Families of Baker domains in a subclass of $\mathcal{K}$

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Families of Baker domains in a subclass of $\mathcal{K}$

Para la obtención del grado de Doctor
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A mis ***padres****, mi primera motivación,*
quienes a su manera
me heredaron parte de lo que soy.

I can't go to a restaurant and
order food because I keep looking
at the fonts on the menu.
Donald Knuth

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A todos los que la presente vieren y entendieren.

Inicio de las Leyes Orgánicas. Juan Carlos I

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Introduction

When a man lies he murders some part
of the world.
These are the pale deaths which men
miscall their lives.

Clifford Lee Burton.

At the beginning of the XX century, P. Fatou in [Fa1] and G. Julia in [Ju] developed, independently, the theory of iteration of rational functions, as part of the competition for the *Great Prize of Mathematical Science* of 1918, see [Au]. After his treatment on rational functions, Fatou in [Fa], noticed that the ideas in the rational iteration theory, could be applied to the iteration of entire functions (this class of functions is closed under composition).
Given the function

$$f(z) = z + 1 + e^{-z},$$

he proved that the right half-plane is contained in an invariant Fatou component U where $f^n \to \infty$ as $n \to \infty$, proving in this way the existence of a new type of Fatou component different from those already existing for rational functions, since $\infty \in \widehat{\mathbb{C}}$ is an essential singularity for the function $f(z) = z + 1 + e^{-z}$ (recall that there are no essential singularities for a rational function).
When the theory of iteration of rational functions was resumed decades later, the ideas of Fatou were formalized creating the theory of iteration of entire transcendental functions. Great part of this work was done by I.N. Baker (1932 - 2001) in [Ba2] and [Ba4]. Because of this, Eremenko and Lyubich, in a treatment of iteration of entire functions [EL1], called this kind of Fatou components *Baker domains*, where the essential singularity (the point at infinity) acts as *attracting* boundary point of the Fatou component.
When the theory of iteration was extended to transcendental meromorphic functions, the existence of Baker domains was a question mark. Some examples of transcendental entire and meromorphic functions have been given along the last decades. One interesting example is the one given by Baker

et. al. in [BKL], where the authors show that the Fatou set of the function

$$f(z) = \frac{1}{z} - e^z$$

contains a 2-periodic Baker domain. In this case, since there are two Baker domains in the cycle, the point at infinity is not the only *attracting* boundary point, also the origin is an *attracting* boundary point and it is an essential singularity for the second iterate $f^2(z) = f(f(z))$. If we consider the new function given by the second iterate of the above function

$$f_2(z) = f^2(z) = \frac{1}{1/z - e^z} - \exp\left(\frac{1}{z} - e^z\right),$$

we notice that there are two essential singularities: one at the origin and other one at the point at infinity. Using the ideas in [BKL] there exist two *Baker domains* for the function f_2: one associated to the origin and other one associated to the point at infinity, thus the function f_2 is no longer a transcendental meromorphic function in the classical sense.
In the last decades, functions in the class $\mathcal{K}$ of general meromorphic functions are studied in iteration theory. In general terms, a function in class $\mathcal{K}$ has more than one essential singularity (for a formal definition see Definition 1.4.3 in Section 1.4). The iteration theory of this class of functions was formalized in the dissertations [Bo] and [Her], where the authors extend the results of the Fatou-Julia theory.
Some examples of this kind of functions are the Barański maps studied in [KU] and the parametric family studied in [DH].
In the present work, we investigate a subclass of functions in $\mathcal{K}$ with two or more essential singularities where each essential singularity is associated to a family of different Baker domains. Moreover, the dynamics inside these domains, near the boundary point can be semi-conjugated to a translation. We resume the result in the following theorem.

Theorem (Main Theorem)**.** *Let $g : \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental meromorphic or rational function, such that*

$$f(z) = z + \exp(g(z))$$

belongs to class $\mathcal{K}$ of general meromorphic functions, where $A(f) = \overline{g^{-1}(\infty)} = \overline{\{\text{poles of } g\}}$. If $z_0 \in g^{-1}(\infty)$ is a pole of g of order $p \geq 1$, then f has p-families of infinitely many different Baker domains with z_0 as its Baker point. Each family lies in a sector of angle $2\pi/p$ of the disc $D(z_0, \delta)$ for some $\delta > 0$. Moreover, each Baker domain can be classified as parabolic type I according to the Cowen and König classification.

The text is divided in four chapters in the following way. In Chapter 1, we recall the fundamental concepts and results in the iteration of *meromorphic* functions. A new class of *general meromorphic* functions is defined and the fundamental concepts and results are then extended to this new class of functions (see [BDH]). In Chapter 2 we give the principal properties of *Baker domains* including its classification according to its dynamics near the *attracting* boundary point. In Chapter 3 we present the proof of the Main Theorem mentioned above as well as some examples with interesting dynamical properties. Finally, in Chapter 4 lines of future work are presented, together with open questions related to the general functions that appear in the Main Theorem and examples studied in Chapter 3. In particular, for the function $f_0(z) = z + \exp(1/z)$ a conjecture about the dimension of its *Teichmüller space* is presented.

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Chapter 1

Preliminaries

Although the iteration theory of rational functions dates back to the late 19th century with works of Cayley, Schröeder and others [Au], the fundamental concepts and theorems in the theory are attributed to Pierre Fatou (1878-1929) and Gaston Julia (1893-1978). Fatou himself, in [Fa], in part of his work, extends some results for the rational case to transcendental entire functions. Decades later, the theory of rational functions was extended formally to transcendental entire and meromorphic functions. For further references [Ber1] and [Sc] are surveys in this subject.
In the present chapter, we consider some of the fundamental and more important results of iteration theory for transcendental meromorphic functions, which are extensions of the rational case, and carry on these results to a more general class of functions considered in this work.

1.1 Classes of functions $\mathcal{R}$, $\mathcal{E}$, $\mathcal{P}_1$ and $\mathcal{M}$

We recall the definition of a *transcendental meromorphic function* in the classical sense. For a deep analysis, the most prevalent references are [Ah], [Con] and [SS].

Definition 1.1.1. *Let f be a nonconstant mapping, defined in a domain $U \subset \mathbb{C}$ except perhaps in a point $z_0 \in U$. If there exists $R > 0$ such that f is analytic in a punctured disc $D^*(z_0, R) \subset U$, then we say that z_0 is an **isolated singularity** of f. If also,*

$$\lim_{z \to z_0} f(z) \tag{1.1.1}$$

*exists in $\mathbb{C}$, then we call z_0 a **removable singularity** of f. If*

$$\lim_{z \to z_0} f(z) = \infty, \tag{1.1.2}$$

*then we call z_0 a **pole** of f. If such a limit does not exist, we say that z_0 is an **essential singularity** of f.*

When z_0 is a pole of f, we can think of f as a function with values in the Riemann sphere $f : U \to \widehat{\mathbb{C}}$, and define in this case $f(z_0) = \infty$. Then we can consider the following definition.

Definition 1.1.2. *We say that $f : U \to \widehat{\mathbb{C}}$ is a **meromorphic** function in U if the only singularities of f are poles.*

In the case $U = \mathbb{C}$, we say only "f is meromorphic". In this context, it is valid to ask if there exists a way, at least continuous, to define f in the point $z_0 = \infty$. For this, we can consider the following limit:

$$\lim_{z\to 0}[f(1/z)]^{-1}. \tag{1.1.3}$$

Definition 1.1.3. *If f is a meromorphic function and limit (1.1.3) exists in $\widehat{\mathbb{C}}$, then f is in fact a **rational** function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. In other cases, f is called a **transcendental meromorphic** function.*

In this sense, the Laurent Expansion Theorem, allows us to have a local representation of f with respect to a pole z_0 in the following way: if z_0 is a pole of a transcendental meromorphic function f, then there exists $R > 0$ such that f can be written as

$$f(z) = \sum_{m=1}^{p} c_n(z - z_0)^{-m} + \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

where both series are convergent in the punctured disc $D^*(z_0, R)$. From the above expression it is clear that limit (1.1.2) holds. The first sum in the right side of the above expression is called **Principal Part** of f with respect to z_0. The natural number p in this term is called **order** of z_0.
The above leads us to consider the poles of functions, as *pre-images* of the point at infinity of the Riemann sphere: {poles of f}$=f^{-1}(\infty)$.
There exists a classification over the meromorphic functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$ depending on the limit (1.1.3) and the nature of pre-images of infinity as follows:

- $\mathcal{R}$:={$f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$: f is **rational**}
- $\mathcal{E}$:={$f : \mathbb{C} \to \mathbb{C}$: f has no poles in $\mathbb{C}$ and (1.1.3) does not exist, f is **transcendental entire**}
- $\mathcal{P}_1$:={$f : \mathbb{C} \to \widehat{\mathbb{C}}$: f has a pole $z_0 \in \mathbb{C}$ which is an omitted value and limit (1.1.3) does not exist}
- $\mathcal{M}$:={$f : \mathbb{C} \to \widehat{\mathbb{C}}$: f has at least one pole in $\mathbb{C}$ which is not an omitted value and limit (1.1.3) does not exist}.

In the last two cases, the point z_0 is an **omitted value** if $z_0 \notin f(\mathbb{C})$. These are the classes of functions where the classical theory of iteration of functions is based.

1.2 Results in iteration theory

In this section, we give some preliminary results on meromorphic functions. We consider especially those results that we would like to extend for a more general class of functions, the surveys [Ber1] and [Sc] mentioned at the beginning of the chapter are again good references. For the rest of this chapter, we will refer to a meromorphic function as a function belonging to any of the above classes and when necessary we will mention the class to which belongs. Let $f : \Omega \to \widehat{\mathbb{C}}$ be a meromorphic function (where $\Omega \in \{\mathbb{C}, \widehat{\mathbb{C}}\}$ depending on the class), we define the n-iterated f^n inductively as

$$f^0(z) = z \qquad \text{and} \qquad f^n(z) = f(f^{n-1}(z)),\ n \geq 1, \tag{1.2.1}$$

always this is well defined. We are interested in the behavior of the discrete dynamical system generated by f. The study of this behavior generates a dichotomy in the Riemann sphere given by the Fatou and Julia sets of the function f, denoted by $\mathcal{F}(f)$ and $\mathcal{J}(f)$ respectively. In general terms, the first set is where the family of iterates behaves in a predictable way, while in the second, we have a behavior of chaotic type. Formally we write

$$\mathcal{F}(f) := \{z \in \widehat{\mathbb{C}} : \{f^n\}_{n\in\mathbb{N}} \text{ is well defined and normal in a neighborhood of } z\}$$

and

$$\mathcal{J}(f) := \widehat{\mathbb{C}} \backslash \mathcal{F}(f),$$

where *normal* is taken in the sense of Montel. From the above definition we have that $\mathcal{F}(f)$ is open while $\mathcal{J}(f)$ is closed.
Note that for $f \in \mathcal{R}$, f^n is well defined in the whole Riemann sphere, so this condition can be omitted in the definition of $\mathcal{F}(f)$. In other cases, we can see that $\infty \in \mathcal{J}(f)$ (which is not necessary the case when $f \in \mathcal{R}$, for example if f is a polynomial then $\infty \in \mathcal{F}(f)$). From the above and the definition of Fatou set $\mathcal{F}(f)$, we can obtain the following theorem.

Theorem 1.2.1. *If f is a meromorphic function, then $\mathcal{J}(f) \neq \emptyset$.*

For the case $f \in \mathcal{R}$, using an argument by contradiction, it is enough to consider the normality in the Montel sense and the Argument Principle, and then conclude that $\mathcal{F}(f)$ cannot be the whole sphere $\widehat{\mathbb{C}}$.

For a point $z_0 \in \widehat{\mathbb{C}}$ we define the **positive orbit** as

$$O^+(z_0) = \bigcup_{n\geq 0} f^n(z_0),$$

when this is well defined. We define the **negative orbit** as

$$O^-(z_0) = \bigcup_{n\geq 1} f^{-n}(z_0),$$

when this is well defined. Finally, we define the **orbit** of z_0 as

$$O(z_0) = O^+(z_0) \cup O^-(z_0).$$

We say that a point z_0 is **exceptional** if $|O^-(z_0)| < \infty$. Denote by E_f the set of exceptional points of f. Note that in the case $f \in \mathcal{M}$, applying *Picard's Great Theorem* (PGT), $|O^-(\infty)| = \infty$, in fact $f^{-3}(\infty)$ is already a set of infinite cardinality. In this case f^n is well defined in $\widehat{\mathbb{C}}$, except in a countable set determined by the poles of f, f^2,..., f^{n-1}. Hence, $\{f^n\}_{n\geq 0}$ is well defined in the open set $\widehat{\mathbb{C}}\backslash\overline{O^-(\infty)}$. Since $f(\widehat{\mathbb{C}}\backslash\overline{O^-(\infty)}) \subset \widehat{\mathbb{C}}\backslash\overline{O^-(\infty)}$ and $|O^-(\infty)| = \infty$, by Montel's Theorem we have

$$\mathcal{F}(f) = \widehat{\mathbb{C}}\backslash\overline{O^-(\infty)} \qquad \text{and} \qquad \mathcal{J}(f) = \overline{O^-(\infty)}. \tag{1.2.2}$$

This is a particular characteristic of functions $f \in \mathcal{M}$, which does not hold in cases $\mathcal{E}$ and $\mathcal{P}_1$. When $f \in \mathcal{R}$, this characteristic may be possible, in fact, in general terms, we have the following theorem.

Theorem 1.2.2. *If f is a meromorphic function and $z_0 \in \mathcal{J}(f)$ is not an exceptional value, then $\mathcal{J}(f) = \overline{O^-(z_0)}$.*

Some other properties of the Fatou and Julia sets, that can be derived directly from the definitions are the following.

Theorem 1.2.3. *If f belongs to class $\mathcal{R}$, $\mathcal{E}$ or $\mathcal{P}_1$, then $\mathcal{F}(f) = \mathcal{F}(f^n)$ and $\mathcal{J}(f) = \mathcal{J}(f^n)$ for every $n \geq 2$.*

In the case $f \in \mathcal{M}$, this property is not well defined, since f^n is no longer a meromorphic function. This is one of the reasons to consider an extension of this theory to more general functions, where f^n still belongs to the same class. In the other cases, we say that these classes are closed under composition.

We say that a set $A \subset \widehat{\mathbb{C}}$ is *completely invariant* (with respect to a function f) if and only if, for $z \in A$, $f^{-1}(z) \subset A$ and $f(z) \in A$ unless $f(z)$ is not defined.

Theorem 1.2.4. *If f is meromorphic, then $\mathcal{F}(f)$ and $\mathcal{J}(f)$ are completely invariant.*

Although we know that $\mathcal{J}$ may not be empty while $\mathcal{F}$ may be.

Theorem 1.2.5. *If the interior of $\mathcal{J}(f)$ is not empty, then $\mathcal{J}(f) = \widehat{\mathbb{C}}$.*

1.3 Julia set and periodic points

One of the greatest contributions of Fatou and Julia (done independently), was to give a dynamical characterization of the Julia set in terms of the *repelling periodic* points.

Definition 1.3.1. *Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function and $z_0 \in \mathbb{C}$. If there exists a positive integer $p \geq 1$ such that $f^p(z_0) = z_0$, we say that z_0 is a* ***periodic point*** *of f, the minimal p with this property is called the* ***period*** *of z_0. If $p = 1$, we just say that z_0 is a* ***fixed point*** *of f.*

There exists a classification of periodic points as follows:

Definition 1.3.2. *Let z_0 be a periodic point of period p and $\lambda = (f^p)'(z_0)$, called the* ***multiplier*** *of z_0. Then*

- *if $(0 =)0 < |\lambda| < 1$, we say that z_0 is a* ***(super)attracting periodic point****,*
- *if $|\lambda| > 1$, we say that z_0 is a* ***repelling periodic point****,*
- *if $|\lambda| = 1$ and $\lambda^n = 1$ for some $n \in \mathbb{N}$, we say that z_0 is a* ***rationally indifferent periodic point****,*
- *if $|\lambda| = 1$ and $\lambda^n \neq 1$ for every $n \in \mathbb{N}$, we say that z_0 is an* ***irrationally indifferent periodic point****.*

There exists a linearization for f^p, locally with respect to z_0, depending on the nature of z_0 and its multiplier λ (see [CG] and [Mi]).
Meromorphic functions may have no fixed points (for example, $f(z) = z+e^z$), for periods of large order, we have the following result, see [Ber1] as reference.

Theorem 1.3.3. *If $f \in \mathcal{M}$ and $n \geq 2$, then f has infinitely many periodic points of minimal period n.*

If a set is nonempty, closed and contains no isolated points, then we call it **perfect**.

Theorem 1.3.4. *If f is a meromorphic function, then $\mathcal{J}(f)$ is perfect.*

The above results, serve as tools to prove the dynamical characterization theorem of the Julia set.

Theorem 1.3.5. *If f is a meromorphic function, then $\mathcal{J}(f)$ is the closure of the repelling periodic points.*

1.4 $\mathcal{K}$, a more general class

We recall that the extension of the theory of iteration of meromorphic functions (in the classical sense) to this new class of meromorphic functions, was given first by Bolsch [Bo] and Herring [Her] in their respective dissertations.

As we saw in the previous section, there exists a property that class $\mathcal{M}$ does not hold as in the other classes, class $\mathcal{M}$ is not closed under composition, i.e. its iterates no longer belong to this class. Take the function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ as an example (studied in [BKL]) given by

$$f(z) = \frac{1}{z} - e^z.$$

In this case, we have $z_1 = 0$ is a pole of f and $z_0 = \infty$ is an essential singularity. If we compute the second iterate of f

$$f^2(z) = f(f(z)) = \frac{1}{\frac{1}{z} - e^z} - \exp\left(\frac{1}{z} - e^z\right),$$

from the last term of the above expression, we have that f^2 has **two** essential singularities $z_0 = \infty$ and $z_1 = 0$, the poles have turned into essential singularities. From the first term of the right side of the last equality and solving the equation

$$0 = \frac{1}{z} - e^z$$

we have that f^2 has infinitely many poles. Consequently, f^3 will have infinitely many essential singularities.
Keeping in mind the ideas behind the definition of a meromorphic function in the classical sense, we consider the following definitions.

Definition 1.4.1. *Let f be a nonconstant mapping defined over a domain $U \subset \widehat{\mathbb{C}}$, except perhaps at a point $z_0 \in U$. If there exists $R > 0$ such that f is analytic in the punctured disc $D^*(z_0, R) \subset U$, then we say that z_0 is an* ***isolated singularity*** *of f. Also, if the limit*

$$\lim_{z \to z_0} f(z) \tag{1.4.1}$$

exists in $\widehat{\mathbb{C}}$, we say that z_0 is a ***removable singularity*** *of f in U. If such a limit does not exist in the sense that it is not well defined, we say that z_0 is an* ***essential singularity*** *of f. We denote by $Ess(f)$ the set of essential singularities of f in $\widehat{\mathbb{C}}$.*

We leave the concept of *pole* only for classical meromorphic functions, where it makes more sense. The above definition is almost the same as the one used in the previous sections. The difference lies now in the fact that we only ask if the limit (1.4.1) exists. In the classical definition of a meromorphic function $\infty \in \widehat{\mathbb{C}}$ is the only possible essential singularity. In Definition 1.4.1, which is more general, essential singularities may be any point in $\widehat{\mathbb{C}}$ (recall that in the above example, f^3 has infinitely many essential singularities in $\widehat{\mathbb{C}}$).
In the case that the limit (1.4.1) exists, we extend f to z_0 as

$$f(z_0) = \lim_{z \to z_0} f(z). \tag{1.4.2}$$

In this way, we are able to consider the following definition.

Definition 1.4.2. *We say that $f : U \subseteq \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a* ***general meromorphic*** *function in U if the only isolated singularities in U are removable. Moreover, we said that f is* ***general transcendental meromorphic*** *if*

$$Ess(f) \neq \emptyset.$$

We are now in position to define the class of functions we will work on.

Definition 1.4.3. *We say that a function f belongs to class $\mathcal{K}$ of general meromorphic functions if there exists a countable compact set $A(f) \subset \widehat{\mathbb{C}}$ such that f is an analytic function in $\widehat{\mathbb{C}} \backslash A(f)$ but in no other superset.*

From the above definitions we have the following characterization for $A(f)$.

Theorem 1.4.4 (Bolsch [Bo], Theorem 1.2)**.** *If $f \in \mathcal{K}$, then $A(f)$ is the closure of isolated essential singularities of f.*

We are able to characterize the following sub-classes of functions depending on the nature of the set $A(f)$:

- $\mathcal{R} = \{f \in \mathcal{K}\colon\ A(f) = \emptyset\}$.
- $\mathcal{E} = \{f \in \mathcal{K}\colon\ A(f) = \{\infty\}$ and $f^{-1}(A(f)) = \emptyset\}$.
- $\mathcal{P}_1 = \{f \in \mathcal{K}\colon\ A(f) = \{\infty\}$, $f^{-1}(A(f)) = \{z_0\}$ and z_0 is omitted$\}$.
- $\mathcal{P}_2 = \{f \in \mathcal{K}\colon\ A(f) = \{z_0, \infty\}$ and $f^{-1}(A(f)) = \emptyset\}$.
- $\mathcal{M} = \{f \in \mathcal{K}\colon\ A(f) = \{\infty\}$ and some $z_0 \in f^{-1}(A(f)) \neq \emptyset$ is not omitted$\}$.

The more general case to consider is when $|A(f)| \geq 2$ and $f^{-1}(A(f)) \neq \emptyset$. We know that in the above *sub-classes*, the theory of iteration is well defined, as it was seen in the previous sections.
Given $f \in \mathcal{K}$, we define the n-th iterate inductively as

$$f^1 = f := \widehat{\mathbb{C}} \backslash A(f) \to \widehat{\mathbb{C}}, \text{ and } f^n := f(f^{n-1}) : \widehat{\mathbb{C}} \backslash \bigcup_{j=0}^{n-1} f^{-j}(A(f)) \to \widehat{\mathbb{C}}.$$

The following result is fundamental for the extension of iteration theory.

Theorem 1.4.5 (Herring [Her], Theorem 3.1.1)**.** *If $f \in \mathcal{K}$, then for each $n \geq 1$, f^n is single valued and general meromorphic in*

$$R_n = \widehat{\mathbb{C}} \backslash \bigcup_{j=0}^{n-1} f^{-j}(A(f)),$$

*with a **natural boundary** of f^n defined by*

$$A_n(f) = A(f^n) := \bigcup_{j=0}^{n-1} f^{-j}(A(f)).$$

Moreover, every set $A_n(f)$ is compact and countable.

Now, we are able to define the fundamental set of iteration theory.

Definition 1.4.6. *Given $f \in \mathcal{K}$, we say that a point $z_0 \in \widehat{\mathbb{C}}$, belongs to the **Fatou set** of f, denoted by $\mathcal{F}(f)$, if there exists a neighborhood of z_0 where $\{f^n\}_{n\geq 0}$ is well defined and forms a normal family in the sense of Montel. We define the **Julia set** as the complement in $\widehat{\mathbb{C}}$ of the Fatou set, denoted by $\mathcal{J}(f) = \widehat{\mathbb{C}} \backslash \mathcal{F}(f)$.*

In the same way, directly from the above definition, we have that $\mathcal{F}(f)$ is open while $\mathcal{J}(f)$ is a closed set (in fact compact in $\widehat{\mathbb{C}}$). Note that if $f \notin \{\mathcal{R}, \mathcal{E}, \mathcal{P}_1, \mathcal{P}_2\}$, then $|O^-(A(f))| = \infty$, hence (as in the case for $f \in \mathcal{M}$) we have

$$\mathcal{F}(f) = \widehat{\mathbb{C}} \backslash \overline{O^-(A(f))} \quad \text{and} \quad \mathcal{J}(f) = \overline{O^-(A(f))}. \tag{1.4.3}$$

So, in view of Theorem 1.4.5, we have the property about iterates of f that we do not have in the classical meromorphic case $\mathcal{M}$.

Proposition 1.4.7 (Bolsch [Bo], Proposition 1.3). *If $f \in \mathcal{K}$, then $f^2 \in \mathcal{K}$ with $A(f^2) = A_2(f) = A(f) \cup f^{-1}(A(f))$.*

Inductively, if $f \in \mathcal{K}$, then $f^n \in \mathcal{K}$ for $n \geq 2$.
A great part of Herring's work in [Her], was to prove that properties of Fatou and Julia sets mentioned in the previous section hold for functions in the class $\mathcal{K}$ in the general setting (in fact, Herring's definition of class $\mathcal{K}$ considers $A(f)$ as a totally disconnected compact set, instead of a countable compact set). Such properties are listed in the following theorem.

Theorem 1.4.8 (Herring [Her], Theorem 3.1.3). *If $f \in \mathcal{K}$, then the following statements hold:*

1. *$\mathcal{F}(f)$ is completely invariant, i.e. $z \in \mathcal{F}(f)$ if and only if $f(z) \in \mathcal{F}(f)$; so, $z \in \mathcal{J}(f) \backslash A(f)$ if and only if $f(z) \in \mathcal{J}(f)$ and $\mathcal{J}(f)$ is completely invariant also.*
2. *For each positive integer p, $\mathcal{F}(f^p) = \mathcal{F}(f)$ and $\mathcal{J}(f^p) = \mathcal{J}(f)$.*
3. *If ϕ is a Möbius transformation and $f_\phi = \phi f \phi^{-1}$, then $\mathcal{F}(f_\phi) = \phi(\mathcal{F}(f))$ and $\mathcal{J}(f_\phi) = \phi(\mathcal{J}(f))$.*
4. *$\mathcal{J}(f)$ is a perfect set.*

5. *For every $w \in \mathcal{J}(f)$ with $w \notin E_f = \{z : O^-(z)$ is a finite set$\}$, the set $O^-(w)$ is dense in $\mathcal{J}(f)$.*

6. *If $\mathcal{J}(f)$ contains a nontrivial disc, then $\mathcal{F}(f) = \emptyset$.*

Since iteration is well defined in class $\mathcal{K}$, the concepts of periodic (fixed) points are directly extended to this class of functions, as well as its classification.
As was mentioned before, one of the most important contribution of Fatou and Julia, was the description of the Julia set for rational functions in a dynamical sense, involving repelling periodic points. Baker in [Ba1], covers the transcendental entire case, while Baker, Kotus and Lü did the same for the meromorphic (classical) case in [BKL].
The following result was prove by Bolsch in his Dissertation.

Theorem 1.4.9 (Bolsch [Bo], Theorem 1.12). *If $f \in \mathcal{K}$, then $\mathcal{J}(f)$ is the closure of its repelling periodic points.*

For completeness we give a brief sketch of the proof of Theorem 1.4.9.

Sketch of proof. First, using Marty's criterion it is proved that repelling periodic points belong to $\mathcal{J}(f)$. The proof is now divided in two cases:

a. $O^-(A(f))$ contains at most two points, this is the case for $f \in \{\mathcal{R}, \mathcal{E}, \mathcal{P}_1, \mathcal{P}_2\}$. For a disc D meeting $\mathcal{J}(f)$, using Zalcman's lemma, it is possible to find a sequence $\{w_n\}$ tending to some $z^* \in D \cap \mathcal{J}(f)$, and subsequence $\{f^{n_m}\}$ with

$$f^{n_m}(w_n) = w_n \qquad \text{and} \qquad (f^{n_m})'(w_n) \to \infty, \text{ as } n \to \infty.$$

The first condition of the above conclusion, defines a sequence of periodic points, while the second condition states that these periodic points are repelling.

b. $O^-(A(f))$ contains at least three points. First, using Montel's Theorem, we have that

$$\mathcal{J}(f) = \overline{O^-(A(f))}.$$

So, it is enough to prove that for each essential singularity $z^* \in O^-(A(f))$, there are repelling periodic points tending to it. This is possible with the following lemma.

Lemma 1.4.10 (Bolsch [Bo], Lemma 1.9). *Let g be meromorphic in $\{0 < |z| < r\}$ for some $r > 0$, with an essential singularity at 0. Then there are $\{w_1, w_2, w_3, w_4\} \in \widehat{\mathbb{C}}$:*
If h is meromorphic in any open subset G of $\widehat{\mathbb{C}}$, ψ analytic in any

neighborhood of 0 *and* $w \in G\backslash\{w_1, ..., w_4\}$, $h(w) = \psi(0)$, $h'(w) \neq 0$, *then there is a sequence* $\{\omega_n\}$, $0 < |\omega_n| < r$, *such that*

$$(h \circ g)(\omega_n) = \psi(\omega_n), \qquad \textit{for all } n,$$

$$\omega_n \to 0, \qquad (h \circ g)'(\omega_n) \to \infty, \qquad \textit{as } n \to \infty.$$

Taking $h(z) := f(z) - z^*$, $g(z) := f^m(z + z^*)$ and $\psi(z) = z$, we have that

$$f^{m+1}(\omega_n + z^*) = \omega_n + z^* \qquad \text{with} \qquad (f^{m+1})'(\omega_n + z^*) \to \infty, \text{ as } n \to \infty.$$

Since $\omega_n \to 0$, we construct a sequence of repelling periodic points $(z_n = \omega_n + z^*)$ tending to z^* which concludes the proof of the theorem.

The proof of Theorem 1.4.9 for the case $f \in \mathcal{M}$ is based on *Ahlfors' Five Island Theorem*, Bolsch uses Marty's criterion and the Zalcman Lemma as fundamental tools for his proof, we refer to Appendix B for more details on this results. Both results play an important role in the modern theory of normal families.

Chapter 2

Baker domains

Fatou, in part of his work, discovered that iteration theory for rational functions may be extended to transcendental entire functions (entire functions are closed under composition). In fact, he showed that the function

$$f(z) = z + 1 + e^{-z}$$

has an invariant component U, containing the right half-plane, where

$$f^n(z) \to \infty, \text{ for all } z \in U.$$

Dynamics seems to be similar to Leau domains, but in this case, the limit point is an essential singularity instead of a periodic point.
When iteration theory is extended formally to entire functions, this kind of components may exist (as well as wandering domains, absent in the rational case). Baker, in [Ba2] and [Ba3], proved several properties that this kind of domains possess. Because of this, A. Eremenko and M. Lyubich were first to call them Baker domains to these kind of Fatou components, see [EL1].
In this chapter we consider the classification of Fatou components in the general meromorphic case $\mathcal{K}$, focusing our attention in *Baker domains*. We present results of Cowen and König related with the classification of Baker domains depending on the local semi-conjugacy near the essential singularity, as it is the case for the other Fatou components. Also, we consider the condition of Rippon and Stallard in [RS] for the existence of families of Baker domains.

2.1 Fatou components

Let $f \in \mathcal{K}$ and $U \subset \mathcal{F}(f)$ a connected component of the Fatou set of f. From Theorem 1.4.8 (Conclusion 1), there exists $U_1 \subset \mathcal{F}(f)$ connected component such that $f(U) \subset U_1$. In general, there exists a connected component $U_n \subset \mathcal{F}(f)$ such that $f^n(U) \subset U_n$. Analogously to the periodic point definition, it is possible to classify the Fatou components in the following way.

Definition 2.1.1. *Let $f \in \mathcal{K}$ and $U \subset \mathcal{F}(f)$ be a connected component. If there exists $p \in \mathbb{N}$ such that $f^p(U_m) \subset U_m$ for some $m \geq 0$, we say that U is a* ***pre-periodic component*** *of f. If in addition $m = 0$, U is called a* ***periodic component*** *of f and $\{U = U_0,\ U_1, ...,\ U_{p-1}\}$ is called* ***cycle*** *of U. If $p = 1$, we called U* ***forward invariant component****. Otherwise, we called U a* ***wandering domain****.*

Since $\mathcal{J}(f)$ is always a nonempty and perfect set, each connected component $U \subset \mathcal{F}(f)$ is a hyperbolic domain, that is, its universal covering is conformal to the unit disc $\mathbb{D}$. Hence, the action of $f^p : U \to U$ over a periodic component is lifted to an action $F : \mathbb{D} \to \mathbb{D}$. In this way, the following classification on periodic components of the Fatou set, is extended directly from the rational and meromorphic case (through the Denjoy-Wolff Theorem, which will be stated in Section 2.4). See [CG] and [Mi] for further reference. From Theorem 1.4.8 (conclusion 2), we may assume U is an invariant component.

Theorem 2.1.2 (Herring [Her], Theorem 4.1.1)**.** *Let $f \in \mathcal{K}$ and $U \subset \mathcal{F}(f)$ be an invariant component. Then, the following (mutually disjoint) cases occur:*

- *There exists an (super)attracting fixed point $z_0 \in U$, such that $f^n(z) \to z_0$ for all $z \in U$. U is called* ***(super)attracting component****.*
- *There exists a rationally indifferent (often called* ***parabolic****) fixed point $z_0 \in \partial U$, such that $f^n(z) \to z_0$ for all $z \in U$. U is called a parabolic component or* ***Leau domain****.*
- *There exists an irrationally indifferent fixed point $z_0 \in U$ and a homeomorphism $\varphi : \mathbb{D} \to U$ such that $\varphi^{-1} \circ f \circ \varphi(z) = \lambda z$ with $\lambda = f'(z_0) = e^{2\pi i\theta}$, for some $\theta \in \mathbb{R}$ irrational. U is called a* ***Siegel disc****.*
- *There exists a homeormorphism $\varphi : A_{1,\rho} \to U$, where $A_{1,\rho} = \{z : 1 < |z| < \rho\}$, such that $\varphi^{-1} \circ f \circ \varphi(z) = \lambda z$ with $\lambda = e^{2\pi i\theta}$, for some $\theta \in \mathbb{R}$ irrational. U is called a* ***Herman ring****.*
- *There exists an essential singularity $z_0 \in \partial U$, such that $f^n(z) \to z_0$ for all $z \in U$. U is called a* ***Baker domain****. The essential singularity z_0 is called the* ***Baker point*** *of U.*

If $f \in \mathcal{R}$, f has no Baker domains by definition. From now on we will focus on the existence of Baker domains in class $\mathcal{K}$.

2.2 Examples

We present now some examples of Baker domains and techniques to determine their existence under some conditions.

Example 2.2.1. *Let*

$$f(z) = z + 1 + e^{-z}.$$

As was mentioned before, this was the first example given by Fatou, where it is possible to see that the right half-plane is contained in an invariant set and $f^n(1) \to \infty$ as $n \to \infty$. Hence, the right half-plane is contained in a component U of $\mathcal{F}(f)$ (see Figure 2.1), using the Denjoy-Wolff Theorem, it can be proved that $f^n(z) \to \infty$ for all $z \in U$. So U is a Baker domain of the function. Since f is a transcendental entire function, we have that the essential singularity is $z_0 = \infty$.

In fact, it can be shown that analytic variations of this function possess a Baker domain containing some right half-plane $H_R = \{z : \Re(z) > R > 0\}$ or part of it.

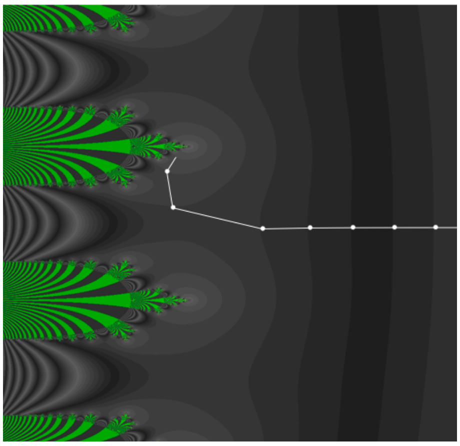

Figure 2.1: Dynamical plane for $f(z) = z + 1 + e^{-z}$

Example 2.2.2. *Let*

$$f(z) = \frac{1}{z} - e^z.$$

In [BKL], Baker et. al. show the existence of a 2-periodic Baker domain with cycle $\{U_0, U_1\}$ where $f^{2n}(z) \to \infty$ if $z \in U_0$ and $f^{2n}(z) \to 0$ if $z \in U_1$, see Figure 2.2.

In the previous chapter, we showed that f^2 has **two** essential singularities at $z_0 = 0$ and $z_1 = \infty$, which can be easily seen from the convergence on U_0 and U_1.

Example 2.2.3. *Let*

$$f(z) = \frac{1}{z} + ae^{-z} + b, \;\; a, b \in \mathbb{R}.$$

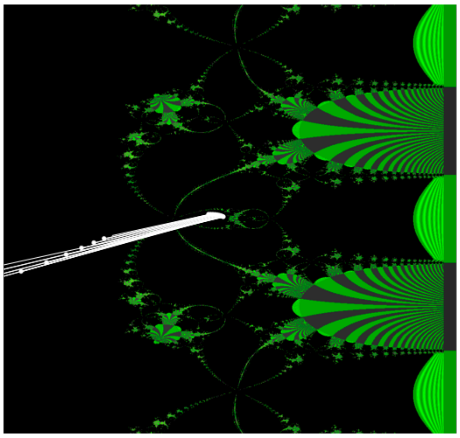

Figure 2.2: Dynamical plane for $f(z) = \frac{1}{z} - e^z$

In [Ko], König shows that for suitable values of a and b, there exists a 3-periodic Baker domain with cycle $\{U_0, U_1, U_2\}$ *where* $f^{3n}(z) \to \infty$ *if* $z \in U_0$, $f^{3n}(z) \to b$ *if* $z \in U_1$ *and* $f^{3n}(z) \to 0$ *if* $z \in U_2$. *Besides,* U_0 *contains an invariant right half-plane under* f^3.

As in the previous example, there exist more than one essential singularity for f^3. In general, if U is a p-periodic Baker domain with cycle $\{U = U_0, U_1, ..., U_{p-1}\}$, then p-Baker points for f^p may exist, see Theorem 13 and Corollaries 1 and 2 in [Ber1] for a formal result on this.
The three examples above use the same technique to determine the existence of a Baker domain, which are listed below.

- Determine the existence of an invariant hyperbolic component (under f or f^p in general).

- Show the existence of a convergent orbit to an essential singularity (under f or f^p in general).

Another technique to determine the existence of Baker domains, called **logarithmic lift**, see [Ber2] and [EL1], consists of the following steps:

- Let $g : \mathbb{C}^* \to \mathbb{C}^*$ be an analytic function, with a dynamical property over 0 or ∞ (being attracting fixed point for example).

- Let $\pi(z) = e^{az}$, $a \neq 0$, be a projective mapping from $\mathbb{C}$ to $\mathbb{C}^*$.

- There exists $f : \mathbb{C} \to \mathbb{C}$ entire such that $\pi(f(z)) = g(\pi(z))$.

- In this setting $\mathcal{J}(f) = \pi^{-1}(\mathcal{J}(g))$ and f is called the logarithmic lift of g.

Example 2.2.4. *Let*

$$g(w) = e^{2\pi i\theta} w e^{w},$$

for some suitable $\theta \in (0,1)$*,* g *has a Siegel disc around* 0*. If* $\pi(z) = e^z$*, we have*

$$g(\pi(z)) = g(e^z) = e^{2\pi i\theta} e^z \exp(e^z) = \exp(2\pi i\theta + z + e^z) = \pi(f(z)),$$

with $f(z) = 2\pi i\theta + z + e^z$*. In this way, in [Hrm], Herman proves that* f *has a Baker domain* U *where* $f^n(z) \to \infty$ *for* $z \in U$*. The domain* U *contains a left half-plane.*

2.3 Local linearization

If we observe the classification of Fatou components, given in Theorem 2.1.2, it is possible to observe that two domains have a predictable dynamic (up to homeomorphic conjugation), these are: Siegel discs and Herman rings. In fact, from this characteristic, both components are called *rotation domains*. Since (super)attracting and parabolic domains are associated with a fixed point, the *Taylor Series Expansion* about this point allow us to determine a "local dynamic" for these kind of Fatou components also. This is usually called *linearization* (except the case of a super-attracting domain, where the local dynamic is not linear) and is given from the following theorems. We cite the books [CG] and [Mi] as principal references.
The following theorem is due to G. Koenigs (1884).

Theorem 2.3.1. *Suppose* f *has an attracting fixed point in* z_0 *with multiplier* λ *satisfying* $0 < |\lambda| < 1$*. Then, there exists a conformal mapping* $\zeta = \varphi(z)$ *from a neighborhood of* z_0 *over a neighborhood of* 0 *conjugating* $f(z)$ *with linear mapping* $g(\zeta) = \lambda\zeta$*. Conjugation is unique up to multiplication by scalar.*

In other words, for some neighborhoods U of z_0 and V of the origin, we have the following commutative diagram.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"'] & U \arrow[d, "\varphi"] \\ V \arrow[r, "\zeta\mapsto\lambda\zeta"] & V \end{tikzcd}$$

This way, we can think of f, locally around z_0 as a mapping $z \mapsto \lambda z$.
The following theorem is due to L.E. Boettcher (1904).

Theorem 2.3.2. *Suppose* f *has a superattracting fixed point in* z_0 *with*

$$f(z) = z_0 + a_p(z - z_0)^p + \ldots, \qquad a_p \neq 0,\ p \geq 2.$$

Then there exists a conformal mapping $\zeta = \varphi(z)$ *from a neighborhood of* z_0 *over a neighborhood of* 0 *conjugating* $f(z)$ *with* $g(\zeta) = \zeta^p$. *Conjugation is unique up to multiplication by a* $(p-1)$*-th root of unity.*

In the same way as the previous case, we have the following commutative diagram.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"'] & U \arrow[d, "\varphi"] \\ V \arrow[r, "\zeta\mapsto\zeta^p"] & V \end{tikzcd}$$

For parabolic fixed points, the theorem is more elaborated: there exist cases to consider, depending on the Taylor series expansion around the fixed point z_0. For this case, there is no neighborhood of conjugation around z_0. The conjugation is about mapping $z \mapsto z+1$ near ∞. So, there are two directions to consider, one "attracting" and one "repelling".
The following development is based on Milnor's book [Mi].
We take the basic case when $z_0 = 0$ (which is possible after translation) with the *simplest form of its multiplier*. Then, f has a local representation around $z_0 = 0$ given by

$$f(z) = z + az^{n+1} + ... \qquad a \neq 0, n \geq 1. \tag{2.3.1}$$

Definition 2.3.3. *A complex number* $\boldsymbol{v}$ *is called* ***repelling vector*** *for* f *at the origin if* $na\boldsymbol{v} = +1$ *and* ***attracting vector*** *for* f *if* $na\boldsymbol{v} = -1$. *If* $\boldsymbol{v}_j$ *is an attracting vector at the origin for* f, *the open set* $\mathcal{P} \subset \widehat{\mathbb{C}}$ *will be called* ***attracting petal*** *for* f *about vector* $\boldsymbol{v}_j$ *if*

1. f *maps* $\mathcal{P}$ *to itself, and*
2. *orbit* $w_0 \mapsto w_1 \mapsto ...$ *under* f *is eventually absorbed by* $\mathcal{P}$ *if and only if it converges to the origin in the* $\boldsymbol{v}_j$ *direction.*

So, we have the following linearization theorem due to L. Leau (1897).

Theorem 2.3.4 (Milnor [Mi], Theorems 10.7 and 10.9)**.** *If* f *has a fixed point at* $z_0 = 0$ *and its written in form (2.3.1) around the origin. Then, inside any neighborhood of the origin there exist petals* $\mathcal{P}_j$, $j = 0, 1, ..., 2n-1$, *each* $\mathcal{P}_j$ *being attracting or repelling if* j *is even or odd. Moreover, for each attracting or repelling petal, there exists conformal embedding* $\alpha : \mathcal{P} \to \mathbb{C}$ *conjugating* $f(z)$ *with* $g(\zeta) = \zeta + 1$. *Embedding is unique up to translation of the form* $z \mapsto z + b$, $b \in \mathbb{C}$.

Again, we have a commutative diagram. Recall we are considering a particular case depending on the multiplier of the fixed point.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\alpha"'] & U \arrow[d, "\alpha"] \\ \mathcal{P} \arrow[r, "\zeta\mapsto\zeta+1"] & \mathcal{P} \end{tikzcd}$$

Questions about Baker domains arise naturally: It is possible to linearize Baker domains? What is the local behavior of f in a Baker domain near an essential singularity? The answer to last question is not unique. For the first question, there are some cases where a positive answer is possible, and is based on Cowen's work [Cow]. In [Ko], König determines the necessary conditions for the existence of linearization of Baker domain. So, we are able to classify Baker domains according to its linearization.

2.4 Classification of Baker domains

From Theorem 1.4.8 (Conclusion 2) of the previous chapter, we consider an invariant Baker domain U of a mapping $f \in \mathcal{K}$. After a Möbius transformation we may assume that the Baker point for U is $z_0 = \infty$. Then $f : U \to U$ is analytic and $f^n(z) \to \infty$ as $n \to \infty$. Moreover, since $\mathcal{J}(f)$ is nonempty and uncountable, we have that U is hyperbolic, the action of f in U can be lifted to an action in $\mathbb{H}_+ = \{z : \Re(z) > 0\}$. For actions on $\mathbb{H}_+$ to itself, we have the celebrated Denjoy-Wolff Theorem (1926).

Theorem 2.4.1 ([CG], Theorem IV.3.1)**.** *Let $\tilde{f} : \mathbb{H}_+ \to \mathbb{H}_+$ be analytic, and assume $\tilde{f}$ is not an elliptic Möbius transformation nor the identity. Then there is $\alpha \in \overline{\mathbb{H}}_+$ such that $\tilde{f}^n(z) \to \alpha$ as $n \to \infty$ for all $z \in \mathbb{H}_+$.*

So, for an invariant Baker domain U of a mapping f, we have the following semi-commutative diagram

$$\begin{tikzcd} \mathbb{H}_+ \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & \mathbb{H}_+ \arrow[d, "\pi"] \\ U \arrow[r, "f"] & U \end{tikzcd}$$

Where π is a covering map of U from its universal covering $\mathbb{H}_+$. Since $f^n(z) \to \infty$, the Denjoy-Wolff Theorem tells us (after a Möbius transformation) that $\tilde{f}^n(z) \to \infty$ also. When U is simply connected, π is actually a Riemann map and f is conformally conjugated to $\tilde{f}$. This is the case for example when f is entire, since Baker in [Ba2] proves that Baker domains for entire functions are simply connected. In general cases, U may be multiply connected (in fact infinitely connected [Bo]).

Cowen in [Cow], obtained a linearization for mappings $g : \mathbb{H}_+ \to \mathbb{H}_+$. He defined the following sets (called *fundamental* in his paper):

Definition 2.4.2. *Let U be a domain in $\widehat{\mathbb{C}}$ and let $f : U \to U$ be a holomorphic map. A domain $V \subset U$ is called **absorbing** in U for f if: $f(V) \subset V$ and for every compact set $K \subset U$ there exists $N = N(K)$ such that $f^N(K) \subset V$.*

Note 2.4.3. *Every immediate basin of attraction of a (super)attracting fixed point is an absorbing domain, as well as attracting petals for parabolic fixed points. In fact, even Siegel discs and Herman rings are absorbing domains.*

Theorem 2.4.4 (Cowen [Cow], Theorem 3.2)**.** *Let $g : \mathbb{H}_+ \to \mathbb{H}_+$ be a holomorphic map such that $g^n \to \infty$ as $n \to \infty$. Then there exists a simply connected domain $V \subset \mathbb{H}_+$, a domain $\Omega \in \{\mathbb{H}_+, \mathbb{C}\}$, a holomorphic map $\varphi : \mathbb{H}_+ \to \Omega$ and a Möbius transformation T mapping Ω into itself, such that:*

a. *V is absorbing in $\mathbb{H}_+$ for g,*

b. *$\varphi(V)$ is absorbing in Ω for T,*

c. *$\varphi \circ g = T \circ \varphi$ on $\mathbb{H}_+$,*

d. *φ is univalent on V.*

Moreover, φ, T depend only on g. In fact (up to a conjugation of T by a Möbius transformation preserving Ω), one of the following cases holds:

- *$\Omega = \mathbb{C}$, $T(w) = w + 1$ (**parabolic type I**),*
- *$\Omega = \mathbb{H}_+$, $T(w) = w \pm i$ (**parabolic type II**),*
- *$\Omega = \mathbb{H}_+$, $T(w) = aw$, $a > 1$ (**hyperbolic type**).*

Sometimes, the triple $\{\varphi,\ T,\ \Omega\}$ is called *conformal conjugacy* for g, denoted by $g \sim T$.
So, in the case g is an entire function or the Baker domain U is simply connected, the above theorem can be applied and the Baker domain is linearized by φ in the three different cases mentioned above, giving a *classification* to the Baker domains.
When U is an invariant Baker domain of a meromorphic function (in $\mathcal{M}$ for example), U may not be simply connected, in [MSK] authors show examples of functions in $\mathcal{M}$ with periodic multiply connected Baker domains. So, the above theorem cannot be applied directly. In [Ko], König considers a more general form of the above theorem and states some dynamical conditions for classification.

Theorem 2.4.5 (König [Ko], Theorem 3)**.** *Let U be a hyperbolic domain in $\mathbb{C}$ and let $g : U \to U$ be a holomorphic map such that $g^n \to \infty$ as $n \to \infty$. Suppose that for every closed curve $\gamma \subset U$ there exists $n > 0$ such that $g^n(\gamma)$ is contractible. Then, the conclusion of Theorem 2.4.4 holds for U.*
Moreover, if conformal conjugacy exists for g, then

- *U is parabolic type I if and only if*

$$\lim_{n\to\infty} \frac{|g^{n+1}(z) - g^n(z)|}{\operatorname{dist}(g^n(z), \partial U)} = 0, \quad \textit{for every } z \in U,$$

- *U is parabolic type II if and only if*

$$\liminf_{n\to\infty} \frac{|g^{n+1}(z) - g^n(z)|}{\operatorname{dist}(g^n(z), \partial U)} > 0, \quad \textit{for every } z \in U,$$

and

$$\inf_{z\in U} \limsup_{n\to\infty} \frac{|g^{n+1}(z) - g^n(z)|}{\operatorname{dist}(g^n(z), \partial U)} = 0,$$

- *U is hyperbolic if and only if*

$$\inf_{z\in U} \inf_{n\geq 0} \frac{|g^{n+1}(z) - g^n(z)|}{\operatorname{dist}(g^n(z), \partial U)} > 0.$$

Since the previous theorems consider holomorphic maps $f : U \to U$ over a hyperbolic domain U, both can be applied for invariant Baker domains U for a function $f \in \mathcal{K}$ (Theorem 2.4.4 if U is simply connected and Theorem 2.4.5 in the other case). So, classification of Baker domains exists if they satisfy conditions in Theorem 2.4.5, and if such conformal conjugacy exists, we are able to classify them according to the conditions in this Theorem.

2.5 Families of Baker domains

Consider the function $g(w) = we^{-w}$, in [BD], Baker and Domínguez proved the following properties of g:

- g has a parabolic fixed point at $z_0 = 0 \in \mathcal{J}(g)$,
- g has a parabolic component $V \subset \mathcal{F}(g)$ containing $\mathbb{R}_+$ and
- $\mathbb{R}_- \subset \mathcal{J}(g)$.

Applying the *logarithmic lift* technique mentioned above, we lift these properties of g by $\pi(z) = e^z$, it can be deduced that $\pi^{-1}(V)$ contains infinitely many components U_k, $k \in \mathbb{Z}$, all congruent under translation by integer multiples of $2\pi i$, such that

$$U_k \subset \{z : (2k-1)\pi < \Im(z) < (2k+1)\pi\},$$

and ∂U_k is asymptotic to horizontal lines $y = (2k \pm 1)\pi$. Also, for each $k \in \mathbb{Z}$, $f^n \to \infty$ and $\Re(f^n) \to \infty$ in U_k and lines $y = (2k \pm 1)\pi$ belongs to

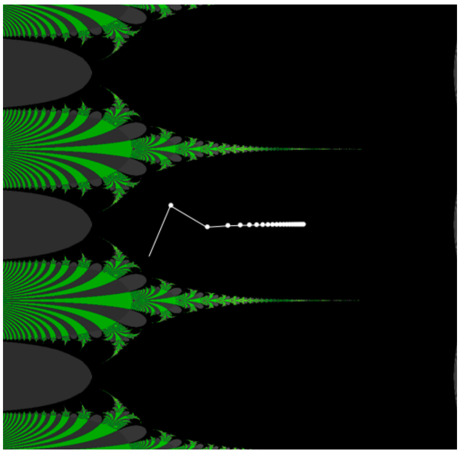

Figure 2.3: Dynamical plane for $f(z) = z + e^{-z}$

$\mathcal{J}(f)$. Then each U_k is an invariant Baker domain of the lifted function (see Figure 2.3)

$$f(z) = z + e^{-z},$$

In other words, f has a **family of Baker domains** U_k, in fact infinitely many different Baker domains U_k, each one contained in a horizontal band of width 2π.
In [RS], P. Rippon and G. Stallard show that there exists a larger class of entire functions (even meromorphic) that exhibit a family of infinitely many Baker domains. Although examples in [RS] are entire functions, the theorem applies to meromorphic functions in class $\mathcal{M}$. The existence of these families is based on the behavior of f over horizontal lines. This condition is stated as follows:

$$\sup\{|\text{Arg }((f(z)-z)e^{z})| : z \in R(t,s)\} \to 0, \text{ as } t \to \infty, \tag{2.5.1}$$

for each $s > 0$ where $R(t,s) = \{z : \Re(z) \geq t > 0,\ |\Im(z)| \leq s\}$, and Arg denotes the principal argument.
The theorem is stated in the following form.

Theorem 2.5.1 (Rippon and Stallard [RS], Theorem 1)**.** *If f is a meromorphic function which satisfies Condition (2.5.1), then:*

- *for each $k \in \mathbb{Z}$, there is an invariant Baker domain U_k of f such that, for $0 < \theta < \pi$, U_k contains a set of the form*

$$V_k(\theta) = \{x + iy : x > v_k(\theta) > 0, |y - 2k\pi| < \theta\},$$

- *the U_k are distinct Baker domains,*

- *if $z \in U_k$, then $|\Im(f^n(z)) - 2k\pi| \to 0$ and $\Re(f^n(z)) \to \infty$ as $n \to \infty$,*

- *each Baker domain U_k contains at least one singularity of f^{-1}.*

See Appendix C for reference on singularities of f^{-1}.
Note that function $f(z) = z + e^{-z}$ satisfies condition (2.5.1) trivially. In fact, Theorem 2.5.1 can be generalized for a function f to have p-families of Baker domains associated to ∞ (see Theorem 6.3 of [Ri]). With this theorem on hand, it is proved that functions with asymptotic form

$$f(z) = z + az^k e^{-z}(1 + o(1)), \text{ as } \Re(z) \to \infty, \tag{2.5.2}$$

$k \in \mathbb{Z}$ and $a > 0$, contain a family of Baker domains. As a corollary, they obtain the following.

Corollary 2.5.2. *For $p \in \mathbb{N}$, let $f(z) = z(1 + e^{-z^p})$, then f has infinitely many Baker domains in each sector $\{z : |Arg\ z - 2j\pi/p| < \pi/p\}$, $j = 0, 1, ..., p-1$.*

For completeness we give a sketch of the proof of the Theorem 2.5.1. The proof is based on a sequence of lemmas determining conditions listed in the theorem.
Sketch of proof. Put $h(z) = (f(z) - z)e^z$, from condition (2.5.1), we have that $|\text{Arg } h(z)| \to 0$ as $\Re(z) \to \infty$ in $R(s,t)$. From here, we deduce that $|f(z) - z| = e^{u(z) - \Re(z)}$ for some bounded harmonic function $u(z)$.

Lemma 2.5.3. *For each $s > 0$*

$$\sup\{|f(z) - z| : z \in R(t,s)\} \to 0 \qquad \text{as } t \to \infty.$$

With the above lemma, it is easy to prove the following result.

Lemma 2.5.4. *For each $s > 0$ there exists $t > 0$ such that f is univalent in $R(t,s)$.*

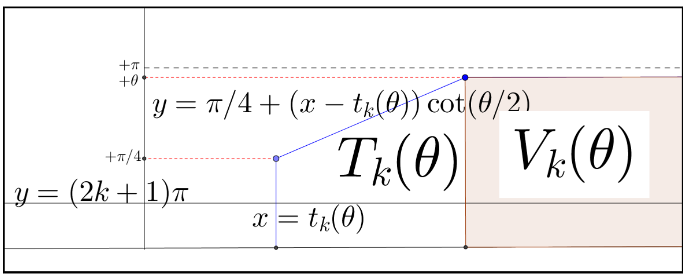

Figure 2.4: Set $V_k(\theta)$ contained in $T_k(\theta)$, which is symmetric with respect $y = (2k+1)\pi$.

For $3\pi/4 < \theta < \pi$, and $k \in \mathbb{Z}$, define the set

$$T_k(\theta) = \{x + iy : x > t_k(\theta), |y - 2k\pi| < \min\{\pi/4 + (x - t_k)\cot(\theta/2), \theta\}\}.$$

From Condition (2.5.1) and previous lemmas, we have

$$|f(z) - z| < \pi/8 \text{ and}$$

$$|\arg\,(f(z) - z) + \Im(z) - 2k\pi| < (\pi - \theta)/2,\ \ z \in T_k(\theta),$$

with these conditions, we can prove that $f(z) \in T_k(\theta)$. Figure 2.4 shows the shape of sets $T_k(\theta)$ and $V_k(\theta)$.

Lemma 2.5.5. *For each $k \in \mathbb{Z}$ and $3\pi/4 < \theta < \pi$, there exists $t(\theta)$ such that f is univalent in $T_k(\theta)$ and $f(T_k(\theta)) \subset T_k(\theta)$.*

The following lemma, allows us to prove that $T_k(\theta)$ is contained in a Baker domain U_k.

Lemma 2.5.6. *If $z \in T_k(7\pi/8)$ for some $k \in \mathbb{Z}$, then $|\Im(f^n(z)) - 2k\pi| \to 0$ and $\Re(f^n(z)) \to \infty$ as $n \to \infty$.*

Finally, with the above lemma (using hyperbolic metric arguments),

Lemma 2.5.7. *The Baker domain U_k contains a singularity of f^{-1}, for each $k \in \mathbb{Z}$.*

Note 2.5.8. *One interesting question after the existence of Baker domains is on the relation of Baker domains and the set of singularities of f^{-1}. Lemma 2.5.7 gives us this relation in this case. See Appendix C for more details on the set of singularities of f^{-1}.*

Chapter 3

Results

At the end of the previous chapter, we consider conditions for the existence of families of Baker domains for meromorphic functions for both classes $\mathcal{E}$ and $\mathcal{M}$. In the present chapter we present a generalized condition for Theorem 2.5.1 which can be applied to functions in the more general class $\mathcal{K}$. Also, the proof of the Main Theorem is given, which uses asymptotic analysis techniques (see Appendix A for basic concepts).
For completeness, at the end of the chapter we present some examples considering two special cases: when $|A(f)| = 1$ (then $f \in \mathcal{M}$ up to a Möbius transformation) and $|A(f)| = 2$ with a one-parametric family, in this case $A(f)$ depends on the parameter. Since $f \in \mathcal{M}$ for the first case, we studied this function in detail proving some interesting properties about it.

3.1 Preliminary results

If we analyze thoroughly the proof of Theorem 2.5.1, it is not difficult to observe that f should not necessary be a meromorphic function (in the classical sense), since conditions are of "local" type. We use quotations to denote that "local" actually means near ∞ in the Riemann sphere $\widehat{\mathbb{C}}$.
In Remark after Theorem 6.1 of [Ri], Rippon considers the possibility that the real axis is not the bisector of the *central* Baker domain, after a change of variable. This possibility can be consider without a change of variable. Thus, Theorem 2.5.1 is then written in the following form.

Theorem 3.1.1. *Let f be a nonconstant complex valued function, analytic in the right half plane $H_0 = \{x + iy : x > x_0 > 0\}$ and suppose that, for each $s > 0$*

$$\sup\{|\mathrm{Arg}((f(z) - z)e^z e^{i\alpha})| : z \in R(t, s)\} \to 0, \text{ as } t \to \infty, \tag{3.1.1}$$

$\alpha \in \mathbb{R}$, where $R(t, s) = \{z : \Re(z) \geq t, |\Im(z)| \leq s\}$ and Arg *denotes the principal argument in $(-\pi, \pi]$. If $\infty \in \widehat{\mathbb{C}}$ is an essential singularity of f, then*

a) for each $k \in \mathbb{Z}$, there exists a Baker domain U_k of f such that, for each $0 < \theta < \pi$, U_k contains a set of the form

$$V_k(\theta) = \{x + iy : x > v_k(\theta) > 0, |y - (2k\pi - \alpha)| < \theta\};$$

b) the U_k are different Baker domains,

c) if $z \in U_k$, then $|\Im(f^n(z)) - (2k\pi - \alpha)| \to 0$, $(f^{n+1}(z) - f^n(z)) \to 0$ y $\Re(f^n(z)) \to \infty$ as $n \to \infty$;

d) each U_k contains a singularity of f^{-1}.

In this general setting, the real number α is the bisector for the *central* Baker domain.
By the time Theorem 3.1.1 was published in [RS], there was no centainty of the existence of simply connected absorbing domains (see Section 2.4) for Baker domains of this type. Recent work, especially [BFJK], has considered conditions for the existence of absorbing domains for analytic automorphisms of a domain $U \subset \mathbb{C}$ with $\infty \in \partial U$.
We present now some results from [BFJK], which allow us to classify Baker domains obtained from Theorem 3.1.1.

Definition 3.1.2. *Let ζ be an isolated point of the boundary of a domain U in $\widehat{\mathbb{C}}$, i.e. there exists a neighborhood V of ζ such that $V \backslash \{\zeta\} \subset U$. If $f(U) \subset U$, from Picard's Theorem, f extends to $V \cup \{\zeta\}$ holomorphically. If $f(\zeta) = \zeta$ we say that ζ is an* ***isolated fixed point*** *of f.*

Theorem 3.1.3 (Barański et. al. [BFJK], Theorem A)**.** *Let U be a hyperbolic domain in $\mathbb{C}$ and let $f : U \to U$ be a holomorphic map without fixed points and without isolated boundary fixed points. Then the following statements are equivalent:*

a. U is parabolic type I.

b. $\rho_U(f^{n+1}(z), f^n(z)) \to 0$ as $n \to \infty$ for some $z \in U$.

c. $\rho_U(f^{n+1}(z), f^n(z)) \to 0$ as $n \to \infty$ almost uniformly on U.

d. $|f^{n+1}(z) - f^n(z)|/\mathrm{dist}(f^n(z), \partial U) \to 0$ as $n \to \infty$ for some $z \in U$.

e. $|f^{n+1}(z) - f^n(z)|/\mathrm{dist}(f^n(z), \partial U) \to 0$ as $n \to \infty$ almost uniformly on U.

Where $\rho_U(\cdot, \cdot)$ denotes the hyperbolic metric on U.
Part of the proof of the Main Theorem is based on asymptotic analysis. We refer for Appendix A to basic concepts and results.

3.2 Main result

Given a general meromorphic function $f \in \mathcal{K}$, we know that there may be several essential singularities for f in $\widehat{\mathbb{C}}$. For Examples 2.2.2 and 2.2.3, it was necessary to consider iterations of f and establish conditions for the existence of Baker domains for the given iteration of f. It is possible to give conditions for an essential singularity $z_0 \in \widehat{\mathbb{C}}$ of a function $f \in \mathcal{K}$ to be a Baker point?
In this section we present the Main Theorem, which establish the form of a function in $\mathcal{K}$ containing a family of Baker domains over each of its essential singularities. We now provide the setting for the proof of the Main Theorem.

Given $p \geq 1$, we define the following sectors

$$A_j := \{z \in \mathbb{C}^* : \frac{(2j-1)\pi}{p} \leq \arg z < \frac{(2j+1)\pi}{p}\},\ j = 0, 1, ..., p-1,$$

("arg" not necessary on the principal branch, but the branch such that A_0 has the positive real axis as its bisector) and rectangular semi-bands contained in each A_j

$$R_j(t,s) := \{e^{\frac{2\pi j}{p}i} z : \Re(z) \geq t > 0,\ |\Im(z)| \leq s \text{ and } \tan s/t = \pi/p\},$$

as Figure 3.1 shows.

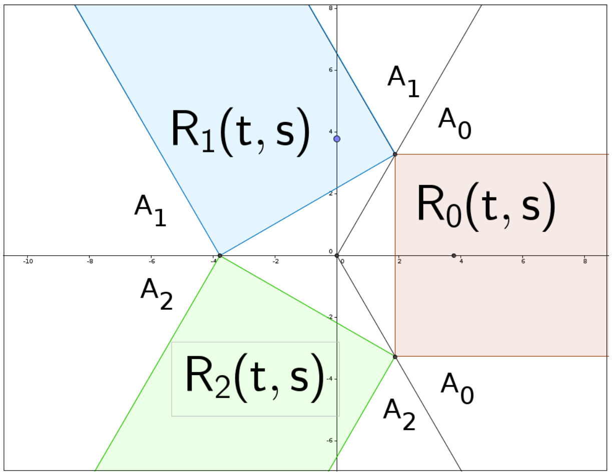

Figure 3.1: Sectors and semi-bands for the case $p = 3$.

Proposition 3.2.1. *Let G be a rational or transcendental meromorphic function, with the following representation in a neighborhood of ∞*

$$G(w) = -w^p + b_{p-1}w^{p-1} + ... + b_1 w + b_0 + c_1 w^{-1} + c_2 w^{-2} + ...$$

Then in each sector A_j and for each $0 < s < \infty$

$$\lim_{R_j(t,s)\ni w\to\infty} \Re(G(w)) = -\infty.$$

Proof. First, note that

$$\lim_{w\to\infty} \frac{G(w)}{-w^p} = 1$$

that is, $G(w) \sim -w^p$ as $w \to \infty$.
Let $w \in R_j(t,s) \subset A_j$ and $\theta = \arg w$. Since the width of semi-bands $R_j(t,s)$ is bounded by $2s < \infty$, we have $\arg w = \theta \to \frac{2\pi j}{p}$ as $w \to \infty$ over semi-band R_j, but

$$w^p = (|w|e^{i\theta})^p = |w|^p e^{pi\theta},$$

hence, $\arg w^p = p\theta$. So, if $w \to \infty$ in R_j, then $\arg w^p = p \cdot \arg w \to 2\pi j$. The above relation and the fact that $G(w) \sim -w^p$ as $w \to \infty$ gives the assertion on the limit and the proposition is proved. ■

The following lemma is also needed for the proof of the main theorem.

Lemma 3.2.2. *Let $f(z) = z + \exp(g(z))$ be a function in the class $\mathcal{K}$, where $g : \mathbb{C} \to \widehat{\mathbb{C}}$ is a transcendental meromorphic or rational function, then f has no isolated boundary fixed points.*

Proof. It is clear that f has no fixed points, except when $g(\infty) \in \mathbb{C}$. In this case we have

$$\lim_{z\to\infty} f(z) = \infty,$$

so $z_1 = \infty$ is a fixed point. After conjugation with $z \mapsto 1/z$, the function f becomes

$$F(w) = \frac{w}{1 + w exp(g(1/w))}, \text{ with } F'(w) = \frac{1 + g'(1/w)exp(g(1/w))}{(1 + w exp(g(1/w))^2}.$$

Since $g(\infty) \in \mathbb{C}$, g is rational. If $g(z) = \frac{P(z)}{Q(z)}$, where P and Q are polynomials without common factors, then $\deg P \leq \deg Q$ $(1 \leq \deg Q)$, so $g'(\infty) = 0$ and we have that ∞ is a parabolic fixed point for F, which is not an isolated point in the boundary of the Leau domain. Then, f has no isolated boundary fixed points. ■

Theorem 3.2.3 (Main Theorem)**.** *Let $g : \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic or rational function, such that*

$$f(z) = z + \exp(g(z))$$

belongs to class $\mathcal{K}$ of general meromorphic functions, where $A(f) = \overline{g^{-1}(\infty)} = \overline{\{\text{poles of } g\}}$. If $z_0 \in g^{-1}(\infty)$ is a pole of g of order $p \geq 1$, then $f(z)$ has p-families of infinitely many different Baker domains with z_0 as its Baker point.

Each family lies in a sector of angle $2\pi/p$ of the disc $D(z_0, \delta)$ for some $\delta > 0$.

Moreover, each Baker domain can be classified as parabolic type I according to the Cowen and König classification.

Proof. We start by analyzing the function $g(z)$.
Since z_0 is a pole of order $p \geq 1$, from Laurent expansion of g around z_0, we have

$$g(z) = b_p(z - z_0)^{-p}(1 + a_1(z - z_0) + a_2(z - z_0)^2 + ...), \tag{3.2.1}$$

where b_p is the $p-$th coefficient of the Principal Part of Laurent expansion. From the above expression it is clear that

$$g(z) = b_p(z - z_0)^{-p}(1 + o(1)), \qquad z \to z_0. \tag{3.2.2}$$

Consider the Möbius transformation $M(z) = -\frac{b}{z-z_0}$ ($\Rightarrow$ $M^{-1}(w) = z_0 - b/w$), and set

$$G(w) = g \circ M^{-1}(w) \qquad \text{and} \qquad F(w) = M \circ f \circ M^{-1}(w).$$

Calculating $G(w)$, we obtain

$$\begin{aligned} G(w) &= g \circ M^{-1}(w) \\ &= g(z_0 - b/w) \\ &= b_p(-b/w)^{-p}(1 + a_1(-b/w) + a_2(-b/w)^2 + ...) \\ &= \tfrac{b_p}{(-b)^p} w^p(1 + a_1(-b/w) + a_2(-b/w)^2 + ...), \end{aligned}$$

if we choose b such that

$$b_p(-b)^p = -1 \tag{3.2.3}$$

(using the principal branch of argument), then

$$G(w) = -w^p(1 + a_1(-b/w) + a_2(-b/w)^2 + ...), \tag{3.2.4}$$

which yields

$$G(w) = -w^p(1 + o(1)) \qquad w \to \infty. \tag{3.2.5}$$

Note that the above relation is the analogous of (3.2.2) for g in a neighborhood of infinity.
On the other hand, the computation for F becomes

$$\begin{aligned} F(w) &= M \circ f \circ M^{-1}(w) \\ &= M \circ f(z_0 - \tfrac{b}{w}) \\ &= M(z_0 - \tfrac{b}{w} + \exp(g \circ M^{-1}(w))) \\ &= M(z_0 - \tfrac{b}{w} + \exp(G(w))) \\ &= -\frac{b}{z_0 - \frac{b}{w} + \exp(G(w)) - z_0} \\ &= \frac{w}{1 - \frac{w}{b}\exp(G(w))}. \end{aligned}$$

We know from Proposition 3.2.1, that over each sector A_j, and for every $0 < s < \infty$, we have

$$\lim_{R_j(t,s) \ni w \to \infty} \Re(G(w)) = -\infty.$$

Hence, given $r < 1$ fixed, for each $0 < s < \infty$, there exists $t_s = t(s) > 0$ such that

$$\left| \frac{w}{b} \exp(G(w)) \right| < r, \quad w \in R_j(t_s, s).$$

So, we define the subset $V_j := \bigcup_{s>0} R_j(t_s, s) \subset A_j$ and then

$$V_j \subset \{w \in A_j : \left| \frac{w}{b} \exp(G(w)) \right| < r < 1\}.$$

Also, it is clear that V_j is nonempty and with an unbounded real part along A_j. The geometric series for $\frac{w}{b}\exp(G(w))$ is uniformly convergent inside V_j, thus we can express $F(w)$ as

$$\begin{aligned} F(w) &= \tfrac{w}{1-\frac{w}{b}\exp(G(w))} \\ &= w(1 + (\tfrac{w}{b}\exp(G(w))) + (\tfrac{w}{b}\exp(G(w)))^2 + ... \\ &= w + \tfrac{w^2}{b}\exp(G(w))(1 + (\tfrac{w}{b}\exp(G(w))) + (\tfrac{w}{b}\exp(G(w)))^2 + ...), \end{aligned}$$

for $w \in V_j$. In this way, we have the following asymptotic form of F in $R_j(t, s) \subset A_j$

$$F(w) = w + \frac{w^2}{b}\exp(G(w))(1 + o(1)), \quad R_j(t, s) \ni w \to \infty. \tag{3.2.6}$$

We fix a region A_j under the following consideration:
If $\theta_j = \frac{2\pi j}{p}$ represents the argument of the bisector of A_j with respect to the real axis, then the change of variable $w \mapsto e^{i\theta_j}w$, which is conformal, transforms A_0 into A_j. With this in mind, we focus on the fixed sector

$$A = A_0 = \{w \in \mathbb{C}^* : \frac{-\pi}{p} \leq \arg w < \frac{\pi}{p}\},$$

the region

$$V = V_0 \subset \{w \in A : \left| \frac{w}{b} \exp(G(w)) \right| < r < 1\},$$

and the semi-band

$$R(t, s) = R_0(t, s) = \{w : \Re(w) \geq t, |\Im(w)| \leq s, \text{ and } \tan \frac{s}{t} = \frac{\pi}{p}\} \subset A.$$

So, we rewrite (3.2.6) as

$$F(w) = w + \frac{w^2}{b}\exp(G(w))(1 + o(1)), \qquad \Re(w) \to \infty, \ w \in R(t, s). \tag{3.2.7}$$

Let $\phi^{-1}(z) = z^{1/p}$ be the unbranched inverse of $w^p = \phi(w)$ defined in the interior of sector A. Recall that in A we have

$$\lim_{R(t,s)\ni w\to\infty} \Re(G(w)) = -\infty. \tag{3.2.8}$$

Since $V \subset A$, ϕ is well defined and single valued in V, so we consider the following conjugation

$$\mathbf{F} := \phi \circ F \circ \phi^{-1} : \mathbf{V} = \phi(V) \to \mathbb{C}\backslash\mathbb{R}_-,$$

which is well defined and single valued. Now, we define rectangular semi-band $\mathbf{R}(t,s)$ analogously as $R(t,s)$ by

$$\mathbf{R}(t,s) = \{z \in \mathbf{V} : \Re(z) \geq t > 0,\ |\Im(z)| \leq s\}.$$

First, note that since $\mathbf{R}(t,s) \subset \mathbf{V}$, then $\phi^{-1}(\mathbf{R}(t,s)) \subset V$, so, for $z \in \mathbf{R}(t,s)$

$$F(\phi^{-1}(z)) = z^{1/p} + z^{1/p}H(z)(1 + H(z) + (H(z))^2 + ...),$$

where $H(z) = (\frac{z^{1/p}}{b}\exp(G(z^{1/p})))$. Also, we have

$$G(z^{1/p}) = -z(1 - a_1bz^{-1/p} + a_2bz^{-2/p} \mp ...)$$

consequently

$$G(z^{1/p}) = -z(1 + o(1)) \qquad \Re(z) \to \infty,\ z \in \mathbf{R}(t,s). \tag{3.2.9}$$

Combining (3.2.7), (3.2.8) and (3.2.9) we obtain

$$F(\phi^{-1}(z)) = z^{1/p} + \frac{z^{2/p}}{b}\exp(-z(1 + o(1))(1 + o(1)).$$

Then, for $z \in \mathbf{R}(t,s)$

$$\begin{aligned}
\mathbf{F}(z) &= \left(z^{1/p} + \tfrac{z^{2/p}}{b}\exp(-z(1 + o(1))(1 + o(1))\right)^p \\
&= z + \tfrac{z^2}{b^p}\exp(-pz(1 + o(1))(1 + o(1))^{p+1} \\
&= z + \tfrac{z^2}{b^p}\exp(-pz(1 + o(1))(1 + o(1)), \quad \Re(z) \to \infty.
\end{aligned}$$

Since $p \in \mathbb{N}$, the mapping $z \mapsto pz$ does not change the form of $\mathbf{R}(t,s)$, we can consider the conjugation $pF(\frac{z}{p})$, denoted again by $\mathbf{F}$ for simplicity. We obtain

$$\mathbf{F}(z) = z + \frac{z^2}{pb^p}\exp(-z(1 + o(1))(1 + o(1)), \tag{3.2.10}$$

as $\Re(z) \to \infty$, $z \in \mathbf{R}(t,s)$. So, we have

$$(\mathbf{F}(z) - z)e^z = \frac{z^2}{pb^p}\exp(zo(1))(1 + o(1)).$$

Calculating its argument

$$\arg\ (\mathbf{F}(z)-z)e^z = \arg\ \left(\frac{z^2}{pb^p}\exp(zo(1))(1+o(1))\right).$$

If $\alpha = -\arg\ b^{-p}$, rewriting the last expression, we obtain

$$\arg\ (\mathbf{F}(z)-z)e^z e^{i\alpha} = \arg\ \left(z^2\exp(zo(1))(1+o(1))\right),$$

which implies

$$\arg\ (\mathbf{F}(z)-z)e^z e^{i\alpha} = (2\arg\ z + \Im(z)o(1))(1+o(1)) \to 0,$$

as $\Re(z) \to \infty$, $z \in \mathbf{R}(t,s)$ ($\Im(z)$ is bounded). Taking supremum, we conclude that $\mathbf{F}$ satisfies condition (3.1.1) of Theorem 3.1.1. Then $\mathbf{F}$ possesses a family of infinitely many different Baker domains with ∞ as its Baker point, each element of the family is contained in the half-plane $\mathbb{H}_+$. Also, each Baker domain satisfies the properties listed in Theorem 3.1.1.
Now, to prove that each Baker domain U_k is parabolic type I, note that, from Property (c) of Theorem 3.1.1,

$$\mathbf{F}^{n+1}(z) - \mathbf{F}^n(z) \to 0 \qquad \text{and} \qquad |\Im(\mathbf{F}^n(z)) - (2k\pi - \alpha)| \to 0,$$

also, by Property (a) of Theorem 3.1.1 each Baker domain contains a set of the form

$$V_k(\theta) = \{x+iy : x > v_k(\theta) > 0, |y-(2k\pi-\alpha)| < \theta\},$$

which implies that $\mathrm{dist}(\mathbf{F}^n(z), \partial U_k)$ is bounded away from 0. Then

$$\frac{|\mathbf{F}^{n+1}(z) - \mathbf{F}^n(z)|}{\mathrm{dist}(\mathbf{F}^n(z), \partial U_k)} \to 0 \text{ as } n \to \infty.$$

So, by Theorem 3.1.3, we conclude that each Baker domain is parabolic type I.
Now, we return to f in the following way:
Since

$$\mathbf{F}(z) = \phi \circ F \circ \phi^{-1}(z),$$

and $\phi^{-1}(z)$ is conformal over $\mathbb{C}\backslash\mathbb{R}_-$, F also satisfies conclusions of $\mathbf{F}$ over the interior of sector A.
In the same way, the map $w \mapsto e^{i\theta_j}w$ is conformal between A and each A_j, so F satisfies the same in each of the sectors.
Finally, since

$$F(w) = M \circ f \circ M^{-1}(w),$$

and $M(z)$ is a Möbius transformation, f also satisfies the arguments of the proof, and then f has p-families of infinitely many different Baker domains, all of them with z_0 as their Baker point. Each family contained in a sector of angle $2\pi/p$ of a small disc around z_0, this sectors correspond to the sectors A_j through $M(z)$.

■

3.3 Example: a uni-parametric family

We now present an example of a family of functions (over a parameter $c \in \mathbb{C}$), with the property that one of the elements of the family belongs to class $\mathcal{M}$, while all other elements belong to class $\mathcal{K}$. In this way we can think of a function $f_c \in \mathcal{K}$ ($c \neq 0$) as a complex perturbation of a fixed function f_0 in class $\mathcal{M}$.
For $c \in \mathbb{C}$, we consider the rational function

$$R_c(z) = \frac{z}{z^2 - c^2}. \tag{3.3.1}$$

Note that for $c = 0$ we have the rational function (actually a Möbius transformation)

$$R_0(z) = \frac{1}{z}. \tag{3.3.2}$$

We take R_c as the function g in Theorem 3.1.1, obtaining in this way the function

$$f_c(z) = z + \exp\left(\frac{z}{z^2 - c^2}\right), \tag{3.3.3}$$

in class $\mathcal{K}$, with $A(f_c) = \{\pm c\}$, which are the poles of the function R_c.

Note 3.3.1. *For $c = 0$ we have the function $f_0(z) = z + \exp(1/z)$, which technically belongs to class $\mathcal{K}$, but, since $R_c \in \mathcal{R}$, the limit*

$$\lim_{z \to \infty} f_c(z) = \infty$$

is well defined, then f_c has only one essential singularity. So, after conjugation by $z \mapsto -1/z$, we obtain a function in class $\mathcal{M}$.

3.3.1 The case in $\mathcal{M}$

We start by proving that the initial function $f_0 \in \mathcal{M}$ possesses infinitely many Baker domains. For $c = 0$ we have the function

$$f_0(z) = z + \exp(1/z). \tag{3.3.4}$$

Since $f_0(z)$ has only one essential singularity, we consider the conformal conjugation by $M(z) = -1/z$ setting $f_0(z)$ as a transcendental meromorphic mapping $F_0 : \mathbb{C} \to \widehat{\mathbb{C}}$. We obtain:

$$\begin{aligned} F_0(z) &= M \circ f_0 \circ M^{-1}(z) \\ &= M \circ f_0(-1/z) \\ &= M(-1/z + \exp(-z)) \\ &= -\tfrac{1}{-1/z + \exp(-z)}. \end{aligned}$$

Function F_0 is then given by

$$F_0(z) = \frac{z}{1 - ze^{-z}}. \tag{3.3.5}$$

If $z := x + iy$, then $|ze^{-z}| = |z|e^{-x}$, so it is clear that $|ze^{-z}| < 1$ in the open region $V = \{x + iy : |y| < \sqrt{e^{2x} - x^2}\}$. Moreover, it is also clear that

$$|ze^{-z}| \to 0,\ \Re(z) \to \infty,\ \Im(z) \text{ bounded}. \tag{3.3.6}$$

From the above equation, it is clear that V is contained in a right half-plane with unbounded real part, see Figure 3.2. So, for $0 < r < 1$ there exists a region $V_r = \{z : |ze^{-z}| < r\} \subset V$, and then for $z \in V_r$ we can express $F_0(z)$ as uniformly convergent series

$$F_0(z) = \frac{z}{1 - ze^{-z}} = z(1 + ze^{-z} + (ze^{-z})^2 + ...).$$

Hence,

$$F_0(z) = \frac{z}{1 - ze^{-z}} = z + z^2e^{-z}(1 + o(1)), \tag{3.3.7}$$

as $\Re(z) \to \infty,\ \Im(z)$ bounded. If we consider

$$(F_0(z) - z)e^z = z^2(1 + o(1)),\ \text{as } \Re(z) \to \infty,\ z \in V.$$

Then F_0 satisfies Condition (3.1.1) of Theorem 3.1.1. We conclude that F_0

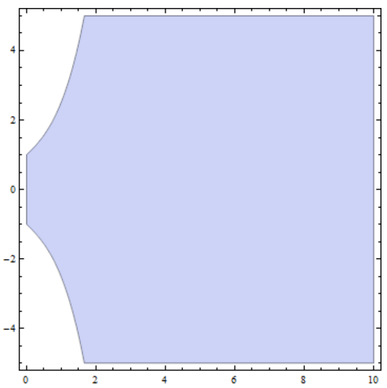

Figure 3.2: Region V for the case $c = 0$.

contains a family of infinitely many different Baker domains. Since $M(z) = -1/z$ is a Möbius transformation, then f_0 contains a family of Baker domains with $z = 0$ as its Baker point, as Figure 3.3 shows. From Theorem 3.1.1 we know that each Baker domain is parabolic type I.

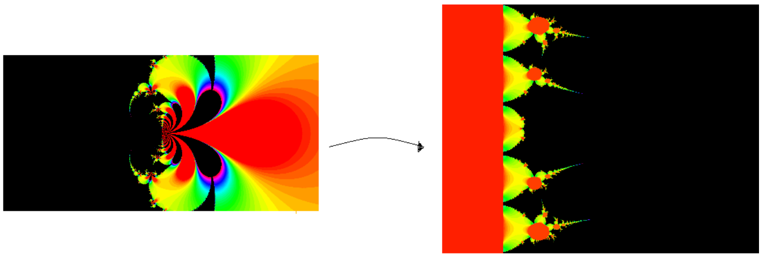

Figure 3.3: Family of Baker domains in $z_0 = 0$ (left) and the conjugacy to $w = \infty$ (right).

3.3.2 The case in $\mathcal{K}$

We consider now the case when $c \neq 0$. The function

$$f_c(z) = z + \exp\left(\frac{z}{z^2 - c^2}\right),$$

may be thought of as a complex singular perturbation of the mapping f_0 studied above. We know that such a function has essential singularities at the points $\{\pm c\} \subset \mathbb{C}$, so $f_c \in \mathcal{K}$.

As in the previous case, we consider the conjugacy of $f_c(z)$ mapping one of the essential singularities to infinity, via a Möbius transformation sending $(-c, c, \infty) \mapsto (1/4c, \infty, 0)$. Then $M(z) = -\frac{1}{2(z-c)}$ and $M^{-1}(w) = c - \frac{1}{2w}$. So

$$\begin{aligned} F_c(w) &= M \circ f \circ M^{-1}(w) \\ &= M \circ f(c - \tfrac{1}{2w}) \\ &= M(c - \tfrac{1}{2w} + \exp(\tfrac{c - \frac{1}{2w}}{(c - \frac{1}{2w})^2 - c^2}) \\ &= \frac{-1}{2\left(c - \frac{1}{2w} + \exp\left(\frac{2w(2cw-1)}{1-4cw}\right)\right)}. \end{aligned}$$

Hence

$$F_c(w) = \frac{w}{1 - 2w exp\left(\frac{2w(2cw-1)}{1-4cw}\right)}. \tag{3.3.8}$$

We need to analyze the expression $2w\exp\left(\frac{2w(2cw-1)}{1-4cw}\right)$.

Let $Q(w) = \frac{2w(2cw-1)}{1-4cw}$, setting $w := u + iv$, we have:

$$\Re(Q(w)) = \frac{4a(u^2 - v^2) - 8buv - 2u - 16|c|^2|w|^2u - 8a|w|^2}{1 - 8\Re(cw) + 16|c|^2|w|^2},$$

$$\Im(Q(w)) = \frac{8auv - 12bv^2 - 2v - 4bu^2 - 16|c|^2|w|^2v}{1 - 8\Re(cw) + 16|c|^2|w|^2}.$$

From where the following limits hold for each $|v| < \infty$:

$$\lim_{u \to \infty} \Re(Q(w)) = -\infty, \tag{3.3.9}$$

$$\lim_{u\to\infty} \Im(Q(w)) = -\frac{b}{4|c|^2} - v. \tag{3.3.10}$$

Hence, by Equation (3.3.9) there exists $u(v) = u_v > 0$, such that

$$|2w \cdot exp(Q(w))| = 2|w|exp(\Re(Q(w))) < r < 1, \tag{3.3.11}$$

for $w \in R(u_v, v)$, and $R(u_v, v)$ is a semiband as defined in Theorem 3.1.1. We define

$$V_r' = \bigcup_{v>0} \mathrm{R}(u_v, v).$$

Now, analogously to the above case, from (3.3.11), for $w \in V_R'$ (see Figure

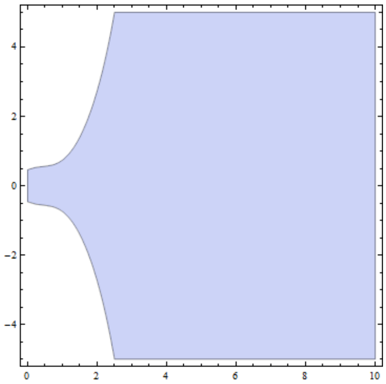

Figure 3.4: Region V_r' for the case $c = 3$.

3.4) we can express $F_c(w)$ as follows:

$$F_c(w) = \frac{w}{1 - 2wexp\left(\frac{2w(2cw-1)}{1-4cw}\right)} = w\left(1 + 2we^{Q(w)} + \left(2we^{Q(w)}\right)^2 + ...\right).$$

So, we have

$$\begin{aligned} F_c(w) &= w(1 + 2we^{Q(w)} + \left(2we^{Q(w)}\right)^2 + ...) \\ &= w + 2w^2e^{Q(w)} + w\left(2we^{Q(w)}\right)^2 + ... \\ &= w + 2w^2e^{Q(w)}(1 + 2we^{Q(w)} + \left(2we^{Q(w)}\right)^2 + ...) \\ &= w + 2w^2e^{Q(w)}(1 + o(1)) \end{aligned}$$

as $u \to \infty$, v bounded.
Now, since $\Im(Q(w)) \to -\frac{b}{4|c|^2} - v$, as $u \to \infty$, rewriting

$$F_c(w) = w + 2w^2e^{Q(w)+w}e^{-w}(1 + o(1)),$$

for $\phi(w) = (F_c(w) - w)e^w = 2w^2e^{Q(w)+w}(1 + o(1))$ we have that

$$\arg\ \phi(w) = \arg\ \left(2w^2e^{Q(w)+w}\right) = 2\arg\ (w) + \Im(Q(w) + w),$$

since $\arg(w) \to 0$ and $\Im(Q(w)) \to -\frac{b}{4|c|^2} - v$ if $\Re(w) \to \infty$, then it is clear that $\arg \phi(w)e^{\alpha} \to 0$ with $\frac{b}{4|c|^2} = \alpha$.
So $F_c(w)$ meets the hypothesis of the Theorem 3.1.1. Proving then that $F_c(w)$ (respectively $f_c(z)$) possess infinitely many different Baker domains associated to infinity (respectively $c \in \mathbb{C}$), as Figure 3.5 shows. For $-c \in \mathbb{C}$ the approach is analogous.

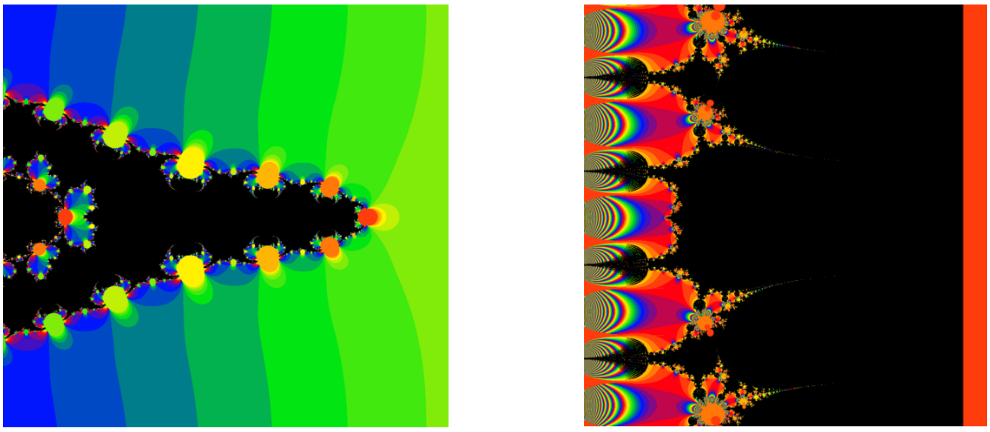

Figure 3.5: Family of Baker domains for the parameter $c = 2$ (left) and the conjugacy at $w = \infty$ (right).

Note 3.3.2. *From Equation (3.2.3), it is clear that the complex number b for the Möbius transformation $M(z) = -\frac{b}{z-z_0}$, depends on the coefficient b_p of the Laurent Series Expansion (3.2.1) of function g. In the case that the pole z_0 is simple ($p = 1$), this coefficient b_p is actually the residue of g with respect to z_0. For both examples above, function g is a rational function, for which it is possible to compute this residue. Then, we obtain the Möbius transformations $M(z) = -1/z$, for $g(z) = 1/z$, and $M(z) = -\frac{1}{2(z-z_0)}$ for $g(z) = \frac{z}{z^2-c^2}$.*

3.4 Distribution of critical points and connectivity results

Given function $f_0(z) := z + e^{1/z}$ we already know that this function contains a family of Baker domains. Also, the point at infinity is a parabolic fixed point. One of the immediate questions about this function is: does there exist another kind of Fatou component? In this case, it is not difficult to see that the only possible *asymptotic value* (see Appendix C, for basic concepts on it) are the essential singularities themself, so there are no asymptotic values for this functions. Then, one way to prove that the answer to the last question is no, is proving that each critical value is associated to each Baker domain and Leau domain at infinity, in this case $sing(f^{-1}) = VCrit(f)$. In this section we prove that this is the case. We summarize this with the following proposition.

Proposition 3.4.1. *Let $f_0 \in \mathcal{M}$ be a function given by*

$$f_0(z) := z + e^{1/z}.$$

Then the Fatou set $\mathcal{F}(f_0)$ is composed only by

- *the family $\{U_k\}_{k\in\mathbb{Z}}$ of different invariant Baker domains associated to the Baker point $z_0 = 0$, and*
- *the Leau domain associate to $z_1 = \infty$.*

As well as all of its pre-images.

The proof of the above proposition will be a consequence of a lemma on critical points, whose proof is based on the Rouché-Estermann Theorem [Es].

Theorem 3.4.2 (Rouché-Estermann). *Let f and g be analytic functions in a neighbourhood of a domain U, without zeros over ∂U, if*

$$|f(z) - g(z)| < |f(z)| + |g(z)|, \quad z \in \partial U,$$

then f and g have the same number of zeros in U.

3.4.1 Critical points

Proposition 3.4.3. *Let $f_0 \in \mathcal{M}$ be a function given by*

$$f_0(z) := z + e^{1/z}.$$

Then each critical point is associated to one and only one of the elements of the family of Baker domains $\{U_k\}_{k\in\mathbb{Z}}$ associate to $z_0 = 0$ or the Leau domain associate to $z_1 = \infty$.

Proof. We consider the function $F_0(z) = M \circ f_0 \circ M^{-1}(z)$, instead of the original function f_0. Given

$$F_0(z) = \frac{z}{1 - ze^{-z}},$$

computing its derivative we obtain

$$F_0'(z) = \frac{1 - z^2e^{-z}}{(1 - ze^{-z})^2}.$$

From which it is clear that $z = 0$ is a parabolic fixed point with multiplier $\lambda = 1$.
We want to compute the solutions of the following equation

$$F_0'(z) = \frac{1 - z^2 e^{-z}}{(1 - ze^{-z})^2} = 0, \tag{3.4.1}$$

which is equivalent to solving

$$e^z - z^2 = 0. \tag{3.4.2}$$

Let $z := x+iy$, expanding the real and imaginary parts of the above equation, we obtain the following system of equations:

$$x^2 + y^2 = e^x \tag{3.4.3}$$
$$x^2 - y^2 = e^x \cos y \tag{3.4.4}$$
$$2xy = e^x \sin y, \tag{3.4.5}$$

where $|e^z| = e^x$.
First, note that if $y = 0$, the system is reduced to equation $x^2 = e^x$. If $h(x) = e^x - x^2$, we have $h'(x) = e^x - 2x > 0$, which implies $h(x)$ is increasing, since $h(0) = 1 > 0$ and $h(-1) = e^{-1} - 1 < 0$, then $h(x_0) = 0$ for a unique $x_0 \in (-1, 0)$.
For $\Re(z) < 0$, from (3.4.3) we have $|z| < 1$. If in addition $y > 0$, from (3.4.5) $\sin y < 0$, so $y \in ((2k-1)\pi, 2k\pi)$, $k \geq 1$ and then $y > \pi$ which is a contradiction since $|z| > 1$. Analogously for $y < 0$ (this case $y \in (2k\pi, (2k+1)\pi)$, $k \leq -1$).
As a first conclusion, Equation (3.4.2) has only one (real) solution in the left half-plane.
For $\Re(z) = 0$, combining equations (3.4.3)-(3.4.5) arises a contradiction. It remains to consider the case $\Re(z) > 0$.
Now the case is reduced to $\Re(z) > 0$, we proceed to prove that equation (3.4.2) has solutions distributed over the bands

$$B_k = \{z : x > 0,\ |y - 2k\pi| < \pi\},\ k \in \mathbb{Z}.$$

We associate Equation (3.4.2) to another known equation using the Rouché-Estermann Theorem.
We apply the theorem to functions $f(z) = e^z - z^2$ and $g(z) = e^z - z$, which reduce to prove

$$|z - z^2| < |e^z - z^2| + |e^z - z| \tag{3.4.6}$$

over the boundary ∂R_k where $R_k = B_k \cap \{x + iy : x \leq C_k,\ 1 << C_k\}$.
Since

$$F_0(\overline{z}) = \frac{\overline{z}}{1 - \overline{z}e^{-\overline{z}}} = \overline{\left(\frac{z}{1 - ze^{-z}}\right)} = \overline{F_0(z)},$$

i.e. F_0 is symmetrical with respect to the real axis, it is enough to prove this inequality in the first quadrant $\{x+iy : x>0,\ y>0\}$.
First we prove that critical points are not on the boundary of each B_k. Suppose that $z_0 = x_0 + iy_0 \in Crit(F_0)$ and $z_0 \in \partial R_k$. There are three cases.

1) $x_0 = 0$,

2) $y_0 = (2k+1)\pi,\ k \geq 0$ or $y = 0$,

3) $x_0 = C_k$.

In case 1): from (3.4.3) we have $y_0^2 = 1$, while from (3.4.5) $\sin y_0 = 0$, which is a contradiction.
In case 2): if $y_0 = (2k+1)\pi$, from (3.4.5) $x_0 = 0$, which has been proved can not happen. In the above subsection, we see that if $y_0 = 0$ then $x < 0$, which is not the case.
In case 3): since $|y|$ is bounded over each band B_k, it is enough to choose $C_k >> 1$ such that $e^{C_k} > C_k^2 + y^2$. So, from (3.4.3) the case is eliminated.
To prove (3.4.6) consider the function

$$h(z) = |e^z - z^2| + |e^z - z| - |z - z^2|. \tag{3.4.7}$$

Written in its real form:

$$\begin{aligned} h(x,y) &= \sqrt{e^{2x} + 2(y^2 - x^2)e^x \cos y - 4xye^x \sin y + x^4 + y^4 + 2x^2y^2} \\ &+ \sqrt{e^{2x} + x^2 + y^2 - 2xe^x \cos y - 2ye^x \sin y} \\ &- \sqrt{x^4 + y^4 + x^2 + y^2 - 2x^3 + 2x^2y^2 - 2xy^2}. \end{aligned}$$

Then, we have to prove that $h(x,y)$ is positive in the cases mentioned above:

1) $x_0 = 0$,

2) $y_0 = (2k+1)\pi,\ k \geq 0$ or $y = 0$ and

3) $x_0 = C_k$.

In case 1): since $x = 0$,

$$h(0,y) = h(y) = \sqrt{y^4 + 2y^2 \cos y + 1} + \sqrt{y^2 - 2y \sin y + 1} - \sqrt{y^4 + y^2}.$$

Now,

$$h(y) > \sqrt{y^4 - 2y^2 + 1} + \sqrt{y^2 - 2y + 1} - \sqrt{y^4 + y^2} = y^2 + y - 2 - \sqrt{y^4 + y^2}.$$

Take $g(y) = y^2 + y - 2 - \sqrt{y^4 + y^2}$, then we have

$$\begin{aligned} g'(y) &= 2y + 1 - \frac{2y^2+1}{\sqrt{y^2+1}} \\ &> 2y + 1 - \frac{y^2+1}{\sqrt{y^2+1}} \\ &= 2y + 1 - \sqrt{y^2+1} \\ &> 2y + 1 - (y+1) \\ &= y > 0. \end{aligned}$$

Hence, $g(y)$ is increasing, since $g(3) = 12 - 3\sqrt{10} > 0$, then $h(y) > 0$ for $y \geq 3$. Remains the case $y \in [0, 3]$.
Since $y \in [0, 3]$, by properties of trigonometric functions, it is easy to see that a minimum value is close to $y = \pi/2$, and

$$h(\pi/2) = \sqrt{\pi^4/16 + 1} + \sqrt{\pi^2/4 - \pi + 1} - \pi/2\sqrt{\pi^2/4 + 1} \approx 0.3 > 0.$$

From which follows that $h(y) > 0$ for $y \in [0, 3]$. Then, (3.4.6) holds in this case.
In case 2): considering first $y = 0$,

$$h(x, 0) = h(x) = \sqrt{e^{2x} - 2x^2e^x + x^4} + \sqrt{e^{2x} + x^2 - 2xe^x} - \sqrt{x^4 + x^2},$$

on one hand, for $x \in [0, 1] \cup [2, \infty]$, $e^{2x} - 2x^2e^x > 0$ and $e^{2x} - 2xe^x > 0$, so $h(x) > 0$ in such intervals. Hence, for $x \in [1, 2]$, if we consider $g_1(x) = x^4 - 2x^2e^x$ and $g_2(x) = x^2 - 2xe^x$, computing derivatives we have $g_1'(x) = 4x^3 - 4xe^x - 2x^2e^x < 0$ and $g_2'(x) = 2x - 2e^x - 2xe^x < 0$, hence g_1 and g_2 are decreasing in $[1, 2]$, but $h(1) \approx 2 > 0$ and $h(2) \approx 4.3 > 0$, so $h(x) > 0$ for $x \in [0, \infty)$. Then (3.4.6) holds also in this case.
For $y = (2k + 1)\pi$, set $K = (2k + 1)$

$$\begin{aligned} h(x, K\pi) = h(x) &= \sqrt{e^{2x} + 2(x^2 - K^2\pi^2)e^x + x^4 + K^4\pi^4 + 2x^2K^2\pi^2} \\ &+ \sqrt{e^{2x} + x^2 + K^2\pi^2 + 2xe^x} \\ &- \sqrt{x^4 + K^4\pi^4 + x^2 + K^2\pi^2 - 2x^3 + 2x^2K^2\pi^2 - 2xK^2\pi^2}. \end{aligned}$$

Again, we have that $h(x)$ is bounded below by

$$\begin{aligned} h(x, K\pi) = h(x) &= \sqrt{e^{2x} + 2(x^2 - K^2\pi^2)e^x + x^4 + K^4\pi^4 + 2x^2K^2\pi^2} \\ &+ \sqrt{e^{2x} + x^2 + K^2\pi^2 + 2xe^x} \\ &- \sqrt{x^4 + K^4\pi^4 + x^2 + K^2\pi^2 + 2x^2K^2\pi^2}. \end{aligned}$$

Considering the argument of the last term of the above expression $g(x) =$

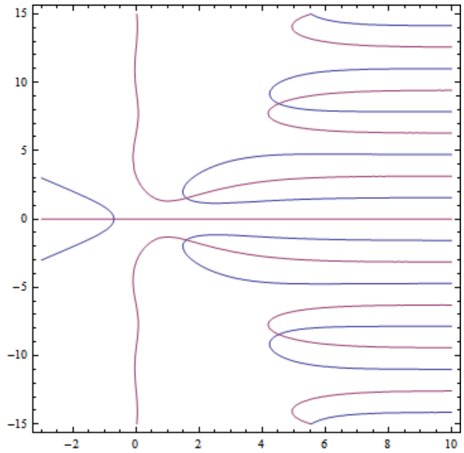

Figure 3.6: Sketch of critical points distribution for $F_0(z)$. See Note 3.4.4.

$x^4 + K^4\pi^4 + x^2 + K^2\pi^2 + 2x^2K^2\pi^2$, we have

$$\begin{aligned} g(x) &= x^4 + K^4\pi^4 + x^2 + K^2\pi^2 + 2x^2K^2\pi^2 \\ &= (x^2 + K^2\pi^2)^2 + (x^2 + K^2\pi^2) \\ &= (x^2 + K^2\pi^2)(x^2 + K^2\pi^2 + 1) < (x^2 + K^2\pi^2 + 1)^2. \end{aligned}$$

Hence,

$$\begin{aligned} h(x) &> \sqrt{e^{2x} + 2(x^2 - K^2\pi^2)e^x + x^4 + K^4\pi^4 + 2x^2K^2\pi^2} \\ &+ \sqrt{e^{2x} + x^2 + K^2\pi^2 + 2xe^x} \\ &- (x^2 + K^2\pi^2 + 1) \\ &> 0. \end{aligned}$$

This way, (3.4.6) holds for this case.
In case 3), note that $h(x, y)$ is bounded below by the function

$$\begin{aligned} h_0(x, y) &= \sqrt{e^{2x} + 2(y^2 - x^2)e^x \cos y - 4xye^x \sin y + x^4 + y^4 + 2x^2y^2} \\ &+ \sqrt{e^{2x} + x^2 + y^2 - 2xe^x \cos y - 2ye^x \sin y} \\ &- \sqrt{x^4 + y^4 + x^2 + y^2 + 2x^2y^2}, \end{aligned}$$

and for $x = C_k >> 1$, $e^{2x} + 2(y^2 - x^2)e^x \cos y - 4xye^x \sin y > 0$ and $e^{2x} - 2xe^x \cos y - 2ye^x \sin y > 0$, so $h_0(x, y)$ is also bounded below by

$$h_1(x, y) = \sqrt{x^4 + y^4 + 2x^2y^2} + \sqrt{x^2 + y^2} - \sqrt{x^4 + y^4 + x^2 + y^2 + 2x^2y^2},$$

which is strictly positive, then $h(x, y) > 0$ for this case.
So, applying the Rouché-Estermann Theorem, $f(z) = e^z - z^2$ and $g(z) = e^z - z$ has the same number of zeros over each band B_k. Since we know $g(z)$

has only one zero in each band B_k for $kn \neq 0$, and two zeros in B_0, the same happens for $f(z)$.
Consequently, under the property *d*) of Theorem 3.1.1, we conclude that each Baker domain U_k ($k \neq 0$) has one and only one critical point and then one critical value. Meanwhile, U_0 has two critical points (Figure 3.6). ■

Proof. (***of Proposition 3.4.1***) By Proposition 3.4.3, each critical value is contained in a Baker domain U_k or the Leau domain of $z = 0$, consequently, by Theorem 7 in [Ber1], there is no other Fatou component. ■

Note 3.4.4 (On Figures 3.6-3.8). *In Figures 3.6-3.8, we can appreciate two sets of curves (the real axis belongs to one of this sets). Each set represent the* cero-*level lines for the real and imaginary part of function $F_0'(z)$. So, the points of intersection, represents each critical point of function $F_0(z)$.*

3.4.2 Connectivity

Since there exists a uniform distribution of the critical values over the Fatou components (Baker and Leau domains) for F_0, it is possible to compute the connectivity of invariant Fatou components through the Riemann-Hurwitz Formula.

Theorem 3.4.5 (Steinmetz [St], Riemann-Hurwitz Formula). *Suppose that f is a proper map of degree k of some m-connected domain D onto some n-connected domain G, f having exactly r critical points in D, counted with multiplicities. Then*

$$m - 2 = k(n - 2) + r.$$

The following results are also needed to compute the connectivity of the Fatou components.

Proposition 3.4.6 (Beardon [Be], Proposition 5.1.7). *Let $\{D_\alpha\}$ be a collection of simply connected domains that is linearly ordered by inclusion. Then $\bigcup_\alpha D_\alpha$ is a simply connected domain.*

Since the ambiguity of the first definition of an absorbing domain (recall that was first defined for mappings $f : \mathbb{H}_+ \to \mathbb{H}_+$), a particular definition was given in [BFJK].

Definition 3.4.7. *An absorbing domain W in a domain U in $\mathbb{C}$ for a holomorphic map $f : U \to U$ with $f^n \to \infty$ as $n \to \infty$ is called **nice** if*

a. $\overline{W}\backslash\{\infty\} \subset U$,

b. $f^n(\overline{W}\backslash\{\infty\}) = \overline{f^n(W)}\backslash\{\infty\} \subset f^{n-1}(W)$ *for every* $n \geq 1$,

c. $\bigcap_{n=1}^{\infty} f^n(\overline{W}\backslash\{\infty\}) = \emptyset$.

The following theorem will be used to prove the connectivity property for this function.

Theorem 3.4.8 (Barański et. al. [BFJK], Theorem B)**.** *Let $U \subset \mathbb{C}$ be a hyperbolic domain and $f : U \to U$ be a holomorphic map, such that $f^n \to \infty$ as $n \to \infty$ in U and ∞ is not an isolated boundary fixed point of U in $\widehat{\mathbb{C}}$. If U is parabolic type I, then there exists a simply connected nice absorbing domain $W \subset U$ for f.*

So, now we can use this to compute the connectivity of some of the Fatou components of the function F_0 (in fact, all of them except the Baker domain U_0).

Theorem 3.4.9. *Let $U \subset \mathcal{F}(F_0)$ be a forward invariant component, with $U \neq U_0$. Then U is simply connected.*

Proof. We know that an attracting petal is a simply connected nice absorbing domain for the Leau domain. Also, by Theorem 3.4.8, there exist a simply connected nice absorbing domain for each U_k, $k \in \mathbb{Z}$.
In this way, let $V \subset U$ be a simply connected nice absorbing domain for F_0 in U. From (c) of Definition 3.4.7, we can take V (or an iterate $F_0^d(V)$ for a finite $d \geq 0$), such that F_0 is univalent in V.
Set $V_0 = V$ and take $V_1 \subset F_0^{-1}(V_0)$ such that $V_0 \cap V_1 \neq \emptyset$. Inductively, define the subset $V_l \subset F_0^{-1}(V_{l-1})$, such that $V_l \cap V_{l-1} \neq \emptyset$, $l \geq 2$. Then, we can apply the Riemann Hurwitz formula (3.4.5) to

$$F_0 : V_l \to V_{l-1},$$

and we obtain

$$m - 2 = k(n - 2) + r.$$

where m is the connectivity of V_l, n is the connectivity of V_{l-1}, and $r = 1$ (the number of critical points on U, Theorem 3.4.3). For $l = 1$, we know that $n = 1$, which implies

$$m - 3 = -k, \tag{3.4.8}$$

but $k \geq 2$, so $m \leq 1$, which implies $m = 1$. Inductively, we have that V_l is simply connected for $l \geq 1$.
By definition $U = \bigcup_n V_n$, then Proposition 3.4.6 implies that U is simply connected. ∎

Note 3.4.10. *For the case $U = U_0$, it was shown in Proposition 3.4.3 the existence of two critical points in U_0. Using the same argument as in the proof of Theorem 3.4.9, we will have the relation*

$$m - 4 = -k$$

instead of (3.4.8), which implies that $m \leq 2$. So, in the following step of induction, we will have $m \leq 4$, so the connectivity of each V_l will be only bounded, instead of be equal one. In other words, there is no control on the connectivity of domains V_l, and then the proof cannot be applied in this case.

3.5 More examples

In this section we present some examples for different type of functions $g : \mathbb{C} \to \widehat{\mathbb{C}}$ in the setting of the class of functions describe in this work, namely

$$f(z) = z + \exp(g(z)) \in \mathcal{K}.$$

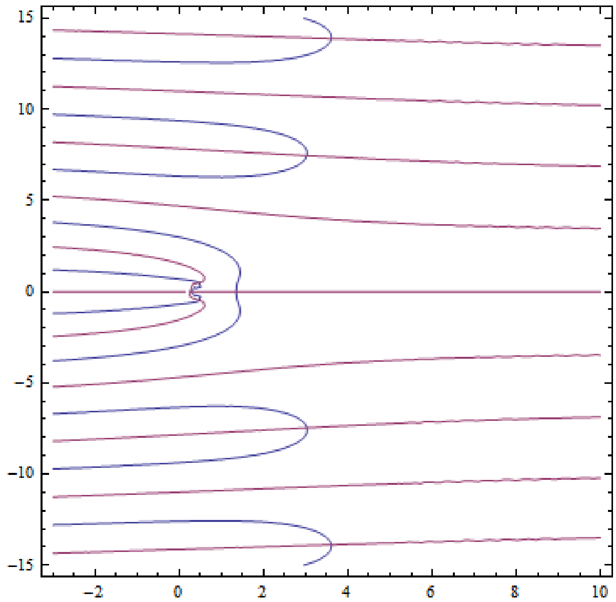

Figure 3.7: Sketch of critical points distribution for $F_1(w)$. See Note 3.4.4.

Example 3.5.1 (**A rational case**). *Previously in this chapter we proved that the function $f_c \in \mathcal{K}$, $c \in \mathbb{C}^*$ given by*

$$f_c(z) = z + \exp\left(\frac{z}{z^2 - c^2}\right),$$

with $A(f_c) = \{\pm c\}$, has a family of Baker domains $\{U_k\}$ for each essential singularity in $A(f_c)$. As in the case for $f_0 \in \mathcal{M}$, $f_0(z) = z + exp(1/z)$, the point at infinity $\infty \in \widehat{\mathbb{C}}$ is again a parabolic fixed point for f_c, $c \neq 0$.
If we consider the conjugation $F_c(w) = M \circ f_c \circ M^{-1}(w)$, given by

$$F_c(w) = \frac{w}{1 - 2w^2 \exp\left(\frac{4cw^2 - 2w}{1 - 4cw}\right)},$$

setting $Q(w) = \frac{4cw^2-2w}{1-4cw}$, *then*

$$F_c'(w) = \frac{1 + 2w^2 Q'(w) e^{Q(w)}}{(1 - 2w^2 e^{Q(w)})^2}.$$

The figures 3.7 and 3.8 show numerical experiments of the distributions of critical points for $c = 1$ *(associate to* ∞ *for* F_1 *and thus associate to* $c = 1$ *for* f_c*).*
It seems that there is a nice distribution of critical points for the function F_1*.*

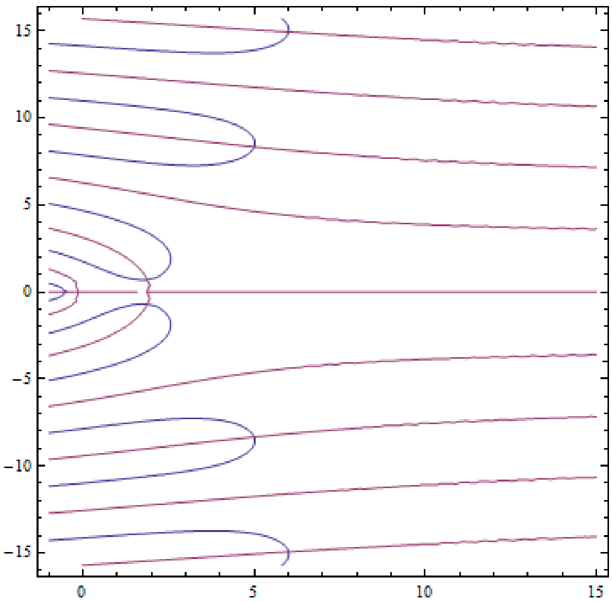

Figure 3.8: Sketch of critical points distribution for $F_{-1}(w)$. See Note 3.4.4.

So we could assume (as a conjecture) that there is no other Fatou components other than the parabolic basin and the families of Baker domains, as in the case for f_0*.*

Example 3.5.2 (**An order 3 pole**). *The following example shows a configuration for a pole of order 3 of a meromorphic function.*
Consider the function $g : \mathbb{C} \to \widehat{\mathbb{C}}$ *given by*

$$g(z) = \frac{1}{z^3} + e^{-z}.$$

It is clear that $z_0 = 0$ *is a pole of order 3. Numerical experiments seems to imply that there is no other Fatou components other than the 3-families of Baker domains associated to the Baker point* $z_0 = 0$ *for* $f(z) = z + exp(g(z))$*.*

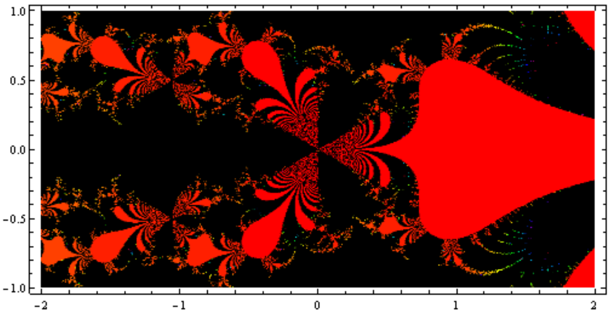

Figure 3.9: Dynamical plane for $g(z) = \frac{1}{z^3} + e^{-z}$.

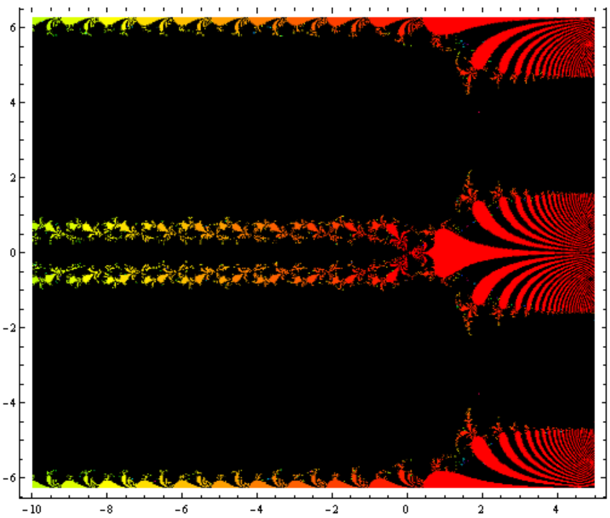

Figure 3.10: Dynamical plane for $g(z) = \frac{1}{z^3} + e^{-z}$.

Chapter 4

Future work and open questions

4.1 Open questions

As it was mentioned at the end of the previous chapter, there are some aspects that have not been possible to determine about functions in the family f_c. In this section we set some open questions about Fatou components of some functions studied in the previous chapter.
Given the function

$$f_0(z) = z + \exp(1/z), \tag{4.1.1}$$

in class $\mathcal{M}$, we proved in previous chapter that each invariant Fatou component is simply connected except the Baker domain U_0, the one with the real axis as bisector. The existence of two critical points does not allow the use of Riemann-Hurwitz formula for the computation of the connectivity.

Question 1. What is the connectivity of the Baker domain U_0 of function (4.1.1)?

For each $c \in \mathbb{C}^*$ the function

$$f_c(z) = z + \exp\left(\frac{z}{z^2 - c^2}\right), \tag{4.1.2}$$

belongs to class $\mathcal{K}$ with $A(f) = \{\pm c\}$. The one-parametric family $\{f_c\}$ can be thought of as a *complex singular perturbation* of function f_0 (4.1.1). Similar to function f_0 (4.1.1), each function f_c has two families of different Baker domains $U_{c,k}$ (since $|A(f)| = 2$), each one associated to essential singularity c or $-c$. Since

$$g(z) = \frac{z}{z^2 - c^2}$$

is a rational function, the point at infinity, is a parabolic fixed point, as in the case for f_0. Figures 3.7 and 3.8 show a numerical distribution of critical points for $c = 1$. Since the distribution is similar to the one for the function f_0 (4.1.1), there are two direct questions.

Question 2. How many critical points has each invariant Fatou component $U \subset \mathcal{F}(f_c)$ for function (4.1.2)?

Question 3. Are there other Fatou components different from the Leau domain for the point at infinity and the families of Baker domains associated to each essential singularity?

More generally, for a function $f \in \mathcal{K}$ given by

$$f(z) = z + \exp(g(z)), \tag{4.1.3}$$

with $g \in \mathcal{M}$, the point at infinity is no longer a parabolic fixed point but an essential singularity.
Question 4. Are there other Fatou components $U \subset \mathcal{F}(f)$ different from the families of Baker domains associated to each essential singularity in $A(f)$?

4.2 Future work

4.2.1 The Teichmüller space

A great part of this work was inspired by Fagella and Henriksen's paper [FH1] where one of the results is the existence of a rigid entire map $f(z) = z + e^{-z}$, i.e. any quasiconformal deformation of f is affinely conjugated to f.
Since the work of D. Sullivan on the proof of non-existence of wandering domains for rational functions [Su], *quasiconformal mappings* and *Teichmüller spaces* have become a strong tool in the study of *deformation spaces* of dynamical systems.
The best way to study the deformation space of a given function f, is studying its *Teichmüller space*. After the Sullivan's paper [Su] the most important reference on Teichmüller spaces of rational functions is the framework by McMullen and Sullivan [MS], where the Teichmüller space of a rational function is defined. After that, Fagella and Henriksen in [FH2] use this ideas and results to define the Teichmüller space of an entire function.
We give some results on Teichmüller spaces for rational and entire functions that can be applied to our function $f_0(z) = z + \exp(1/z)$ as in the example in the previous chapter. We refer to frameworks [MS] and [FH2] and books [Hu] and [IT] for basic concepts and fundamental theorems in the theory of Teichmm̈uller spaces.

Although Teichmüller spaces were defined for Riemann surfaces, it is possible to define them for a function f in the following way.

Theorem 4.2.1 ([MS], Theorem 6.1). *Suppose every component of the one-dimensional manifold V is hyperbolic, $f : V \to V$ is a covering map, and the grand orbit relation of f is discrete. If V/f is connected then V/f is a Riemann surface and*

$$\mathcal{T}(f, V) = \mathcal{T}(V/f).$$

For a rational function R, we can take $V \subset \mathcal{F}(R)$. In the framework [MS], the Teichmüller space of a rational function is then well defined (recall that for rational functions there are no Baker domains).

Theorem 4.2.2 ([MS], Theorem 6.2). *The Teichmüller space of a rational function f of degree d is naturally isomorphic to*

$$\mathcal{T}(f, \Omega^{fol}) \times \mathcal{T}(\Omega^{disc}/f) \times \mathcal{B}_1(f, \mathcal{J}(f)),$$

where Ω^{disc}/f is a complex manifold.

The idea in [FH2] is to extend all results in [MS] to entire functions with Baker domains. Since every Baker domain for an entire function is simply connected, they can be easily classified. Then the Theorem 4.2.1 can be applied to Baker domains.

From Theorem 3.1.1 we know that function $f(z) = z + e^{-z}$ has a family of Baker domains. In fact, there are no other Fatou components. Moreover, we know that each Baker domain is parabolic type I. Then, the following theorem gives a characterization of Teichmüller space restricted to a Baker domain of an entire function.

Theorem 4.2.3 (FH, Main Theorem). *Let U be a proper Baker domain of an entire function f and $\mathcal{U}$ its grand orbit. Denote by S the set of singular values of f in U, and by $\widehat{S}$ the closure of the grand orbit of S taken in $\mathcal{U}$. Then $\mathcal{T}(f,\mathcal{U})$ is infinite dimensional except if U is parabolic type I and the cardinality of $\widehat{S}/f$ is finite. In that case the dimension of $\mathcal{T}(f,\mathcal{U})$ equals $|\widehat{S}/f| - 1$.*

It is not difficult to see that $f(z) = z + e^{-z}$ is semi-conjugated to $g(z) = ze^{-z}$ via the projective map $\pi(z) = e^z$. So, it is possible to conclude that $\mathcal{J}(f)$ has measure zero since the same happens for g (see [Ber2]). The proof for the measure of $\mathcal{J}(g)$ is based on a theorem in [EL1].

Proposition 4.2.4. *The map $f(z) = z + e^{-z}$ is rigid, i.e. if $\tilde{f}$ is a holomorphic map which is quasiconformally conjugated to f, then f is conjugated to $\tilde{f}$ by an affine map.*

For $k \in \mathbb{Z}$, let $U_k \subset \mathcal{F}(f)$ represent an element of the family of Baker domains for function $f(z) = z + e^{-z}$, and let $\mathcal{U}_k$ represent their grand orbit. Then, for $f(z) = z + e^{-z}$ we have

$$\mathcal{T}(f, \mathbb{C}) = \mathcal{T}(f, \cup\mathcal{U}_k) \times \mathcal{B}_1(f, \mathcal{J}(f)) = \prod \mathcal{T}(f, \mathcal{U}_k) \times \mathcal{B}_1(f, \mathcal{J}(f)).$$

It follows from Theorem 4.2.1 that each $\mathcal{T}(f, \mathcal{U}_k)$ is trivial. The measure zero on $\mathcal{J}(f)$ implies $\mathcal{B}_1(f, \mathcal{J}(f))$ is trivial, then the Teichmüller space of f is trivial.

4.2.2 Conjecture

Given function

$$f_0(z) = z + \exp(1/z),$$

we know the following from the analysis in previous chapter:

- f_0 contains a family of Baker domains U_k, $k \in \mathbb{Z}$,
- each U_k is parabolic type I,
- each U_k, $k \neq 0$ contains only one singular value, and U_0 contains two,
- f_0 contains a Leau domain L at the parabolic fixed point $z = \infty$,
- L contains one singular value,
- there are no other Fatou components.

So, if we define the restricted Teichmüller space over each Fatou component, it is easy to see that it would be trivial, except for U_0, in which case we have

$$\dim \mathcal{T}(f, U_0) = 1.$$

We can then split the Teichmüller space for f as follows

$$\mathcal{T}(f_0, \widehat{\mathbb{C}}) = \prod_{k \neq 0} \mathcal{T}(f_0, U_k) \times \mathcal{T}(f_0, U_0) \times \mathcal{B}_1(f, \mathcal{J}(f_0)). \tag{4.2.1}$$

From the above relation, we have the following proposition.

Proposition 4.2.5. *For $f_0(z) = z + exp(1/z)$, the dimension of the Teichmüller space is given by*

$$\dim \mathcal{T}(f_0, \widehat{\mathbb{C}}) = 1 + \dim \mathcal{B}_1(f_0, \mathcal{J}(f_0)). \tag{4.2.2}$$

Several authors have proved that it is possible to approximate dynamically the exponential function (see for example [BDHRGH] and [Kr]) through polynomial functions of the form

$$P_d(z) = \left(1 + \frac{z}{d}\right)^d,$$

with the approximation given as $d \to \infty$. In the same way, it is possible to consider a rational approximation of the function $f_0(z) = z + \exp(1/z)$ given by

$$R_d(z) = z + \left(1 + \frac{1}{dz}\right)^d.$$

For the rational function R_d we have the following immediate properties:

- $z_0 = \infty$ is a parabolic fixed point of R_d with one attracting petal.
- $z_d = -\frac{1}{d}$ is a parabolic fixed point of R_d with $d-1$ attracting petals
- There are no other Fatou components.

In this case, the parabolic fixed point $z_d \to 0$ as $d \to \infty$ and the number of petals converge to infinity, also seems that these petals turn into Baker domains of function f_0 (another conjecture).
It is a well known conjecture (see Section 9 in [MS]) that

$$\dim \mathcal{B}_1(R_d, \mathcal{J}(R_d)) = 0, \ d < \infty,$$

since R_d is a rational function. Then we consider the following conjecture.

Conjecture 4.2.6. *For f_0 as above,*

$$\dim \mathcal{B}_1(f_0, \mathcal{J}(f_0)) < \infty,$$

and then

$$\dim \mathcal{T}(f_0, \widehat{\mathbb{C}}) < \infty.$$

Appendix A

Asymptotic analysis

The existence of an essential singularity, prevents the analytic representation for a function around it. When such an essential singularity is a *Baker point*, we have uniform convergence on the iterates of a function f inside the Baker domain U in the sphere. One way to have a local representation for a function f is using asymptotic analysis.
The present appendix contains basic concepts and some results relevant in the proof ot the Main Theorem 3.2.3 of this work. For further references see [Mlr]

A.1 Big-oh

The principal idea is to compare general functions with known functions.

Definition A.1.1. *Let f and g be two complex-valued functions defined in some set D of the complex plane. Then we write*

$$f(z) = O(g(z)), \qquad z \in D,$$

if we can find a constant $K > 0$ such that

$$|f(z)| \leq K|g(z)| \qquad \text{whenever } z \in D.$$

That is, f is bounded in magnitude by a fixed constant multiple of g for all z in the set D.

The following is a local definition for the *big-oh* operator.

Definition A.1.2. *Let f and g be two complex-valued functions defined in some set D of the complex plane and let z_0 be a limit point of D. Then we write*

$$f(z) = O(g(z)), \qquad \text{as } z \to z_0 \text{ from } D$$

if there is a number $\delta > 0$ such that

$$f(z) = O(g(z)), \qquad z \in D \text{ with } 0 < |z - z_0| < \delta,$$

in the sense of Definition A.1.1. That is, f is bounded in magnitude by a fixed constant multiple of g for all $z \in D$ that lie close enough to z_0.

The same definition can be apply when point z_0 is the point at infinity, in other words, if D is an unbounded set in the complex plane.

Definition A.1.3. *Let f and g be two complex-valued functions defined in an unbounded set D of the complex plane. Then we write*

$$f(z) = O(g(z)) \qquad \text{as } z \to \infty \text{ from } D$$

if there is a number $M > 0$ such that

$$f(z) = O(g(z)), \qquad z \in D \text{ with } |z| > M,$$

in the sense of Definition A.1.1. That is, f is bounded in magnitude by a fixed constant multiple of f for all $z \in D$. that are large enough.

Example A.1.4. *For any positive exponent $p > 0$, however small, we have*

$$\log(z) = O(z^p) \qquad \text{as } z \to \infty \text{ with } z \text{ real and positive.}$$

To prove this, suppose that we pick $M = 1$. Then for $z > M$,

$$|\log(z)| = \log(z) = \int_1^z \frac{dt}{t} \le \int_1^z \frac{t^p}{t} dt \le \int_0^z \frac{t^p}{t} dt = \frac{1}{p} z^p = \frac{1}{p} |z^p|.$$

A.2 Little-oh

Instead of trying to bound f by a fixed multiple of $|g|$, we can try to bound f by *all* constant multiples of $|g|$, at least if we look nearer to z_0 (or infinity) whenever we wish to have a bound involving a smaller multiple of $|g|$.

Definition A.2.1. *Let f and g be two complex-valued functions defined in some set D of the complex plane and let z_0 be a limit point of D. Then we write*

$$f(z) = o(g(z)), \qquad \text{as } z \to z_0 \text{ from } D$$

if for any given $\epsilon > 0$, however small, we can find a corresponding $\delta(\epsilon) > 0$ such that

$$|f(z)| \le \epsilon |g(z)| \qquad \text{whenever } z \in D \text{ and } 0 < |z - z_0| < \delta(\epsilon).$$

That is, f is smaller in magnitude than any multiple of g for $z \in D$ close enough to z_0 (how close depends on which multiple of g is being considered as the bound).

Similarly for asymptotic near infinity, we have the following.

Definition A.2.2. *Let f and g be two complex-valued functions defined in an unbounded set D of the complex plane. Then we write*

$$f(z) = o(g(z)) \qquad \text{as } z \to \infty \text{ from } D$$

if for any given $\epsilon > 0$, however small, we an find a corresponding $M(\epsilon) > 0$ such that

$$|f(z)| \leq \epsilon |g(z)| \qquad \text{whenever } z \in D \text{ and } |z| > M(\epsilon).$$

A.3 Operation rules

We now list some rules for manipulating *oh*-terms.

- **Constants in *oh*-terms.** If C is a positive constant, then the estimate $f(z) = O(Cg(z))$ is equivalent to $f(z) = O(g(z))$. In particular, the estimate $f(z) = O(C)$ is equivalent to $f(z) = O(1)$. The same holds for o-estimates.

- **Transitivity.** O-estimates are transitive, in the sense that if $f(z) = O(g(z))$ and $g(z) = O(h(z))$ then $f(z) = O(h(z))$ when the three estimates are in the same domain and limit. The same holds for o-estimates.

- **Multiplication of *oh*-terms.** If $f_i(z) = O(g_i(z))$, for $i = 1, 2$, then $f_1(z)f_2(z) = O(g_1(z)g_2(z))$ when the three estimates are in the same domain and limit. The same holds for o-estimates.

- **Pulling out factors.** If $f(z) = O(g(z)h(z))$ then $f(z) = g(z)O(h(z))$, equivalently $f(z)/g(z) = O(h(z))$, when the three estimates are in the same domain and limit. The same holds for o-estimates. This property allow us to factor out main terms from *oh*-expressions.

Appendix B

Modern Trends in Complex Analysis

For a rational function Theorem 1.4.9 was proved by Fatou and Julia. Although methods in their proofs were different, both use properties of rational functions: existence of periodic points, existence of repelling or indifferent periodic points and the bounded number of nonrepelling periodic points for example. Some of these properties can be applied to transcendental meromorphic functions, but not in general.
For transcendental meromorphic functions, several authors have had to use other tools, one of these tools is the celebrated *Ahlfors' Five Island Theorem.* The best reference for this, is the survey by W. Bergweiler in [Ber3].

Theorem B.0.1 (AFIT [Ber1], Lemma 5)**.** *Let f be a transcendental meromorphic function, and let D_1, D_2,..., D_5 be five simply connected domains in $\mathbb{C}$ with disjoint closures. Then there exists $j \in \{1, 2, ..., 5\}$ and, for any $R > 0$, a simply connected domain $G \subset \{z \in \mathbb{C} : |z| > R\}$ such that f is a conformal map of G onto D_j. If f has only finitely many poles, then "five" may be replaced by "three".*

From which is possible to deduce the following.

Theorem B.0.2 ([Ber1] Lemma 6)**.** *Suppose that $f \in \mathcal{M}$ and that $z_1, z_2, ...,$ $z_5 \in O^-(\infty)\backslash\{\infty\}$ are distinct. Define n_j by $f^{n_j}(z_j) = \infty$. Then there exists $j \in \{1, 2, ..., 5\}$ such that z_j is a limit point of repelling periodic points of minimal period $n_j + 1$. If f has only finitely many poles then "five" may be replaced by "three".*

Which leads us to the conclusion of Theorem 1.4.9 for $f \in \mathcal{M}$.
For function in class $\mathcal{K}$, the existence of singularities different from the point at infinity, makes the extension of Theorem B.0.1 complicated. Instead of considering the complex plane $\mathbb{C}$, it is possible to consider the extended complex plane $\widehat{\mathbb{C}}$. For this, we have the following derivative.

Lemma B.0.3. *Let f be meromorphic in a domain D. Then for all $z_0 \in D$ the limit*

$$f^{\#}(z_0) := \lim_{z \to z_0} \frac{\sigma(f(z), f(z_0))}{|z - z_0|}$$

exists, where $\sigma(\cdot, \cdot)$ denotes the spherical distance, and we have

$$f^{\#}(z_0) = \begin{cases} \frac{|f'(z_0)|}{1+|f(z_0)|^2} & \text{if } f(z_0) \neq \infty. \\ \lim_{z \to z_0} \frac{|f'(z_0)|}{1+|f(z_0)|^2} & \text{if } f(z_0) = \infty. \end{cases}$$

We call $f^{\#}$ the **spherical derivative** of f. It is continuous on D. With this on hand, we have the following criterion for normal families in $\widehat{\mathbb{C}}$.

Theorem B.0.4 (Marty's Criterion)**.** *A family F of meromorphic functions in a domain D is normal if and only if the family $\{f^{\#} : f \in F\}$ of the respective spherical derivatives is locally uniformly bounded, that is, if for every $z_0 \in D$ there exists a neighborhood $U \subset D$ of z_0 and a constant $M < \infty$ such that*

$$f^{\#} \leq M \qquad \text{for all } z \in U \text{ and all } f \in F.$$

Finally, we have the rescaling lemma used in the proof of Theorem 1.4.9, which was prove by L. Zalcman in [Za] and extended by X.C. Pang in [Pa].

Theorem B.0.5 (Zalcman-Pang Lemma)**.** *Let F be a family of functions meromorphic in the unit disc $\mathbb{D}$ all of whose zeros have multiplicity at least m and all of whose poles have multiplicity at least p. Assume that $-p < \alpha < m$. If F is not normal at $z_0 \in \mathbb{D}$, then there exist sequences $\{f_n\}_n \subset \mathrm{F}$, $\{z_n\}_n \subset \mathbb{D}$ and $\{\varrho_n\}_n \subset (0,1)$ such that $\lim_{n\to\infty} \varrho_n = 0$, $\lim_{z \to z_0} z_n = z_0$ and the sequence $\{g_n\}_n$ defined by*

$$g_n(\zeta) := \frac{1}{\varrho_n^{\alpha}} \cdot f_n(z_n + \varrho_n \zeta)$$

converges locally uniformly in $\mathbb{C}$ (with respect to the spherical metric) to a nonconstant function g which is meromorphic in $\mathbb{C}$ and satisfies

$$g^{\#}(\zeta) \leq g^{\#}(0) = 1 \qquad \text{for all } \zeta \in \mathbb{C}.$$

Appendix C

Fatou components and singularities of f^{-1}

In general terms, the set of singularities of f^{-1} is the set of points where each branch of the inverse function f^{-1} is not well defined. In the case of a rational function, this set consists of the critical values of function f. When we consider a transcendental meromorphic function (class $\mathcal{E}$ or $\mathcal{M}$, for example), since the point at infinity is an essential singularity, we have to add another points to this set. This points are the called finite *asymptotic values.*

Definition C.0.1. *Given a function $f \in \{\mathcal{E}, \mathcal{P}_1, \mathcal{M}\}$, we called a point $a \in \mathbb{C}$ a finite* ***asymptotic value*** *of f if there exists a path $\gamma : [0,1) \to \mathbb{C}$ with $\gamma(t) \to \infty$ as $t \to 1$, such that $f(\gamma(t)) \to a$ as $t \to 1$.*

As a simple example consider the function

$$f(z) = e^z,$$

and the path $\gamma : [0,1) \to \mathbb{C}$ given by

$$\gamma(t) = \frac{1}{t-1}.$$

It is clear that the point $a = 0$ is finite asymptotic value for $f(z) = e^z$. It is well know that function $\log z$ (in any branch) it not analytic in $z = 0$.
In the case of class $\mathcal{K}$, the definition needs to be modified, since we have more essential singularities. We have the following generalized definition.

Definition C.0.2. *Given a function $f \in \mathcal{K}$, we called a point $a \in \widehat{\mathbb{C}} \backslash Ess(f)$ an asymptotic value of f if there exist a path $\gamma : [0,1) \to \widehat{\mathbb{C}}$ and an essential singularity $e \in Ess(f)$ with $\gamma(t) \to e$ as $t \to 1$, such that $f(\gamma(t)) \to a$ as $t \to 1$.*

As an example, consider the function

$$f(z) = \exp\left(\frac{z}{z^2-9}\right)$$

in class $\mathcal{K}$. In this case $Ess(f) = \{\pm 3\}$, and the only asymptotic value is $a = 0$ (from the properties of the exponential function).
We denote by $Asym(f)$ to the set of asymptotic values of a function $f \in \mathcal{K}$.

Definition C.0.3. *Given a function $f \in \mathcal{K}$, we define the set of singularities of f^{-1} as*

$$sing(f^{-1}) := \overline{VCrit(f) \cup Asym(f)},$$

where naturally, $VCrit(f)$ is the set of critical values of function f.
And the postcritical set of f by

$$P(f) = \bigcup_{n=0}^{\infty} f^n(sing(f^{-1}) \backslash Ess(f^n).$$

Some texts consider the closure of this set or the starting index $n = 1$. In the case of rational maps, recall that there are no Baker domains in this case, it is well know that attractive and rotation domains has a close relation with the set of singularities of f^{-1}. This relation is given in the following result.

Theorem C.0.4 ([Ber1], Theorem 7)**.** *Let f be a meromorphic function and let $C = \{U_0, U_1, ..., U_{p-1}\}$ be a periodic cycle of components of $\mathcal{F}(f)$.*

- *If C is a cycle of immediate attractive basins of Leau domains, then $U_j \cap sing(f^{-1}) \neq \emptyset$ for some $j \in \{0, 1, ..., p-1\}$. More precisely, there exists $j \in \{0, 1, ..., p-1\}$ such that $U_j \cap sing(f^{-1}) \neq \emptyset$ contains a point which is not preperiodic or such that U_j contains a periodic critical point (in which case C is a cycle of superattractive basins).*
- *If C is a cycle of Siegel discs or Herman rings, then $\partial U_j \subset \overline{P(f)}$ for all $j \in \{0, 1, ..., p-1\}$.*

From the above theorem we can deduce that the number of immediate attractive basins or Leau domains does not exceed the number of singularities of f^{-1}. Which in the case of rational function is finite, giving a bound for the number of immediate attractive basins and Leau domains. The bound for rotation domains was sharpened by M. Shishikura in [Sh].
For Baker domains the relation with singularities of f^{-1} is not clear. There are examples of Baker domains without singularities of f^{-1} (*univalent Baker domains*) and there are examples with infinitely many singularities of f^{-1} in it (see Examples in [Ri]).
The more accurate relation is given by Bergweiler in [Ber4].

Theorem C.0.5. *Let f be a transcendental entire function with an invariant Baker domain U and $U \cap sing(f^{-1}) = \emptyset$, then there exists a sequence $\{p_n\}_n$ such that $p_n \in \overline{O^+(sing(f^{-1}))}$, $|p_n| \to \infty$, $|p_{n+1}/p_n| \to 1$ and $dist(p_n, U) = o(|p_n|)$ as $n \to \infty$.*

The above theorem, tell us that a Baker domain cannot lie too far from the set $\overline{P(f)}$.

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