha_q, \beta_q \in T_q^*M$ that under $\mathcal{B}_q$ go to u_q and v_q. Using this we can describe ω_{Σ_q}:

$$\omega_{\Sigma_q}(q)(u_q, v_q) = \pi_q(\alpha_q, \beta_q) = \langle \alpha_q, v_q \rangle = -\langle \beta_q, u_q \rangle. \tag{2.5.4}$$

As the rank varies, so do the dimensions of the symplectic leaves of the foliation.

***Definition* 2.5.3** *A Poisson manifold M is said to be complete if every Hamiltonian vector field on M is complete.*

Notice that M is complete if and only if every symplectic leaf is bounded in the sense that its closure is compact.

***Definition* 2.5.4** *Let M be a Poisson manifold. A function $h \in C^\infty(M)$ is called a* Casimir *if $\{h, g\} = 0$ for every $g \in C^\infty(M)$. Equivalently $\mathcal{B}(dh) = 0$.*

Linearization of Poisson structures

***Definition* 2.5.5** *A mapping $\phi : (M_1, \{\cdot,\cdot\}_1) \to (M_2, \{\cdot,\cdot\}_1)$ between two Poisson manifolds is called a* Poisson mapping *if for every $f, g \in C^\infty(M_2)$:*

$$\{f \circ \phi, g \circ \phi\}_1 = \{f, g\}_2 \circ \phi.$$

If ϕ is also a diffeomorphism it will be called a Poisson equivalence.

***Definition* 2.5.6** *Let x_o be a point in a Poisson manifold $(M, \{\cdot,\cdot\})$ such that $\{f,g\}(x_o) = 0$ for every $f, g \in C^\infty(M)$. Then $T^*_{x_o}(M)$ becomes a Lie algebra with the Lie bracket:*

$$[df, dg]_{xo} := d\{f, g\}_{x_o}.$$

It is called the isotropy Lie algebra *at x_o, and will be denoted by $\mathfrak{g}_{x_o}$. This also induces a Poisson structure on the tangent space $T_{x_o} = \mathfrak{g}^*_{x_o}$.*

Note that at a point $x \in M$, the isotroppy algebra coincides with $\mathrm{Ker}(\mathcal{B}_x)$.

***Definition* 2.5.7** *A Poisson manifold $(M, \{\cdot,\cdot\})$ is said to be* linearizable *at a point $x_o \in M$ if there is a Poisson equivalence $\phi : U \to V$ from a neighborhood $U \subset M$ of x_o to a neighborhood $V \subset T_{x_o}(M)$ of 0.*

If such equivalence ϕ exists, the Poisson structure is said to be linearizable around x_o.

***Theorem* 2.5.8 (Conn)** *[14] Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold with a point $x_o \in M$ where $\{f,g\}(x_o) = 0$ for every $f, g \in C^\infty(M)$. If the isotropy Lie algebra $\mathfrak{g}_{x_o}$ is semisimple of compact type, then $\{\cdot,\cdot\}$ is linearizable around x_o.*

Integrability of Poisson structures

Given a finite dimensional real Lie algebra $\mathfrak{g}$ the integrability problem was solved by Lie's third Theorem [27]. It gives the existence of a Lie group G such that $Lie(G) \cong \mathfrak{g}$.

For Poisson structures the integrability problem is the existence of a Lie group integrating the Lie algebra given by the Poisson bracket. A Poisson bracket also gives an structure of Lie algebroid to the cotangent bundle $T^*(M)$. The integrability of the Poisson manifold is then about the integrability of that Lie algebroid. We will not give a deep exposition on this subject, we only set the idea of this problem. Roughly speaking, one seeks an associating space that encodes all geometric information of the structure of the Poisson manifold.

Denote by $P : T^*M \to M$ the canonical projection. A *cotangent path* in M is a path $a : [0, 1] \to T^*M$ such that:

$$\frac{d}{dt}P(a(t)) = \mathcal{B}(a(t)).$$

If $x \in M$, the *monodromy group* is:

$$\mathcal{N}_x = \{v \in \mathbf{Z}(\mathfrak{g}_x) | a(t) = v \text{is homotopic to the zero path}\}$$

where $\mathbf{Z}(\mathfrak{g}_x)$ denotes the center of the isotropy Lie algebra.

The *Weinstein grupoid* is denoted by $\Sigma(M, x)$, the set of equivalence classes of cotangent paths with base points in M, starting at x.

M. Crainic and R. L. Fernandes found obstructions for the integrability of Lie algebroids. Their main theorem in [16] establishes that a Lie algebroid A over a manifold M is integable if and only the monodromy groups are discrete and the inferior limit of the distance of the monodromy groups to 0 is positive.

***Theorem* 2.5.9 (Crainic-Fernandes)** *[16] Let $(M, \{\cdot,\cdot\})$ a Poisson manifold. If $Ker(\mathcal{B}_q)$ is a Lie semi-simple algebra, then M is integrable.*

Following [17] we have a different description of the integability given by Crainic-Fernandes. There is a long exact sequence

$$... \longrightarrow \pi_2(L,x) \longrightarrow G(\nu_x(L)) \longrightarrow \Sigma(M,x) \longrightarrow \pi_1(L,x)$$

where L denotes the symplectic leaf through x, $G(\mathfrak{g}_{x_o})$ the corresponding Lie group of the isotropy Lie algebra at x.

Crainic and Fernandes showed that the monodromy groups are discrete if and only if $\Sigma(M,x)$ is a Lie group [16]. In particular a Poisson bracket is integrable (by a symplectic Lie grupoid) if and only if the monodromy groups are locally discrete. They also proved that this holds if and only if $\Sigma(M,x)$ is a Lie group. They obtain in [16]:

***Proposition* 2.5.10** *Any Poisson manifold where the symplectic leaves have vanishing second homotopy groups is integrable.*

2.6 Wrinkled and broken Lefschetz fibrations

In this section we will give some preliminaries about singularity theory. In particular we are interested in broken Lefschetz and wrinkled fibrations.

Let $f\colon M \to X$ be a smooth map between two smooth manifolds with $\dim(M) \geq \dim(X)$ and differential map $df\colon TM \to TX$. A point $p \in M$ is called a *regular point* if the rank of df_p is maximal. In this case f is a submersion at p. If $\mathrm{Rank}(df_p) < \dim(X)$, then a point $p \in M$ is called a *singularity of f*. Let $k = \dim(X) - \mathrm{Rank}(df_p)$ denote the corank of f. The set:

$$\Sigma_k = \{p \in M \mid \mathrm{corank}(df_p) = k \geq 1\}.$$

is known as the *singularity set* or singular locus of f. For generic maps, Σ_k are submanifolds of M. As we can see from the definition, there can be different singularity sets depending on the corank of f. In this work we will focus on singularities of corank 1. The elements of the set Σ_1 satisfying $T_p\Sigma_1(f) \oplus \ker(df_p) = T_pM$ are called *fold* singularities of f.

A mapping $f\colon M \to X$ is then known as a *submersion with folds*, if it is a submersion outside the set of fold singularities. In particular, a submersion with folds restricts to an immersion on its fold locus (see Lemma 4.3 p.87 [26]). Submersions with folds are related to stable maps. By a stable f we mean that any nearby map $\tilde{f} \in C^\infty(M,X)$ is equivalent to f after a smooth change of coordinates in the domain and range. Folds are locally modelled by real coordinate charts $\mathbb{R}^n \to \mathbb{R}^q$ with $n > q$ and coordinates

$$(x_1,\ldots,x_n) \to \left(x_1,\ldots,x_{q-1}, \pm x_q^2 \pm x_{q+1}^2 \pm \cdots \pm x_n^2\right)$$

As we can see from the above parametrization, when $q = 1$, submersions with folds correspond precisely to Morse functions on M. It is well known that Morse functions are dense in the set of smooth mappings from any n-dimensional manifold M to $\mathbb{R}$. There is an equivalent statement for maps with a 2-dimensional target space. Assumming that f is generic then Σ_1 is a submanifold, and the restriction of f at Σ_1 gives a smooth map between manifolds that can also have generic singularities. When the target map is of dimension 2, there is one extra type of generic singularity called cusp. *Cusps* are points $p \in \Sigma_1$ such that $T_p\Sigma_1(f) = \ker(df_p)$, and they are parametrized by real charts

$$\mathbb{R}^n \to \mathbb{R}^2 \qquad (x_1, \ldots, x_n) \to \left(x_1, x_2^3 + x_1 \cdot x_2 \pm x_3^2 \pm \cdots \pm x_n^2\right).$$

***Definition* 2.6.1** *On a smooth* 4*–manifold* X*, a* broken Lefschetz fibration *is a smooth map* $f\colon X \to S^2$ *that is a submersion outside a singularity set. The allowed singularities are of the following type:*

1. *Lefschetz singularities: finitely many points*

   $$\{p_1, \ldots, p_k\} \subset X,$$

   which are locally modelled by complex charts

   $$\mathbb{C}^2 \to \mathbb{C}, \qquad (z_1, z_2) \mapsto z_1^2 + z_2^2,$$

2. *Indefinite fold singularities, also called broken, contained in the smooth embedded 1-dimensional submanifold* $\Gamma \subset X \setminus \{p_1, \ldots, p_k\}$*, which are locally modelled by the real charts*

   $$\mathbb{R}^4 \to \mathbb{R}^2, \qquad (t, x_1, x_2, x_3) \mapsto (t, -x_1^2 + x_2^2 + x_3^2).$$

The term indefinite in (ii) refers to the fact that the quadratic form $-x_1^2 + x_2^2 + x_3^2$ is neither negative nor positive definite. In the language of singularity theory, these subsets are known as fold singularities of corank 1. Since X is closed, Γ is homeomorphic to a collection of disjoint circles. On the other hand, we can only assert that $f(\Gamma)$ is a union of immersed curves. In particular, the images of the components of Γ need not be disjoint, and the image of each component can self-intersect.

The existence of broken Lefschetz fibrations has been proved by Baykur [4], by Akbulut and Karakurt [2], and by Lekili [39] on any closed oriented smooth 4−manifold.

The notion of *wrinkled fibration* on a smooth 4-manifold was introduced by Lekili [39]. He showed that these wrinkled fibrations exist in every closed oriented smooth 4−manifold. Broken Lefschetz fibrations are not stable maps. In contrast, wrinkled fibrations are stable. So if one is interested in perturbations of broken Lefschetz fibrations, one is led naturally to the study of wrinkled fibrations.

***Definition* 2.6.2** *Let* X *be a closed* 4*–manifold, and* Σ *be a* 2*–dimensional surface. A map* $f : X \to \Sigma$ *is said to have a* cusp singularity *at a point* p *in* X*, if around* p*,* f *is locally modelled in oriented charts by the map:*

$$(t, x, y, z) \mapsto (t, x^3 - 3xt + y^2 - z^2).$$

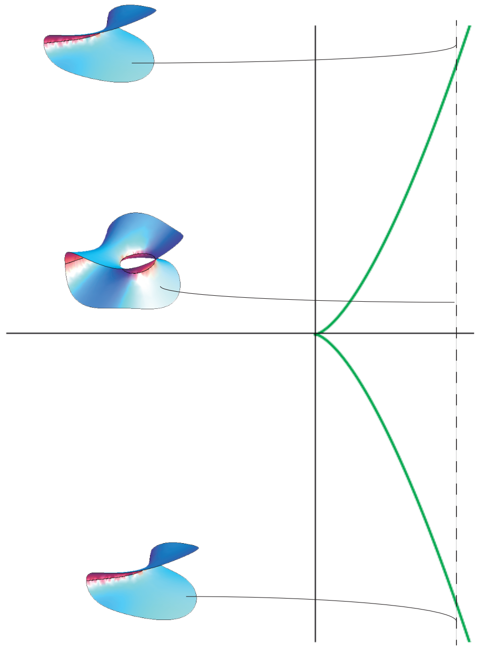

Figure 2.1: A diagram depicting a fibration with a cusp singularity along the critical values depicted in green, which is a subset of the base of this fibration. The black lines indicate the points over which each of the fibres lie.

The critical point set is a smooth arc, $\{x^2 = t, y = 0, z = 0\}$, the critical value set is a cusp given by $\{(t, s) : 4t^3 = s^2\}$ (see figure 2.1).

Following Lekili we state:

***Definition* 2.6.3** *A* wrinkled fibration *on a closed* 4*–manifold* X *is a smooth map* f *to a closed surface which is a broken fibration when restricted to* $X \setminus C$*, where* C *is a finite set such that around each point in* C*,* f *has cusp singularities. We say that a fibration is purely wrinkled if it has no isolated Lefschetz-type singularities.*

Note that folds and cusps are the singularities of a wrinkled fibration.

Wrinkled fibrations may be obtained from broken Lefschetz fibrations by performing wrinkling moves. These eliminate a Lefschetz type singularity and introduce a wrinkled fibration structure. Conversely, it is possible to modify a wrinkled fibration locally by smoothing out the cusp singularity by introducing a Lefschetz type singularity, so obtaining a broken fibration (see [39, 62]).

As Lekili, by a *deformation of wrinkled fibrations* we mean a one-parameter family of maps which is a wrinkled fibration for all but finitely many values. One of Lekili's major contributions in [39] was to show that any one-parameter family deformation of a purely wrinkled fibration is homotopic (relative endpoints) to one which realizes a sequence of births, merges, flips, their inverses, and isotopies staying within the class of purely wrinkled fibrations. Moreover, these moves do not change the diffeomorphism type of the 4–manifold X on which they take place.

Let us briefly describe these moves, readers may consult both [39, 62] for the corresponding descriptions in terms of how the fibres change and how these moves can be described using handlebodies. The pictures what we present here originally appeared in [53]. The moves we are interested in are to be considered as maps $\mathbf{R} \times \mathbb{R}^3 \to \mathbb{R}$, given by the following equations, each depending on a real parameter s:

Move 1 (Birth, fig. 2.2)

$$b_s(x, y, z, t) = (t, x^3 - 3x(t^2 - s) + y^2 - z^2)$$

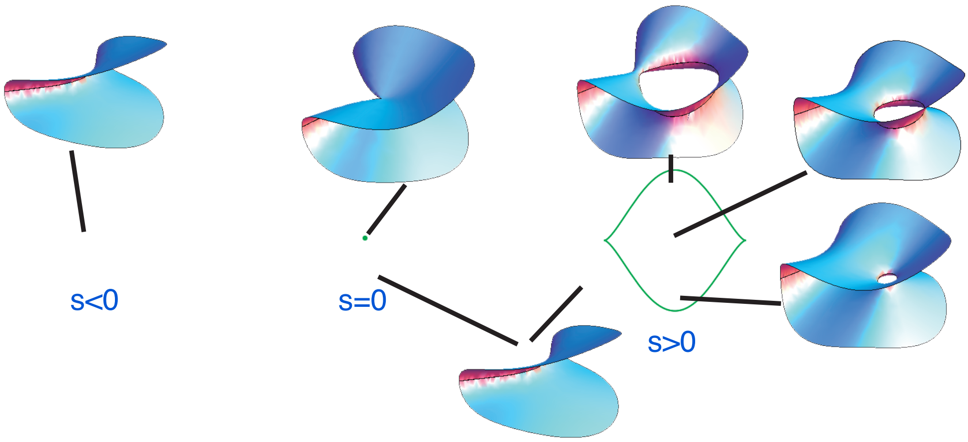

Figure 2.2: **Birth Move:** $b_s(x, y, z, t) = (t, x^3 - 3x(t^2 - s) + y^2 - z^2)$. For $s < 0$ the critical set is empty. Then when $s = 0$ the fibre above the critical point is shown to develop a singularity. As s becomes postive the wrinkled critical set appears, here depicted by the green line, which is a subset of the base of this fibration. The black lines indicate the points over which each of the fibres lies.

Move 2 (Merging, fig. 2.3)

$$m_s(x, y, z, t) = (t, x^3 - 3x(s - t^2) + y^2 - z^2)$$

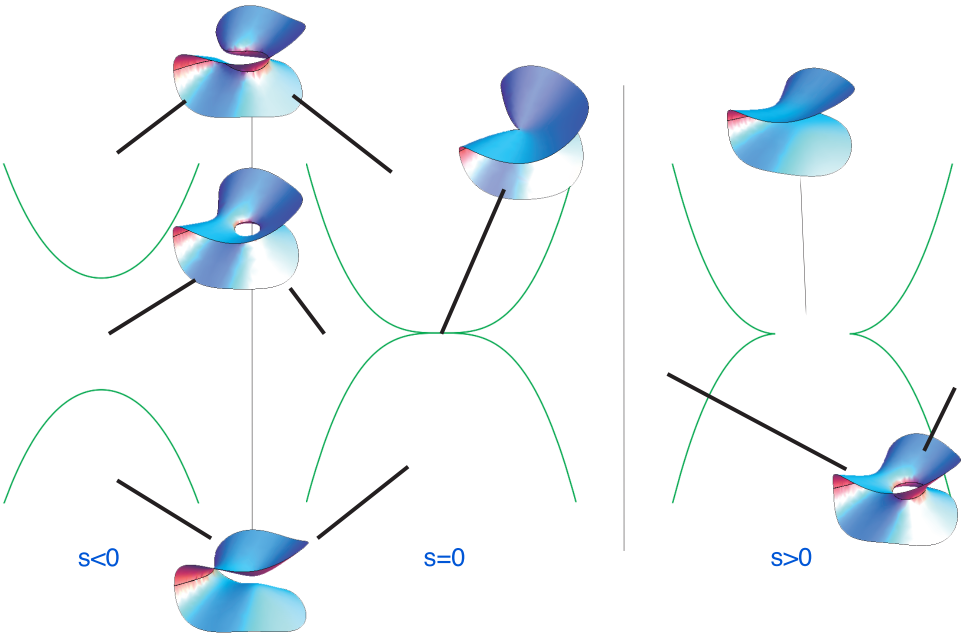

Figure 2.3: **Merging Move:** $m_s(x,y,z,t) = (t, x^3 - 3x(s-t^2) + y^2 - z^2)$. For $s < 0$ the critical set shows two connected components. Then when $s = 0$ these two components touch, and the fibre above the critical point is shown to develop a singularity. As s becomes postive the critical set (shown in green) separates again. The black lines indicate the points over which each of the fibres lies.

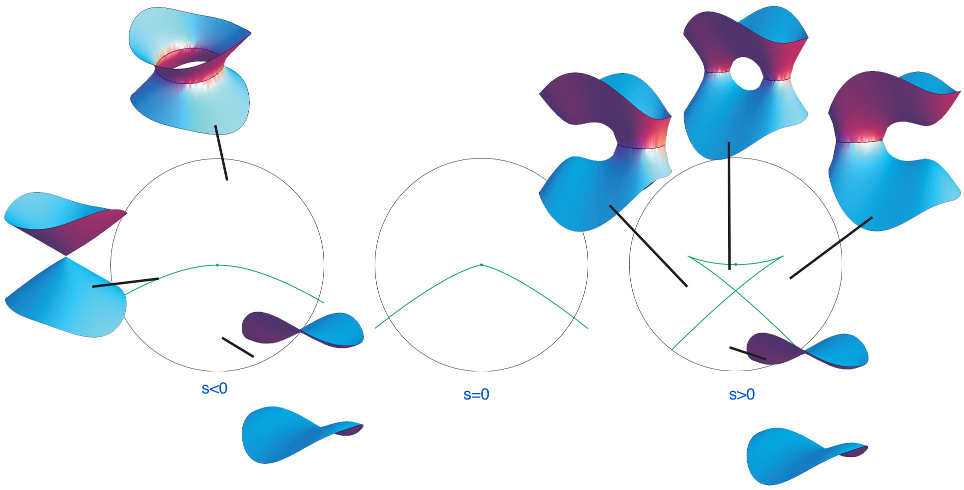

Figure 2.4: **Flipping Move:** $f_s(x,y,z,t) = (t, x^4 - x^2 s + xt + y^2 - z^2)$. For $s < 0$ the critical set corresponds to that of a broken Lefschetz fibration. As s becomes postive the critical set (shown in green) crosses itself. The black lines indicate the points over which each of the fibres lies.

Move 3 (Flipping, fig. 2.4)

$$f_s(x,y,z,t) = (t, x^4 - x^2 s + xt + y^2 - z^2)$$

Move 4 (Wrinkling)

$$w_s(x,y,z,t) = (t^2 - x^2 + y^2 - z^2 + st, 2tx + 2yz)$$

The following theorem by Golubitsky and Guillemin shows that generic maps from any n–dimensional manifold to a 2–dimensional base have folds and cusps.

***Theorem* 2.6.4** *[26] A generic smooth map $M^n \to N^2$ has folds and cusps singularities.*

In this context, wrinkled fibrations are generic maps defined on a smooth 4-manifold with image on the 2-sphere, and broken Lefschetz fibrations are submersions with folds and Lefschetz singularities. It was shown by Donaldson, there is a correspondence between symplectic 4-manifolds and Lefschetz fibrations. Yet, Lefschetz singularities are not stable from the point of view of singularity theory. Lekili showed that Lefschetz singularities can be transformed into cusps yielding to a wrinkled fibration. As a consequence, we can modify a broken Lefschetz fibration into a submersion with folds and cusps, which are stable and dense.

Chapter 3

Ricci solitons

In the 1980's Richard Hamilton [28] introduced the Ricci Flow as an approach to prove Thurston's Geometrization Conjecture. A solution to the Ricci flow is a 1-parameter family of Riemannian metrics $g(t)$ satisfying the partial differential equation:

$$\frac{\partial g(t)}{\partial t} = \frac{2}{n} rg - 2Ric(g), \tag{3.0.1}$$

where r is the average of the scalar curvature R:

$$r = \frac{\int R d\mu}{\int d\mu},$$

and Ric is the Ricci curvature on a Riemannian manifold (M, g).

Einstein metrics are fixed points for the Ricci flow, since fixed points must have constant average scalar and Ricci curvatures. In order to understand the deformation progress under Ricci flow research has focused on the singularities, due to the Hamilton short-time existence theorem [28].

Among the different types of solutions for the Ricci flow we are interested in natural generalizations of Einstein metrics, namely Ricci solitons. They often arise as dilation limits of singularities in the Ricci flow, they can also be viewed as generalized fixed points of the Ricci flow, on the space of Riemannian metrics modulo diffeomorphisms and scalings.

Formally:

***Definition* 3.0.5 (Ricci Soliton)** *Let (M^n, g) be a Riemannian manifold of dimension n such that*

$$-2Ric(g) = \mathcal{L}_X g + 2\lambda g \tag{3.0.2}$$

holds for some constant λ and some vector field X on M^n. We say that g is a Ricci soliton, *where $\mathcal{L}_X g$ is the Lie derivative of the metric in the direction of X.*

Depends on the sign of λ, negative, zero or positive it corresponds to ***shrinking, steady,*** *or* ***expanding*** *Ricci soliton, respectively.*

If $X \equiv 0$, then $Ric(g) = -\lambda g$. It means that any Ricci soliton may be regarded as a generalization of an Einstein metric.

In [21] DeTurck proved short-time existence and uniqueness of solutions to the Ricci flow by a modified Ricci flow:

$$\frac{\partial}{\partial t} g(t) = -2Ric(g(t)) + \mathcal{L}_X g(t)$$

where X is a certain vector field. This flows is parabolic, and so admits solutions over some short-time interval, given an initial metric $g_o = g(0)$ (see Section 5.2 [56]). On a closed manifold uniqueness also follows, and solutions for this flow produces solutioins to the Ricci flow. In some sense the idea is to flow along Ricci solitons to obtain existence of solutions to the Ricci flow. For a detailed and clear exposition we refer to [56].

Recall that the Hessian of a smooth function is given by:

$$Hess(f)(X,Y) = Y(X(f)) - \nabla_X Y(f).$$

If $X = \nabla(-f)$, i.e. X is the gradient of f,

$$\begin{aligned}
(\mathcal{L}_X g)(Y,Z) &= g(\nabla_Y X, Z) + g(\nabla_Z X, Y) \\
&= g(\nabla_Y(\nabla(-f)), Z) + g(\nabla_Z(\nabla(-f)), Y) \\
&= Yg(\nabla(-f), Z) - g(\nabla_Y Z, \nabla(-f)) + Zg(\nabla(-f), Y) - g(\nabla_Z Y, \nabla(-f)) \\
&= -Y(Z(f)) + \nabla_Y Z(f) - Z(Y(f)) + \nabla_Z Y(f) \\
&= [Y,Z](f) - 2Y(Z(f)) - [Y,Z](f) + 2\nabla_Y Z(f) \\
&= -2Y(Z(f)) + 2\nabla_Y Z(f) \\
&= -2Hess(f)(Y,Z).
\end{aligned}$$

Hence, the equation (3.0.2) becomes:

$$Hess(f) = Ric(g) + \lambda g$$

We say that g is a **gradient Ricci soliton**.

Hamilton [30] and Ivey [34] proved that in compact steady or expanding gradient Ricci solitons are necessarily Einstein. Perelman showed that any compact Ricci soliton is a gradient soliton [46]. Therefore, the study of compact Ricci non-Einstein solitons leads to study potential functions which describe gradient shrinking solitons.

***Definition* 3.0.6** *Let M be a complex manifold. A Riemannian metric g on M is an* Hermitian metric *if it is invariant under the almost complex structure J on M:*

$$g(JX, JY) = g(X,Y)$$

for every $X, Y \in \chi(M)$.

There is an associated differential 2−form ω_g to a Hermitian metric g,

$$\omega_g(X, Y) = g(JX, Y),$$

which is is closed if and only if J is parallel. That is, if ∇ is the Levi-Civita connection of g

$$\nabla_X JY = J\nabla_X Y.$$

***Definition* 3.0.7** *A Hermitian manifold M with Riemannian metric g and almost complex structure J is called a* Kähler manifold *if its associated form ω_g is closed.*

***Definition* 3.0.8** *The* Ricci form *of a Kähler manifold is a closed form $Ric(\omega_g)$ defined by:*

$$Ric(\omega_g) = Ric_g(JX, Y)$$

Let us review known examples of Ricci solitons.

Examples

a) **Hamilton's Cigar soliton [29]**

Let $g = dx^2 + dy^2$ be the standard metric on $\mathbb{R}^2$. If g_f is a conformal metric to g, $g_f = e^{2f} \cdot g$, recall that in dimension 2, their scalar curvatures are related by:

$$S_{g_f} e^{2f} = S_g + 2\triangle_g f$$

Let $f = Log(\rho)$, where $\rho^2 = \dfrac{1}{1 + x^2 + y^2}$. On $\mathbb{R}^2$ with the metric $g_o = \rho^2(dx^2 + dy^2)$. It follows that the Gauss curvature is

$$K_g = -\frac{1}{\rho^2}\triangle Log(\rho)$$

Then $Ric(g_o) = Kg_o$. Indeed, we compute $K = \frac{2}{1+x^2+y^2}$. If we define a radial vector field $Y := -2(x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y})$. We obtain:

$$\mathcal{L}_X g_o = -2Ric(g_o)$$

Hence g_o is a steady soliton, called ***Hamiltons Cigar Soliton***. Historically it has been distinguished as part of one of the posible limits that we can find after scaling the Ricci flow in dimension 3 near singularities that appear in finite time.

The Cigar soliton is the unique gradient soliton that is rotationally symmetric, up to homothetical changes. It has positive curvature and is asymptotic at infinity to a cylinder of finite circumference.

b) **Gaussian soliton**

Consider $(\mathbb{R}^n, g_o)$ with the Euclidean metric. We may construct a gradient expanding or shrinking soliton.

If we take as potential function $f = \dfrac{||x||^2}{4}$, we obtain the shrinking Gaussian soliton. If $f = -\dfrac{||x||^2}{4}$, we obtain the expanding Gaussian soliton.

c) **Bryant soliton [8]**

Let g_o^n be the round metric $n-$sphere. Let g be the warped product metric $g = dr^2 + \phi(r)g_o^n$ defined on $(0,\infty) \times S^n$, with φ a radial function on S^n. The Bryant soliton is a rotationally symmetric gradient steady soliton in $\mathbb{R}^n$ $(n \geq 3)$ of positive sectional curvature. If f is the potential function, then a complete soliton is obtained by the solution to the system:

$$f'' = (n-1)\frac{\phi''}{\phi}, \qquad \phi\phi' f' = -(n-2)(1-(\phi')^2) + \phi\phi''.$$

d) **Koiso-Cao soliton**

Koiso, and independently Cao constructed a non-Einstein shrinking soliton in $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ with symmetry $U(2)$ and positive Ricci curvature ([36, 10]). It is the only known cohomogeneity one Ricci soliton in dimension 4.

More generally, they constructed $U(n)-$invariant Kähler-Ricci solitons on twisted projective line bundles over $\mathbb{CP}^{n-1}$, $n \geq 2$. Cao's construction finds a Kähler potential $U(n)-$invariant, $u : (-\infty, \infty) \to \mathbb{R}$, and a $U(n)-$invariant smooth function $f : (-\infty, \infty) \to \mathbb{R}$ such that:

$$Ric(\omega_g) - \omega_g = Hess(f-u).$$

It requieres additionally that ∇f is holomorphic, and u satisfies some asymptotic conditions at $-\infty$ and ∞. He proved that the existence of such a pair of functions is equivalent to the existence of the following differential equation:

$$\frac{u'''}{u''} + \left(\frac{n-1}{u'} + c\right) u'' = n - u'$$

for some constant $-1 < c < 0$.

e) **Wang-Zhu soliton**

Wang and Zhu found a gradient Kähler Ricci soliton on $\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$ with $U(1)\times U(1)$ symmetry [61]. They determine the metric by solving a complex Monge-Ampère equation.

f) Since compact homogeneous Ricci solitons are Einstein, the next case to study are cohomogeneity one Ricci solitons. They were studied by Dancer, Hall and Wang in [19] and [20]. In fact, Dancer and Wang [19] produced examples of gradient shrinking, steady and expanding Kähler solitons on bundles over the product Kähler-Einstein manifolds of positive Chern class. Their construction generalizes Koiso and Cao's examples.

Chapter 4

Contact and symplectic solitons

The aim of this and the following sections is to present some deductions based on Cho's results on contact solitons. In particular, we discuss their extension to certain Riemannian products and symplectic manifolds. It is a work in progress.

4.1 Contact solitons

If we fix some vector field, then we try to find a metric and a potential scalar which satisfy the soliton equation. This idea has been explored for contact manifolds by J. T. Cho [12], and previously by R. Sharma [52]. They used the Reeb vector field.

Recall that a contact manifold is denoted by (M, η, ξ), with η the contact form and ξ the Reeb vector field.

***Theorem* 4.1.1** *Let (M, η, ξ) be a contact manifold. Then there are a Riemannian metric g and a $(1,1)$ tensor field $\varphi : \chi(M) \to \chi(M)$ such that:*

$$\eta(X) = g(X, \xi) \qquad d\eta(X, Y) = g(X, \varphi Y) \qquad \varphi^2 = -I + \eta \otimes \xi.$$

For all vector fields X, Y.

For a proof we refer to[6].

Such a manifold (M, η, ξ) with a Riemannian metric g and a $(1,1)$-tensor field φ is called a ***contact metric manifold*** [6]. We will denote it by $(M, \eta, \xi, g, \varphi)$.

***Definition* 4.1.2** *A* contact metric manifold *$(M, \eta, \xi, g, \varphi)$ is Sasakian if and only if:*

$$(\nabla_X \varphi) = g(X, Y)\xi - \eta(Y)X$$

for every $X, Y \in \chi(M)$.

In the work of Cho [12, 13] he states:

***Definition* 4.1.3** *A contact metric manifold* $(M, \eta, \xi, g, \varphi)$ *is a* contact soliton *if* g *is a Ricci soliton defined by* ξ, *and some scalar* $\lambda \in \mathbb{R}$:

$$-2Ric(g) = \mathcal{L}_\xi g + 2\lambda g.$$

It will be denoted by (M, g, ξ, λ).

A transversal soliton *to a contact soliton is a Ricci soliton defined by* ν, *a complete vector field orthogonal to* ξ.

The following results appear on[12]. He proved that a compact Ricci soliton is Sasaki-Einstein:

***Theorem* 4.1.4** *A contact Ricci soliton is shrinking and is Einstein. The Reeb vector field* ξ *is a Killing field.*

Corollary 4.1.5 *A compact contact soliton is Sasaki-Einstein.*

On the other hand, in [13] Cho proved:

***Theorem* 4.1.6** *A contact* 3*-manifold* M *admitting a transversal Ricci soliton* (g, ν, λ) *is either Sasakian or locally isometric to one of the following Lie groups with a left invariant metric:* $SU(2)$, $SL(2, \mathbb{R})$, $E(2)$ *(the group of rigid motions of the Euclidean* 2*-space),* $E(1,1)$ *(the group of rigid motions of the Minkowski* 2*-space).*

***Definition* 4.1.7** *Let* (M, g_M) *and* (N, g_N) *be Riemannian manifolds. Let* π *and* σ *the respective projections of* $M \times N$ *to* M *and* N. *Over* $M \times N$ *we put the product metric:*

$$g = \pi^*(g_M) + \sigma^*(g_N)$$

We say that $(M \times N, g)$ *is a* Riemannian product.

Let $B = M \times N$ be a product manifold. The tangent space of B splits as:

$$T_b B = T_p M \oplus T_q N$$

for every $b \in B$, $b = (p, q)$. Hence, a vector field X on B can be written uniquely as a *horizontal vector field*, which pointwise is the lift of a tangent vector in a horizontal direction to the product, plus a *vertical vector field*, the lift of a tangent vector in a vertical direction. We write this as $X = X^h + X^v$.

We follow [44], where further facts about Riemannian products can be consulted.

***Proposition* 4.1.8** *Let* (M, g_M, X, λ_1) *and* (N, g_N, Y, λ_1) *be two Ricci solitons. Then* $M \times N$ *is a Ricci soliton with constant* λ *if and only* $\lambda_1 = \lambda_2 = \lambda$.

Proof: Let $g = g_M + g_N$ and. The Ricci curvature on a Riemannian product is the sum of the Ricci curvatures: $Ric_g = Ric_{g_M} + Ric_{g_N}$. Since g_M and g_N are Ricci solitons:

$$\begin{aligned} \mathcal{L}_{X+Y} g &= \mathcal{L}_X g_M + \mathcal{L}_Y g_N \\ &= -2Ric_{g_M} + 2\lambda_1 g_M - 2Ric_{g_N} + 2\lambda_2 g_N \\ &= -2Ric_g + 2\lambda_1 g_M + 2\lambda_2 g_N \end{aligned}$$

In order to obtain a soliton on the Riemannian product $M \times N$, the equation

$$\mathcal{L}_{X+Y} g = -2Ric_g + \lambda g$$

must holds. It implies that:

$$(\lambda - \lambda_1) g_M + (\lambda - \lambda_2) g_N = 0.$$

Therefore $\lambda = \lambda_1 = \lambda_2$. □

***Proposition* 4.1.9** *Let (N, g_N) a Ricci soliton and (M, g_M) a Riemannian manifold. If $g_M \times g_N$ is a Ricci soliton, then g_M is a Ricci soliton.*

Proof: Let $X_1, X_2 \in \chi(M)$. Then $Ric_{g_N}(X_1, X_2) = \mathcal{L}_Y g_N(X_1, X_2) = g_N(X_1, X_2) = 0$. It follows that:

$$\begin{aligned} \mathcal{L}_{X+Y} g(X_1, X_2) &= \mathcal{L}_X g_M(X_1, X_2) + \mathcal{L}_Y g_N(X_1, X_2) \\ &= \mathcal{L}_X g_M(X_1, X_2) \\ &= -2Ric_{g_M+g_N}(X_1, X_2) + 2\lambda(g_M + g_N)(X_1, X_2) \\ &= -2Ric_{g_M}(X_1, X_2) + 2\lambda g(X_1, X_2). \end{aligned}$$

Therefore g_M is a soliton. □

Let (Y, g, ξ, λ) be a contact soliton. We want to extend Cho's results to dimension 4. Begin with $Y^3 \times S^1$. It is natural to take the vector field of rotations θ on S^1. Nevertheless, the vector field $\xi + \theta$ does not produce a soliton.

Since S^1 is Ricci flat, and θ is a Killing vector field, then S^1 is a steady soliton. If $Y^3 \times S^1$ were a soliton with the vector field $\xi + \theta$, the equation:

$$Ric_{g+d\theta^2} = 2\lambda g = 0$$

must hold.

This implies that the contact soliton must be Ricci flat. But $3-$contact metric manifolds whose Reeb vector field ξ is Killing must satisfy $Ric(\xi, \xi) = 2$ (See Theorem 7.1 in [6]). In fact, $Y^3 \times S^1$ doest not admit a Ricci soliton metric.

We have that S^1 has to be a soliton of the same type as Y^3, that is, a shrinking soliton, with associated constant $\lambda < 0$, and

$$\mathrm{Hess}_{g_o^1}(f) = \lambda g_o^1.$$

But it was proven that this implies that $f = \frac{\lambda |x|^2}{2}$ ([48] Proposition 3.1). Then $\mathrm{Hess}(f) = 0$.

By theorem 4.1.4 we obtain the following:

***Proposition* 4.1.10** *Let (M, g, ξ, λ) be a contact soliton of dimension n, for $n \geq 3$. Consider the euclidean metric on $\mathbb{R}$ and the vector field $\zeta = \frac{d}{dt}\frac{\lambda|x|^2}{2}$. Then $M \times \mathbb{R}$ is a Ricci soliton with the vector field $\xi + \zeta$.*

If in addition M admits a transversal soliton, with transversal vector field τ, then M is a product of a Sasakian manifold with $\mathbb{R}$ or locally isometric to a product of $\mathbb{R}$ with some of the following Lie groups: $SU(2)$, $SL(2, \mathbb{R})$, $E(2)$ y $E(1, 1)$.

On (S^2, g_o), the round sphere define $g(t) = r(t)^2 g_o$, where r is a positive function. Then $g(t)$ is a solution to the unormalized Ricci flow

$$\frac{\partial}{\partial t} g(t) = -2Ric(g(t))$$

with initial condition $g(0) = g_o$ if and only if $r(t) = \sqrt{1 - 2t}$. Note that the solution exists for any $t \in (-\infty, 1/2)$.

Let $\widetilde{\lambda} = \frac{d}{dt}\frac{r^2(t)}{2} = -1$. Take θ the vector field of rotations on S^2. It is a Killing vector field. Therefore:

$$\begin{aligned} \mathcal{L}_\theta g_o + 2Ric(g_o) + 2\widetilde{\lambda} g_o &= 2Ric(g) + 2\widetilde{\lambda} g \\ &= 2 - 2 = 0. \end{aligned}$$

Therefore, it defines a shrinking soliton on S^2.

On the other hand, Bowden-Crowley-Stipsicz [7] proved that if a closed manifold M admits a contact structure, so does the product $M \times S^2$. This allows us to think in extend Cho's results for these products.

By the Proposition 4.1.8 and Cho theorem 4.1.6 we obtain:

***Proposition* 4.1.11** *Let (M, g, ξ, λ) be a contact soliton. Let (S^2, g_o), with g_o be the shrinking soliton determined by the vector field of rotations θ. Then $B = M \times S^2$ is a contact manifold that admits a soliton determined by the vector field (ξ, θ).*

If M also admits a transversal soliton with transversal vector field τ, then B is a product of a Sasakian manifold with S^2 or locally isometric to a product of S^2 with some of the Lie groups: $SU(2)$, $SL(2, \mathbb{R})$, $E(2)$ and $E(1, 1)$.

4.2 Symplectic solitons

4.2.1 Hamiltonian case

We want to proceed analogously to the contact case. We will study symplectic manifolds with a Ricci soliton structure determined by Hamiltonian vector fields. We introduce Hamiltonian solitons, those solitons determined by Hamiltonian vector fields.

***Definition* 4.2.1** *Let (M, ω) be a symplectic manifold with a Riemannian metric compatible with ω, with almost complex structure J. Let X_H be a Hamiltonian vector field associated to a smooth function $H : M \to \mathbb{R}$. Suppose that X_H is holomorphic ($\mathcal{L}_{X_H} J = 0$). We will say X_H determines a* Hamiltonian soliton *if there exists a scalar λ such that:*

$$-2Ric(g) = \mathcal{L}_{X_H} g + 2\lambda g.$$

***Lemma* 4.2.2** *Let (M, ω) be a symplectic manifold compatible with a metric g. Let X be a holomorphic vector field on M. If $Z \in \chi(M)$, write $W = JZ$. Then:*

$$(\mathcal{L}_X\omega)(Y, Z) = (\mathcal{L}_X g)(Y, W).$$

Proof: Let $X_1, X_2 \in \chi(M)$. Recall that the Lie derivative of the complex structure J is given by:

$$\begin{aligned} (\mathcal{L}_{X_1} J)(X_2) &= \mathcal{L}_{X_1} J(X_2) - J(\mathcal{L}_{X_1} X_2) \\ &= [X_1, JX_2] - J([X_1, X_2]). \end{aligned}$$

Then, if X is holomorphic:

$$[X, J(Y)] - J([X, Y]) = 0$$

for any $Y \in \chi(M)$. We compute:

$$\begin{aligned} (\mathcal{L}_X\omega)(Y, Z) &= X_H\omega(Y, Z) - \omega([X, Y], Z) + \omega([X, Z], Y) \\ &= X\omega(Y, Z) - \omega([X, Y], Z) - \omega(Y, [X, Z]) \\ &= Xg(Y, W) - g([X, Y], W) - g(Y, J([X, Z])) \\ &= g(\nabla_X Y, W) + g(\nabla_X W, Y) - g([X, Y], W) - g(Y, J([X, Z])) \\ &= g(\nabla_X Y - [X, Y], W) + g(\nabla_X W - J([X, Z]), Y) \\ &= g(\nabla_Y X, W) + g(\nabla_X W - J([X, Z]), Y) \\ &= g(\nabla_Y X, W) + g([X, W] + \nabla_W X - J([X, Z]), Y) \\ &= g(\nabla_Y X, W) + g(\nabla_W X, Y) \\ &= (\mathcal{L}_X g)(Y, W). \end{aligned}$$

□

***Proposition* 4.2.3** *A Hamiltonian soliton g is Einstein.*

Proof: Let $Y, W, Z \in \chi(M)$ with $J(Z) = W$. Then, since X_H is holomorphic by the previous lemma 4.2.2 we have that:

$$(\mathcal{L}_{X_H}\omega)(Y, Z) = (\mathcal{L}_{X_H} g)(Y, W).$$

By 2.4.1, $\mathcal{L}_{X_H} g = 0$. Thus, g is Einstein. □

4.2.2 Liouville case

***Definition* 4.2.4** *Let (M, ω) be a symplectic manifold. A vector field V is a* Liouville field *if:*

$$\mathcal{L}_V\omega = \omega$$

Hence, analogous to the previous case we define:

***Definition* 4.2.5** *Let (M, ω) be a symplectic manifold with a Riemannian metric compatible with ω, with almost complex structure J. Let V be a holomorphic Liouville vector field. We say V determines a* Liouville soliton *if there exists a scalar λ such that:*

$$-2Ric(g) = \mathcal{L}_V g + 2\lambda g$$

***Proposition* 4.2.6** *Liouville solitons are Einstein metrics*

Proof: Let $Y, W, Z \in \chi(M)$ with $J(Z) = W$. Since V is a holomorphic Liouville vector field, by the Lemma 4.2.2:

$$\begin{aligned}(\mathcal{L}_V\omega)(Y,Z) &= \omega(Y,Z) \\ &= g(Y,W) \\ &= (\mathcal{L}_V g)(Y,W).\end{aligned}$$

Then:

$$(\mathcal{L}_V g)(Y,W) = \omega(Y,Z) = g(Y,W).$$

If g is a soliton it follows:

$$-2Ric(g) = (1+2\lambda)g$$

□

It is known that one may *"simplectize"* a contact manifold and that it admits a canonical Liouville vector field. Formally:

***Lemma* 4.2.7** *[6] Let (M, η, ξ) be a contact $3-$ manifold. Let $B = M \times \mathbb{R}$ and $\omega = d(e^t\eta)$. Then ω is a symplectic form and $\frac{\partial}{\partial t}$ is a Liouville vector field for ω.*

Therefore we obtain:

***Proposition* 4.2.8** *Let (M, g, η, λ) be a contact soliton of dimension 3. Then $\lambda = -1$. Consider the euclidean metric on $\mathbb{R}$ and the vector field $\zeta = -\frac{d}{dt}\frac{|x|^2}{2}$. Then $M \times \mathbb{R}$ is a trivial Ricci soliton with the vector field $\frac{\partial}{\partial t} + \zeta$.*

Chapter 5

Yamabe equation of the Koiso-Cao soliton

5.1 Construction of the Koiso-Cao soliton

Koiso [36] and Cao [10] constructed $U(n)-$invariant gradient shrinking Kähler-Ricci solitons on twisted projective line bundles over $\mathbb{CP}^{n-1}$ for $n \geq 2$. Cao's construction consists in giving conditions of the Kähler potential corresponding to a $U(n)-$invariant Kähler metric defined on $\mathbb{C}^n \setminus \{0\}$ to obtain a compact shrinking Ricci soliton. The Kähler potential is a smooth function defined on $(-\infty, \infty)$ with certain asymptotic conditions at $-\infty$ and ∞.

In this section we will review this construction from a different point of view for $n = 2$, which is the Koiso-Cao soliton. This will help us to study the Yamabe equation. For further details see [35], where a similar construction is performed for the Page metric. For $n = 2$, the Koiso-Cao construction give a soliton on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ (which is a S^2-bundle over S^2).

***Definition* 5.1.1** *Let (M, g) be a Riemannian manifold which admits a compact Lie group action by isometries. The action is said to be of* cohomogeneity one *if the principal orbits are hypersurfaces in M, that is, the codimension of the principal orbits is one.*

All cohomogeneity one actions in dimension 4 are classfied by J. Parker [45]. Furthermore, in this case it is known that the orbit space is the circle S^1, or an interval I. The only singular orbits appear at the endpoints of I. When the underlying manifold has finite fundamental group, the orbit space is an interval.

Shrinking Ricci solitons have finite fundamental group [24]. Thus, the orbit space of a cohomogeneity one shrinking Ricci soliton is an interval.

The Page metric was the first non-homogeneous Einstein metric constructed. It is a cohomogeneity one Einstein metric on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, with the action of the unitary group of dimension 2, $U(2)$, with positive Ricci curvature λ_o. The orbit space is an interval, we denote the parameter of the orbits by t. An explicit construction can be found in [5]. This metric can be obtained using the Hopf fibration as a Riemannian submersion, by making a conformal change of the spheres. It is not a Kähler metric.

It is obtained from the solution of the equations for the functions f and h:

$$\begin{aligned} - \quad & \frac{f''}{f} - 2\frac{h''}{h} = \lambda_o, \\ - \quad & \frac{f''}{f} - 2\frac{f'h'}{fh} + 2\frac{f^2}{h^4} = \lambda_o, \\ - \quad & \frac{h''}{h} - \frac{f'h'}{fh} - \frac{h'^2}{h^2} + \frac{4}{h^2} - 2\frac{f^2}{h^4} = \lambda_o. \end{aligned}$$

The metric can be written as:

$$g = dt^2 + f^2\theta^2 + \frac{1}{4}h^2 g_o^2$$

for a 1-form θ on S^1.

By numerical computations one may obtain that $\lambda_o = 3.73282$ and that the Yamabe constant is $Y(\mathbb{CP}^2\#\overline{\mathbb{CP}^2}, g) = 2.829 < Y(S^n, g_o) = 8\pi\sqrt{6}$.

A $U(2)-$invariant metric $(\mathbb{CP}^2\#\overline{\mathbb{CP}^2}, \mathbf{g})$ can be described in the following way. The regular orbits of the $U(2)-$action is an open dense subset diffeomorphic to $S^3 \times (\alpha, \beta)$, and $U(2)$ acts on S^3. There are two singular orbits diffeomorphic to S^2. The invariant metric g on $S^3 \times (\alpha, \beta)$ is written as $g = dt^2 + g_t$, where g_t is a $U(2)-$invariant metric on S^3.

Let f and h be positive smooth functions defined on (α, β). For each $t \in (\alpha, \beta)$ the $U(2)-$invariant metric g_t is such that the principal $(S^1, f^2(t)g_o)-$bundle with projection $\pi : (S^3, g_t) \longrightarrow (S^2, h^2(t)g_o^2)$ is a Riemannian submersion. Here $\pi : S^3 \longrightarrow S^2$ is the Hopf fibration, and g_o and g_o^2 are the round metrics on S^1 and S^2, respectively.

The metric g can be extended to a smooth metric $\mathbf{g}$ on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ provided the following asympthotic conditions hold:

$$\begin{aligned} f(\alpha) = f(\beta) = 0, \quad f'(\alpha) = -f'(\beta) = 1, \\ h(\alpha) \neq h(\beta) \neq 0, \quad h'(\alpha) = h'(\beta) = 0, \\ f^{2k}(\alpha) = f^{2k}(\beta) = h^{2k+1}(\alpha) = h^{2k+1}(\beta) = 0. \end{aligned} \tag{5.1.1}$$

If we let X, Y and Z be $SU(2)$–left invariant vector fields on S^3 given by:

$$X : \begin{pmatrix} v \\ w \end{pmatrix} \to \begin{pmatrix} iv \\ -iw \end{pmatrix}, \quad Y : \begin{pmatrix} v \\ w \end{pmatrix} \to \begin{pmatrix} w \\ -v \end{pmatrix}, \quad Z : \begin{pmatrix} v \\ w \end{pmatrix} \to \begin{pmatrix} iw \\ iv \end{pmatrix}.$$

Let $H = \frac{\partial}{\partial t}$, then $E = \left\{ H, \frac{X}{f}, \frac{Y}{h}, \frac{Z}{h} \right\}$ is an orthonormal frame on $(S^3 \times (\alpha, \beta), g)$.

We have the following commuting relations:

$$[X, Y] = 2Z \qquad [Y, Z] = 2X \qquad [Z, X] = 2Y \qquad [H, T] = 0$$

for every $T \in E$. Then the Levi-Civita connection induced by g can be computed using Koszul's formula:

$$2g(\nabla_A B, C) = Ag(B,C)+Bg(A,C)-Cg(A,B)+g([A,B],C)-g([B,C],A)+g([C,A],B).$$

We start by computing $\nabla_{\mathbf{X}}\mathbf{Y}$. For any T:

$$2g(\nabla_X Y, T) = Xg(Y,T)+Yg(X,T)-Tg(X,Y)+g([X,Y],T)-g([Y,T],X)+g([T,X],Y).$$

Note that X, Y, Z and H are orthogonal, and $g(Y,Y)$ and $g(X,X)$ are constant in the direction of X and Y respectively. It follows that the first three terms of the Koszul's formula vanish. Then we obtain:

$$2g(\nabla_X Y, T) = g([X,Y],T) - g([Y,T],X) + g([T,X],Y).$$

We use the relations (5.1). Put $T = X$, then:

$$\begin{aligned} 2g(\nabla_X Y, X) &= g([X,Y],X) - g([Y,X],X) + g([X,X],Y) \\ &= g(2Z,X) - g(-2Z,X) + g(0,Y) = 0. \end{aligned}$$

If $T = Y$:

$$2g(\nabla_X Y, Y) = g(2Z,Y) - g(0,X) + g(-2Z,Y) = 0.$$

Similarly, if $T = Z$:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y) \\ &= g(2Z,Z) - g(2X,X) + g(2Y,Y) \\ &= 2(h^2 - f^2 + h^2) = -2f^2 + 4h^2. \end{aligned}$$

Finally, if $T = H$

$$\begin{aligned} 2g(\nabla_X Y, H) &= g([X,Y],T) - g([Y,T],X) + g([T,X],Y) \\ &= g(2Z,Y) - g(0,X) + g(0,Y) = 0. \end{aligned}$$

Therefore $\nabla_X Y$ is parallel to Z. Since $g(Z,Z) = h^2$ then:

$$\nabla_X Y = \frac{2h^2(t) - f^2(t)}{h^2(t)} Z.$$

In order to compute $\nabla_{\mathbf{Y}}\mathbf{X}$ we use that $\nabla_Y X = \nabla_X Y - [X,Y]$. Then:

$$\nabla_Y X = \frac{2h^2(t) - f^2(t)}{h^2(t)} Z - 2Z = -\frac{f^2(t)}{h^2(t)} Z.$$

Now, to compute $\nabla_{\mathbf{X}}\mathbf{Z}$. We have that:

$$2g(\nabla_X Z, T) = Xg(Z,T)+Zg(X,T)-Tg(X,Z)+g([X,Z],T)-g([Z,T],X)+g([T,X],Z).$$

Note that $g(Z,Z) = h^2(t)$ y $g(X,X) = f^2(t)$ are constant in the direction of X and Z respectively. We then obtain:

$$2g(\nabla_X Z, T) = g([X,Z],T) - g([Z,T],X) + g([T,X],Z).$$

If $T = Y$:

$$\begin{aligned} 2g(\nabla_X Z, Y) &= g([X,Z],Y) - g([Z,Y],X) + g([Y,X],Z) \\ &= g(-2Y,Y) - g(-2X,X) + g(-2Z,Z) \\ &= 2(-h^2 + f^2 - h^2) = 2f^2 - 4h^2. \end{aligned}$$

For the remaining cases, all of the terms in Koszul's formula vanish. Since $g(Y,Y) = h^2(t)$, if follows:

$$\nabla_X Z = \frac{f^2(t) - 2h^2(t)}{h^2(t)} Y.$$

Similarly, to compute $\nabla_{\mathbf{Z}}\mathbf{X}$:

$$\begin{aligned} \nabla_Z X &= \nabla_X Z - [X,Z] \\ &= \left(\frac{f^2(t) - 2h^2(t)}{h^2(t)} + 2 \right) Y \\ &= \frac{f^2(t)}{h^2(t)}. \end{aligned}$$

To compute $\nabla_{\mathbf{H}}\mathbf{X}$, begin with:

$$2g(\nabla_H X, T) = Hg(X,T) + Xg(H,T) - Tg(H,X) + g([H,X],T) - g([X,T],H) + g([T,H],X).$$

Proceeding as before, we have that $g(H,X) = 0$, and $Xg(H,T) = 0$ for every T. Since we also have that $[H,T] = 0$ for any T:

$$2g(\nabla_H X, T) = Hg(X,T) - g([X,T],H) + g([T,H],X).$$

If $T = X$:

$$\begin{aligned} 2g(\nabla_H X, X) &= Hg(X,X) - g([X,X],H) + g([X,H],X) \\ &= Hg(X,X) = 2ff'. \end{aligned}$$

If $T = Y$:

$$\begin{aligned} 2g(\nabla_H X, Y) &= Hg(X,Y) - g([X,Y],H) + g([Y,H],X) \\ &= -g(2Z,H) = 0. \end{aligned}$$

If $T = Z$:

$$\begin{aligned} 2g(\nabla_H X, Z) &= Hg(X,Z) - g([X,Z],H) + g([Z,H],X) \\ &= -g(-2Y,H) = 0. \end{aligned}$$

If $T = H$:

$$2g(\nabla_H X, T) = Hg(X,H) - g([X,H],H) + g([H,H],X) = 0.$$

Since $g(X,X) = f^2$:

$$\nabla_H X = \frac{f'}{f} X.$$

By the commutation relations (5.1), $\nabla_X H - \nabla_H X = [X, H] = 0$:

$$\nabla_X H = \frac{f'}{f} X.$$

Now, for $\nabla_{\mathbf{Y}} \mathbf{Z}$. Koszul's formula says that for any T:

$$2g(\nabla_Y Z, T) = Yg(Z,T) + Zg(Y,T) - Tg(Y,Z) + g([Y,Z],T) - g([Z,T],Y) + g([T,Y],Z).$$

First three terms are zero. Then:

$$2g(\nabla_Y Z, T) = g([Y,Z],T) - g([Z,T],Y) + g([T,Y],Z).$$

If $T = X$:

$$\begin{aligned} 2g(\nabla_Y Z, X) &= g([Y,Z],X) - g([Z,X],Y) + g([X,Y],Z) \\ &= g(2X,X) - g(2Y,Y) + g(2Z,Z) \\ &= 2(f^2 - h^2 + h^2) = 2f^2. \end{aligned}$$

If $T = Y$:

$$\begin{aligned} 2g(\nabla_Y Z, Y) &= g([Y,Z],Y) - g([Z,Y],Y) + g([Y,Y],Z) \\ &= g(2X,Y) - g(-2X,Y) = 0. \end{aligned}$$

If $T = Z$:

$$\begin{aligned} 2g(\nabla_Y Z, Z) &= g([Y,Z],Z) - g([Z,Z],Y) + g([Z,Y],Z) \\ &= g(2X,Z) + g(-2X,Z) = 0. \end{aligned}$$

If $T = H$:

$$\begin{aligned} 2g(\nabla_Y Z, H) &= g([Y,Z],H) - g([Z,H],Y) + g([H,Y],Z) \\ &= g(2X,H) = 0. \end{aligned}$$

Since $g(X,X) = f^2$, we obtain:

$$\nabla_Y Z = X.$$

From $\nabla_Z Y = \nabla_Y Z - [Y,Z]$, it follows:

$$\nabla_Z Y = -X.$$

To compute $\nabla_{\mathbf{Y}} \mathbf{H}$, we have in this case that:

$$2g(\nabla_Y H, T) = Yg(H,T) + Hg(Y,T) - Tg(Y,H) + g([Y,H],T) - g([H,T],Y) + g([T,Y],H).$$

Then:

$$2g(\nabla_Y H, T) = Hg(Y,T) - g([H,T],Y) + g([T,Y],H).$$

If $T = X$:

$$\begin{aligned} 2g(\nabla_Y H, X) &= Hg(Y,X) - g([H,X],Y) + g([X,Y],H) \\ &= g(2Z,H) = 0. \end{aligned}$$

If $T = Y$, similarly as before, we obtain:

$$\begin{aligned} 2g(\nabla_Y H, Y) &= Hg(Y,Y) - g([H,Y],Y) + g([Y,Y],H) \\ &= Hg(Y,Y) = 2hh'. \end{aligned}$$

If $T = Z$:

$$\begin{aligned} 2g(\nabla_Y H, Z) &= Hg(Y,Z) - g([H,Z],Y) + g([Z,Y],H) \\ &= g(-2X,H) = 0. \end{aligned}$$

If $T = H$:

$$2g(\nabla_Y H, H) = Hg(Y,H) - g([H,H],Y) + g([H,Y],H) = 0.$$

Therefore:

$$\nabla_Y H = \frac{h'}{h} Y.$$

Then:

$$\nabla_H Y = \frac{h'}{h} Y.$$

Similarly, to compute $\nabla_{\mathbf{Z}}\mathbf{H}$. We have

$$2g(\nabla_Z H, T) = Zg(H,T) + Hg(Z,T) - Tg(Z,H) + g([Z,H],T) - g([H,T],Z) + g([T,Z],H).$$

It is the same as:

$$2g(\nabla_Z H, T) = Hg(Z,T) - g([H,T],Z) + g([T,Z],H).$$

The only non-vanishing term is obtained if $T = Z$:

$$\begin{aligned} 2g(\nabla_Z H, Z) &= Hg(Z,Z) - g([H,Z],Z) + g([Z,Z],H) \\ &= Hg(Z,Z) = 2hh'. \end{aligned}$$

It follows:

$$\nabla_Z H = \frac{h'}{h} Z.$$

Then:

$$\nabla_H Z = \frac{h'}{h} Z.$$

It remains to compute $\nabla_{\mathbf{X}}\mathbf{X}, \nabla_{\mathbf{Y}}\mathbf{Y}, \nabla_{\mathbf{Z}}\mathbf{Z}$ and $\nabla_{\mathbf{H}}\mathbf{H}$. For both cases Koszul's formula reads, for a fixed S, and for any T:

$$2g(\nabla_S S, T) = Sg(S,T) + Sg(S,T) - Tg(S,S) + g([S,S],T) - g([S,T],S) + g([T,S],S).$$

Then:

$$2g(\nabla_S S, T) = Sg(S,T) + Sg(S,T) - Tg(S,S) - g([S,T],S) + g([T,S],S).$$

There are two cases. First, if $T = S$:

$$\begin{aligned} 2g(\nabla_S S, S) &= Sg(S,S) + Sg(S,S) - Sg(S,S) - g([S,S],S) + g([S,S],S) \\ &= Sg(S,S) = 0. \end{aligned}$$

If $T \neq S$:

$$\begin{aligned}2g(\nabla_S S, T) &= Sg(S,T) + Sg(S,T) - Tg(S,S) - g([S,T],S) + g([T,S],S)\\ &= -Tg(S,S) - 2g([S,T],S).\end{aligned}$$

In the special case $S = X$, we obtain that when $T = H$:

$$\begin{aligned}2g(\nabla_X X, H) &= -Hg(X,X) - 2g([X,H],X)\\ &= -Hg(X,X) = -2ff'.\end{aligned}$$

The other products are zero. Hence

$$\nabla_X X = -ff'H.$$

In the case $S = Y$, when $T = H$

$$\begin{aligned}2g(\nabla_Y Y, H) &= -Hg(Y,Y) - 2g([Y,H],Y)\\ &= -Hg(Y,Y) = -2hh'.\end{aligned}$$

and the other products are zero. Hence:

$$\nabla_Y Y = -hh'H.$$

Similarly, we have:

$$\nabla_Z Z = -hh'H.$$

and:

$$\nabla_H H = 0.$$

The following Table contains the computations in terms of the functions f and h.

∇	X	Y	Z	H
X	$-ff'H$	$\frac{2h^2-f^2}{h^2}Z$	$\frac{(f^2-2h^2)}{h^2}Y$	$\frac{f'}{f}X$
Y	$-\frac{f^2}{h^2}Z$	$-hh'H$	X	$\frac{h'}{h}Y$
Z	$\frac{f^2}{h^2}Y$	$-X$	$-hh'H$	$\frac{h'}{h}Z$
H	$\frac{f'}{f}X$	$\frac{h'}{h}Y$	$\frac{h'}{h}Z$	$0.$

Table 5.1: Levi-Civita connection

On the other hand, the almost complex structure J of $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ restricted to $S^3 \times (\alpha,\beta)$ is given by:

$$J(H) = \frac{X}{f}, \qquad J(\frac{X}{f}) = -H, \qquad J(Y) = Z, \qquad J(Z) = -Y.$$

Recall that the Hermitian metric g is Kähler if and only if

$$\nabla_X JY = J\nabla_X Y$$

for every X, Y vector fields on $S^3 \times (\alpha,\beta)$. From Table (5.1) we obtain the computations of $J\nabla_A B$, for every $A, B \in E$:

$J\nabla$	X	Y	Z	H
X	$-f'X$	$\frac{f^2-2h^2}{h^2}Y$	$\frac{(f^2-2h^2)}{h^2}Z$	$-f'H$
Y	$\frac{f^2}{h^2}Y$	$-\frac{hh'}{f}X$	$-fH$	$\frac{h'}{h}Z$
Z	$\frac{f^2}{h^2}Z$	fH	$-\frac{hh'}{f}X$	$-\frac{h'}{h}Y$
H	$-f'H$	$\frac{h'}{h}Z$	$-\frac{h'}{h}Y$	$0.$

Table 5.2: $J\nabla_A B$

We also compute:

∇	$JX = -fH$	$JY = Z$	$JZ = -Y$	$JH = \frac{X}{f}$
X	$-f'X$	$\frac{(f^2-2h^2)}{h^2}Y$	$\frac{(f^2-2h^2)}{h^2}Z$	$-f'H$
Y	$-\frac{fh'}{h}Y$	X	$hh'H$	$-\frac{f}{h^2}Z$
Z	$-\frac{fh'}{h}Z$	$-hh'H$	X	$\frac{f}{h^2}Y$
H	$-f'H$	$\frac{h'}{h}Z$	$-\frac{h'}{h}Y$	$0.$

Table 5.3: $\nabla_A J(B)$

By comparing Tables (5.3) and (5.2) we obtain that the corresponding equation to have a Kähler metric are the following:

$$\begin{aligned} -\frac{fh'}{h} &= \tfrac{f^2}{h^2} \\ -\frac{f^2}{h^2} &= \tfrac{fh'}{h} \\ -\frac{hh'}{f} &= 1 \\ -f &= hh' \\ -hh' &= f. \end{aligned}$$

We deduce that the metric g is a Kähler metric if:

$$f = -hh'. \tag{5.1.2}$$

This and conditions (5.1.1) imply:

$$h(\alpha)h''(\alpha) = -h(\beta)h''(\beta) = -1. \tag{5.1.3}$$

Therefore, for any positive function h defined on $[\alpha, \beta]$ satisfying $h'(\alpha) = h'(\beta) = 0$, and (5.1.3), $(S^3 \times (\alpha, \beta), g)$ is a Kähler metric which extends to a Kähler metric $\mathbf{g}$ on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$.

The constant λ for a soliton can be normalized to be $1, 0$ or -1. We have that g is a $U(2)-$invariant gradient shrinking Ricci soliton if there exists a $U(2)-$invariant function u such that the equation

$$Ric(g) + Hess(u) - g = 0 \tag{5.1.4}$$

holds.

Since the Kähler metric g is in particular Hermitian, it satisfies $g(JU, JV) = g(U, V)$ for every U, V vector fields, and $Ric(JU, JV) = Ric(U, V)$. This implies that, if in addition g is a gradient Ricci soliton, $Hess(u)(JU, JV) = Hess(u)(U, V)$.

Let ψ be a smooth function on $S^3 \times (\alpha, \beta)$ invariant by the action of $U(2)$, and simply write $\psi : (\alpha, \beta) \to \mathbb{R}$. We compute the Hessian of ψ using Table (5.1), since for any $A, B \in E$, $Hess(\varphi)(A, B) = A(B(\varphi)) - \nabla_A B(\varphi)$.

In the orthonormal basis $\left\{H, \frac{X}{f}, \frac{Y}{h}, \frac{Z}{h}\right\}$, the Hessian of ψ diagonal and

$$\begin{aligned} Hess(\psi)\left(\frac{X}{f}, \frac{X}{f}\right) &= \frac{f'\psi'}{f} \\ Hess(\psi)\left(\frac{Y}{h}, \frac{Y}{h}\right) &= \frac{h'\psi'}{h} \\ Hess(\psi)\left(\frac{Z}{h}, \frac{Z}{h}\right) &= \frac{h'\psi'}{h} \\ Hess(\psi)(H, H) &= \psi''. \end{aligned} \tag{5.1.5}$$

If u is a potential function determining a gradient shrinking Ricci soliton g, the Hessian of u is $J-$invariant. This happens if and only if

$$\begin{aligned} Hess(u)(H, H) &= Hess(u)(JH, JH) = Hess(u)\left(\frac{X}{f}, \frac{X}{f}\right) \\ Hess(u)\left(\frac{Y}{h}, \frac{Y}{h}\right) &= Hess(u)\left(J\frac{Y}{h}, J\frac{Y}{h}\right) = Hess(u)\left(J\frac{Z}{h}, J\frac{Z}{h}\right), \end{aligned}$$

that is, if and only if

$$u'' = \frac{f'u'}{f}.$$

Solving for u' we obtain:

$$u' = cf$$

for some constant c. Note that $c \neq 0$, if g is a non-trivial Ricci soliton. Substituting (5.1.2) we obtain:

$$u' = -chh' = -\frac{c}{2}\frac{d}{dt}h^2.$$

Then

$$u = \frac{-ch^2}{2} + d$$

for some constants c, d, with $c \neq 0$. Without loss of generality, we may set $d = 0$, and so:

$$u = \frac{-ch^2}{2} \tag{5.1.6}$$

On the other hand, the Ricci curvature is given by

$$Ric(H,H) = -\frac{f''}{f} - 2\frac{h''}{h} \tag{5.1.7}$$

$$Ric\left(\frac{X}{f},\frac{X}{f}\right) = -\frac{f''}{f} - 2\frac{f'h'}{fh} + 2\frac{f^2}{h^4} \tag{5.1.8}$$

$$Ric\left(\frac{Y}{h},\frac{Y}{h}\right) = Ric\left(\frac{Z}{h},\frac{Z}{h}\right) = -\frac{h''}{h} - \frac{f'h'}{fh} - \frac{h'^2}{h^2} + \frac{4}{h^2} - 2\frac{f^2}{h^4} \tag{5.1.9}$$

and $Ric(A,B) = 0$ whenever $A \neq B$, for every $A, B \in E$. Since also Ricci curvature has to be J–invariant, $Ric(H,H) = Ric(\frac{X}{f},\frac{X}{f})$. Hence:

$$\frac{h''}{h} - \frac{f^2}{h^4} - \frac{f'h'}{fh} = 0.$$

Now we write the soliton equation using the previous calculations for each direction, for some constant c.

In the direction of H, by (5.1.2) we have:

$$\begin{aligned}
0 &= Ric(H,H) + Hess(u)(H,H) - g(H,H) \\
&= -\frac{f''}{f} - 2\frac{h''}{h} + u'' - 1 \\
&= -\frac{f''}{f} - 2\frac{h''}{h} - ch'^2 - chh'' - 1 \\
&= -\frac{5h''}{h} - \frac{h'''}{h'} - 1 - c(hh'' + h'^2).
\end{aligned}$$

Then we obtain the equation:

$$\frac{h'''}{h'} + \frac{5h''}{h} + 1 + c(hh'' + h'^2) = 0. \tag{5.1.10}$$

Proceeding similarly in the direction of X,

$$\begin{aligned}
0 &= Ric(X,X) + Hess(u)(X,X) - g(X,X) \\
&= f^2\left(-\frac{f''}{f} - 2\frac{f'h'}{fh} + 2\frac{f^2}{h^4}\right) + cf^2f' - f^2 \\
&= -5hh'^2h'' - h^2h'h''' - h^2h'^2 - ch^2h'^2(hh'' + h'^2),
\end{aligned}$$

and then

$$h^2h'h''' + 5hh'^2h'' + h^2h'^2(1 + c(hh'' + h'^2)) = 0, \tag{5.1.11}$$

which is equivalent to the equation (5.1.10).

In the direction of Y,

$$\begin{aligned}
0 &= Ric(Y,Y) + Hess(u)(Y,Y) - g(Y,Y) \\
&= h^2\left(-\frac{h''}{h} - \frac{f'h'}{fh} - \frac{h'^2}{h^2} + \frac{4}{h^2} - 2\frac{f^2}{h^4}\right) + hh'u' - h^2 \\
&= h^2\left(-\frac{h''}{h} - \frac{f'h'}{fh} - \frac{h'^2}{h^2} + \frac{4}{h^2} - 2\frac{f^2}{h^4}\right) - ch^2h'^2 - h^2 \\
&= -2hh'' - 4h'^2 + 4 - ch^2h'^2 - h^2.
\end{aligned}$$

We then obtain:

$$2hh'' + 4h'^2 - 4 + h^2(1 + ch'^2) = 0. \tag{5.1.12}$$

If we differentiate equation (5.1.12) with respect to t, we obtain:

$$\begin{aligned}
0 &= 2(hh''' + h'h'') + 8h'h'' + h^2(2ch'h'') + 2hh'(1 + ch'^2) \\
&= 2hh''' + 10h'h'' + 2ch^2h'h'' + 2hh' + 2chh'^3.
\end{aligned}$$

Then

$$hh''' + 5h'h'' + hh'(1 + c(hh'' + h'^2)) = 0.$$

It follows that if h solves (5.1.12), it also solves (5.1.10) and (5.1.11). We have therefore shown:

***Proposition* 5.1.2** *The $U(2)-$invariant Kähler metric $g = dt^2 + g_t$ defined on $S^3 \times (\alpha, \beta)$ by the function h, extends to a gradient shrinking Kähler-Ricci soliton* $\mathbf{g}$ *on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ if h is a smooth positive function which, for a constant $c \in \mathbb{R}$, solves the equation:*

$$2hh'' + 4h'^2 - 4 + h^2(1 + ch'^2) = 0 \tag{5.1.13}$$

with $h'(\alpha) = h'(\beta) = 0$, and $h(\alpha)h''(\alpha) = -h(\beta)h''(\beta) = -1$. It follows that $h(\alpha) = \sqrt{6}$ and $h(\beta) = \sqrt{2}$.

Without loss of generality we may choose $\alpha = 0$.

***Proposition* 5.1.3** *There exists a unique constant c_o such that for the solution of (5.1.13) satisfying $h_{c_o}(0) = \sqrt{6}$, $h'_{c_o}(0) = 0$ there exists $\beta > 0$ such that $h'_{c_o} < 0$ on $(0, \beta)$, $h'_{c_o}(\beta) = 0$ and $h_{c_o}(\beta) = \sqrt{2}$. If there exists $\beta > 0$ for which $h'_c < 0$ on $(0, \beta)$, $h'_c(\beta) = 0$ and $h_c(\beta) > \sqrt{2}$ (or $h_c(\beta) < \sqrt{2}$) then $c_o < c$ (resp. $c < c_o$). The constant c_o can be computed numerically, its aproximation is $c_o = -0.5276195198969626$.*

Proof: Given a constant $c \neq 0$ and $s > 0$ we have a solution $h_{c,s}$ to (5.1.13) satisfying $h_{c,s}(0) = s$, ${h_{c,s}}'(0) = 0$. Note that if $h_{c,s}(t)$ is solution to (5.1.13), with ${h_{c,s}}'(0) = 0$, then $h_{c,s}(-t)$ is also a solution. Or more generally, a solution to (5.1.13) will be symmetric around any critical point.

Let $v = h'$, then from equation (5.1.13) we obtain the following non-linear autonomous system:

$$\begin{aligned} h' &= v \\ v' &= \frac{2}{h} - 2\frac{v^2}{h} - \frac{h}{2}(1 + cv^2). \end{aligned} \tag{5.1.14}$$

We may depict its corresponding phase portrait in coordinates (h, v) in the right-half plane $\mathbb{R}^+ \times \mathbb{R} \subset \mathbb{R} \times \mathbb{R}$. Observe that $(2, 0)$ is the only critical point of the system.

Note that if $h_{c,s}$ has a maximum and a minimum, then it is periodic and the corresponding integral curve of (5.1.14) is closed. Note also that $h_{c,s}$ has a maximum at 0 if $s > 2$, and if it has its first minimum at $\beta_{c,s} > 0$, then $0 < h_{c,s}(\beta_{c,s_1}) < 2$. In this case ${h_{c,s}}' < 0$ on $(0, \beta_{c,s})$.

Let d and c, with $c < d$, and $s_1, s_2 > 2$, such that the solutions h_{c,s_1} and h_{d,s_2} to (5.1.13), with initial conditions ${h_{c,s_1}}'(0) = {h_{d,s_2}}'(0) = 0$, have first minimums at β_{c,s_1} and β_{d,s_2}, respectively. Denote by $\gamma_c^{s_1}$ and $\gamma_d^{s_2}$ the corresponding integral curves, they are closed curves.

If $\gamma_c^{s_1}$ and $\gamma_d^{s_2}$ intersect in the lower-half plane, there exist $t_1, t_2 > 0$ such that $h_{c,s_1}(t_2) = h_{d,s_2}(t_1) > 0$, and ${h_{c,s_1}}'(t_2) = {h_{d,s_2}}'(t_1) < 0$. Then we have two equations:

$$2h_{c,s_1}(t_2)h''_{c,s_1}(t_2) + 4h'^2_{c,s_1}(t_2) - 4 + h^2_{c,s_1}(t_2)(1 + ch'^2_{c,s_1}(t_2)) = 0, \tag{5.1.15}$$

$$2h_{d,s_2}(t_1)h''_{d,s_2}(t_1) + 4h'^2_{d,s_2}(t_1) - 4 + h^2_{d,s_2}(t_1)(1 + dh'^2_{d,s_2}(t_1)) = 0. \tag{5.1.16}$$

Also we obtain:

$$2h_{d,s_2}(t_1)h''_{c,s_1}(t_2) + 4h'^2_{d,s_2}(t_1) - 4 + h^2_{d,s_2}(t_1)(1 + ch'^2_{d,s_2}(t_1)) = 0. \tag{5.1.17}$$

If we substract (5.1.17) from (5.1.16) we find:

$$2h_{d,s_2}(t_1)(h''_{d,s_2}(t_1) - h''_{c,s_1}(t_2)) + h^2_{d,s_2}(t_1)h'_{d,s_2}(t_2)(d - c) = 0.$$

Therefore, since $h_{d,s_2}(t_1) > 0$ and $h'_{d,s_2}(t_2) < 0$, we obtain $h''_{d,s_2}(t_1) > h''_{c,s_1}(t_2)$. It follows that, since

$$\frac{d}{dt}\gamma_c^{s_1} = (h'_{c,s_1}, h''_{c,s_1}) = (h'_{c,s_1}, \frac{2}{h_{c,s_1}} - 2\frac{{h'_{c,s_1}}^2}{h_{c,s_1}} - \frac{h_{c,s_1}}{2}(1 + c{h'_{c,s_1}}^2)),$$

and

$$\frac{d}{dt}\gamma_d^{s_2} = (h'_{d,s_2}, h''_{d,s_2}) = (h'_{d,s_2}, \frac{2}{h_{d,s_2}} - 2\frac{{h'_{d,s_2}}^2}{h_{d,s_2}} - \frac{h_{d,s_2}}{2}(1 + d{h'_{d,s_2}}^2)),$$

if $\gamma_c^{s_1}$ and $\gamma_d^{s_2}$ intersect in the lower-half plane, after the crossing point $\gamma_c^{s_1}$ goes above $\gamma_d^{s_2}$. It follows that if h_{c,s_1} has a minimum, so does h_{d,s_2}, and $h_{d,s_2}(\beta_{d,s_2}) > h_{c,s_1}(\beta_{c,s_1})$.

Now, take $s_1 = s_2 = \sqrt{6}$, $c < d$ as before, and such that $h_c = h_{c,\sqrt{6}}$ and $h_d = h_{d,\sqrt{6}}$, both have positive minimums m_c and m_d, respectively. We have their corresponding integral curves γ_c and γ_d.

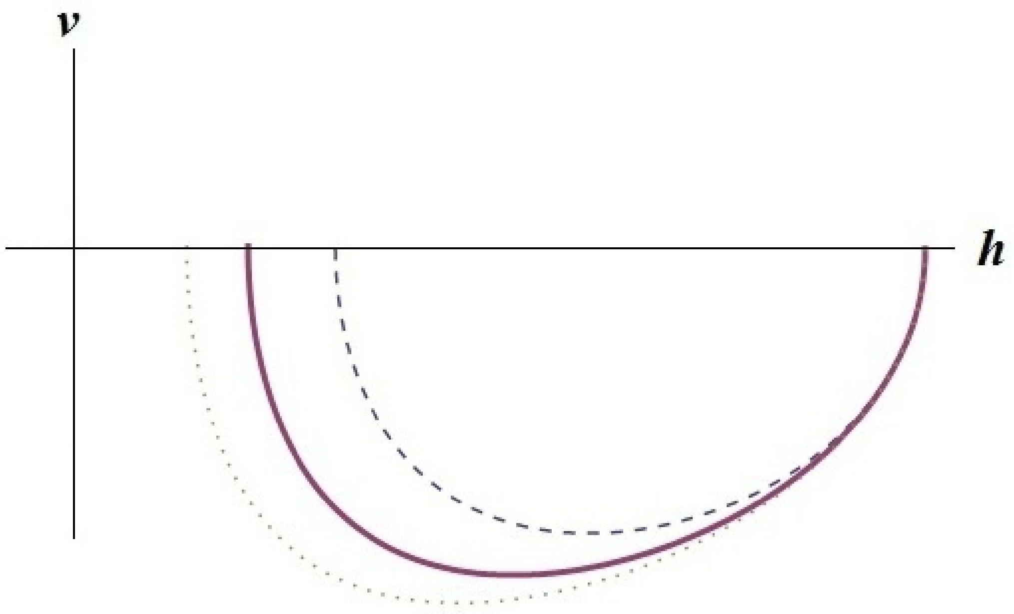

Figure 5.1: A diagram depicting the behavior of three solutions of the system (5.1.14) with same initial conditions. For $c < c_o < d$, the solid line corresponds to $h_{c_o,\sqrt{6}}$, the dashed one to $h_{d,\sqrt{6}}$ and the dotted one to $h_{c,\sqrt{6}}$.

Define the function $min : (c,d) \to (0,2)$ which assigns to each $a \in (c,d)$ the first minimum value of the solution $h_a^{\sqrt{6}}$. We will prove that the function min is well defined, continuous and increasing.

If γ_c goes under γ_d, it has to intersect some γ_d^s, with $2 < s < \sqrt{6}$. But then γ_c also intersects every integral curve γ_d^t, with $t < s$. Take $\gamma_d^{\frac{s+\sqrt{6}}{2}}$, for instance. Since $d > c$, by previous discussion, γ_c stays below $\gamma_d^{\frac{s+\sqrt{6}}{2}}$ and cannot intersect γ_d^s. Then, in particular $m_c < m_d$. Same argument applies to each $a \in (c,d)$, the solution $h_{a,\sqrt{6}}$ has a minimum. Then the associated integral curve is closed, since $\sqrt{6} > 2$. We have that $m_d < min(a) < m_c$.

Therefore, it can only exist one $c_o \in (c,d)$ such that the solution $h_{c_o,\sqrt{6}}$ to (5.1.13) with $h'_{c_o,\sqrt{6}}(0) = 0$ has as minimum value $\sqrt{2}$.

Finally, the value of c_o can be computed numerically by solving the equation (5.1.13) for different parameters c, and initial conditions $h(0) = \sqrt{6}$ and $h'(0) = 0$. By the claim, if $c < d$ are two values for which the solutions h_c and h_d each have minimums m_c, m_d, and $m_d < \sqrt{2} < m_c$ then we know that $c < c_o < d$. Then c_o can be approximated by interpolation.

Cao also obtained the value of c_o as root of the function:

$$k(x) = e^{2x}(2 - 4x + 3x^2) - 2 + x^2.$$

See Lemma 4.1. in [10]. □

5.2 Curvature of the Koiso-Cao soliton

In this section we will write the Yamabe equation on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ for a $U(2)-$invariant function. Recall that $\mathbf{g}$ has positive Ricci curvature [10]. We will use this to see that $S_{\mathbf{g}}$ is a decreasing function of t on $S^3 \times (\alpha, \beta)$. For the sake of completeness we give a proof of the positivity of Ricci curvature using our description of $\mathbf{g}$.

***Proposition* 5.2.1** *The Koiso- Cao soliton* $\mathbf{g}$ *has positive Ricci curvature.*

Proof: For a gradient shrinking soliton we have the equation (5.1.4). Note that since $Hess(u)$ is diagonal in the orthonormal basis E, the Ricci curvature of $\mathbf{g}$ is diagonal. If $\mathbf{g}$ is Kähler the Ricci curvature is invariant by the action of J,

$$Ric\left(\frac{X}{f}, \frac{X}{f}\right) = Ric(H, H)$$
$$Ric\left(\frac{Z}{h}, \frac{Z}{h}\right) = Ric\left(\frac{Y}{h}, \frac{Y}{h}\right).$$

Then we only have to show that the functions $Ric(H, H)$ and $Ric(\frac{Y}{h}, \frac{Y}{h})$ are positive. Using the Hessian (5.1.5) and the expression of u (5.1.6) we obtain

$$\begin{aligned} Ric(H, H) &= g(H, H) - Hess(u)(H, H) \\ &= 1 - u'' \\ &= 1 + c(hh'' + h'^2), \end{aligned}$$

and

$$\begin{aligned} Ric\left(\frac{Y}{h}, \frac{Y}{h}\right) &= g\left(\frac{Y}{h}, \frac{Y}{h}\right) - Hess(u)\left(\frac{Y}{h}, \frac{Y}{h}\right) \\ &= 1 - \frac{h'u'}{h} \\ &= 1 + ch'^2. \end{aligned}$$

First we will lead with the Ricci curvature in the direction of Y. From equation (5.1.13) we rewrite

$$(hh'' + h'^2) + h'^2 - 2 + \frac{h^2}{2}(1 + ch'^2) = 0$$

which is the same as:

$$f' = h'^2 - 2 + \frac{h^2}{2}(1 + ch'^2) \tag{5.2.1}$$

Let $A(x) = x - 2 + \frac{b^2}{2}(1 + cx)$. If we differentiate with respect to x,

$$\frac{d}{dx}A = 1 + c\frac{b^2}{2}$$

The critical points of A occurs when $\frac{b^2}{2} = -1/c$.

Claim: $1 + ch'^2 > 0$.

We have a critical point of the function $(1 + ch'^2)$ if $h' = 0$ or $h'' = 0$. Observe that that $h' = 0$ and $h'' = 0$ can not happen simultaneously, else the value of the critical points would be 2. This is not possible since critical values have to be $\sqrt{2}$ and $\sqrt{6}$. If $h' = 0$, the claim follows and then $Ric(\frac{Y}{h}, \frac{Y}{h}) > 0$ at 0 and β. For those points where $h'' = 0$, from equation (5.1.13):

$$h'^2 = \frac{4 - h^2}{4 + ch^2}. \tag{5.2.2}$$

There are two cases.

i) If $3 > \frac{h^2}{2} \geq -1/c$. Then, $4 - h^2 < 4 + 2/c$. From the value of $c = -0.5276195198969626$ we have $4 + 2/c < 1$. Also, $4 + 6c < 4 + ch^2$. If follows that

$$h'^2 < \frac{4 + 2/c}{4 + 6c} < \frac{1}{4 + 6c} < -1/c,$$

and then $1 + ch'^2 > 0$.

ii) If $1 < \frac{h^2}{2} < -1/c$, we have $4 - h^2 < 2$ and $2 < 4 + ch^2$. Then:

$$h'^2 = \frac{4 - h^2}{4 + ch^2} < 1.$$

The claim is proved, and then $Ric(\frac{Y}{h}, \frac{Y}{h}) > 0$.

On the other hand, for the Ricci curvature in the direction of H, note that, since $-1 < c < 1$ and $h(0)h''(0) = -h(\beta)h''(\beta) = 1$, $Ric(H, H)$ is positive at 0 and β. Then it remains to prove that $1 + c(hh'' + h'^2) > 0$ for every $t \in (0, \beta)$.

Since $c < 0$, we have to prove $f' = (hh'' + h'^2) > 1/c \approx -1.8$. It is enough to see $f' > -1$. Hence we will show the bound for f' in the two previous cases.

Using equation (5.2.1), the fact that $Ric(\frac{Y}{h}, \frac{Y}{h}) > 0$ and the value of c:

i) If $\frac{h^2}{2} \geq -1/c$ we have,

$$\begin{aligned} f' &= h'^2 - 2 + \frac{h^2}{2}(1 + ch'^2) \\ &\geq h'^2 - 2 - \frac{1}{c}(1 + ch'^2) \\ &= -2 - 1/c > -1. \end{aligned}$$

ii) If $1 < \frac{h^2}{2} < -1/c$, first note that $(1 + \frac{ch^2}{2}) > 0$, and since $1 < \frac{h^2}{2}$:

$$\begin{aligned} f' &= h'^2(1 + \frac{ch^2}{2}) + \frac{h^2}{2} - 2 \\ &> \frac{h^2}{2} - 2 > -1. \end{aligned}$$

Therefore, $f' > -1$, and so $Ric(H,H) > 0$. $\square$

Let ψ be a smooth function on $S^3 \times (0,\beta)$ invariant by the action of $U(2)$, then by (5.1.5) we get

$$\begin{aligned} \triangle_{\mathbf{g}}\psi &= -\frac{f'\psi'}{f} - 2\frac{h'\psi'}{h} - \psi'' \\ &= -\psi'\left(\frac{hh'' + h'^2}{hh'} + 2\frac{h'}{h}\right) - \psi'' \\ &= -\psi'\left(\frac{h''}{h'} + 3\frac{h'}{h}\right) - \psi''. \end{aligned} \tag{5.2.3}$$

Then for $u = -\frac{ch^2}{2}$ we have:

$$\begin{aligned} \triangle_{\mathbf{g}}u &= \frac{d}{dt}\left(\frac{ch^2}{2}\right)\left(\frac{h''}{h'} + 3\frac{h'}{h}\right) + \frac{d^2}{dt^2}\left(\frac{ch^2}{2}\right) \\ &= chh'' + 3ch'^2 + chh'' + ch'^2 \\ &= 4ch'^2 + 2chh''. \end{aligned}$$

Now, taking the trace of the soliton equation (5.1.4) we obtain

$$S_{\mathbf{g}} - \triangle_{\mathbf{g}}u = 4$$

and therefore the scalar curvature is:

$$S_{\mathbf{g}} = 4ch'^2 + 2chh'' + 4. \tag{5.2.4}$$

From Proposition 5.1.2 and Proposition 5.1.3 we obtain:

$$\begin{aligned} S_{\mathbf{g}}(0) &= 4 - 2c > 0 \\ S_{\mathbf{g}}(\beta) &= 4 + 2c > 0. \end{aligned} \tag{5.2.5}$$

***Proposition* 5.2.2** *The scalar curvature of the Koiso-Cao soliton* $\mathbf{g}$ *is decreasing as a function of* $t \in [0,\beta]$.

Proof: Take the derivative with respect to t of (5.2.4), then:

$$S' = 8ch'h'' + 2c(hh''' + h'h'') = 10ch'h'' + 2chh'''.$$

But,

$$\begin{aligned} Ric_{\mathbf{g}}(H,H) &= -\frac{f''}{f} - 2\frac{h''}{h} \\ &= -\frac{hh''' + 5h'h''}{hh'}. \end{aligned}$$

Since $Ric_{\mathbf{g}}(H,H) > 0$, we have that $hh''' + 5h'h'' > 0$. Therefore, since $c < 0$, $S' < 0$ on $(0,\beta)$. $\square$

We can also compute the volume of the soliton.

$$\begin{aligned}\mathrm{Vol}(\mathbb{CP}^2\#\overline{\mathbb{CP}^2},\mathbf{g}) := \mathrm{Vol}(\mathbf{g}) &= 2\pi^2\int_\alpha^\beta fh^2dt\\ &= 2\pi^2\int_\alpha^\beta(-h^3h')dt\\ &= -2\pi^2\frac{h^4}{4}|_\alpha^\beta\\ &= 16\pi^2.\end{aligned}$$

Hence, the Hilbert-Einstein functional for the soliton is:

$$\begin{aligned}\frac{\int_\alpha^\beta S}{\mathrm{Vol}(\mathbf{g})^{1/2}} &= \frac{\int_\alpha^\beta(\triangle u+4)fh^2dt}{\mathrm{Vol}(\mathbf{g})^{1/2}}\\ &= \frac{\int_\alpha^\beta 4fh^2dt}{\mathrm{Vol}(\mathbf{g})^{1/2}}\\ &= 16\pi\sim 50.26548246\\ &< 8\pi\sqrt{6}\sim 61.5624.\end{aligned}$$

5.3 Uniqueness of invariant solutions to the Yamabe equation

We have considered the non-trivial Ricci soliton on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ constructed by Koiso and Cao. It was constructed as a Kähler metric invariant by the $U(2)$ action on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$. Now we will study its Yamabe equation and prove it has exactly one $U(2)-$invariant solution, up to homothecy.

Let φ be a smooth function invariant under the action of $U(2)$ on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$. Then we identify φ with a function $\phi:[0,\beta]\to\mathbb{R}$ such that $\phi'(0)=\phi'(\beta)=0$. We use previous calculation of the Laplacian (5.2.3) to obtain the Yamabe equation of the Koiso-Cao soliton:

$$6\phi'\left(\frac{h''}{h'}+3\frac{h'}{h}\right)+6\phi''+S_{\mathbf{g}}\phi=\lambda\phi^3. \tag{5.3.1}$$

Since the Koiso-Cao soliton has positive scalar curvature, then λ has to be positive. We fix $\lambda=1$.

***Theorem* 5.3.1** *There exists a unique $U(2)$-invariant solution to the Yamabe equation on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ with the Koiso-Cao metric.*

Proof: Let ϕ be a positive $U(2)-$invariant function on $S^3\times(0,\beta)$ with the Kähler metric g. In order to $\phi^2\cdot\mathbf{g}$ be a Riemannian metric on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, ϕ must satisfy $\phi'(0)=\phi'(\beta)=0$. Additionally, the metric $\phi^2\cdot\mathbf{g}$ has constant scalar curvature equals 1 if ϕ satisfies the Yamabe equation (5.3.1):

$$6\phi'\left(\frac{h''}{h'}+3\frac{h'}{h}\right)+6\phi''+S_{\mathbf{g}}\phi=\phi^3.$$

Since:

$$\begin{aligned}\lim_{t\to 0}\phi'\frac{h''}{h'} &= \lim_{t\to 0}\left(\frac{\phi' h'''}{h''}+\phi''\right)\\ &= \phi''(0)\end{aligned}$$

then we have:

$$12\phi''(0)=\phi(0)(\phi^2(0)-S_{\mathbf{g}}(0)).$$

For $t\in(0,\beta)$ with $\phi'(t)=0$,

$$6\phi''(t)=\phi(t)(\phi^2(t)-S_{\mathbf{g}}(t)).$$

Claim: $\phi^2(0)\le S(0)=4-2c$ (by (5.2.5)).

If $\phi^2(0)>S(0)$ then 0 is a local minimum. ϕ increases and S decreases. After 0, at the next critical point of ϕ we would again have that $\phi^2>S$ and it would be a minimum, which is a contradiction.

Then, same argument shows that there is no local minimums on $[0,\beta)$. Therefore, 0 is a local maximum and β a local minimum, and there is no more critical points of ϕ on $(0,\beta)$. Then ϕ is decreasing.

We have:

$$\sqrt{S(\beta)}\le\phi(\beta)\le\phi(0)\le\sqrt{S(0)}.$$

It is known that on any compact Riemannian manifold (M,g) that admits the action of a compact Lie group G, there exists a conformal G-invariant metric to g of constant scalar curvature [31]. Hence, in the case of the Koiso-Cao soliton there exists a $U(2)$-invariant solution to the Yamabe problem. It remains to prove uniqueness. If $s\in(\sqrt{S(\beta)},\sqrt{S(0)}]$, let ϕ_s be a solution to the Yamabe equation (5.3.1) such that $\phi_s(0)=s$. Let s_1, $s_2\in(\sqrt{S(\beta)},\sqrt{S(0)}]$, $s_1<s_2$. We have 0 is a local maximum for both solutions ϕ_{s_1} and ϕ_{s_2}. Define $F(t):=\phi_{s_2}-\phi_{s_1}$.

Claim: F has a local minimum at 0.

Note that $F(0)>0$. Since 0 is a critical point of ϕ_{s_1} and ϕ_{s_2}, so it is for F. Since ϕ_{s_1} and ϕ_{s_2} satisfy (5.3.1), it follows that:

$$F''(0)=\frac{1}{12}\left(\phi_{s_2}^3(0)-\phi_{s_2}(0)S(0)-(\phi_{s_1}^3(0)-\phi_{s_1}(0)S(0)\right).$$

Consider the function $v(x)=x^3-S(0)x$. Then:

$$v'(x)=3x^2-S(0).$$

Note that $x_o:=\sqrt{\frac{S(0)}{3}}$ is the only critical point, and v is increasing on $(\sqrt{\frac{S(0)}{3}},\sqrt{S(0)})$.

We have, by (5.2.5) and the value of c,

$$S(0) = 4 - 2c = 5.0552 \quad \text{and} \quad S(\beta) = 4 + 2c = 2.9447.$$

then

$$\sqrt{S(0)} = 2.2483, \quad \sqrt{S(\beta)} = 1.716, \quad \text{and} \quad \sqrt{\frac{S(0)}{3}} = 1.2981.$$

Hence, since

$$\sqrt{\frac{S(0)}{3}} < \sqrt{S(\beta)} < \phi_{s_1}(0) < \phi_{s_2}(0) \leq \sqrt{S(0)},$$

it follows that $F''(0) > 0$, and so 0 is a local minimum of F.

Hence, F is positive and increasing on $(0, \varepsilon)$, for $\varepsilon > 0$ enough small. Assume that there exists $t_o \in (\varepsilon, \beta]$ such that $F'(t_o) = 0$, and take the first critical point. The discussion above applies for t_o, that is, we have:

$$F''(t_o) = \frac{1}{6}\left(\phi_{s_2}^3(t_o) - \phi_{s_2}(t_o)S(t_o) - (\phi_{s_1}^3(t_o) - \phi_{s_1}(t_o)S(t_o)\right).$$

Similarly as before, consider the function $g(x) = x^3 - S(t_o)x$. Then $x_1 := \sqrt{\frac{S(t_o)}{3}}$ is its only critical point, which is a minimum. From the choosing of t_o it follows :

$$\sqrt{\frac{S(t_o)}{3}} < S(\beta) < \phi_{s_1}(t_o) < \phi_{s_2}(t_o).$$

Then t_o is a local minimum of F. But this implies there are another $t_1 \in (0, t_o)$ satisfying $F'(t_1) = 0$, which is a contradiction, since t_o was the first point.

We have proved that $F' > 0$ on $(0, \beta]$ and in particular $F'(\beta) = \phi_{s_2}(\beta) - \phi_{s_1}(\beta) > 0$. Then it cannot happen that $\phi'_{s_1}(\beta) = \phi'_{s_2}(\beta) = 0$. This proves uniqueness. $\square$

Chapter 6

Singular Poisson structures in Wrinkled Fibrations in dimension 4

In this chapter we give explicit Poisson bivectors for a certain type of singular fibrations. We obtain Poisson structures whose characteristic distributions are singular.

The following Proposition was shown to hold by L. García-Naranjo, P. Suárez-Serrato and R. Vera in [25]. It provides a way to construct Poisson integrable structures, and a class of Poisson structures from a given one by multiplying with a non-vanishing function.

***Proposition* 6.0.2**

1. *If π is a bivector field on M whose characteristic distribution is integrable and has rank less than or equal to two at each point, then π is Poisson.*
2. *Let π be a Poisson structure on M whose rank at each point is less than or equal to two. Then $\pi_1 := k\pi$ is also a Poisson structure where $k \in C^\infty(M)$ is an arbitrary non-vanishing function.*

It was also shown in [25]:

***Theorem* 6.0.3** *Let M be an orientable n-manifold, N an orientable $n-2$ manifold, and $f : M \to N$ a smooth map. Let μ and Ω be orientations of M and N respectively. The bracket on M defined by*

$$\{g, h\}\mu = k\, dg \wedge dh \wedge f^*\Omega \tag{6.0.1}$$

where k is any non-vanishing function on M is Poisson. Moreover, its symplectic leaves are

(i) the 2-dimensional leaves $f^{-1}(s)$ where $s \in N$ is a regular value of f,

(ii) the 2-dimensional leaves $f^{-1}(s) \setminus \{\text{Critical Points of } f\}$ where $s \in N$ is a singular value of f.

(iii) the 0-dimensional leaves corresponding to each critical point.

Formula (6.0.1) appeared in [18]. It is attributed to H. Flaschka and T. Ratiu.

6.1 Poisson structures on wrinkled fibrations in dimension 4

The aim of this section is to obtain:

***Theorem* 6.1.1** *Let X be a closed, orientable, smooth 4-manifold equipped with a wrinkled fibration, or for a fixed s a fibration given by one of the birth, merging, flipping, or wrinkiling moves described above. Then there exists a complete Poisson structure whose symplectic leaves correspond to the fibres of the given fibration structure, and the singularities of both the fibration and the Poisson structures coincide.*

The proof of the existence is given by explicit construction. It relies on the Flaschka-Ratiu formula. We will give the local expressions for the Poisson bivectors for each type of singularity and their corresponding symplectic forms.

Furtheremore by 2.5.10, since $\pi_2(S^2) = \mathbb{Z}$, it directly follows:

***Proposition* 6.1.2** *A Poisson structure associated to a wrinkled fibration structure as in Theorem 6.1.1, or to a broken Lefschetz fibration as in [25], none of whose symplectic leaves are, or contain, 2–spheres, is integrable.*

The following result allows us to obtain algebraic conditions for the existence of Poisson structures that can be fed to a computer.

***Proposition* 6.1.3** *Let π be a bivector field in $\mathbb{R}^4$. Assume that in local coordinates we can write $\pi^{ij} = p(x,y,z,t)x^{ij}$ for a function $p : \mathbb{R}^4 \to \mathbb{R}$. Then the Jacobi identity for π holds if and only if the following equation is satisfied:*

$$\nabla(px^{ij}) \cdot (x^{1k}, x^{2k}, x^{3k}, x^{4k}) + \nabla(px^{jk}) \cdot (x^{1i}, x^{2i}, x^{3i}, x^{4i}) + \nabla(px^{ki}) \cdot (x^{1j}, x^{2j}, x^{3j}, x^{4j}) = 0 \tag{6.1.1}$$

Proof: An equivalent formulation of the Jacobi identity for a Poisson bracket can be given in local coordinates in the following way (see [37]):

$$\sum_{l=1}^{4} \left(\pi^{lk} \frac{\partial \pi^{ij}}{\partial x_l} + \pi^{li} \frac{\partial \pi^{jk}}{\partial x_l} + \pi^{lj} \frac{\partial \pi^{ki}}{\partial x_l} \right) = 0$$

For all i, j, and k. Here, the term x_l stands for the $l-$th coordinate of $\{x, y, z, t\}$. Since $\pi^{ij} = p(x,y,z,t)x^{ij}$, substituting in the previous equation we obtain:

$$\sum_{l=1}^{4} p \left(x^{lk} \frac{\partial x^{ij}}{\partial x_l} + x^{li} \frac{\partial x^{jk}}{\partial x_l} + x^{lj} \frac{\partial x^{ki}}{\partial x_l} \right) + \frac{\partial p}{\partial x_l} \left(x^{lk}x^{ij} + x^{li}x^{jk} + x^{lj}x^{ki} \right) = 0$$

Developing one of the terms we observe that:

$$\begin{aligned}\sum_{l=1}^{4} p\left(x^{lk}\frac{\partial x^{ij}}{\partial x_l} + x^{li}\frac{\partial x^{jk}}{\partial x_l} + x^{lj}\frac{\partial x^{ki}}{\partial x_l}\right) = \; & p((x^{1k}, x^{2k}, x^{3k}, x^{4k}) \cdot \nabla x^{ij} \\ & + (x^{1i}, x^{2i}, x^{3i}, x^{4i}) \cdot \nabla x^{jk} \\ & + (x^{1j}, x^{2j}, x^{3j}, x^{4j}) \cdot \nabla x^{ki})\end{aligned}$$

A similar approach to the other term yields:

$$\begin{aligned}\sum_{l=1}^{4} \frac{\partial p}{\partial x_l}\left(x^{lk}x^{ij} + x^{li}x^{jk} + x^{lj}x^{ki}\right) = \; & x^{ij}\nabla p \cdot (x^{1k}, x^{2k}, x^{3k}, x^{4k}) \\ & + x^{jk}\nabla p \cdot (x^{1i}, x^{2i}, x^{3i}, x^{4i}) \\ & + x^{ki}\nabla p \cdot (x^{1j}, x^{2j}, x^{3j}, x^{4j})\end{aligned}$$

Therefore the following equality holds:

$$\begin{aligned}(p\nabla x^{ij} + x^{ij}\nabla p) \cdot (x^{1k}, x^{2k}, x^{3k}, x^{4k}) + (p\nabla x^{jk} + x^{jk}\nabla p) \cdot (x^{1i}, x^{2i}, x^{3i}, x^{4i}) \\ + (p\nabla x^{ki} + x^{ki}\nabla p) \cdot (x^{1j}, x^{2j}, x^{3j}, x^{4j}) = 0\end{aligned}$$

We find that the Jacobi identity will hold if and only if the next equation is satisfied:

$$\nabla(px^{ij}) \cdot (x^{1k}, x^{2k}, x^{3k}, x^{4k}) + \nabla(px^{jk}) \cdot (x^{1i}, x^{2i}, x^{3i}, x^{4i}) + \nabla(px^{ki}) \cdot (x^{1j}, x^{2j}, x^{3j}, x^{4j}) = 0 \tag{6.1.2}$$

□

6.1.1 Local forumlæ for the Poisson bivectors.

We will now construct explicit expressions for the Poisson structure and the corresponding symplectic forms in a neighbourhood of cusp singularities of wrinkled fibrations $X \to \Sigma$, as well as for all the possible moves described above. All of the expressions that we will give depend upon a choice of a non-vanishing function $k \in C^\infty(X)$ (see [25]).

Before proceeding we will describe the general strategy employed to find the local bivectors.

Step 1: Consider the coordinate functions C_1, C_2 that describe each fibration as Casimir functions for the Poisson structure that we want to find.

Step 2: Calculate the differentials dC_1, dC_2 of the Casimirs C_1, C_2.

Step 3: We use formula 6.0.1 to compute the skew-symmetric matrix with entries:

$$\pi^{ij} = \{x^i, x^j\}\mu = dx^i \wedge dx^j \wedge dC_1 \wedge dC_2.$$

This matrix will then annihilate dC_1, dC_2, as this matrix is to be the endomorphism $\mathcal{B}$ associated to a Poisson structure with dC_1 and dC_2 as Casimirs. The components of the bivector field will be given by:

$$\{x^i, x^j\} = \det\left(\epsilon^i, \epsilon^j, dC_1, dC_2\right)$$

Here ϵ^i is the 4×1 canonical basis column vector, whose i-th component is 1 and all others are zero.

For the cusp singularity and the birth, merging, and flipping moves, t is a Casimir, so we only have to compute

$$\{x,y\}, \quad \{x,z\} \quad \text{and} \quad \{y,z\}.$$

In fact, for these four cases if we denote by dC_2^i the components of the column vector dC_2 we obtain

$$\begin{aligned} \{x,y\} &= -dC_2^3 \\ \{x,z\} &= dC_2^2 \\ \{y,z\} &= -dC_2^1 \end{aligned}$$

Step 4: According to Proposition 6.0.2 (ii), we write the Poisson bivector using the skew-symmetric matrix entries.

Near a wrinkling move the Poisson bivector will be obtained using formula 6.0.1. For the other cases the bivector admits a general expression given by:

$$\pi = k(x,y,z,t)\left[-dC_2^3\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + dC_2^2\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} - dC_2^1\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\right] \tag{6.1.3}$$

for any non-vanishing smooth function k.

The corresponding results for neighborhoods of Lefschetz singularities and broken singular circles were obtained in [25]. We proceed to describe the results this general strategy yields for cusp singularities and the moves described above.

Local expressions near a cusp singularity.

The local coordinate model around a cusp singularity is given by:

$$(x,y,z,t) \mapsto (C_1(x,y,z,t), C_2(x,y,z,t)) = (t, x^3 - 3xt + y^2 - z^2)$$

The differentials dC_1 and dC_2 are therefore:

$$\begin{aligned} dC_1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \\ dC_2 &= \begin{pmatrix} 3x^2 - 3t & 2y & -2z & -3x \end{pmatrix} \end{aligned}$$

The following matrix annihilates dC_1 and dC_2 and its entries satisfy the Jacobi identity:

$$\begin{pmatrix} 0 & 2kz & 2ky & 0 \\ -2kz & 0 & k(3t-3x^2) & 0 \\ -2ky & k(3x^2-3t) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Which means that the Poisson bivector in the local coordinates of a cusp singularity is described by:

$$\pi = k(x,y,z,t)\left[2z\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} + (3t-3x^2)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\right] \tag{6.1.4}$$

Local expressions near a birth move.

The local coordinate model around a birth move is given by:

$$b_s(x,y,z,t) = (C_1(x,y,z,t), C_2(x,y,z,t)) = (t, x^3 - 3x(t^2-s) + y^2 - z^2)$$

The differentials dC_1 and dC_2 are therefore:

$$\begin{array}{rcccc} dC_1 = & (0 & 0 & 0 & 1) \\ dC_2 = & (3x^2 - 3(t^2-s) & 2y & -2z & -6xt) \end{array}$$

From which we can obtain the following matrix:

$$\begin{pmatrix} 0 & 2kz & 2ky & 0 \\ -2kz & 0 & k\left(3\left(t^2-s\right)-3x^2\right) & 0 \\ -2ky & k\left(3x^2-3\left(t^2-s\right)\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence the Poisson bivector near a birth move has the form:

$$\pi_s = k(x,y,z,t)\left[2z\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} - 3(s-t^2+x^2)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\right] \tag{6.1.5}$$

Local expressions near a merging move.

The local coordinate model around a merging move is given by:

$$m_s(x,y,z,t) = (C_1(x,y,z,t), C_2(x,y,z,t)) = (t, x^3 - 3x(s-t^2) + y^2 - z^2)$$

The differentials dC_1 and dC_2 are therefore:

$$\begin{array}{rcccc} dC_1 = & (0 & 0 & 0 & 1) \\ dC_2 = & (3x^2 - 3(s-t^2) & 2y & -2z & 6xt) \end{array}$$

The associated matrix is then:

$$\begin{pmatrix} 0 & 2kz & 2ky & 0 \\ -2kz & 0 & k\left(3\left(s-t^2\right)-3x^2\right) & 0 \\ -2ky & k\left(3x^2-3\left(s-t^2\right)\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So the Poisson bivector in a neighbourhood of a merging move is described as:

$$\pi_s = k(x,y,z,t)\left[2z\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} - 3(s-t^2-x^2)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\right] \tag{6.1.6}$$

Local expressions near a flipping move.

The local coordinate model around a flipping move is given by:

$$f_s(x,y,z,t) = (C_1(x,y,z,t), C_2(x,y,z,t)) = (t, x^4 - x^2 s + xt + y^2 - z^2)$$

The differentials dC_1 and dC_2 are therefore:

$$\begin{array}{rcccc} dC_1 = & (0 & 0 & 0 & 1) \\ dC_2 = & (4x^3 - 2xs + t & 2y & -2z & x) \end{array}$$

The corresponding matrix is:

$$\begin{pmatrix} 0 & 2kz & 2ky & 0 \\ -2kz & 0 & k\left(-4x^3 + 2sx - t\right) & 0 \\ -2ky & k\left(4x^3 - 2sx + t\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The Poisson bivector in a neighborhood of a flipping move can then be written in the following way:

$$\pi_s = k(x,y,z,t)\left[2z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + 2y\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} - (t - 2sx + 4x^3)\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}\right] \tag{6.1.7}$$

Local expressions near a wrinkling move.

The local coordinate model around a wrinkling move is given by:

$$w_s(x,y,z,t) = (C_1(x,y,z,t), C_2(x,y,z,t)) = (t^2 - x^2 + y^2 - z^2 + st, 2tx + 2yz)$$

The differentials dC_1 and dC_2 are therefore:

$$\begin{array}{rcccc} dC_1 = & (-2x & 2y & -2z & 2t + s) \\ dC_2 = & (2t & 2z & 2y & 2x) \end{array}$$

The matrix we are interested in is given by:

$$\begin{pmatrix} 0 & k(sy + 2ty + 2xz) & k(2xy - (s+2t)z) & -2k\left(y^2 + z^2\right) \\ -k(sy + 2ty + 2xz) & 0 & k\left(st + 2\left(t^2 + x^2\right)\right) & k(2tz - 2xy) \\ k((s+2t)z - 2xy) & -k\left(st + 2\left(t^2 + x^2\right)\right) & 0 & 2k(ty + xz) \\ 2k\left(y^2 + z^2\right) & 2k(xy - tz) & -2k(ty + xz) & 0 \end{pmatrix}$$

The expression for the Poisson bivector in a neighbourhood of a wrinkling move is then:

$$\begin{aligned} \pi_s = k(x,y,z,t)\,[&(-2sy - 4ty - 4xz)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + (-4xy + 2sz + 4tz)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} \\ + \quad & (4y^2 + 4z^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial t}(2st + 4t^2 + 4x^2)\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} \\ + \quad & 4(xy - tz)\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial t} - 4(ty + xz)\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial t}] \end{aligned} \tag{6.1.8}$$

6.1.2 Linearization

We follow chapters 3 and 4 of [23], where more details and examples may be found. Let $\mathfrak{l}$ be a finite-dimensional Lie algebra. Denote by $\mathfrak{r}$ the radical of $\mathfrak{l}$, i.e., the maximal solvable ideal of $\mathfrak{l}$. Then $\mathfrak{g} = \mathfrak{l}/\mathfrak{r}$ is a semi-simple Lie algebra. The Levi-Malcev theorem states that $\mathfrak{l}$ can be decomposed as a semi-direct product:

$$\mathfrak{l} = \mathfrak{g} \ltimes \mathfrak{r}.$$

In analogy with this Levi-Malcev decomposition we have a Levi decomposition for Poisson structures. Let π be a Poisson structure and denote by π_0 its linear part. By definitions we obtain that π_0 generates a Lie algebra $\mathfrak{l}$. We take the Levi-Malcev decomposition of $\mathfrak{l}$, with the previous notation. Let $\{x_1, ..., x_m, y_1, ..., y_m\}$ be a basis for $\mathfrak{l}$, such that $\mathrm{Span}\{x_1, ..., x_m\} = \mathfrak{g}$, and $\mathrm{Span}\{y_1, ..., y_m\} = \mathfrak{r}$, a complement of $\mathfrak{g}$ with respect to the adjoint action of $\mathfrak{g}$ on $\mathfrak{l}$.

A Levi decomposition for π at 0, with $\pi(0) = 0$, provides local coordinates such that;

$$\{x_i, x_j\} \in \mathfrak{g} \quad \text{and} \quad \{x_i, y_j\} \in \mathfrak{r}.$$

Any analytic Poisson structure π, which vanishes at 0, admits a Levi decomposition in a neighborhood of 0.

Now we will focus on the expressions for the bivectors obtained in equations (7.2.6), (6.1.5), (6.1.6), (6.1.7), and (6.1.8). We fix $k \equiv 1$. It can be seen that in the case of cusp singularities, the linear part of the corresponding Poisson structure (7.2.6) defines a Lie algebra through the commutation relations:

$$[x, z] = 2y \qquad [x, y] = 2y \qquad [y, z] = 3t$$

For the Birth, Merge and Flipping moves, corresponding to the bivectors (6.1.5), (6.1.6), and (6.1.7), respectively, their linear part in all these cases is generated by:

$$[x, z] = 2y \qquad [x, y] = 2z$$

Notice that this Lie algebra contains a nonzero Abelian ideal, hence it is not semi-simple.

So in all these cases the linear part of the Poisson structure admits a decomposition of the form $\mathbb{R} \times L_3$, where L_3 is a semi-direct product of Lie algebras:

$$L_3 = \mathbb{R} \ltimes_A \mathbb{R}^2$$

Here $\mathbb{R}$ acts on $\mathbb{R}^2$ linearly by the matrix:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

For the case of wrinkled fibrations, corresponding to (6.1.8), we see that all the commutation relations are trivial. Therefore the corresponding linear part of its Poisson structure spans an Abelian Lie algebra, which is not semi-simple.

We have that Conn's theorem 2.5.8 asserts that, provided the linear part of an analytic Poisson structure π that vanishes at 0, corresponds to a semi-simple Lie algebra, π

admits a local analytic linearizaton at 0 [14]. Hence, in the spirit of Conn's theorem, the linearization of all these Poisson structures is not guaranteed. Moreover, the semi-direct product L_3 is degenerate (formally, analytically, and smoothly).

When k is a non vanishing smooth function, we obtain other Poisson structures. Then it is natural to ask:

Question 6.1.4 *Does there exist a function k such that an expression given by one of the bivectors (7.2.6), (6.1.5), (6.1.6), (6.1.7) or (6.1.8) is linearizable?*

6.1.3 Equations for the symplectic forms on the leaves near singularities.

***Proposition* 6.1.5** *Let $q \in B^4 \backslash \{0\}$ and let π be the Poisson structure near a cusp singularity. Then the symplectic form induced by π on the symplectic leaf Σ_q through $q = (x, y, z, t)$ at the point q is given by*

$$\omega_{\Sigma_q} = \frac{1}{3\,(x^2 - t)} \omega_{Area}(q) \tag{6.1.9}$$

here ω_{Area} is the area form on Σ_q induced by the euclidean metric on B^4.

Proof: If u_q, v_q are tangent vectors to the leaves there exist co-vectors $\alpha_q, \beta_q \in T_q^* M$ such that $\mathcal{B}_q(\alpha_q) = u_q$ and $\mathcal{B}_q(\beta_q) = v_q$, where $\mathcal{B}_q$ is given by the rule:

$$\mathcal{B}_q(\alpha)(\cdot) = \pi_q(\cdot, \alpha)$$

Finding two tangent vectors to the symplectic leaves is equivalent to detecting vectors annihilated simultaneously by the differential of two Casimir functions for the corresponding Poisson structure. Note that the characteristic distribution has rank 2.

In this case we find that the vectors:

$$u_q = -\frac{1}{3\,(t - x^2)}(2z\frac{\partial}{\partial x} + 3\left(t - x^2\right)\frac{\partial}{\partial z})$$

$$v_q = \frac{1}{3(t - x^2)}(2y\frac{\partial}{\partial x} + 3(t - x^2)\frac{\partial}{\partial y})$$

are tangent to Σ_q at q. Using the local expression of the Poisson structure for a cusp singularity given by equation (7.2.6), one can check that $\mathcal{B}_q(\alpha_q) = u_q$, for

$$\alpha_q = -\frac{dy}{k(x, y, z, t)3(t - x^2)}.$$

Similarly, $\mathcal{B}_q(\beta_q) = v_q$, for

$$\beta_q = \frac{1}{k(x, y, z, t)}(-\frac{1}{2z}dx + \frac{y}{3(t - x^2)z}dy).$$

A direct calculation now implies that the symplectic form is:

$$\omega_{\Sigma_q}(q)(u_q, v_q) = \langle \alpha_q, v_q \rangle = \frac{1}{3(x^2 - t)}\omega_{Area}(q)$$

$\square$

***Proposition* 6.1.6** *Let $q \in B^4\backslash\{0\}$ and π_s be the Poisson structure near a birth move. The symplectic form induced by π_s on the symplectic leaf Σ_q through $q=(x,y,z,t)$ at the point q is given by*

$$\omega_{\Sigma_q} = \frac{1}{k(x,y,z,t)(3(s-t^2+x^2)}\omega_{Area}(q) \tag{6.1.10}$$

here ω_{Area} is the area form on Σ_q induced by the euclidean metric on B^4.

Proof: Making use of the corresponding Casimir functions for the Poisson structure associated to a birth move we obtain that the vectors

$$u_q = \frac{1}{3\left(s-t^2+x^2\right)}(2z\frac{\partial}{\partial x}+3\left(s-t^2+x^2\right)\frac{\partial}{\partial z})$$

$$v_q = -\frac{1}{3(s-t^2+x^2)}(2y\frac{\partial}{\partial x}+3(s-t^2+x^2)\frac{\partial}{\partial y})$$

are tangent to Σ_q at q. Using the local expression (6.1.5) of the Poisson structure one can check that $\mathcal{B}_q(\alpha_q)=u_q$, for

$$\alpha_q = \frac{dy}{k(x,y,z,t)3(s-t^2+x^2)}.$$

Similarly, $\mathcal{B}_q(\beta_q)=v_q$, for

$$\beta_q = \frac{1}{k(x,y,z,t)}(-\frac{1}{2z}dx - \frac{y}{3(s-t^2+x^2)z}dy)$$

The expression for the symplectic form follows from:

$$\omega_{\Sigma_q}(q)(u_q,v_q) = \langle\alpha_q, v_q\rangle = -\langle\beta_q, u_q\rangle$$

□

***Proposition* 6.1.7** *Let $q \in B^4\backslash\{0\}$ and let π_s be the Poisson structure near a merging move. The symplectic form induced by π_s on the symplectic leaf Σ_q through $q=(x,y,z,t)$ at the point q is given by*

$$\omega_{\Sigma_q} = \frac{1}{3(t^2-s+x^2)}\omega_{Area}(q) \tag{6.1.11}$$

here ω_{Area} is the area form on Σ_q induced by the euclidean metric on B^4.

Proof: We proceed similarly to the previous cases above. We find that the vectors

$$u_q = \frac{1}{3\left(s-t^2-x^2\right)}(-2z\frac{\partial}{\partial x}+3\left(s-t^2-x^2\right)\frac{\partial}{\partial z})$$

$$v_q = \frac{1}{3(s-t^2-x^2)}(2y\frac{\partial}{\partial x}+3(s-t^2-x^2)\frac{\partial}{\partial y})$$

are tangent to Σ_q at q. Using the local expression (6.1.6) of the corresponding Poisson structure one can check that $\mathcal{B}_q(\alpha_q) = u_q$, for

$$\alpha_q = -\frac{dy}{k(x,y,z,t)3(s-t^2-x^2)}.$$

Similarly, $\mathcal{B}_q(\beta_q) = v_q$, for

$$\beta_q = \frac{1}{k(x,y,z,t)}\left(-\frac{1}{2z}dx + \frac{y}{3(s-t^2-x^2)z}dy\right).$$

As before the symplectic form is obtained by computing:

$$\omega_{\Sigma_q}(q)(u_q, v_q) = \langle \alpha_q, v_q\rangle = -\langle \beta_q, u_q\rangle$$

□

***Proposition* 6.1.8** *Let $q \in B^4 \backslash \{0\}$ and let π_s be the Poisson structure near a flipping move. The symplectic form induced by π_s on the symplectic leaf Σ_q through $q = (x,y,z,t)$ at the point q is given by*

$$\omega_{\Sigma_q} = \frac{1}{k(x,y,z,t)(t-2sx+4x^3)}\omega_{Area}(q) \tag{6.1.12}$$

here ω_{Area} is the area form on Σ_q induced by the euclidean metric on B^4.

Proof: In this case, the following vectors:

$$u_q = \frac{1}{t-2sx+4x^3}\left(2z\frac{\partial}{\partial x} + t - 2sx + 4x^3\frac{\partial}{\partial z}\right)$$

$$v_q = \frac{1}{t-2sx+4x^3}\left(-2y\frac{\partial}{\partial x} + t - 2sx + 4x^3\frac{\partial}{\partial y}\right)$$

are tangent to Σ_q at q. Using equation (6.1.7) we can check that $\mathcal{B}_q(\alpha_q) = u_q$, for

$$\alpha_q = -\frac{dy}{k(x,y,z,t)(-t+2sx-4x^3)}.$$

Similarly, $\mathcal{B}_q(\beta_q) = v_q$, for

$$\beta_q = \frac{1}{k(x,y,z,t)}\left(-\frac{1}{2z}dx - \frac{y}{(t-2sx+4x^3)z}dy\right).$$

A straightforward calculation shows that:

$$\omega_{\Sigma_q} = \frac{1}{k(x,y,z,t)(t-2sx+4x^3)z}\omega_{Area}(q)$$

□

***Proposition* 6.1.9** *Let $q \in B^4 \backslash \{0\}$ and let π_s be the Poisson structure near a wrinkling move. The symplectic form induced by π_s on the symplectic leaf Σ_q through $q = (x,y,z,t)$ at the point q is given by*

$$\omega_{\Sigma_q} = -\frac{1}{2(ty+xz)}\omega_{Area}(q) \tag{6.1.13}$$

here ω_{Area} is the area form on Σ_q induced by the euclidean metric on B^4.

Proof: Using the corresponding Poisson structure for a wrinkling move given in equation (6.1.8) we obtain:

$$u_q = -\frac{1}{2(ty+xz)}((2xy - sz - 2tz)\frac{\partial}{\partial x} + (st + 2t^2 + 2x^2)\frac{\partial}{\partial y} - 2(ty + xz)\frac{\partial}{\partial t})$$

$$v_q = -\frac{1}{(ty+xz)}((y^2 + z^2)\frac{\partial}{\partial x} + (xy - tz)\frac{\partial}{\partial y} - (ty + xz)\frac{\partial}{\partial z})$$

and makes v_q an unitary vector. These vectors are tangent to Σ_q at q. Using the local expression of the Poisson structure in (6.1.8) we check that $\mathcal{B}_q(\alpha_q) = u_q$, for

$$\alpha_q = \frac{1}{k(x,y,z,t)}(\frac{1}{2(y^2+z^2)}dx - \frac{-2xy + sz + 2tz}{4(ty+xz)(y^2+z^2)}dt)$$

Similarly, $\mathcal{B}_q(\beta_q) = v_q$, for

$$\beta_q = \frac{dt}{2(ty+xz)}$$

The proposition is shown as in the previous cases by calculating the symplectic form explicitly using the above equations. □

Chapter 7

Singular Poisson structures in Generalized Wrinkled Fibrations

We start by defining a generalization of wrinkled fibrations on dimension 6 based on singularity theory. As previously we construct Poisson structures that match the singularities of the fibration and give their local models. Before presenting these constructions, we briefly recall the notion of a generalized broken Lefschetz fibration, which serves as a reference for the definition of generalized wrinkled fibrations. Our first observation appears after combining the definition of generalized broken Lefschetz fibration [60] together with the results of [18] and [25] on Poisson structures.

***Definition* 7.0.10** *Let M, X be smooth manifolds of dimensions $2n$ and $2n-2$. By a* generalized broken Lefschetz fibration *we mean a submersion $f\colon M \to X$ with two types of singularities:*

1. Indefinite fold singularities*, locally modeled by:*

$$\mathbb{R}^{2n} \to \mathbb{R}^{2n-2}$$
$$(t_1, \ldots, t_{2n-3}, x_1, x_2, x_3) \mapsto (t_1, \ldots, t_{2n-3}, -x_1^2 + x_2^2 + x_3^2)$$

The fold locus is an embedded codimension 3 submanifold. We denote it by Z. Singular fibres have again at most one singularity on each fibre, but this time crossing Z changes the genus of the regular fibre by one.

Throughout this part of the work we assume that the singular fibres do not intersect each other.

2. Lefschetz-type singularities*, locally modelled by:*

$$\mathbb{C}^n \to \mathbb{C}^{n-1}$$
$$(z_1, \ldots, z_n) \to (z_1, \ldots z_{n-2}, z_{n-1}^2 + z_n^2)$$

These singularities are contained in codimension 4 submanifolds cross a Lefschetz singular point. We denote the set of Lefschetz-type singularities by C. Each singular fibre presents at most one singularity on each fibre. On a piece of the fibre, this can be depicted as a local cone that collapses at the origin where $z_{n-1}^2 + z_n^2 = 0$. Nearby fibres are *smooth. In the local description on a piece of a fibre, the cone opens up again and it is convex.*

Stable maps from M^n to X^q are dense in $C^\infty(M, X)$ if and only if the pair (n, q) satisfies certain conditions depending on the dimension q of the target manifold X and the difference $(n-q)$. We refer the reader to [26, 42] for a detailed account. In particular, in the case of $M^6 \to X^4$ we have the following characterization.

***Theorem* 7.0.11** *[26, 42] A generic smooth map $M^6 \to N^4$ has folds, cusps, swallowtails, and butterflies singularities.*

This suggests the following definition.

***Definition* 7.0.12** *On a smooth 6-manifold M a* generalized wrinkled fibration $f\colon M \to X$ *is a submersion to a smooth closed 4-manifold X with the following four indefinite singularities each locally modelled by real charts $\mathbb{R}^6 \to \mathbb{R}^4$*

1. folds

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, -x_1^2 + x_2^2 + x_3^2)$$

2. cusps

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^3 - 3t_1 \cdot x_1 + x_2^2 - x_3^2)$$

3. swallowtails

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^4 + t_1 x_1^2 + t_2 x_1 + x_2^2 - x_3^2)$$

4. butterflies

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^5 + t_1 x_1^3 + t_2 x_1^2 + t_3 x_1 + x_2^2 - x_3^2)$$

The aim of this section is to show the existence of Poisson structures on generalized wrinkled fibrations.

***Theorem* 7.0.13** *Let M be a closed, orientable, smooth* 6*–manifold equipped with a generalized wrinkled fibration $f\colon M \to X$ over a smooth 4-manifold X. Then there exists a complete Poisson structure whose symplectic leaves correspond to the fibres of the given fibration structure, and the singularities of both the fibration and the Poisson structures coincide. Moreover, for each singularity, the Poisson bivector are given by the following equations:*

Folds	*Poisson bivectors (7.2.1),* (7.2.4), (7.2.5), (7.5.3),
Cusps	*Poisson bivector (7.2.6),* (7.2.9), (7.2.10), (7.5.5),
Swallowtail	*Poisson bivector (7.2.11),* (7.2.14), (7.2.15), (7.5.7),
Butterfly	*Poisson bivector (7.2.16),* (7.2.19), (7.2.20), (7.5.9).

For each singularity, the induced symplectic form on the leaves are given by the following equations:

Folds	*Symplectic forms* (7.5.1), (7.5.2), (7.5.3),
Cusps	*Symplectic forms (7.5.4),* (7.5.5),
Swallowtail	*Symplectic forms (7.5.6),* (7.5.7),
Butterfly	*Symplectic forms (7.5.8),* (7.5.9).

If none of its symplectic leaves are, or contain, 2*–spheres, then this Poisson structure is integrable.*

***Theorem* 7.0.14** *Let M and X be closed oriented smooth manifolds with* $\dim(M) = 2n, \dim(X) = 2n-2$, *and $f\colon M \to X$ a generalized broken Lefschetz fibration. Then there is a complete singular Poisson structure of rank 2 whose associated bivector vanishes on the singularity set of f. If none of its symplectic leaves are, or contain, 2–spheres, then this Poisson structure is integrable.*

The existence is given by a direct application of Theorem 6.0.3 proved in [25]. The integrability follows from Proposition 2.5.10.

7.1 General criterion for constructing Poisson bivectors on singularities

We will now give explicit local descriptions for the Poisson structures and the corresponding symplectic forms in a neighbourhood singularities of generalized wrinkled fibrations in dimension 6. All of the expressions that we will give depend abstractly on an arbitrary choice of a non-vanishing function k in $C^\infty(M)$. See Proposition 6.0.2.

Before proceeding we will describe the general strategy employed to find the local bivectors.

Step 1: Consider the coordinate functions C_1, C_2, C_3, C_4 that describe each fibration as Casimir functions for the Poisson structure that we want to find.

Step 2: Calculate the differentials dC_i, $i = 1, 2, 3, 4$.

Step 3: We use formula 6.0.1 to compute the skew-symmetric matrix with entries:

$$\pi^{ij} = \{x^i, x^j\}\mu = \; dx^i \wedge dx^j \wedge dC_1 \wedge dC_2 \wedge dC_3 \wedge dC_4.$$

This matrix will then annihilate dC_i, $i = 1, 2, 3, 4$. It will give the endomorphism $\mathcal{B}$ associated to a Poisson structure with dC_i, $i = 1, 2, 3, 4$, as Casimirs. The components of the bivector field will be given by:

$$\{x^i, x^j\} = \det\left(\epsilon^i, \epsilon^j, dC_1, dC_2, dC_3, dC_4\right).$$

Here ϵ^i is the 6×1 canonical basis column vector, whose i-th component is 1 and all others are zero.

Step 4: We then write the Poisson bivector using the skew-symmetric matrix entries.

We extend the previous strategy to manifolds of dimension $2n$ when we have a singular submersion with singularities of corank 1. The following construction will describe a procedure that can be used to compute local expressions of Poisson structures and their corresponding symplectic forms. We will implement this scheme to study the 6−dimensional case.

***Proposition* 7.1.1** *Let q be a point that either has complex coordinates $q = (z_1, z_2, \dots z_n)$ or real coordinates $(t_1, t_2, \dots, t_{2n-4}, t_{2n-3}, x_1, x_2, x_3)$. Let f be a smooth map given as either $f\colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ or $f\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n-2}$ such that*

$$f(q) = (z_1, \dots, z_{n-2}, f_o(z_{n-1}, z_n))$$

or

$$f(q) = (t_1, t_2, \ldots, t_{2n-4}, t_{2n-3}, f_o(t_1, \ldots, t_{2n-3}, x_1, x_2, x_3)),$$

respectively. Here f_o is a smooth map which depends only on the last coordinates z_{n-1}, z_n or x_1, x_2, x_3. Then we can produce a Poisson structure associated to the local model given by f. The Poisson bivector has the form:

$$\pi = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & \pi^{11} & \pi^{12} & \pi^{13} & \pi^{14} \\ 0 & \cdots & 0 & \pi^{21} & \pi^{22} & \pi^{23} & \pi^{24} \\ 0 & \cdots & 0 & \pi^{31} & \pi^{32} & \pi^{33} & \pi^{34} \\ 0 & \cdots & 0 & \pi^{41} & \pi^{42} & \pi^{43} & \pi^{44} \end{pmatrix}$$

where π^{ij} is the Poisson bivector of the map f_o. Then $\pi^{ii} = 0$ and $\pi^{ij} = \pi^{ji}$. Therefore the Poisson bivector has the local form:

$$\pi(x) = \sum_{i,j=1}^{4} \left[\pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \right].$$

Proof: In the case when f is a complex map we use the real and imaginary parts of each coordinate function as a Casimir function for the Poisson structure that we want to find. That is, we will have $2n-2$ Casimir functions:

$$\begin{aligned} C_i &= Re(z_i) \quad 1 \leq i \leq n-2 \\ C_{i+n-2} &= Im(z_i) \quad 1 \leq i \leq n-2 \\ C_{2n-3} &= Re(f_o(z_{n-1}, z_n)) \\ C_{2n-2} &= Im(f_o(z_{n-1}, z_n)). \end{aligned}$$

Now we compute the differential matrix of the map. It gives a matrix with a 2×4-block corresponding to the derivatives of the real and complex part of f_o and ones on the principal diagonal.

$$D = \begin{pmatrix} 1 & \ldots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & \frac{\partial C_{2n-3}}{\partial t_{2n-3}} & \frac{\partial C_{2n-2}}{\partial t_{2n-3}} \\ 0 & \cdots & 0 & \frac{\partial C_{2n-3}}{\partial x_1} & \frac{\partial C_{2n-2}}{\partial x_1} \\ 0 & \cdots & 0 & \frac{\partial C_{2n-3}}{\partial x_2} & \frac{\partial C_{2n-2}}{\partial x_2} \\ 0 & \cdots & 0 & \frac{\partial C_{2n-3}}{\partial x_3} & \frac{\partial C_{2n-2}}{\partial x_3} \end{pmatrix}. \tag{7.1.1}$$

According to the formula (6.0.1) the coefficients of the bivector matrix are given by

$$\pi^{ij} = Det\left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \epsilon_i^1 & \epsilon_j^1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \epsilon_i^{2n-4} & \epsilon_j^{2n-4} \\ 0 & \cdots & 0 & \frac{\partial C_{2n-3}}{\partial t_{2n-3}} & \frac{\partial C_{2n-2}}{\partial t_{2n-3}} & \epsilon_i^{2n-3} & \epsilon_j^{2n-3} \\ 0 & \cdots & 0 & \frac{\partial C_{2n-3}}{\partial x_1} & \frac{\partial C_{2n-2}}{\partial x_1} & \epsilon_i^{2n-2} & \epsilon_j^{2n-2} \\ 0 & \cdots & 0 & \frac{\partial C_{2n-3}}{\partial x_2} & \frac{\partial C_{2n-2}}{\partial x_2} & \epsilon_i^{2n-1} & \epsilon_j^{2n-1} \\ 0 & \cdots & 0 & \frac{\partial C_{2n-3}}{\partial x_3} & \frac{\partial C_{2n-2}}{\partial x_3} & \epsilon_i^{2n} & \epsilon_j^{2n} \end{pmatrix}\right]$$

where ϵ_i^k and ϵ_j^k are canonical basis column vectors, whose $i-$th and $j-$th component, respectively is 1 and all others are zero. Note that it contains a identity matrix of dimension $(2n-4)\times(2n-4)$. Therefore the determinant is the same as of the following matrix

$$\begin{pmatrix} 0 & 0 & \epsilon_i^{2n-4} & \epsilon_j^{2n-4} \\ \frac{\partial C_{2n-3}}{\partial t_{2n-3}} & \frac{\partial C_{2n-2}}{\partial t_{2n-3}} & \epsilon_i^{2n-3} & \epsilon_j^{2n-3} \\ \frac{\partial C_{2n-3}}{\partial x_1} & \frac{\partial C_{2n-2}}{\partial x_1} & \epsilon_i^{2n-2} & \epsilon_j^{2n-2} \\ \frac{\partial C_{2n-3}}{\partial x_2} & \frac{\partial C_{2n-2}}{\partial x_2} & \epsilon_i^{2n-1} & \epsilon_j^{2n-1} \\ \frac{\partial C_{2n-3}}{\partial x_3} & \frac{\partial C_{2n-2}}{\partial x_3} & \epsilon_i^{2n} & \epsilon_j^{2n} \end{pmatrix}$$

which gives the coordinates of the Poisson bivector associated to f_o. Recall that f_o depends only on the las coordinates.

When f is a real map, we take the coordinates functions as Casimir functions:

$$\begin{aligned} C_i &= t_i \quad 1 \le i \le 2n-3 \\ C_{2n-2} &= f_o(t_1, \ldots, t_{2n-3}, x_1, x_2, x_3). \end{aligned}$$

The differential matrix of the map is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{\partial C_{2n-2}}{\partial t_1} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & \frac{\partial C_{2n-2}}{\partial t_{2n-4}} \\ 0 & \cdots & 0 & 1 & \frac{\partial C_{2n-2}}{\partial t_{2n-3}} \\ 0 & \cdots & 0 & 0 & \frac{\partial C_{2n-2}}{\partial x_1} \\ 0 & \cdots & 0 & 0 & \frac{\partial C_{2n-2}}{\partial x_2} \\ 0 & \cdots & 0 & 0 & \frac{\partial C_{2n-2}}{\partial x_3} \end{pmatrix}. \tag{7.1.2}$$

Then, the coefficients of the corresponding bivector matrix are given by

$$\pi^{ij} = Det\left[\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{\partial C_{2n-2}}{\partial t_1} & \epsilon_i^1 & \epsilon_j^1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & \frac{\partial C_{2n-2}}{\partial t_{2n-4}} & \epsilon_i^{2n-4} & \epsilon_j^{2n-4} \\ 0 & \cdots & 0 & 1 & \frac{\partial C_{2n-2}}{\partial t_{2n-3}} & \epsilon_i^{2n-3} & \epsilon_j^{2n-3} \\ 0 & \cdots & 0 & 0 & \frac{\partial C_{2n-2}}{\partial x_1} & \epsilon_i^{2n-2} & \epsilon_j^{2n-2} \\ 0 & \cdots & 0 & 0 & \frac{\partial C_{2n-2}}{\partial x_2} & \epsilon_i^{2n-1} & \epsilon_j^{2n-1} \\ 0 & \cdots & 0 & 0 & \frac{\partial C_{2n-2}}{\partial x_3} & \epsilon_i^{2n} & \epsilon_j^{2n} \end{pmatrix}\right].$$

We note that $\pi^{ij} = 0$ for $1 \leq i \leq 2n-4$ and $1 \leq j \leq 2n-4$. The rest of the coefficients can be computed with the following

$$Det \left[\begin{pmatrix} 1 & 0 & \epsilon_i^{2n-3} & \epsilon_j^{2n-3} \\ 0 & \frac{\partial C_{2n-2}}{\partial x_1} & \epsilon_i^{2n-2} & \epsilon_j^{2n-2} \\ 0 & \frac{\partial C_{2n-2}}{\partial x_2} & \epsilon_i^{2n-1} & \epsilon_j^{2n-1} \\ 0 & \frac{\partial C_{2n-2}}{\partial x_3} & \epsilon_i^{2n} & \epsilon_j^{2n} \end{pmatrix} \right].$$

In fact, the only nonzero coefficients are:

$$\begin{aligned} \pi^{23} &= \frac{\partial C_{2n-2}}{\partial x_3} \\ \pi^{24} &= \frac{\partial C_{2n-2}}{\partial x_2} \\ \pi^{34} &= \frac{\partial C_{2n-2}}{\partial x_3}. \end{aligned}$$

The result follows. □

7.2 Poisson structures on generalized wrinkled fibrations in dimension 6.

We apply the general criterion presented above to the case of wrinkled fibrations on 6-manifolds. Let $q \in M$ be a point, and $k : M \to X, k(t_1, t_2, t_3, x_1, x_2, x_3)$, be a non-vanishing smooth function.

Poisson bivector near a fold singularity.

Indefinite fold
The local coordinate model around a fold singularity is given by the map:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, -x_1^2 + x_2^2 + x_3^2).$$

Considering each coordinate function as a Casimir function for the Poisson bivector that we want to find, we compute the differential matrix of the map

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2x_1 \\ 0 & 0 & 0 & 2x_2 \\ 0 & 0 & 0 & 2x_3 \end{pmatrix}.$$

The resulting Poisson structure of a fold singularity is given by:

$$\pi = k \left[2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} - 2x_1 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right]. \quad (7.2.1)$$

Definite fold

In addition, we also compute the Poisson bivector for definite singularities for each wrinkled fibration. In this case, they are locally modeled by:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^2 + x_2^2 + x_3^2) \tag{7.2.2}$$

and

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, -x_1^2 - x_2^2 - x_3^2). \tag{7.2.3}$$

In the first case 7.2.2, following the same computations as above, the Poisson matrix is then:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2x_3 & -2x_2 \\ 0 & 0 & 0 & -2x_3 & 0 & 2x_1 \\ 0 & 0 & 0 & 2x_2 & -2x_1 & 0 \end{pmatrix}$$

It follows that the Poisson bivector is given by:

$$\pi = k\left[2x_3\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + 2x_1\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2}\right]. \tag{7.2.4}$$

For the case when the map is 7.2.3 the Poisson bivector is given by:

$$\pi = k\left[-2x_3\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + 2x_1\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2}\right]. \tag{7.2.5}$$

Poisson bivector near a cusp singularity.

Indefinite cusp

The local coordinate model around a cusp singularity is given by:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^3 - 3t_1x_1 + x_2^2 - x_3^2).$$

The differential matrix of the map is

$$\begin{pmatrix} 1 & 0 & 0 & -3x_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3x_1^2 - 3t_1 \\ 0 & 0 & 0 & 2x_2 \\ 0 & 0 & 0 & -2x_3 \end{pmatrix}.$$

Then, the Poisson bivector in the local coordinates of a cusp singularity is given by:

$$\pi = k\left[-2x_3\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + 3(x_1^2 - t_1)\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2}\right]. \tag{7.2.6}$$

Definite cusp

For definite singularities in cusps, we obtain in each case (7.2.7) and (7.2.8):

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^3 - 3t_1x_1 + x_2^2 + x_3^2), \tag{7.2.7}$$

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^3 - 3t_1x_1 - x_2^2 - x_3^2). \tag{7.2.8}$$

The corresponding bivectors are, respectively:

$$\pi = k\left[2x_3\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + 3(x_1^2 - t_1)\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2}\right] \tag{7.2.9}$$

and

$$\pi = k\left[-2x_3\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + 3(x_1^2 - t_1)\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2}\right]. \tag{7.2.10}$$

Poisson bivector near a swallowtail singularity.

Indefinite swallowtail
The local coordinate model around a swallowtail singularity is given by the map:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^4 + t_1x_1^2 + t_2x_1 + x_2^2 - x_3^2).$$

Its differential matrix is:

$$\begin{pmatrix} 1 & 0 & 0 & x_1^2 \\ 0 & 1 & 0 & x_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4x_1^3 + 2t_1x_1 + t_2 \\ 0 & 0 & 0 & 2x_2 \\ 0 & 0 & 0 & -2x_3 \end{pmatrix}.$$

Then, the Poisson bivector in the local coordinates of a swallowtail singularity is described by:

$$\pi = k\left[-2x_3\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + (4x_1^3 + 2t_1x_1 + t_2)\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2}\right]. \tag{7.2.11}$$

Definite swallowtail
For definite singularities:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^4 + t_1x_1^2 + t_2x_1 + x_2^2 + x_3^2), \tag{7.2.12}$$

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^4 + t_1x_1^2 + t_2x_1 - x_2^2 - x_3^2). \tag{7.2.13}$$

The corresponding bivectors are, respectively:

$$\pi = k\left[2x_3\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + (4x_1^3 + 2t_1x_1 + t_2)\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2}\right] \tag{7.2.14}$$

and

$$\pi = k\left[-2x_3\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + (4x_1^3 + 2t_1x_1 + t_2)\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2}\right]. \tag{7.2.15}$$

Poisson bivector near a butterfly singularity.

Indefinite butterfly
The local coordinate model around a buttterfly singularity is given by:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^5 + t_1 x_1^3 + t_2 x_1^2 + t_3 x_1 + x_2^2 - x_3^2)$$

The differential of the map is given by:

$$\begin{pmatrix} 1 & 0 & 0 & x_1^3 \\ 0 & 1 & 0 & x_2^2 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 5x_1^4 + 3t_1 x_1^2 + 2t_2 x_1 + t_3 \\ 0 & 0 & 0 & 2x_2 \\ 0 & 0 & 0 & -2x_3 \end{pmatrix}.$$

The Poisson bivector in the local coordinates of a butterfly singularity is described by:

$$\pi = k\left[-2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + (5x_1^4 + 3t_1 x_1^2 + 2t_2 x_1 + t_3) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2}\right]. \tag{7.2.16}$$

Definite butterfly The singularity is modeled by the coordinates:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^5 + t_1 x_1^3 + t_2 x_1^2 + t_3 x_1 + x_2^2 + x_3^2), \tag{7.2.17}$$

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^5 + t_1 x_1^3 + t_2 x_1^2 + t_3 x_1 - x_2^2 - x_3^2). \tag{7.2.18}$$

The corresponding bivectors are, respectively:

$$\pi = k\left[2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + (5x_1^4 + 3t_1 x_1^2 + 2t_2 x_1 + t_3) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2}\right] \tag{7.2.19}$$

and

$$\pi = k\left[-2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + (5x_1^4 + 3t_1 x_1^2 + 2t_2 x_1 + t_3) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2}\right]. \tag{7.2.20}$$

7.3 Poisson bivectors on higher dimensional type $2n$ generalized wrinkled fibrations.

As we described, Lekili showed that any 1-parameter family deformation of a purely wrinkled fibration is homotopic (relative endpoints) to one which realises a sequence of births, merges, flips, their inverses, and isotopies staying within the class of purely wrinkled fibrations. For higher dimensions, we will introduce a generalized form of the deformations given by Lekili in dimension 4. We will use them to give local expressions for the associated Poisson bivectors and symplectic forms near singularities described by the deformations.

Consider the following maps $\mathbb{R} \times \mathbb{R}^{2n-1} \to \mathbb{R}^{2n-2}$, given by the equations below, and each depending on a real parameter s:

$$b_s(t_1, \dots, t_{2n-3}, x_1, x_2, x_3) = (t_1, \dots, t_{2n-3}, x_1^3 - 3x_1(t_{2n-3}^2 - s) + x_2^2 - x_3^2) \tag{7.3.1}$$

$$m_s(t_1, \dots, t_{2n-3}, x_1, x_2, x_3) = (t_1, \dots, t_{2n-3}, x_1^3 - 3x_1(s - t_{2n-3}^2) + x_2^2 - x_3^2) \tag{7.3.2}$$

$$f_s(t_1, \dots, t_{2n-3}, x_1, x_2, x_3) = (t_1, \dots, t_{2n-3}, x_1^4 - x_1^2 s + x_1 t_{2n-3} + x_2^2 - x_3^2) \tag{7.3.3}$$

$$w_s(t_1, \dots, t_{2n-3}, x_1, x_2, x_3) = (t_1, \dots, t_{2n-4}, t_{2n-3}^2 - x_1^2 + x_2^2 - x_3^2 + s t_{2n-3}, 2t_{2n-3}x_1 + 2x_2x_3). \tag{7.3.4}$$

We will also need a generalized winkled fibration for dimensions greater than 6.

***Definition* 7.3.1** *Let M be a smooth $2n$–manifold, and X be a smooth closed $2n-2$–manifold. A* type $2n$-wrinkled fibration *is a smooth map $f : M \to X$ that is a sumbersion with the following four indefinite singularities each locally modelled by real charts $\mathbb{R}^{2n} \to \mathbb{R}^{2n-2}$:*

1. *folds*
$$(t_1, \dots, t_{2n-3}, x_1, x_2, x_3) \mapsto (t_1, \dots, t_{2n-3}, -x_1 + x_2^2 + x_3^2),$$
2. *cusps*
$$(t_1, \dots, t_{2n-3}, x_1, x_2, x_3) \mapsto (t_1, \dots, t_{2n-3}, x_1^3 - 3t_1x_1 + x_2^2 - x_3^2),$$
3. *swallowtails*
$$(t_1, \dots, t_{2n-3}, x_1, x_2, x_3) \mapsto (t_1, \dots, t_{2n-3}, x_1^4 + t_1x_1^2 + t_2x_1 + x_2^2 - x_3^2),$$
4. *butterflies*
$$(t_1, \dots, t_{2n-3}, x_1, x_2, x_3) \mapsto (t_1, \dots, t_{2n-3}, x_1^5 + t_1x_1^3 + t_2x_1^2 + t_3x_1 + x_2^2 - x_3^2).$$

We use the general strategy to compute the Poisson structures associated to all these different singularity models. Then we obtain the following Corollary:

Corollary 7.3.2 *For a non-vanishing smooth function k in $C^\infty(M)$ we have the following consequences:*
(1) *Let X be a closed smooth oriented and connected $2n$-manifold, and $f : M \to X$ a generalized broken Lefschetz fibration. The Poisson structures in a neighborhood of the two type of singularities can be computed to obtain Poisson bivectors near the following singularities:*

Lefschetz-type singularity

$$\begin{aligned}\pi = \quad & k \quad \left[(x_2^2+x_3^2)\frac{\partial}{\partial t_{2n-3}}\wedge\frac{\partial}{\partial x_1}+(x_1x_2-t_{2n-3}x_3)\frac{\partial}{\partial t_{2n-3}}\wedge\frac{\partial}{\partial x_2}\right.\\ - & \quad (t_{2n-3}x_2+x_1x_3)\frac{\partial}{\partial t_{2n-3}}\wedge\frac{\partial}{\partial x_3}+(t_{2n-3}x_2+x_1x_3)\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}\\ + & \quad \left.(x_1x_2-t_{2n-3}x_3)\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_3}+(t_{2n-3}^2+x_1^2)\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_3}\right].\end{aligned}$$

Indefinite fold singularity

$$\pi = k\left[x_1\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_3}+x_2\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_3}-x_3\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}\right].$$

(2) *Let M be a closed, orientable, smooth $2n$-manifold endowed with a type $2n$-wrinkled fibration f to a closed $(2n-2)$ manifold X. Then a complete Poisson structure is given by the following bivectors near the corresponding singularities:*

Fold

$$\pi = k\left[2x_3\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_1}-2x_2\frac{\partial}{\partial x_3}\wedge\frac{\partial}{\partial x_1}-2x_1\frac{\partial}{\partial x_3}\wedge\frac{\partial}{\partial x_2}\right]$$

Cusp

$$\pi = k\left[-2x_3\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_1}-2x_2\frac{\partial}{\partial x_3}\wedge\frac{\partial}{\partial x_1}+3(x_1^2-t_{2n-5})\frac{\partial}{\partial x_3}\wedge\frac{\partial}{\partial x_2}\right]$$

Swallowtail

$$\pi = k\left[-2x_3\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_1}-2x_2\frac{\partial}{\partial x_3}\wedge\frac{\partial}{\partial x_1}+3(4x_1^3+2t_{2n-5}x_1+t_{2n-4})\frac{\partial}{\partial x_3}\wedge\frac{\partial}{\partial x_2}\right]$$

Butterfly

$$\pi = k\left[-2x_3\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_1}-2x_2\frac{\partial}{\partial x_3}\wedge\frac{\partial}{\partial x_1}+(5x_1^4+3t_{2n-5}x_1^2+2t_{2n-4}x_1+t_{2n-3})\frac{\partial}{\partial x_3}\wedge\frac{\partial}{\partial x_2}\right].$$

The following equations depend on a real parameter s. Near a singularity locally modeled by the b_s, m_s, f_s, and w_s the corresponding Poisson bivectors are:

Map b_s

$$\pi_s = k\left[2x_3\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}+2x_2\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_3}-3(s-t_{2n-3}^2+x_1^2)\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_3}\right]$$

Map m_s

$$\pi_s = k\left[2x_3\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}+2x_2\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_3}-3(s-t_{2n-3}^2-x_1^2)\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_3}\right]$$

Map f_s

$$\pi_s = k\left[2x_3\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}+2x_2\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_3}-(t_{2n-3}-2sx_1+4x_1^3)\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_3}\right]$$

Map w_s

$$\pi_s = k\Bigg[(-2sx_2 - 4t_{2n-3}x_2 - 4x_1x_3)\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2} + (-4x_1x_2 + 2sx_3 + 4t_{2n-3}x_3)\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_3}$$
$$+(4x_2^2 + 4x_3^2)\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial t_{2n-3}} - (2st_{2n-3} + 4t_{2n-3}^2 + 4x_1^2)\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_3}$$
$$+4(x_1x_2 - t_{2n-3}x_3)\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial t_{2n-3}} - 4(t_{2n-3}x_2 + x_1x_3)\frac{\partial}{\partial x_3}\wedge\frac{\partial}{\partial t_{2n-3}}.\Bigg]$$

7.4 General criterion for constructing symplectic forms on leaves of generalized wrinkled fibrations

***Theorem* 7.4.1** *Under the hypothesis of Proposition 7.1.1, the symplectic form induced by the Poisson structure π on the symplectic leaf Σ_q through $q \neq 0$ is completely determined by the Poisson structure of the map f_o. That is, if u_q, v_q are tangent vectors to the leaves, then:*

$$\omega_{\Sigma_q}(u_q, v_q) = \omega_o(\tilde{u}_q, \tilde{v}_q)$$

where ω_o is the symplectic structure of f_o, and $\tilde{u}_q, \tilde{v}_q$ are the tangent vectors u_q and v_q restricted to the last 4 coordinates.

Proof: First, we have to obtain vectors tangent to the leaves. That is, we want to find vectors such that they are annhilated simultaneously by the $2n-2$ Casimir functions. Then we transpose the matrix and compute its null space.

In the case when f is a complex map, we used its real and imaginary parts of each coordinate function as Casimir functions. We obtained the matrix (7.1.1) whose transpose matrix is:

$$D^T = \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & \frac{\partial C_{2n-3}}{\partial t_{2n-3}} & \frac{\partial C_{2n-3}}{\partial x_1} & \frac{\partial C_{2n-3}}{\partial x_2} & \frac{\partial C_{2n-3}}{\partial x_3} \\ 0 & \cdots & 0 & \frac{\partial C_{2n-2}}{\partial t_{2n-3}} & \frac{\partial C_{2n-2}}{\partial x_1} & \frac{\partial C_{2n-2}}{\partial x_2} & \frac{\partial C_{2n-2}}{\partial x_3} \end{pmatrix}.$$

Note that its left upper block is an identity matrix of dimension $2n-4$.
Let

$$\partial C_{2n-3} : = \left(0, \ldots 0, \frac{\partial C_{2n-3}}{\partial t_{2n-3}}, \frac{\partial C_{2n-3}}{\partial x_1}, \frac{\partial C_{2n-3}}{\partial x_2}, \frac{\partial C_{2n-3}}{\partial x_3}\right),$$
$$\partial C_{2n-3} : = \left(0, \ldots 0, \frac{\partial C_{2n-2}}{\partial t_{2n-3}}, \frac{\partial C_{2n-2}}{\partial x_1}, \frac{\partial C_{2n-2}}{\partial x_2}, \frac{\partial C_{2n-2}}{\partial x_3}\right).$$

Then a vector $a = (a_1, a_2, \ldots, a_{2n})$ belongs to $Ker(D^T)$ if and only if:

$$\langle \partial C_{2n-3}, a\rangle = 0$$
$$\langle \partial C_{2n-2}, a\rangle = 0.$$

Observe that the first $2n-4$ entries of a equal zero. Then, $a \in Ker(D^T)$ if

$$a = (0, 0, \dots, 0, a_{2n-3}, a_{2n-2}, a_{2n-1}, a_{2n}),$$

where the coefficients $a_{2n-3}, a_{2n-2}, a_{2n-1}, a_{2n}$ are determined by the equations:

$$\begin{cases} a_{2n-3}\frac{\partial C_{2n-3}}{\partial t_{2n-3}} + a_{2n-2}\frac{\partial C_{2n-3}}{\partial x_1} + a_{2n-1}\frac{\partial C_{2n-3}}{\partial x_2} + a_{2n}\frac{\partial C_{2n-3}}{\partial x_3} = 0 \\ a_{2n-3}\frac{\partial C_{2n-2}}{\partial t_{2n-3}} + a_{2n-2}\frac{\partial C_{2n-2}}{\partial x_1} + a_{2n-1}\frac{\partial C_{2n-2}}{\partial x_2} + a_{2n}\frac{\partial C_{2n-2}}{\partial x_3} = 0 \end{cases}. \tag{7.4.1}$$

Since the rank of the matrix D is $2n-2$, it has nullity 2. Therefore there exist two vectors u_q and v_q that generate all solutions to the previous system. We may assume they are orthogonal. Now, we have to find vectors α_q, β_q such that $\mathcal{B}_q(\alpha_q) = u_q$ and $\mathcal{B}_q(\beta_q) = v_q$.

To compute the symplectic form it is enough to find α_q. In order to compute β_q we may proceed similarly. We know that α is the solution to the equation $\mathcal{B}_{\text{II}}(\alpha)(\cdot) = \pi(\cdot, \alpha) = u_q$.

It is equivalent to consider the system $\pi \cdot \alpha_q = u_q$ and solve for α_q. By the previous discussion and recalling the form of the Poisson matrix, if u_q, α_q and v_q have coordinates:

$$\begin{aligned} u_q &= (0, 0, \dots, u_{2n-3}, u_{2n-2}, u_{2n-1}, u_{2n}) \\ v_q &= (0, 0, \dots, u_{2n-3}, v_{2n-2}, v_{2n-1}, v_{2n}) \\ \alpha_q &= (\alpha_1, \alpha_2, \dots, \alpha_{2n}). \end{aligned}$$

This system is reduced to:

$$\begin{cases} u_{2n-3} = \alpha_{2n-2}\pi^{12} + \alpha_{2n-1}\pi^{13} + \alpha_{2n}\pi^{14} \\ u_{2n-2} = -\alpha_{2n-3}\pi^{12} + \alpha_{2n-1}\pi^{23} + \alpha_{2n}\pi^{24} \\ u_{2n-1} = -\alpha_{2n-3}\pi^{13} - \alpha_{2n-2}\pi^{23} + \alpha_{2n}\pi^{34} \\ u_{2n} = -\alpha_{2n-3}\pi^{14} - \alpha_{2n-2}\pi^{24} - \alpha_{2n-1}\pi^{34}. \end{cases} \tag{7.4.2}$$

Therefore the symplectic form will be given by

$$\omega_{\Sigma_q}(q) = \langle \alpha_q, v_q \rangle,$$

here α_q is the solution to the system (7.4.2), and v satisfies the system (7.4.1). Note that we may choose α with the first $2n-4$ coordinates equal zero.

When the map f is real we obtained the matrix (7.1.2). Its transpose is:

$$D^T = \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ \frac{\partial C_{2n-2}}{\partial t_1} & \cdots & \frac{\partial C_{2n-2}}{t_{2n-4}} & \frac{\partial C_{2n-2}}{t_{2n-3}} & \frac{\partial C_{2n-2}}{x_1} & \frac{\partial C_{2n-2}}{x_2} & \frac{\partial C_{2n-2}}{x_3} \end{pmatrix}.$$

Its left upper block is an identity matrix of dimension $2n-3$. Then $a \in Ker(D^T)$ if $a = (0, 0, \dots, 0, 0, a_{2n-2}, a_{2n-1}, a_{2n})$, where the coefficients $a_{2n-2}, a_{2n-1}, a_{2n}$ are determined by the equation:

$$a_{2n-2}\frac{\partial C_{2n-2}}{x_1} + a_{2n-1}\frac{\partial C_{2n-2}}{x_2} + a_{2n}\frac{\partial C_{2n-2}}{x_3} = 0.$$

We can give the explicit solutions, they are generated by the vectors:

$$u = \{0, 0, \dots, 0, -\frac{\frac{\partial C_{2n-2}}{x_2}}{\frac{\partial C_{2n-2}}{x_1}}, 1, 0\}, \quad v = \{0, 0, \dots, 0, -\frac{\frac{\partial C_{2n-2}}{x_3}}{\frac{\partial C_{2n-2}}{x_1}}, 0, 1\} \tag{7.4.3}$$

Let $u_q = u$ and $v_q = proj_u(v)$, the orthogonal projection of v over u. Then u_q and v_q are orthogonal and generate all solutions to the previous system. As before, we know that α_q is the solution to the equation $\mathcal{B}_q(\alpha)(\cdot) = \pi(\cdot, \alpha) = u_q$.

This is equivalent to solving the system $\pi \cdot \alpha_q = u_q$ for α_q. If α_q has coordinates:

$$\alpha_q = (\alpha_1, \alpha_2, \dots, \alpha_{2n})$$

this system is reduced to

$$\begin{cases} -\frac{\frac{\partial C_{2n-2}}{x_2}}{\frac{\partial C_{2n-2}}{x_1}} & = & -\alpha_{2n-3}\pi^{12} + \alpha_{2n-1}\pi^{23} + \alpha_{2n}\pi^{24} \\ 1 & = & -\alpha_{2n-3}\pi^{13} - \alpha_{2n-2}\pi^{23} + \alpha_{2n}\pi^{34} \\ 0 & = & -\alpha_{2n-3}\pi^{14} - \alpha_{2n-2}\pi^{24} - \alpha_{2n-1}\pi^{34}. \end{cases} \tag{7.4.4}$$

Therefore the symplectic form will be given by

$$\omega_{\Sigma_q}(q) = \langle \alpha_q, v_q \rangle$$

where α_q is the solution to the system (7.4.4), and v_q has the form (7.4.3). Note that we may choose α with the first $2n-4$ coordinates equal zero. □

7.5 Symplectic forms on the leaves of generalized wrinkled fibrations in dimension 6

As a corollary of the previous theorem we obtain the following result in dimension 6.

Corollary 7.5.1 . *Let M be a closed, orientable, smooth 6–manifold equipped with a generalized wrinkled fibration $f\colon M \to X$ on a smooth 4-manifold X. Let $(U, (t_1, t_2, t_3, x_1, x_2, x_3))$ be a coordinate neighbourhood of $q \in \mathrm{Crit}_f$, an element of the singularity set of f . Then, there is a symplectic form on U induced by π on the symplectic leaf Σ_q through q near each of the singularities of the fibration with the following expressions:*

Folds

Indefinite Fold

$$\omega_{\Sigma_q} = \frac{x_1^2}{2k(q)(x_1^2 + x_3^2)^{1/2}} \omega_{Area}(q) \tag{7.5.1}$$

where ω_{Area} is the area form on Σ_q induced by the euclidean metric on B^6.

Definite Fold

For the definite definite fold singularities described by the equations (7.2.2) and (7.2.3) we obtain the symplectic forms

$$\omega_{\Sigma_q} = -\frac{x_1^2}{2(x_1^2 + x_3^2}) ^{1/2} \omega_{Area}(q) \tag{7.5.2}$$

and

$$\omega_{\Sigma_q} = \frac{x_1^2}{2(x_1^2 + x_3^2}\big)^{1/2}\omega_{Area}(q) \tag{7.5.3}$$

respectively.

Cusps

Indefinite Cusp

$$\omega_{\Sigma_q} = \frac{3x_2\,(t_1 - x_1^2)}{k(q)(9\,(t_1 - x_1^2)^2 + 4x_3^2)^{1/2}}\omega_{Area}(q) \tag{7.5.4}$$

where ω_{Area} is the area form on Σ_q induced by the euclidean metric on B^6.

Definite Cusps

The definite singularities modelled by the parametrizations (7.2.7) and (7.2.8) have the corresponding symplectic form which coincides in both cases:

$$\omega_{\Sigma_q} = \frac{3(t_1 - x_1^2)x_2}{(9(t_1 - x_1^2)^2 + 4x_3^2)^{1/2}}\omega_{Area}(q). \tag{7.5.5}$$

Swallotail

Indefinite Swallowtail

$$\omega_{\Sigma_q} = -\frac{t_2 + 2t_1x_1 + 4x_1^3}{k(q)((t_2 + 2t_1x_1 + 4x_1^3)^2 + 4x_3^2)^{1/2}}\omega_{Area}(q) \tag{7.5.6}$$

here ω_{Area} is the area form on Σ_q induced by the euclidean metric on B^6.

Definite Swallowtail

The definite swallowtails modelled by the parametrizations (7.2.12) and (7.2.12) have the corresponding symplectic form which coincides in both cases:

$$\omega_{\Sigma_q} = -\frac{(t_2 + 2t_1x_1 + 4x_1^3}{((t_2 + 2t_1x_1 + 4x_1^3)^2 + 4x_3^2)^{1/2}}\omega_{Area}(q). \tag{7.5.7}$$

Butterflies

Indefinite Butterfly

$$\omega_{\Sigma_q} = -\frac{t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3)}{k(q)((t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3))^2 + 4x_3^2)^{1/2}}\omega_{Area}(q) \tag{7.5.8}$$

where ω_{Area} is the area form on Σ_q induced by the euclidean metric on B^6.

Definite Butterfly

$$\omega_{\Sigma_q} = -\frac{(t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3)}{(t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3))^2 + 4x_3^2)^{1/2}}\omega_{Area}(q). \tag{7.5.9}$$

Proof: As we described in the general proccedure, if u_q, v_q are tangent vectors to the leaves there exist co-vectors $\alpha_q, \beta_q \in T_q^*M$ such that $\mathcal{B}_q(\alpha_q) = u_q$ and $\mathcal{B}_q(\beta_q) = v_q$, where the map $\mathcal{B}_q$ is given by:

$$\mathcal{B}_q(\alpha)(\cdot) = \pi_q(\cdot, \alpha)$$

Therefore, if we want to find two tangent vectors to the symplectic leaves we have to give vectors annihilated simultaneously by the differential of four Casimir functions for the corresponding Poisson structure.

Folds

Indefinite Fold

A straightforward calculation yields that the vectors,

$$u_q = \frac{x_3 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_3}}{(x_1^2 + x_3^2)^{1/2}}$$

$$v_q = \frac{x_1^2 x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - x_1 x_2 x_3 \frac{\partial}{\partial x_3}}{(x_1^2 + x_3^2)^{1/2}}$$

are tangent to Σ_q at q, and orthogonal with respect to the euclidean metric

$$ds^2 = dt_1^2 + dt_2^2 + dt_3^2 + dx_1^2 + dx_2^2 + dx_3^2$$

on B^6. Using the local expression of the Poisson structure for a fold singularity given by equation (7.2.1), one can check that $\mathcal{B}_q(\alpha_q) = u_q$, for

$$\alpha_q = \frac{x_3 dx_1 + x_1 dx_2}{k(q)(x_1^2 + x_3^2)^{1/2}}.$$

Similarly, $\mathcal{B}_q(\beta_q) = v_q$, for

$$\beta_q = \frac{-x_1 x_2 x_3 dx_2 - x_1(x_1^2 + x_3^2)}{2k(q)(x_1^2 + x_3^2)}.$$

A direct calculation now implies that the symplectic form is given by 7.5.1.

Definite Fold

For definite singularities described by the equations (7.2.2) and (7.2.3) we obtain the symplectic forms 7.5.2 and 7.5.3respectively. This follows directly with the same computations of the previous case. Tangent vectors to the leaves u_q and v_q are slightly different, one component changes its sign. This creates a change of sign on one of the components of the corresponding vectors α_q and β_q.

Cusps

Indefinite Cusps

In this case we find that the vectors,

$$u_q = \frac{-2x_3 \frac{\partial}{\partial x_1} + 3\left(t_1 - x_1^2\right) \frac{\partial}{\partial x_3}}{(9\left(t_1 - x_1^2\right)^2 + 4x_3^2)^{1/2}}$$

$$v_q = \frac{(2x_2 - 8x_2x_3^2)\frac{\partial}{\partial x_1} + 3(t_1 - x_1^2)(9(t_1 - x_1^2)^2 + 4x_3^2)\frac{\partial}{\partial x_2} + 12(t_1 - x_1^2)x_2x_3\frac{\partial}{\partial x_3}}{9(t_1 - x_1^2)^2 + 4x_3^2}$$

are tangent to Σ_q at q, and orthogonal with respect to the euclidean metric

$$ds^2 = dt_1^2 + dt_2^2 + dt_3^2 + dx_1^2 + dx_2^2 + dx_3^2$$

on B^6. Using the corresponding local expression of the bivector (7.2.6), we check that $\mathcal{B}_q(\alpha_q) = u_q$, for

$$\alpha_q = \frac{3(t_1 - x_1^2)dx_1 + x_3dx_3}{2k(q)(9t_1^2 - 18t_1x_1^2 + 9x_1^4 + 4x_3^2)^{1/2}}.$$

Similarly, $\mathcal{B}_q(\beta_q) = v_q$, for

$$\beta_q = \frac{6(t_1 - x_1^2)x_2x_3dx_1 - 9(t_1 - x_1^2)^2x_2dx_3}{k(q)(9(t_1 - x_1^2)^2 + 4x_3^2)}.$$

Now a direct calculation gives that the symplectic form is 7.5.4.

Definite Cusps

Definite singularities modelled by the equations (7.2.7) and (7.2.8). The corresponding symplectic forms on the leaves coincide with the previous one:

$$\omega_{\Sigma_q} = \frac{3(t_1 - x_1^2)x_2}{(9(t_1 - x_1^2)^2 + 4x_3^2)^{1/2}}\omega_{Area}(q).$$

This last equality follows from very similar computations as in the previous case, up to a sign, as in the fold case.

Swallowtails

Indefinite Swallowtail

We find that the vectors,

$$u_q = \frac{2x_3\frac{\partial}{\partial x_1} + (t_2 + 2t_1x_1 + 4x_1^3)\frac{\partial}{\partial x_3}}{((t_2 + 2t_1x_1 + 4x_1^3)^2 + 4x_3^2)^{1/2}}$$

$$v_q = \frac{(-2x_2(t_2 + 2t_1x_1 + 4x_1^3)^2 + 4x_3^2 + 8x_3^2)\frac{\partial}{\partial x_1} + (t_2 + 2t_1x_1 + 4x_1^3)\frac{\partial}{\partial x_2}}{(t_2 + 2t_1x_1 + 4x_1^3)^2 + 4x_3^2}$$
$$+\frac{4(t_2 + 2t_1x_1 + 4x_1^3)x_2x_3\frac{\partial}{\partial x_3}}{(t_2 + 2t_1x_1 + 4x_1^3)^2 + 4x_3^2}$$

are tangent to Σ_q at q, and orthogonal with respect to the euclidean metric

$$ds^2 = dt_1^2 + dt_2^2 + dt_3^2 + dx_1^2 + dx_2^2 + dx_3^2$$

on B^6. Using the local expression of the Poisson structure for a fold singularity given by equation (7.2.1), one can check that $\mathcal{B}_q(\alpha_q) = u_q$, for

$$\alpha_q = \frac{(t_2 + 2t_1x_1 + 4x_1^3)dx_1 - 2x_3dx_3}{2x_2k(q)((t_2 + 2t_1x_1 + 4x_1^3)^2 + 4x_3^2)^{1/2}}.$$

Similarly, $\mathcal{B}_q(\beta_q) = v_q$, for

$$\beta_q = \frac{1}{k}\left(\frac{2x_3(t_2+2t_1x_1+4x_1^3)dx_1}{t_2^2+4t_1t_2x_1+4t_1^2x_1^2+8t_2x_1^3+16t_1x_1^4+16x_1^6+4x_3^2}\right.$$
$$\left.+\left(1-\frac{4x_3^2}{(t_2+2t_1x_1+4x_1^3)^2+4x_3^2}\right)dx_3\right).$$

A direct calculation now implies that the symplectic form is 7.5.6.

Definite Swallowtail

On definite singularities we obtain that the symplectic forms on the leaves coincide in both cases (7.2.12) and (7.2.13):

$$\omega_{\Sigma_q} = -\frac{t_2+2t_1x_1+4x_1^3}{((t_2+2t_1x_1+4x_1^3)^2+4x_3^2)^{1/2}}\omega_{Area}(q).$$

Analogously to the last cases, we proceed changing the corresponding signs.

Butterflies

Indefinite Butterfly

We find that the vectors,

$$u_q = \frac{2x_3\frac{\partial}{\partial x_1}+(t_3+x_1(2t_2+3t_1x_1+5x_1^3))\frac{\partial}{\partial x_3}}{((t_3+x_1(2t_2+3t_1x_1+5x_1^3))^2+4x_3^2)^{1/2}}$$

$$v_q = \left(\frac{8x_2x_3^2}{(t_3+x_1(2t_2+3t_1x_1+5x_1^3))^2+4x_3^2}-2x_2\right)\frac{\partial}{\partial x_1}$$
$$+\left(t_3+x_1(2t_2+3t_1x_1+5x_1^3)\right)\frac{\partial}{\partial x_2}$$
$$+\frac{4(t_3+x_1(2t_2+3t_1x_1+5x_1^3))x_2x_3}{(t_3+x_1(2t_2+3t_1\alpha_1+5x_1^3))^2+4x_3^2}\frac{\partial}{\partial x_3}$$

are tangent to Σ_q at q, and orthogonal with respect to the euclidean metric

$$ds^2 = dt_1^2+dt_2^2+dt_3^2+dx_1^2+dx_2^2+dx_3^2$$

on B^6. Using the local expression of the Poisson structure for a fold singularity given by equation (7.2.1), one can check that $\mathcal{B}_q(\alpha_q) = u_q$, for

$$\alpha_q = \frac{(t_3+2t_2x_1+3t_1x_1^2+5x_1^4)dx_1-2x_3dx_3}{2x_2k(q)((t_3+x_1(2t_2+3t_1x_1+5x_1^3))^2+4x_3^2)^{1/2}}.$$

Similarly, $\mathcal{B}_q(\beta_q) = v_q$, for

$$\beta_q = \frac{2x_3(t_3+x_1(2t_2+3t_1x_1+5x_1^3))}{(t_3+x_1(2t_2+3t_1x_1+5x_1^3))^2+4x_3^2}dx_1$$
$$+\left(1-\frac{4x_3^2}{(t_3+x_1(2t_2+3t_1x_1+5x_1^3))^2+4x_3^2}\right)dx_3.$$

A direct calculation now implies that the symplectic form is given by 7.5.8.

Definite Butterfly

We have that for the corresponding butterfly definite singularities the symplectic form is in both cases:

$$\omega_{\Sigma_q} = -\frac{(t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3)}{(t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3))^2 + 4x_3^2)^{1/2}}\omega_{Area}(q).$$

□

7.6 Symplectic forms on higher dimensional type $2n$ generalized wrinkled fibrations

Corollary 7.6.1 *For a non-vanishing smooth function $k \in C^\infty(M)$ we have the following consequences:*

(1) *Let M be a closed smooth oriented and connected $2n$-manifold, and $f : M \to X$ a generalized broken Lefschetz fibration. The symplectic forms induced by the corresponding Poisson structures on the symplectic leaves Σ_q through a point $q = (t_1, \dots, t_{2n-3}, x_1, x_2, x_3)$ have the following local expressions:*

Lefschetz-type singularity

Let $q \in B^{2n}\backslash\{0\}$. Near Lefschetz-type singularities the symplectic form is given by:

$$\omega_{\Sigma_q} = \frac{1}{k(q)(t_{2n-3}^2 + x_1^2 + x_2^2 + x_3^2)}\omega_{Area}(p).$$

Indefinite fold singularity

Near indefinite fold singularities Z the symplectic form is locally described by:

$$\omega_{\Sigma_q} = \frac{1}{k(q)\sqrt{x_1^2 + x_2^2 + x_3^2}}\omega_{Area}(q).$$

where $\omega_{Area}(q)$ is the area form on Σ_q induced by the metric

$$ds^2 = dt_1^2 + \cdots + dt_{2n-3}^2 + dx_1^2 + dx_2^2 + dx_3^2$$

on $Z \times B^3$.

(2) *Let M be a closed, orientable, smooth $2n$-manifold endowed with a type $2n$-wrinkled fibration f to a closed $2n-2$ manifold X. Let $q \in B^{2n}\backslash\{0\}$. Then the symplectic forms associated to the complete Poisson structure are given by the following expressions near the corresponding singularities:*

Fold

$$\omega_{\Sigma_q} = \frac{x_1^2}{2k(q)(x_1^2 + x_3^2)^{1/2}}\omega_{Area}(q)$$

Cusp

$$\omega_{\Sigma_q} = \frac{3x_2\,(t_{2n-5} - x_1^2)}{k(q)(9\,(t_1 - x_1^2)^2 + 4x_3^2)^{1/2}}\omega_{Area}(q)$$

Swallowtail

$$\omega_{\Sigma_q} = -\frac{t_{2n-4} + 2t_{2n-5}x_1 + 4x_1^3}{k(q)((t_{2n-4} + 2t_{2n-5}x_1 + 4x_1^3)^2 + 4x_3^2)^{1/2}}\omega_{Area}(q)$$

Butterfly

$$\omega_{\Sigma_q} = -\frac{t_{2n-3} + x_1(2t_{2n-4} + 3t_{2n-5}x_1 + 5x_1^3)}{k(q)((t_{2n-3} + x_1(2t_{2n-4} + 3t_{2n-5}x_1 + 5x_1^3))^2 + 4x_3^2)^{1/2}}\omega_{Area}(q)$$

The following equations depend on a real parameter s. Near a singularity locally modeled by the maps b_s (7.3.1), m_s (7.3.2), f_s(7.3.3), and w_s (7.3.4) the corresponding symplectic forms are

Map b_s

$$\omega_{\Sigma_q} = \frac{(s - t_{2n-3}^2 + x_1^2)}{k(q)((s - t_{2n-3}^2 + x_1^2)^2(9(s - t_{2n-3}^2 + x_1^2)^2 + 4(x_2^2 + x_3^2)))^{1/2}}\omega_{Area}(q)$$

Map m_s

$$\omega_{\Sigma_q} = -\frac{(s - t_{2n-3}^2 - x_1^2)}{k(q)((s - t_{2n-3}^2 - x_1^2)^2(9(s - t_{2n-3}^2 - x_1^2)^2 + 4(x_2^2 + x_3^2)))^{1/2}}\omega_{Area}(q)$$

Map f_s

$$\omega_{\Sigma_q} = \frac{(t_{2n-3} - 2sx_1 + 4x_1^3)}{k(q)((t_{2n-3} - 2sx_1 + 4x_1^3)^2((t_{2n-3} - 2sx_1 + 4x_1^3)^2 + 4(x_2^2 + x_3^2)))^{1/2}}\omega_{Area}(q)$$

Map w_s

$$\begin{aligned}\omega_{\Sigma_q} &= \frac{1}{2\mu k(q)} \cdot (t_{2n-3}x_2 + x_1x_3)\\ &\quad \frac{((st_{2n-3} + 2(t_{2n-3}^2 + x_1^2))^2 + (x_3(s + 2t_{2n-3}) - 2x_1x_2)^2 + 4(t_{2n-3}x_2 + x_1x_3))}{((st_{2n-3} + 2(t_{2n-3}^2 + x_1^2))^2 + (x_3(s + 2t_{2n-3}) - 2x_1x_2)^2 + 4(t_{2n-3}x_2 + x_1x_3)^2)^{1/2}}\omega_{Area}(q)\end{aligned}$$

here ω_{Area} is the area form on Σ_q induced by the euclidean metric on B^{2n}, and

$$\begin{aligned}\mu^2 =&(t_{2n-3}x_2 + x_1x_3)^2(s^2(t_{2n-3}^2 + x_2^2 + x_3^2) + 4st_{2n-3}(t_{2n-3}^2 + x_1^2 + x_2^2 + x_3^2)\\ &+ 4(t_{2n-3}^2 + x_1^2 + x_2^2 + x_3^2)^2)(s^2(t_{2n-3}^2 + x_3^2) + 4(t_{2n-3}^2 + x_1^2)(t_{2n-3}^2 + x_1^2 + x_2^2 + x_3^2)\\ &+ 4s(t_{2n-3}^3 - x_1x_2x_3 + t_{2n-3}(x_1^2 + x_3^2))).\end{aligned}$$

For all these cases $\omega_{Area}(q)$ is the area form induced by the euclidean metric on B^{2n}.

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