CIMAT

Centro de Investigación en Matemáticas

SINGULARITIES IN PRIME CHARACTERISTIC

Thesis

in degree of

Master in Science

with speciality in

Basic Mathematics

by

Veasna Chum

Advisors:

Dr. Luis Núñez Betancourt

and

Dr. Xavier Gómez-Mont Ávalos

May 24, 2017 Guanajuato, Gto. México

Acknowledgments

First of all I would like to thank my supervisors Dr. Luis Núñez-Betancourt and Dr. Xavier Gómez-Mont. In recent semester, I have worked in commutative algebra. From Luis and Xavier, I have gotten an interesting scope of commutative algebra, in particular, how it is closely related to geometry. I have understood that commutative algebra and algebraic geometry are extremely rich and well developed fields. Luis gave me the problem and has guided me, with his help I have understood the concepts in my thesis. Luis spent over large amount of time meeting with me during my third and fourth semesters. He was often able to explain me and detail the points that I didn't understand. Whereas, Xavier questioned me by giving some examples concerning the problems and concepts that were not clear. In addition, he gave me other explanations about them. These aspects are very important for me to make more sense each concept. After Luis gave me the problem about F-threshold technique for an ideal in Noetherian ring with prime characteristic, Xavier tried to observe what is related to my thesis.

Second, I would like to thank CONACYT, CIMAT, CIMPA and IMU. CIMPA's advice and support led me to CIMAT. In particular, CIMPA workes closely to IMU to help push my potential in mathematics. CIMAT has given me an opportunity to study in Mexico for two and half years. Before the courses started, CIMAT gave me Spanish classes. In that time, I have learned a lot of things about Mexico. I also thank CONACYT for its support via an international scholarship.

Moreover, I would like to thank all the professors in each courses that I have taken in CIMAT, in particular, CIMAT staffs. They are very kind and helpful. They always answer my question even if my Spanish is not very good. I also thank my English teachers, Stephanie Dunbar and Janet Mary Izzo. Stephanie very kindly helped me correct my thesis draft and guided me to speak English correctly. For two years at CIMAT, I have learned a lot of things from her, either English or the living life. Whereas, Janet gave me a regular English class for three semesters. She is a very good teacher. From her courses, I have learned a lot of grammar and vocabulary. She always helps me to improve my English speaking.

Finally, I would like to thank my family and my friends. My parents have looked after me carefully and supported me in every way until I could fly. They have advised me about what I'm going to do and encourage me to do it. They taught me the about life. All of my brothers always help me do a lot of things while I am away from the family. Indeed, I have learned many things from my elder brothers. I also thank my friends that gave me the experiences and positive thinking about the road I'm walking.

I hope that all of them will have good luck in their life and have the happiness in their own family always.

Contents

Abstract

In this thesis, we discuss invariants in prime characteristic inspired by objects in birational complex geometry. We study the test ideal, the F-jumping exponent, and the F-threshold of an ideal. The F-jumping exponents are the points where the test ideal change. We discuss the proof that the F-jumping exponents are rational numbers and there are finitely many in every bounded interval for polynomial rings. We also introduce the F-thresholds for every Noetherian ring. We compute the F-threshold of a maximal ideal in a Stanley-Riesner ring using properties of combinatorial commutative and integral closure.

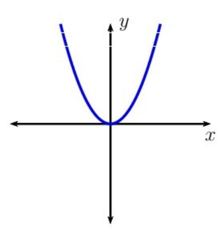
Chapter 1

Introduction

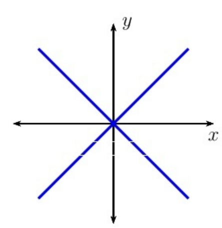
The purpose of this thesis is to provide some concepts of commutative algebra which are related to geometry. The main tool that we use is the F- threshold of an ideal in a ring with prime characteristic p. This concept is worked with the use of a Frobenius map (see Chapter 2). This map gives a vast arsenal of techniques in commutative algebra, algebraic geometry, and representation theory.

Let us discuss a relationship between algebra and geometry. Suppose that we are working in the complex space $\mathbb{C}^n$. We define $\mathcal{V}(f_1, ..., f_\ell)$ as the set of all solutions to the system of polynomial equations $f_1(x) = 0, ..., f_\ell(x) = 0$. This set is called algebraic variety. We can make sense of the concepts of dimension, irreducibility, and smoothness of $\mathcal{V}(f_1, \dots, f_\ell)$ by using the algebra associated to $f_1, \dots, f_\ell$. The mathematical area that studies this interaction is algebraic geometry, and its algebraic side is dominated by commutative algebra.

We consider, for instance, a polynomial f over a real field, f vanishes at $x_0 \in \mathbb{C}^n$. We say that x_0 has a *mutiplicity of at least* n in f if $(\frac{\partial^{\alpha_1} f}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n} f}{\partial x_n^{\alpha_n}})(x_0) = 0$ for every $\alpha_1 + \ldots + \alpha_n \leq n-1$. We say that x_0 is a *singular* point in $\mathcal{V}(f)$ if all the derivations of f vanish at x_0; in particular, x has multiplicity at least 2. If x_0 is not singular, we say that it is *smooth* or *regular*.

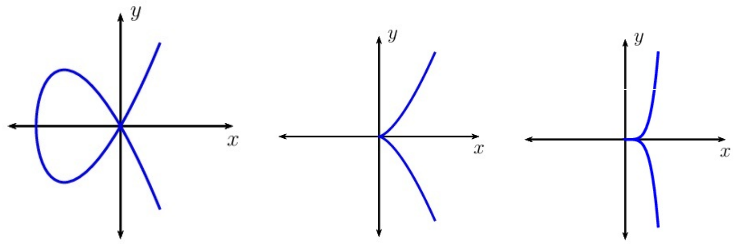


f smooth at $(0, 0)$



f singular at $(0, 0)$

There are several invariants used to detect the singularity, notably the Hilbert-Samuel multiplicity (which was described above using differential operators). However, this method is not good for measuring a singularity. For example, the polynomials $y^2 - x^3 - x^2$, $y^2 - x^3$ and $y^2 - x^9$ have multiplicity 2, but the shape of curves is very different.



$y^2 - x^3 - x^2 = 0 \qquad y^2 - x^3 = 0 \qquad y^2 - x^9 = 0$

In order to measure the singularity, we use an analytic approach. We study the following function changes

$$\varphi : \mathbb{C}^N \setminus V(f) \longrightarrow \mathbb{R}$$
$$z \longrightarrow \frac{1}{\mid f(z) \mid^{2\lambda}},$$

where f is a polynomial and $\lambda \in \mathbb{R}_{\geq 0}$. We observe that φ_1 does not belong to L^2 because its integral in the neighborhood of the vanishing point is not convergent. In this set, we look for a positive large value one λ in which the $\int \frac{1}{|f|^{2\lambda}}$ is finite in some neighborhoods of the vanishing point of f. This value is called the *log-canonical threshold* or the *complex singularity exponent* of f at a vanishing point x_0 and is defined by

$$lct_{x_0}(f) = \sup\{\lambda \in \mathbb{R}| \textit{ there exists a neighborhood } B \textit{ at } x \textit{ such that } \int_B \frac{1}{\mid f \mid^{2\lambda}} < \infty\}.$$

These invariants measure the sharpness of a curve at a point x_0. For instance, the log-canonical thresholds of the curves above are $1, 5/6$ and $11/18$. The previous definition can be extended to any ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$. Furthermore, one can use the resolution of singularities to define the log canonical threshold over other fields with characteristic zero. However, we can use neither integrals nor the resolution of singularities in prime characteristic.

In order to study singularities in prime characteristic, one turns to the Frobenious map. This is motivated by Kunz's Theorem [Kun69], which states that a ring is regular if and only if the Frobenius map is faithfully flat. Let R denote a finitely generated $\mathbb{K}$-algebra over a field of prime characteristic. The F-*threshold* of an ideal (f) is defined by $c^{\mathbf{m}}(f) = \lim_{e\to\infty} \frac{\nu_f^{\mathbf{m}}(p^e)}{p^e}$ where $\nu_f^{\mathbf{m}}(p^e) = \max\{r \in \mathbb{N}| \ f^r \not\subseteq \mathbf{m}^{[p^e]}\}$ and $\mathbf{m}$ is the maximal ideal which defines a point. If $R = K[x_1, \ldots, x_n]$, then the F-threshold is called the F-pure threshold and denoted by $\operatorname{fpt}(I)$. If the ideal I comes from an ideal defined over $\mathbb{Z}[x_1, \ldots, x_n]$, the F-pure thresholds and the log canonical thresholds can be compared as follows $\lim_{p\to\infty} \operatorname{fpt}(I \bmod p) = lct(I)$ [MTW05, HY03].

One can study higher F-thresholds via a class of ideals called the test ideals. These have become a fundamental tool in the study of birational geometry in prime characteristic [ST12]. The *generalized test ideals* $\tau(\mathbf{a}^\lambda)$ are defined via Frobenius fractional powers, $\mathbf{a}^{[\frac{1}{p^e}]}$. Since $\tau(\mathbf{a}^\lambda)$ is parameterized by a real number λ, we can consider the points where they change. These are called the F-jumping exponents of an ideal. It tuns out that the F-thresholds and F-jumping exponents coincide with a polynomial ring.

Motivated by the behavior over polynomial rings, one may wonder if this can be resembled over rings with mild singularities. It turns out that the F-thresholds are defined in great generality. The only assumption needed is that the ring is Noetherian. In this manuscript, we discuss examples related to combinatorial commutative algebra.

In summary, in this work we work with the test ideals, the F-jumping exponents, and the F-thresholds of a ring in prime characteristic motivated which is by invariants used in birational geometry in characteristic zero. In addition, we discuss computations of these invariants for a combinatorial setting and discuss open questions. We now give a brief description of each chapter in this thesis.

Chapter 2 gives an introduction to the Frobenius map. In particular, we show how it works and state a few basic properties. The chapter also contains material about integral closure of ideals. We also discuss a few properties that are crucial for the study of F-thresholds. In particular, this closure operation is very helpful to prove the main theorem in Chapter 5. The main reference for methods in prime characteristic is Huneke's book on tight closure [Hun96] and the main reference for integral closure is Swanson and Huneke's book [HS06].

Chapter 3 gives the introduction and properties for the generalized test ideals, F-jumping exponents and F-thresholds in a polynomial ring. Our study is based on the work of Blickle, Mustata and Smith [BMS08]. At the end of the chapter, we conclude that in the polynomial ring, F-jumping exponents and F-threshold values are the same. Furthermore, we discuss the proof that the set of F-jumping exponent numbers is a discrete subset of the rational numbers.

Chapter 4 extends the notion of F-thresholds to any Noetherian ring with prime characteristic. We define the F-threshold for an ideal $\mathbf{a}$ with respect to any ideal J where $\mathbf{a} \subseteq \sqrt{J}$. In particular, we focus on F-thresholds with respect to a maximal ideal. In addition, we give the F-threshold properties as the techniques to compute the F-threshold values. The main references are [DSNBP, MTW05].

Chapter 5 gives examples of F-thresholds which are related to combinatorial commutative algebra. We give an introduction to Stanley-Reisner theory based on the book by Miller, Sturmfels [MS05]. In particular, we point out a correspondence between simplicial complexes and squarefree monomial ideals. In this chapter, we explicitly compute the diagonal F-threshold, $c^{\mathbf{m}}(\mathbf{m})$, of a Stanley-Reisner ring. This result may already be known to the experts, but it has not been recorded in the literature.

In this thesis is assumed basic knowledge of commutative algebra (eg [Eis95, AM69]).

Chapter 2

Background

In this chapter we introduce the basic regarding the Frobenius map. This is the main tool to study singularities in prime characteristic. We also discuss integral closure. This is an important operation on ideals, which will be helpful for the main theorem in Chapter 5.

All rings in this manuscript are commutative, Noetherian, and of the prime characteristics p.

2.1 Frobenius map

Since our ring has prime characteristic p, we have that

$$(r_1 + r_2)^p = (r_1)^p + (r_2)^p \quad \& \quad (r_1 \cdot r_2)^p = {r_1}^p \cdot {r_2}^p.$$

We define the Frobenius map by

$$\begin{aligned} F : R &\longmapsto R^p \\ r &\longmapsto r^p. \end{aligned}$$

We denote F^e by $F \circ F \circ ... \circ F$, the $e-$iteration of the Frobenius map. Then, $F^e(r) = r^{p^e}$. If R is reduced, then R and $Im(F^e)$ are isomorphic under e^{th}-power of the Frobenius map.
Let I and J be two ideals of R, then their *ideal quotient* is defined by

$$(I : J) = \{r \in R |\ rJ \subseteq I\},$$

for any R-algebra S. Since R is Noetherian, J is finitely generated so we suppose that $J = (g_1, ..., g_l)$.

Claim 2.1.1. *Let S be an R-algebra. If S is free as an R-module, then $(I : J)S = (IS : JS)$.*

Proof. Define a morphism

$$\begin{aligned} \varphi : R &\longrightarrow (R/I)^\ell \\ r &\longrightarrow ([rg_1], ..., [rg_l]). \end{aligned}$$

If $\varphi(r) = 0$ then $([rg_1], ..., [rg_l]) = 0$. This means that $rg_i \in I$ for all $i = 1, ..., \ell$. We have that $ker\varphi = (I : J)$. Then we obtain a natural exact sequence

$$0 \longmapsto (I : J) \longmapsto R \longmapsto (R/I)^\ell.$$

Since S is a free R-module, we obtain an exact sequence

$$0 \longmapsto (I : J)S \overset{Id^{\oplus n}}{\longmapsto} S \overset{\varphi^{\oplus n}}{\longmapsto} ((S/IS)^\ell).$$

From the last exact sequence, we obtain $(I :_R J)S = (IS :_S JS)$. □

We set $q = p^e$ where e is a positive integer. If J is an ideal of R, we denote that $J^{[q]}$ is an ideal of R that is generated by the q^{th}- power of the elements in J,

$$J^{[q]} = (f^q | f \in J).$$

We observe that $J^{[q]} \subseteq J^q$ for any ideal J. We note that $J^{[q]} = F^e(J)R$. Since F^e is an isomorphism between R and $F^e(R)$, we obtain

$$F^e(I :_R J) = (F^e(I) :_{F^e(R)} F^e(J)).$$

Proposition 2.1.2. *If $R = \mathbb{K}[x_1, ..., x_n]$, then R is R^{p^e}-free.*

Proof. For the sake of clarity, we assume that $\mathbb{K}$ is perfect. The general proof follows from the fact that $\mathbb{K}$ is a $\mathbb{K}^{p^e}$-vector space.

We observe that R is an R^{p^e}-module, because R^{p^e} is a subring. Let e is a positive integer and $f = \sum_{0 \leq \alpha_i \leq m_i} a_{\alpha_1, \cdots, \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $m_i \in \mathbb{N}$ and $a_{\alpha_1, \cdots, \alpha_n} \in R$.

If x^α is a monomial term in f, such that $\alpha_i \geq p^e$, we can apply division algorithm to obtain $\beta \in p^e\mathbb{N}^n$ and $\theta \in \mathbb{N}^n$ such that $x^\alpha = x^\beta x^\theta$ and $\theta_i \leq p^e - 1$ for every i.

Then, we have

$$f = \sum_{0 \leq \alpha_i \leq p^e - 1} b_{\alpha_1, \cdots, \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where $b_{\alpha_1, \cdots, \alpha_n} \in R^{p^e}$ are uniquely determined by f. This implies that R is R^{p^e}-free with a basis

$$\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} | \, 0 \leq \alpha_i \leq p^e - 1, \; \textit{for any } i = 1, 2, ..., n\}.$$

□

Proposition 2.1.3. *If R is R^q-free, $q = p^e$, then*

$$(I :_R J)^{[q]} = (I^{[q]} :_{R^q} J^{[q]}).$$

In particular, this holds if R is a polynomial ring over the field with prime characteristic.

Proof. We note that $F^e(I :_R J) = (F^e(I) :_{F^e(R)} F^e(J))$. Since R is R^{p^e}-free, then it follows that $F^e(I : J)R = (F^e(I) :_{F^e(R)} F^e(J))R = (F^e(I)R : F^e(J)R)$ by Claim 2.1.1. Hence, $(I :_R J)^{[q]} = (I^{[q]} :_{R^q} J^{[q]})$. □

2.2 Integral closure of ideal

In this section, we introduce the basic properties and definitions of integral closure. The integral closure of an ideal is an important closure operation in commutative algebra and algebraic geometry. We now recall a few definitions and result about integral closure. We then use this construction to prove the main result in Chapter 5. We refer to the book by Swanson and Huneke [HS06] for proofs and further details.

Definition 2.2.1. Let I be an ideal of ring R. An element $r \in R$ is an *integral over* I if there exist an integer n and an element $a_i \in I^i$, $i = 1, 2, ..., n$ such that

$$r^n + a_1 r^n - 1 + \cdots + a_{n-1} r + a_n = 0.$$

The set of all elements that are integral over I is called *integral closure of ideal*, denoted by $\overline{I}$. In addition, I is said to be *integrally closed* if $\overline{I} = I$.

Observation 2.2.2. Let I, J be ideals of R. Then,

(1) $I \subseteq \overline{I}$;

(2) $\overline{I} \cdot \overline{J} \subseteq \overline{IJ}$;

(3) if $I \subseteq J$, then $\overline{I} \subseteq \overline{J}$;

(4) $\overline{\overline{I}} = \overline{I}$.

Proposition 2.2.3. *If I is radical, then $\overline{I} = I$.*

Proof. We observe that $I \subseteq \overline{I}$. Let $r \in \overline{I}$. Then there exists $t \in \mathbb{N}$, $a_i \in I^i$ such that

$$r^t + a_1 r^{t-1} + \cdots + a_{t-1} r + a_t = 0.$$

It implies that

$$r^t = -(a_1 r^{t-1} + \cdots + a_{t-1} r + a_t) \in I.$$

Then $r \in \sqrt{I} = I$, and so $\overline{I} \subseteq I$. This completes the proof. □

Examples 2.2.4. Let $R = \mathbb{K}[x, y]$.

(1) $\overline{(x)} = (x)$ because it is a prime ideal.

(2) $\overline{(x, y)} = (x, y)$ because it is a maximal ideal.

(3) $\overline{(x^d, y^d)} \supseteq (x, y)^d$, because if $r = x^i y^{d-i}$, we take

$$a_d = x^{id} y^{d^2 - id} = (x^d)^i (y^d)^{d-i} \in (x^d, y^d)^d$$

and $a_j = 0, j = 1, 2, ..., d - 1$ to obtain $r^d + a_d = 0$. In fact, one can check that $\overline{(x^d, y^d)} = (x, y)^d$.

Proposition 2.2.5. *The integral closure of an ideal is an ideal.*

Proposition 2.2.6. *Let I be an ideal of R. Then there exists positive integer m such that $\overline{I}^{m+\ell} \subseteq I^\ell$ for every $\ell \in \mathbb{N}$.*

Theorem 2.2.7. *Let R be a d-dimensional standard graded* $\mathbb{K}$-algebra *with $\mathbb{K}$ an infinite field. Let $\boldsymbol{m}$ be a maximal homogeneous ideal. Then there exists $J \subseteq R$ such that J is generated by d elements and $\overline{J} = \boldsymbol{m}$.*

Chapter 3

F-threshold and Test ideal in polynomial ring

In this chapter we review the definition and properties of test ideals. Our main reference is the paper by Blickle, Mustata, and Smith [BMS08].

3.1 Generalized test ideals and F-thresholds

In this section we construct the generalized a test ideal in the form $\mathbf{a}^{[\frac{1}{p^e}]}$. This ideal characterizes the F-thresholds for the ideals in a polynomial ring.

3.1.1 The ideals $\mathbf{a}^{[\frac{1}{q}]}$

Definition 3.1.1. Let $\mathbf{a}$ be an ideal of R and $q = p^e$, and e be a positive integer. Let $\mathbf{a}^{[\frac{1}{q}]}$ be the smallest ideal of J such that $\mathbf{a} \subseteq J^{[q]}$.

By the definition of ideal which is generated by q^{th}-power, $\mathbf{a}^{[\frac{1}{p^0}]} = \mathbf{a}^{[1]} = \mathbf{a}$. The following proposition shows that $\mathbf{a}^{[\frac{1}{p^e}]}$ always exists.

Proposition 3.1.2. *Given $\boldsymbol{a}$ that is an ideal of R, we have that $\boldsymbol{a}^{[\frac{1}{p^e}]} = \bigcap_{\boldsymbol{a} \subseteq J^{[q]}} J$.*

Proof. Since R is free over R^{p^e}, we have

$$(\bigcap_{\mathbf{a} \subseteq J^{[q]}} J)^{[q]} = F^e(\bigcap_{\mathbf{a} \subseteq J^{[q]}} J)R = (\bigcap_{\mathbf{a} \subseteq J^{[q]}} F^e(J))R = \bigcap_{\mathbf{a} \subseteq J^{[q]}} F^e(J)R = \bigcap_{\mathbf{a} \subseteq J^{[q]}} J^{[q]}.$$

From the construction of $\mathbf{a}^{[\frac{1}{p^e}]}$, we obtain the $\mathbf{a} \subseteq \bigcap_{\mathbf{a} \subseteq J^{[q]}} J^{[q]} = (\bigcap_{\mathbf{a} \subseteq J^{[q]}} J)^{[q]}$. Then, $\mathbf{a}^{[\frac{1}{q}]} \subseteq \bigcap_{\mathbf{a} \subseteq J^{[q]}} J$ by Definition 3.1.1. In addition, $\mathbf{a} \subseteq (\mathbf{a}^{[\frac{1}{q}]})^{[q]}$, and so $\cap_{\mathbf{a} \subseteq J^{[q]}} J \subseteq \mathbf{a}^{[\frac{1}{p^e}]}$ by Definition 3.1.1. □

We note that $\mathbf{a}^{[\frac{1}{q}]} \supseteq \mathbf{a}$ because $\mathbf{a} \subseteq J^{[q]} \subseteq J$ and $\cap_{\mathbf{a} \subseteq J^{[q]}} J = \mathbf{a}^{[\frac{1}{q}]}$.

We now present a result that gives a criterion, based on Frobenius, to decide whether or not an element belongs to an ideal.

Lemma 3.1.3.

(1) If $u \in R$, then $u^{p^e} \in J^{[p^e]}$ if and only if $u \in J$.

(2) If $\exists c \neq 0$, $u \in R$, then $cu^{p^e} \in J^{[p^e]}$ for $e >> 0$ if and only if $u \in J$.

Proof.

(1) To prove this is true we sufficiently prove that if $u^{p^e} \in J^{[p^e]}$ implies that $u \in J$ is true. We pick a morphism of $R^{p^e}-$modules

$$\varphi : R \longrightarrow R^{p^e}$$

such that $\varphi(1) = 1$. This is possible because R is R^{p^e}-free and 1 is part of the basis.

Suppose that $J = (g_1, ..., g_m)$. Now we see that, $\varphi(u^{p^e}) = u^{p^e} \in F^e(R)$. Since $F^e(J)$ is an ideal of R^{p^e}, then

$$u^{p^e} = \sum_{1 \leq i \leq m} b_i^{p^e} g_i^{p^e} \Rightarrow (u - \sum_{1 \leq i \leq m} b_i g_i)^{p^e} = 0,$$

where $b \in R$. Hence, $u = \sum_{1 \leq i \leq m} b_i g_i \in J$.

(2) To prove this argument is true we just prove that if $cu^{p^e} \in J^{[p^e]}$ implies that $u \in J$. Suppose that $cu^{p^e} \in J^{[p^e]}$, $c \neq 0$. Let $d = \deg c < p^e$. Since R is free over R^{p^e}, then

$$c = \sum_{0 \leq \alpha_i \leq d} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where $a_\alpha \in k \setminus \{0\}$. Define a morphism

$$\begin{aligned} \pi : & R \longrightarrow R^{p^e} \\ & u \longrightarrow projection \;\; on \;\; x_1^{\alpha_1} \cdots x_n^{\alpha_n} \end{aligned}$$

of R^{p^e}-modules. Then,

$$\pi(cu^{p^e}) = a_\alpha u^{p^e}.$$

Moreover,

$$J^{[p^e]} = (g_1^{p^e}, ..., g_m^{p^e}) = \bigoplus_{0 \leq \alpha_i < p^e} F^e(J) x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

And $\pi(g_i^{p^e} x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = g_i^{p^e}$. This is, $\pi(J^{p^e}) = F^e(J) \subseteq R^{p^e}$.

Then

$$\pi(cu^{p^e}) = a_\alpha u^{p^e} \in F^e(J).$$

Since a_α is a unit, then $a_\alpha^{-1} \cdot \pi(cu^{p^e}) = u^{p^e} \in F^e(J)$.

Then, $u \in J$ because F^e is injective.

□

Lemma 3.1.4. *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be the ideals of R. Let $q = p^e$ and $q' = p^{e'}$, where e, e' are positive integers. Then the following statements hold.*

(1) $\boldsymbol{a} \subseteq (\boldsymbol{a}^{[\frac{1}{q}]})^{[q]}$.

(2) If $\boldsymbol{a} \subseteq \boldsymbol{b}$, *then* $\boldsymbol{a}^{[\frac{1}{q}]} \subseteq \boldsymbol{b}^{[\frac{1}{q}]}$.

(3) $(\boldsymbol{a} \cap \boldsymbol{b})^{[\frac{1}{q}]} \subseteq \boldsymbol{a}^{[\frac{1}{q}]} \cap \boldsymbol{b}^{[\frac{1}{q}]}$.

(4) $(\boldsymbol{a} + \boldsymbol{b})^{[\frac{1}{q}]} = \boldsymbol{a}^{[\frac{1}{q}]} + \boldsymbol{b}^{[\frac{1}{q}]}$.

(5) $(\boldsymbol{a} \cdot \boldsymbol{b})^{[\frac{1}{q}]} \subseteq \boldsymbol{a}^{[\frac{1}{q}]} \cdot \boldsymbol{b}^{[\frac{1}{q}]}$.

(6) $(\boldsymbol{b}^{[q']})^{[\frac{1}{q}]} = \boldsymbol{b}^{[\frac{q'}{q}]} \subseteq (b^{[\frac{1}{q}]})^{[q']}$.

(7) $\boldsymbol{b}^{[\frac{1}{qq'}]} \subseteq (\boldsymbol{b}^{[\frac{1}{q}]})^{[\frac{1}{q'}]}$.

(8) $\boldsymbol{b}^{[\frac{1}{q}]} \subseteq (\boldsymbol{b}^{q'})^{[\frac{1}{q'q}]}$.

Proof.

(1) Since R is R^{p^e}-free, we have $\mathbf{a} \subseteq \bigcap_{\mathbf{a} \subseteq J^{[q]}} J^{[q]} = (\bigcap_{\mathbf{a} \subseteq J^{[q]}} J)^{[q]} = (\mathbf{a}^{[\frac{1}{q}]})^{[q]}$.

(2) Since $\mathbf{a} \subseteq \mathbf{b} \subseteq (\mathbf{b}^{[\frac{1}{q}]})^{[q]}$. This implies that $\mathbf{a}^{[\frac{1}{q}]} \subseteq \mathbf{b}^{[\frac{1}{q}]}$.

(3) The statements immediately follow from (2).

(4) For the containment, we have to prove that $(\mathbf{a} + \mathbf{b})^{[\frac{1}{q}]} \subseteq \mathbf{a}^{[\frac{1}{q}]} + \mathbf{b}^{[\frac{1}{q}]}$. We apply (1) to be obtained that

$$\mathbf{a} + \mathbf{b} \subseteq (\mathbf{a}^{[\frac{1}{q}]})^{[q]} + (\mathbf{b}^{[\frac{1}{q}]})^{[q]} = (\mathbf{a}^{[\frac{1}{q}]} + \mathbf{b}^{[\frac{1}{q}]})^{[q]}.$$

By the definition, $(\mathbf{a} + \mathbf{b})^{[\frac{1}{q}]} \subseteq \mathbf{a}^{[\frac{1}{q}]} + \mathbf{b}^{[\frac{1}{q}]}$. For now we show that $\mathbf{a}^{[\frac{1}{q}]} + \mathbf{b}^{[\frac{1}{q}]} \subseteq (\mathbf{a} + \mathbf{b})^{[\frac{1}{q}]}$. By using (ii) we have $(\mathbf{a} + \mathbf{b})^{[\frac{1}{q}]} \supseteq \mathbf{a}^{[\frac{1}{q}]} + \mathbf{b}^{[\frac{1}{q}]}$.

(5) We apply (1), then $\mathbf{a} \cdot \mathbf{b} \subseteq (\mathbf{a}^{[\frac{1}{q}]})^{[q]} \cdot (\mathbf{b}^{[\frac{1}{q}]})^{[q]} = (\mathbf{a}^{[\frac{1}{q}]} \cdot \mathbf{b}^{[\frac{1}{q}]})^{[q]}$. Hence,$(\mathbf{a} \cdot \mathbf{b})^{[\frac{1}{q}]} \subseteq \mathbf{a}^{[\frac{1}{q}]} \cdot \mathbf{b}^{[\frac{1}{q}]}$.

(6) Suppose that $q = p^e, q' = p^{e'}$ with $e >> e'$. By (1) then

$$\mathbf{b} \subseteq (\mathbf{b}^{[\frac{1}{q}]})^{[q]} = (\mathbf{b}^{[\frac{1}{p^e}]})^{[p^e]} = (\mathbf{b}^{[\frac{1}{p^e}]})^{[\frac{p^{e+e'}}{p^{e'}}]} = ((\mathbf{b}^{[\frac{1}{p^e}]})^{[p^{e'}]})^{[p^{e-e'}]}.$$

We obtain that, $\mathbf{b}^{[\frac{q'}{p}]} \subseteq (\mathbf{b}^{[\frac{1}{q}]})^{[q']}$.

By Definition 3.1.1, we recall $(\mathbf{b}^{[p^{e'}]})^{[\frac{1}{p^e}]}$ is the smallest ideal of J such that

$$\mathbf{b}^{[p^{e'}]} \subseteq J^{[p^e]}.$$

We now show that $\mathbf{b}^{[p^{e'}]} \subseteq J^{[p^e]}$ if and only if $\mathbf{b} \subseteq J^{[p^{e-e'}]}$.

Let $f \in \mathbf{b}$. We have that $f^{p^{e'}} \in J^{[p^e]}$ and so $f \in J^{[p^{e-e'}]}$. So this means that $\mathbf{b} \subseteq J^{[p^{e-e'}]}$. It is also true for the inverse because if $\mathbf{b} \subseteq J^{[p^{e-e'}]}$ implies that $\mathbf{b}^{[p^{e'}]} \subseteq J^{[p^e]}$. For this reason, we may say that $\mathbf{b}^{[\frac{1}{p^{e-e'}}]}$ is the smallest ideal in J. Therefore, $(\mathbf{b}^{[q']})^{[\frac{1}{q}]} = \mathbf{b}^{[\frac{q'}{q}]}$.

(7) By (1) and (6), we obtain

$$\mathbf{b} \subseteq (\mathbf{b}^{[\frac{1}{q}]})^{[q]} = (\mathbf{b}^{[\frac{1}{p^e}]})^{[p^e]} = (\mathbf{b}^{[\frac{1}{p^e}]})^{[\frac{p^{e+e'}}{p^{e'}}]} \subseteq ((\mathbf{b}^{[\frac{1}{p^e}]})^{[\frac{1}{p^{e'}}]})^{[p^{e+e'}]}.$$

Hence, $\mathbf{b}^{[\frac{1}{qq'}]} \subseteq (\mathbf{b}^{[\frac{1}{q}]})^{[\frac{1}{q'}]}$.

(8) By (6), it follows that

$$\mathbf{b} = \mathbf{b}^{[\frac{q'}{q'}]} = (\mathbf{b}^{[q']})^{[\frac{1}{q'}]} \subseteq (\mathbf{b}^{q'})^{[\frac{1}{q'}]} = (\mathbf{b}^{q'})^{[\frac{q}{qq'}]} \subseteq ((\mathbf{b}^{q'})^{[\frac{1}{qq'}]})^{[q]}.$$

Then $\mathbf{b}^{[\frac{1}{q}]} \subseteq (\mathbf{b}^{q'})^{[\frac{1}{qq'}]}$.

□

Proposition 3.1.5. *Let $e_1, e_2, ..., e_N$ be a free basis of R over R^q. Let $h_1, h_2, ..., h_s$ be the generators of an ideal* $\mathbf{b}$ *of R, and for every i=1,...,s*

$$h_i = \sum_{1 \leq j \leq N} {a_{ij}}^q e_j$$

with $a_{ij} \in R$. Then $\mathbf{b}^{[\frac{1}{q}]} = (a_{ij} | i \leq s, j \leq N)$.

Proof. We first show that $\mathbf{b}^{[\frac{1}{q}]} \subseteq (a_{ij} | i \leq s, j \leq N)$. From the hypothesis, we have $h_i \in ({a^q}_{ij} | i \leq s, j \leq N) = (a_{ij} | i \leq s, j \leq N)^{[q]}$, $\forall i$. Then $\mathbf{b} = (h_1, h_2, ..., h_N) \subseteq (a_{ij} | i \leq s, j \leq N)^{[q]}$ and by the Definition 3.1.1, we have

$$\mathbf{b}^{[\frac{1}{q}]} \subseteq (a_{ij} | i \leq s, j \leq N).$$

We now show that $(a_{ij} | i \leq s, j \leq N) \subseteq \mathbf{b}^{[\frac{1}{q}]}$. Recall that $\mathbf{b}^{[\frac{1}{q}]}$ is the smallest of J such that $\mathbf{b} \subseteq J^{[q]}$ and by Proposition 3.1.2, $\mathbf{b}^{[\frac{1}{q}]} = \cap_{\mathbf{b} \subseteq J^{[q]}} J$. So now from these we have $\mathbf{b} \subseteq J^{[q]}$ and let $J = (f_1, f_2, ..., f_m)$, we get $J^{[q]} = ({f_1}^q, {f_2}^q, ..., {f_m}^q)$. Then we express

$$h_i = \sum_{1 \leq j \leq m} c_{ij} {f_j}^q$$

for all $i = 1, ..., N$ with $c_{ij} \in R$.

We consider in the dual space $Hom_{R^q}(R, R^q)$ which has $e_1^*, e_2^*, ..., e_N^*$ as a basis. We know that

$$e_j^*(e_i) = \delta_{ij}.$$

Then, $e_j^*(h_i) = a_{ij}^q$. On the other hand,

$$e_j^*(h_i) = \sum_{1 \le j \le m} e_j^*(c_{ij}) f_j^q \in J^{[q]}, \quad \text{for all } J \text{ such that } \mathbf{b} \subseteq J^{[q]}.$$

Hence, $a_{ij}^q \in J^{[q]}$ for all J and from the Lemma 3.1.3, $a_{ij} \in J$ for all J such that $\mathbf{b} \subseteq J^{[q]}$. Then,

$$(a_{ij} | i \le s, \quad j \le N) \subseteq \cap_{\mathbf{b} \subseteq J^{[q]}} J = \mathbf{b}^{[\frac{1}{q}]},$$

which completes the proof. □

3.1.2 Generalized test ideals

Lemma 3.1.6. *Let $\boldsymbol{a}$ be an ideal of R. If r, r', e and e' are positive integers such that $\frac{r}{p^e} \ge \frac{r'}{p^{e'}}$ and $e' \ge e$, then*

$$(\boldsymbol{a}^r)^{[\frac{1}{p^e}]} \subseteq (\boldsymbol{a}^{r'})^{[\frac{1}{p^{e'}}]}.$$

Proof. From the hypothesis, we have $\frac{r}{p^e} \ge \frac{r'}{p^{e'}}$ and $e' \ge e$, then we obtain $r' \le \frac{rp^{e'}}{p^e} = rp^{e'-e}$. Then

$$\mathbf{a}^r = (\mathbf{a}^r)^{[\frac{p^{e'-e}}{p^{e'-e}}]} = (\mathbf{a}^{r[p^{e'-e}]})^{[\frac{1}{p^{e'-e}}]} \subseteq (\mathbf{a}^{rp^{e'-e}})^{[\frac{p^e}{p^{e'}}]}$$
$$\subseteq ((\mathbf{a}^{rp^{e'-e}})^{[\frac{1}{p^{e'}}]})^{[p^e]} \subseteq ((\mathbf{a}^{r'})^{[\frac{1}{p^{e'}}]})^{[p^e]}.$$

By Lemma 3.1.4, we obtain $(\mathbf{a}^r)^{[\frac{1}{p^e}]} \subseteq (\mathbf{a}^{r'})^{[\frac{1}{p^{e'}}]}$. □

We observe that for any e that is a positive integer and $c > 0$,

$$\frac{\lceil cp^e \rceil}{p^e} = \frac{\lceil cp^e \rceil p}{p^e p} \ge \frac{\lceil cp^{e+1} \rceil}{p^{e+1}}.$$

Now we apply Lemma 3.1.6, we obtain a sequence of ideals of R as follows

$$(\mathbf{a}^{\lceil cp^e \rceil})^{[\frac{1}{p^e}]} \subseteq (\mathbf{a}^{\lceil cp^{e+1} \rceil})^{[\frac{1}{p^{e+1}}]} \subseteq (\mathbf{a}^{\lceil cp^{e+2} \rceil})^{[\frac{1}{p^{e+2}}]} \subseteq \cdots$$

Definition 3.1.7. Let $\mathbf{a}$ be an ideal of R and $c > 0$, we define the generalized test ideal of $\mathbf{a}$ with an exponent c as follows

$$\tau(\mathbf{a}^c) = \bigcup_{e \in \mathbb{N}} (\mathbf{a}^{\lceil cp^e \rceil})^{[\frac{1}{p^e}]}.$$

Since R is a Noetherian ring then when $e >> 0$ the sequence $\{(\mathbf{a}^{\lceil cp^e \rceil})^{[\frac{1}{p^e}]}\}_{e \in \mathbb{N}}$ stabilizes. Therefore, $\tau(\mathbf{a}^c) = (\mathbf{a}^{\lceil cp^e \rceil})^{[\frac{1}{p^e}]}$, for $e >> 0$.

Proposition 3.1.8. *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two ideals of R.*

(1) If $c_1 < c_2$, then $\tau(\boldsymbol{a}^{c_2}) \subseteq \tau(\boldsymbol{a}^{c_1})$.

(2) *If* $\boldsymbol{a} \subseteq \boldsymbol{b}$*, then* $\tau(\boldsymbol{a}^c) \subseteq \tau(\boldsymbol{b}^c)$*.*

(3) $\tau((\boldsymbol{a} \bigcap \boldsymbol{b})^c) \subseteq \tau(\boldsymbol{a}^c) \bigcap \tau(\boldsymbol{b}^c)$ *and* $\tau(\boldsymbol{a}^c) + \tau(\boldsymbol{b}^c) \subseteq \tau((\boldsymbol{a}+\boldsymbol{b})^c)$*.*

(4) $\tau((\boldsymbol{a} \cdot \boldsymbol{b})^c) \subseteq \tau(\boldsymbol{a}^c) \cdot \tau(\boldsymbol{b}^c)$*.*

Proof.

(1) By definition of generalized test ideal, we have

$$\tau(\mathbf{a}^{c_1}) = \bigcup_{e>0} (\mathbf{a}^{\lceil c_1 p^e \rceil})^{[\frac{1}{p^e}]} \quad \text{and} \quad \tau(\mathbf{a}^{c_2}) = \bigcup_{e>0} (\mathbf{a}^{\lceil c_2 p^e \rceil})^{[\frac{1}{p^e}]}.$$

By (2) of Lemma 3.1.4, we have $(\mathbf{a}^{\lceil c_2 p^e \rceil})^{[\frac{1}{p^e}]} \subseteq (\mathbf{a}^{\lceil c_1 p^e \rceil})^{[\frac{1}{p^e}]}$ for all e. Thus, $\tau(\mathbf{a}^{c_2}) \subseteq \tau(\mathbf{a}^{c_1})$.

(2) Since $\mathbf{a} \subseteq \mathbf{b}$, then $\mathbf{a}^{\lceil cp^e \rceil} \subseteq \mathbf{b}^{\lceil cp^e \rceil}$. Applying Lemma 3.1.4 (2), we have

$$\bigcup_{e>0} (\mathbf{a}^{\lceil cp^e \rceil})^{[\frac{1}{p^e}]} \subseteq \bigcup_{e>0} (\mathbf{b}^{\lceil cp^e \rceil})^{[\frac{1}{p^e}]},$$

which completes the proof.

(3) We apply (2) to obtain $\tau((\mathbf{a} \cap \mathbf{b})^c) \subseteq \tau(\mathbf{a}^c) \bigcap \tau(\mathbf{b}^c)$. Moreover,

$$\tau(\mathbf{a}^c) \subseteq \tau((\mathbf{a}+\mathbf{b})^c) \quad \text{and} \quad \tau(\mathbf{b}^c) \subseteq \tau((\mathbf{a}+\mathbf{b})^c).$$

Thus, $\tau(\mathbf{a}^c) + \tau(\mathbf{b}^c) \subseteq \tau((\mathbf{a}+\mathbf{b})^c)$.

(4) Since $\mathbf{a} \cdot \mathbf{b} \subseteq \mathbf{a}$ and $\mathbf{a} \cdot \mathbf{b} \subseteq \mathbf{b}$, then we use (2) to obtain

$$\tau((\mathbf{a} \cdot \mathbf{b})^c) \subseteq \tau(\mathbf{a}^c) \quad \text{and} \quad \tau((\mathbf{a} \cdot \mathbf{b})^c) \subseteq \tau(\mathbf{b}^c).$$

Hence,

$$\tau((\mathbf{a} \cdot \mathbf{b})^c) \subseteq \tau(\mathbf{a}^c) \cdot \tau(\mathbf{b}^c).$$

□

Proposition 3.1.9. *If* $\boldsymbol{a}$ *is a ideal of* R *and* c *is a non-negative real number, there exists an* $\epsilon > 0$ *such that* $\tau(\boldsymbol{a}^c) = (\boldsymbol{a}^r)^{[\frac{1}{p^e}]}$ *whenever* $c < \frac{r}{p^e} < c + \epsilon$*. This is,* $\tau(\boldsymbol{a}^c) = \tau(\boldsymbol{a}^{c'})$ *where* c' *is a rational number of the form* $\frac{r}{p^e}$ *that approximates* c *from above sufficiently well.*

Proof. By definition of the generalized test ideal, $\tau(\mathbf{a}^c) = \bigcup_{e>0} (\mathbf{a}^{\lceil cp^e \rceil})^{[\frac{1}{p^e}]}$ and

$$(\mathbf{a}^{\lceil cp^e \rceil})^{[\frac{1}{p^e}]} \subseteq (\mathbf{a}^{\lceil cp^{e+1} \rceil})^{[\frac{1}{p^{e+1}}]}, \quad \text{for all } e.$$

Since R is a Noetherian ring and when $e >> 0$, this sequence is stabilized. This is,

$$\tau(\mathbf{a}^c) = (\mathbf{a}^{\lceil cp^e \rceil})^{[\frac{1}{p^e}]} = (\mathbf{a}^r)^{[\frac{1}{p^e}]},$$

where e is a sufficient large. Now we check that there are $\epsilon > 0$ and r is a positive integer such that $c < \frac{r}{p^e} < c + \epsilon$. If cp^e is not an integer. From the properties of integer part, we obtain

$$cp^e \lneqq \lceil cp^e \rceil \lneqq cp^e + 1.$$
$$c < \frac{\lceil cp^e \rceil}{p^e} < c + \frac{1}{p^e}.$$

Take $\epsilon > 0$ when e is large enough $c + \frac{1}{P^e} < c + \epsilon$ and $r = \lceil cp^e \rceil$.

If cp^e is an integer, then $\lceil cp^e \rceil = cp^e$ and $c < c + \frac{1}{p^e} < c + \epsilon$, ϵ as the previous case and $r = cp^e + 1$. We now show that $\tau(\mathbf{a}^c) = (\mathbf{a}^{(c+\frac{1}{p^e})p^e})^{[\frac{1}{p^e}]}$.

$$I = (\mathbf{a}^{cp^e+1})^{[\frac{1}{p^e}]} \subseteq (\mathbf{a}^{cp^e})^{[\frac{1}{p^e}]} = \tau(\mathbf{a}^c)$$

For the reverse inclusion, we choose another positive integer e' such that $e' \geq e$ then $c < \frac{cp^{e'}+1}{p^{e'}} = c + \frac{1}{p^{e'}} \leq c + \frac{1}{p^e}$. Since R is Noetherian and in case e is large enough, we have

$$(\mathbf{a}^{cp^e+1})^{[\frac{1}{p^e}]} = (\mathbf{a}^{cp^{e'}+1})^{[\frac{1}{p^{e'}}]} \quad \text{and} \quad \tau(\mathbf{a}^c) = (\mathbf{a}^{cp^e})^{[\frac{1}{p^e}]} = (\mathbf{a}^{cp^{e'}})^{[\frac{1}{p^{e'}}]}.$$

Now we have, $\mathbf{a}^{cp^{e'}+1} \subseteq I^{[p^{e'}]} = (I^{[p^e]})^{[p^{e'-e}]}$.

Let $u \in \mathbf{a}^{cp^e}$. Then for any $v \in \mathbf{a}$, $v \neq 0$ we have $vu^{p^{e'-e}} \in \mathbf{a}^{cp^{e'}+1} \subseteq (I^{[p^e]})^{[p^{e'-e}]}$. By Lemma 3.1.3, we obtain $u \in I^{[p^e]}$, i.e. $\mathbf{a}^{cp^e} \subseteq I^{[p^e]}$. Thus, $\tau(\mathbf{a}^c) \subseteq I$. □

Corollary 3.1.10. *If m is a positive integer, then for every $c \in \mathbb{R}_{\geq 0}$ we have*

$$\tau((\boldsymbol{a}^m)^c) = \tau(\boldsymbol{a}^{cm}).$$

Proof. If c is an positive integer number, then this statement is true. If c is not an integer, as R is a Noetherian ring, then

$$\tau((\mathbf{a}^m)^c) = ((\mathbf{a}^m)^{\lceil cp^e \rceil})^{[\frac{1}{p^e}]} = (\mathbf{a}^{\lceil cp^e \rceil m})^{[\frac{1}{p^e}]} \quad \text{and} \quad \tau(\mathbf{a}^{cm}) = (\mathbf{a}^{\lceil cmp^e \rceil})^{[\frac{1}{p^e}]},$$

where e is large enough. We choose $e >> 0$. Then there exists $\epsilon > 0$ with $cm < \frac{\lceil cp^e \rceil m}{p^e} < cm + \epsilon$. When e is large enough then $\frac{\lceil cp^e \rceil m}{p^e}$ is closed to cm and by Proposition 3.1.9, we obtain $\tau(\mathbf{a}^{cm}) = (\mathbf{a}^{\lceil cp^e \rceil m})^{[\frac{1}{p^e}]} = \tau((\mathbf{a}^c)^m)$. □

Corollary 3.1.11. *For every ideal $\boldsymbol{a}$ in R and a nonnegative real number c, there exists $\epsilon > 0$ such that $\tau(\boldsymbol{a}^c) = \tau(\boldsymbol{a}^{c'})$ for every $c' \in [c, c + \epsilon)$.*

Proof. If e is large enough, $\tau(\mathbf{a}^c) = (\mathbf{a}^{\lceil cp^e \rceil})^{[\frac{1}{p^e}]}$ and $\tau(\mathbf{a}^{c'}) = (\mathbf{a}^{\lceil c'p^e \rceil})^{[\frac{1}{p^e}]}$. From the previous proposition, take $\epsilon > 0$ we have $\tau(\mathbf{a}^c) = (\mathbf{a}^r)^{[\frac{1}{p^e}]}$ whenever $c < \frac{r}{p^e} < c + \epsilon$. By hypothesis $c \leq c' < c + \epsilon$ then $c \leq \frac{\lceil c'p^e \rceil}{p^e} < c + \epsilon$.

If $c = c'$ there is nothing to prove.

If $c < \frac{\lceil c'p^e \rceil}{p^e} < c + \epsilon$, then we apply Proposition 3.1.9 to obtain

$$\tau(\mathbf{a}^c) = (\mathbf{a}^{\lceil c'p^e \rceil})^{[\frac{1}{p^e}]} = \tau(\mathbf{a}^{c'}).$$

The proof is completed. □

3.1.3 Skoda's theorem

Proposition 3.1.12. *If $\boldsymbol{a}$ is an ideal of R which is generated by m elements, then for every $c \geq m$ we have*
$$\tau(\boldsymbol{a}^c) = \boldsymbol{a} \cdot \tau(\boldsymbol{a}^{c-1}).$$

Proof. Let e be large enough, we have
$$\tau(\mathbf{a}^c) = (\mathbf{a}^{\lceil cp^e \rceil})^{[\frac{1}{p^e}]} \quad \text{and} \quad \tau(\mathbf{a}^{c-1}) = (\mathbf{a}^{\lceil (c-1)p^e \rceil})^{[\frac{1}{p^e}]} = (\mathbf{a}^{\lceil cp^e \rceil - p^e})^{[\frac{1}{p^e}]}.$$

It is sufficient to show that $(\mathbf{a}^{\lceil cp^e \rceil})^{[\frac{1}{p^e}]} = \mathbf{a} \cdot (\mathbf{a}^{\lceil cp^e \rceil - p^e})^{[\frac{1}{p^e}]}$. Let $r = \lceil cp^e \rceil$. We focus on the first containment. We note that for all $r \geq p^e$,
$$\mathbf{a}^{r-p^e} \cdot \mathbf{a}^{[p^e]} \subseteq \mathbf{a}^r \subseteq ((\mathbf{a}^r)^{[\frac{1}{p^e}]})^{[p^e]}.$$

Then
$$\mathbf{a}^{r-p^e} \subseteq (((\mathbf{a}^r)^{[\frac{1}{p^e}]})^{[p^e]} : \mathbf{a}^{[p^e]}) = ((\mathbf{a}^r)^{[\frac{1}{p^e}]} : \mathbf{a})^{[p^e]}.$$

It follow that,
$$(\mathbf{a}^{r-p^e})^{[\frac{1}{p^e}]} \subseteq ((\mathbf{a}^r)^{[\frac{1}{p^e}]} : \mathbf{a}).$$

This says that,
$$\mathbf{a} \cdot (\mathbf{a}^{r-p^e})^{[\frac{1}{p^e}]} \subseteq (\mathbf{a}^r)^{[\frac{1}{p^e}]}.$$

For the reverse inclusion, suppose that $\mathbf{a} = (f_1, ..., f_m)$. We have $r = \lceil cp^e \rceil \geq mp^e$, then $r-1 \geq m(p^e-1)$. It is true that $\mathbf{a}^{r-p^e} \cdot \mathbf{a}^{[p^e]} \subseteq \mathbf{a}^r$ from the first proof. On the other hand, $\mathbf{a}^r = \{f_1^{n_1} \cdots f_m^{n_m} | \, n_1 + ... + n_m = r \geq m(p^e-1)+1\}$ and $\mathbf{a}^{[p^e]} = (f_1^{p^e}, ..., f_m^{p^e})$. Show that $\mathbf{a}^{r-p^e} \cdot \mathbf{a}^{[p^e]} \supseteq \mathbf{a}^r$ it is sufficient to prove that there exists $i \in \{1, 2, ..., m\}$ such that $n_i \geq p^e$.

Suppose $n_i \leq p^e - 1$ for all i, then
$$m(p^e-1)+1 \leq n_1 + ... + n_m < m(p^e-1).$$

It is equivalent to
$$m(p^e-1)+1 \leq m(p^e-1) \Rightarrow 1 \leq 0.$$

Then it is a contradiction. It follows that $f_1^{n_1} \cdots f_m^{n_m} = f_1^{n_1} \cdots f_i^{n_i - p^e} \cdots f_m^{n_m} \cdot f_i^{p^e} \in \mathbf{a}^{[p^e]} \cdot \mathbf{a}^{r-p^e}$. Now we have, $\mathbf{a}^r \subseteq \mathbf{a}^{[p^e]} \cdot \mathbf{a}^{r-p^e}$. Observe that,
$$\mathbf{a}^r \subseteq \mathbf{a}^{[p^e]} \cdot \mathbf{a}^{r-p^e} \subseteq \mathbf{a}^{[p^e]} \cdot ((\mathbf{a}^{r-p^e})^{[\frac{1}{p^e}]})^{[p^e]} = (\mathbf{a} \cdot (\mathbf{a}^{r-p^e})^{[\frac{1}{p^e}]})^{[p^e]}$$

Hence, $(\mathbf{a}^r)^{[\frac{1}{p^e}]} \subseteq \mathbf{a} \cdot (\mathbf{a}^{r-p^e})^{[\frac{1}{p^e}]}$, which completes the proof. □

Lemma 3.1.13. $\tau(\boldsymbol{a}^c) = \tau(\overline{\boldsymbol{a}}^c)$ *for every c that is a positive real number.*

Proof. Since $\mathbf{a} \subseteq \overline{\mathbf{a}}$ then $\tau(\mathbf{a}^c) \subseteq \tau(\overline{\mathbf{a}}^c)$. For reverse inclusion, we know that there exists m a positive integer such that $\overline{\mathbf{a}}^{m+\ell} \subseteq \mathbf{a}^\ell$ for every ℓ. By Corollary 3.1.11, we have $\tau(\mathbf{a}^c) = \tau(\mathbf{a}^{c'})$ and $\tau(\overline{\mathbf{a}}^c) = \tau(\overline{\mathbf{a}}^{c'})$ for every $c' \in [c, c+\epsilon)$. We see that
$$\tau(\overline{\mathbf{a}}^c) = \tau(\overline{\mathbf{a}}^{c'}) = \tau((\overline{\mathbf{a}}^{m+\ell})^{\frac{c'}{m+\ell}}) \subseteq \tau((\mathbf{a}^\ell)^{\frac{c'}{m+\ell}}) \subseteq \tau(\mathbf{a}^c),$$

for ℓ big enough such that $c < \frac{\ell c'}{m+\ell}$. This is, $\tau(\overline{\mathbf{a}}^c) \subseteq \tau(\mathbf{a}^c)$. □

3.1.4 F-jumping exponent and F-thresholds

Definition 3.1.14. A positive real number c is a $F-jumping\ \ exponent$ for ideal $\mathbf{a}$ if $\tau(\mathbf{a}^c) \neq \tau(\mathbf{a}^{c-\epsilon})$ for every positive ϵ.

Definition 3.1.15. Let $\mathbf{a}$ be an ideal of R. For a fixed ideal J in R such that $\mathbf{a} \subseteq \sqrt{J}$ and for an integer $e > 0$, we define $\nu_{\mathbf{a}}^J(p^e)$ to be the largest nonnegative integer r such that $\mathbf{a}^r \nsubseteq J^{[p^e]}$.

Claim 3.1.16. *If $\boldsymbol{a}^r \nsubseteq J^{[p^e]}$, then $\boldsymbol{a}^{rp} \nsubseteq J^{[p^{e+1}]}$.*

Proof. We suppose that there exists p which satisfies $\mathbf{a}^{rp} \subseteq J^{[p^{e+1}]} = (J^{[p^e]})^{[p]}$. Now we have $\mathbf{a}^{rp} = (\mathbf{a}^r)^p \subseteq (J^{[p^e]})^p$. Recall Lemma 3.1.3, for any $u \in \mathbf{a}^r$, we observe that $u^p \in (J^{[p^e]})^p$ then $u \in J^{[p^e]}$. Hence, $\mathbf{a}^r \subseteq J^{[p^e]}$ is a contradiction. □

Using Definition 3.1.15, we have some properties of $\nu_{\mathbf{a}}^J(p^e)$ as follows:

(1) $\mathbf{a}^{\nu_{\mathbf{a}}^J(p^e)+1} \subseteq J^{[p^e]}$.

(2) $\nu_{\mathbf{a}}^J(p^e) \leq \nu_{\mathbf{a}}^J(p^{e+1})$ and $p \cdot \nu_{\mathbf{a}}^J(p^e) \leq \nu_{\mathbf{a}}^J(p^{e+1})$. Because $\mathbf{a}^{\nu_{\mathbf{a}}^J(p^e)} \nsubseteq J^{[p^e]}$ by Claim 3.1.16, we obtain $\mathbf{a}^{p\nu_{\mathbf{a}}^J(p^e)} \nsubseteq J^{[p^{e+1}]}$.

(3) $\nu_{\mathbf{a}}^{J^{[p]}}(p^e) = \nu_{\mathbf{a}}^J(p^{e+1})$.

Now we have

$$\frac{\nu_{\mathbf{a}}^J(p^e)}{p^e} = \frac{p \cdot \nu_{\mathbf{a}}^J(p^e)}{p^{e+1}} \leq \frac{\nu_{\mathbf{a}}^J(p^{e+1})}{p^{e+1}}.$$

The sequence $\{\nu_{\mathbf{a}}^J(p^e)\}_{e\in\mathbb{N}}$ is increasing. Note that if $\mathbf{a}$ is generated by m elements, then from the proof of Proposition 3.1.12, we have $\mathbf{a}^{m(p^e-1)+1} \subseteq \mathbf{a}^{[p^e]}$. Since $\mathbf{a} \subseteq \sqrt{J}$, there exists ℓ a positive integer such that $\mathbf{a}^\ell \subseteq J$. We have $\mathbf{a}^{\ell(m(p^e-1)+1)} \subseteq \mathbf{a}^{\ell[p^e]} \subseteq J^{[p^e]}$. From Definition 3.1.15, we observe that $\nu_{\mathbf{a}}^J(p^e) \leq \ell(m(p^e-1)+1) - 1$ for every e. Now we let e approach infinity, then the value of the sequence is closed to ℓm. We define F-Threshold of $\mathbf{a}$ with respect to J as

$$c^J(\mathbf{a}) = \lim_{e\to\infty} \frac{\nu_{\mathbf{a}}^J(p^e)}{p^e} = \sup_{e\in\mathbb{N}} \frac{\nu_{\mathbf{a}}^J(p^e)}{p^e}.$$

Proposition 3.1.17. *Let $\boldsymbol{a}$ be an ideal in R.*

(1) If J is an ideal in R such that $\boldsymbol{a} \subseteq \sqrt{J}$, then

$$\tau(\boldsymbol{a}^{c^J(\boldsymbol{a})}) \subseteq J.$$

(2) If c is a nonnegative real number, then $\boldsymbol{a} \subseteq \sqrt{\tau(\boldsymbol{a}^c)}$ and

$$c^{\tau(\boldsymbol{a}^c)}(\boldsymbol{a}) \leq c.$$

Proof.

(1) By Corollary 3.1.11, there exists $\epsilon > 0$ and $c' \in [c^J(\mathbf{a}), c^J(\mathbf{a}) + \epsilon)$ we have $\tau(\mathbf{a}^{c^J(\mathbf{a})}) = \tau(\mathbf{a}^{c'})$. We take $e >> 0$, such that

$$\tau(\mathbf{a}^{c'}) = (\mathbf{a}^{\lceil c'p^e \rceil})^{[\frac{1}{p^e}]}.$$

Since $c' > c^J(\mathbf{a}) \geq \frac{\nu_{\mathbf{a}}^J(p^e)}{p^e}$ for all e, $\lceil c'p^e \rceil \geq \nu_{\mathbf{a}}^J(p^e) + 1$. We obtain

$$\mathbf{a}^{\lceil c'p^e \rceil} \subseteq \mathbf{a}^{\nu_{\mathbf{a}}^J(p^e)+1} \subseteq J^{[p^e]}.$$

Then, $(\mathbf{a}^{\lceil c'p^e \rceil})^{[\frac{1}{p^e}]} \subseteq J$ which completes the proof.

(2) Since $\tau(\mathbf{a}^c) = (\mathbf{a}^{\lceil cp^e \rceil})^{[\frac{1}{p^e}]}$ whenever e is large enough, then by Definition 3.1.1 we have $\mathbf{a}^{\lceil cp^e \rceil} \subseteq (\tau(\mathbf{a}^c))^{[p^e]} \subseteq \tau(\mathbf{a}^c)$. Then $\mathbf{a} \subseteq \sqrt{\tau(\mathbf{a}^c)}$.

We now show another claim, by the the property (i) of Lemma 3.1.4 we observe that

$$\mathbf{a}^{\lceil cp^e \rceil} \subseteq ((\mathbf{a}^{\lceil cp^e \rceil})^{[\frac{1}{p^e}]})^{[p^e]} = \tau(\mathbf{a}^c).$$

This means that, $\nu_{\mathbf{a}}^J(p^e) \leq \lceil cp^e \rceil - 1 < cp^e$. By the definition of F-threshold then

$$c^J(\mathbf{a}) = \sup_{e \in \mathbb{N}} \frac{\nu_{\mathbf{a}}^J(p^e)}{p^e} \leq c.$$

□

Corollary 3.1.18. *For every ideal* $\mathbf{a}$ *in* R*, the set of* F-jumping exponents *for* $\mathbf{a}$ *is equal to the set of* F- thresholds *of* $\mathbf{a}$ *(as we range over all possible ideals* J*).*

Proof. Let

$$A = \{\alpha |\ \alpha \text{ is a } F\text{-jumping exponent for } \mathbf{a}\}$$

and

$$B = \{c^J(\mathbf{a}) | J \subseteq R \ \text{ and } \ \mathbf{a} \subseteq \sqrt{J}\}.$$

We now show that $A = B$ by double containment.

We first show that $A \subseteq B$. Take $c \in A$, then $\tau(\mathbf{a}^c) \neq \tau(\mathbf{a}^{c-\epsilon})$ for every $\epsilon > 0$. Let $J = \tau(\mathbf{a}^c)$ then by (ii) of Proposition 3.1.17, $\mathbf{a} \subseteq \sqrt{J}$ and $c^J(\mathbf{a}) \leq c$. So we have

$$\tau(\mathbf{a}^c) \subseteq \tau(\mathbf{a}^{c^J(\mathbf{a})}) \subseteq J = \tau(\mathbf{a}^c).$$

By F-jumping, we get $c = c^J(\mathbf{a})$, that is $c \in B$.

We now show that $B \subseteq A$. Let $\alpha = c^J(\mathbf{a}) \in B$ is an F-threshold for an ideal J of R such that $\mathbf{a} \subseteq \sqrt{J}$. Suppose there exists $\alpha \notin A$, then there exists $\alpha' < \alpha$ with $\tau(\mathbf{a}^{\alpha'}) = \tau(\mathbf{a}^{\alpha}) \subseteq J$. If e is large enough, we have

$$\tau(\mathbf{a}^{\alpha'}) = (\mathbf{a}^{\lceil \alpha' p^e \rceil})^{[\frac{1}{p^e}]} \subseteq J.$$

It means that,

$$\mathbf{a}^{\lceil \alpha' p^e \rceil} \subseteq J^{[p^e]}.$$

By Definition 3.1.15, we have

$$\nu_{\mathbf{a}}^{J}(p^e) \leq \lceil \alpha' p^e \rceil - 1 < \alpha' p^e.$$

From the definition of F-threshold, we obtain that

$$\alpha = c^J(\mathbf{a}) = \sup_{e \in \mathbb{N}} \frac{\nu_{\mathbf{a}}^{J}(p^e)}{p^e} \leq \alpha',$$

which is a contradiction. Hence, $\alpha \in A$. □

3.2 Discreteness and rationality

Proposition 3.2.1. *Let $\boldsymbol{a}$ be an ideal in $R = \mathbb{K}[x_1, ..., x_n]$, where $\mathbb{K}$ is a field of characteristic p such that $[\mathbb{K} : \mathbb{K}^p] < \infty$. If $\boldsymbol{a}$ can be generated by polynomials of degree at most d then, for every non-negative real number c, the ideal $\tau(\boldsymbol{a}^c)$ can be generated by polynomials of degree at most $\lfloor cd \rfloor$.*

Proof. We note that when e is large enough, then $\tau(\mathbf{a}^c) = (\mathbf{a}^{\lceil cp^e \rceil})^{[\frac{1}{p^e}]}$. Let $r = \lceil cp^e \rceil$. Suppose that $\mathbf{a} = (f_1, ..., f_m| \quad \deg(f_i) \leq d, \quad \forall i = 1, 2, ..., m)$, then $\mathbf{a}^r = (f_1^r, ..., f_m^r| \quad \deg(f_i) \leq d, \quad \forall i = 1, 2, ..., m) = (h_1, ..., h_m)$ where $deg(h_i)$ at most rd. Since $[\mathbb{K} : \mathbb{K}^p] < \infty$, we let $b_1, ..., b_s$ to be a basic of $\mathbb{K}$ over $\mathbb{K}^{p^e}$. The polynomial R can be expressed by

$$R = \sum_{\alpha_l \geq 0} (\sum_{1 \leq i \leq s} \mathbb{K}^{p^e} b_i) x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

We consider R over R^{p^e}, then

$$R = \sum_{\substack{0 \leq i \leq s \\ 0 < \alpha_j < p^e - 1}} R^{p^e} \cdot b_i \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Thus, the basic of R over R^{p^e} is

$$\{b_i x_1^{\alpha_1} \cdots x_n^{\alpha_n} | 1 \leq i \leq s, \quad 0 \leq \alpha_j \leq p^e - 1\}.$$

The generators of $\mathbf{a}^r$ can be expressed as

$$h_l = \sum_{\substack{0 \leq i \leq s \\ 0 < \alpha_j < p^e - 1}} c_{i,l,\alpha_1,...,\alpha_n}^{p^e} \cdot b_i \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n} \qquad (1),$$

for all $l = 1, 2, ..., m$, with $c_{i,l,\alpha_1,...,\alpha_n} \in R$. By Proposition 3.1.5, we have

$$(\mathbf{a}^r)^{[\frac{1}{p^e}]} = (c_{i,l,\alpha_1,...,\alpha_n} | 1 \leq s, 0 \leq \alpha_j \leq p^e - 1, \forall j, l = 1, 2, ..., m).$$

From (1), we see that

$$\deg(c_{i,l,\alpha_1,...,\alpha_n}^{p^e}) \leq deg(h_l) \leq rd.$$

Then
$$\deg\left(c_{i,l,\alpha_1,\ldots,\alpha_n}\right) \leq \frac{rd}{p^e} = \frac{\lceil cp^e \rceil d}{p^e}.$$
If cp^e is an integer, then
$$\deg\left(c_{i,l,\alpha_1,\ldots,\alpha_n}\right) \leq \frac{rd}{p^e} = \frac{cp^e d}{p^e} = cd.$$
In this case, we have $\deg\left(c_{i,l,\alpha_1,\ldots,\alpha_n}\right) \leq \lfloor cd \rfloor$. If cp^e is not an integer, then we obtain that
$$\deg\left(c_{i,l,\alpha_1,\ldots,\alpha_n}\right) \leq \left\lfloor \frac{rd}{p^e} \right\rfloor = \left\lfloor \frac{\lceil cp^e \rceil d}{p^e} \right\rfloor.$$
If e is large enough, $\tau(\mathbf{a}^c) = (\mathbf{a}^{\lceil cp^e \rceil})^{[\frac{1}{p^e}]}$, where $c < \frac{\lceil cp^e \rceil}{p^e}$ and $\frac{\lceil cp^e \rceil}{p^e}$ is very close to c. Hence, $\lfloor cd \rfloor = \left\lfloor \frac{\lceil cp^e \rceil d}{p^e} \right\rfloor$, which is a contradiction. □

Proposition 3.2.2. *Let **a** be an ideal in a regular F-finite ring R.*

*(1) If α is an F-jumping exponent for **a**, then also $p\alpha$ is an F-jumping exponent.*

*(2) If **a** can be generated by m elements and if $\alpha > m$ is an F-jumping exponent for **a**, then also $\alpha - 1$ is an F-jumping exponent.*

Proof.

(1) By Corollary 3.1.18 , there is an ideal J such that $\mathbf{a} \subseteq Rad(J)$ with $\alpha = c^J(\mathbf{a})$.
$$p\alpha = pc^J(\mathbf{a}) = \lim_{e\to\infty} \frac{p\nu_{\mathbf{a}}^J(p^e)}{p^e} = \lim_{e\to\infty} \frac{\nu_{\mathbf{a}}^{J^{[p]}}(p^e)}{p^e} = \lim_{e\to\infty} \frac{\nu_{\mathbf{a}}^J(p^{e+1})}{p^e} = c^{J^{[p]}}(\mathbf{a}).$$
By Corollary 3.1.18, $p\alpha$ is an F-jumping exponent for **a**.

(2) Suppose that $\alpha - 1$ is not an F-jumping exponent for **a**. Then there exists $\epsilon > 0$ such that $\tau(\mathbf{a}^{\alpha-1}) = \tau(\mathbf{a}^{\alpha-1-\epsilon})$ and $\alpha - \epsilon > m$. By Proposition 3.1.12,
$$\tau(\mathbf{a}^{\alpha}) = \mathbf{a}\cdot\tau(\mathbf{a}^{\alpha-1}) = \mathbf{a}\cdot\tau(\mathbf{a}^{\alpha-1-\epsilon}) = \tau(\mathbf{a}^{\alpha-\epsilon}).$$
This contradiction completes this proof.

□

Theorem 3.2.3. *Let k be a field of characteristic $p > 0$ and let R be a regular F-finite ring, essentially of finite type over k. Suppose that **a** is an ideal in R.*

*(1) The set of F-jumping exponents of **a** is discrete (i.e. in every finite interval there are only finitely many such numbers).*

*(2) Every F-jumping exponents of **a** is a rational number.*

Proof.

(1) Let A be the set of F-jumping exponent of $\mathbf{a}$. To prove that A is discrete it just shows that A has no accumulation point. Suppose that there exists a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ converges to $\alpha \in A$. Recall Corollary 3.1.11, there exists $\epsilon > 0$ such that $\tau(\mathbf{a}^\alpha) = \tau(\mathbf{a}^{\alpha'})$ for all $\alpha' \in [\alpha, \alpha+\epsilon)$. Then, $\alpha_n < \alpha$ for all $n \in \mathbb{N}$. Now we reduce this sequence to its subsequence $\{\alpha_m\}_{m\in\mathbb{N}}$ such that $\alpha_m < \alpha_{m+1} < \alpha$. Assume that $\mathbf{a}$ is finitely generated by polynomials with degree at most d. By the Proposition 3.2.1, we obtain that $\tau(\mathbf{a}^{\alpha_m})$ is finitely generated by polynomials with degree at most $\lfloor \alpha_m d \rfloor < \lfloor \alpha d \rfloor$. If e is large enough, we observe that $\{\tau(\mathbf{a}^{\alpha_m})\}_{m\in\mathbb{N}}$ is a strictly decreasing sequence. We take $V = \{f \in R | deg(f) \leq \lfloor \alpha d \rfloor\}$; it is a vector subspace of R over k. Let $V_m = \tau(\mathbf{a}^{\alpha_m}) \cap V$. It is a vector subspace of R and V_m with finite dimension. We now have a sequence $V_m \supseteq V_{m-1} \supseteq \cdots$. By strictly decreasing of vector subspace, then the dimension vector subspace is strictly decreasing, this is, $dimV_m \geq dimV_{m-1} \geq ....$ Let e approach infinity, then the dimension of vector subspace goes to negative. This is a contradiction.

(2) Let $\alpha \in A$. Then by Proposition 3.2.2 (1) we have $p^e\alpha \in A$. If $p^e\alpha \in \mathbb{N}$ then $\alpha \in \mathbb{Q}$. If $p^e\alpha \notin \mathbb{N}$. Suppose that $\mathbf{a}$ is generated by m polynomials, there are some $e_1 >> 0$ such that $p^{e_1}\alpha > m$. Then by Proposition 3.2.2 (2), we obtain that $p^{e_1}\alpha - 1$ is an F-jumping exponent for $\mathbf{a}$ and $p^{e_1}\alpha - 1 > m-1$. In this case, there exists $\ell \geq 1$ is a positive integer such that $p^e\alpha - \ell + 1 > m$ is an F-jumping exponent and $p^{e_1}\alpha - \ell_1 \in [m-1, m)$. By Proposition 3.2.2 (2), $p^{e_1}\alpha - \ell_1$ is also an F-jumping exponent. Since $e \in \mathbb{N}$, $e >> 0$ and from the previous (i), there are finitely many F-jumping exponents in $[m-1, m)$. Then there exist ℓ_2 and ℓ_1 are integers such that $p^{e_1}\alpha - \ell_1 = p^{e_2}\alpha - \ell_2 \in [m-1, m)$. We now obtain that $\alpha = \frac{\ell_2 - \ell_1}{p^{e_2} - p^{e_1}}$.

Hence, $\alpha \in \mathbb{Q}^+$. These complete the proof of this theorem.

□

Corollary 3.2.4. *Let $\mathbf{a}$ be an ideal of R. Then for every F-threshold element of $\mathbf{a}$ is a rational number at any ideal J such that $\mathbf{a} \subseteq \sqrt{J}$, this is, $c^J(\mathbf{a}) \in \mathbb{Q}^+$.*

Proof. This proof follows from Corollary 3.1.18 and Theorem 3.2.3. □

Proposition 3.2.5. *Let $\mathbf{a}$ be an ideal of R which is generated by m polynomials of degree at most d. If e_0 is such that $p^{e_0} > md$ and $N = \binom{md+n}{n}$, then for every F-jumping exponent α of $\mathbf{a}$ we have $p^r(p^s-1)\alpha \in \mathbb{N}$ for some $r \leq e_0 + N$ and $s \leq N$.*

Proof. Take α as an F-jumping exponent of $\mathbf{a}$. If $\alpha < \frac{1}{d} < m$, then $\tau(\mathbf{a}^\alpha)$ is generated by polynomials of degree at most $\lfloor \alpha d \rfloor$. Since $\alpha d < 1$, we obtain $\lfloor \alpha d \rfloor = 0$. Hence, $\tau(\mathbf{a}^\alpha) = (c_1, ..., c_n) = (1) = R$ or $\tau(\mathbf{a}^\alpha) = 0$ because $c_i \in k$. In this case, we take $0 < \epsilon < \frac{1}{d}$, then $\tau(\mathbf{a}^\alpha) = \tau(\mathbf{a}^{\alpha-\epsilon})$; it is a contradiction. Now we have $\alpha \geq \frac{1}{d}$, then $p^e\alpha \geq p^{e_0}\frac{1}{d} > m$ for all $e \in [e_0, e_0 + N]$. By the proof of Theorem 3.2.3 (2), we have $p^{e_1}\alpha - \ell_1 = p^{e_2}\alpha - \ell_2$ for $e_0 \leq e_1 < e_2 \leq e_0 + N$. Then $p^{e_1}(p^{e_2-e_1} - 1)\alpha = \ell_2 - \ell_1 \in \mathbb{N}$. This completes the proof. □

Example 3.2.6. Given $R = \mathbb{K}[x_1, ..., x_n]$, where $\mathbb{K}$ is a field with a prime characteristic p. Show that

$$\tau(\mathbf{m}^\lambda) = \begin{cases} R & \text{if} \quad \lambda < n \\ \mathbf{m}^{\lfloor \lambda \rfloor - n + 1} & \text{if} \quad \lambda \geq n \end{cases},$$

where $\mathbf{m} = (x_1, ..., x_n)$.

Proof. We note that $\tau(\mathbf{m}^\lambda) = \cup_{e>0}(\mathbf{m}^{\lceil \lambda p^e \rceil})^{[\frac{1}{p^e}]} = (\mathbf{m}^{\lceil \lambda p^e \rceil})^{[\frac{1}{p^e}]}$, when $e >> 0$.

- *Case* $\lambda < n$: Take $\lambda_e = n(1 - \frac{1}{p^e}) < n$, for $e \in \mathbb{N}$. Then we have $\tau(\mathbf{m}^\lambda) = (\mathbf{m}^{n(1-\frac{1}{n})p^e})^{[\frac{1}{p^e}]} = (\mathbf{m}^{np^e - n})^{[\frac{1}{p^e}]}$. We realize that

$$\mathbf{m}^{np^e - n} = (x_1^{\alpha_1} \cdots x_n^{\alpha_n} | \alpha_1 + \cdots + \alpha_n = n(p^e - 1)).$$

  Since $\mathbf{m}^{np^e-n} \subseteq R$ and R is R^{p^e}-free with a basis $B = \{x_1^{\beta_1} \cdots x_n^{\beta_n} | 0 \leq \beta_i \leq p^e - 1, \quad \textit{for} \quad 1 \leq i \leq n\}$, then we observe that when $\alpha_i = p^e - 1$ for all $i = 1, ..., n$, $x_1^{p^e-1} \cdots x_n^{p^e-1}$ is an element of B. By Proposition 3.1.5, we have $1 \in (\mathbf{m}^{np^e-n})^{[\frac{1}{p^e}]}$. Hence, $\tau(\mathbf{m}^{\lambda_e}) = R$. We notice that for all $\lambda < n$, there exists $e \in \mathbb{N}$ such that $\lambda < \lambda_e < n$. Then it follows that $\tau(\mathbf{m}^{\lambda_e}) \subseteq \tau(\mathbf{m}^\lambda)$ by Proposition 3.1.8 (i). Thus, $\tau(\mathbf{m}^\lambda) = R$.

- *Case* $\lambda = n$: We obtain $\tau(\mathbf{m}^\lambda) = (\mathbf{m}^{np^e})^{[\frac{1}{p^e}]}$, and $p^e \geq n$ for $e >> 0$. It is similar to the first case

$$\mathbf{m}^{np^e} = (x_1^{\alpha_1} \cdots x_n^{\alpha_n} | \alpha_1 + \cdots + \alpha_n = np^e).$$

  If $\alpha_{i_0} = 2p^e - 1$, $\alpha_{i_0+1} = n - 1$ and $\alpha_i = p^e - 1$ for $i \neq i_0, i_0 + 1$. Then we have

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} = x_{i_0}^{2p^e-1} \Pi_{i \neq i_0, i_0+1} x_i^{p^e-1} \cdot x_{i_0+1}^{n-1} = x_{i_0}^{p^e-1} \Pi_{i \neq i_0+1} x_i^{p^e-1} \cdot x_{i_0+1}^{n-1}.$$

  Since $\Pi_{i \neq i_0+1} x_i^{p^e-1} \cdot x_{i_0+1}^{n-1} \in B$, then $x_{i_0} \in (\mathbf{m}^{np^e})^{[\frac{1}{p^e}]}$. We take $i_{i_0} \in \{1, 2, ..., n\}$, it follows that $\mathbf{m} = (x_1, ..., x_n) \subseteq (\mathbf{m}^{np^e})^{[\frac{1}{p^e}]}$. On the other hand, $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ where $\alpha_1 + \cdots + \alpha_n = np^e$ there exists $\alpha_i \geq p^e$. It concludes that $\mathbf{m}^{np^e} \subseteq \mathbf{m}^{[p^e]}$. Then $(\mathbf{m}^{np^e})^{[\frac{1}{p^e}]} \subseteq \mathbf{m}$. Hence, it holds for this case.

- *Case* $\lambda > n$: Let $k \in \mathbb{N}$ such that $n + k - 1 < \lambda \leq n + k$. We proceed by induction on k. Suppose that $k = 0$. Then $n < \lambda \leq n + 1$, and $\tau(\mathbf{m}^\lambda) = \mathbf{m} \cdot \tau(\mathbf{m}^{\lambda-1})$. By the second case $\tau(\mathbf{m}^{\lambda-1}) = R$. We obtain $\tau(\mathbf{m}^\lambda) = \mathbf{m}R = \mathbf{m} = \mathbf{m}^{\lfloor \lambda \rfloor - n + 1}$. We now assume our claim for k and prove it is true for $k + 1$. By Skoda's theorem, we observe that

$$\begin{aligned}(\mathbf{m}^\lambda) &= \mathbf{m} \cdot \tau(\mathbf{m}^{\lambda-1}) \\ &= \mathbf{m} \cdot \mathbf{m}^k \quad \text{, by hypothesis of induction} \\ &= \mathbf{m}^{k+1} \\ &= \mathbf{m}^{\lfloor \lambda \rfloor - n + 1}.\end{aligned}$$

Therefore, it completes the proof. ∎

Chapter 4

F-thresholds for all Noetherian rings

In the previous sections, we have discussed the *F-thresholds* of an ideal in a polynomial ring with prime characteristic p. Now in this section, we present the *F- threshold* of an ideal for all Noetherian rings with prime characteristic p. In particular, for a singular ring. Part of this work is based on a recent paper by De Stefani, Núñez-Betancourt, and Pérez [DSNBP]. We also generalize some properties from the paper by Mustat, Takagi, and Watanable [MTW05].

4.1 F-threshold of an ideal

Definition 4.1.1. Let R be a ring of prime characteristic p. Let $\mathbf{a}$ and J be ideals of R such that $\mathbf{a} \subseteq \sqrt{J}$, we define

$$\nu_{\mathbf{a}}^{J}(p^e) = \max\{r \in \mathbb{N} |\ \mathbf{a}^r \nsubseteq J^{[P^e]}\}.$$

Notation 4.1.2. If $\mathbf{a}$ is an ideal of a ring R, we define $\mu(\mathbf{a})$ as the minimum number of generators of $\mathbf{a}$.

Lemma 4.1.3. *Let R be a Noetherian ring with a prime characteristic p and $\boldsymbol{a}$ be an ideal. Then for every $r \geq (\mu(\boldsymbol{a}) + s - 1)p^e$, we have that $\boldsymbol{a}^r = \boldsymbol{a}^{r-sp^e} \cdot (\boldsymbol{a}^{[p^e]})^s$.*

Proof. We proceed by induction. Let $u = \mu(\mathbf{a})$ and let $\mathbf{a} = (f_1, ..., f_u)$. We note that $\mathbf{a}^r = (f_1^{\alpha_1} \cdot f_2^{\alpha_2} \cdots f_u^{\alpha_u} |\ \alpha_1 + ... + \alpha_u = r)$. If $s = 1$, then $r \geq up^e$. Similar to the proof in Proposition 3.1.12, there exists $\alpha_i \geq p^e$ for some $i \in \{1, 2, ..., u\}$. This follows that

$$f_1^{\alpha_1} \cdot f_2^{\alpha_2} \cdots f_u^{\alpha_u} = f_1^{\alpha_1} \cdot f_2^{\alpha_2} \cdots f_i^{\alpha_i - p^e} \cdots f_u^{\alpha_u} \cdot f_i^{p^e} \in \mathbf{a}^{r-p^e} \cdot \mathbf{a}^{[p^e]}.$$

Hence, $\mathbf{a}^r \subseteq \mathbf{a}^{r-p^e} \cdot \mathbf{a}^{[p^e]}$ and we already have that $\mathbf{a}^r = \mathbf{a}^{r-p^e} \cdot \mathbf{a}^{p^e} \supseteq \mathbf{a}^{r-p^e} \cdot \mathbf{a}^{[p^e]}$. Suppose that it is true for $s \geq 1$. We now prove that it is true for $s+1$. In this case, we have $r \geq (u + (s+1) - 1)p^e = (u+s)p^e$. Then,

$$\begin{aligned}
\mathbf{a}^{r-(s+1)p^e} \cdot (\mathbf{a}^{[p^e]})^{s+1} &= \mathbf{a}^{(r-sp^e)-p^e} \cdot (\mathbf{a}^{[p^e]})^s \cdot \mathbf{a}^{[p^e]} \\
&= \mathbf{a}^{(r-sp^e)-p^e} \cdot \mathbf{a}^{[p^e]} \cdot (\mathbf{a}^{[p^e]})^s \\
&= \mathbf{a}^{r-sp^e} \cdot (\mathbf{a}^{[p^e]})^s = \mathbf{a}^r, \text{by the induction hypothesis.}
\end{aligned}$$

□

Lemma 4.1.4. *Let R be a Noetherian ring of prime characteristic p. Let $\boldsymbol{a}, J \subseteq R$ be ideals such that $\boldsymbol{a} \subseteq \sqrt{J}$. Then*

$$\frac{\nu_{\boldsymbol{a}}^J(p^{e_1+e_2})}{p^{e_1+e_2}} - \frac{\nu_{\boldsymbol{a}}^J(p^{e_1})}{p^{e_1}} \leq \frac{\mu(\boldsymbol{a})}{p^{e_1}},$$

for every $e_1, e_2 \in \mathbb{N}$.

Proof. It is sufficient to prove that $\nu_{\mathbf{a}}^J(p^{e_1+e_2}) \leq p^{e_2} \cdot \nu_{\mathbf{a}}^J(p^{e_1}) + p^{e_2} \cdot \mu(\mathbf{a})$. We apply Lemma 4.1.3 by taking $r = p^{e_2} \cdot \nu_{\mathbf{a}}^J(p^{e_1}) + p^{e_2} \cdot \mu(\mathbf{a}) + 1$ and $s = \nu_{\mathbf{a}}^J(p^{e_1})$, then

$$\mathbf{a}^{p^{e_2} \cdot \nu_{\mathbf{a}}^J(p^{e_1}) + p^{e_2} \cdot \mu(\mathbf{a}) + 1} = \mathbf{a}^{p^{e_2} \cdot \mu(\mathbf{a}) + 1} \cdot (\mathbf{a}^{[p^{e_2}]})^{\nu_{\mathbf{a}}^J(p^{e_1})}.$$

We now prove that $\mathbf{a}^{p^{e_2} \cdot \mu(\mathbf{a}) + 1} \subseteq \mathbf{a}^{[p^{e_2}]}$. Suppose that it fails. Then for any $f_1^{\alpha_1} \cdot f_2^{\alpha_2} \cdots f_{\mu(\mathbf{a})}^{\alpha_u}$ such that $\alpha_1 + \alpha_2 + ... + \alpha_{\mu(\mathbf{a})} = p^{e_2} \cdot \mu(\mathbf{a}) + 1$, we obtain $\alpha_i < p^{e_2}$ for all $i = 1, 2, ..., \mu(\mathbf{a})$. Hence, $\alpha_1 + \alpha_2 + ... + \alpha_{\mu(\mathbf{a})} < \mu(\mathbf{a}) p^{e_2}$. It is a contradiction. For now we have

$$\begin{aligned} \mathbf{a}^{p^{e_2} \cdot \nu_{\mathbf{a}}^J(p^{e_1}) + p^{e_2} \cdot \mu(\mathbf{a}) + 1} &\subseteq \mathbf{a}^{[p^{e_2}]} \cdot (\mathbf{a}^{[p_2^e]})^{\nu_{\mathbf{a}}^J(p^{e_1})} \\ &= (\mathbf{a}^{[p^{e_2}]})^{\nu_{\mathbf{a}}^J(p^{e_1}) + 1} \\ &= (\mathbf{a}^{\nu_{\mathbf{a}}^J(p^{e_1}) + 1})^{[p^{e_2}]} \in (J^{[p^{e_1}]})^{[p^{e_2}]} = J^{[p^{e_1+e_2}]}. \end{aligned}$$

Therefore, $\nu_{\mathbf{a}}^J(p^{e_1+e_2}) \leq p^{e_2} \cdot \nu_{\mathbf{a}}^J(p^{e_1}) + p^{e_2} \cdot \mu(\mathbf{a})$. It completes the proof. □

Theorem 4.1.5. *Let R be a ring of prime characteristic p. If $\boldsymbol{a}, J \subseteq R$ are ideals such that $\boldsymbol{a} \subseteq \sqrt{J}$, then $\lim_{e \longmapsto \infty} \frac{\nu_{\boldsymbol{a}}^J(p^e)}{p^e}$ exists.*

Proof. From Lemma 4.1.4 we have

$$\frac{\nu_{\mathbf{a}}^J(p^{e_1+e_2})}{p^{e_1+e_2}} \leq \frac{\mu(\mathbf{a})}{p^{e_1}} + \frac{\nu_{\mathbf{a}}^J(p^{e_1})}{p^{e_1}},$$

for every $e_1, e_2 \in \mathbb{N}$. Then,

$$\lim_{e \longmapsto \infty} \sup \frac{\nu_{\mathbf{a}}^J(P^e)}{p^e} = \lim_{e_2 \longmapsto \infty} \sup \frac{\nu_{\mathbf{a}}^J(P^{e_1+e_2})}{p^{e_1+e_2}} \leq \frac{\mu(\mathbf{a})}{p^{e_1}} + \frac{\nu_{\mathbf{a}}^J(p^{e_1})}{p^{e_1}} \quad \text{for all} \quad e_1 \in \mathbb{N}.$$

Hence,

$$\begin{aligned} \lim_{e \longmapsto \infty} \sup \frac{\nu_{\mathbf{a}}^J(p^e)}{p^e} &\leq \lim_{e_1 \longmapsto \infty} \inf \left(\frac{\mu(\mathbf{a})}{p^{e_1}} + \frac{\nu_{\mathbf{a}}^J(p^{e_1})}{p^{e_1}}\right) \\ &= \lim_{e_1 \longmapsto \infty} \inf \frac{\mu(\mathbf{a})}{p^{e_1}} + \lim_{e_1 \longmapsto \infty} \inf \frac{\nu_{\mathbf{a}}^J(p^{e_1})}{p^{e_1}} \\ &= \lim_{e \longmapsto \infty} \inf \frac{\nu_{\mathbf{a}}^J(p^e)}{p^e}. \end{aligned}$$

We include that $\lim_{e \longmapsto \infty} \frac{\nu_{\mathbf{a}}^J(p^e)}{p^e}$ exists. □

After Theorem 4.1.5, we can define an F-threshold in full generality.

Definition 4.1.6. Let R be a ring of prime characteristic p. If $\mathbf{a}, J \subseteq R$ are ideals such that $\mathbf{a} \subseteq \sqrt{J}$. We define the F-threshold of $\mathbf{a}$ with respect to J by

$$c^J(\mathbf{a}) = \lim_{e\to\infty} \frac{\nu_{\mathbf{a}}^J(p^e)}{p^e}.$$

In what follows we fix the ideal $\mathbf{a}$, and study the F-thresholds which appear for various J. We record in the next propositions for some properties which deal with the variation J.

Proposition 4.1.7. *Let R be a ring of prime characteristic p. Let $\boldsymbol{a}, I, J$ be ideals of R. Then*

(1) If $J \subseteq I$, and $\boldsymbol{a} \subseteq \sqrt{J}$, then $c^I(\boldsymbol{a}) \leq c^J(\boldsymbol{a})$.

(2) If $\boldsymbol{a} \subseteq \sqrt{J}$, then $c^{J^{[p]}}(\boldsymbol{a}) = p \cdot c^J(\boldsymbol{a})$.

(3) If $\boldsymbol{a} = (f_1, ..., f_d)$, then $c^{\boldsymbol{a}}(\boldsymbol{a}) \leq d$.

Proof.

(1) We observe that $\mathbf{a}^{\nu_{\mathbf{a}}^I(p^e)} \nsubseteq I^{[p^e]}$. Since $J \subseteq I$, then $\mathbf{a}^{\nu_{\mathbf{a}}^I(p^e)} \nsubseteq J^{[p^e]}$. Hence, $\nu_{\mathbf{a}}^I(p^e) \leq \nu_{\mathbf{a}}^J(p^e)$. By Definition 4.1.6, we have $c^I(\mathbf{a}) \subseteq c^J(\mathbf{a})$.

(2) By Definition 4.1.1, we note that

$$\begin{aligned}
\nu_{\mathbf{a}}^{J^{[p]}}(p^e) &= \max\{r|\ \ \mathbf{a}^r \nsubseteq (J^{[p]})^{[p^e]}\} \\
&= \max\{r|\ \ \mathbf{a}^r \nsubseteq J^{[p^{e+1}]}\} \\
&= \nu_{\mathbf{a}}^J(p^{e+1}).
\end{aligned}$$

By Definition 4.1.6, then

$$\begin{aligned}
c^{J^{[p]}}(\mathbf{a}) &= \lim_{e\to\infty} \frac{\nu_{\mathbf{a}}^{J^{[p]}}(p^e)}{p^e} \\
&= \lim_{e\to\infty} \frac{\nu_{\mathbf{a}}^J(p^{e+1})}{p^e} \\
&= \lim_{e\to\infty} p \cdot \frac{\nu_{\mathbf{a}}^J(p^{e+1})}{p^{e+1}} \\
&= p \cdot c^J(\mathbf{a}).
\end{aligned}$$

(3) We observe that

$$\begin{aligned}
\nu_{\mathbf{a}}^{\mathbf{a}}(p^e) &= \max\{r \in \mathbb{N}|\ \ \mathbf{a}^r \nsubseteq \mathbf{a}^{[p^e]}\} \\
&= \max\{r \in \mathbb{N}|\ \ \ (f_1^{\alpha_1} \cdots f_d^{\alpha_d}) \nsubseteq (f_1^{p^e}, ..., f_d^{p^e}),\ \ \alpha_1 + \cdots + \alpha_d = r\}.
\end{aligned}$$

We see that $0 \leq \alpha_i \leq p^e - 1$, then $\nu_{\mathbf{a}}^{\mathbf{a}}(p^e) \leq d(p^e - 1)$. Thus, it follows that $c^{\mathbf{a}}(\mathbf{a}) \leq d$ by definition of an F-threshold.

□

Proposition 4.1.8. *Let R be a Noetherian local ring of prime characteristic p. Let $\boldsymbol{a}$, $\boldsymbol{b}$, J be ideals of R such that $\boldsymbol{a}$, $\boldsymbol{b} \subseteq \sqrt{J}$. Then*

(1) If $\boldsymbol{a} \subseteq \boldsymbol{b}$, then $c^J(\boldsymbol{a}) \leq c^J(\boldsymbol{b})$.

(2) $c^J(\boldsymbol{a}^s) = \frac{c^J(\boldsymbol{a})}{s}$ for every positive integer s.

(3) If $\boldsymbol{a} \subseteq J^s$ and J can be generated by m elements, then $c^J(\boldsymbol{a}) \leq \frac{m}{s}$.

(4) If $\overline{\boldsymbol{a}}$ is the integral closure of $\boldsymbol{a}$, then $c^J(\boldsymbol{a}) = c^J(\overline{a})$.

(5) $c^J(\boldsymbol{a}+\boldsymbol{b}) \leq c^J(\boldsymbol{a}) + c^J(\boldsymbol{b})$.

Proof.

(1) By the previous definition, we have $\mathbf{a}^{\nu^J_{\mathbf{a}}(p^e)} \nsubseteq J^{[p^e]}$. Since $\mathbf{a} \subseteq \mathbf{b}$, then $\mathbf{b}^{\nu^J_{\mathbf{a}}(p^e)} \nsubseteq J^{[p^e]}$. Then $\nu^J_{\mathbf{a}}(p^e) \leq \nu^J_{\mathbf{b}}(p^e)$. By Definition 4.1.6, we obtain $c^J(\mathbf{a}) \leq c^J(\mathbf{b})$.

(2) Given s is a positive integer, we obtain $(\mathbf{a}^s)^{\nu^J_{\mathbf{a}^s}(p^e)} \nsubseteq J^{[p^e]}$. Then $s \cdot \nu^J_{\mathbf{a}^s}(p^e) \leq \nu^J_{\mathbf{a}}(p^e)$. Hence, $\nu^J_{\mathbf{a}^s}(p^e) \leq \frac{\nu^J_{\mathbf{a}}(p^e)}{s}$. On the other hand, we have $\mathbf{a}^{\nu^J_{\mathbf{a}}(p^e)} \nsubseteq J^{[p^e]}$, which means that $\mathbf{a}^{s \cdot \left\lfloor \frac{\nu^J_{\mathbf{a}}(p^e)}{s} \right\rfloor} \nsubseteq J^{[p^e]}$. Hence, $\nu^J_{\mathbf{a}^s}(p^e) \geq \left\lfloor \frac{\nu^J_{\mathbf{a}}(p^e)}{s} \right\rfloor$. By Definition 4.1.6, we have $c^J(\mathbf{a}^s) \geq \frac{c^J(\mathbf{a})}{s}$.

(3) We have $\mathbf{a} \subseteq J^s$, then by (1) and (2) it follows that $c^J(\mathbf{a}) \leq c^{J^s}(p^e) = \frac{c^J(\mathbf{a})}{s}$. For all elements $g_1^{\alpha_1} \cdots g_m^{\alpha_m}$ in $J^{\nu^J_{\mathbf{a}}(p^e)}$ such that $\alpha_1 + ... + \alpha_m = \nu^J_{\mathbf{a}}(p^e)$, we observe that $\alpha_i \leq p^e - 1$ for all $i - 1, 2, ..., m$. Then $\nu^J_{\mathbf{a}}(p^e) \leq m(p^e - 1)$. We now apply Definition 4.1.6 to obtain $c^J(\mathbf{a}) \leq m$. Therefore, $c^J(\mathbf{a}) \leq \frac{m}{s}$.

(4) Since $\mathbf{a} \subseteq \overline{\mathbf{a}}$, then it follows that $c^J(\mathbf{a}) \leq c^J(\overline{a})$, from (1). On the other hand, we note that there exists $m \in \mathbb{N}$ such that $\overline{\mathbf{a}}^{m+\ell} \subseteq \mathbf{a}^\ell$ for any $\ell \in \mathbb{N}$. By (1), we have $c^J(\overline{\mathbf{a}}^{m+\ell}) \leq c^J(\mathbf{a}^\ell)$. Use (2) to obtain that $\frac{c^J(\overline{\mathbf{a}})}{m+\ell} \leq \frac{c^J(\mathbf{a})}{\ell}$ for any $\ell \in \mathbb{N}$. Then $c^J(\overline{\mathbf{a}}) \leq \inf_{\ell \in \mathbb{N}} \frac{m+\ell}{\ell} c^J(\mathbf{a}) = c^J(\mathbf{a})$.

(5) Let $u = \nu^J_{\mathbf{a}}(p^e) + \nu^J_{\mathbf{b}}(p^e) + 1$. Then $(\mathbf{a}+\mathbf{b})^u = \sum_{0 \leq i \leq u} \mathbf{a}^i \mathbf{b}^{u-i}$. There are two cases.

Case 1: if $i \geq \nu^J_{\mathbf{a}}(p^e) + 1$, then $\mathbf{a}^i\mathbf{b}^{u-i} \subseteq \mathbf{a}^i \subseteq J^{[p^e]}$.

Case 2: if $i \leq \nu^J_{\mathbf{a}}(p^e)$, then $u - i \geq \nu^J_{\mathbf{b}}(p^e) + 1$. Hence, $\mathbf{a}^i\mathbf{b}^{u-i} \subseteq \mathbf{b}^{u-i} \subseteq J^{[p^e]}$.

Therefore, $(\mathbf{a}+\mathbf{b})^u \subseteq J^{[p^e]}$. We now have $\nu^J_{\mathbf{a}+\mathbf{b}}(p^e) \leq \nu^J_{\mathbf{a}}(p^e) + \nu^J_{\mathbf{b}}(p^e)$. By Definition 4.1.6, it completes the proof.

□

Notation 4.1.9. If $\mathbf{a} = (f)$, we simply write $\nu_f^J(p^e)$ and $c^J(f)$.

Proposition 4.1.10. *If $J = \bigcap_{\lambda\in\Gamma} J_\lambda$, then*

$$c^J(\boldsymbol{a}) = \sup_{\lambda\in\Gamma} c^{J_\lambda}(\boldsymbol{a}).$$

Proof. We know that

$$\begin{aligned}
\nu_{\mathbf{a}}^J(p^e) &= \max\{r \in \mathbb{N} |\ \mathbf{a}^r \nsubseteq J^{[p^e]} = \bigcap_{\lambda\in\Gamma} J_\lambda^{[p^e]}\} \\
&= \max\{r \in \mathbb{N} |\ \mathbf{a}^r \nsubseteq J_\lambda^{[p^e]}, \text{ for some } \ \lambda \in \Gamma\} \\
&\leq \sup_{\lambda\in\Gamma} \max\{r \in \mathbb{N} |\ \mathbf{a}^r \nsubseteq J_\lambda^{[p^e]}\} \\
&= \sup_{\lambda\in\Gamma} \nu_{\mathbf{a}}^{J_\lambda}(p^e).
\end{aligned}$$

Then, $c^J(\mathbf{a}) \leq \sup_{\lambda\in\Gamma} c^{J_\lambda}(\mathbf{a})$. On the other hand, since $J \subseteq J_\lambda$, by Proposition 4.1.7 (1), we have $c^{j_\lambda}(\mathbf{a}) \leq c^J(\mathbf{a})$ for all $\lambda \in \Gamma$. The last statement follows that

$$\sup_{\lambda\in\Gamma} c^{J_\lambda}(\mathbf{a}) \leq c^J(\mathbf{a}).$$

□

Proposition 4.1.11. *If $I, \boldsymbol{a}, J$ are the ideals of R such that $\boldsymbol{a} \subseteq \sqrt{J}$. Let $T = R/I$, then $c^{JT}(\boldsymbol{a}T) \leq c^J(\boldsymbol{a})$.*

Proof. We observe that

$$\begin{aligned}
\nu_{\mathbf{a}T}^{JT}(p^e) &= \max\{r \in \mathbb{N} |\ (\mathbf{a}T)^r \nsubseteq (JT)^{[p^e]}\} \\
&= \max\{r \in \mathbb{N} |\ (\mathbf{a} + I)^r \nsubseteq (J + I)^{[p^e]}\} \\
&= \max\{r \in \mathbb{N} |\ (\mathbf{a}^r + I) \nsubseteq J^{[p^e]} + I\}.
\end{aligned}$$

Now, we have $\mathbf{a}^r + I \nsubseteq J^{[p^e]} + I$, then $\mathbf{a}^r \nsubseteq J^{[p^e]}$. It implies that $\nu_{\mathbf{a}T}^{JT}(p^e) \leq \nu_{\mathbf{a}}^J(p^e)$. Hence, $c^{JT}(\mathbf{a}T) \leq c^J(\mathbf{a})$ by definition of an F-threshold. □

Theorem 4.1.12. [HMTW08] *Let $I \subseteq S = \mathbb{K}[x_1, ..., x_n]$ be a homogeneous prime ideal and $d = \dim(R)$ where $R = S/I$. Let $J = (x_1, .., x_d)$ be an ideal of S such that $\dim(R/J) = 0$. Then,*

(1) $c^J(I) < d$ if and only if $I \subseteq \overline{J}$.

(2) If $J \subseteq I$, then $I \subseteq \overline{J}$ if and only if $c^J(I) = d$.

Chapter 5

Examples and open questions

In this chapter we compute the diagonal F-threshold of a Stanley-Reisner ring. We direct the interested reader to the book of Miller and Sturmfels [MS05] for details surrounding the algebra of general monomial ideals as well as the combinatorial topology of simplicial complex.

For the questions about determinental rings, we refer the interested reader to the book of Bruns [BV88].

5.1 Squarefree monomial ideals

In this subsection $\mathbb{K}$ denotes a field of prime characteristic and $S = \mathbb{K}[x_1, ..., x_n]$ denotes the polynomial ring over $\mathbb{K}$.

Definition 5.1.1. A *monomial* in $\mathbb{K}[x_1, ..., x_n]$ is a product $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for a vector $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}^n$ of the non-negative integers. An ideal $I \subseteq \mathbb{K}[x_1, ..., x_n]$ is called a *monomial ideal* if it is generated by monomials.

Let $S_\alpha = \mathbb{K}x^\alpha$ be the a vector subspace of S spanned by the monomial x^α. We observe that $S = \bigoplus_{\alpha \in \mathbb{N}} S_\alpha$ and $S_\alpha \cdot S_\beta = S_{\alpha+\beta}$. Then we say that S is $\mathbb{N}^n$*-graded* $\mathbb{K}$*-algebra*. Moreover, I can be expressed as a direct sum, namely $I = \oplus_{x_\alpha \in I} \mathbb{K}\{x^{\alpha\}}$.

Lemma 5.1.2. *Every monomial ideal of S has a unique minimal set of monomial generators, and this set is finite.*

Proof. By the Hilbert Basis Theorem, we observe that S is a Noetherian ring. If I is a monomial ideal of S, it follows that $I = (x^{\alpha_1}, ..., x^{\alpha_m})$ where $\alpha_1, ..., \alpha_m \in \mathbb{N}$. Since I can be expressed by a direct sum, any polynomial f lies inside I if and only if each term of f is divided by one of the given generators x^{α_i}. This is, I is generated by minimal monomials. □

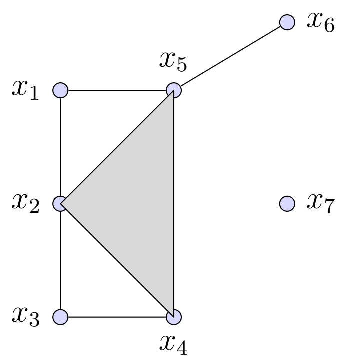
Definition 5.1.3. A monomial x^α is *square free* if every coordinate of α is 0 or 1. A monomial ideal is *square free* if it is generated by square free monomials.

Definition 5.1.4. A *simplicial complex* Δ on the *vertex set* $\{1, ..., n\}$ is a collection of subsets called *faces* or *simplices*, closed under talking subsets; that is, if $\sigma \in \Delta$

is a face and $\tau \subseteq \sigma$, then $\tau \in \Delta$. An element $\sigma \in \Delta$ is called *facet* or *maximal* if it is not contained in the other faces. A simplex $\sigma \in \Delta$ of cardinality $|\sigma| = i + 1$, where $\dim \Delta = i$, we say that σ is a maximal face of Δ. The *dimension* $\dim(\Delta)$ is the maximum of the dimensions of its faces, or it is $-\infty$ if $\Delta = \{\}$ is the *void complex*, which has no face.

Notation 5.1.5. The empty set $\{\varnothing\}$ is the only simplicial complex with dimension -1, and the void complex $\varnothing$ has dimension $-\infty$. We frequently identify the vertex set $\{1, 2, ..., n\}$ with the variable $\{x_1, x_2, ..., x_n\}$, as our next example, or as $\{a, b, c, ...\}$.

Example 5.1.6. The simplicial complex Δ on $\{1, 2, 3, 4, 5, 6, 7\}$ consisting of all subsets of the sets $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{4, 2, 5\}$,$\{1,5\}$,$\{7\}$ and $\{5, 6\}$ is appears below:



We observe that all the points, segments and an area of the picture are the faces of simplicial complex Δ. Note that Δ is completely specified by its *facets* or maximal faces.

Remark 5.1.7. If Δ_1 and Δ_2 are simplicial complexes, then $\Delta_1 \cup \Delta_2$ and $\Delta_1 \cap \Delta_2$ are also simplicial complexes.

Notation 5.1.8. For each $\sigma \in \{1, 2, ..., n\}$ we associate a square free monomial by $x^\sigma = \Pi_{i\in\sigma} x_i$.

Definition 5.1.9. The *Stanley-Reisner ideal* of the simplicial complex Δ is the square free monomial ideal is defined by

$$I_\Delta = (x^\tau | \tau \notin \Delta),$$

where τ is a nonface of Δ. In addition, the *Stanley-Reisner ring* of Δ is the quotient ring S/I_Δ.

Notation 5.1.10. We write $\mathbf{p}_\sigma$ for the prime ideal $(x_i \mid i \in \sigma)$ and $\overline{\sigma}$ for the set $\{1, 2, ..., n\} \setminus \sigma$.

Proposition 5.1.11. *Let Δ_1 and Δ_2 be two simplicial complexes. Then*

(1) $I_{\Delta_1 \cup \Delta_2} = I_{\Delta_1} \cap I_{\Delta_2}$.

(2) $I_{\Delta_1 \cap \Delta_2} = I_{\Delta_1} + I_{\Delta_2}$.

(3) If $\Delta = \Delta_1 \cup \cdots \cup \Delta_m$, *where* $\Delta_i = 2^{\sigma_i}$ and σ_i *is maximal,* $i \leq m \in \mathbb{N}$, *then* $I_\Delta = I_{\Delta_1} \cap \cdots \cap I_{\Delta_m}$ *and* $I_{\Delta_i} = \boldsymbol{p}_{\overline{\sigma_i}}$.

Proof.

(1) By Definition 5.1.9, we have

$$\begin{aligned} I_{\Delta_1 \cup \Delta_2} &= (x^\tau | \ \ \tau \notin \Delta_1 \cup \Delta_2) \\ &= (x^\tau | \ \ \tau \notin \Delta_1 \ \ \textit{and} \ \ \tau \notin \Delta_2) \\ &= (x^\tau | \ \ \tau \notin \Delta_1) \cap (x^\tau | \ \ \tau \notin \Delta_2) \\ &= I_{\Delta_1} \cap I_{\Delta_2}. \end{aligned}$$

(2) Similarly, we observe that

$$\begin{aligned} I_{\Delta_1 \cap \Delta_2} &= (x^\tau | \ \ \tau \notin \Delta_1 \ \ \textit{or} \ \ \tau \notin \Delta_2) \\ &= I_{\Delta_1} + I_{\Delta_2}. \end{aligned}$$

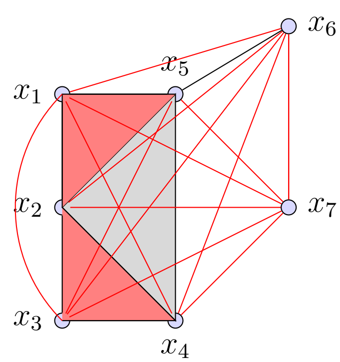
(3) It follows that $I_\Delta = I_{\Delta_1} \cap \cdots \cap I_{\Delta_m}$ by (1). In the case $\Delta_i = 2^{\sigma_i}$, σ_i is maximal, we observe that $I_{\Delta_i} = (x^\tau | \ \ \tau \notin \Delta_i) = (x^\tau \mid \tau \cap \overline{\sigma_i} \neq \varnothing) = \mathbf{p}_{\overline{\sigma_i}}$.

□

We note that the last part of the previous proposition is usually proven using that $I_{\Delta_i} = \bigcap_{\sigma \in \Delta_i} \mathbf{p}_{\overline{\sigma}}$..

Example 5.1.12. Let Δ be a simplicial complex as in Example 5.1.6. We obtain

$$\begin{aligned} I_\Delta = (&\{x_1x_2x_5, x_2x_3x_4, x_1x_4, x_3x_5, x_1x_3\} \cup \{x_ix_6 | \ \ i = 1,2,3,4\} \\ &\cup \{x_ix_7 | \ \ i = 1,2,3,4,5,6\}). \end{aligned}$$



Corollary 5.1.13. *Every squarefree ideal is an intersection of monomial prime ideals.*

Proof. This follows immediately from Proposition 5.1.11(4). □

Theorem 5.1.14. *The correspondence $\Delta \rightsquigarrow I_\Delta$ constitutes a bijection from a simplicial complex on the vertices $\{1, 2, ..., n\}$ to squarefree monomial ideals inside $S = \mathbb{K}[x_1, ..., x_n]$.*

Proof. Given a square free monomial ideal I, we take

$$\Delta_I = \{\sigma \subset \{1, \ldots, n\} \mid x^\sigma \notin I\}.$$

Given a square free monomial ideal we have that We have that

$$I_{\Delta_I} = (x^\sigma \mid \sigma \notin \Delta_I) = (x^\sigma \mid x^\sigma \in I) = I$$

Given a simplicial complex Δ, we have that

$$\Delta_{I_\Delta} = \{\sigma \subset \{1, \ldots, n\} \mid x^\sigma \notin I_\Delta\} = \{\sigma \subset \{1, \ldots, n\} \mid x^\sigma \in \Delta\} = \Delta$$

Then, $I \rightsquigarrow \Delta_I$ is the inverse function of $\Delta \rightsquigarrow I_\Delta$. □

Proposition 5.1.15. *Let R be a Stanley-Reisner ring with an infinite field and $\boldsymbol{m}$ be the maximal homogeneous ideal. Then*

$$c^{\boldsymbol{m}}(\boldsymbol{m}) \leq \dim(R).$$

Proof. Let $d = \dim(R)$. By Theorem 2.2.7, we take I as an ideal of R which is generated by d elements called $f_1, ..., f_d$ such that $\overline{I} = \mathbf{m}$. By Proposition 4.1.7 (c) and Proposition 4.1.8 (4), we have $c^{\mathbf{m}}(\mathbf{m}) = c^{\mathbf{m}}(\overline{I}) = c^{\mathbf{m}}(I) \leq d = \dim(R)$. □

Lemma 5.1.16. *Let R be a finite generated $\mathbb{K}$-algebra and $S = R \otimes_{\mathbb{K}} \overline{\mathbb{K}}$. Let $\boldsymbol{a}, J$ be ideals of R such that $\boldsymbol{a} \subseteq \sqrt{J}$. Then $c^J(\boldsymbol{a}) = c^{J(R \otimes_{\mathbb{K}} \overline{\mathbb{K}})}(\boldsymbol{a}(R \otimes_{\mathbb{K}} \overline{\mathbb{K}}))$.*

Proof. We first have to prove that $\nu^J_{\mathbf{a}}(p^e) = \nu^{JS}_{\mathbf{a}S}(p^e)$. We observe that

$$\begin{aligned}\nu^{JS}_{\mathbf{a}S}(p^e) &= \max\{r \in \mathbb{N} \mid (\mathbf{a}S)^r \nsubseteq (JS)^{[p^e]}\} \\ &= \max\{r \in \mathbb{N} \mid \mathbf{a}^r S \nsubseteq J^{[p^e]} S\}.\end{aligned}$$

Since S is a free R-module, we deduce that $\mathbf{a}^r \subseteq J^{[p^e]}$ if and only if $\mathbf{a}^r S \subseteq J^{[p^e]} S$. Then

$$\begin{aligned}\nu^{JS}_{\mathbf{a}S}(p^e) &= \max\{r \in \mathbb{N} \mid \mathbf{a}^r \nsubseteq J^{[p^e]}\} \\ &= \nu^J_{\mathbf{a}}(p^e).\end{aligned}$$

Hence, $c^J(\mathbf{a}) = c^{J(R \otimes_{\mathbb{K}} \overline{\mathbb{K}})}(\mathbf{a}(R \otimes_{\mathbb{K}} \overline{\mathbb{K}}))$ by the definition of F-thresholds. □

Corollary 5.1.17. *Let R be an Stanley-Reisner ring and $\boldsymbol{m}$ be the maximal homogeneous ideal. Then*

$$c^{\boldsymbol{m}}(\boldsymbol{m}) \leq \dim(R).$$

Proof. We observe that

$$\begin{aligned} c^{\mathbf{m}}(\mathbf{m}) &= c^{\mathbf{m}(R\otimes_{\mathbb{K}}\overline{\mathbb{K}})}(\mathbf{m}(R\otimes_{\mathbb{K}}\overline{\mathbb{K}})) \text{ by Lemma 5.1.16} \\ &\leq \dim(R\otimes_{\mathbb{K}}\overline{\mathbb{K}}) \text{ by Proposition 5.1.15} \\ &= \dim(R). \end{aligned}$$

□

Theorem 5.1.18. *If $R = \frac{\mathbb{K}[x_1,\ldots,x_n]}{I}$, where I is a squarefree monomial ideal and $\boldsymbol{m}$ is a maximal ideal, then*

$$c^{\boldsymbol{m}}(\boldsymbol{m}) = \dim(R).$$

Proof. By the Corollary 5.1.17, we have $c^{\mathbf{m}}(\mathbf{m}) \leq \dim(R)$. We now just prove that $c^{\mathbf{m}}(\mathbf{m}) \geq \dim(R)$. By Proposition 5.1.11, there exists a simplicial complex Δ of vertex $\{1, 2, \cdots, n\}$ which satisfy $I = I_\Delta = \bigcap_{\sigma\in\Delta} \mathbf{p}_{\overline{\sigma}}$. We note that $\mathbf{p}_{\overline{\sigma}}$ is a prime ideal which is generated by variables. In this case, we observe that $c^{\mathbf{m}}_R(\mathbf{m}) \geq c^{\mathbf{m}}_{R/\mathbf{p}_{\overline{\sigma}}}(\mathbf{m})$ by Proposition 4.1.11 for all $\sigma \in \Delta$. Since $R/\mathbf{p}_{\overline{\sigma}}$ is isomorphic to a polynomial ring, then $c^{\mathbf{m}}_{R/\mathbf{p}_{\overline{\sigma}}}(\mathbf{m}) = \dim(R/\mathbf{p}_{\overline{\sigma}})$. We observe that the simplicial complex Δ contains finitely many simplices σ. Hence, there also are finite numbers of prime ideals $\mathbf{p}_{\overline{\sigma}}$. Since $\dim(R) = \max\{\dim(R/\mathbf{p}_{\overline{\sigma}})|\ \sigma \in \Delta\}$, there exists $\sigma_0 \in \Delta$ such that $\dim(R/\mathbf{p}_{\overline{\sigma_0}}) = \dim(R)$. Therefore,

$$c^{\mathbf{m}}_R(\mathbf{m}) \geq c^{\mathbf{m}}_{R/\mathbf{p}_{\overline{\sigma_0}}}(\mathbf{m}) = \dim(R).$$

This completes the proof. □

5.2 Open questions

In the previpus subsection we computed the the F-threshold $c^{\mathbf{m}}(\mathbf{m})$. We now discuss a few open cases for rings that have a combinatorial structure. We first focus on rings araising from determinental matices.

Let $X = (x_{i,j})$ a matrix of $n \times m$, where $n \leq m$. We consider the polynomial ring $S = \mathbb{K}[X]$. Set $t \in \mathbb{N}$ such that $t \leq n$. Let $I_t(X)$ denote the ideal generated by the $t \times t$-minors of the matrix X. We set $R_{n,m,t} = S/I_t(X)$.

Question 5.2.1. Is there a formula for $c^{\mathbf{m}}(\mathbf{m})$ in terms of n, m and t?

In the case where $t = 2$, there exists a formula for $c^{\mathbf{m}}(\mathbf{m})$ using the fact that $R_{n,m,t}$ is a toric variety [Hir09].

For the case where $n = m = t$ computational evidence suggeste the following conjecture.

Conjecture 5.2.2. If $n = m = t$, then $c^{\mathbf{m}}(\mathbf{m}) = n^2 - n$.

It is worth pointing out that this result is true if $n = 2$ [Hir09].

We now focus on ings defined by graphs. Given a simple graph G with vertex set $\{1, \ldots, n\}$, the binomial edge ideal [HHH+10, Oht11], $J_G \subseteq S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$, associated to G is defined by

$$J_G = (x_i y_j - x_j y_i \mid \{i, j\} \in G) \subseteq K[x_1, \ldots, x_n, y_1, \ldots, y_n].$$

We set $R_G = S/J_G$.

Question 5.2.3. In R_G is there a formula for $c^{\mathbf{m}}(\mathbf{m})$ in terms of the structure of the graph?

Motivated by recent result by on connectivity of graph and binomial edge idea [BNnB17], we expect that the answer to the previous question will relate to the connectivity of G.

Bibliography

[AM69] Michael F. Atiyah and Ian G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969. 9

[BMS08] Manuel Blickle, Mircea Mustaţă, and Karen E. Smith. Discreteness and rationality of F-thresholds. *Michigan Math. J.*, 57:43–61, 2008. Special volume in honor of Melvin Hochster. 9, 13

[BNnB17] Arindam Banerjee and Luis Núñez Betancourt. Graph connectivity and binomial edge ideals. *Proc. Amer. Math. Soc.*, 145(2):487–499, 2017. 37

[BV88] Winfried Bruns and Udo Vetter. *Determinantal rings*, volume 1327 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988. 32

[DSNBP] Alessandro De Stefani, Luis Núñez-Betancourt, and Felipe Pérez. On the existence of F-thresholds and related limits. *Transactions of the American Mathematical Society*. To appear. 9, 27

[Eis95] David Eisenbud. *Commutative algebra: with a view toward algebraic geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. 9

[HHH$^+$10] Jürgen Herzog, Takayuki Hibi, Freyja Hreinsdóttir, Thomas Kahle, and Johannes Rauh. Binomial edge ideals and conditional independence statements. *Adv. in Appl. Math.*, 45(3):317–333, 2010. 36

[Hir09] Daisuke Hirose. Formulas of F-thresholds and F-jumping coefficients on toric rings. *Kodai Math. J.*, 32(2):238–255, 2009. 36

[HMTW08] Craig Huneke, Mircea Mustaţă, Shunsuke Takagi, and Kei-ichi Watanabe. F-thresholds, tight closure, integral closure, and multiplicity bounds. *Michigan Math. J.*, 57:463–483, 2008. Special volume in honor of Melvin Hochster. 31

[HS06] Craig Huneke and Irena Swanson. *Integral closure of ideals, rings, and modules*, volume 336 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006. 9, 12

[Hun96] Craig Huneke. *Tight closure and its applications*, volume 88 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996. With an appendix by Melvin Hochster. 9

[HY03] Nobuo Hara and Ken-Ichi Yoshida. A generalization of tight closure and multiplier ideals. *Trans. Amer. Math. Soc.*, 355(8):3143–3174, 2003. 8

[Kun69] Ernst Kunz. Characterizations of regular local rings for characteristic p. *Amer. J. Math.*, 91:772–784, 1969. 8

[MS05] Ezra Miller and Bernd Sturmfels. *Combinatorial commutative algebra*, volume 227 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005. 9, 32

[MTW05] Mircea Mustaţă, Shunsuke Takagi, and Kei-ichi Watanabe. F-thresholds and Bernstein-Sato polynomials. pages 341–364, 2005. 8, 9, 27

[Oht11] Masahiro Ohtani. Graphs and ideals generated by some 2-minors. *Comm. Algebra*, 39(3):905–917, 2011. 36

[ST12] Karl Schwede and Kevin Tucker. A survey of test ideals. In *Progress in commutative algebra 2*, pages 39–99. Walter de Gruyter, Berlin, 2012. 8