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Expected Discounted Penalty Functions and Wiener-Hopf factorization for two-sided jumps Lévy risk processes

Tesis

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P R E S E N T A:

Ehyter Matías Martín González

Director:

Dra. Ekaterina Todorova Kolkovska

Guanajuato, Guanajuato, México

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Integrantes del Jurado.

Dr. Victor Rivero Mercado (Presidente)
Dr. José Luis Ángel Pérez Garmendia (Secretario)
Dr. Antonio Murillo Salas
Dr. José Villa Morales
Dra. Ekaterina Todorova Kolkovska (Vocal y director de tesis)

Asesor:

____

Dra. Ekaterina Todorova Kolkovska.

Sustentante:

____

M.C. Ehyter Matías Martín González.

To my beloved children, Harry and Fabiana, whose pressence made this work possible.

Abstract

Let us consider the class of Lévy processes $\mathcal{X} = \{\mathcal{X}(t), t \geq 0\}$ defined by the equation

$$\mathcal{X}(t) = ct + \gamma \mathcal{B}(t) + \mathcal{Z}(t) - \mathcal{S}(t),\ \gamma \geq 0,$$

where $c \geq 0$ is a drift term, $\mathcal{B} = \{\mathcal{B}(t), t \geq 0\}$ is a brownian motion with zero mean, $\mathcal{Z} = \{\mathcal{Z}(t), t \geq 0\}$ is a compound Poisson process whose jumps have a probability distribution with rational Laplace transform and $\mathcal{S} = \{\mathcal{S}(t), t \geq 0\}$ is a pure positive jumps Lévy process. The processes $\mathcal{B}$, $\mathcal{Z}$ and $\mathcal{S}$ are assumed to be independent.

We study the Wiener-Hopf factorization of this class of processes, particularly focusing on the distribution of the negative Wiener-Hopf factor (the factor given by the infimum of $\mathcal{X}$ stopped at a random exponential time). We present explicitly a subordinator such that the negative Wiener-Hopf factor is equal in distribution to this subordinator, and then use this result to obtain an expression for the Laplace transform of this negative Wiener-Hopf factor. We invert this Laplace transform to obtain an explicit expression for its probability density. This probability density is in terms of functions which depend only on the parameters of the process, and in terms of the derivative of a q-scale function of an associated spectrally negative Lévy process. Due to the importance of q-scale functions, we apply our techniques to study two important cases of spectrally negative Lévy processes and obtain explicit expressions for their corresponding q-scale functions.

Furthermore, we use our result about the density of the negative Wiener-Hopf factor to obtain an explicit expression for the Generalized Expected Discounted Penalty Function (Generalized EDPF, for short) of a process of the form $u + \mathcal{X}$, where $u \geq 0$. This model corresponds to a two-sided jumps Lévy risk process with rational positive jumps and general negative jumps, allowing the case when there exists a random factor (a perturbation) which models random gains or losses. The two-sided jumps Lévy risk process with an α-stable perturbation, which generalizes the model in Furrer [1998], arises as a particular case. This model is studied in full detail and we present several results about its corresponding EDPF. We also obtain asymptotic expressions for its corresponding ruin probability and the joint tail of the severity of ruin and the surplus prior to ruin for this case. The asymptotic results for this joint tail are also stated for the classical risk process perturbed by an α-stable motion.

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Contents

Introduction

Lévy processes are one of the most studied class of stochastic processes in several branches of applied probability, such as Mathematical Finance, Theory of Branching Processes, Insurance Mathematics and many others.

In particular, their Wiener-Hopf factors, which represent the supremum and the infimum of the Lévy process stopped at an independent exponential time, have been intensively studied. These factors are called, respectively, the positive and the negative Wiener-Hopf factor. The positive Wiener-Hopf factor, for instance, allows one to solve the optimal stopping problem corresponding to the pricing of a perpetual call option, in the case when the market model is generated by a Lévy process. Similarly, the negative Wiener-Hopf factor is useful in solving the optimal stopping problem corresponding to the pricing of a perpetual put option, when the market model is generated by a Lévy process. This negative Wiener-Hopf factor can also be applied in Insurance Risk Mathematics to study the classical ruin problem.

While the distribution of the positive Wiener-Hopf factor has been studied by many authors (see, for instance, Kuznetsov [2010b], Kuznetsov and Peng [2012], Lewis and Mordecki [2008] and the references therein), the distribution of the negative one has been studied only in a few particular cases (see, for instance, Kuznetsov [2010a] and the references therein).

In this thesis we obtain the probability density of the negative Wiener-Hopf factor for the following class of Lévy processes:

Let us take $\mathcal{X} = \{\mathcal{X}(t), t \geq 0\}$ as processes defined by the equation

$$\mathcal{X}(t) = ct + \mathcal{Z}(t) + \gamma\mathcal{B}(t) - \mathcal{S}(t),\ \ t, \gamma \geq 0, \tag{1}$$

where $c \geq 0$ is a drift term, $\mathcal{B} = \{\mathcal{B}(t), t \geq 0\}$ is a Brownian motion with zero mean, $\mathcal{S} = \{\mathcal{S}(t), t \geq 0\}$ is a pure jumps Lévy process with only positive jumps (therefore, its dual $-\mathcal{S}$ has only negative jumps) and $\mathcal{Z} = \{\mathcal{Z}(t), t \geq 0\}$ is a compound Poisson process with Lévy measure $\lambda_1 f_1(x)dx$, such that f_1 is a probability density with

Laplace transform

$$\widehat{f}_1(r) = \frac{Q(r)}{\prod_{i=1}^{N}(q_i + r)^{m_i}}, \tag{2}$$

where $N, m_i \in \mathbb{N}$ with $m_1 + m_2 + \cdots + m_N = m$ and $0 < q_1 < q_2 < \ldots q_m$ and $Q(r)$ is a polynomial function of degree $m-1$ or less. The processes $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{Z}$ are all assumed to be independent. Since every nonnegative distribution can be obtained as a limit of a sequence of combinations of exponential distributions (see Dufresne [2007]), the restriction imposed on f_1 allows for numerical approximations of the general case of f_1. We point out that the explicit distribution of the positive Wiener-Hopf factor of this class of Lévy processes was obtained in Lewis and Mordecki [2008].

In Insurance Mathematics, a process of the form of $u + \mathcal{X}$, for $u \geq 0$ represents the surplus of an insurance company up to time t, in the case when its initial capital is u. The classical risk model, which is an spectrally negative Lévy risk process with negative jumps given by the dual of a compound Poisson process, only models the case when the insurance company begins its service with initial capital u, earns a fixed amount of money per time unit (modeled by the drift term $c \geq 0$) and has to pay for random claims that appear at random times (these are modeled by the dual of a compound Poisson process).

A process with the structure of $\mathcal{X}$ is more realistic than the classical risk model, since it allows the possibility that the insurance company earns money, for instance by investing in the stock market (this is modeled by the process $\mathcal{Z}$) and it also considers the case when there are random fluctuations corresponding to gains or losses. These random fluctuations are represented by the brownian component $\mathcal{B}$ and the process $\mathcal{S}$, when $\mathcal{S}$ has unbounded variation. When $\mathcal{S}$ is a subordinator other than a Compound Poisson Process, we have the case of random claims which appear very often in each time interval.

We also study the Expected Discounted Penalty Function (EDPF for short) of the class of two-sided jumps Lévy processes defined by equation (1). The standard version of this function was introduced by Gerber and Shiu [1998] and it is defined as follows:

We let $\tau_0 = \min\{t \geq 0 : u + \mathcal{X}(t) < 0\}$ be the first passage time of $u + \mathcal{X}$ below zero. In the classical ruin problem studied in Insurance Mathematics, this first passage time is known as the time to ruin of the process $u + \mathcal{X}$. We consider a function $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, which is nonnegative, real-valued and such that $\omega(0+, 0+) = \lim_{(x,y)\to(0,0)} \omega(x, y)$ exists. Then, the EDPF for $u + \mathcal{X}$, denoted by $\phi(u)$, is defined as

$$\phi(u) = \mathbb{E}_u \left[e^{-\delta\tau_0} \omega \left(|\mathcal{X}(\tau_0)|, \mathcal{X}(\tau_0-) 1_{\{\tau_0<\infty\}} \right) \right], \tag{3}$$

where $\delta \geq 0$ represents a discounted force of interest, $|\mathcal{X}(\tau_0)|, \mathcal{X}(\tau_0-)$ are known in Insurance Mathematics, respectively, as the severity of ruin (overshoot at first passage) and the surplus prior to ruin (undershoot at first passage). The function ω is known as a penalty function.

The EDPF presented above renders as particular cases many important risk measures, which arise by changing the choose of δ and ω. For instance, for $\delta = 0$ and $\omega(x, y) \equiv 1$ for all $x, y > 0$, the EDPF reduces to the probability of ruin of the process $u + \mathcal{X}$. If $\omega(x, y) \equiv 1$ and $\delta > 0$, ϕ reduces to the Laplace transform of the time to ruin τ_0. This EDPF has been widely studied, specially in the case when $\mathcal{X}$ is spectrally negative (see, for instance, Gerber and Shiu [1998] and Biffis and Kyprianou [2009]). There are also some results on the case when $\mathcal{X}$ is a two-sided jumps Lévy risk process, for instance in Albrecher et al. [2010] (and the references therein).

In general, the study of the aforementioned class of two-sided jumps Lévy processes is not easy. For instance, standard tools such as first step analysis cannot be applied when the process $\mathcal{S}$ has unbounded variation, because of the infinite jumps that occur in $\mathcal{S}$ in any finite interval. Moreover, the fact that many unbounded variation Lévy processes do not have a probability density with closed form (for instance, the α-stable motion) also makes the analysis much harder than in the case when $\mathcal{S}$ is simply a compound Poisson process. Another difficulty to study this kind of processes comes from the positive jumps process $\mathcal{Z}$. Many of the results for the spectrally negative case, even when $\mathcal{S}$ has unbounded variation, are in terms of the recently studied q-scale functions, which are known to exist only in the spectrally negative case, and so far, there is no analogue of them for the two-sided jumps case.

In this thesis we obtain a series of results that solve this problem. First we study in great detail the EDPF of an important particular case of the process $u + \mathcal{X}$: we suppose $\mathcal{X}$ has no brownian component and take $\mathcal{S}$ as the sum of a compound Poisson process and an α-stable motion with only positive jumps (again, this implies that $-\mathcal{S}$ has only negative jumps). We refer to this process as the classical two-sided jumps risk process perturbed by an α-stable motion. We deal with the lack of a closed expression for the α-stable density by constructing approximating sequences based on compound Poisson processes such that they converge weakly to the α-stable process of interest. This approach allows us to use first step analysis, and other techniques, to study our process of interest. In Theorem 7, Chapter 2 we obtain an expression for the Laplace transform of the EDPF for the classical two-sided jumps risk process perturbed by an α-stable motion. In Theorem 8, Chapter 2, we invert this Laplace transform and present a renewal equation satisfied by ϕ. This results provide a natural extension of a celebrated result by Furrer [1998], which was stated and proved only for the ruin probability of a classical risk process perturbed by an α-stable motion, which is a particular case of a spectrally negative Lévy risk process.

In Theorem 9, Chapter 2 we obtain the asymptotic behavior of the ruin probability corresponding to the classical two-sided jumps risk process perturbed by an α-stable motion, while in Theorem 10, Chapter 2, we obtain asymptotic formulae for the joint bivariate tail of the overshoot and undershoot at first passages below zero of the classical two-sided jumps risk process perturbed by an α-stable motion, i.e. the joint bivariate tail of the severity of ruin and the surplus prior to ruin. We also present the corresponding asymptotic formula for the joint bivariate tail of the surplus prior to ruin and the severity of ruin in the case of a classical risk process with an α-stable perturbation.

In Chapter 3 we consider the more general case of $\mathcal{X}$ when we have a brownian component and $\mathcal{S}$ is any pure positive jumps Lévy process, other than a compound Poisson process, and study the Wiener-Hopf factorization of this kind of processes. In Theorem 13, we present an explicit expression for the probability density of the negative Wiener-Hopf factor for the class of two-sided jumps Lévy risk processes defined by (1). The result in this theorem is in terms of known functions, which depend only on the parameters of the process, and also in terms of the q-scale function of an associated spectrally negative Lévy processes, which is given explicitly. This complements a previous result given in Lewis and Mordecki [2008], where only the distribution of the positive Wiener-Hopf factor is obtained.

In Theorem 14 we obtain an expression for a generalized version of the EDPF defined in 3. This Generalized EDPF was introduced and studied in Biffis and Morales [2010] and Biffis and Kyprianou [2009], only for the spectrally negative case. Hence, our result in Theorem 14 extends these previous results to the two-sided jumps case. Finally, we apply the techniques developed in the previous chapters to study two important cases of spectrally negative Lévy processes, and obtain explicit expressions for their corresponding q-scale functions.

This work is organized as follows: in Chapter 1 we give some preliminary concepts and notation that are used throughout this thesis, and introduce the concept of weak convergence of stochastic processes. In Chapter 2 we study the EDPF of the particular case of $\mathcal{X}$ when $\mathcal{S}$ equals the sum of a compound Poisson process and an α-stable process, and there is no brownian component. This chapter also contains our asymptotic results.

In Chapter 3 we study the more general case of $\mathcal{X}$ in which there is a brownian component and $\mathcal{S}$ is a general pure positive jumps Lévy process, other than a compound Poisson process. We provide explicit expressions for the density of the corresponding negative Wiener-Hopf and also present explicitly a Lévy subordinator which is equal in distribution to the aforementioned negative Wiener-Hopf factor. We use this result, among others, to obtain an explicit expression for the Generalized EDPF associated to this class of Lévy processes, from which we get to see that, as in the spectrally negative case, such Generalized EPDF is strongly related to the positive and negative Wiener-Hopf factors of the corresponding Lévy process. In the final section of this chapter, we present a few important and non trivial examples of the application of our results.

Finally, in Chapter 4 we obtain explicit expressions for the q-scale functions of two important cases of spectrally negative Lévy processes.

Many technical proofs are given in an Appendix.

Chapter 1

Preliminaires

In this section we present some basic tools and notation that are used in the next chapters. In general we use the notations $i = \sqrt{-1}$, $\mathbb{C}_+ := \{z \in \mathbb{C} : Re(z) \geq 0\}$, $\mathbb{C}_{++} := \{z \in \mathbb{C} : Re(z) > 0\}$ and denote by $|\cdot|$ the usual norm of complex numbers.

1.1 Laplace transforms, convolutions, Dickson and Hipp operator and α-stable distribution

We denote the Laplace transform of a measurable function f as

$$\widehat{f}(r) = \int_{-\infty}^{\infty} e^{-rx} f(x)dx,\ r \geq 0.$$

for each r (real or complex) such that the integral above exists and is finite. If F is a distribution function with finite first moment μ, and $F(0) = 0$, we define its integrated tail distribution F_I by

$$F_I(x) = \frac{1}{\mu}\int_0^x \overline{F}(y)\,dy,\ x \geq 0.$$

Clearly this function has a density given by $f_I(x) = \frac{1}{\mu}\overline{F}(x)$, and it is easily proved that, if F has a density f, then

$$\mu\widehat{f}_I(r) = \frac{1 - \widehat{f}(r)}{r}. \tag{1.1}$$

For any two nonnegative measurable functions h, g, we define its convolution $h * g$ as

$$h * g(x) = \int_0^x h(x-y)g(y)dy = \int_0^x g(x-y)h(y)dy,$$

for each x when the integral above exists.

We denote by g^{*n}, the nth-convolution of the function g with itself, with $g^{*0}(x) = 1_{\{0\}}(x)$.

We use the property $\widehat{f*g}(r) = \widehat{f}(r)\widehat{g}(r)$ for all $r \geq 0$ for which the latter Laplace transforms exist. If F and G are two distribution functions such that $F(0) = G(0) = 0$, its convolution $F*G(x)$ is defined as $F*G(x) = \int_0^x F(x-y)G(dy)$.

The following is a translation operator introduced in Dickson and Hipp [2001].

Definition 1. *For any integrable function $f \geq 0$ and $r \in \mathbb{C}_+$, the Dickson-Hipp translation operator $T_r f$ is defined by the equation*

$$T_r f(x) = \int_x^\infty e^{-r(y-x)} f(y)dy, x > 0. \tag{1.2}$$

It can be easily proved that $T_r f$ satisfies the equalities

$$T_r f(0) = \widehat{f}(r) \text{ and } \widehat{T}_{r_2} f(r_1) = \frac{\widehat{f}(r_1) - \widehat{f}(r_2)}{r_2 - r_1}, \tag{1.3}$$

for $r, r_1, r_2 \in \mathbb{C}_+$. We can also define the operator above when $f(x)dx$ is replaced by a measure $\nu(dx)$. In this case we have

$$T_r \nu(x) = \int_x^\infty e^{-r(y-x)} \nu(dy), x > 0. \tag{1.4}$$

and

$$\widehat{T}_{r_2}\nu(r_1) = \frac{\int_{0+}^\infty \left(e^{-r_1 x} - e^{-r_2 x}\right)\nu(dx)}{r_2 - r_1}, \tag{1.5}$$

Now let us denote by $S_\alpha(\sigma, \beta, \mu)$, the α-stable distribution with parameters $0 < \alpha \leq 2$ (index of stability), $\sigma > 0$ (scale), $-1 \leq \beta \leq 1$, (skewness) $-\infty < \mu < \infty$ (shift), and density $g_{\alpha,\beta,\sigma,\mu}(x)$. According to Theorem C 3 in Zolotarev [1986], this probability density has characteristic function given by

$$\mathbb{E}[e^{i\theta X}] = \begin{cases} e^{\sigma^\alpha (i\mu\theta - |\theta|^\alpha exp\{-i(\pi/2)\beta K(\alpha) sgn(\theta)\})} & \text{for } \alpha \neq 1, \\ e^{\sigma[i\mu\theta - |\theta|(\pi/2 + i\beta log|\theta| sgn(\theta))]} & \text{for } \alpha = 1, \end{cases} \tag{1.6}$$

where $K(\alpha) = \alpha - 1 + sgn(1-\alpha)$ and $sgn(\theta) = 1_{\{\theta>0\}} + \theta 1_{\{\theta=0\}} - 1_{\{\theta<0\}}$.

It is known (see Zolotarev [1986]) that the α-stable density $g_{\alpha,1,\sigma,\mu}$ has a Laplace transform given by

$$\widehat{g}_{\alpha,1,\sigma,\mu}(r) = \begin{cases} e^{\sigma^\alpha[-\mu r - sgn(1-\alpha) r^\alpha]} & \alpha \neq 1, \\ e^{\sigma[-\mu r + r log r]} & \alpha = 1. \end{cases} \tag{1.7}$$

In what follows, we denote by $g_{\alpha,\beta}$ the α-stable density $g_{\alpha,\beta,1,1}$, and we write $\widehat{g}_{\alpha,\beta}$ for its corresponding Laplace transform.

1.2 Lévy processes, Wiener-Hopf factors and scale functions

A stochastic process $X = \{X(t), t \geq 0\}$ is a Lévy process if it satisfies the following conditions:

- $X(0) = 0$ a.s.
- X has $\mathbb{P}$-a.s. right-continuous paths with left limits (càdlàg trajectories)
- For $0 \leq s \leq t$, $X(t) - X(s) \stackrel{d}{=} X(t-s)$ and $X(t) - X(s)$ is independent of $\{X(u), u \leq s\}$

where $\stackrel{d}{=}$ denotes equality in distribution.

In the particular case when

$$X(t) - X(s) \sim S_\alpha\left[(t-s)^{1/\alpha}, \beta, \mu\right],$$

for $0 \leq s < t < \infty$, X is called **α-stable Levy motion**. It is called **standard α-stable motion** when $\sigma = 1, \mu = 0$. If $1 < \alpha < 2$, the moments of X with order smaller than α, are finite, and when $\beta = 1$, only positive jumps of W_α are possible. For $\alpha = 2$, we obtain the Brownian motion $\{\sqrt{2}B(t), t \geq 0\}$.

From the Lévy-Khintchine formula, if X is a real-valued Lévy process, we can write its characteristic exponent $\Psi_X^{[c]}(r) = \log \mathbb{E}[e^{irX(1)}]$ as

$$\Psi_X^{[c]}(r) = air - \frac{1}{2}\sigma^2 r^2 - \int\limits_{\mathbb{R}\backslash\{0\}} \left(1 - e^{iry} + iry1_{\{|y|<1\}}\right) \nu_X(dy), \quad r \in \mathbb{R},$$

where ν_X is a measure such that $\int\limits_{\mathbb{R}\backslash\{0\}} (y^2 \wedge 1)\, \nu_X(dy)$. This formula characterizes Lévy processes as a Brownian motion with variance σ^2 plus a drift $a \in \mathbb{R}$ and a pure jumps process with characteristic measure ν_X. Let $\mathcal{S} = \{\mathcal{S}(t), t \geq 0\}$ be a spectrally positive Lévy process with $\sigma = 0$ with characteristic exponent $\Psi_{\mathcal{S}}^{[c]}(r)$. We call S a subordinator if S has nondecreasing paths. In this case we have $\int\limits_{0+}^{\infty} (y \wedge 1)\, \nu_X(dy) < \infty$ and $\Psi_{\mathcal{S}}^{[c]}(r)$ can be written as:

$$\Psi_{\mathcal{S}}^{[c]}(r)(r) = a_0 ir + \int\limits_{0+}^{\infty} \left(1 - e^{iry}\right) \nu_{\mathcal{S}}(dy),$$

where $a_0 = a + \int\limits_{0+}^{1} y\nu_{\mathcal{S}}(dy) \leq 0$. In the case when $\int\limits_{0+}^{1} y\nu_{\mathcal{S}}(dy) = \infty$, we say that the process S has paths of unbounded variation. For further information and properties about Lévy processes, we refer the reader to Bertoin [1996] and Kyprianou [2006].

Now we let X be a Lévy process with characteristic exponent $\Psi_X^{[c]}(r)$, and set

$$S_t = \sup_{0 \leq s \leq t} X(s) \text{ and } I_t = \inf_{0 \leq s \leq t} X(s).$$

According to Bertoin [1996], the Wiener-Hopf factors of X are given by $\mathbb{E}[e^{irS_{e_q}}]$ and $\mathbb{E}[e^{-irI_{e_q}}]$, and satisfy the equalities

$$\mathbb{E}\left[e^{irX(e_q)}\right] = \frac{q}{q - \Psi_X^{[c]}(r)} = \mathbb{E}\left[e^{irS_{e_q}}\right]\mathbb{E}\left[e^{-irI_{e_q}}\right],\; q > 0, \tag{1.8}$$

where e_q is an exponential random variable with mean $1/q$ independent of X.

We end this section with the definition of q-scale function for a spectrally negative Lévy process (i.e. a Lévy process with only negative jumps). These class of functions is used in the statement of two of our main results.

Let Y be a spectrally negative Lévy process with Laplace exponent $\Psi_Y(r) = \log \mathbb{E}[e^{rY(1)}]$. It is known (see, for instance, Kyprianou [2006]) that for any $q \geq 0$, there exists a function $\mathfrak{W}^{(q)} : \mathbb{R} \to [0, \infty)$ such that $\mathfrak{W}^{(q)}(x) = 0$ for $x < 0$ and $\mathfrak{W}^{(q)}$ is characterized on $[0, \infty)$ as the unique strictly increasing and right-continuous function whose Laplace transform satisfies:

$$\int_0^\infty e^{-rx}\mathfrak{W}^{(q)}(x)dx = \frac{1}{\Psi_Y(r) - q}, \text{ for all } r > v(q). \tag{1.9}$$

where $v(q)$ is the biggest solution of $\Psi_Y(r) - q = 0$. The case when $q = 0$ is denoted as $\mathfrak{W}(x)$.

As stated and proved in Cohen et al. [2012], this scale function is $C^1(0, \infty)$ when the process Y has unbounded variation, it has a nonzero Gaussian component or the tail of its Lévy measure is continuous. In the case of unbounded variation, it satisfies $\mathfrak{W}^{(q)}(0) = 0$ for all $q \geq 0$.

Assuming that Y has unbounded variation, we write $\mathfrak{W}^{(q)\prime}(x)$ for the derivative of $\mathfrak{W}^{(q)}$ (with respect to x) for $q > 0$ and $\mathfrak{W}'(x)$ when $q = 0$. Using $\mathfrak{W}(0) = 0$ we obtain for $q = 0$:

$$\widehat{\mathfrak{W}'}(r) = \frac{r}{\Psi_Y(r)}, r > 0. \tag{1.10}$$

1.3 Divided differences

Let us consider a function f which is m-times differentiable.

Definition 2. *We define the divided differences of f at m different points $x_1, \dots, x_m$ as*

$$f[x_1] = f(x_1),\ f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1},\ f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$
$$\vdots$$
$$f[x_1, \dots, x_m] = \frac{f[x_2, \dots, x_m] - f[x_1, \dots, x_{m-1}]}{x_m - x_1}$$

Under certain assumptions, the divided differences can be extended to the case when some of the $x_1, \dots, x_m$ are repeated, and they satisfy the following property: we suppose we have different numbers $x_1, x_1, x_2, \dots, x_k$ repeated, respectively $m_1, m_2, \dots, m_k$ times, and define for each $i = 0, 1, \dots, k$:

$$(x_i)_{m_i} = \underbrace{(x_i, x_i, \dots, x_i)}_{m_i - \text{times}}.$$

We let $x_1^j, x_2^j(n) \dots, x_{m_j}^j(n)$ for $j = 1, 2, \dots, k$ be $m = \sum_j m_j$ distinct numbers such that $\lim_{n\to\infty} x_{l_j}^j(n) \to x_1^j$ for each $j = 1, 2, \dots, k$ and $l_j = 2, \dots, m_j$, then the divided differences with repeated numbers, denoted by $f\left[(x_1^1)_{m_1}, \dots, (x_1^k)_{m_k}\right]$, satisfy the equalities

$$f\left[(x_1^1)_{m_1}, \dots, (x_1^k)_{m_k}\right] := \lim_{n\to\infty} f[x_1^1, \dots, x_{m_1}^1(n), x_1^2, \dots, x_{m_2}^2(n), \dots, x_1^k, \dots, x_{m_k}^k(n)]. \tag{1.11}$$

and (see, for instance Labbé et al. [2011], Corollary A.2):

$$f[(x_1^1)_{m_1}, \dots, (x_1^k)_{m_k}] = (-1)^{m-1} \sum_{j=1}^{k} \frac{(-1)^{m-m_j}}{(m_j - 1)!} \frac{\partial^{m_j - 1}}{\partial s^{m_j - 1}} \left[\frac{f(s)(x_1^j - s)^{m_j}}{\prod_{l=1}^{k} (x_1^l - s)^{m_l}} \right]_{s = x_1^j} \tag{1.12}$$

The formula above also implies the following equality for the case of different numbers $x_1, x_2, \ldots, x_k$:

$$f[x_1, \ldots, x_k] = (-1)^{k-1} \sum_{j=1}^{k-1} \frac{f(x_j)}{\prod_{l \neq j} (x_l - x_j)} \tag{1.13}$$

1.4 Weak convergence and the space $D_{\mathbb{E}}[0,\infty)$

Let us take a metric space (S,d) with Borel σ-algebra $\mathcal{S} := \mathcal{B}(S)$, and define $\mathcal{P}(S)$ as the family of Borel probability measures on S. We also define $C_b(S)$ as the set of continuous and bounded functions $f : S \to \mathbb{R}$. If $\{P_n\}$ is a sequence of probability measures in $\mathcal{P}(S)$, we say that P_n converges weakly to a probability measure $P \in \mathcal{P}(S)$ if for all $f \in C_b(S)$, we have the equality $\lim_{n\to\infty} P_n f = Pf$, where $Pf = \int f dP$. The space of probability measures $\mathcal{P}(S)$ can be topologized using Prohorov's metric (see, for instance, Billingsley [1999]), which we denote by ρ. In the case when (S,d) is separable, it can be proved (see Ethier and Kurtz [1986]) that $(\mathcal{P}(S), \rho)$ is separable. If, in addition, (S,d) is complete, then $(\mathcal{P}(S), \rho)$ is also complete.

It is proved in Ethier and Kurtz [1986], Theorem 3.1, Chapter 3, that if (S,d) is separable and $X, X_1, X_2, \ldots$ are S-valued random variables defined on the same probability space, with distributions $P, P_1, P_2, \ldots$, then the weak convergence $X_n \Rightarrow X$ is equivalent to $\lim_{n\to\infty} \rho(P_n, P) = 0$, where again ρ is the Prohorov's metric. If we do not have separability, then we only have that $\lim_{n\to\infty} \rho(P_n, P) = 0$ implies $X_n \Rightarrow X$.

We denote by $D_E := D_{\mathbb{E}}[0,\infty)$ the space of all the functions $x : [0,\infty) \to E$ such that $\lim_{s\to t+} x(s) = x(t)$ and the limit $\lim_{s\to t-} x(s) \equiv x(t-)$ exists. This is the space of all càdlàg functions from $[0,\infty)$ to E. It can be proved that if $x \in D_E$, then the set of discontinuities of x is at most countable.

Now we consider the space (D_E, d_S), where d_S is Skorokhod's metric (see, for instance, Billingsley [1999], Ethier and Kurtz [1986], and the references therein). It is proved, for instance, in Ethier and Kurtz [1986] that if E is separable then D_E is separable, and furthermore, if (E,r) is complete then (D_E, d_S) is complete.

If $\{X_n, n \geq 1\}$ is a sequence of stochastic processes such that their trajectories are elements of D_E, we say that X_n converges weakly to some X with trajectories in D_E, denoted by $X_n \Rightarrow X$, if $\lim_{n\to\infty} \mathbb{E}[f(X_n)] = \mathbb{E}[f(X)]$ for any $f : E \to \mathbb{R}$ continuous, bounded and measurable.

There are many criteria for this weak convergence to hold. We refer to Ethier and Kurtz [1986] and Billingsley [1999] for further reading on this topic. Here we only present the following result, which is used in the next chapter. The standard notations $\stackrel{d}{=}$ and $\stackrel{P}{\to}$ are used to denote, respectively, equality in distribution and convergence in probability.

Theorem 1. *Let $X, X_1, X_2, \ldots, X_n, \ldots$ be Lévy processes in $\mathbb{R}^d, d \geq 1$ such that $X_n(1) \stackrel{d}{\to} X(1)$, then there exists processes $\widetilde{X}_n \stackrel{d}{=} X_n$ such that $(\widetilde{X}_n - X)^*_t \stackrel{P}{\to} 0$, for all $t \geq 0$, where $(X)^*_t = \sup_{0\leq s\leq t} X(s)$.*

For the proof of the above theorem, we refer the reader to Theorem 15.17 in Kallenberg [2002]. We also need the following result, which can be found in Billingsley [1999], page 26. This is the corresponding Mapping Theorem for sequences of stochastic processes.

Theorem 2. *Let $X, X_1, \ldots, X_n, \ldots$ be stochastic processes taking values on some measurable space $(R, \mathcal{R})$. Let $h : R \to R'$ be $\mathcal{R}/\mathcal{R}'$-measurable, where $\mathcal{R}'$ denotes the σ-algebra associated to the space R'. Let D_h be the set of discontinuity points of h. Then, if $X_n \Rightarrow X$ and $\mathbb{P}[X \in D_h] = 0$, we have $h(X_n) \Rightarrow h(X)$.*

Chapter 2

EDPF for the α-stable case

In this chapter we consider a particular case of the process defined in (2.1). Let $V_\alpha = \{V_\alpha(t), t \geq 0\}$ be the Lévy risk process defined by

$$V_\alpha(t) = u + ct + Z_1(t) - Z_2(t) - \eta \mathcal{W}_\alpha(t), \eta \geq 0, \tag{2.1}$$

where $u, c \geq 0$, $Z_1 = \{Z_1(t), t \geq 0\}$ is a compound Poisson process with Lévy measure $\lambda_1 f_1(x)dx$, $\lambda_1 > 0$, $Z_2 = \{Z_2(t), t \geq 0\}$ is an independent compound Poisson process with Lévy measure $\lambda_2 f_2(x)dx$, $\lambda_2 > 0$, where f_2 is the density function of a nonnegative random variable, and $\mathcal{W}_\alpha = \{\mathcal{W}_\alpha(t), t \geq 0\}$ is an α-stable process with $\alpha \in (1, 2)$ and only positive jumps. Z_1, Z_2 and $\mathcal{W}_\alpha$ are assumed to be independent.

2.1 Weak approximations of V_α and convergence of Lundberg equations

We construct a sequence of classical two-sided jumps risk processes which converges weakly to V_α in the Skorokhod space $D_{\mathbb{R}}$, and then prove that the corresponding EDPF converge to the EDPF of V_α. We assume the following:

Hypothesis 1.

a) $\mathbb{E}\left[V_\alpha(1) - u\right] > 0$.

b) The upward density f_1 has a Laplace transform of the form (2).

c) There exists a positive constant B such that $\omega(x, y) \leq B$ for all $x, y \geq 0$.

d) If D_ω denotes the set of discontinuities of the function $\omega(x, y)$, then

$$\mathbb{P}\left[\Big(|V_\alpha(\tau_0)|, V_\alpha(\tau_0-)\Big) \in D_\omega\right] = 0.$$

Condition b) will be relaxed later. Many relevant penalty functions which satisfy the above assumptions arise as particular instances of ω in the following way:

1. If $\omega(x, y) \equiv a$ for some constant $a > 0$ we obtain that $\phi(u) = a\varphi_\delta(u)$, where $\varphi_\delta(u) = \mathbb{E}(e^{-\delta\tau_0} 1_{\{\tau_0<\infty\}})$ is the Laplace transform of the time to ruin when $\delta > 0$, and if $\delta = 0$ we obtain $\phi(u) = a\psi(u)$, where $\psi(u)$ is the ruin probability.
2. Putting $\omega(x, y) = 1_{\{x>a,y>b\}}$ for some constants $a, b > 0$ and $\delta = 0$, we obtain that ϕ is the joint tail of the severity of ruin $|V_\alpha(\tau_0)|$ and the surplus prior to ruin $V_\alpha(\tau_0-)$.
3. When $\delta > 0$ and $\omega(x, y) = e^{-sx-ty}$ for fixed constants $s, t \geq 0$, then ϕ is the trivariate Laplace transform of the time of ruin τ_0, the severity of ruin $|V_\alpha(\tau_0)|$ and the surplus before ruin $V_\alpha(\tau_0-)$.
4. If $\delta = 0$ and $\omega(x, y) = 1_{\{x+y>a\}}$ for some constant $a > 0$, then ϕ is the tail of the distribution of the claim that causes ruin.
5. If $\omega(x, y) = \max\{K - e^{a-y}, 0\}$ for some constants $K, a > 0$, then ϕ is a particular case of a payoff function in option pricing.

The penalty functions in examples 1, 3 and 5 are continuous, hence they satisfy Hypothesis 1 d) and they are clearly bounded. The penalty functions in examples 2 and 4 are also bounded, and it can be proved that $\mathbb{P}\left[|V_\alpha(\tau_0)| = a, V_\alpha(\tau_0-) = b\right] = 0$, for any $a, b > 0$ and $\mathbb{P}\left[|V_\alpha(\tau_0)| + V_\alpha(\tau_0-) = a\right] = 0$ for any $a > 0$.

Now we proceed to construct the sequence $\{V_n, n \geq 0\}$ of two-sided jumps classical risk processes such that $V_n \Rightarrow V_\alpha$, and prove afterward that the EDPF of V_n converges to the corresponding functional of V_α.

First, in order to avoid unnecessary technical complications due to the drift $c \geq 0$, we construct a sequence of processes $\{V_{n,k}, n, k \geq 0\}$ for which the prime c is 0, and a sequence of processes $\{V_n, n \geq 0\}$ with prime $c \geq 0$, such that $V_{n,k} \Rightarrow V_n$ for each fixed n. Next, we prove that the constructed sequence $\{V_n, n \geq 1\}$ satisfies the convergence $V_n \Rightarrow V_\alpha$.

The sequence $\{V_{n,k}(t),\ t \geq 0\}$ comes from the following result.

Theorem 3. *Let* $Z_1(t) = \sum_{i=1}^{N_1(t)} Y_{i1}$ *and* $Z_2(t) = \sum_{i=1}^{N_2(t)} Y_{i2}$. *For each* $k \in \mathbb{N}$ *and any fixed constant* $c > 0$, *let us define* $A(k) = 1 - (k+1)^{-1}$, $\lambda_1(k) = \lambda_1/(1 - A(k))$,

$b(k)=\lambda_1/[c(1-A(k))]$ and

$$p_k^*(x)=\begin{cases} A(k)b(k)e^{-b(k)x}+(1-A(k))f_1(x) & \textit{for } x>0,\\ 0 & \textit{for } x\le 0.\end{cases}$$

Consider the sequence of processes

$$V_k^{[c]}(t)=u+\sum_{i=1}^{N_{1,k}(t)}Y_{ik}^*-\sum_{i=1}^{N_2(t)}Y_{i2}:=u+Z_{1,k}(t)-Z_2(t),\quad k=1,2,\dots, \tag{2.2}$$

where $\{Z_{1,k}(t),\, t\ge 0\}$ is a compound Poisson process with intensity $\lambda_1(k)$, which is independent of Z_2, and $\{Y_{ik}^*,\, i=1,2,\dots\}$ is a sequence of independent and identically distributed random variables with common density function p_k^*. Then, as $k\to\infty$, $V_k^{[c]}\Rightarrow u+ct+Z_1(t)-Z_2(t)$.

Proof. For fixed t, let $\xi_{1,k}$, ξ_1 and ξ_2 be the characteristic functions of the random variables $V_k^{[c]}(t), X_{11}$, and X_{12}, respectively. Then,

$$\xi_{1,k}(s)=\exp\left\{uis+\lambda_1(k)t\left(\frac{A(k)b(k)}{b(k)-is}+(1-A(k))\xi_1(s)-1\right)+\lambda_2 t\left(\xi_2(-s)-1\right)\right\},$$

Since $A(k)=1-(k+1)^{-1}=k(k+1)^{-1}$, we have:

$$\begin{aligned}\lim_{k\to\infty}\lambda_1(k)\left(\frac{A(k)b(k)}{b(k)-is}-1\right) &= \frac{\lambda_1}{1-A(k)}\left(\frac{A(k)\frac{\lambda_1}{c(1-A(k))}}{\frac{\lambda_1}{c(1-A(k))}-is}-1\right) \qquad (2.3)\\ &=\lim_{k\to\infty}\frac{\lambda_1}{(k+1)^{-1}}\left(\frac{\frac{k}{k+1}\frac{\lambda_1(k+1)}{c}}{\frac{\lambda_1(k+1)}{c}-is}-1\right)\\ &=\lim_{k\to\infty}\frac{\lambda_1}{(k+1)^{-1}}\left(\frac{\lambda_1 k}{\lambda_1(k+1)-ics}-1\right)\\ &=\lim_{k\to\infty}\frac{\lambda_1}{(k+1)^{-1}}\left(\frac{\lambda_1 k-\lambda_1(k+1)+ics}{\lambda_1(k+1)-ics}-\right)\\ &=\lim_{k\to\infty}\frac{\lambda_1}{(k+1)^{-1}}\left(\frac{-\lambda_1+ics}{\lambda_1(k+1)-ics}-\right)\\ &=\lim_{k\to\infty}\lambda_1\left(\frac{-\lambda_1+ics}{\lambda_1-ics(k+1)^{-1}}-\right)\\ &=cis-\lambda_1. \qquad (2.4)\end{aligned}$$

Hence we obtain $\lim_{k\to\infty} \xi_{1,k}(s) = \exp\{uis + ctis + \lambda_1 t(\xi_1(s)-1) + \lambda_2 t(\xi_2(-s)-1)\}$, which is the characteristic function of $u + ct + Z_1(t) - Z_2(t)$. Now the result follows from Theorem 15.17 in Kallenberg (2002). ∎

Now we construct a sequence of processes $\{V_{n,k}, n, k \geq 0\}$ for which the prime c is 0, and a sequence of processes $\{V_n, n \geq 0\}$ with prime $c \geq 0$, such that $V_{n,k} \Rightarrow V_n$ for each fixed n and prove that $V_n \Rightarrow V_\alpha$.

Theorem 4. *For any fixed $n \in \mathbb{N}$, we set $c_n = c + n^{1-1/\alpha}\eta^\alpha$, and let the sequence of risk processes $V_{n,k} = \{V_{n,k}(t),\ t \geq 0\}$ be defined by*

$$V_{n,k}(t) = V_k^{[c_n]}(t) - \frac{1}{n^{1/\alpha}} \sum_{i=1}^{M(nt)} W_i,$$

where $\{V_k^{[c_n]}, k = 1, 2, \dots\}$ is defined as in (2.2) with $b(k) = \lambda_1/[c_n(1 - A(k))] := b_n(k)$ and $\sum_{i=1}^{M(nt)} W_i$ is a compound Poisson process independent of $V_k^{[c_n]}$. The Poisson process M has intensity η^α and $W_1, W_2, \dots$ are independent and identically distributed random variables with common distribution $S_\alpha(1,1,1)$. We also define the sequence of processes $V_n = \{V_n(t), t \geq 0\}$ by

$$V_n(t) = u + c_n t + Z_1(t) - Z_2(t) - \frac{1}{n^{1/\alpha}} \sum_{i=1}^{M(nt)} W_i. \tag{2.5}$$

Then $V_{n,k} \Rightarrow V_n$ for each $n \in \mathbb{N}$, and $V_n \Rightarrow V_\alpha$.

Proof. The first convergence follows from Theorem 3 and the independence of $\{W_i\}$, $\{Z_{1,k}\}$ and Z_2. For the proof of the second convergence, we note that since $W_i, i = 1, 2, ...$ have common distribution $S_\alpha(1,1,1)$, from (1.6) it follows that for each $n \geq 1$, $\frac{1}{n^{1/\alpha}} \sum_{k=1}^n (W_k - 1) \stackrel{d}{=} W_\alpha(1,1,0)$. Hence equality (3) in Furrer et al. (1997) holds with $\phi(n) = n^{1/\alpha}$, and since in our case $c^{(n)} = c + \eta^\alpha n^{1-1/\alpha}$, $\lambda = \eta^\alpha$, the hypothesis in Theorem 1 in Furrer et al. (1997) are fulfilled, and it follows $u + c_n t - n^{-1/\alpha}\sum_{i=1}^{M(nt)} W_i \Rightarrow u + ct - \eta W(t)$. Using now the independence of W, Z_1 and Z_2, we obtain the result. ∎

For any $1 < \alpha \leq 2$, let us denote by $\phi_{n,k}, \phi_n$ and ϕ the EDPF of the processes

$V_{n,k}, V_n$ and V_α, respectively, with corresponding Laplace transforms $\widehat{\phi}_{n,k}, \widehat{\phi}_n$ and $\widehat{\phi}$. The following result can be proved similarly as in Furrer (1998).

Theorem 5. *Under Hypothesis 1,* $\lim_{k\to\infty} \phi_{n,k}(u) = \phi_n(u)$ *for all* $u \geq 0$, *and each* $n \in \mathbb{N}$. *Moreover,* $\lim_{n\to\infty} \phi_n(u) = \phi(u)$.

Using partial fraction decomposition, it can be proved that, when f_1 satisfies (2), it admits the representation

$$f_1(x) = \sum_{i=1}^{N}\sum_{j=1}^{m_i} \beta_{ij} \frac{x^{j-1} q_i^j e^{-q_i x}}{(j-1)!},\ x > 0,\ \text{where } \beta_{ij} = \frac{1}{q_i^j (m_i - j)!} \frac{d^{m_i - j}}{dr^{m_i - j}} \left\{ \prod_{k=1, k\neq i}^{N} \frac{Q(r)}{q_k + r} \right\}\Bigg|_{r=-q_i},$$

hence

$$\widehat{f}_1(r) = \sum_{i=1}^{N}\sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i + r)^j}. \tag{2.6}$$

We also consider the following interpolation identity:

Lemma 1. *For each* $m \geq 1$ *and for any different non-zero complex numbers* $x_1, \ldots, x_{m+1}$,

$$\sum_{j=1}^{m+1} \frac{x_j^{-1}}{\prod\limits_{l=1, l\neq j}^{m+1} (x_l - x_j)} = \frac{1}{\prod\limits_j x_j}.$$

For $r \neq q_i$ we define the Generalized Lundberg functions associated to the processes $V_{n,k}, V_n$ and V_α, respectively by

$$\begin{aligned} L_{\alpha,n,k}(r) &= \lambda_2 \widehat{f}_2(r) + \lambda_1(k) A(k) \frac{b_n(k)}{b_n(k) - r} + \lambda_1(k)\left(1 - A(k)\right) \sum_{i=1}^{N}\sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} \\ &\quad + n\eta^\alpha \exp\left\{ -\frac{r}{n^{1/\alpha}} + \frac{r^\alpha}{n} \right\} - (\lambda_1(k) + \lambda_2 + n\eta^\alpha + \delta), \quad r \neq b_n(k), \quad (2.7) \\ L_{\alpha,n}(r) &= \lambda_2 \widehat{f}_2(r) + \lambda_1 \sum_{i=1}^{N}\sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} + \left(c + \eta^\alpha n^{1-1/\alpha}\right) r \quad (2.8) \\ &\quad + n\eta^\alpha \exp\left\{ -\frac{r}{n^{1/\alpha}} + \frac{r^\alpha}{n} \right\} - \left(n\eta^\alpha + \lambda_1 + \lambda_2 + \delta\right), \quad (2.9) \\ L_\alpha(r) &= \lambda_2 \widehat{f}_2(r) + \lambda_1 \sum_{i=1}^{n}\sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} + cr + (\eta r)^\alpha - (\lambda_1 + \lambda_2 + \delta). \quad (2.10) \end{aligned}$$

We denote

$$Q_1(r) = \prod_{k=1}^{N}(q_i - r)^{m_i}, \quad \mathbb{C}_+ = \{z \in \mathbb{C} : Re(z) \geq 0\}, \quad \text{and} \quad \mathbb{C}_{++} := \{z \in \mathbb{C} : Re(z) > 0\}.$$

For $\rho \in \mathbb{C}_+$ and $d > 0$ we put

$$B_d(\rho) = \{r \in \mathbb{C}_+ : |r - \rho| \leq d\}. \tag{2.11}$$

Recall that $s \in \mathbb{C}_+$ is a root of a function L of multiplicity $m \geq 1$ if $L(s) = \frac{dL}{dr}(s) = \cdots = \frac{d^{m-1}L}{dr^{m-1}}(s) = 0$ and $\frac{d^m L}{dr^m}(s) \neq 0$. We have the following results.

Lemma 2. *The function $P(r) = ar + br^\alpha - c$, for $a \geq 0$, $b, c > 0$ and $\alpha \in (1, 2)$, has exactly one positive and real root.*

Proof. Let us suppose that there exists a root s of $P(r)$ such that $Re(s) \geq 0$, $Im(s) \neq 0$ and $arg(s) = \theta$. Then by De Moivre's formula we obtain

$$a|s|\cos(\theta) + b|s|^\alpha \cos(\alpha\theta) - c = 0,$$

$$a|s|\sin(\theta) + b|s|^\alpha \sin(\alpha\theta) = 0. \tag{2.12}$$

We will see that (2.12) is only possible for $\theta = 0$. By the assumption that $Re(s) \geq 0$, we have $\theta \in [-\pi/2, \pi/2]$, hence if $0 < \theta \leq \pi/2$ we obtain $\alpha\theta \in (0, \pi)$, which implies $\sin(\alpha\theta) > 0$, hence $a|s|\sin(\alpha\theta) > 0$, and analogously for the case $\theta \in (-\pi/2, 0)$. Hence all the possible roots of P are real. Since for $r \geq 0$ we have

$$\frac{d}{dr}P(r) = a + b\alpha r^{\alpha-1} > 0 \textit{ and } \frac{d^2}{dr^2}P(r) = b\alpha(\alpha-1)r^{\alpha-2} > 0, \text{ for all } r > 0,$$

$P(r)$ is strictly increasing in the nonnegative real line, and noting that $P(0) = -c < 0$, we obtain the result. ∎

Proposition 1. *a) For all sufficiently large $n \in \mathbb{N}$ and $r \neq b_n(k)$, $\lim_{k\to\infty} L_{\alpha,n,k}(r) = L_{\alpha,n}(r)$ uniformly in r in sets of the form (2.11).*

b) Moreover, $\lim_{n\to\infty} L_{\alpha,n}(r) = L_\alpha(r)$, uniformly in sets of the form (2.11).

c) For $\delta \geq 0$, the functions L_α, $L_{\alpha,n}$ and $L_{\alpha,n,k}$ have exactly one root of multiplicity one in the interval $[0, q_1)$, which is equal to zero if and only if $\delta = 0$. We denote these roots by $\rho_{1,\delta}, \rho_{1,\delta}(n)$ and $\rho_{1,\delta}(n,k)$, respectively.

d) For $\delta \geq 0$ and $c+\eta > 0$, the function L_α has exactly $m+1$ roots $\rho_{1,\delta}, \rho_{2,\delta}, \ldots, \rho_{m+1,\delta}$ in $\mathbb{C}_+$. For $\delta > 0$ these roots are in $\mathbb{C}_{++}$, and if $\delta = 0$, $\rho_{1,\delta} = 0$ is the only root on the imaginary axis. Moreover, for all sufficiently large n and k, and all $\delta \geq 0$, the functions $L_{\alpha,n}$ and $L_{\alpha,n,k}$ also have $m+1$ roots in $\mathbb{C}_+$, which we denote respectively by $\rho_{1,\delta}(n), \ldots, \rho_{m+1,\delta}(n)$ and $\rho_{1,\delta}(n,k), \ldots, \rho_{m+1,\delta}(n,k)$. When $\delta > 0$ all these roots are in $\mathbb{C}_{++}$, and when $\delta = 0$, $\rho_{1,0}(n,k) = \rho_{1,0}(n) = 0$ are the only roots of $L_{n,k}$ and L_n, respectively, lying on the imaginary axis.

e) Let $c+\eta > 0$. For any $j \in \{1, 2, \ldots, m+1\}$ there exists $l \in \{1, 2, \ldots, m+1\}$ such that $\lim_{k\to\infty} \rho_{j,\delta}(n,k) = \rho_{l,\delta}(n)$ and $\lim_{n\to\infty} \rho_{j,\delta}(n) = \rho_{l,\delta}$.

f) We have $\lim_{\delta\to 0} \rho_{1,\delta} = \rho_{1,0} = 0$.

Proof. a) It suffices to consider the closed complex semicircle $B_d := B_d(0)$. For any $r \neq b_n(k)$ and $n \in \mathbb{N}$, $\lim_{k\to\infty} L_{\alpha,n,k}(r) = L_{\alpha,n}(r)$ due to (2.4). We will show that this convergence is uniform in B_d.

For $r \in B_d$ and $k > c_n d - \lambda_1$ we have $\lambda_1(k+1) - c_n r > 0$, and since

$$\begin{aligned}\left| c_n r - \lambda_1 - \lambda_1(k) A(k) \frac{b_n(k)}{b_n(k) - r} + \lambda_1(k) \right| &= \left| c_n r - \lambda_1 + \lambda_1 \frac{\lambda_1(k+1) - c_n(k+1) r}{\lambda_1(k+1) - c_n r} \right| \\ &= \left| \frac{c_n r(\lambda_1 - c_n r)}{\lambda_1(k+1) - c_n r} \right|,\end{aligned}$$

we obtain

$$\begin{aligned}\left| c_n r - \lambda_1 - \lambda_1(k) A(k) \frac{b_n(k)}{b_n(k) - r} + \lambda_1(k) \right| &\leq \frac{c_n d |\lambda_1 - c_n r|}{|\lambda_1(k+1) - c_n r|} \\ &\leq \frac{c_n d \lambda_1 + c_n^2 d^2}{\lambda_1(k+1) - c_n d},\end{aligned}$$

and the result follows.

b) First we prove that $\lim_{n\to\infty} L_{\alpha,n}(r) = L_\alpha(r)$. Using the series expansion of the exponential function, it follows that

$$n\eta^\alpha \widehat{g}_{\alpha,1}(r/n^{1/\alpha}) = n\eta^\alpha \exp\{-r/n^{1/\alpha} + r^\alpha/n\} = n\eta^\alpha - n^{1-1/\alpha}\eta^\alpha r + \eta^\alpha r^\alpha + a_n(r),$$

where $a_n(r) = n\eta^\alpha \sum_{k=2}^{\infty} \frac{\left(-\frac{r}{n^{1/\alpha}}+\frac{r^\alpha}{n}\right)^k}{k!}$. Hence we have

$$L_{\alpha,n}(r) = \lambda_2 \widehat{f}_2(r) + \lambda_1 \sum_{i=1}^{N}\sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} + cr + \eta^\alpha r^\alpha + a_n(r) - (\lambda_1 + \lambda_2 + \delta).$$

For sufficiently large n and $r \in B_d$, we have

$$\begin{aligned} |a_n(r)| = n\eta^\alpha \left| \sum_{k=2}^{\infty} \frac{\left(-\frac{r}{n^{1/\alpha}} + \frac{r^\alpha}{n}\right)^k}{k!} \right| &< n\eta^\alpha \sum_{k=2}^{\infty} \frac{|r^\alpha + r|^k}{n^{k/\alpha}} \leq n\eta^\alpha \sum_{k=2}^{\infty} \frac{C(d)^k}{n^{k/\alpha}} \\ &= n\eta^\alpha \frac{C(d)}{n^{1/\alpha}} \sum_{k=1}^{\infty} \frac{C(d)^k}{n^{k/\alpha}} = \eta^\alpha \frac{C(d)^2/n^{2/\alpha-1}}{1 - C(d)/n^{1/\alpha}}, \end{aligned}$$

where $C(d) = 2\max\{d^\alpha, d\}$ is a constant depending on d. Since $\alpha \in (1,2)$, the right-hand side in the above inequality converges to 0 as $n \to \infty$ uniformly in B_d, and the result is obtained.

c) We will prove that $L_{\alpha,n}$ has one real nonnegative root in $[0, q_1]$; the cases for the functions L and $L_{\alpha,n,k}$ can be handled in a similar way. We compute $\frac{d}{dr}L_{\alpha,n}(r)$:

$$\begin{aligned} \frac{d}{dr}L_{\alpha,n}(r) = -\lambda_2 \int_0^\infty xe^{-rx} f_2(x)dx + \lambda_1 \sum_{i=1}^{N}\sum_{j=1}^{m_i} \frac{\beta_{ij} j q_i^j}{(q_i - r)^{j+1}} + c + n^{1-1/\alpha}\eta^\alpha \\ + n\eta^\alpha \left(-\frac{1}{n^{1/\alpha}} + \frac{\alpha r^{\alpha-1}}{n}\right) \exp\left\{-\frac{r}{n^{1/\alpha}} + \frac{r^\alpha}{n}\right\}. \end{aligned}$$

From the equality above Hypothesis 1 we get $\frac{dL_{\alpha,n}}{dr}(0) = c + n^{1-1/\alpha}\eta^\alpha + \lambda_1\mu_1 -$

$\lambda_2\mu_2 > 0$. Moreover,

$$\frac{d^2 L_{\alpha,n}}{dr^2}(r) = \lambda_2 \int_0^\infty x^2 e^{-rx} f_2(x)dx + \lambda_1 \sum_{i=1}^{N} \sum_{j=1}^{m_i} \frac{\beta_{ij} j(j+1) q_i^j}{(q_i - r)^{j+2}}$$
$$+ n\eta^\alpha \left[\left(-\frac{1}{n^{1/\alpha}} + \frac{\alpha r^{\alpha-1}}{n} \right)^2 + \frac{\alpha(\alpha-1) r^{\alpha-2}}{n} \right] \exp\left\{ -\frac{r}{n^{1/\alpha}} + \frac{r^\alpha}{n} \right\} > 0,$$

for $r < q_1$, hence $L_{\alpha,n}(r)$ is increasing in $[0, q_1)$ with $L_{\alpha,n}(0) = -\delta$, and the result follows.

d) Let us define the functions $L^*(r) = Q_1(r) L_{\alpha,\delta}(r)$ and

$$L^{**}(r) = Q_1(r) \left[cr + \eta^\alpha r^\alpha - (\lambda_1 + \lambda_2 + \delta) \right].$$

Now we take $\delta > 0$ and consider, for fixed $s > 0$, the contour C_s as the imaginary axis together with a semicircle of radius s, moving clockwise from $-is$ to is. We note that

$$|L^*(r) - L^{**}(r)| = \left| Q_1(r) \left(\lambda_2 \widehat{f}_2(r) + \lambda_1 \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} \right) \right|.$$

Since $\lim_{|r|\to\infty} |L^{**}(r)| = \infty$ for any $c \geq 0$, for r in the semicircle and s sufficiently large, we have

$$\left| \lambda_2 \widehat{f}_2(r) + \lambda_1 \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} \right| \leq \lambda_1 + \lambda_2 < |cr + \eta^\alpha r^\alpha - (\lambda_1 + \lambda_2 + \delta)|, \tag{2.13}$$

for $c \geq 0$. For r in the imaginary axis, we have $r = i|r|\sin(\pi/2)$ and $r^\alpha = |r|^\alpha [\cos(\alpha\pi/2) + i\sin(\alpha\pi/2)]$. Using that $\alpha \in (1,2)$, it follows that $\cos(\alpha\pi/2) = \cos(-\alpha\pi/2) < 0$ and

$$\begin{aligned} &|cr + \eta^\alpha r^\alpha - (\lambda_1 + \lambda_2 + \delta)| \\ &= \sqrt{(\eta^\alpha |Re\,(r^\alpha)| + \lambda_1 + \lambda_2 + \delta)^2 + (cIm(r) + \eta^\alpha Im\,(r^\alpha))^2} \\ &> \lambda_1 + \lambda_2. \end{aligned} \tag{2.14}$$

which holds for $c \geq 0$. From (2.13) and (2.14) we obtain for sufficiently large s that $|L^*(r) - L^{**}(r)| < |L^{**}(r)|$. Now for $r \in \mathbb{R}\backslash\{0\}$, using that $sin(\theta) = -sin(-\theta)$ we obtain for $c \geq 0$:

$$|L(ir)| = \sqrt{[R_0 - \lambda_1 - \lambda_2 - \delta + \eta^\alpha |r|^\alpha cos(\alpha\pi/2)]^2 + [c|r| + \eta^\alpha |r|^\alpha sin(\alpha\pi/2) + I_0]^2}, \tag{2.15}$$

where

$$R_0 = Re(\lambda_2 \widehat{f}_2(ir)) + Re\left(\lambda_1 \sum_{j=1}^{N}\sum_{k=1}^{m_j} \beta_{jk}\left(\frac{q_j}{q_j - ir}\right)^k\right),$$
$$I_0 = Im(\lambda_2 \widehat{f}_2(ir)) + Im\left(\lambda_1 \sum_{j=1}^{N}\sum_{k=1}^{m_j} \beta_{jk}\left(\frac{q_j}{q_j - ir}\right)^k\right).$$

Since $\lambda_1 + \lambda_2 \geq |R_0|$ and $cos(\alpha\pi/2) < 0$ for $\alpha \in (1, 2)$, it follows that the right-hand side of (2.15) equals $\lambda_1 + \lambda_2 + \delta - R_0 - \eta^\alpha |r|^\alpha cos(\alpha\pi/2)$ and this is bounded from below by $\lambda_1 + \lambda_2 + \delta - R_0 - \eta^\alpha |r|^\alpha cos(\alpha\pi/2) > 0$. Hence we conclude that $|L_{\alpha,\delta}(ir)| > 0$ for all $r \neq 0$ and for $\delta \geq 0$, which implies that there are no roots on the imaginary axis when $\delta > 0$ and using c) we conclude that, when $\delta = 0$, the only root on the imaginary axis is $\rho_{1,\delta} = 0$, and such a root has multiplicity one.

Now applying Rouche's theorem we conclude that $L^*(r)$ has the same number of roots as $L^{**}(r)$ in C_s. Letting s tend to infinity we obtain the result for $\mathbb{C}_{++}$. Taking $P(r) = cr + \eta^\alpha r^\alpha - \lambda_1 - \lambda_2 - \delta$, by Lemma 2 we conclude that $L^{**}(r)$ has $m + 1$ roots in $\mathbb{C}_{++}$ for $c \geq 0$.

Now we prove the result about the number of roots of $L_{\alpha,n}(r)$. We take $L^{**}(r)$ as before and

$$L_n^*(r) = Q_1(r)\left[\lambda_2 \widehat{f}_2(r) + \lambda_1 \sum_{i=1}^{N}\sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} + cr + \eta^\alpha n^{1-1/\alpha} r + n\eta^\alpha e^{-\frac{r}{n^{1/\alpha}} + \frac{r^\alpha}{n}}\right.$$
$$\left. - n\eta^\alpha - (\lambda_1 + \lambda_2 + \delta)\right].$$

Then, for r in a semicircle with sufficiently large radius s, $0 < \varepsilon < \delta$ and n

sufficiently large:

$$\begin{aligned}
|L_n^*(r) - L^{**}(r)| &= |Q_1(r)| \left| \lambda_2 \widehat{f}_2(r) + \lambda_1 \sum_{i=1}^{N} \sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} + \eta^\alpha n^{1-1/\alpha} r \right. \\
&\left. + n\eta^\alpha e^{-\frac{r}{n^{1/\alpha}} + \frac{r^\alpha}{n}} - n\eta^\alpha - \eta^\alpha r^\alpha \right| \\
&\leq |Q_1(r)| \left| \lambda_2 \widehat{f}_2(r) + \lambda_1 \sum_{i=1}^{N} \sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} \right| + |Q_1(r)| \left| \eta^\alpha n^{1-1/\alpha} r \right. \\
&\left. + n\eta^\alpha e^{-\frac{r}{n^{1/\alpha}} + \frac{r^\alpha}{n}} - n\eta^\alpha - \eta^\alpha r^\alpha \right| \\
&\leq |Q_1(r)| \left(\lambda_1 + \lambda_2 + \varepsilon \right) < |Q_1(r)| \left| cr + \eta^\alpha r^\alpha - (\lambda_1 + \lambda_2 + \delta) \right| \\
&= L^{**}(r), \qquad (2.16)
\end{aligned}$$

where the last inequality follows for sufficiently large n using the uniform convergence of $\eta^\alpha n^{1-1/\alpha} r + \eta^\alpha e^{-\frac{r}{n^{1/\alpha}} + \frac{r^\alpha}{n}} - n\eta^\alpha$ to $\eta^\alpha r^\alpha$ in B_d.

Now for r in the imaginary axis we use (2.14) to obtain

$$\begin{aligned}
|L^{**}(r)| &> |Q_1(r)| \left(\lambda_1 + \lambda_2 + \delta \right) > |Q_1(r)| \left(\lambda_1 + \lambda_2 + \varepsilon \right) \\
&> |Q_1(r)| \left| \lambda_2 \widehat{f}_2(r) + \lambda_1 \sum_{i=1}^{N} \sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} + cr \right. \\
&\left. + \eta^\alpha n^{1-1/\alpha} r + \eta^\alpha e^{-\frac{r}{n^{1/\alpha}} + \frac{r^\alpha}{n}} - n\eta^\alpha - (\lambda_1 + \lambda_2 + \delta) \right| \\
&= \left| L_n^*(r) - L^{**}(r) \right|,
\end{aligned}$$

and the result follows by Rouche's theorem. The proof for $L_{\alpha,n,k}$ is analogous.

e) If $\lim_{k\to\infty} \rho_{j,\delta}(n,k) = r_j$ then from part a), $\lim_{k\to\infty} L_{\alpha,n,k}(\rho_{j,\delta}(n,k)) = L_{\alpha,n}(r_j) = 0$, hence r_j is a root of $L_{\alpha,n}(r)$. The second limit is obtained in the same way.

f) By the weak convergence of the stochastic processes with Laplace exponent $L_\alpha(r)$ with $\delta > 0$, to the stochastic process with Laplace exponent $L_\alpha(r)$ when $\delta = 0$, we know that $\lim\limits_{\delta\to 0} \rho_{1,\delta}$ exists. Let us suppose that $\lim\limits_{\delta\to 0} \rho_{1,\delta} = s_0 \in [0, q_1)$. Since

$L_{\alpha,\delta}(r) \to L_{\alpha,0}(r)$ uniformly when $\delta \to 0$ on $r \in [0, q_1)$, we obtain $s_0 = 0$ since $\rho_{1,0} = 0$ is the only root of L_α in $[0, q_1)$.

■

2.2 The Laplace Transform of ϕ

The following lemmas are needed.

Lemma 3. *For each $n \in \mathbb{N}$ and $r \in \mathbb{C}_+$, the integral*

$$I_n = n^{1+1/\alpha} \int_{-\infty}^{0} \left(1 - e^{-rx} - rx\right) g_{\alpha,1}(n^{1/\alpha}x)\, dx$$

exists and satisfies $\lim_{n\to\infty} |I_n| = 0$.

Proof. See Appendix A. ■

Lemma 4. *Under Hypothesis 1, the integral $K_0(n,k,r) = \lambda_1(k) \int_0^\infty \int_0^\infty e^{-ru} \phi_{n,k}(u + x) p_k^*(x)\, dx\, du$ is finite for all $r > 0$, and has the equivalent expression*

$$K_0(n,k,r) = \frac{P_{1,k}(r)}{Q_{1,k}(r)} \widehat{\phi}_{n,k}(r) - \frac{P_{2,k}(r)}{Q_{1,k}(r)},$$

where

$$P_{1,k}(r) = Q_{1,k}(r) \left[\frac{\lambda_1(k) A(k) b_n(k)}{b_n(k) - r} + \lambda_1 \sum_{i=1}^{N} \sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} \right],$$

$$P_{2,k}(r) = Q_{1,k}(r) \left[\frac{\lambda_1(k)(1 - A(k)) b_n(k)}{b_n(k) - r} \widehat{\phi}_{n,k}(b_n(k)) + \lambda_1 \sum_{i=1}^{N} \sum_{j=1}^{m_i} \beta_{ij} q_i^j \sum_{l=0}^{j-1} \frac{(q_i - r)^l \gamma_{l,i}(n,k)}{l!(q_i - r)^j} \right],$$

$$\gamma_{l,i}(n,k) = \int_0^\infty \phi_{n,k}(z) e^{-q_i z} z^l dz, \qquad Q_{1,k}(r) = (b_n(k) - r) \prod_{j=1}^{N} (q_j - r)^{m_j}.$$

Proof. See Appendix A. ∎

Let us define

$$M_{\alpha,n}(r) = n^{1+1/\alpha}\eta^{\alpha}\int_0^{\infty}\int_u^{\infty} e^{-ru}\omega(x-u,u)g_{\alpha,1}(n^{1/\alpha}x)dxdu,$$

$$M_{\alpha}(r) = \frac{\eta^{\alpha}\alpha(\alpha-1)}{\Gamma(2-\alpha)}\int_0^{\infty}\int_u^{\infty} e^{-ru}\omega(x-u,u)x^{-1-\alpha}dxdu, \quad 1<\alpha<2,$$

$$N_{\omega}(r) = \lambda_2\int_0^{\infty}\int_u^{\infty} e^{-ru}\omega(x-u,u)f_2(x)dxdu, \tag{2.1}$$

and note that $\frac{1}{\lambda_2}N_{\omega}(r) = \widehat{\xi_{\omega}}(r)$, where $\xi_{\omega}(u) = \int\limits_u^{\infty}\omega(x-u,u)f_2(x)dx$.

Lemma 5. *For any two complex numbers $r_1, r_2 \in \mathbb{C}_+$ with $r_1 \neq r_2$, there holds*

$$\lim_{n\to\infty} M_{\alpha,n}(r_1) - M_{\alpha,n}(r_2) = M_{\alpha}(r_1) - M_{\alpha}(r_2) \tag{2.2}$$

and

$$\lim_{\alpha\to 2}\frac{M_{\alpha}(r_1) - M_{\alpha}(r_2)}{r_2 - r_1} = \omega(0,0). \tag{2.3}$$

Proof. See Appendix A. ∎

In order to obtain simpler expressions for the Laplace transforms $\widehat{\phi}_{n,k}$ and for $\widehat{\phi}$, we impose the following condition.

Hypothesis 2. *For any $\delta \geq 0$ and $c \geq 0$, all roots of $L_{\alpha}(r)$ in $\mathbb{C}_+$ have multiplicity 1.*

Due to Proposition 1 e), Hypothesis 2 implies that, for all sufficiently large n and k, the roots of $L_{\alpha,n,k}$ and $L_{\alpha,n}$ have multiplicity 1.

We need the following functions:

$$\widehat{g_{\alpha,1}}(r/n^{1/\alpha}) = \int_{-\infty}^{0} e^{-rx/n^{1/\alpha}}g_{\alpha,1}(x)dx,$$

$$
\begin{aligned}
\widehat{g}^{+}_{\alpha,1}(r/n^{1/\alpha}) &= \int_0^\infty e^{-rx/n^{1/\alpha}} g_{\alpha,1}(x)dx, \\
T(\phi_{n,k}) &= n^{1+1/\alpha}\eta^\alpha \int_{-\infty}^0 \int_0^{-x} \phi_{n,k}(z) g_{\alpha,1}(n^{1/\alpha}x)dzdx, \\
A_n(r) &= n^{1+1/\alpha}\eta^\alpha \int_{-\infty}^0 \int_0^{-x} \left(e^{-r(x+z)} - 1\right) \phi_{n,k}(z) g_{\alpha,1}(n^{1/\alpha}x)dzdx \qquad (2.4) \\
K(n,r) &= n^{1+1/\alpha}\eta^\alpha \int_{-\infty}^0 \int_{-x}^\infty e^{-r(x+z)} \phi_{n,k}(z) g_{\alpha,1}(n^{1/\alpha}x)dzdx.
\end{aligned}
$$

Notice that $\phi_{n,k}(r) \leq B$ due to Hypothesis 1 c), hence

$$
A_n(r) \leq B\frac{n^{1+1/\alpha}\eta^\alpha}{r} \int_{-\infty}^{0} \left(1 - e^{-rx} - rx\right) g_{\alpha,1}(n^{1/\alpha}x)dx,
$$

and from by Lemma 2 it follows that $\lim_{n\to\infty} A_n(r) = 0$.

In the next theorem we obtain an expression for $\widehat{\phi}_{n,k}$.

Theorem 6. *Assume that Hypothesis 1 holds and that $(c,\eta) \neq (0,0)$. Then the Laplace transform of the EDPF $\phi_{n,k}$ of $V_{n,k}$ admits the representation*

$$
L_{\alpha,n,k}(r)\widehat{\phi}_{n,k}(r) = \frac{P_{1,k}(r)}{Q_{1,k}(r)} - N_\omega(r) - M_{\alpha,n}(r) - T(\phi_{n,k}) - A_n(r). \qquad (2.5)
$$

Moreover, under Hypothesis 1 and 2, we have for all $\delta \geq 0$,

$$
\begin{aligned}
&L_{\alpha,n,k}(r)\widehat{\phi}_{n,k}(r) \\
&= \sum_{l=1}^{m+1} \frac{Q_1(\rho_{l,\delta}(n,k))}{Q_{1,k}(r)} \frac{\prod_{i=1,i\neq l}^{m+1} (\rho_{i,\delta}(n,k) - r)}{\prod_{i=1,i\neq l}^{m+1} (\rho_{i,\delta}(n,k) - \rho_{l,\delta}(n,k))} \Big[(b_n(k) - \rho_{l,\delta}(n,k)) N_\omega(\rho_{l,\delta}(n,k)) \\
&- (b_n(k) - r)N_\omega(r) + (b_n(k) - \rho_{l,\delta}(n,k))A_n(\rho_{l,\delta}(n,k)) - (b_n(k) - r)A_n(r) \\
&+ (b_n(k) - \rho_{l,\delta}(n,k))M_{\alpha,n}(\rho_{l,\delta}(n,k)) - (b_n(k) - r)M_{\alpha,n}(r) \Big], \qquad (2.6)
\end{aligned}
$$

Proof. We consider a small time interval $(0, h)$ and condition on the first jump time and first claim size of $V_{n,k}$. This gives the equation

$$\begin{aligned}
\phi_{n,k}(u) = e^{-(\lambda_n+\delta)h}\phi_{n,k}(u) + \lambda_1(k)\int_0^h\int_0^\infty e^{-(\lambda_1(k)+\delta)t}\phi_{n,k}(u+x)p_k^*(x)dxdt \\
+ \lambda_2 \int_0^h\int_0^u e^{-(\lambda_2+\delta)t}\phi_{n,k}(u-x)f_2(x)dxdt \\
+ \lambda_2 \int_0^h\int_u^\infty e^{-(\lambda_2+\delta)t}\omega(x-u,u)f_2(x)dxdt \\
+ n^{1+1/\alpha}\eta^\alpha \int_0^h\int_0^u e^{-(n\eta^\alpha+\delta)t}\phi_{n,k}(u-x)g_{\alpha,1}(n^{1/\alpha}x)dxdt \\
+ n^{1+1/\alpha}\eta^\alpha \int_0^h\int_{-\infty}^0 e^{-(n\eta^\alpha+\delta)t}\phi_{n,k}(u-x)g_{\alpha,1}(n^{1/\alpha}x)dxdt \\
+ n^{1+1/\alpha}\eta^\alpha \int_0^h\int_u^\infty e^{-(n\eta^\alpha+\delta)t}\omega(x-u,u)g_{\alpha,1}(n^{1/\alpha}x)dxdt,
\end{aligned}$$

where $\lambda_n = \lambda_1(k) + \lambda_2 + n\eta^\alpha$.

Using a Taylor expansion in the exponential function in $e^{-(\lambda_n+\delta)h}\phi_{n,k}(u)$, we obtain:

$$\frac{\phi_{n,k}(u)}{h}$$
$$= \frac{1}{h}\Bigg((1-(\lambda_n+\delta)h + o(h))\phi_{n,k}(u) + \lambda_1(k)\int_0^h\int_0^\infty e^{-(\lambda_1(k)+\delta)t}\phi_{n,k}(u+x)p_k^*(x)dxdt$$

$$
\begin{aligned}
&+ \lambda_2 \int_0^h \int_0^u e^{-(\lambda_2+\delta)t} \phi_{n,k}(u-x) f_2(x) dx dt + \lambda_2 \int_0^h \int_u^\infty e^{-(\lambda_2+\delta)t} \omega(x-u,u) f_2(x) dx dt \\
&+ n^{1+1/\alpha}\eta^\alpha \int_0^h \int_0^u e^{-(n\eta^\alpha+\delta)t} \phi_{n,k}(u-x) g_{\alpha,1}(n^{1/\alpha}x) dx dt \\
&+ n^{1+1/\alpha}\eta^\alpha \int_0^h \int_{-\infty}^0 e^{-(n\eta^\alpha+\delta)t} \phi_{n,k}(u-x) g_{\alpha,1}(n^{1/\alpha}x) dx dt \\
&+ n^{1+1/\alpha}\eta^\alpha \int_0^h \int_u^\infty e^{-(n\eta^\alpha+\delta)t} \omega(x-u,u) g_{\alpha,1}(n^{1/\alpha}x) dx dt \Bigg).
\end{aligned}
$$

Letting $h \to 0$ and taking Laplace transforms, we obtain

$$
\begin{aligned}
&(\lambda_n + \delta)\widehat{\phi}_{n,k}(r) \\
&= \lambda_1(k) \int_0^\infty \int_0^\infty e^{-ru} \phi_{n,k}(u+x) p_k^*(x) dx du + \lambda_2 \int_0^\infty \int_0^u e^{-ru} \phi_{n,k}(u-x) f_2(x) dx du \\
&+ \lambda_2 \int_0^\infty \int_u^\infty e^{-ru} \omega(x-u,u) f_2(x) dx du + n^{1+1/\alpha}\eta^\alpha \int_0^\infty \int_{-\infty}^u e^{-ru} \phi_{n,k}(u-x) g_{\alpha,1}(n^{1/\alpha}x) dx du \\
&+ n^{1+1/\alpha}\eta^\alpha \int_0^\infty \int_u^\infty e^{-ru} \omega(x-u,u) g_{\alpha,1}(n^{1/\alpha}x) dx du \\
&= K_0(n,k,r) + \lambda_2 \widehat{\phi}_{n,k}(r)\widehat{f}_2(r) + N_\omega(r) + n\eta^\alpha \widehat{\phi}_{n,k}(r)\widehat{g}^+_{\alpha,1}(r/n^{1/\alpha}) + K(n,r) + M_{\alpha,n}(r).
\end{aligned}
\tag{2.7}
$$

Next, we obtain a more explicit expression for the function $K(n,r)$ defined in (2.4). Changing the order of integration and setting $z = u - x$ in (2.4) yields

$$
\begin{aligned}
K(n,r) &= n^{1+1/\alpha}\eta^{\alpha} \int_{-\infty}^{0} \int_{-x}^{\infty} e^{-r(x+z)} \phi_{n,k}(z) g_{\alpha,1}(n^{1/\alpha}x) dz dx \\
&\pm n^{1+1/\alpha}\eta^{\alpha} \int_{-\infty}^{0} \int_{0}^{-x} e^{-r(x+z)} \phi_{n,k}(z) g_{\alpha,1}(n^{1/\alpha}x) dz dx \\
&= n\eta^{\alpha}\widehat{\phi}_{n,k}(r)\widehat{g_{\alpha,1}^{-}}(r/n^{1/\alpha}) \\
&- n^{1+1/\alpha}\eta^{\alpha} \int_{-\infty}^{0} \int_{0}^{-x} e^{-r(x+z)} \phi_{n,k}(z) g_{\alpha,1}(n^{1/\alpha}x) dz dx \\
&= n\eta^{\alpha}\widehat{\phi}_{n,k}(r)\widehat{g_{\alpha,1}^{-}}(r/n^{1/\alpha}) - A_n(r) - T(\phi_{n,k}).
\end{aligned}
$$

From the last equality we get

$$
n\eta^{\alpha}\widehat{\phi}_{n,k}(r)\widehat{g}^{+}_{\alpha,1}(r/n^{1/\alpha}) + K(n,r) = n\eta^{\alpha}\widehat{\phi}_{n,k}(r)\widehat{g}_{\alpha,1}(r/n^{1/\alpha}) + T(\phi_{n,k}) + A_n(r),
$$

which, together with (2.7) and Lemma 4, yields

$$
\begin{aligned}
(\lambda_n + \delta)\widehat{\phi}_{n,k}(r) = \frac{P_{1,k}(r)}{Q_{1,k}(r)}\widehat{\phi}_{n,k}(r) - \frac{P_{2,k}(r)}{Q_{1,k}(r)} + \lambda_2\widehat{\phi}_{n,k}(r)\widehat{f}_2(r) \\
+ N_{\omega}(r) + n\eta^{\alpha}\widehat{\phi}_{n,k}(r)\widehat{g}_{\alpha,1}(r/n^{1/\alpha}) \\
- M_{\alpha,n}(r) - T(\widehat{\phi}_{n,k}) - A_n(r).
\end{aligned}
\tag{2.8}
$$

Since the function $L_{\alpha,n,k}$ has the equivalent expression

$$
L_{\alpha,n,k}(r) = \lambda_2\widehat{f}_2(r) + \frac{P_{1,k}(r)}{Q_{1,k}(r)} + n\eta^{\alpha}\widehat{g}(r/n^{1/\alpha}) - (\lambda_n + \delta),
$$

then (2.5) follows from (2.8).

Because of Hypothesis 2, all roots $\rho_{j,\delta}(n,k)$, $j = 1, \ldots, m+1$, have multiplicity 1. Hence, substituting $r = \rho_{j,\delta}(n,k)$ in (2.8) and using Lagrange interpolation we get

$$P_{2,k}(r) = \sum_{l=1}^{m+1} \frac{Q_{1,k}(\rho_{l,\delta}(n,k)) \prod_{i=1,i\neq l}^{m+1} (\rho_{i,\delta}(n,k) - r)}{\prod_{i=1,i\neq l}^{m+1} (\rho_{i,\delta}(n,k) - \rho_{l,\delta}(n,k))} \Big[N_\omega(\rho_{l,\delta}(n,k)) + M_{\alpha,n}(\rho_{l,\delta}(n,k)) + T(\widehat{\phi}_{n,k}) + A_n(\rho_{l,\delta}(n,k))\Big].$$

Hence from (2.5),

$$\begin{aligned}
&L_{\alpha,n,k}(r)\widehat{\phi}_{n,k}(r) \\
&= \sum_{l=1}^{m+1} \frac{Q_{1,k}(\rho_{l,\delta}(n,k))}{Q_{1,k}(r)} \frac{\prod_{i=1,i\neq l}^{m+1} (\rho_{i,\delta}(n,k) - r)}{\prod_{i=1,i\neq l}^{m+1} (\rho_{i,\delta}(n,k) - \rho_{l,\delta}(n,k))} (N_\omega(\rho_{l,\delta}(n,k)) + M_{\alpha,n}(\rho_{l,\delta}(n,k))) \\
&+ \sum_{l=1}^{m+1} \frac{Q_{1,k}(\rho_{l,\delta}(n,k))}{Q_{1,k}(r)} \frac{\prod_{i=1,i\neq l}^{m+1} (\rho_{i,\delta}(n,k) - r)}{\prod_{i=1,i\neq l}^{m+1} (\rho_{i,\delta}(n,k) - \rho_{l,\delta}(n,k))} \Big(T(\widehat{\phi}_{n,k}) + A_n(\rho_{l,\delta}(n,k))\Big) \\
&- \Big(N_\omega(r) + M_{\alpha,n}(r) + T(\widehat{\phi}_{n,k}) + A_n(r)\Big).
\end{aligned}$$

Using Lagrange interpolation and recalling that $Q_{1,k}(r) = (b_n(k) - r)Q_1(r)$, we get

$$\sum_{l=1}^{m+1} (b_n(k) - \rho_{l,\delta}(n,k)) Q_1(\rho_{l,\delta}(n,k)) \frac{\prod_{i=1,i\neq l}^{m+1} (\rho_{i,\delta}(n,k) - r)}{\prod_{i=1,i\neq l}^{m+1} (\rho_{i,\delta}(n,k) - \rho_{l,\delta}(n,k))} = (b_n(k) - r)Q_1(r).$$

Plugging this into the above equality we obtain

$$
\begin{aligned}
&L_{\alpha,n,k}(r)\widehat{\phi}_{n,k}(r)\\
&= \sum_{l=1}^{m+1} \frac{Q_1(\rho_{l,\delta}(n,k))}{Q_{1,k}(r)} \frac{\prod_{i=1,i\neq l}^{m+1}(\rho_{i,\delta}(n,k)-r)}{\prod_{i=1,i\neq l}^{m+1}(\rho_{i,\delta}(n,k)-\rho_{l,\delta}(n,k))} \Big[(b_n(k)-\rho_{l,\delta}(n,k))N_\omega(\rho_{l,\delta}(n,k))\\
&- (b_n(k)-r)N_\omega(r) + (b_n(k)-\rho_{l,\delta}(n,k))A_n(\rho_{l,\delta}(n,k)) - (b_n(k)-r)A_n(r)\\
&+ (b_n(k)-\rho_{l,\delta}(n,k))M_{\alpha,n}(\rho_{l,\delta}(n,k)) - (b_n(k)-r)M_{\alpha,n}(r)\Big],
\end{aligned}
$$

and (2.6) follows. ■

From Theorem 6 we obtain our main result in this section:

Theorem 7. *(Main Theorem I). Suppose Hypothesis 1 and 2 hold, and* $(c,\eta)\neq(0,0)$*. Then for all* $\delta\geq 0$ *the Laplace transform of the EDPF of the perturbed risk process* V_α *is given by*

$$
\widehat{\phi}(r) = \frac{\sum_{j=1}^{m+1} \frac{Q_1(\rho_{j,\delta})\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}-r)}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}-\rho_{j,\delta})}\left[N_\omega(\rho_{j,\delta})-N_\omega(r)+M_\alpha(\rho_{j,\delta})-M_\alpha(r)\right]}{L_\alpha(r)\left[\sum_{j=1}^{m+1}\frac{Q_1(\rho_{j,\delta})\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}-r)}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}-\rho_{j,\delta})}\right]}, \tag{2.9}
$$

or equivalently, by

$$
\widehat{\phi}(r) = \frac{\sum_{j=1}^{m+1} \frac{Q_1(\rho_{j,\delta})}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}-\rho_{j,\delta})}\left[\frac{N_\omega(\rho_{j,\delta})-N_\omega(r)}{\rho_{j,\delta}-r}+\frac{M_\alpha(\rho_{j,\delta})-M_\alpha(r)}{\rho_{j,\delta}-r}\right]}{L_\alpha(r)\left[\sum_{j=1}^{m+1}\frac{Q_1(\rho_{j,\delta})}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}-\rho_{j,\delta})(\rho_{j,\delta}-r)}\right]}. \tag{2.10}
$$

Proof. From Theorem 5, in order to obtain an expression for $\widehat{\phi}(r)$ we have to take limits in (2.6) firstly when $k \to \infty$ and afterward when $n \to \infty$. Since $\lim_{k\to\infty} b_n(k) = \infty$ implies

$$\lim_{k\to\infty} \frac{Q_1(\rho_{j,\delta}(n))(b_n(k) - \rho_{j,\delta}(n))}{Q_{1,k}(r)} = \lim_{k\to\infty} \frac{Q_1(\rho_{j,\delta}(n))(b_n(k) - \rho_{j,\delta}(n))}{Q_1(r)(b_n(k) - r)} = \frac{Q_1(\rho_{j,\delta}(n))}{Q_1(r)},$$

from (2.6) and Proposition 1 we get

$$L_{\alpha,n}(r)\widehat{\phi}_n(r) = \frac{\sum_{j=1}^{m+1} Q_1(\rho_{j,\delta}(n)) \frac{\prod_{i=1,i\neq j}^{m+1}(\rho_i(n)-r)}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}(n)-\rho_{j,\delta}(n))} \left[N_\omega(\rho_{j,\delta}(n)) + M_{\alpha,n}(\rho_{j,\delta}(n))\right]}{\left[\sum_{j=1}^{m+1} Q_1(\rho_{j,\delta}(n)) \frac{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}(n)-r)}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}(n)-\rho_{j,\delta}(n))}\right]}$$

$$- \frac{\sum_{j=1}^{m+1} Q_1(\rho_{j,\delta}(n)) \frac{\prod_{i=1,i\neq j}^{m+1}(\rho_i(n)-r)}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}(n)-\rho_{j,\delta}(n))} \left[N_\omega(r) - M_{\alpha,n}(r)\right]}{\left[\sum_{j=1}^{m+1} Q_1(\rho_{j,\delta}(n)) \frac{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}(n)-r)}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}(n)-\rho_{j,\delta}(n))}\right]}$$

$$- \frac{\sum_{j=1}^{m+1} Q_1(\rho_{j,\delta}(n)) \frac{\prod_{i=1,i\neq j}^{m+1}(\rho_i(n)-r)}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}(n)-\rho_{j,\delta}(n))} \left[A_n(\rho_{i,\delta}(n,k)) - A_n(r)\right]}{\left[\sum_{j=1}^{m+1} Q_1(\rho_{j,\delta}(n)) \frac{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}(n)-r)}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}(n)-\rho_{j,\delta}(n))}\right]}.$$

Identity (2.9) follows now by letting $n \to \infty$ in the above equality, and using Proposition 1e) and (2.2). The equality (2.10) follows immediately from (2.9) after multiplying and dividing by $\rho_{j,\delta} - r$ the j-th term in the sums in the numerator and denominator of (2.9). ∎

Remark 1. *Let us suppose that f_1 is the hyperexponential distribution with density*

$$f_1(x) = \sum_{l=1}^{m} A_l q_l e^{-q_l x}, \quad x > 0,$$

with $A_l > 0$ and $\sum_{l=1}^{m} A_l = 1$. In this case the roots of the Lundberg function L_α are all real and different; the proof of this fact is similar to that in Bowers et al. (1997), p. 422. If in addition $\eta = 0$, Theorem 7 above gives the result in Albrecher et al. (2010). In the case when $\widehat{f}_1(r) = q^m/(q+r)^m$ and $\eta = 0$, Theorem 7 gives Corollary 6.2 in Labbé et al *(2011) when $c > 0$.*

2.3 A Renewal Equation for ϕ

In this section we obtain expressions for the EDPF ϕ by inverting its Laplace transform, given in Theorem 7. The expressions we obtain are in terms of the operator T_r introduced in Dickson and Hipp [2001], defined in (1.2). This allows us to obtain a renewal equation for ϕ, which is of interest in Actuarial Mathematics.

For $r_1, r_2 \in \mathbb{C}_{++}$ and $r_1 \neq r_2$ we have:

$$\frac{M_\alpha(r_1) - M_\alpha(r_2)}{r_2 - r_1} = \eta^\alpha \widehat{T}_{r_2} m_\alpha(r_1) \tag{2.1}$$

where $m_\alpha(u) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_u^\infty \omega(x-u,u) x^{-1-\alpha}\, dx$. We also define

$$E(\rho_{j,\delta}) = \frac{Q_1(\rho_{j,\delta})}{\prod\limits_{l\neq j} (\rho_{l,\delta} - \rho_{j,\delta})}, \ j = 1, 2, \ldots, m+1.$$

The following corollary is a direct consequence of (2.10) and the definition of $N_\omega(r)$.

Corollary 1. *Assume that Hypothesis 1 and 2 hold. Then*

$$\phi(r) = h_{\alpha,\delta,\omega} * W_\delta(u), \ u > 0, \tag{2.2}$$

where

$$h_{\alpha,\delta,\omega}(u) = \sum_{j=1}^{m+1} E(\rho_{j,\delta}) T_{\rho_{j,\delta}} \left[\lambda_2 \xi_\omega + \eta^\alpha m_\alpha\right](u) \tag{2.3}$$

and $W_\delta(u), u > 0$ is the function with Laplace transform

$$\widehat{W}_\delta(r) = \left(-L_\alpha(r) \sum_{j=1}^{m+1} \frac{Q_1(\rho_{j,\delta})}{\prod_{i=1, i\neq j}^{m+1} (\rho_{i,\delta} - \rho_{j,\delta})(\rho_{j,\delta} - r)} \right)^{-1}. \tag{2.4}$$

Our next step is to show that the function $\widehat{W}_\delta(r)$ is related to the Laplace transform of the time to ruin when $\delta > 0$ and to the ruin probability when $\delta = 0$, and that it is the Laplace transform of some function $W_\delta(u)$ whose explicit form is given in Proposition 3 below. We recall that for $c > 0$ and $\alpha \in (1, 2)$, the tail of the extremal stable distribution $\zeta_{\alpha,c}$ is given by

$$\overline{\zeta}_{\alpha,c}(x) = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(1 + \{\alpha - 1\}n)} x^{n(\alpha-1)}, \quad x > 0,$$

and denote the density of $\zeta_{\alpha,c}$ by $z_{\alpha,c}$. Due to Lemma 1 in Furrer (1998), $\widehat{z}_{\alpha,c}(r)$ exists for all $r \geq 0$ and is given by

$$\widehat{z}_{\alpha,c}(r) = c/(c + r^{\alpha-1}).$$

Since $\rho_{j,\delta}$, $j = 1, 2, \ldots, m+1$ appear in conjugate pairs, it follows that for $\delta > 0$ we have $\prod_{j=2}^{m+1} \rho_{j,\delta} > 0$. Using the change of variables $\rho^*_{j,\delta}(r) = \rho_{j,\delta} - r$ and Lemma 1, one can show that if $\rho_{1,\delta}, \ldots, \rho_{m+1,\delta}$ are different complex numbers, and $P_l(x) =$

$a_l x^l + a_{l-1}x^{l-1} + \cdots + a_1 x + q_0$ is a polynomial of degree l, then for all $l \geq 1$,

$$\sum_{j=1}^{m+1} \frac{P_l(\rho_{j,\delta})}{\prod_{l=1,l\neq j}^{m+1}(\rho_{l,\delta} - \rho_{j,\delta})} = \begin{cases} 0 & \text{if } l = 0, 1, \ldots, m-1 \\ \\ (-1)^m a_m & \text{if } l = m. \end{cases} \tag{2.5}$$

The following two lemmas can be proved using Lemma 1 and (2.5), and the fact that the roots $\{\rho_{j,\delta}\}$ of L_α are in conjugate pairs.

Lemma 6. *For $\delta > 0$,*

$$\sum_{j=1}^{m+1} E(\rho_{j,\delta}) = 1 \quad \textit{and} \quad \sum_{j=1}^{m+1} E(\rho_{j,\delta})\rho_{j,\delta}^{-1} = \frac{\prod_{l=1}^{N} q_l^{m_l}}{\prod_{k=1}^{m+1} \rho_{k,\delta}}.$$

Lemma 7. *For any function $K : (0,\infty) \to [0,\infty)$ and all $\delta \geq 0$ the functions*

$$x \mapsto \sum_{j=1}^{m+1} E(\rho_{j,\delta}) T_{\rho_{j,\delta}} K(x), \qquad \textit{and} \qquad x \mapsto \sum_{j=1}^{m+1} E(\rho_{j,\delta})\rho_{j,\delta} T_{\rho_{j,\delta}} K(x), \quad x > 0,$$

are real-valued.

We define the function

$$\ell_\alpha(u) := \frac{(\alpha-1)u^{-\alpha}}{\Gamma(2-\alpha)}, \quad u > 0.$$

Although $\ell_\delta(u)$ is not integrable, the function $T_r \ell_\alpha(x)$ exists and is finite for all $x > 0$ and $r > 0$.

For all complex numbers $r_1, r_2 \in \mathbb{C}_+$, such that $r_1 \neq r_2$ and $\alpha \in (1,2)$, (see Zolotarev (1989), p. 10) it can be proved by integration by parts that

$$\int_0^\infty \left[e^{-r_1 x} - e^{-r_2 x}\right] x^{-\alpha} dx = \frac{\Gamma(2-\alpha)}{\alpha - 1}\left[r_2^{\alpha-1} - r_1^{\alpha-1}\right]. \tag{2.6}$$

It follows that

$$\widehat{T}_{r_1}\ell_\alpha(r_2) = \int_0^\infty e^{-r_2 x}\int_x^\infty e^{-r_1(y-x)}\ell_\delta(y)dydx = \frac{r_1^{\alpha-1}-r_2^{\alpha-1}}{r_1-r_2},\ r_1 \neq r_2,$$

and

$$\widehat{T}_{r_1}\ell_\alpha(r_1) = \int_0^\infty e^{-r_1 x}\int_x^\infty e^{-r_1(y-x)}\ell_\delta(y)dydx = (\alpha-1)r_1^{\alpha-2},\ r_1 \neq 0.$$

Let us define $f_{\alpha,\delta}(x) = \sum_{j=1}^{m+1} E(\rho_{j,\delta})\rho_{j,\delta}T_{\rho_{j,\delta}}\ell_\alpha(x)$ and $g_\delta(x) = \lambda_2 \sum_{j=1}^{m+1} E(\rho_{j,\delta})T_{\rho_{j,\delta}}f_2(x)$, for $x > 0$. Due to Lemma 7 the functions $h_{\alpha,\delta,\omega}$, $f_{\alpha,\delta}$ and g_δ are real-valued. In the sequel we assume the following condition.

Hypothesis 3. *The functions $h_{\alpha,\delta,\omega}$, $f_{\alpha,\delta}$ and g_δ defined above are nonnegative.*

It is straightforward to prove that Hypothesis 3 holds in the case when f_1 is a hyperexponential distribution and f_2 is a general density function, because in such case $E(\rho_{j,\delta})$ and $\rho_{j,\delta}$ are nonnegative numbers.

In the following proposition we obtain an alternative representation of $\widehat{W}_\delta$, which allows us to calculate its inverse Laplace transform.

Proposition 2. *Under Hypothesis 1, 2 and 3, we have*

a) For $\eta > 0$ and $c \geq 0$,

$$\widehat{W}_\delta(r) = \frac{\frac{1}{\eta^\alpha\theta_\delta}\widehat{\nu}_{\alpha,\delta}(r)}{1-\frac{1}{\theta_\delta}\left[\kappa_\delta\widehat{\nu}_{\alpha,\delta}(r)+\eta^{-\alpha}\widehat{g}_\delta(r)\widehat{\nu}_{\alpha,\delta}(r)\right]}, \tag{2.7}$$

where $\kappa_\delta = \frac{1}{\eta^\alpha}\widehat{g}_\delta(0) + \widehat{f}_{\alpha,\delta}(0)$ and $\theta_\delta = c/\eta^\alpha + \kappa_\delta$ are constants, and

$$\widehat{\nu}_{\alpha,\delta}(r) = \frac{\widehat{z}_{\alpha,\theta_\delta}(r)}{1+\frac{1}{\theta_\delta}\widehat{f}_{\alpha,\delta}(r)\widehat{z}_{\alpha,\theta_\delta}(r)}. \tag{2.8}$$

b) The function $\widehat{W}_\delta(r)$ is related to the time to ruin and the probability of ruin $\psi(u)$

by the following equalities:

$$\widehat{\varphi}_\delta(r) = \frac{1}{r} - \frac{\delta}{r}\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=1}^{m+1}\rho_{j,\delta}}\widehat{W}_\delta(r), \quad \delta > 0, \tag{2.9}$$

and

$$\widehat{\psi}(r) = \frac{1}{r} - \frac{c+\lambda_1\mu_1-\lambda_2\mu_2}{r}\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}\widehat{W}_0(r), \quad \delta = 0, \tag{2.10}$$

where $\varphi_\delta(u) = \mathbb{E}\left[e^{-\delta\tau_0}1_{\{\tau_0<\infty\}}|V_\alpha(0)=u\right]$ *is the Laplace transform of the ruin time for* $\delta > 0$.

Proof. See Appendix A. ∎

Corollary 2. *We also note from the results in Proposition 2 b) that, for* $u > 0$, *both* $\psi(u)$ *and* $\varphi_\delta(u)$ *are tails of probability distributions with respective densities* $\psi'(u) = (c+\lambda_1\mu_1-\lambda_2\mu_2)\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}W_0(u)$ *and* $\varphi'_\delta(u) = \delta\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=1}^{m+1}\rho_{j,\delta}}W_\delta(u)$. *Hence, from (2.2) the EDPF is given by the expressions*

$$\phi(u) = \begin{cases} \left[(c+\lambda_1\mu_1-\lambda_2\mu_2)\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}\right]^{-1} h_{\alpha,0,\omega} * \psi'(u) & \text{for } \delta = 0, \\ \left[\delta\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=1}^{m+1}\rho_{j,\delta}}\right]^{-1} h_{\alpha,\delta,\omega} * \varphi_\delta(u) & \text{for } \delta > 0. \end{cases}$$

Now we are ready to give a representation of W_δ as a series of convolutions of the functions $f_{\alpha,\delta}$, $g_\delta, \nu_{\alpha,\delta}$ defined above.

Proposition 3. *Under Hypothesis 1, 2 and 3 the following properties hold.*

a) For $r \geq 0$, *the function* $\widehat{\nu}_{\alpha,\delta}(r)$ *defined in (2.8) is the Laplace transform of the function*

$$\nu_{\alpha,\delta}(u) = z_{\alpha,\theta_\delta} * \sum_{n=0}^{\infty}\left[-\frac{1}{\theta_\delta}\right]^n \left[f_{\alpha,\delta} * z_{\alpha,\theta_\delta}\right]^{*n}(u). \tag{2.11}$$

b) For $u \geq 0$, the function $\widehat{W}_\delta(u)$ defined in (2.4) is the Laplace transform of the functions

$$W_\delta(u) = \frac{1}{\eta^\alpha \theta_\delta} \nu_{\alpha,\delta} * \sum_{n=0}^{\infty} \theta_\delta{}^{-n} \left[\kappa_\delta \nu_{\alpha,\delta} + \eta^{-\alpha} g_\delta * \nu_{\alpha,\delta}\right]^{*n}(u),$$

c) $\nu_{\alpha,\delta}(u)$ and $\frac{1}{\theta_\delta \eta^\alpha} g_\delta * \nu_{\alpha,\delta}(u) + \frac{\kappa_\delta}{\theta_\delta} \nu_{\alpha,\delta}(u)$ are defective density functions.

Proof. a) Since $0 < \frac{\widehat{f}_{\alpha,\delta}(0)}{\theta_\delta} < 1$, Hypothesis 3 implies $0 < \frac{\widehat{f}_{\alpha,\delta}(r)}{\theta_\delta} < 1$ for all $r \geq 0$. Hence the series

$$\widehat{z}_{\alpha,\theta_\delta}(r) \sum_{n=0}^{\infty} \left[-\frac{1}{\theta_\delta}\right]^n \left[\widehat{f}_{\alpha,\delta}(r) \widehat{z}_{\alpha,\theta_\delta}(r)\right]^n \tag{2.12}$$

is absolutely convergent for $r \geq 0$, and it equals (2.8). We set

$$\mathcal{I}(u) := \int_0^\infty \sum_{n=0}^{\infty} \left[\frac{1}{\theta_\delta}\right]^n z_{\alpha,\theta_\delta} * \left[f_{\alpha,\delta} * z_{\alpha,\theta_\delta}\right]^{*n}(u) du,$$

and apply monotone convergence theorem to obtain:

$$\begin{aligned}
\mathcal{I}(u) &= \lim_{m\to\infty} \sum_{n=0}^{m} \int_0^\infty \left[\frac{1}{\theta_\delta}\right]^n z_{\alpha,\theta_\delta} * \left[f_{\alpha,\delta} * z_{\alpha,\theta_\delta}\right]^{*n}(u) du \\
&\leq \lim_{m\to\infty} \sum_{n=0}^{m} \left| \left[-\frac{1}{\theta_\delta}\right]^n \widehat{z}_{\alpha,\theta_\delta}(0) \left[\widehat{f}_{\alpha,\delta}(0) \widehat{z}_{\alpha,\theta_\delta}(0)\right]^n \right| \\
&= \sum_{n=0}^{\infty} \left| \left[-\frac{1}{\theta_\delta}\right]^n \widehat{z}_{\alpha,\theta_\delta}(0) \left[\widehat{f}_{\alpha,\delta}(0) \widehat{z}_{\alpha,\theta_\delta}(0)\right]^n \right| < \infty.
\end{aligned}$$

This implies that the series

$$\sum_{n=0}^{\infty} \left[-\frac{1}{\theta_\delta}\right]^n z_{\alpha,\theta_\delta} * \left[f_{\alpha,\delta} * z_{\alpha,\theta_\delta}\right]^{*n}(u)$$

converges absolutely. Hence, we obtain (2.11) by inverting (2.12).

b) By Hypothesis 3 and the definition of $\widehat{\nu}_{\alpha,\delta}$ we have $\widehat{\nu}_{\alpha,\delta}(0) < 1$, hence $\nu_{\alpha,\delta}$ is a defective density function. From Proposition 2 and (2.7) we obtain

$$\widehat{\phi}(r) = \frac{\frac{1}{\eta^\alpha\theta_\delta}\widehat{\nu}_{\alpha,\delta}(r)}{-L_\alpha(r)\left[\sum_{j=1}^{m+1}\frac{Q_1(\rho_{j,\delta})}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}-\rho_{j,\delta})(\rho_{i,\delta}-r)}\right]\frac{\widehat{\nu}_{\alpha,\delta}(r)}{\eta^\alpha\theta_\delta}} = \frac{\frac{1}{\eta^\alpha\theta_\delta}\widehat{\nu}_{\alpha,\delta}(r)}{1-\frac{1}{\theta_\delta}\left[\kappa_\delta\widehat{\nu}_{\alpha,\delta}(r)+\eta^{-\alpha}\widehat{g}_\delta(r)\widehat{\nu}_{\alpha,\delta}(r)\right]}.$$

Hence, for all $r \geq 0$ we obtain the equality of denominators:

$$-L_\alpha(r)\sum_{j=1}^{m+1}\frac{Q_1(\rho_{j,\delta})}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}-\rho_{j,\delta})(\rho_{i,\delta}-r)}\frac{\widehat{\nu}_{\alpha,\delta}(r)}{\eta^\alpha\theta_\delta} = 1-\frac{\kappa_\delta\widehat{\nu}_{\alpha,\delta}(r)+\eta^{-\alpha}\widehat{g}_\delta(r)\widehat{\nu}_{\alpha,\delta}(r)}{\theta_\delta}.$$

Putting $r = 0$ in the above equality and using the second equality in (2.5), it follows that

$$\begin{aligned}1-\frac{1}{\theta_\delta}\left[\eta^{-\alpha}\widehat{g}_\delta(0)+\kappa_\delta\right]\widehat{\nu}_{\alpha,\delta}(0) &= -\frac{L_\alpha(0)\widehat{\nu}_{\alpha,\delta}(0)}{\eta^\alpha\theta_\delta}\sum_{j=1}^{m+1}\frac{Q_1(\rho_{j,\delta})}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}-\rho_{j,\delta})\rho_{j,\delta}} \\ &= \frac{\widehat{\nu}_{\alpha,\delta}(0)}{\eta^\alpha\theta_\delta}\frac{\delta\prod_{i=1}^{N}q_i^{m_i}}{\prod_{j=1}^{m+1}\rho_{j,\delta}} > 0,\end{aligned}$$

From the inequality above and the fact that $\widehat{\nu}_{\alpha,\delta}(0)$ and $\widehat{g}_\delta(0)+\kappa_\delta$ are always positive, it follows that $\frac{1}{\theta_\delta}\left[\eta^{-\alpha}\widehat{g}_\delta(0)+\kappa_\delta\right]\widehat{\nu}_{\alpha,\delta}(0) < 1$. Now using Hypothesis 3 we obtain

$$\frac{1}{\theta_\delta}\left[\eta^{-\alpha}\widehat{g}_\delta(r)+\kappa_\delta\right]\widehat{\nu}_{\alpha,\delta}(r) < 1$$

for all $r \geq 0$, which implies that $\frac{1}{\theta_\delta\eta^\alpha}g_\delta * \nu_{\alpha,\delta}(u)+\frac{\kappa_\delta}{\theta_\delta}\nu_{\alpha,\delta}(u)$ is a defective density function. The proof of this result for $\nu_{\alpha,\delta}(u)$ is analogous.

■

From (2.2) and Proposition 3 we obtain the main result in this section, in which we give a representation of $\phi(u)$ in terms of an infinite series of convolutions of $h_{\alpha,\delta,\omega}$ and the functions g_δ and $\nu_{\alpha,\delta}$ defined above, and the corresponding defective renewal equation for ϕ.

Theorem 8. *(Main Theorem II). Assume Hypothesis 1, 2 and 3. Then, for* $\eta > 0$, *the EDPF satisfies the defective renewal equation*

$$\phi(u) = \frac{1}{\theta_\delta}\int_0^u \phi(u-y)\left[\kappa_\delta \nu_{\alpha,\delta}(y) + \frac{1}{\eta^\alpha} g_\delta * \nu_{\alpha,\delta}(y)\right] dy + \frac{1}{\eta^\alpha \theta_\delta} h_{\alpha,\delta,\omega} * \nu_{\alpha,\delta}(u),$$

whose solution is given by

$$\phi(u) = \frac{1}{\eta^\alpha \theta_\delta} h_{\alpha,\delta,\omega} * \sum_{n=0}^{\infty} \nu_{\alpha,\delta}^{*(n+1)} * \left[\frac{\kappa_\delta}{\theta_\delta} + \frac{1}{\eta^\alpha \theta_\delta} g_\delta\right]^{*n}(u).$$

We note from Corollary 1 that the only dependence of ϕ on the penalty function ω appears in $h_{\alpha,\delta,\omega}(u)$, hence in order to obtain a formula for $\phi(u)$ for different penalty functions, we only need to calculate the corresponding function $h_{\alpha,\delta,\omega}$. Let us take $\omega(x,y) = e^{-sx-ty}$ for $s, t \geq 0$. Using (2.6), we obtain that, in this case, the function $h_{\alpha,\delta,\omega}$ defined before has the form

$$h_{\alpha,\delta,\omega}(u) = \sum_{j=1}^{m+1} E(\rho_{j,\delta}) T_{\rho_{j,\delta}} \left(\eta^\alpha f_{1,s,t} + \lambda_2 f_{2,s,t}\right)(u),$$

where $f_{1,s,t}(x) = e^{-tx}\ell_\alpha(x) - se^{-tx}T_s\ell_\alpha(x)$ and $f_{2,s,t}(x) = e^{-tx}T_s f_2(x)$.

Since $-\frac{\partial}{\partial s} e^{-sx-ty}|_{s=t=0} = x$, $-\frac{\partial}{\partial t} e^{-sx-ty}|_{s=t=0} = y$ and $\frac{\partial^2}{\partial s \partial t} e^{-sx-ty}|_{s=t=0} = xy$, for $\delta > 0$ the results of the previous theorem can be extended to the cases of penalty functions

$$\omega(x,y) = x, \quad \omega(x,y) = y \quad \text{and} \quad \omega(x,y) = xy, \tag{2.13}$$

which are not bounded. This can be shown by applying the dominated convergence theorem and calculating the corresponding derivatives of $h_{\alpha,\delta,\omega}$. In this way we obtain the following result.

Corollary 3. *Let* $\delta > 0$ *and*

$$h_{\alpha,\delta,\omega}(u) = \begin{cases} \sum_{j=1}^{m+1} E(\rho_{j,\delta}) T_{\rho_{j,\delta}} \left(\eta^\alpha \Lambda_\alpha + \lambda_2 \mu_2 \overline{F}_{2,I}\right)(u) & \text{if } \omega(x,y) = x, \\ \sum_{j=1}^{m+1} E(\rho_{j,\delta}) T_{\rho_{j,\delta}} \left((\alpha-1)\eta^\alpha \Lambda_\alpha + \lambda_2 G\right)(u) & \text{if } \omega(x,y) = y, \\ \sum_{j=1}^{m+1} E(\rho_{j,\delta}) T_{\rho_{j,\delta}} \left(\eta^\alpha \Lambda^*_\alpha + \lambda_2 G^*\right)(u) & \text{if } \omega(x,y) = xy, \end{cases} \tag{2.14}$$

where $G(u) = u\overline{F_2}(u), \Lambda_\alpha(u) = \int_u^\infty \ell_\delta(x)dx, \Lambda^*_\alpha(u) = uT_0\ell_\delta(u),\ G^*(u) = u\int_u^\infty (z-u)f_2(z)dz$ *and* $F_{2,I}$ *is the integrated tail distribution of* F_2. *Then Theorem 8 holds also for the penalty functions (2.13), with the same functions* g_δ *and* $\nu_{\alpha,\delta}$, *and corresponding functions* $h_{\alpha,\delta,\omega}$ *given by (2.14)*

2.4 Examples and conclusions

Here we illustrate how to obtain the above results for two particular cases of risk processes. We assume that $\lambda_1 = \lambda_2 = \eta = 1$, $c > 0$, $1 < \alpha \leq 2$, the penalty function ω is such that Hypothesis 1 holds.

Example 1. For given positive constants $a, b,$, let $f_1(x) = ae^{-ax}, x > 0$ and $f_2(x) = be^{-bx}$, $x > 0$. In this case the Lundberg's equation $L_\alpha(r) - \delta = 0$ is given by $\frac{b}{b+r} + \frac{a}{a-r} + cr + r^\alpha - 2 - \delta = 0$, and it has two roots in $\mathbb{C}_{++}$, denoted by ρ_1 and ρ_2.

These roots are real and satisfy the inequalities $\rho_1 < a < \rho_2 < b$. In order to obtain the EDPF ϕ for general penalty function ω from (2.2), we need to calculate the functions $h_{\alpha,\delta,\omega}$ and $W_{\alpha,\delta}$. We have

$$h_{\alpha,\delta,\omega}(u) = \frac{a-\rho_1}{\rho_2-\rho_1}\int_u^\infty e^{-\rho_1(x-u)}\int_x^\infty \omega(y-x,x)\left(\lambda_2 b e^{-by} + \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}y^{-1-\alpha}\right)dydx$$
$$+\frac{\rho_2-a}{\rho_2-\rho_1}\int_u^\infty e^{-\rho_2(x-u)}\int_x^\infty \omega(y-x,x)\left(\lambda_2 b e^{-by} + \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}y^{-1-\alpha}\right)dydx, \tag{2.15}$$

and from (A.19) we obtain

$$\widehat{W}_\delta(r) = \frac{1}{c + r^{\alpha-1} - \frac{b}{b+r}\frac{a+b}{(b+\rho_1)(b+\rho_2)} + \frac{a-\rho_1}{\rho_2-\rho_2}\rho_1\frac{\rho_1^{\alpha-1}-r^{\alpha-1}}{\rho_1-r} + \frac{\rho_2-a}{\rho_2-\rho_1}\rho_2\frac{\rho_2^{\alpha-1}-r^{\alpha-1}}{\rho_2-r}}. \tag{2.16}$$

Since $\alpha < 2$, the above formula does not admit a simple decomposition in partial fractions as in the case when when $\alpha = 2$. However, using the formula in Proposition 3 b) we obtain an expression for the inverse of $\widehat{W}_\delta$. This results to:

$$W_\delta(u) = \frac{1}{\eta^\alpha\theta_\delta}\nu_{\alpha,\delta} * \sum_{n=0}^\infty \theta_\delta{}^{-n}\left[\kappa_\delta\nu_{\alpha,\delta} + \int_0^\cdot e^{-bx}\nu_{\alpha,\delta}(\cdot - x)dx\right]^{*n}(u),$$

where $\kappa_\delta = \frac{a+b}{(b+\rho_1)(b+\rho_2)} + \frac{a-\rho_1}{\rho_2-\rho_1}\rho_1^{\alpha-1} + \frac{\rho_2-a}{\rho_2-\rho_1}\rho_2^{\alpha-1}$, $\theta_\delta = c + \frac{a+b}{(b+\rho_1)(b+\rho_2)} + \frac{a-\rho_1}{\rho_2-\rho_1}\rho_1^{\alpha-1} + \frac{\rho_2-a}{\rho_2-\rho_1}\rho_2^{\alpha-1}$, and by Proposition 3 the function $\nu_{\alpha,\delta}$ is given by

$$\nu_{\alpha,\delta}(u) = z_{\alpha,\theta_\delta} * \sum_{n=0}^\infty \left[-\frac{1}{\theta_\delta}\right]^n \left[\frac{a-\rho_1}{\rho_2-\rho_1}\rho_1\int_0^\cdot z_{\alpha,\theta_\delta}(\cdot - y)\int_y^\infty e^{-\rho_1(z-y)}\frac{(\alpha-1)z^{-\alpha}}{\Gamma(2-\alpha)}dzdy\right.$$
$$\left.+\frac{\rho_2-a}{\rho_2-\rho_1}\rho_2\int_0^\cdot z_{\alpha,\theta_\delta}(\cdot - y)\int_y^\infty e^{-\rho_2(z-y)}\frac{(\alpha-1)z^{-\alpha}}{\Gamma(2-\alpha)}dzdy\right]^{*n}(u). \tag{2.17}$$

Note that using (2.16), the formulae for the Laplace transforms of the ruin probability and of the Laplace transform of the ruin time given in Proposition 2b) are simple in this case.

Example 2. Now we assume that $f_1(x)$ is as in the previous example, and $f_2(x) = b^2 e^{-bx} x, x > 0$ (f_2 is an Erlang density with shape parameter $k = 2$ and scale parameter $b > 0$). In this case the Lundberg's equation is $\left(\frac{b}{b+r}\right)^2 + \frac{a}{a-r} + cr + r^\alpha - 2 - \delta = 0$, which has two roots in $\mathbb{C}_{++}$, denoted as ρ_1, ρ_2, which are real and satisfy the inequalities $\rho_1 < a < \rho_2$. In this case $\widehat{W}_\delta(r) = \frac{1}{\widetilde{L}_\alpha(r)}$, where

$$\begin{aligned}\widetilde{L}_\alpha(r) = c + r^{\alpha-1} - \Bigg\{ & \frac{a-\rho_1}{\rho_2-\rho_1}\left[\frac{b^2}{b+\rho_1}\frac{1}{(b+r)^2} + \frac{b^2}{(b+\rho_1)^2}\frac{1}{b+r}\right] \\ & + \frac{\rho_2-a}{\rho_2-\rho_1}\left[\frac{b^2}{b+\rho_2}\frac{1}{(b+r)^2} + \frac{b^2}{(b+\rho_2)^2}\frac{1}{b+r}\right]\Bigg\} \\ & + \frac{a-\rho_1}{\rho_2-\rho_1}\rho_1\frac{\rho_1^{\alpha-1} - r^{\alpha-1}}{\rho_1 - r} + \frac{\rho_2-a}{\rho_2-\rho_1}\rho_2\frac{\rho_2^{\alpha-1} - r^{\alpha-1}}{\rho_2 - r} \qquad (2.18)\end{aligned}$$

Again, this expression does not admit a partial fraction decomposition as in the case when $\alpha = 2$. Hence we use Proposition 3 b) to obtain:

$$\begin{aligned} & W_\delta(u) \\ & = \frac{1}{\eta^\alpha \theta_\delta}\nu_{\alpha,\delta} * \sum_{n=0}^{\infty} \theta_\delta{}^{-n} \Bigg\{ \kappa_\delta \nu_{\alpha,\delta} + \int_0^{\cdot} \Bigg(\frac{a-\rho_1}{\rho_2-\rho_1}\left[\frac{b^2}{b+\rho_1}e^{-bx}x + \frac{b^2}{(b+\rho_1)^2}e^{-bx}\right] \\ & + \frac{\rho_2-b}{\rho_2-\rho_1}\left[\frac{b^2}{b+\rho_2}e^{-bx}x + \frac{b^2}{(b+\rho_2)^2}e^{-bx}\right]\Bigg) \nu_{\alpha,\delta}(\cdot - x)dx \Bigg\}^{*n}(u),\end{aligned}$$

where $\kappa_\delta = \frac{a-\rho_1}{\rho_2-\rho_1}\left[\frac{1}{b+\rho_1} + \frac{b}{(b+\rho_1)^2}\right] + \frac{\rho_2-a}{\rho_2-\rho_1}\left[\frac{1}{b+\rho_2} + \frac{b}{(b+\rho_2)^2}\right] + \frac{a-\rho_1}{\rho_2-\rho_1}\rho_1^{\alpha-1} + \frac{\rho_2-a}{\rho_2-\rho_1}\rho_2^{\alpha-1}$ and $\theta_\delta = c + \frac{a-\rho_1}{\rho_2-\rho_1}\left[\frac{1}{b+\rho_1} + \frac{b}{(b+\rho_1)^2}\right] + \frac{\rho_2-a}{\rho_2-\rho_1}\left[\frac{1}{b+\rho_2} + \frac{b}{(b+\rho_2)^2}\right] + \frac{a-\rho_1}{\rho_2-\rho_1}\rho_1^{\alpha-1} + \frac{\rho_2-a}{\rho_2-\rho_1}\rho_2^{\alpha-1}$. The functions $h_{\alpha,\delta,\omega}$ and $v_{\alpha,\delta}$ have the same expressions as in (2.15) and (2.17), with the corresponding roots ρ_1 and ρ_2.

Although the formulae for $W_\delta(u)$ presented in these two examples are difficult to work with in general, the formulae for $\widehat{W}_\delta(r)$ are rather simple and their inverse Laplace transforms can be calculated by using numerical methods.

Since $h_{\alpha,\delta,\omega}$ and the constants in the formulae above can be calculated explicitly by knowing the process V_α and choosing δ and ω, the function W_δ becomes the most interesting object of study. For instance, the formulae given in Proposition 3 a) and b) allow the use of theoretical tools to obtain asymptotic expressions for $\nu_{\alpha,\delta}$ and W_δ.

These results can be used to obtain asymptotic expressions for the ruin probability, the Laplace transform of the time to ruin, the joint tail of the severity of ruin and the surplus prior to ruin and some other important cases of EDPFs. The asymptotic expressions for the ruin probability and the joint tail of the severity of ruin and the surplus prior to ruin are the main topic in the next section.

Finally, the function W_δ is related to the density of the negative Wiener-Hopf factor of the Lévy process V_α, which we study in further detail in Chapter 3.

2.5 Asymptotic results

We begin this section recalling some notation and results from previous sections. Again, $\psi(u)$ denotes the probability of ruin $\mathbb{P}[\tau_0 < \infty | V_\alpha(0) = u]$.

We denote by $\Upsilon_{a,b}(u)$ the joint tail of the severity of ruin and surplus prior to ruin of V_α, i.e. $\Upsilon_{a,b}(u) = \mathbb{P}[V_\alpha(\tau_0)| > a, V_\alpha(\tau_0-) > b, \tau_0 < \infty | V_\alpha(0) = u]$ for $u \geq 0,\ a, b > 0$.

We recall that $\psi(u)$ and $\Upsilon_{a,b}(u)$ are particular cases of $\phi(u)$ when $\delta = 0$ and, respectively, $\omega(x, y) = 1$ and $\omega(x, y) = 1_{\{x>a,y>b\}}$. We also recall, as it was proved in the previous section, that the Generalized Lundberg equation

$$cr + \eta^\alpha r^\alpha + \lambda_1 \widehat{f}_1(-r) + \lambda_2 \widehat{f}_2(r) - (\lambda_1 + \lambda_2) = 0,$$

has exactly $m + 1$ roots in the right-half complex plane $\{z \in \mathbb{C} : Re(z) > 0\}$, and $\rho_1 = 0$ is a root of the above equation with multiplicty 1.

Again we impose the conditions

a) The upward distribution F_1 has a density f_1, whose Laplace transform has the form (2).

b) For $c \geq 0$ we have the Net Profit Condition $\mathbb{E}[V_\alpha(1) - u] > 0$.

c) The $m + 1$ roots of the Generalized Lundberg equation when $\delta = 0$, denoted by

$\rho_1, \ldots, \rho_{m+1}$, are all different.

For $a > 0$, we denote again by $z_{\alpha;a}(u)$ the density of the Mittag-Leffler distribution with tail $\overline{\zeta}_{\alpha,a}(u)$ and Laplace transform $\widehat{z}_{\alpha;a}(r) = \frac{a}{a+r^{\alpha-1}}$. We recall that

$$E(\rho_{j,0}) := \frac{\prod_{l=1}^{N}(q_l-\rho_{j,0})^{m_l}}{\prod_{l\neq j}(\rho_l-\rho_{j,0})} \quad \text{and} \quad g_0(u) = \lambda_2 \sum_{j=1}^{m+1} E(\rho_{j,0}) T_{\rho_{j,0}} f_2(x)$$

where $T_r f$ is the Dickson-Hipp operator defined in (1.2), We denote by $f_\alpha(u)$ the function with Laplace transform

$$\widehat{f}_\alpha(r) = \sum_{j=2}^{m+1} E(\rho_{j,0})\rho_{j,0} \frac{\rho_{j,0}^{\alpha-1} - r^{\alpha-1}}{\rho_{j,0} - r},$$

and $\nu_\alpha(u)$ is the function whose Laplace transform satisfies the equality

$$\widehat{\nu}_\alpha(r)\left(1 + \frac{1}{\theta}\widehat{f}_\alpha(r)\widehat{z}_{\alpha,\theta}(r)\right) = \widehat{z}_{\alpha,\theta}(r), \tag{2.19}$$

where $\theta = c/\eta^\alpha + \kappa$ and $\kappa = \frac{1}{\eta^\alpha}\widehat{g}_0(0) + \widehat{f}_\alpha(0)$.

We assume f_α, g_0 and ν_α are nonnegative. This follows at least in the case of a convex sum of exponential densities with positive coefficients, as it was mentioned in the previous section.

From Proposition 2 b) we have the the Laplace transform of the ruin probability $\psi(u)$ satisfies the equality

$$\widehat{\psi}(r) = \frac{1}{r} - \frac{(c + \lambda_1\mu_1 - \lambda_2\mu_2)}{r} \frac{\prod_{j=1}^{N} q_j^{m_j}}{\prod_{j=2}^{m+1} \rho_{j,0}} \widehat{W_\delta}(r). \tag{2.20}$$

where $W_\delta(u)$ is the function with Laplace transform given in (2.7)

Since the roots of Lundberg's equation appear in conjugate pairs, we obtain that $\frac{\prod_{j=1}^{N} q_j^{m_j}}{\prod_{j=2}^{m+1} \rho_{j,0}} > 0$.

We recall the following definitions:

Let F be a distribution such that $F(0) = 0$, If there exist $c_1, c_2 > 0$ such that

$\overline{F}(x) \leq c_1 e^{-c_2 x}$ for all $x > 0$, we say that F is a light-tailed distribution. Otherwise, we say that F is a heavy-tailed distribution and write $F \in \mathcal{H}$.

If $\lim_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2$, we say that F belongs to the class of subexponential distributions, denoted by $F \in \mathcal{S}$.

We say that F belongs to the class $\mathcal{L}$ if for any $y \geq 0$, we have $\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1$.

F belongs to the class $\mathcal{R}_c$ for $c \geq 0$ if F has a density f and $\lim_{x\to\infty} \frac{f(x)}{\overline{F}(x)} = c$.

We say that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a regularly varying function of x at ∞, with order $a \in \mathbb{R}$, if $\lim_{x\to\infty} \frac{f(xt)}{f(x)} = t^a$, and write $f \in RV_a$. In the particular case when $a = 0$, we say that f is a slowly varying function of x at ∞.

If f is regularly varying of order a, then it can characterized as $f(x) = x^a L(x)$, for some slowly varying function $L(x)$.

If F is a distribution function such that $\overline{F}(x) \approx x^a L(x)$, we write $F \in \overline{RV}_a$ and we have the inclusions (see Rolski et al. [1999]):

$$\overline{RV}_a \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{H} \text{ and } \mathcal{R}_0 \subset \mathcal{L}.$$

For given functions $f : \mathbb{R} \to \mathbb{R},\ g : \mathbb{R} \to \mathbb{R}$, we say that $f(x) \approx cg(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = c$ for some $c \in (0, \infty)$. We write $f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

The following result is required.

Lemma 8. *Let $H = F_1 * F_2$ be the convolution of two distribution functions such that $F_i(0) = 0, i = 1, 2$.*

a) If $F_2 \in \mathcal{S}$ and $\overline{F}_1(x) = o(\overline{F}_2(x))$ as $x \to \infty$, then $H \in \mathcal{S}$. Moreover, $\overline{H}(x) \approx \overline{F}_2(x)$.

b) If $\overline{F}_i(x) \approx x^{-\delta} L_i(x)$ for $i = 1, 2$, where $L_1(x), L_2(x)$ are slowly varying functions, then $\overline{H}(x) \approx x^{-\delta} (L_1(x) + L_2(x))$ as $x \to \infty$.

c) If $\overline{F}_2(x) \approx c\overline{F}_1(x)$ for some $c \in (0, \infty)$, then $F_1 \in \mathcal{S}$ if and only if $F_2 \in \mathcal{S}$ and $\overline{H} \approx (1 + c)\overline{F}_2(x)$.

d) If $\beta \in (0,1)$ and $K(x) = (1-\beta)\sum_{n=0}^{\infty} \beta^n F_1^{*n}(x)$ then the following are equivalent:

$$K \in \mathcal{S}, \quad F_1 \in \mathcal{S}, \quad \overline{K}(x) \approx \tfrac{\beta}{1-\beta}\overline{F}_1(x).$$

Proof. For a) and d), see Proposition 1a) and Theorem 3, respectively, in Embrechts et al. [1979]. For b) see Feller [1971], p. 278. The proof of c) is given in Lemmas 2.5.2. and 2.5.4. in Rolski et al. [1999].

■

We also need the following result which can be found, for instance, in Feller [1971].

Lemma 9. *Let $\ell(x)$ be a slowly varying function and $U(x)$ a nondecreasing right-continuous function on $\mathbb{R}$ with density $u(x)$ and such that $U(x) = 0$ for all $x < 0$. If $c \geq 0$, $\beta \geq 0$ and $u(x)$ is monotone in some interval (x_0, ∞), then:*

$$\lim_{x\to\infty} \frac{u(x)}{\ell(x)x^{\beta-1}} = \frac{c}{\Gamma(\beta)} \quad \textit{if and only if} \quad \lim_{r\downarrow 0} \frac{r\widehat{U}(r)}{\ell(1/r)r^{-\beta}} = c$$

The case $c = 0$ is equivalent to

$$u(x) = o\left(\ell(x)x^{\beta-1}\right) \quad \textit{if and only if} \quad \lim_{r\downarrow 0} \frac{r\widehat{U}(r)}{\ell(1/r)r^{-\beta}} = 0 \tag{2.21}$$

2.5.1 Asymptotic behavior of the ruin probability

For $x > 0$ we define the functions

$$F_\alpha(x) = \frac{1}{\mathcal{C}_F}\int_{0+}^{x} f_\alpha(y)dy, \ \overline{F}_\alpha(x) = 1 - F_\alpha(x) = \frac{1}{\mathcal{C}_F}\int_{x}^{\infty} f_\alpha(y)dy,$$

$$G_0(x) = \frac{1}{\mathcal{C}_G}\int_{0+}^{x} g_0(y)dy, \ \overline{G}_0(x) = 1 - G_0(x) = \frac{1}{\mathcal{C}_G}\int_{x}^{\infty} g_0(y)dy,$$

$$U_\alpha(x) = \frac{1}{\mathcal{C}_U} \int\limits_{0+}^{x} \nu_\alpha(y)dy, \ \overline{U}_\alpha(x) = 1 - U_\alpha(x),$$

where $\mathcal{C}_F = \widehat{f}_\alpha(0)$, $\mathcal{C}_G = \widehat{g}_0(0)$ and $\mathcal{C}_U = \widehat{\nu}_\alpha(0)$.

Let F be any distribution function with density f and tail $\overline{F}$. Throughout this section we will use the equalities

$$\widehat{F}(r) = \frac{\widehat{f}(r)}{r} \text{ and } \widehat{\overline{F}}(r) = \frac{1 - \widehat{f}(r)}{r} \tag{2.22}$$

In what follows, $F_{2,I}(x)$ denotes the tail distribution of F_2 defined as $F_{2,I}(x) = \frac{1}{\mu_2} \int\limits_0^x \overline{F}_2(y)dy$.

We use Lemma 8 and Lemma 9 to prove the following result.

Proposition 4. *$F_\alpha, U_\alpha \in \mathcal{S}$ and*

a) We have

$$\lim_{x\to\infty} \frac{\overline{F}_\alpha(x)}{\frac{x^{1-\alpha}}{\Gamma(2-\alpha)}} = \frac{1}{\mathcal{C}_F} \left(1 - \frac{\prod\limits_{i=1}^{N} q_i^{m_i}}{\prod\limits_{j=2}^{m+1} \rho_{j,0}} \right). \tag{2.23}$$

b) If $F_2 \in \mathcal{R}_0$, then

$$\lim_{x\to\infty} \frac{\overline{G}_0(x)}{\overline{F}_{2,I}(x)} = \frac{\lambda_2\mu_2}{\mathcal{C}_G} \frac{\prod\limits_{i=1}^{N} q_i^{m_i}}{\prod\limits_{j=2}^{m+1} \rho_{j,0}}. \tag{2.24}$$

Moreover, if $F_{2,I} \in \mathcal{S}$, then $G_0 \in \mathcal{S}$.

c) If $\overline{F}_2(x) = o(x^{-\alpha})$, then $\overline{G}_0(x) = o(x^{1-\alpha})$.

d) There holds

$$\lim_{x\to\infty}\frac{\overline{U}_\alpha(x)}{\overline{\zeta}_{\alpha,\theta}(x)}=\mathcal{C}_U\frac{\prod\limits_{i=1}^{N}q_i^{m_i}}{\prod\limits_{j=2}^{m+1}\rho_{j,0}}. \tag{2.25}$$

Proof. See Appendix A. ∎

Throughout this section we consider the following cases for the claims distribution F_2 :

$$\begin{aligned}
&\text{Case 1}: && \overline{F}_2(x)=o(x^{-\alpha}),\\
&\text{Case 2}: && \overline{F}_2(x)\approx\kappa x^{-\alpha}\ \text{for some}\ \ \kappa>0,\\
&\text{Case 3}: && x^{-\alpha}=o\left(\overline{F}_2(x)\right)\ \text{for}\ F_{2,I}\in\mathcal{S}\ \text{and}\ F_2\in\mathcal{R}_0.
\end{aligned} \tag{2.26}$$

Now we are ready to obtain the asymptotic expressions for the probability of ruin.

Theorem 9.

a) In case 1:

$$\psi(u)\approx\frac{\eta^\alpha}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\,\Gamma(2-\alpha)}u^{1-\alpha}, \tag{2.27}$$

b) In case 2:

$$\psi(u)\approx\frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)}+\frac{\lambda_2\kappa}{\alpha-1}\right]u^{1-\alpha}, \tag{2.28}$$

c) In case 3:

$$\psi(u)\approx\frac{\lambda_2\mu_2}{c+\lambda_1\mu_1-\lambda_2\mu_2}\overline{F}_{2,I}(u), \tag{2.29}$$

and in all cases $\Phi\in\mathcal{S}$.

Proof. Case 1. We define the function $G_0^*(x)=\int\limits_0^x\overline{G}_0(y)dy, x>0$. By (2.22) we have $\widehat{G}_0^*(r)=\frac{1-\widehat{g}_0(r)}{r^2}$. By Proposition 4 c) and the assumption that $\overline{F}_2(x)=o(x^{-\alpha})$

we have $\overline{G}_0(x) = o\left(\overline{\zeta}_{\alpha,\theta}(x)\right)$, hence (2.21) and the equality $\widehat{G}_0^*(r) = \frac{1-\widehat{g}_0(r)}{r^2}$ imply

$$0 = \lim_{r\downarrow 0} \frac{r\widehat{G}_\alpha^*(r)}{r^{\alpha-2}} = \lim_{r\downarrow 0} \frac{1 - \dfrac{\widehat{g}_0(r)}{\mathcal{C}_G}}{r^{\alpha-1}}. \tag{2.30}$$

Let us denote by $\psi(\infty)$ the limit $\lim\limits_{u\to\infty} \psi(u)$. By the final value theorem for Laplace transforms we have $\psi(\infty) = \lim\limits_{r\downarrow 0} r\widehat{\psi}(r)$, which, by (2.20), implies $\psi(\infty) = 1 - \frac{\prod_{j=1}^N q_j^{m_j}}{\prod_{j=1}^R \rho_{j,\delta}^{k_j}} \widehat{W}_\delta(0)$.

Since $\psi(u)$ is the tail of a probability distribution, we know that $\psi(\infty) = 0$, hence the equality above implies $\widehat{W}_\delta(0) = \frac{1}{\frac{\prod_{j=1}^N q_j^{m_j}}{\prod_{j=1}^R \rho_{j,\delta}^{k_j}}}$. Setting $r = 0$ in (2.7) we obtain

$$\frac{1}{\frac{\prod_{j=1}^N q_j^{m_j}}{\prod_{j=1}^R \rho_{j,\delta}^{k_j}}} = \frac{\frac{1}{\eta^\alpha\theta}\widehat{\nu}_\alpha(0)}{1 - \frac{1}{\theta}\left[\kappa\widehat{\nu}_\alpha(0) + \frac{1}{\eta^\alpha}\widehat{g}_0(0)\widehat{\nu}_\alpha(0)\right]} = \frac{\frac{1}{\eta^\alpha\theta}C_U}{1 - \frac{1}{\theta}\left[\kappa C_U + \frac{1}{\eta^\alpha}C_G C_U\right]}.$$

or equivalently

$$\frac{\prod_{j=1}^N q_j^{m_j}}{\prod_{j=1}^R \rho_{j,\delta}^{k_j}} = \frac{1 - \frac{1}{\theta}\left[\kappa C_U + \frac{1}{\eta^\alpha}C_G C_U\right]}{\frac{1}{\eta^\alpha\theta}C_U}. \tag{2.31}$$

Now we set $\psi^*(u) = \int\limits_0^u \psi(y)dy$. By (2.22), (2.20) and (2.31) we have

$$\widehat{\psi}^*(r) = \frac{1 - \left[\frac{1-\frac{1}{\theta}\left[\kappa C_U + \frac{1}{\eta^\alpha}C_G C_U\right]}{\frac{1}{\eta^\alpha\theta}C_U}\right] \frac{\frac{1}{\eta^\alpha\theta}\widehat{\nu}_\alpha(r)}{1-\frac{1}{\theta}\left[\kappa\widehat{\nu}_\alpha(r) + \frac{1}{\eta^\alpha}\widehat{g}_0(r)\widehat{\nu}_\alpha(r)\right]}}{r^2}$$

$$= \frac{1 - \frac{1}{\theta}\left[\kappa\widehat{\nu}_\alpha(r) + \frac{1}{\eta^\alpha}\widehat{g}_0(r)\widehat{\nu}_\alpha(r)\right] - \left[\frac{1-\frac{1}{\theta}\left[\kappa C_U + \frac{1}{\eta^\alpha}C_G C_U\right]}{C_U}\right]\widehat{\nu}_\alpha(r)}{r^2\left(1 - \frac{1}{\theta}\left[\kappa\widehat{\nu}_\alpha(r) + \frac{1}{\eta^\alpha}\widehat{g}_0(r)\widehat{\nu}_\alpha(r)\right]\right)} \tag{2.32}$$

It follows that:

$$\begin{aligned}
&\lim_{r\downarrow 0}\frac{r\widehat{\psi}^*(r)}{r^{\alpha-2}} \\
&= \lim_{r\downarrow 0}\frac{1}{r^{\alpha-1}}\left(\frac{1 - \frac{\kappa\widehat{\nu}_{\alpha,0}(r)+\eta^{-\alpha}\widehat{g}_0(r)\widehat{\nu}_{\alpha,0}(r)}{\theta} - \left[\frac{1-\frac{1}{\theta}\left[\kappa\mathcal{C}_U + \eta^{-\alpha}\mathcal{C}_G\mathcal{C}_U\right]}{\mathcal{C}_U}\right]\widehat{\nu}_{\alpha,0}(r)}{1 - \frac{1}{\theta}\left[\kappa\widehat{\nu}_{\alpha,0}(r) + \eta^{-\alpha}\widehat{g}_0(r)\widehat{\nu}_{\alpha,0}(r)\right]}\right) \\
&= \lim_{r\downarrow 0}\frac{1}{r^{\alpha-1}}\left(\frac{1 - \frac{\widehat{\nu}_{\alpha,0}(r)}{C_U} + \frac{\mathcal{C}_G}{\eta^\alpha\theta}\left[1 - \frac{\widehat{g}_0(r)}{\mathcal{C}_G}\right]\widehat{\nu}_{\alpha,0}(r)}{1 - \frac{1}{\theta}\left[\kappa\widehat{\nu}_{\alpha,0}(r) + \eta^{-\alpha}\widehat{g}_0(r)\widehat{\nu}_{\alpha,0}(r)\right]}\right) \qquad (2.33) \\
&= \frac{\frac{1}{\theta}\frac{\prod_{i=1}^{N} q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}\mathcal{C}_U}{1 - \frac{1}{\theta}\left[\kappa\mathcal{C}_U + \eta^{-\alpha}\mathcal{C}_G\mathcal{C}_U\right]}.
\end{aligned}$$

where the last equality follows from (A.37), (A.39) and (2.30).

From (2.31) we obtain

$$\frac{\frac{1}{\theta}\frac{\prod_{i=1}^{N} q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}\mathcal{C}_U}{\frac{c+\lambda_1\mu_1-\lambda_2\mu_2}{\eta^\alpha\theta}\frac{\prod_{i=1}^{N} q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}\mathcal{C}_U} = \frac{\eta^\alpha}{c+\lambda_1\mu_1-\lambda_2},$$

and hence equality (2.33) is equivalent to $\lim_{r\downarrow 0}\frac{r\widehat{\psi}^*(r)}{r^{\alpha-2}} = \frac{\eta^\alpha}{c+\lambda_1\mu_1-\lambda_2}$.

Now the asymptotic formula (2.27) follows from Lemma 9. Since (2.27) implies that $\Phi(u)$ has a regularly varying tail, we conclude $\Phi(u) \in \mathcal{S}$ in this case.

Case 2. We consider $\psi^*(u)$ and $G_0^*(x)$ as before. Since $F_2 \in \overline{RV}_{-\alpha}$, and we also have $F_2 \in \mathcal{R}_0$, hence we obtain from Proposition 4 b) that $\overline{G}_0(x) \approx \frac{\lambda_2\mu_2}{\mathcal{C}_G} \frac{\prod_{i=1}^{N} q_i^{m_i}}{\prod_{j=2}^{m+1} \rho_{j,0}} \overline{F}_{2,I}(x)$. By the assumption that $\overline{F}_2(x) \approx \kappa x^{-\alpha}$, an application of L'Hopital's rule to $\lim_{x\to\infty} \frac{\overline{F}_{2,I}(x)}{x^{1-\alpha}}$ yields $\overline{F}_{2,I}(x) \approx \kappa \frac{x^{1-\alpha}}{(\alpha-1)\mu_2}$. Hence

$$\overline{G}_0(x) \approx \frac{\lambda_2\kappa}{\mathcal{C}_G(\alpha-1)} \frac{\prod_{i=1}^{N} q_i^{m_i}}{\prod_{j=2}^{m+1} \rho_{j,0}} x^{1-\alpha} \tag{2.34}$$

By (2.34) and Lemma 9 applied to $G_0^*(x)$, it follows that

$$\frac{\lambda_2\mu_2\kappa}{\mathcal{C}_G(\alpha-1)} \frac{\prod_{i=1}^{N} q_i^{m_i}}{\prod_{j=2}^{m+1} \rho_{j,0}} = \lim_{r\downarrow 0} \frac{r\widehat{G}_\alpha^*(r)}{r^{\alpha-2}} = \lim_{r\downarrow 0} \frac{1-\dfrac{\widehat{g}_0(r)}{\mathcal{C}_G}}{r^{\alpha-1}}. \tag{2.35}$$

From (2.33), we have:

$$\lim_{r\downarrow 0} \frac{r\widehat{\psi}^*(r)}{r^{\alpha-2}} = \lim_{r\downarrow 0} \frac{1}{r^{\alpha-1}} \left(\frac{1-\dfrac{\widehat{\nu}_{\alpha,0}(r)}{C_U} + \frac{\mathcal{C}_G}{\eta^\alpha\theta}\left[1-\dfrac{\widehat{g}_0(r)}{\mathcal{C}_G}\right]\widehat{\nu}_{\alpha,0}(r)}{1-\frac{1}{\theta}\left[\kappa\widehat{\nu}_{\alpha,0}(r)+\eta^{-\alpha}\widehat{g}_0(r)\widehat{\nu}_{\alpha,0}(r)\right]} \right),$$

hence, we apply (A.37), (A.39) and (2.35) in the above equality, and obtain

$$\begin{aligned} \lim_{r\downarrow 0} \frac{r\widehat{\psi}^*(r)}{r^{\alpha-2}} &= \frac{\dfrac{1}{\theta} \dfrac{\prod_{i=1}^{N} q_i^{m_i}}{\prod_{j=2}^{m+1} \rho_{j,0}} \mathcal{C}_U}{1-\frac{1}{\theta}\left[\kappa\mathcal{C}_U + \eta^{-\alpha}\mathcal{C}_G\mathcal{C}_U\right]} \left[1+\frac{\lambda_2\kappa\Gamma(2-\alpha)}{\eta^\alpha(\alpha-1)}\right] \\ &= \frac{\eta^\alpha(\alpha-1)+\lambda_2\kappa\Gamma(2-\alpha)}{(c+\lambda_1\mu_1-\lambda_2\mu_2)(\alpha-1)}. \end{aligned}$$

Applying Lemma 9 again and using that $\Phi(u)$ has a regularly varying tail in this case, we obtain the result.

Case 3. We define $\beta = \frac{1}{\theta}\left[\kappa \mathcal{C}_U + \eta^{-\alpha}\mathcal{C}_G\mathcal{C}_U\right]$ and

$$\begin{aligned} K(x) &= (1-\beta)\sum_{n=0}^{\infty}\beta^n\left(\frac{1}{\theta\beta}\left[\kappa\mathcal{C}_U U_\alpha + \eta^{-\alpha}\mathcal{C}_G\mathcal{C}_U G_0 * U_\alpha\right]\right)^{*n}(x) \\ &= (1-\beta)\sum_{n=0}^{\infty}\beta^n K_0^{*n}(x). \end{aligned}$$

where $K_0(x) = \left(\frac{1}{\theta\beta}\left[\kappa\mathcal{C}_U U_\alpha + \eta^{-\alpha}\mathcal{C}_G\mathcal{C}_U G_0 * U_\alpha\right]\right)(x)$.

By Proposition 3 b) we have the equality

$$W_\delta(x) = \frac{1}{\eta^\alpha\theta}\nu_\alpha * \sum_{n=0}^{\infty}\frac{1}{\theta^n}\left[\kappa\nu_\alpha + \frac{1}{\eta^\alpha}g_0 * \nu_\alpha\right]^{*n}(x).$$

From (2.20) we note that $\frac{\prod_{j=1}^N q_j^{m_j}}{\prod_{j=1}^R \rho_{j,\delta}^{k_j}}W_\delta(x)$ is the density of the probability of survival $\Phi(x)$, hence it is integrable and by dominated convergence theorem we can show that

$$\Phi(x) = \frac{1}{\eta^\alpha\theta}\frac{\prod_{j=1}^N q_j^{m_j}}{\prod_{j=1}^R \rho_{j,\delta}^{k_j}}C_U U_\alpha * \sum_{n=0}^{\infty}\frac{1}{\theta^n}\left(\kappa C_U U_\alpha + \frac{1}{\eta^\alpha}C_G C_U G_0 * U_\alpha\right)^{*n}(x)$$

Using $\beta = \frac{1}{\theta}\left[\kappa \mathcal{C}_U + \eta^{-\alpha}\mathcal{C}_G\mathcal{C}_U\right]$ and (2.31) in the equality above, we obtain

$$\begin{aligned} \Phi(x) &= (1-\beta)U_\alpha * \sum_{n=0}^{\infty}\frac{\beta^n}{\theta^n}\left[\frac{1}{\beta}\left(\kappa C_U U_\alpha + \frac{1}{\eta^\alpha}C_G C_U G_0 * U_\alpha\right)\right]^{*n}(x) \\ &= U_\alpha * K(x) \end{aligned} \tag{2.36}$$

This implies that $\Phi(x)$ is the convolution of the distribution functions $U_\alpha(x)$ and $K(x)$. In view of this, we need to study the asymptotic behaviour of

$\overline{K}(x)$. The assumption that $x^{-\alpha} = o(\overline{F}_2(x))$ and an application of L'Hopital's rule, imply that $x^{1-\alpha} = o\left(\overline{F}_{2,I}(x)\right)$. Assuming that $F_2 \in \mathcal{R}_0$, Proposition 4 b) yields $\overline{G}_0(x) \approx \frac{\lambda_2\mu_2}{\mathcal{C}_G}\frac{\prod_{i=1}^{N} q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}\overline{F}_{2,I}(x)$, hence $x^{1-\alpha} = o\left(\overline{G}_0(x)\right)$ and by Proposition 1d) $\overline{U}_\alpha(x) = o\left(\overline{G}_0(x)\right)$.

Since U_α and G_0 are distribution functions and $\beta = \frac{1}{\theta}\left[\kappa\mathcal{C}_U + \eta^{-\alpha}\mathcal{C}_G\mathcal{C}_U\right]$, we have that $K_0(x) = \left(\frac{1}{\theta\beta}\left[\kappa\mathcal{C}_U U_\alpha + \eta^{-\alpha}\mathcal{C}_G\mathcal{C}_U G_0 * U_\alpha\right]\right)(x)$ is a distribution function.

Then, it t follows from the equalities above and Lemma 8 a) that $1 - K_0(x) \approx \frac{\eta^{-\alpha}\mathcal{C}_G\mathcal{C}_U}{\theta\beta}\overline{G}_0(x)$.

By the assumption that $F_{2,I} \in \mathcal{S}$ and Proposition 4 b), we have $G_0 \in \mathcal{S}$, hence Lemma 8 d) gives:

$$\overline{K}(x) \approx \frac{\beta}{1-\beta}\frac{\eta^{-\alpha}\mathcal{C}_G\mathcal{C}_U}{\theta\beta}\overline{G}_0(x) \approx \frac{\lambda_2\mu_2\frac{\mathcal{C}_U}{\eta^\alpha\theta}\frac{\prod_{i=1}^{N} q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}}{\frac{c+\lambda_1\mu_1-\lambda_2\mu_2}{\eta^\alpha\theta}\frac{\prod_{i=1}^{N} q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}\mathcal{C}_U}\overline{F}_{2,I}(x). \quad (2.37)$$

Simplifying the coefficient in the right-hand side of (2.37) results to:

$$\overline{K}(x) \approx \frac{\lambda_2\mu_2}{c+\lambda_1\mu_1-\lambda_2\mu_2}\overline{F}_{2,I}(x). \quad (2.38)$$

Using (2.36) together with $\overline{U}_\alpha(x) = o(\overline{F}_{2,I}(x))$, (2.38) and Lemma 8 a), we obtain (2.29) and $\Phi \in \mathcal{S}$.

∎

The following lemma is immediate.

Lemma 10. *Let f_1, f_2 be two nonnegative functions such that $f_1 \approx g_1$ and $f_2 \approx g_2$, for some functions g_1, g_2 such that $\lim_{x\to\infty} \frac{g_1(x)}{g_2(x)} = c \in [0, \infty]$. Then*

a) $f_1 + f_2 \approx g_1 + g_2$,

b) If $c \neq 1$, $f_1 - f_2 \approx g_1 - g_2$.

Theorem 9 and Lemma 10 give the following corollary.

Corollary 4. *For all the cases in (2.26) the ruin probability $\psi(u)$ has the asymptotic expression:*

$$\psi(u) \approx \frac{\eta^\alpha}{(c + \lambda_1\mu_1 - \lambda_2\mu_2)\,\Gamma(2-\alpha)} u^{1-\alpha} + \frac{\lambda_2\mu_2}{c + \lambda_1\mu_1 - \lambda_2\mu_2} \overline{F}_{2,I}(u) \tag{2.39}$$

In particular, if $\overline{F}_2(u) \approx L_1(u)u^{-\alpha}$ for some slowly varying function L_1, and $\overline{F}_2$ satisfies any of the cases in (2.26), we have:

$$\psi(u) \approx \frac{1}{c + \lambda_1\mu_1 - \lambda_2\mu_2} \left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2}{\alpha - 1} L_1(u) \right] u^{1-\alpha} \tag{2.40}$$

Proof. We obtain (2.39) from Theorem 9 and Lemma 10 a). To obtain (2.40) we consider the three cases in (2.26)

Case 1. We have $\lim_{u\to\infty} \frac{L_1(u)u^{-\alpha}}{u^{-\alpha}} = \lim_{u\to\infty} \frac{\overline{F}_2(u)}{u^{-\alpha}} \frac{L_1(u)u^{-\alpha}}{\overline{F}_2(u)} = 0$. Hence:

$$\lim_{u\to\infty} \frac{\psi(u)}{\frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2\mu_2}{\alpha-1} L_1(u)\right] u^{1-\alpha}}$$

$$= \lim_{u\to\infty} \frac{\frac{\psi(u)}{u^{1-\alpha}}}{\frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2\mu_2}{\alpha-1} \frac{L_1(u)u^{1-\alpha}}{u^{1-\alpha}}\right]} = 1$$

where in the last equality we used (2.27).

Case 2. We set $C = \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2\kappa}{\alpha-1}\right]$.

Using the equality $\lim_{u\to\infty} L_1(u) = \kappa$ and (2.28) we obtain:

$$\lim_{u\to\infty} \frac{\psi(u)}{\frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2}{\alpha-1}L_1(u)\right]u^{1-\alpha}}$$
$$= \lim_{u\to\infty} \frac{\frac{\psi(u)}{Cu^{1-\alpha}}}{\frac{1}{C}\frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2}{\alpha-1}\frac{L_1(u)u^{1-\alpha}}{u^{1-\alpha}}\right]} = 1$$

Case 3. We have that $u^{-\alpha} = o\left(\overline{F}_2(u)\right)$ implies $u^{1-\alpha} = o\left(\overline{F}_{2,I}(u)\right)$. Now, using Karamata's theorem (see, for instance, Bingham et al. [1987], Proposition 1.5.10) we obtain $\lim_{u\to\infty} \frac{\overline{F}_{2,I}(u)}{L_1(u)u^{1-\alpha}} = \frac{\alpha-1}{\mu_2}$. Hence

$$\lim_{u\to\infty} \frac{\psi(u)}{\frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2\mu_2}{\alpha-1}L_1(u)\right]u^{1-\alpha}}$$
$$= \lim_{u\to\infty} \frac{\frac{\psi(u)}{\frac{\lambda_2\mu_2}{c+\lambda_1\mu_1-\lambda_2\mu_2}\overline{F}_{2,I}(u)}}{\frac{\eta^\alpha}{\Gamma(2-\alpha)}\frac{u^{1-\alpha}}{\lambda_2\mu_2\overline{F}_{2,I}(u)} + \frac{1}{\alpha-1}\frac{L_1(u)u^{1-\alpha}}{\overline{F}_{2,I}(u)}} = 1$$

■

2.5.2 Asymptotic behavior of $\Upsilon_{a,b}(u)$

Now we study the asymptotic behavior of $\Upsilon_{a,b}(u)$, defined as

$$\Upsilon_{a,b}(u) = \mathbb{P}\left[|V_\alpha(\tau_0)| > a, V_\alpha(\tau_0-) > b, \tau_0 < \infty | V_\alpha(0) = u\right].$$

For fixed $\beta > 0$ and $a \geq 0$, we define the function:

$$B(x;\beta,a) := \int_x^\infty e^{-\beta(y-x)}\left(\lambda_2\overline{F}_2(y+a) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(y+a)^{-\alpha}\right)dy. \qquad (2.41)$$

In order to obtain the corresponding asymptotic expressions for $\Upsilon_{a,b}(u)$ when

$\Delta = \max\{u, b\} \to \infty$, the following three results are needed.

Lemma 11.

a) $B(x;\beta,a) \le \lambda_2\mu_2 + \frac{\eta^\alpha}{\Gamma(2-\alpha)}a^{1-\alpha}$

b) In any of the cases considered in (2.26) we have $B(x;\beta,a) = o(\psi(x+a))$.

Proof. a) Since $e^{-\beta(y-x)} \le 1$ for every $y \ge x$ and $\overline{F}_2(y+a) \le \overline{F}_2(y)$, we have

$$\begin{aligned}
B(x;\beta,a) &\le \int_x^\infty \left(\lambda_2\overline{F}_2(y+a) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(y+a)^{-\alpha}\right)dy \\
&\le \int_x^\infty \left(\lambda_2\overline{F}_2(y) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(y+a)^{-\alpha}\right)dy \\
&\le \lambda_2\mu_2 + \frac{\eta^\alpha}{\Gamma(2-\alpha)}a^{1-\alpha}.
\end{aligned}$$

b) Using $\overline{F}_2(y+a) \le \overline{F}_2(x+a)$ and $(y+a)^{-\alpha} \le (x+a)^{-\alpha}$, for all $y \ge x$, we obtain:

$$\begin{aligned}
B(x;\beta,a) &\le \int_x^\infty e^{-\beta(y-x)}\left(\lambda_2\overline{F}_2(x+a) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(x+a)^{-\alpha}\right)dy \\
&\le \left(\lambda_2\overline{F}_2(x+a) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(x+a)^{-\alpha}\right)\int_x^\infty e^{-\beta(y-x)}dy \\
&= \frac{1}{\beta}\left(\lambda_2\overline{F}_2(x+a) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(x+a)^{-\alpha}\right) \qquad (2.42)
\end{aligned}$$

In cases 1 and 2, the limit $\lim\limits_{x\to\infty}\frac{\overline{F}_2(x)}{x^{-\alpha}}$ equals a constant $d \in [0,\infty)$, hence in any of these two cases we obtain:

$$\lim_{x\to\infty} \frac{\frac{1}{\beta}\left(\lambda_2\overline{F}_2(x+a)+\frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(x+a)^{-\alpha}\right)}{(x+a)^{1-\alpha}} = \lim_{x\to\infty} \frac{\frac{1}{\beta}\left(\lambda_2\frac{\overline{F}_2(x+a)}{(x+a)^{-\alpha}}+\frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}\right)}{x+a} = 0. \tag{2.43}$$

By (2.27) and (2.28), in cases 1 and 2 we also have $\psi(u) \approx Au^{1-\alpha}$, for some constant $A > 0$. This and (2.43) imply:

$$\lim_{x\to\infty} \frac{\frac{1}{\beta}\left(\lambda_2\overline{F}_2(x+a)+\frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(x+a)^{-\alpha}\right)}{\psi(x+a)} = \lim_{x\to\infty} \frac{\dfrac{\frac{1}{\beta}\left(\lambda_2\overline{F}_2(x+a)+\frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(x+a)^{-\alpha}\right)}{(x+a)^{1-\alpha}}}{\dfrac{\psi(x+a)}{(x+a)^{1-\alpha}}} = 0.$$

Hence we obtain the result in these two cases by dividing by $\psi(x+a)$ both sides of (2.42), and letting $x \to \infty$.

In case 3, the assumption that $F_2 \in \mathcal{R}_0$ and L'Hopital's rule imply that $\overline{F}_{2,I} \in \mathcal{R}_0$. We also have in this case that $\psi(u) \approx A_2\overline{F}_{2,I}(u)$ for some constant $A_2 > 0$, and from the proof of Theorem 9 c), $x^{1-\alpha} = o(\overline{F}_{2,I}(x))$. Using these two results together with $\overline{F}_{2,I} \in \mathcal{R}_0$ we obtain:

$$\lim_{x\to\infty} \frac{\frac{1}{\beta}\left(\lambda_2\overline{F}_2(x+a)+\frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(x+a)^{-\alpha}\right)}{\psi(x+a)} = \lim_{x\to\infty} \frac{\dfrac{\frac{1}{\beta}\left(\lambda_2\overline{F}_2(x+a)+\frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(x+a)^{-\alpha}\right)}{\overline{F}_{2,I}(x+a)}}{\dfrac{\psi(x+a)}{\overline{F}_{2,I}(x+a)}} = 0. \tag{2.44}$$

Hence the result follows again dividing by $\psi(x+a)$ and taking limits when $x \to \infty$ in (2.42).

∎

Lemma 12. *For $\omega(x,y) = 1_{\{x>a,y>b\}}$ and $\delta = 0$ we have*

$$\Upsilon_{a,b}(u) = h_\alpha * W_\delta(u), \tag{2.45}$$

where

$$h_\alpha(u) = \frac{\prod_{i=1}^{N} q_i^{m_i}}{\prod_{j=2}^{m+1} \rho_{j,0}} \int_u^\infty \left[\lambda_2 \overline{F}_2(a+z) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(a+z)^{-\alpha}\right] 1_{\{z>b\}} dz + I_{a,b}(u),$$

and

$$I_{a,b}(x) = \sum_{j=2}^{m+1} E(\rho_{j,0}) \int_x^\infty e^{-\rho_{j,0}(y-x)} \left(\lambda_2 \overline{F}_2(y+a) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(y+a)^{-\alpha}\right) 1_{\{y>b\}} dy.$$

Moreover, if F_2 and $F_{2,I}$ satisfy cases 1, 2 or 3, then

$$\int_0^u I_{a,b}(u-y)\Phi(dy) = o(\psi(u)). \tag{2.46}$$

Proof. Formula (2.45) follows from Corollary 1. To prove (2.46) we first note that

$$|I_{a,b}(x)| \le \sum_{j=2}^{m+1} |E(\rho_{j,0})| \int_x^\infty e^{-Re(\rho_{j,0})(y-x)} \left(\lambda_2 \overline{F}_2(y+a) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(y+a)^{-\alpha}\right) dy,$$

which by (2.41), is equivalent to:

$$|I_{a,b}(x)| \leq \sum_{j=2}^{m+1} |E(\rho_{j,0})| B\left[x; Re(\rho_{j,0}), a\right] \tag{2.47}$$

By Lemma 11 b), there exists an u_0 such that for all $u > u_0$ we have the inequality $\sum_{j=2}^{m+1} |E(\rho_{j,0})| B\left[u; Re(\rho_{j,0}), a\right] < \varepsilon\psi(u)$.

Hence using this and (2.47), we obtain

$$\begin{aligned} \left| \frac{\int_0^u I_{a,b}(u-y)\Phi(dy)}{\psi(u)} \right| &< \frac{\varepsilon \int_0^{u-u_0} (1-\Phi(u-y))\Phi(dy)}{\psi(u)} + \frac{\int_{u-u_0}^{u} |I_{a,b}(u-y)|\Phi(dy)}{\psi(u)} \\ &\leq \frac{\varepsilon \int_0^{u} (1-\Phi(u-y))\Phi(dy)}{\psi(u)} + \frac{\int_{u-u_0}^{u} |I_{a,b}(u-y)|\Phi(dy)}{\psi(u)}. \end{aligned}$$

By Lemma 11 a), the right-hand side of (2.47) is bounded above by

$$c_0 = \sum_{j=2}^{m+1} |E(\rho_{j,0})| \left(\lambda_2\mu_2 + \frac{\eta^\alpha}{\Gamma(2-\alpha)} a^{1-\alpha} \right).$$

Hence

$$\begin{aligned} \left| \frac{\int_0^u I_{a,b}(u-y)\Phi(dy)}{\psi(u)} \right| &< \frac{\varepsilon\left(\Phi(u) - \Phi * \Phi(u)\right)}{\psi(u)} + c_0 \frac{\Phi(u) - \Phi(u-u_0)}{\psi(u)} \\ &= \frac{\varepsilon\left(1 - \Phi * \Phi(u)) - \psi(u)\right)}{\psi(u)} + c_0 \frac{\psi(u-u_0) - \psi(u)}{\psi(u)} \end{aligned}$$

Using that $\Phi(u) \in \mathcal{S}$, we obtain the result by letting $u \to \infty$ (and $\varepsilon \downarrow 0$). ■

The asymptotic expressions for $\Upsilon_{a,b}(u)$ are given in the next result.

Theorem 10. *The joint tail of the severity of ruin and the surplus prior to ruin, $\Upsilon_{a,b}(u)$, has the following asymptotic expressions.*

a) In case 1:

$$\Upsilon_{a,b}(u) \approx \frac{\eta^{\alpha}}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\,\Gamma(2-\alpha)}(a+\Delta)^{1-\alpha}$$

b) In case 2:

$$\Upsilon_{a,b}(u) \approx \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^{\alpha}}{\Gamma(2-\alpha)}+\frac{\lambda_2\kappa}{\alpha-1}\right](a+\Delta)^{1-\alpha}$$

c) In case 3:

$$\Upsilon_{a,b}(u) \approx \frac{\lambda_2\mu_2}{c+\lambda_1\mu_1-\lambda_2\mu_2}\overline{F}_{2,I}(a+\Delta)$$

Proof. Since

$$h_{\alpha,\delta,\omega}(u) = \frac{\prod_{i=1}^{N} q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}\int_u^{\infty}\left[\lambda_2\overline{F}_2(a+z)+\frac{\eta^{\alpha}(\alpha-1)}{\Gamma(2-\alpha)}(a+z)^{-\alpha}\right]1_{\{z>b\}}dz + I_{a,b}(u),$$

by (2.45) and (2.20), we have:

$$\Upsilon_{a,b}(u) = \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\int_0^{u}\int_{u-y}^{\infty}\left[\lambda_2\overline{F}_2(a+z)+\frac{\eta^{\alpha}(\alpha-1)}{\Gamma(2-\alpha)}(a+z)^{-\alpha}\right]1_{\{z>b\}}dz\Phi(dy)$$
$$+\frac{\frac{\prod_{i=1}^{N} q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}}{c+\lambda_1\mu_1-\lambda_2\mu_2}\int_0^{u} I_{a,b}(u-y)\Phi(dy) \tag{2.48}$$

By (2.46) we only need to study the asymptotic behavior of

$$\Upsilon^*(u,a,b) := \frac{\int\limits_0^u \int\limits_{u-y}^\infty \left[\lambda_2 \overline{F}_2(a+z) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(a+z)^{-\alpha}\right] 1_{\{z>b\}} dz \Phi(dy)}{c+\lambda_1\mu_1 - \lambda_2\mu_2}$$

with respect to $\psi(u+a)$. First we suppose $\Delta = u$, and define

$$\Upsilon_0(u,a) := \frac{\int\limits_0^u \left[\lambda_2\mu_2 \overline{F}_{2,I}(a+u-y) + \frac{\eta^\alpha}{\Gamma(2-\alpha)}(a+u-y)^{1-\alpha}\right] \Phi(dy)}{c+\lambda_1\mu_1 - \lambda_2\mu_2}.$$

We have:

$$\begin{aligned} \Upsilon^*(u,a,b) &\leq \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2} \int\limits_0^u \int\limits_{u-y}^\infty \left[\lambda_2 \overline{F}_2(a+z) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(a+z)^{-\alpha}\right] dz \Phi(dy) \\ &= \Upsilon_0(u,a) \end{aligned} \quad (2.49)$$

and

$$\begin{aligned} \Upsilon^*(u,a,b) &\geq \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2} \int\limits_0^u \int\limits_{u}^\infty \left[\lambda_2 \overline{F}_2(a+z) + \frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(a+z)^{-\alpha}\right] dz \Phi(dy) \\ &= \frac{\lambda_2\mu_2\overline{F}_{2,I}(a+u) + \frac{\eta^\alpha}{\Gamma(2-\alpha)}(a+u)^{1-\alpha}}{c+\lambda_1\mu_1-\lambda_2\mu_2}\Phi(u). \end{aligned} \quad (2.50)$$

Clearly, (2.50) implies that

$$\lim_{u\to\infty} \frac{\Upsilon^*(u,a,b)}{\psi(u+a)} \geq 1, \quad (2.51)$$

because $\psi(u+a) \approx \dfrac{\lambda_2\mu_2\overline{F}_{2,I}(a+u) + \frac{\eta^\alpha}{\Gamma(2-\alpha)}(a+u)^{1-\alpha}}{c+\lambda_1\mu_1-\lambda_2\mu_2}$ and $\lim\limits_{u\to\infty}\Phi(u) = 1$. Now we will prove that

$$\lim_{u\to\infty} \frac{\Upsilon_0(u,a)}{\psi(u+a)} = 1, \quad (2.52)$$

for all the claim distributions considered in Theorem 9. Indeed, from (2.52) together with (2.49), (2.51), (2.48) and (2.46) we obtain the result.

Now we note that

$$\Upsilon_0(u,a) = \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\lambda_2\mu_2\overline{F}_{2,I}(a)\int_0^u (1-F_{a,I}(u-y))\,\Phi(dy) + \frac{a^{1-\alpha}\eta^\alpha}{\Gamma(2-\alpha)}\int_0^u (1-P_{a,\alpha}(u-y))\,\Phi(dy)\right]$$

where $F_{a,I}(u) = 1 - \dfrac{\overline{F}_{2,I}(u+a)}{\overline{F}_{2,I}(a)}$ and $P_{a,\alpha}(u) = 1 - \left(\dfrac{au}{a+u}\right)^{\alpha-1} u^{1-\alpha}$.

Hence we obtain:

$$\begin{aligned}\Upsilon_0(u,a) &= \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\Bigg(\lambda_2\mu_2\overline{F}_{2,I}(a)\left[\Phi(u)-F_{a,I}*\Phi(u)\right] \\ &+ \frac{a^{1-\alpha}\eta^\alpha}{\Gamma(2-\alpha)}\left[\Phi(u)-P_{a,\alpha}*\Phi(u)\right]\Bigg) \\ &= \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\Bigg(\lambda_2\mu_2\overline{F}_{2,I}(a)\left[\Phi(u)-1+1-F_{a,I}*\Phi(u)\right] \\ &+ \frac{a^{1-\alpha}\eta^\alpha}{\Gamma(2-\alpha)}\left[\Phi(u)-1+1-P_{a,\alpha}*\Phi(u)\right]\Bigg) \\ &= \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\Bigg(\lambda_2\mu_2\overline{F}_{2,I}(a)\left[1-F_{a,I}*\Phi(u)-\Psi(u)\right] \\ &+ \frac{a^{1-\alpha}\eta^\alpha}{\Gamma(2-\alpha)}\left[1-P_{a,\alpha}*\Phi(u)-\Psi(u)\right]\Bigg) \end{aligned} \tag{2.53}$$

Case 1. By Theorem 9 a) we have $\Phi \in \mathcal{S}$ and $\psi(u) \approx \dfrac{\eta^\alpha}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\Gamma(2-\alpha)}u^{1-\alpha}$, hence by Lemma 8 c) and the assumption $\overline{F}_{2,I}(u) = o(u^{1-\alpha})$, we obtain

$$1 - F_{a,I}*\Phi(u) \approx \frac{\eta^\alpha}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\Gamma(2-\alpha)}u^{1-\alpha}.$$

This implies

$$\frac{\lambda_2\mu_2\overline{F}_{2,I}(a)}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\,1-F_{a,I}*\Phi(u)-\psi(u)\right]=o(u^{1-\alpha}). \tag{2.54}$$

By Lemma 8 b),

$$1-P_{a,\alpha}*\Phi(u)\approx\left[\left(\frac{au}{a+u}\right)^{\alpha-1}+\frac{\eta^\alpha}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\Gamma(2-\alpha)}\right]u^{1-\alpha},$$

which by (2.27) and Lemma 10 b) implies:

$$1-P_{a,\alpha}*\Phi(u)-\psi(u)\approx\left(\frac{a}{a+u}\right)^{\alpha-1}. \tag{2.55}$$

Using the expression for $\Upsilon_0(u,a)$ given in (2.53) together with (2.54), (2.55) and Lemma 10 a) we obtain

$$\Upsilon_0(u,a)\approx\frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\lambda_2\mu_2\overline{F}_{2,I}(a+\Delta)+\frac{\eta^\alpha}{\Gamma(2-\alpha)}(a+\Delta)^{1-\alpha}\right],$$

and (2.52) follows.

Case 2. Since $\overline{F}_2(u)\approx\kappa u^{1-\alpha}$ by assumption, by L'Hospital's rule we obtain $\overline{F}_{2,I}(u)\approx\frac{\kappa}{\mu_2(\alpha-1)}u^{1-\alpha}$. Hence $\overline{F}_{a,I}(u)\approx\frac{\kappa}{\mu_2(\alpha-1)\overline{F}_{2,I}(a)}u^{1-\alpha}$.

By (2.28) we have $\Psi(u)\approx Cu^{1-\alpha}$, where $C=\frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)}+\frac{\lambda_2\kappa}{\alpha-1}\right]$. Using this and Lemma 8 b) we obtain $1-F_{a,I}*\Phi(u)\approx\left[C+\frac{\kappa}{\mu_2(\alpha-1)}\right]u^{1-\alpha}$, and applying Lemma 10 b) it follows that

$$1-F_{a,I}*\Phi(u)-\psi(u)\approx\frac{\kappa}{\mu_2(\alpha-1)}u^{1-\alpha}\approx\frac{\kappa}{\mu_2(\alpha-1)}(a+u)^{1-\alpha} \tag{2.56}$$

By Lemma 8 b) and (2.28), it follows that:

$$1 - P_{a,\alpha} * \Phi(u) \approx \left[\left(\frac{au}{a+u} \right)^{\alpha-1} + \frac{\eta^\alpha}{(c+\lambda_1\mu_1 - \lambda_2\mu_2)\Gamma(2-\alpha)} + \frac{\lambda_2\kappa}{c+\lambda_1\mu_1-\lambda_2\mu_2} \right] u^{1-\alpha}. \tag{2.57}$$

This together with (2.28) and Lemma 10 a) implies:

$$1 - P_{a,\alpha} * \Phi(u) - \psi(u) \approx \left(\frac{a}{a+u} \right)^{\alpha-1}. \tag{2.58}$$

Now using (2.56), (2.58) and Lemma 10 b), again, we obtain

$$\Upsilon_0(u,a) \approx \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2} \left[\frac{\eta^\alpha}{\Gamma(2-\alpha)} + \frac{\lambda_2\kappa}{\alpha-1} \right] (a+u)^{1-\alpha}.$$

Case 3. By the assumption that $u^{-\alpha} = o\left(\overline{F}_2(u)\right)$, we apply L'Hospital's rule to obtain $u^{1-\alpha} = o\left(\overline{F}_{2,I}(u)\right)$.

Since $\overline{P}_{a,\alpha}(u) = \left(\frac{au}{a+u}\right)^{\alpha-1} u^{1-\alpha}$ and $\lim_{u\to\infty} \frac{\left(\frac{auy}{a+uy}\right)^{\alpha-1}}{\left(\frac{au}{a+u}\right)^{\alpha-1}} = 1$ for all $y > 0$, we have $\overline{P}_{a,\alpha}(u) \approx u^{1-\alpha}$.

Hence $\overline{P}_{a,\alpha}(u) = o\left(\overline{F}_{2,I}(u)\right)$, and by Corollary 8 and (2.29) we obtain $1 - P_{a,\alpha} * \Phi(u) \approx \frac{\lambda_2\mu_2}{c+\lambda_1\mu_1-\lambda_2\mu_2}\overline{F}_{2,I}(u)$. By (2.29) and Lemma 10, we conclude that $1 - P_{a,\alpha} * \Phi(u) - \psi(u) = o(\overline{F}_{2,I}(u))$.

By Lemma 8 b),

$$1 - F_{a,I} * \Phi(u) \approx \left(\frac{1}{\overline{F}_{2,I}(a)} + \frac{\lambda_2\mu_2}{c+\lambda_1\mu_1-\lambda_2\mu_2} \right) \overline{F}_{2,I}(a+u),$$

and by Lemma 10 b), this implies $1 - F_{a,I} * \Phi(u) - \psi(u) \approx \overline{F}_{a,I}(u)$. Hence, by Lemma 10 a) we obtain (2.52) again.

If $\Delta = b$ we have:

$$\begin{aligned}\Upsilon^*(u,a,b) &= \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\int_0^u\int_{u-y}^{\infty}\left[\lambda_2\overline{F}_2(a+z)+\frac{\eta^\alpha(\alpha-1)}{\Gamma(2-\alpha)}(a+z)^{-\alpha}\right]1_{\{z>b\}}dz\Phi(dy)\\ &= \frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\int_0^u\left[\lambda_2\mu_2\overline{F}_{2,I}(a+b)+\frac{\eta^\alpha}{\Gamma(2-\alpha)}(a+b)^{1-\alpha}\right]\Phi(dy)\\ &= \frac{\lambda_2\mu_2\overline{F}_{2,I}(a+b)+\frac{\eta^\alpha}{\Gamma(2-\alpha)}(a+b)^{1-\alpha}}{c+\lambda_1\mu_1-\lambda_2\mu_2}\Phi(u).\end{aligned}$$

Hence the result follows dividing $\Upsilon_{a,b}(u)$ by $\dfrac{\lambda_2\mu_2\overline{F}_{2,I}(a+b)+\frac{\eta^\alpha}{\Gamma(2-\alpha)}(a+b)^{1-\alpha}}{c+\lambda_1\mu_1-\lambda_2\mu_2}$, then letting $u\to\infty$ and applying cases 1,2,3 with u replaced by b. ∎

Finally, as in the case of Corollary 1, Lemma 10 gives the following result.

Corollary 5. *For any of the cases in (2.26), the joint tail $\Upsilon_{a,b}(u)$ has the asymptotic expression:*

$$\psi(u)\approx\frac{\eta^\alpha}{(c+\lambda_1\mu_1-\lambda_2\mu_2)\,\Gamma(2-\alpha)}(a+\Delta)^{1-\alpha}+\frac{\lambda_2\mu_2}{c+\lambda_1\mu_1-\lambda_2\mu_2}\overline{F}_{2,I}(a+\Delta)$$

In particular, if $\overline{F}_2(u)\approx L_1(u)u^{-\alpha}$ for some slowly varying function L_1, and $\overline{F}_2$ satisfies any of the cases in (2.26), we have:

$$\psi(u)\approx\frac{1}{c+\lambda_1\mu_1-\lambda_2\mu_2}\left[\frac{\eta^\alpha}{\Gamma(2-\alpha)}+\frac{\lambda_2}{\alpha-1}L_1(a+\Delta)\right](a+\Delta)^{1-\alpha}$$

2.5.3 Asymptotic behavior of $\Upsilon_{a,b}(u)$ in the spectrally negative case

This final subsection aims to provide the corresponding result for the joint tail of the time to ruin, the severity of ruin and the surplus prior to ruin, for the case of the Lévy risk process studied in Furrer [1998].

The approach is the same as in the previous section, and all results are stated without proofs. Instead, we refer to Kolkovska and Martín-González [2016] for the fully detailed proofs.

We begin with some basic notation: throughout this section, $\mathcal{X} = \{\mathcal{X}(t), t \geq 0\}$ is the classical risk process with an α-stable perturbation, defined by the equation

$$\mathcal{X}(t) = u + ct - \mathcal{S}(t) - \eta \mathcal{W}_\alpha(t), \tag{2.59}$$

where $u \geq 0$ is the insurance company's initial capital, $c \geq 0$ is a premium per unit time, $\mathcal{S} = \{\mathcal{S}(t), t \geq 0\}$ is a compound Poisson process with Lévy measure $\lambda f(x)dx$, such that f is a probability density, and $\mathcal{W}_\alpha = \{\mathcal{W}_\alpha(t), t \geq 0\}$ is an α-stable process with only positive jumps, with $\alpha \in (1, 2)$, independent of $\mathcal{S}$.

In this case, $\Upsilon_{a,b}(u)$ is defined analogously as in the previous section, as

$$\Upsilon_{a,b}(u) = \mathbb{P}\left[|\mathcal{X}(\tau_0)| > a, \mathcal{X}(\tau_0-) > b, \tau_0 < \infty | \mathcal{X}(0) = u\right],$$

where τ_0 is the first passage time below zero of X (time to ruin of X).

Notice that $\Upsilon(u, a, b)$ is the EDPF of the process $\mathcal{X}$ in the particular case $\omega(x, y) = 1_{\{x>a,y>b\}}$ and $\delta = 0$. Hence using Theorem 1 in Biffis and Kyprianou [2009], for $\omega(x, y) = 1_{\{x>a,y>b\}}$ we obtain:

$$\phi(u) = \int_0^u \left(\mathfrak{W}^{(q)\prime}(u-y) - \rho \mathfrak{W}^{(q)}(u-y)\right) \int_y^\infty e^{-\rho(v-y)} \times \\ \left[\eta^\alpha \frac{\alpha - 1}{\Gamma(2-\alpha)} \frac{1}{(v+a)^\alpha} + \lambda \overline{F}(v+a)\right] 1_{\{v>b\}} dv dy$$

If we set $q = 0$, the formula above yields:

$$\Upsilon(u, a, b) = \int_0^u \mathfrak{W}'(u-y) \int_y^\infty \left[\eta^\alpha \frac{\alpha - 1}{\Gamma(2-\alpha)} \frac{1}{(v+a)^\alpha} + \lambda \overline{F}(v+a)\right] 1_{\{v>b\}} dv dy$$

$$= \int_0^u \left\{ \int_{u-y}^{\infty} \left[\eta^\alpha \frac{\alpha-1}{\Gamma(2-\alpha)} \frac{1}{(v+a)^\alpha} + \lambda \overline{F}(v+a) \right] 1_{\{v>b\}} dv \right\} W'(y) dy \quad (2.60)$$

Now we consider the same three cases from the previous section. For this, we denote by $\overline{F}_I(u)$ the integrated tail corresponding to f.

Case 1. $\overline{F}_I(u) = o(u^{1-\alpha})$

Case 2. $\overline{F} \in RV_{-\alpha}$

Case 3. $F_I \in \mathcal{S}$ and $u^{1-\alpha} = o(\overline{F}_I(u))$

Theorem 11. *Let F satisfy one of the cases above, where in case 2 we assume that $\overline{F}(x) = x^{-\alpha} L(x)$ with $\lim_{x\to\infty} L(x) = C_L \in [0, \infty]$. If b is fixed and $u \to \infty$, or if both $u, b \to \infty$, then*

$$\Upsilon(u,a,b) \approx \frac{\lambda\mu}{c-\lambda\mu} \overline{F}_I(a+\Delta) + \frac{\eta}{(c-\lambda\mu)\Gamma(2-\alpha)} \left(\frac{\eta}{a+\Delta} \right)^{\alpha-1},$$

where $\Delta = \max\{u, b\}$. In cases a) and c) we have, respectively, the approximations

$$\Upsilon(u,a,b) \approx \frac{\eta}{(c-\lambda\mu)\Gamma(2-\alpha)} \left(\frac{\eta}{a+\Delta} \right)^{\alpha-1} \quad \textit{and} \quad \Upsilon(u,a,b) \approx \frac{\lambda\mu}{c-\lambda\mu} \overline{F}_I(a+\Delta).$$

An application of the theorem above yields the following expressions for the tail of the severity of ruin $\Upsilon(u,a,0)$, and the tail of the surplus prior to ruin $\Upsilon(u,0,b)$.

Corollary 6. *Under the conditions of Theorem 11, we have*

$$\Upsilon(u,a,0) \approx \frac{\lambda\mu}{c-\lambda\mu} \overline{F}_I(a+u) + \frac{\eta}{(c-\lambda\mu)\Gamma(2-\alpha)} \left(\frac{\eta}{a+u} \right)^{\alpha-1}, \ \textit{when } u \to \infty,$$

$$\Upsilon(u,0,b) \approx \frac{\lambda\mu}{(c-\lambda\mu)} \overline{F}_I(\Delta) + \frac{\eta}{(c-\lambda\mu)\Gamma(2-\alpha)} \left(\frac{\eta}{\Delta} \right)^{\alpha-1}, \ \textit{when } u, b \to \infty.$$

Chapter 3

Wiener-Hopf factorization

In this chapter we study the negative Wiener-Hopf factor for two-sided jumps Lévy risk processes $\mathcal{X}$, defined by the equation

$$\mathcal{X}(t) = ct + \gamma\mathcal{B}(t) + \mathcal{Z}(t) - \mathcal{S}(t), \ \ , t \geq 0, \tag{3.1}$$

where $c \geq 0$ is a drift term, $\mathcal{Z} = \{\mathcal{Z}(t), t \geq 0\}$ is a compound Poisson process with Lévy measure $\lambda_1 f_1(x)$, where f_1 is a probability density with Laplace transform given by (2), $\mathcal{B} = \{\mathcal{B}(t), t \geq 0\}$ is a Brownian motion and $\mathcal{S} = \{\mathcal{S}(t), t \geq 0\}$ is a pure jumps Lévy process with positive jumps. This process $\mathcal{X}$ includes, as a particular case, the process V_α considered in the previous section in the case when $u = 0$.

We use the results obtained about these Wiener-Hopf factors to calculate the generalized version of the EDPF associated to $u + \mathcal{X}$, for $u \geq 0$.

We also make use of the known result for the probability density of the positive Wiener-Hopf factor for the class of processes defined by $\mathcal{X}$. Such probability density has been previously studied in Lewis and Mordecki [2008].

3.1 Notation and preliminary results

Before proceeding with the statement and proofs of our results, we point out some basic properties of Lévy processes and define some recquired notation.

We set $\Psi_{\mathcal{S}}(r) = \int\limits_{0+}^{\infty} \left(1 - e^{-rx} - rx1_{\{x<1\}}\right) \nu_{\mathcal{S}}(dx)$ and assume that $\mathbb{E}[\mathcal{X}(1)] > 0$. This condition implies that the process $\mathcal{X}$ drifts to infinity and, as stated in Biffis and Kyprianou [2009], besides the usual condition on Lévy measures $\int\limits_{0+}^{\infty} (x^2 \wedge 1)\, \nu_{\mathcal{S}}(dx) < \infty$, we have $\int\limits_{0+}^{\infty} (x^2 \wedge x)\, \nu_{\mathcal{S}}(dx) < \infty$.

In the case when $\mathcal{S}$ is a subordinator, we also have $\int\limits_{0+}^{\infty} (x \wedge 1)\, \nu_{\mathcal{S}}(dx) < \infty$, meaning that the Lévy measure of $\mathcal{S}$ has a finite mean in this case. In the sequel, we denote this mean by $\mu_{\mathcal{S}} = \int\limits_{0+}^{\infty} x \nu_{\mathcal{S}}(dx)$.

The inequality $\int\limits_{0+}^{\infty} (x^2 \wedge x)\, \nu_{\mathcal{S}}(dx) < \infty$ implies that we can rewrite $\Psi_{\mathcal{S}}(r)$ without using the indicator function $1_{\{x<1\}}$. Hence, if $\mathcal{S}$ is a subordinator we have $cr - \Psi_{\mathcal{S}}(r) = (c+\mu_{\mathcal{S}})r - G_{\mathcal{S}}(r)$, where $G_{\mathcal{S}}(r) = \int\limits_{0+}^{\infty} (1 - e^{-rx})\, \nu_{\mathcal{S}}(dx)$. In view of this, we define the Generalized Lundberg Function associated to $\mathcal{X}$ as

$$L_{\mathcal{X}}(r) = cr + \gamma^2 r^2 + \lambda_1 \left(\frac{Q(-r)}{\prod_{j=1}^{N} (q_j - r)^{m_j}} - 1 \right) - \Psi_{\mathcal{S}}(r). \tag{3.2}$$

when $\mathcal{S}$ is a pure positive jumps process but not a subordinator and

$$L_{\mathcal{X}}(r) = cr + \gamma^2 r^2 + \lambda_1 \left(\frac{Q(-r)}{\prod_{j=1}^{N} (q_j - r)^{m_j}} - 1 \right) - G_{\mathcal{S}}(r). \tag{3.3}$$

when $\mathcal{S}$ is a subordinator, assuming that in this case the drift c is of the form $c_0 + \mu_{\mathcal{S}}$ for some c_0 such that $c_0 + \mu_{\mathcal{S}} \geq 0$.

We let $\Psi_{\mathcal{X}}^{[c]}(r) = -\log \mathbb{E}\left[e^{ir\mathcal{X}(1)}\right]$ denote the characteristic exponent of $\mathcal{X}$. Hence, the GLF above is related to $\Psi_{\mathcal{X}}^{[c]}(r)$ by the equality

$$-\Psi_{\mathcal{X}}^{[c]}(-ir) = L_{\mathcal{X}}(r). \tag{3.4}$$

Moreover, we also obtain the equality $L_{\mathcal{X}}(r) = \log \mathbb{E}[e^{r\mathcal{X}(1)}]$ for $r < q_1$, which means that the GLF coincides with the exponent of the moment generating function of $\mathcal{X}(1)$ for all r for which this moment generating function exists.

Let us set $\overline{\mathcal{V}}_{\mathcal{S}}(u) = \int\limits_{u}^{\infty} \nu_{\mathcal{S}}(dx)$. We shall use the following property valid when $\mathcal{S}$ is a subordinator (see, for instance, exercise 2.11 in Kyprianou [2006]):

$$G_{\mathcal{S}}(r) = r\widehat{\overline{\mathcal{V}}}(r). \tag{3.5}$$

We also recall the following property (see Kyprianou [2006], Chapter 7): the Wiener-Hopf factors can be identified through their Laplace exponents by the relations

$$\mathbb{E}\left[e^{-rS^{\mathcal{X}}_{e_q}}\right] = \frac{\kappa_A(q,0)}{\kappa_A(q,r)} \quad \text{and} \quad \mathbb{E}\left[e^{-r\left(-I^{\mathcal{X}}_{e_q}\right)}\right] = \frac{\kappa_D(q,0)}{\kappa_D(q,r)} \tag{3.6}$$

where $\frac{1}{\kappa_A(s,r)}$ and $\frac{1}{\kappa_D(s,r)}$ are the respective bivariate Laplace transforms of the functions $\mathcal{U}_A(dx,dy)$ and $\mathcal{U}_D(dx,dy)$. These functions are, respectively, the potential measures of the associated ascending and descending ladder processes (see Kyprianou [2006], Chapter 7 for further information on this). In Section 3.3 (specifically, in the proof of the main result of this section), we identify two functions related to these potential measures, which are closely related to the distributions of the positive and negative Wiener-Hopf factors.

We denote the q-scale functions for spectrally negative Lévy process as $\mathfrak{W}^{(q)}$, as we did in Chapter 1, and recall the notation $\mathbb{C}_+ = \{z \in \mathbb{C} : Re(z) \geq 0\}$ and $\mathbb{C}_{++} = \{z \in \mathbb{C} : Re(z) > 0\}$. For a given function $L : \mathbb{C} \to \mathbb{C}$, we also recall that $s \in \mathbb{C}$ is a root of the function L, with multiplicity $m \geq 1$, if $L(s) = 0$, $\frac{d^j}{dr^j}L(r)|_{r=s} = 0$ for all $j = 1, 2, \ldots, m-1$ and $\frac{d^m}{dr^m}L(r)|_{r=s} \neq 0$.

We consider the following three cases:

Case A. $c = \gamma = 0$ and $\mathcal{S}$ is a driftless subordinator (other than a compound Poisson process),

Case B. $c > 0$, $\gamma = 0$ and $\mathcal{S}$ is any subordinator,

Case C. Any other case except when $c = \gamma = 0$ and $\mathcal{S}$ is a compound Poisson process.

We use the following result, which follows from Lemma 1.1 in Lewis and Mordecki [2008] and (3.4)

Lemma 13. *If $\delta > 0$:*

a) In case A, $L_{\mathcal{X}}(r) - \delta = 0$ has m roots in $\mathbb{C}_{++}$,

b) In cases B and C, $L_{\mathcal{X}}(r) - \delta = 0$ has $m+1$ roots in $\mathbb{C}_{++}$.

In all the cases above, there is exactly one root $\rho_{1,\delta}$ *in the interval* $(0, q_1)$. *This root is such that* $\lim_{\delta \downarrow 0} \rho_{1,\delta} = 0$ *and* $\rho_{1,0} = 0$, *in cases A, B and C. In all these cases,* $\rho_{1,0} = 0$ *is a simple root.*

In what follows we denote by β_A the function whose value is 1 in case A, and 0 otherwise. We assume that the equation $L_{\mathcal{X}}(r) - \delta = 0$ has R different roots in $\mathbb{C}_{++}$, denoted respectively by $\rho_{1,\delta}, \ldots, \rho_{R,\delta}$ with multiplicities $k_1, k_2, \ldots, k_R$ such that $\sum_{j=1}^{R} k_j = m + 1 - \beta_A$. We let $\rho_{1,\delta}$ be the real root such that $\rho_{1,\delta} \in [0, q_1)$, which implies $k_1 = 1$.

It can be proved that, for all j, the roots $\rho_{j,\delta}$ have a limit when $\delta \downarrow 0$. We denote this limits as ρ_j, i.e.

$$\rho_j := \lim_{\delta \downarrow 0} \rho_{j,\delta} \tag{3.7}$$

Using $L_{\mathcal{X}}(\rho_{1,\delta}) = \delta$, $\lim_{\delta \downarrow 0} \rho_{1,\delta} = 0$ and the fact that $\rho_{1,\delta}$ and 0 are simple roots, respectively, of $L_{\mathcal{X}}(r) - \delta = 0$ and $L_{\mathcal{X}}(r) = 0$, we obtain by L'Hopital's rule

$$\lim_{\delta \downarrow 0} \frac{\delta}{\rho_{1,\delta}} = \lim_{\delta \downarrow 0} \frac{L_{\mathcal{X}}(\rho_{1,\delta})}{\rho_{1,\delta}} = \lim_{\rho_{1,\delta} \to 0} \frac{L_{\mathcal{X}}(\rho_{1,\delta})}{\rho_{1,\delta}} = L'_{\mathcal{X}}(0+) = \mathbb{E}[\mathcal{X}(1)], \tag{3.8}$$

where $L_{\mathcal{X}}(0+) := \frac{d}{dr} L_{\mathcal{X}}(r)|_{r=0}$.

Now we define an operator which is required in our main results.

For $a = 0, 1, \ldots, m+1$, we define $\mathcal{T}_{s;a}$ by the equation

$$\mathcal{T}_{s;a} f(u) = \int_u^{\infty} (y-u)^a e^{-s(y-u)} f(y) dy,$$

for each measurable f and complex s such that the integral above exists and is finite. Clearly this operator is linear. If ν is a measure such that $\int_u^{\infty} (y-u)^a e^{-s(y-u)} \nu(dy)$ exists, we define

$$\mathcal{T}_{s;a} \nu(u) = \int_u^{\infty} (y-u)^a e^{-s(y-u)} \nu(dy) \tag{3.9}$$

for $a = 0, 1, \ldots, m+1$. We denote the corresponding Laplace transforms as $\widehat{\mathcal{T}}_{s;a}f(r)$ and $\widehat{\mathcal{T}}_{s;a}\nu(r)$, for all $r \in \mathbb{C}_+$ such that these Laplace transforms exist.

In the case when $a = 0$, we obtain the operators $T_s f(x)$ and $T_s\nu(x)$ defined in (1.2) and (1.4). The following lemma is easy to prove and it relates the Laplace transforms of T and $\mathcal{T}$.

Lemma 14. *Let f be a function (or a measure) such that $\mathcal{T}_{s,k}f(u)$ exists for every $s \in \mathbb{C}_{++}$, $k \in \mathbb{N} \cup \{0\}$ and $u > 0$, then for each $r \in \mathbb{C}_+$, $s \in \mathbb{C}_{++}$ and $k \in \mathbb{N} \cup \{0\}$, we have $\frac{\partial^k}{\partial s^k}\widehat{T}_s f(r) = (-1)^k \widehat{\mathcal{T}}_{s,k}f(r)$.*

For each $j = 1, 2, \ldots, R$ and $\delta \geq 0$ we set

$$E(j,a,\delta) = \binom{k_j - 1}{a} \frac{(-1)^{1-k_j+a}}{(k_j-1)!} \frac{\partial^{k_j-1-a}}{\partial s^{k_j-1-a}} \left[\frac{\prod_{l=1}^{N} (q_l - s)^{m_l} (\rho_{j,\delta} - s)^{k_j}}{\prod_{l=1}^{R} (\rho_{l,\delta} - s)^{k_l}} \right]_{s=\rho_{j,\delta}},$$

$$E_*(j,a,\delta) = \binom{k_j - 1}{a} \frac{(-1)^{1-k_j+a}}{(k_j-1)!} \frac{\partial^{k_j-1-a}}{\partial s^{k_j-1-a}} \left[\frac{\prod_{l=1}^{N} (q_l - s)^{m_l} (\rho_{j,\delta} - s)^{k_j}}{\prod_{l=1}^{R} (\rho_{l,\delta} - s)^{k_l}} s \right]_{s=\rho_{j,\delta}},$$

and define, for $\delta \geq 0$, the functions

$$\ell_\delta(u) = \sum_{j=1}^{R} \sum_{a=0}^{k_j-1} E(j,a,\delta) \mathcal{T}_{\rho_{j,\delta};a} \nu_{\mathcal{S}}(u), \tag{3.10}$$

$$\mathcal{L}_\delta(u) = \sum_{j=1}^{R} \sum_{a=0}^{k_j-1} E_*(j,a,\delta) \mathcal{T}_{\rho_{j,\delta};a} \overline{\mathcal{V}}_{\mathcal{S}}(u) \tag{3.11}$$

We have the following technical lemma.

Lemma 15. *Let $\nu_{\mathcal{S}}$ be the Lévy measure of a spectrally positive pure jumps Lévy* *process. Then*

$$\frac{\Psi_{\mathcal{S}}(r_1)-\Psi_{\mathcal{S}}(r_2)}{r_2-r_1}=r_2\widehat{T}_{r_2}\overline{\mathcal{V}}_{\mathcal{S}}(r_1)-\frac{\Psi_{\mathcal{S}}(r_1)}{r_1}=r_1\widehat{T}_{r_2}\overline{\mathcal{V}}_{\mathcal{S}}(r_1)-\frac{\Psi_{\mathcal{S}}(r_2)}{r_2} \tag{3.12}$$

for any $r_1, r_2 \in \mathbb{C}_+$ *such that* $r_1 \neq r_2$.

Additionally, if the condition $\int\limits_{0+}^{\infty}(x^2\wedge x)\,\nu_{\mathcal{S}}(dx)<\infty$ *holds, then*

$$-\Psi_{\mathcal{S}}(r)=r\int\limits_{0+}^{\infty}\left(1-e^{-rx}\right)\overline{\mathcal{V}}_{\mathcal{S}}(x)dx \tag{3.13}$$

Proof. See Appendix A. ∎

3.2 The Wiener-Hopf factors of $\mathcal{X}$

In this section we obtain an explicit expression for the probability density of the negative Wiener-Hopf factor $I^{\mathcal{X}}_{e_\delta}$.

In order to simplify our notation, we define the constants

$$a_\delta=\delta\frac{\prod_{j=1}^{N}q_j^{m_j}}{\prod_{j=1}^{R}\rho_{j,\delta}^{k_j}}\text{ when }\delta>0,\text{ and }a_0=\Psi'_{\mathcal{X}}(0+)\frac{\prod_{j=1}^{N}q_j^{m_j}}{\prod_{j=2}^{R}\rho_{j,0}^{k_j}}\text{ when }\delta=0.$$

We also recall the following notation from Chapter 2: $Q_1(r)=\prod\limits_{i=1}^{N}(q_i-r)^{m_i}$ and $\widehat{e}_\delta^{+}(r)=\frac{\prod_{j=1}^{N}(q_j-r)^{m_j}}{\prod_{l=1}^{R}(\rho_{l,\delta}-r)^{k_l}}$, for $r\neq\rho_{j,\delta}$, $j=1,2,\ldots,R$.

The following lemma is key for our main result.

Lemma 16. *The term* $[\delta-L_{\mathcal{X}}(r)]\,\widehat{e}_\delta^{+}(r)$ *has the following equivalent representations, for* $\delta\geq 0$.

$$\begin{array}{ll} a_\delta+G_{\mathcal{S}}(r)+\widehat{\ell}_\delta(0)-\widehat{\ell}_\delta(r), & \textit{in Case A},\\ a_\delta+\widehat{\ell}_\delta(0)-\widehat{\ell}_\delta(r), & \textit{in Case B},\\ a_\delta+\gamma^2 r-\frac{\Psi_{\mathcal{S}}(r)}{r}-\left[\widehat{\mathcal{L}}_\delta(0)-\widehat{\mathcal{L}}_\delta(r)\right], & \textit{in Case C}.\end{array} \tag{3.14}$$

Proof. For the proof of the case $\delta > 0$, see Appendix A. The case $\delta = 0$ follows by taking limits when $\delta \downarrow 0$ and using (3.7) and (3.8). ∎

Let us consider the following functions, for $j = 1, 2, 3$:

$$\chi_{j,\mathcal{S}}(x;\delta)dx = \begin{cases} \nu_{\mathcal{S}}(dx) + \ell_\delta(x)dx & j = 1, \\ \ell_\delta(x)dx & j = 2, \\ \left[\overline{\mathcal{V}}_{\mathcal{S}}(x) - \mathcal{L}_\delta(x)\right] dx & j = 3. \end{cases} \tag{3.15}$$

defined for $x > 0$ and $\delta \geq 0$.

We have the following important result.

Proposition 5. *The function $\chi_{j,\mathcal{S}}(y;\delta)$ satisfies the equality*

$$\int_{0+}^{\infty} \left(1 - e^{-ry}\right) \chi_{j,\mathcal{S}}(y;\delta)dy = \int_{(0,\infty)^2} \left(1 - e^{-ry}\right) e^{-\delta x} \Lambda_j(dx, dy),$$

for $j = 1, 2, 3$, where $\Lambda_j(dx, dy)$ is the Lévy measure of a bivariate subordinator. Therefore, $\chi_{j,\mathcal{S}}(x;\delta)dx$ is the marginal Laplace transform $\int_{(0,\infty)} e^{-\delta x}\Lambda_j(dx, dy)$ and it is also the Lévy measure of some univariate subordinator.

Proof. By Theorem 2.2 in Lewis and Mordecki [2008], we have

$$\mathbb{E}\left[e^{irS^{\mathcal{X}}_{e_\delta}}\right] = \delta^{-1} \frac{\prod_{j=1}^{R} \rho_{j,\delta}^{k_j}}{\prod_{j=1}^{N} q_j^{m_j}} \widehat{e}_\delta^{+}(ir),$$

for $s \in \mathbb{R}$. Therefore (1.8) gives: $\mathbb{E}\left[e^{irI^{\mathcal{X}}_{e_\delta}}\right] = \frac{\delta \frac{\prod_{j=1}^{N} q_j^{m_j}}{\prod_{j=1}^{R} \rho_{j,\delta}^{k_j}}}{\left[\delta + \Psi^{[c]}_{\mathcal{X}}(r)\right] \widehat{e}_\delta^{+}(ir)}$.

An application of the relation $-\Psi_{\mathcal{X}}^{[c]}(-ir) = L_{\mathcal{X}}(r)$, for $r \geq 0$, yields:

$$\mathbb{E}\left[e^{-r\left[-I_{e_\delta}^{\mathcal{X}}\right]}\right] = \frac{\delta \frac{\prod_{j=1}^{N} q_j^{m_j}}{\prod_{j=1}^{R} \rho_{j,\delta}^{k_j}}}{\left[\delta - L_{\mathcal{X}}(r)\right] \widehat{e}_\delta^{+}(r)} \tag{3.16}$$

Since $-I_{e_\delta}^{\mathcal{X}}$ is a nonnegative random variable, it suffices to work with its Laplace transform, as given in (3.16). Now by Lemma 16, (3.16) and the definition of a_δ, we have:

$$\mathbb{E}\left[e^{-r\left[-I_{e_\delta}^{\mathcal{X}}\right]}\right] = \begin{cases} \frac{a_\delta}{a_\delta + G_{\mathcal{S}}(r) + \widehat{\ell}_\delta(0) - \widehat{\ell}_\delta(r)}, & \text{in Case A,} \\ \frac{a_\delta}{a_\delta + \widehat{\ell}_\delta(0) - \widehat{\ell}_\delta(r)}, & \text{in Case B,} \\ \frac{a_\delta}{a_\delta + \gamma^2 r - \frac{\Psi_{\mathcal{S}}(r)}{r} - \left[\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r)\right]}, & \text{in Case C.} \end{cases} \tag{3.17}$$

We apply (3.13) in case C to obtain $-\frac{\Psi_{\mathcal{S}}(r)}{r} = \int_{0+}^{\infty} \left(1 - e^{-rx}\right) \overline{\mathcal{V}}_{\mathcal{S}}(x) dx$. This and the equality $\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r) = \int_{0+}^{\infty} \left(1 - e^{-rx}\right) \mathcal{L}_\delta(x) dx$ give $-\frac{\Psi_{\mathcal{S}}(r)}{r} - \left[\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r)\right] = \int_{0+}^{\infty} \left(1 - e^{-rx}\right) \chi_{3,\mathcal{S}}(x;\delta) dx$. Hence, from (3.17) and (3.15) we obtain

$$\mathbb{E}\left[e^{-r\left[-I_{e_\delta}^{\mathcal{X}}\right]}\right] = \begin{cases} \frac{a_\delta}{a_\delta + \int_{0+}^{\infty} (1 - e^{-rx}) \chi_{1,\mathcal{S}}(x;\delta) dx}, & \text{in case A,} \\ \frac{a_\delta}{a_\delta + \int_{0+}^{\infty} (1 - e^{-rx}) \chi_{2,\mathcal{S}}(x;\delta) dx}, & \text{in case B,} \\ \frac{a_\delta}{a_\delta + \gamma^2 r + \int_{0+}^{\infty} (1 - e^{-rx}) \chi_{3,\mathcal{S}}(x;\delta) dx}, & \text{in case C.} \end{cases} \tag{3.18}$$

We note that $-I_{e_\delta}^{\mathcal{X}}$ is the positive Wiener-Hopf factor of $-\mathcal{X}$, therefore, by the arguments in Kyprianou [2006], p. 165, there exist bivariate Lévy measures $\Lambda_j(dx, dy)$ and subordinators $\mathbb{H}_j = \{\mathbb{H}_j(t), t \geq 0\}$, for $j = 1, 2, 3$ such that

$$b_j r + \int_{0+}^{\infty} \left(1 - e^{-ry}\right) \int_{0+}^{\infty} e^{-\delta x} \Lambda_j(dx, dy) = \kappa_D(\delta, r) - \kappa_{j,D}(\delta, 0), \tag{3.19}$$

and

$$\mathbb{E}\left[e^{-r\left[-I^{\mathcal{X}}_{e_\delta}\right]}\right] = \begin{cases} \mathbb{E}\left[e^{-r\mathbb{H}_1\left(e_{\kappa_{1,D}(\delta,0)}\right)}\right] = \frac{\kappa_{1,D}(\delta,0)}{\kappa_{1,D}(\delta,0)+\left[\kappa_{1,D}(\delta,r)-\kappa_{1,D}(\delta,0)\right]} & \text{in case A} \\ \mathbb{E}\left[e^{-r\mathbb{H}_2\left(e_{\kappa_{2,D}(\delta,0)}\right)}\right] = \frac{\kappa_{2,D}(\delta,0)}{\kappa_{2,D}(\delta,0)+\left[\kappa_{2,D}(\delta,r)-\kappa_{2,D}(\delta,0)\right]} & \text{in case B} \\ \mathbb{E}\left[e^{-r\mathbb{H}_3\left(e_{\kappa_{3,D}(\delta,0)}\right)}\right] = \frac{\kappa_{3,D}(\delta,0)}{\kappa_{3,D}(\delta,0)+\left[\kappa_{3,D}(\delta,r)-\kappa_{3,D}(\delta,0)\right]} & \text{in case C} \end{cases}$$

where $e_{\kappa_{j,D}(\delta,0)}$, for $j = 1, 2, 3$, are exponential random variables with mean $1/\kappa_{j,D}(\delta, 0)$, independent of $\mathbb{H}_j$.

From (3.18) we identify

$$\kappa_{j,D}(\delta, r) - \kappa_{j,D}(\delta, 0) = \begin{cases} \int_{0+}^{\infty} \left(1 - e^{-rx}\right) \chi_{1,\mathcal{S}}(x; \delta)dx & \text{for } j = 1 \text{ in case A} \\ \int_{0+}^{\infty} \left(1 - e^{-rx}\right) \chi_{2,\mathcal{S}}(x; \delta)dx & \text{for } j = 2 \text{ in case B} \\ \gamma^2 r + \int_{0+}^{\infty} \left(1 - e^{-rx}\right) \chi_{3,\mathcal{S}}(x; \delta)dx & \text{for } j = 3 \text{ in case C} \end{cases}$$

Comparing this with (3.19) and using the unicity of the Wiener-Hopf factorization, we conclude that $b_3 = \gamma^2$, $b_1 = b_2 = 0$, and $\int_{0+}^{\infty} e^{-\delta x}\Lambda_j(dx, dy) = \chi_{j,\mathcal{S}}(y; \delta)dy$ for $j = 1, 2, 3$, hence the result follows. ■

Definition 3. *For $j = 1, 2, 3$ we denote by $\mathcal{N}_{j,\delta} = \{\mathcal{N}_{j,\delta}(t), t \geq 0\}$, $\delta \geq 0$, the subordinator with Lévy measure $\chi_{j,\mathcal{S}}(x; \delta)dx$, and define, or $\delta \geq 0$, the function $W_\delta(u), u > 0$ through its Laplace transform:*

$$\widehat{W}_\delta(r) = \frac{1}{\left[\delta - L_{\mathcal{X}}(r)\right] \widehat{e_\delta^+}(r)}. \tag{3.20}$$

Remark 2. *We note that $N_{2,\delta}$ is a compound Poisson process.*

It is easy to prove that $I^{\mathcal{X}}_{e_\delta} \xrightarrow{d} I^{\mathcal{X}}_{\infty}$ when $\delta \downarrow 0$. With this in mind, when we refer to the case $\delta \geq 0$ we understand the case $\delta = 0$ as a limit case. The following is our first main result in this section. It summarizes the results obtained previously.

Theorem 12. *For $\delta \geq 0$, the following assertions hold:*

a) *The random variables $-I^{\mathcal{X}}_{e_\delta}$ for $\delta > 0$ and $-I^{\mathcal{X}}_{\infty}$ satisfy the equalities in distribution:*

$$-I^{\mathcal{X}}_{e_\delta} \stackrel{d}{=} \begin{cases} \mathcal{N}_{1,\delta}(e_{a_\delta}) & \textit{in case A} \\ \mathcal{N}_{2,\delta}(e_{a_\delta}) & \textit{in case B} \\ \gamma^2 e_{a_\delta} + \mathcal{N}_{3,\delta}(e_{a_\delta}) & \textit{in case C} \end{cases} \tag{3.21}$$

where e_{a_δ} is an exponential random variable with mean $1/a_\delta$, for $\delta \geq 0$, independent of $\mathcal{N}_{j,\delta}$ for $j = 1, 2, 3$.

b) *We have*

$$\mathbb{E}\left[e^{-r\left[-I^{\mathcal{X}}_{e_\delta}\right]}\right] = a_\delta \widehat{W}_\delta(r).$$

c) *The function $\widehat{W}_\delta(r)$ satisfies the following equalities:*

$$a_\delta \widehat{W}_\delta(r) = \begin{cases} \dfrac{\mathbb{E}\left[e^{-r\mathcal{S}(e_{a_\delta})}\right]}{1+\frac{1}{a_\delta}\mathbb{E}\left[e^{-r\mathcal{S}(e_{a_\delta})}\right]\left[\widehat{\ell}_\delta(0)-\widehat{\ell}_\delta(r)\right]}, & \textit{in Case A,} \\ \dfrac{\frac{a_\delta}{a_\delta+\widehat{\ell}_\delta(0)}}{1-\frac{1}{a_\delta+\widehat{\ell}_\delta(0)}\widehat{\ell}_\delta(r)}, & \textit{in Case B,} \\ \dfrac{a_\delta \widehat{\mathfrak{W}}'_{\mathcal{Y}}(r)}{1-\widehat{\mathfrak{W}}'_{\mathcal{Y}}(r)\left[\widehat{\mathcal{L}}_\delta(0)-\widehat{\mathcal{L}}_\delta(r)\right]}, & \textit{in Case C,} \end{cases} \tag{3.22}$$

where $\widehat{\mathfrak{W}}'_{\mathcal{Y}}(r)$ is given by

$$\widehat{\mathfrak{W}}'_{\mathcal{Y}}(r) = \frac{r}{a_\delta r + \gamma^2 r^2 - \Psi_{\mathcal{S}}(r)}, \ r, \delta \geq 0$$

In particular, $\widehat{\mathfrak{W}}'_{\mathcal{Y}}(r)$ is the Laplace transform of the derivative of the scale function (for $q = 0$) of the spectrally negative Lévy process $\mathcal{Y}_\delta = \{\mathcal{Y}_\delta(t), t \geq 0\}$ given by

$$\mathcal{Y}_\delta(t) = a_\delta t + \gamma \mathcal{B}(t) - \mathcal{S}(t), \ \delta \geq 0, \gamma \geq 0.$$

Proof. The statements in b) follow from (3.20) and (3.16). For a) and c) we prove all the results for $\delta > 0$. The results for $\delta = 0$ follow by taking limits.

Case A: From (3.20) and Lemma 16 we obtain:

$$a_\delta \widehat{W}_\delta(r) = \frac{a_\delta}{a_\delta + G_{\mathcal{S}}(r) + \widehat{\ell}_\delta(0) - \widehat{\ell}_\delta(r)} = \frac{\frac{a_\delta}{a_\delta + G_{\mathcal{S}}(r)}}{1 + \frac{1}{a_\delta}\frac{a_\delta}{a_\delta + G_{\mathcal{S}}(r)}\left[\widehat{\ell}_\delta(0) - \widehat{\ell}_\delta(r)\right]}. \tag{3.23}$$

Substituting the equality $\mathbb{E}\left[e^{-r\mathcal{S}(e_{a_\delta})}\right] = \frac{a_\delta}{a_\delta + G_{\mathcal{S}}(r)}$ in (3.23), we obtain c). The result in a) follows from $\mathbb{E}\left[e^{-r\left(\mathcal{N}_{1,\delta}(e_{a_\delta})\right)}\right] = \frac{a_\delta}{a_\delta + G_{\mathcal{S}}(r) + \widehat{\ell}_\delta(0) - \widehat{\ell}_\delta(r)}$, and (3.23).

Case B: From (3.20) and Lemma 16 we obtain $a_\delta \widehat{W}_\delta(r) = \frac{a_\delta}{a_\delta + \widehat{\ell}_\delta(0) - \widehat{\ell}_\delta(r)}$, which implies c). The result in b) follows from the equality $\mathbb{E}\left[e^{-r\mathcal{N}_{2,\delta}(e_{a_\delta})}\right] = \frac{a_\delta}{a_\delta + \widehat{\ell}_\delta(0) - \widehat{\ell}_\delta(r)}$.

Case C: Now we let $\mathcal{Y}_\delta = \{\mathcal{Y}_\delta(t), t \geq 0\}$ be as in the statement of this theorem.

By (1.9), the q-scale function (for $q = 0$) $\mathfrak{W}_{\mathcal{Y}}$ of $\mathcal{Y}$ has Laplace transform

$$\widehat{\mathfrak{W}}_{\mathcal{Y}}(r) = \frac{1}{a_\delta r + \gamma^2 r^2 - \Psi_{\mathcal{S}}(r)}.$$

In case C there is at least one process with unbounded variation (either the Brownian motion or the claim process $\mathcal{S}$), hence $\mathfrak{W}_{\mathcal{Y}} \in \mathcal{C}^1(0, \infty)$ and $\mathfrak{W}'_{\mathcal{Y}}(0) = 0$. Therefore, by (1.10) we have

$$\widehat{\mathfrak{W}}'_{\mathcal{Y}}(r) = \frac{r}{a_\delta r + \gamma^2 r^2 - \Psi_{\mathcal{S}}(r)}.$$

Using (3.20), Lemma 16 and the equality $\widehat{\mathfrak{W}}'_{\mathcal{Y}}(r) = \frac{r}{a_\delta r + \gamma^2 r^2 - \Psi_{\mathcal{S}}(r)}$, it follows that:

$$\begin{aligned} a_\delta \widehat{W}_\delta(r) &= \frac{a_\delta}{a_\delta + \gamma^2 r - \frac{\Psi_{\mathcal{S}}(r)}{r} - \left[\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r)\right]} \\ &= \frac{a_\delta \frac{1}{a_\delta + \gamma^2 r - \frac{\Psi_{\mathcal{S}}(r)}{r}}}{1 - \frac{1}{a_\delta + \gamma^2 r - \frac{\Psi_{\mathcal{S}}(r)}{r}}\left[\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r)\right]} \qquad (3.24) \\ &= \frac{a_\delta \frac{r}{a_\delta r + \gamma^2 r^2 - \Psi_{\mathcal{S}}(r)}}{1 - \frac{r}{a_\delta r + \gamma^2 r^2 - \Psi_{\mathcal{S}}(r)}\left[\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r)\right]} \\ &= \frac{a_\delta \widehat{\mathfrak{W}}_{\mathcal{Y}}(r)}{1 - \widehat{\mathfrak{W}}_{\mathcal{Y}}(r)\left[\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r)\right]} \qquad (3.25) \end{aligned}$$

This proves c). Now we note that:

$$\mathbb{E}\left[e^{-r\left(\gamma^2 e_{a_\delta}+N_{3,\delta}(e_{a_\delta})\right)}\right] = \frac{a_\delta}{a_\delta+\gamma^2 r-\frac{\Psi_{\mathcal{S}}(r)}{r}-\left[\widehat{\mathcal{L}}_\delta(0)-\widehat{\mathcal{L}}_\delta(r)\right]}.$$

Therefore (3.24) gives the result in a). ∎

We let $\Theta_\delta(x), \delta \geq 0$ be the distribution of the random variable $\mathcal{S}(e_{a_\delta})$. For simplicity, we assume that this distribution has a density θ_δ, although an extension for the case when such a density does not exist is simple. Under this assumption we have $\widehat{\theta}_\delta(r) = \frac{a_\delta}{a_\delta+G_{\mathcal{S}}(r)}$. We are now ready to state and prove our main result, in which we invert $\widehat{W}_\delta(r)$ for cases A, B and C.

Theorem 13. *(Main Theorem III).*

For $\delta \geq 0$, the random variable $-I^{\mathcal{X}}_{e_\delta}$ has density function $a_\delta W_\delta(u), u > 0$, where:

$$W_\delta(u) = \begin{cases} \frac{1}{a_\delta}\theta_\delta(u) + \frac{1}{a_\delta}\theta_\delta * \sum_{n=1}^{\infty}\left(-\frac{1}{a_\delta}\right)^n \left(\widehat{\ell}_\delta(0)\theta_\delta - \ell_\delta * \theta_\delta\right)^{*n}(u) & \text{in Case A,} \\ \dfrac{1}{a_\delta+\widehat{\ell}_\delta(0)}\delta_0 + \dfrac{1}{a_\delta+\widehat{\ell}_\delta(0)}\sum_{n=1}^{\infty}\left(\dfrac{1}{a_\delta+\widehat{\ell}_\delta(0)}\right)^n \ell_\delta^{*n}(u), & \text{in Case B,} \\ \mathfrak{W}'_{\mathcal{Y}}(u) + \mathfrak{W}'_{\mathcal{Y}} * \sum_{n=1}^{\infty}\left(\widehat{\mathcal{L}}_\delta(0)\mathfrak{W}'_{\mathcal{Y}} - \mathfrak{W}'_{\mathcal{Y}} * \mathcal{L}_\delta\right)^{*n}(u) & \text{in Case C,} \end{cases} \tag{3.26}$$

where δ_0 is Dirac's delta function.

Proof. By Theorem 12 c) and the definition of θ_δ, we have:

In case A:

$$\begin{aligned} \widehat{W}_\delta(r) &= \frac{1}{a_\delta}\widehat{\theta}_\delta(r)\sum_{n=0}^{\infty}\left(-\frac{1}{a_\delta}\mathbb{E}\left[e^{-r\mathcal{S}(e_{a_\delta})}\right]\left[\widehat{\ell}_\delta(0)-\widehat{\ell}_\delta(r)\right]\right)^n \\ &= \frac{1}{a_\delta}\widehat{\theta}_\delta(r) + \frac{1}{a_\delta}\widehat{\theta}_\delta(r)\sum_{n=1}^{\infty}\left(-\frac{1}{a_\delta}\mathbb{E}\left[e^{-r\mathcal{S}(e_{a_\delta})}\right]\left[\widehat{\ell}_\delta(0)-\widehat{\ell}_\delta(r)\right]\right)^n \end{aligned} \tag{3.27}$$

In case B:

$$\widehat{W}_\delta(r) = \frac{1}{a_\delta + \widehat{\ell}_\delta(0)} \sum_{n=0}^{\infty} \left(\frac{1}{a_\delta + \widehat{\ell}_\delta(0)} \widehat{\ell}_\delta(r) \right)^n$$
$$= \frac{1}{a_\delta + \widehat{\ell}_\delta(0)} + \frac{1}{a_\delta + \widehat{\ell}_\delta(0)} \sum_{n=1}^{\infty} \left(\frac{1}{a_\delta + \widehat{\ell}_\delta(0)} \widehat{\ell}_\delta(r) \right)^n \tag{3.28}$$

and, in case C:

$$\widehat{W}_\delta(r) = \widehat{\mathfrak{W}}'_{\mathcal{Y}}(r) \sum_{n=0}^{\infty} \left(\widehat{\mathfrak{W}}'_{\mathcal{Y}}(r) \big[\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r) \big] \right)^n$$
$$= \widehat{\mathfrak{W}}'_{\mathcal{Y}}(r) + \widehat{\mathfrak{W}}'_{\mathcal{Y}}(r) \sum_{n=1}^{\infty} \left(\widehat{\mathfrak{W}}'_{\mathcal{Y}}(r) \big[\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r) \big] \right)^n \tag{3.29}$$

Now the result follows inverting the formulae in the second equality in (3.27), (3.28) and (3.29) for the case $\delta > 0$. The result for $\delta = 0$ follows by taking limits. ■

3.2.1 Two particular cases

We study two important particular subcases of case C. First we show that our formula in Theorem 12 has an alternative expression in the following subcase: Let us suppose that the negative jumps in the process $\mathcal{X}$ are given by the process $-(\mathcal{S} + \mathcal{M})$, where $\mathcal{S} = \{\mathcal{S}(t), t \geq 0\}$ is a subordinator and $\mathcal{M} = \{\mathcal{M}(t), t \geq 0\}$ is a pure jumps process with unbounded variation and only positive jumps.

This means we work with the process $\mathcal{X}$ whose GLF is given by

$$L_{\mathcal{X}}(r) = \lambda_1 \left(\frac{Q(-r)}{\prod_{j=1}^{N} (q_j - r)^{m_j}} - 1 \right) + \gamma^2 r^2 + cr - G_{\mathcal{S}}(r) - \Psi_{\mathcal{M}}(r), \tag{3.30}$$

where $-\Psi_{\mathcal{M}}(r) = -\int_{0+}^{\infty} (1 - e^{-rx} - rx) \nu_{\mathcal{M}}(dx)$ is the Laplace exponent of $\mathcal{M}$ and $\nu_{\mathcal{M}}(dx)$ is its corresponding Lévy measure, $c \geq 0$ and λ_1, $\frac{Q(-r)}{\prod_{j=1}^{N}(q_j - r)^{m_j}}$, γ^2 and $G_{\mathcal{S}}(r)$ are as before. Again we suppose that $\mathbb{E}\left[\mathcal{X}(1)\right] > 0$.

The case we now study includes, as a particular case, the process studied in Chapter 2 where $\mathcal{S}$ is a compound Poisson process and $\mathcal{M}$ is an α-stable process with only positive jumps and $\alpha \in (1,2)$.

In view of Theorem 14 and Corollary 7, we only need to study the measure $a_\delta W_\delta(du)$ corresponding to the distribution of the corresponding negative Wiener-Hopf factor.

Since the case we are considering is a particular case of Case 3 (as defined before), we work with the function $\mathcal{L}_\delta(u)$ defined in (3.11). We denote by $\nu_{\mathcal{S}}$ and $\nu_{\mathcal{M}}$ the Lévy measures of $\mathcal{S}$ and $\mathcal{M}$, respectively, and define the function

$$\mathcal{L}_{\delta,\mathcal{M}}(u) = \sum_{j=1}^{R} \sum_{a=0}^{k_j-1} E_*(j,a,\delta) T_{\rho_{j,\delta};a} \overline{\mathcal{V}}_{\mathcal{M}}(u)$$

We have the following result.

Lemma 17. *Suppose that $\mathcal{X} = \{\mathcal{X}(t), t \geq 0\}$ is given by*

$$\mathcal{X}(t) = ct + \mathcal{Z}(t) + \gamma\mathcal{B}(t) - \mathcal{S}(t) - \mathcal{M}(t), \tag{3.31}$$

where $\mathcal{S}$ is a subordinator and $\mathcal{M}$ is a pure jumps Lévy process with unbounded variation and positive jumps. Then, for $\delta \geq 0$, the term $[L_{\mathcal{X}}(r) - \delta]\, \widehat{e}_\delta^{+}(r)$ has the equivalent representation:

$$[L_{\mathcal{X}}(r) - \delta]\, \widehat{e}_\delta^{+}(r) = c + \gamma^2 D_\delta + \gamma^2 r - \widehat{\ell}_\delta(r) - \frac{\Psi_{\mathcal{M}}(r)}{r} + \widehat{\mathcal{L}}_{\delta,\mathcal{M}}(r), \tag{3.32}$$

where

$$D_\delta = \sum_{l=1}^{R} \frac{(-1)^{1-k_l}}{(k_l-1)!} \frac{\partial^{k_l-1}}{\partial s^{k_l-1}} \left[\frac{\prod_{j=1}^{R} (q_j - s)^{m_j} (\rho_{l,\delta} - s)^{k_l}}{\prod_{j=1}^{R} (\rho_{j,\delta} - s)} s \right]_{s=\rho_{l,\delta}} .$$

Proof. From (A.51) we obtain

$$[\delta - L_{\mathcal{X}}(r)]\,\widehat{e}^{+}_{\delta}(r) = c + \gamma^2 D_\delta + \gamma^2 r - \frac{\Psi_{\mathcal{S}+\mathcal{M}}(r)}{r} + \sum_{l=1}^{R} \frac{(-1)^{1-k_l}}{(k_l-1)!}\frac{\partial^{k_l-1}}{\partial s^{k_l-1}}\left[\frac{\prod_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}} s\widehat{T}_s\overline{\mathcal{V}}_{\mathcal{S}+\mathcal{M}}(r)\right]_{s=\rho_{l,\delta}} \tag{3.33}$$

Since $\Psi_{\mathcal{S}+\mathcal{M}}(r) = G_{\mathcal{S}}(r) + \Psi_{\mathcal{M}}(r)$, we deduce that $\widehat{T}_s\overline{\mathcal{V}}_{\mathcal{S}+\mathcal{M}}(r) = \widehat{T}_s\overline{\mathcal{V}}_{\mathcal{M}}(r) + \widehat{T}_s\overline{\mathcal{V}}_{\mathcal{S}}(r)$.

The two equalities above and (3.56) imply that the expression in (3.33) is equivalent to

$$\begin{aligned}
&[\delta - L_{\mathcal{X}}(r)]\,\widehat{e}^{+}_{\delta}(r) \\
&= c + \gamma^2 D_\delta + \gamma^2 r - \frac{\Psi_{\mathcal{M}}(r)}{r} \\
&+ \sum_{l=1}^{R} \frac{(-1)^{1-k_l}}{(k_l-1)!}\frac{\partial^{k_l-1}}{\partial s^{k_l-1}}\left[\frac{\prod_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}}\left(s\widehat{T}_s\overline{\mathcal{V}}_{\mathcal{M}}(r) + s\widehat{T}_s\overline{\mathcal{V}}_{\mathcal{S}}(r) - \frac{G_{\mathcal{S}}(r)}{r}\right)\right]_{s=\rho_{l,\delta}}
\end{aligned} \tag{3.34}$$

Now we have

$$\begin{aligned}
s\widehat{T}_s\overline{\mathcal{V}}_{\mathcal{S}}(r) - \frac{G_{\mathcal{S}}(r)}{r} &= s\frac{\int_{0+}^{\infty}(e^{-rx}-e^{-sx})\int_{x}^{\infty}\nu_{\mathcal{S}}(dy)dx}{s-r} - \frac{\int_{0+}^{\infty}(1-e^{-rx})\nu_{\mathcal{S}}(dx)}{r} \\
&= s\frac{\int_{0+}^{\infty}\int_{0}^{y}(e^{-rx}-e^{-sx})dx\nu_{\mathcal{S}}(dy)}{s-r} - \frac{\int_{0+}^{\infty}(1-e^{-rx})\nu_{\mathcal{S}}(dx)}{r} \\
&= s\frac{\int_{0+}^{\infty}\left[\frac{1}{r}(1-e^{-rx}) - \frac{1}{s}(1-e^{-sx})\right]\nu_{\mathcal{S}}(dx)}{s-r} - \frac{\int_{0+}^{\infty}(1-e^{-rx})\nu_{\mathcal{S}}(dx)}{r}
\end{aligned}$$

Simplifying the right-hand side in the third equality above we obtain:

$$\begin{aligned}
&s\frac{\int_{0+}^{\infty}\left[\frac{1}{r}(1-e^{-rx})-\frac{1}{s}(1-e^{-sx})\right]\nu_{\mathcal{S}}(dx)}{s-r}-\frac{\int_{0+}^{\infty}(1-e^{-rx})\nu_{\mathcal{S}}(dx)}{r}\\
&=s\frac{\int_{0+}^{\infty}\left[\frac{1}{r}(1-e^{-rx})-\frac{1}{s}(1-e^{-sx})\right]\nu_{\mathcal{S}}(dx)}{s-r}-(s-r)\frac{\int_{0+}^{\infty}(1-e^{-rx})\nu_{\mathcal{S}}(dx)}{(s-r)r}\\
&=\frac{\int_{0+}^{\infty}\left[\frac{s}{r}(1-e^{-rx})-(1-e^{-sx})\right]\nu_{\mathcal{S}}(dx)-\frac{s}{r}\int_{0+}^{\infty}(1-e^{-rx})\nu_{\mathcal{S}}(dx)+\int_{0+}^{\infty}(1-e^{-rx})\nu_{\mathcal{S}}(dx)}{s-r}\\
&=\frac{\int_{0+}^{\infty}\left[e^{-sx}-e^{-rx}\right]\nu_{\mathcal{S}}(dx)}{s-r}=-\widehat{T}_s\nu_{\mathcal{S}}(r)
\end{aligned}$$

where the last equality follows from (1.5).

Applying the equality above and the linearity of the partial derivatives we obtain:

$$\begin{aligned}
\left[\delta-L_{\mathcal{X}}(r)\right]\widehat{e}_{\delta}^{+}(r)&=c+\gamma^2D_{\delta}+\gamma^2r-\frac{\Psi_{\mathcal{M}}(r)}{r}\\
&-\sum_{l=1}^{R}\frac{(-1)^{1-k_l}}{(k_l-1)!}\frac{\partial^{k_l-1}}{\partial s^{k_l-1}}\left[\frac{\prod_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}}\widehat{T}_s\nu_{\mathcal{S}}(r)\right]_{s=\rho_{l,\delta}}\\
&+\sum_{l=1}^{R}\frac{(-1)^{1-k_l}}{(k_l-1)!}\frac{\partial^{k_l-1}}{\partial s^{k_l-1}}\left[\frac{\prod_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}}s\widehat{T}_s\overline{\mathcal{V}}_{\mathcal{M}}(r)\right]_{s=\rho_{l,\delta}}
\end{aligned}\tag{3.35}$$

By definition of $\ell_{\delta}(u)$ and $\mathcal{L}_{\delta,\mathcal{M}}(u)$, the equality above gives the result. ∎

In the next result we show that the function $\widehat{W}_{\delta}(r), r\geq 0$, admits an alternative representation.

Proposition 6. *Suppose $\mathcal{X}$ is the process given in (3.31). Let $\kappa_{\delta}=\widehat{\ell}_{\delta}(0)+\widehat{\mathcal{L}}_{\delta,\mathcal{M}}(0)$.*

Then

$$\widehat{W}_\delta(r) = \frac{\widehat{\vartheta}_\delta(r)}{1 - \left[\kappa_\delta \widehat{\vartheta}_\delta(r) + \widehat{\ell}_\delta(r)\widehat{\vartheta}_\delta(r)\right]}$$

where

$$\widehat{\vartheta}_\delta(r) = \frac{\widehat{\mathfrak{W}}'_{\mathcal{M}^-}(r)}{1 + \widehat{\mathcal{L}}_{\delta,\mathcal{M}}(r)\widehat{\mathfrak{W}}'_{\mathcal{M}^-}},$$

is the Laplace transform of the function

$$\vartheta_\delta(u) = \mathfrak{W}'_{\mathcal{M}^-}(u) + \mathfrak{W}'_{\mathcal{M}^-} * \sum_{n=1}^{\infty} (-1)^n \left(\mathcal{L}_{\delta,\mathcal{M}} * \mathfrak{W}'_{\mathcal{M}^-}\right)^{*n}(u), u > 0$$

and

$$\widehat{\mathfrak{W}}'_{\mathcal{M}^-}(r) = \frac{r}{(c + \gamma^2 D_\delta + \kappa_\delta)\, r + \gamma^2 r^2 - \Psi_{\mathcal{M}}(r)} \tag{3.36}$$

is the Laplace transform of the first derivative of the scale function $\mathfrak{W}_{\mathcal{M}^-}(x), x > 0$ *corresponding to the spectrally negative process* $\mathcal{M}^- = \{\mathcal{M}^-(t), t \geq 0\}$ *defined as*

$$\mathcal{M}^-(t) = \left(c + \gamma^2 D_\delta + \kappa_\delta\right) t + \gamma \mathcal{B}(t) - \mathcal{M}(t).$$

Moreover, the function $\widehat{W}_\delta(r)$ *is the Laplace transform of the function*

$$W_\delta(u) = \vartheta_\delta(u) + \vartheta_\delta * \sum_{n=1}^{\infty} \left(\kappa_\delta \vartheta_\delta + \ell_\delta * \vartheta_\delta\right)^{*n}(u), u > 0$$

Proof. By Lemma 17 and (3.20) we have

$$\widehat{W}_\delta(r) = \frac{1}{c + \gamma^2 D_\delta + \gamma^2 r - \widehat{\ell}_\delta(r) - \frac{\Psi_{\mathcal{M}}(r)}{r} + \widehat{\mathcal{L}}_{\delta,\mathcal{M}}(r)}.$$

We add and subtract k_δ as defined in the statement of this proposition and apply some basic algebra to obtain:

$$\widehat{W}_\delta(r) = \frac{1}{c + \gamma^2 D_\delta \pm \kappa_\delta + \gamma^2 r - \widehat{\ell}_\delta(r) - \frac{\Psi_{\mathcal{M}}(r)}{r} + \widehat{\mathcal{L}}_{\delta,\mathcal{M}}(r)}$$

$$= \frac{\frac{1}{c+\gamma^2 D_\delta+\kappa_\delta+\gamma^2 r-\frac{\Psi_{\mathcal{M}}(r)}{r}+\widehat{\mathcal{L}}_{\delta,\mathcal{M}}(r)}}{1-\left[\kappa_\delta+\widehat{\ell}_\delta(r)\right]\frac{1}{c+\gamma^2 D_\delta+\kappa_\delta+\gamma^2 r-\frac{\Psi_{\mathcal{M}}(r)}{r}+\widehat{\mathcal{L}}_{\delta,\mathcal{M}}(r)}}$$

$$= \frac{\frac{1}{c+\gamma^2 D_\delta+\kappa_\delta+\gamma^2 r-\frac{\Psi_{\mathcal{M}}(r)}{r}}\frac{1}{1+\widehat{\mathcal{L}}_{\delta,\mathcal{M}}(r)\frac{1}{c+\gamma^2 D_\delta+\kappa_\delta+\gamma^2 r-\frac{\Psi_{\mathcal{M}}(r)}{r}}}}{1-\left[\kappa_\delta+\widehat{\ell}_\delta(r)\right]\frac{1}{c+\gamma^2 D_\delta+\kappa_\delta+\gamma^2 r-\frac{\Psi_{\mathcal{M}}(r)}{r}}\frac{1}{1+\widehat{\mathcal{L}}_{\delta,\mathcal{M}}(r)\frac{1}{c+\gamma^2 D_\delta+\kappa_\delta+\gamma^2 r-\frac{\Psi_{\mathcal{M}}(r)}{r}}}}$$

$$= \frac{\frac{r}{(c+\gamma^2 D_\delta+\kappa_\delta)r+\gamma^2 r^2-\Psi_{\mathcal{M}}(r)}\frac{1}{1+\widehat{\mathcal{L}}_{\delta,\mathcal{M}}(r)\frac{r}{(c+\gamma^2 D_\delta+\kappa_\delta)r+\gamma^2 r^2-\Psi_{\mathcal{M}}(r)}}}{1-\left[\kappa_\delta+\widehat{\ell}_\delta(r)\right]\frac{r}{(c+\gamma^2 D_\delta+\kappa_\delta)r+\gamma^2 r^2-\Psi_{\mathcal{M}}(r)}\frac{1}{1+\widehat{\mathcal{L}}_{\delta,\mathcal{M}}(r)\frac{r}{(c+\gamma^2 D_\delta+\kappa_\delta)r+\gamma^2 r^2-\Psi_{\mathcal{M}}(r)}}}$$

The result follows applying (1.10) and the definition of $\widehat{\vartheta}_\delta(r)$, and then inverting the resulting Laplace transform. ∎

In the following proposition we consider another particular subcase of case C, in which we can find an expression for $\mathfrak{W}'_{\mathcal{Y}}$.

Proposition 7. *Let $\mathcal{X}$ satisfy case C in the situation when its negative jumps are given by $-\mathcal{S}$, where $\mathcal{S}$ is a driftless subordinator. Suppose also that $\gamma > 0$, then:*

$$\mathfrak{W}'_{\mathcal{Y}}(u) = \frac{1}{a_\delta+\mu_{\mathcal{S}}} e_{(a_\delta+\mu_{\mathcal{S}})/\gamma^2}(u) + \frac{1}{a_\delta+\mu_{\mathcal{S}}} e_{(a_\delta+\mu_{\mathcal{S}})/\gamma^2} * \sum_{n=1}^{\infty}\left(e_{(a_\delta+\mu_{\mathcal{S}})/\gamma^2} * \overline{\mathcal{V}}_{\mathcal{S}}\right)^{*n}(u),$$

where $\overline{\mathcal{V}}_{\mathcal{S}}(x) = \int_x^\infty \nu_{\mathcal{S}}(dy)$, $\mu_{\mathcal{S}} = \int_0^\infty x\nu_{\mathcal{S}}(dx)$ and $e_{(a_\delta+\mu_{\mathcal{S}})/\gamma^2}(u)du, u > 0$ is an exponential density with mean $\gamma^2/(a_\delta+\mu_{\mathcal{S}})$,

Proof. From (3.22) we have that $\widehat{\mathfrak{W}'_{\mathcal{Y}}}(r) = \frac{r}{a_\delta r+\gamma^2 r^2-\Psi_{\mathcal{S}}(r)}$, or equivalently

$$\widehat{\mathfrak{W}'_{\mathcal{Y}}}(r) = \frac{1}{a_\delta+\gamma^2 r-\frac{\Psi_{\mathcal{S}}(r)}{r}}.$$

We recall that $\frac{\Psi_{\mathcal{S}}(r)}{r} = \mu_{\mathcal{S}}+\int_0^\infty e^{-rx}\overline{\mathcal{V}}_{\mathcal{S}}(x)dx = \mu_{\mathcal{S}}+\widehat{\overline{\mathcal{V}}}_{\mathcal{S}}(r)$, and that $\mu_{\mathcal{S}} < \infty$ because of the assumption that $\mathbb{E}\left[\mathcal{X}(1)\right] > 0$. Using the last equality together with the definition of $\widehat{e}_{(a_\delta+\mu_\mathcal{S})/\gamma^2}$ we obtain:

$$\begin{aligned}\widehat{\mathfrak{W}}'_{\mathcal{Y}}(r) &= \frac{1}{a_\delta+\mu_\mathcal{S}+\gamma^2 r-\widehat{\overline{\mathcal{V}}}_\mathcal{S}(r)} = \frac{\frac{1}{a_\delta+\mu_\mathcal{S}}\frac{a_\delta+\mu_\mathcal{S}}{a_\delta+\mu_\mathcal{S}+\gamma^2 r}}{1-\frac{1}{a_\delta+\mu_\mathcal{S}}\frac{a_\delta+\mu_\mathcal{S}}{a_\delta+\mu_\mathcal{S}+\gamma^2 r}\widehat{\overline{\mathcal{V}}}_\mathcal{S}(r)}\\
&= \frac{1}{a_\delta+\mu_\mathcal{S}}\widehat{e}_{(a_\delta+\mu_\mathcal{S})/\gamma^2}\sum_{n=0}^{\infty}\left(\frac{1}{a_\delta+\mu_\mathcal{S}}\widehat{e}_{(a_\delta+\mu_\mathcal{S})/\gamma^2}\widehat{\overline{\mathcal{V}}}_\mathcal{S}(r)\right)^n\\
&= \frac{1}{a_\delta+\mu_\mathcal{S}}\widehat{e}_{(a_\delta+\mu_\mathcal{S})/\gamma^2}+\frac{1}{a_\delta+\mu_\mathcal{S}}\widehat{e}_{(a_\delta+\mu_\mathcal{S})/\gamma^2}\sum_{n=1}^{\infty}\left(\frac{1}{a_\delta+\mu_\mathcal{S}}\widehat{e}_{(a_\delta+\mu_\mathcal{S})/\gamma^2}\widehat{\overline{\mathcal{V}}}_\mathcal{S}(r)\right)^n
\end{aligned} \tag{3.37}$$

where in the fourth equality we have used that $\frac{1}{a_\delta+\mu_\mathcal{S}}\widehat{\overline{\mathcal{V}}}_\mathcal{S}(r) \le \frac{1}{a_\delta+\mu_\mathcal{S}}\widehat{\overline{\mathcal{V}}}_\mathcal{S}(0) = \frac{\mu_\mathcal{S}}{a_\delta+\mu_\mathcal{S}} < 1$, for $\delta \ge 0$. The result now follows by inverting the Laplace transform in the right-hand side of (3.37). ■

The result above provides an explicit expression for the q-scale function, for $q = 0$ of an spectrally negative Lévy process with nonnegative drift, nonzero brownian component and negative jumps given by the dual of a subordinator. In Chapter 4 we use this technique to obtain explicit expressions for q-scale functions for a wider class of spectrally negative Lévy processes.

3.3 The Generalized EDPF

In this section we show how the Wiener-Hopf factors $S^{\mathcal{X}}_{e_\delta}$ and $I^{\mathcal{X}}_{e_\delta}$ to derive formulae for the expected discounted penalty function (EDPF for short) for the class of Lévy risk processes $\mathcal{X}^u = \{\mathcal{X}^u(t), t \ge 0\}$ defined as $\mathcal{X}^u(t) = u + \mathcal{X}(t), u \ge 0$, where $\mathcal{X}$ is defined in (3.1). The approach presented here is the natural adaptation of the one used in Biffis and Morales [2010], where the case of spectrally negative Lévy risk processes was considered. The main result in this section strongly depends on our result for the probability density of $I^{\mathcal{X}}_{e_\delta}$, which was obtained in the previous section.

In this setting, the value $u \ge 0$ represents the initial capital of the risk process, the constant c and the process Z_1 represent, respectively, a fixed amount that the insurance company gets at each unit of time and a random amount of gains up to time t. The brownian component represents a perturbation due, for instance, to investment in the insurance market or random events which may mean either gains or losses for the insurance company. Finally, the process $\mathcal{S}$ represents the aggregate claim amount that the insurance company has to pay up to time t in the case when $\mathcal{S}$ is a subordinator. In the case when $\mathcal{S}$ does not have mononote paths, it can be interpreted as a the claims that the insurance company has to pay up to time t, perturbed by some random component whose interpretation is similar to that of the brownian component.

We consider the following generalized version of the Expected discounted penalty function (EDPF for short) associated to $\mathcal{X}^u$, considered in Biffis and Kyprianou [2009] and Biffis and Morales [2010].

We set $\tau_0^- = \inf\{t \geq 0 : \mathcal{X}^u(t) < 0\}$ and define the Generalized EDPF associated to $\mathcal{X}^u$ as

$$\phi(u;\delta,\omega) = \mathbb{E}\left[e^{-\delta\tau_0^-}\omega\left(|\mathcal{X}^u(\tau_0^-)|, \mathcal{X}^u(\tau_0^- -), I^{\mathcal{X}^u}_{\tau_0^- -}\right) 1_{\{\tau_0^- < \infty\}} \middle| \mathcal{X}^u(0) = u\right], u \geq 0$$

where $\delta \geq 0$ represents a discounted force of interest, $\omega : \mathbb{R}^3_+ \to \mathbb{R}_+$ is a function known as penalty function such that $\omega_0 = \omega(0+, 0+, 0+)$ exists, and the quantities $|\mathcal{X}^u(\tau_0^-)|, \mathcal{X}^u(\tau_0^- -)$ represent, respectively, the severity of ruin and the surplus immediate before ruin. The quantity $I^{\mathcal{X}}_{\tau_0^- -}$ is the last infimum before the ruin, as defined before.

Let us consider the dual process $\mathcal{X}^* = \{\mathcal{X}^*(t) = -\mathcal{X}(t), t \geq 0\}$ and its first passage time above level u $\tau_u^+ = \{t \geq 0 : \mathcal{X}^*(t) > u\}$, for $u \geq 0$.

We define the following random variables associated to τ_u^+.

- $\overline{\mathcal{G}}_{\tau_u^+} = \sup\left\{s < \tau_u^+ : S_s^{\mathcal{X}^*} = \mathcal{X}^*(s)\right\}$: the time of the last maximum prior to first passage,

- $\tau_u^+ - \overline{\mathcal{G}}_{\tau_u^+}$: the length of the excursion making the first passage,

- $\mathcal{X}^*(\tau_u^+) - u$: the overshoot at first passage,
- $u - \mathcal{X}^*(\tau_u^+ -)$: the undershoot at first passage,
- $u - S^{\mathcal{X}^*}_{\tau_u^+ -}$: the undershoot of the last maximum at first passage.

By definition of $\mathcal{X}^u$ and $\mathcal{X}^*$, we note that

$$\mathcal{X}^*(\tau_u^+) - u = |\mathcal{X}^u(\tau_0^-)|, \quad u - \mathcal{X}^*(\tau_u^+ -) = \mathcal{X}^u(\tau_0^- -), \quad u - S^{\mathcal{X}^*}_{\tau_u^+ -} = I^{\mathcal{X}^u}_{\tau_0^- -},$$

and $\overline{\mathcal{G}}_{\tau_u^+} + \tau_u^+ - \overline{\mathcal{G}}_{\tau_u^+} = \tau_u^+ = \tau_0^-$. Hence we can rewrite ϕ as

$$\phi(u;\delta,\omega) = \mathbb{E}\left[e^{-\delta\overline{\mathcal{G}}_{\tau_u^+} - \delta\left(\tau_u^+ - \overline{\mathcal{G}}_{\tau_u^+}\right)}\omega\left(\mathcal{X}^*(\tau_u^+) - u, u - \mathcal{X}^*(\tau_u^+ -), u - S^{\mathcal{X}^*}_{\tau_u^+ -}\right)1_{\{\tau_u^+ < \infty\}}\right].$$

To deal with the function above we consider the case in which the process $\mathcal{X}^*$ passes above u by a jump or under the event $\{\mathcal{X}^*(\tau_u^+) = u\}$. This last possibility is known as **creeping**, and is due to the presence of the brownian component. It is known that, in this case, we have the equality

$$\left(\mathcal{X}^*(\tau_u^+) - u, u - \mathcal{X}^*(\tau_u^+ -), u - S^{\mathcal{X}^*}_{\tau_u^+ -}\right) = (0,0,0).$$

With this in mind, we split the above expression for $\phi(u;\delta,\omega)$ as

$$\begin{aligned}
&\phi(u;\delta,\omega) \\
&= \mathbb{E}\left[e^{-\delta\overline{\mathcal{G}}_{\tau_u^+} - \delta\left(\tau_u^+ - \overline{\mathcal{G}}_{\tau_u^+}\right)}\omega\left(\mathcal{X}^*(\tau_u^+) - u, u - \mathcal{X}^*(\tau_u^+ -), u - S^{\mathcal{X}^*}_{\tau_u^+ -}\right)1_{\{\tau_u^+ < \infty, \mathcal{X}^*(\tau_u^+) - u > 0\}}\right] \\
&+ \omega_0\mathbb{E}\left[e^{-\delta\overline{\mathcal{G}}_{\tau_u^+} - \delta\left(\tau_u^+ - \overline{\mathcal{G}}_{\tau_u^+}\right)}1_{\{\tau_u^+ < \infty, \mathcal{X}^*(\tau_u^+) = u\}}\right], \quad u \geq 0. \qquad (3.38)
\end{aligned}$$

In order to deal with the two expressions in the right-hand side of (3.38) we use the following lemmas. The first one is merely technical and it is proved in Appendix A.

The first part of the second lemma is due to exercise 6.7 i) in Kyprianou [2006] and the equality $S^{\mathcal{X}^*}_{e_q} = -I^{\mathcal{X}}_{e_q}$, and the second part is the joint law of the random variables $\overline{\mathcal{G}}_{\tau_u^+}, \tau_u^+ - \overline{\mathcal{G}}_{\tau_u^+}, \mathcal{X}^*(\tau_u^+) - u, u - \mathcal{X}^*(\tau_u^+ -), u - S^{\mathcal{X}^*}_{\tau_u^+ -}$, a result which was stated and proved in Doney and Kyprianou [2006]. The version we present here is rewritten analogously to the version presented in Biffis and Morales [2010].

Lemma 18. *The function* $\widehat{e}_\delta^+(r) = \frac{\prod_{l=1}^N (q_l - r)^{m_l}}{\prod_{j=1}^R (\rho_{j,\delta} - r)^{k_j}}$ *has the equivalent expression*

$$\beta_A + \sum_{j=1}^{R} \sum_{a=0}^{k_j - 1} E(j, a, \delta) \frac{a!}{(\rho_{j,\delta} - r)^{a+1}}.$$

Therefore, for $r < \min\{Re(\rho_{j,\delta}), j = 1, 2, \ldots, R\}$, *we have* $\widehat{e}_\delta^+(r) = \int_0^\infty e^{rx} g_*(x) dx$, *where*

$$g_*(x) = \delta_0(x)\beta_A + \sum_{j=1}^{R} \sum_{a=0}^{k_j - 1} E(j, a, \delta) x^a e^{-\rho_{j,\delta} x} 1_{\{x>0\}},$$

and $\delta_0(x)$ *is the Dirac delta function.*

Proof. See Appendix A. ∎

Lemma 19. *Let* $\mathcal{X}$ *and* $\mathcal{X}^*$ *be as before.*

a) We have $\mathbb{E}\left[e^{-q\tau_u^+} 1_{\{\tau_u^+ < \infty\}}\right] = \mathbb{P}\left[-I^{\mathcal{X}}_{e_q} > u\right]$.

b) Suppose $\mathcal{X}^*$ *is not a compound Poisson process. Then, for each* $u > 0$, *the following holds for all* $s, t \geq 0, x > 0, v \geq y$ *and* $y \in [0, u]$.

$$\mathbb{P}\left[\overline{\mathcal{G}}_{\tau_u^+} \in dt, \tau_u^+ - \overline{\mathcal{G}}_{\tau_u^+} \in ds, \mathcal{X}^*(\tau_u^+) - u \in dx, u - \mathcal{X}^*(\tau_u^+ -) \in dv, u - S^{\mathcal{X}^*}_{\tau_u^+ -} \in dy\right]$$
$$= \mathcal{U}_A^*(ds, u - dy)\mathcal{U}_D^*(dt, dv - y)\nu_{\mathcal{X}^*}(dx + v).$$

Here, $\nu_{\mathcal{X}^*}(dx)$ *denotes the Lévy measure of* $\mathcal{X}^*$ *and the functions* $\mathcal{U}_A^*(ds, u - dy), \mathcal{U}_D^*(ds, u - dy)$ *are defined (see Kyprianou [2006] eq. 6.18 and 7.10) through the bivariate Laplace transforms:*

$$\int_{[\infty,0)^2} e^{-rx-sy} \mathcal{U}_A^*(dx, dy) = \frac{1}{\kappa_A^*(r,s)}, \quad \int_{[\infty,0)^2} e^{-rx-sy} \mathcal{U}_D^*(dx, dy) = \frac{1}{\kappa_D^*(r,s)}, \tag{3.39}$$

where κ_A^* and κ_D^* are identified through the Wiener-Hopf factors of $\mathcal{X}^*$ by the equalities

$$\mathbb{E}\left[e^{-rS_{e_q}^{\mathcal{X}^*}}\right] = \frac{\kappa_A^*(q,0)}{\kappa_A^*(q,r)} \quad \textit{and} \quad \mathbb{E}\left[e^{-r\left(-I_{e_q}^{\mathcal{X}^*}\right)}\right] = \frac{\kappa_D^*(q,0)}{\kappa_D^*(q,r)}. \tag{3.40}$$

We recall cases A, B and C considered in the last section and the equation $L_{\mathcal{X}}(r)-\delta = 0$, where $L_{\mathcal{X}}$ is the GLF corresponding to $\mathcal{X}$, which is assumed to have R roots $\rho_{j,\delta} \in \mathbb{C}_{++}, j = 1, 2, \ldots, R$ with their multiplicities $k_1 \equiv 1, k_j,\ j = 2, 3, \ldots, R$, such that $\sum_{j=1}^{R} k_j = m + 1 - \beta_A$.

Remark 3. *Due to the restriction that $\mathcal{X}^*$ cannot be a compound Poisson process, we exclude the case in which, according to (3.1), we have $c = \gamma = 0$ and both $\mathcal{Z}$ and $\mathcal{S}$ are compound Poisson processes. However, explicit expressions for ϕ under this case have been already obtained in Labbé et al. [2011] for the standard EDPF, when both $\mathcal{Z}$ and $\mathcal{S}$ are compound Poisson processes.*

We are now ready to state and prove the main result of this section.

Theorem 14. *(Main Theorem IV)*

For $\delta \geq 0$, the Generalized EDPF ϕ associated to $\mathcal{X}^u = u + \mathcal{X}$ has the expression

$$\phi(u; \delta, \omega) = \omega_0 \gamma^2 W_\delta(u) + H_{\delta,\omega} * W_\delta(u), \tag{3.41}$$

where $a_\delta W_\delta(u)$ is the density of the Wiener-Hopf factor $-I_{e_\delta}^{\mathcal{X}}$, given explicitly in Theorem 13, and $H_{\delta,\omega}$ is defined as

$$H_{\delta,\omega}(u) = K_\omega(u)\beta_A + \sum_{j=1}^{R} \sum_{a=0}^{k_j-1} E(j, a, \delta) J_{\omega,\delta,a,j}(u), \tag{3.42}$$

where

$$K_\omega(y) = \int_{y+}^{\infty} \omega(x - y, y, y)\nu_{\mathcal{S}}(dx)$$

and

$$J_{\omega,\delta,a,j}(y) = \int_{y+}^{\infty} (v-y)^a \, e^{-\rho_{j,\delta}(v-y)} \int_{v}^{\infty} \omega(x-v,v,y)\nu_{\mathcal{S}}(dx)dv.$$

Proof. We prove the result for $\delta > 0$ and obtain the corresponding formulae for $\delta = 0$ by taking limits when $\delta \downarrow 0$. We proceed by steps.

Step 1: We identify the functions

$$\mathcal{U}^*_{A,\delta}(dy) = \int_0^{\infty} e^{-\delta x}\mathcal{U}^*_A(dx,dy) \text{ and } \mathcal{U}^*_{D,\delta}(dy) = \int_0^{\infty} e^{-\delta x}\mathcal{U}^*_D(dx,dy).$$

For this, we note that $S^{\mathcal{X}^*}_{e_\delta} = -I^{\mathcal{X}}_{e_\delta}$ and $I^{\mathcal{X}^*}_{e_\delta} = -S^{\mathcal{X}}_{e_\delta}$. Therefore

$$\tfrac{\kappa^*_A(q,0)}{\kappa^*_A(q,r)} = \mathbb{E}\left[e^{-rS^{\mathcal{X}^*}_{e_q}}\right] = \mathbb{E}\left[e^{-r\left(-I^{\mathcal{X}}_{e_q}\right)}\right] \quad \text{and} \quad \tfrac{\kappa^*_D(q,0)}{\kappa^*_D(q,r)} = \mathbb{E}\left[e^{-r\left(-I^{\mathcal{X}^*}_{e_q}\right)}\right] = \mathbb{E}\left[e^{rS^{\mathcal{X}}_{e_q}}\right] \tag{3.43}$$

By Theorem 12 a) we have:

$$\mathbb{E}\left[e^{-r\left(-I^{\mathcal{X}}_{e_q}\right)}\right] = \begin{cases} \frac{a_\delta}{a_\delta+\Psi_{\mathcal{N}_{1,\delta}(r)}} & \text{in case A,} \\ \frac{a_\delta}{a_\delta+\Psi_{\mathcal{N}_{2,\delta}(r)}} & \text{in case B,} \\ \frac{a_\delta}{a_\delta+\gamma^2 r+\Psi_{\mathcal{N}_{3,\delta}(r)}} & \text{in case C,} \end{cases} \tag{3.44}$$

where e_{a_δ} is, again, an exponential random variable with mean $1/a_\delta$ independent of $\mathcal{N}_{j,\delta}$ and $\Psi_{\mathcal{N}_{j,\delta}}(r)$ is the Laplace exponent of $\mathcal{N}_{j,\delta}$, $j = 1,2,3$. Hence, in view of (3.44), we identify $\kappa^*_A(\delta,r) = a_\delta + \Psi_{\mathcal{N}_{j,\delta}}(r)$ in cases A ($j=1$) and B ($j=2$) and $\kappa^*_A(\delta,r) = a_\delta + \gamma^2 r + \Psi_{\mathcal{N}_\delta}(r)$ in case C $j=3$. In all these cases, $\kappa^*_A(\delta,0) = a_\delta$.

From (3.39), we note that the function $\mathcal{U}^*_{A,\delta}(dy)$ has Laplace transform $\int_0^{\infty} e^{-ry}\mathcal{U}^*_A(dy) = \frac{1}{\kappa^*_A(\delta,r)}$, hence, from (3.44) we deduce that

$$\int_0^{\infty} e^{-ry}\mathcal{U}^*_A(dy) = \frac{1}{a_\delta}\mathbb{E}\left[e^{-r\left(-I^{\mathcal{X}}_{e_q}\right)}\right].$$

By the first equality in Theorem 12 b) we have $\frac{1}{a_\delta}\mathbb{E}\left[e^{-r\left(-I^{\mathcal{X}}_{e_q}\right)}\right] = \widehat{W}_\delta(r)$, and this Laplace transform has an inverse given in Theorem 13. Therefore:

$$\mathcal{U}^*_{A,\delta}(dy) = W_\delta(y)dy. \tag{3.45}$$

By Theorem 2.2 in Lewis and Mordecki [2008] and the second pair of equalities in (3.43) we also deduce that $\kappa^*_D(\delta, r) = \frac{\prod_{j=1}^R(\rho_{j,\delta}-r)^{k_j}}{\prod_{l=1}^N(q_l-r)^{m_l}}$ and $\kappa^*_D(\delta, 0) = \left(\frac{\prod_{j=1}^N q_j^{m_j}}{\prod_{j=1}^R \rho_{j,\delta}^{k_j}}\right)^{-1}$. Now we note that $\mathcal{U}^*_{D,\delta}(dy)$ has Laplace transform

$$\int_0^\infty e^{-ry}\mathcal{U}^*_D(dy) = \frac{1}{\kappa^*_D(\delta, r)} = \frac{\prod_{l=1}^N(q_l - r)^{m_l}}{\prod_{j=1}^R(\rho_{j,\delta} - r)^{k_j}} = \widehat{e}^+_\delta(r).$$

Therefore, using Lemma 18 we obtain:

$$\mathcal{U}^*_{D,\delta}(dy) = \left(\delta_0(y)\beta_A + \sum_{j=1}^R\sum_{a=1}^{k_j} E(j, a, \delta)y^a e^{-\rho_{j,\delta}y}\right) dy. \tag{3.46}$$

Step 2: We define $\phi^*(u)$ as

$$\phi^*(u) = \mathbb{E}\left[e^{-\delta\overline{\mathcal{G}}_{\tau_u^+} - \delta\left(\tau_u^+ - \overline{\mathcal{G}}_{\tau_u^+}\right)}\omega\left(\mathcal{X}^*(\tau_u^+) - u, u - \mathcal{X}^*(\tau_u^+-), u - S^{\mathcal{X}^*}_{\tau_u^+-}\right)\right.$$
$$\left.\times\, 1_{\{\tau_u^+<\infty, \mathcal{X}^*(\tau_u^+)-u>0\}}\right],$$

and set $\mathcal{L}_u(ds, dt, dx, dv, dy) = \mathcal{U}^*_A(ds, u - dy)\mathcal{U}^*_D(dt, dv - y)\nu_{\mathcal{X}^*}(dx + v)$. Now we use Lemma 19 b) to obtain:

$$\phi^*(u) = \int_{0+}^{u}\int_{y}^{\infty}\int_{0+}^{\infty}\int_{0+}^{\infty}\int_{0}^{\infty} e^{-\delta(s+t)}\omega(x, v, y)\mathcal{L}_u(ds, dt, dx, dv, dy) \tag{3.47}$$

$$= \int_{0+}^{u} \int_{y}^{\infty} \int_{0+}^{\infty} \omega(x, v, y) \nu_{\mathcal{X}^*}(dx + v) \mathcal{U}^*_{D,\delta}(dv - y) \mathcal{U}^*_{A,\delta}(u - dy)$$

$$= \int_{0+}^{u} \int_{y}^{\infty} \int_{v+}^{\infty} \omega(z - v, v, y) \nu_{\mathcal{X}^*}(dz) \mathcal{U}^*_{D,\delta}(dv - y) \mathcal{U}^*_{A,\delta}(u - dy), \tag{3.48}$$

where in the third equality we used the change of variable $z = x + v$.

Since $\mathcal{X}^* = -\mathcal{X}$ by definition, its Lévy measure $\nu_{\mathcal{X}^*}(dx)$ is given by

$$\nu_{\mathcal{X}^*}(dx) = \nu_{\mathcal{S}}(dx) 1_{\{x>0\}} + \nu_{\mathcal{Z}}(dx) 1_{\{x<0\}}.$$

Using that $z = x + v$ in (3.48) moves from $v \geq y > 0$ to ∞, the right-hand side of (3.48) becomes

$$\phi^*(u) = \int_{0+}^{u} \int_{y}^{\infty} \int_{v+}^{\infty} \omega(z - v, v, y) \nu_{\mathcal{S}}(dz) \mathcal{U}^*_{D,\delta}(dv - y) \mathcal{U}^*_{A,\delta}(u - dy).$$

Substitution of (3.46) in the above equality yields:

$$\begin{aligned}
\phi^*(u) &= \beta_A \int_{0+}^{u} \int_{y}^{\infty} \int_{v+}^{\infty} \omega(z - v, v, y) \nu_{\mathcal{S}}(dz) \delta_0(v - y) dv \mathcal{U}^*_{A,\delta}(u - dy) \\
&+ \sum_{j=1}^{R} \sum_{a=1}^{k_j} E(j, a, \delta) \int_{0+}^{u} \int_{y}^{\infty} \int_{v+}^{\infty} \omega(z - v, v, y) \nu_{\mathcal{S}}(dz) (v - y)^a e^{-\rho_{j,\delta}(v-y)} dv \mathcal{U}^*_{A,\delta}(u - dy) \\
&= \beta_A \int_{0+}^{u} \mathcal{U}^*_{A,\delta}(u - dy) \int_{y+}^{\infty} \omega(z - y, y, y) \nu_{\mathcal{S}}(dz) \\
&+ \sum_{j=1}^{R} \sum_{a=1}^{k_j} E(j, a, \delta) \int_{0+}^{u} \mathcal{U}^*_{A,\delta}(u - dy) \int_{y}^{\infty} \int_{v+}^{\infty} \omega(z - v, v, y) (v - y)^a e^{-\rho_{j,\delta}(v-y)} \nu_{\mathcal{S}}(dz) dv
\end{aligned}$$

$$= \beta_A \int_{0+}^{u} \mathcal{U}^*_{A,\delta}(u-dy)K_\omega(y) + \sum_{j=1}^{R}\sum_{a=1}^{k_j} E(j,a,\delta) \int_{0+}^{u} \mathcal{U}^*_{A,\delta}(u-dy)J_{\omega,\delta,a,j}(y)$$
$$= \int_{0+}^{u} \mathcal{U}^*_{A,\delta}(u-dy)H_{\delta,\omega}(y)dy,$$

where in the third and fourth equality we used the definitions of K_ω, $J_{\omega,\delta,a,j}$ and $H_{\delta,\omega}$. Finally we substitute (3.45) in the equality above to obtain

$$\phi^*(u) = \int_{0+}^{u} W_\delta(u-y)H_{\delta,\omega}(y)dy = H_{\delta,\omega} * W_\delta(u). \tag{3.49}$$

Step 3: We calculate $\mathbb{E}\left[e^{-\delta\overline{\mathcal{G}}_{\tau_u^+}-\delta\left(\tau_u^+-\overline{\mathcal{G}}_{\tau_u^+}\right)}1_{\{\tau_u^+<\infty,\mathcal{X}^*(\tau_u^+)=u\}}\right]$.

First we note that $\mathbb{E}\left[e^{-\delta\overline{\mathcal{G}}_{\tau_u^+}-\delta\left(\tau_u^+-\overline{\mathcal{G}}_{\tau_u^+}\right)}1_{\{\tau_u^+<\infty,\mathcal{X}^*(\tau_u^+)=u\}}\right]$ does not depend on the penalty function $\omega(x,y,z)$, therefore we consider ϕ when $\omega(x,y,z)\equiv 1$ and use the result in Step 2. For this choice of ω, using (3.38) and Step 2 we obtain

$$\begin{aligned}\phi(u;\delta,\omega) &= \mathbb{E}\left[e^{-\delta\tau_u^+}1_{\{\tau_u^+<\infty\}}\right]\\ &= \mathbb{E}\left[e^{-\delta\overline{\mathcal{G}}_{\tau_u^+}-\delta\left(\tau_u^+-\overline{\mathcal{G}}_{\tau_u^+}\right)}1_{\{\tau_u^+<\infty,\mathcal{X}^*(\tau_u^+)=u\}}\right] + H_{\delta,\omega} * W(u),\end{aligned} \tag{3.50}$$

which yields

$$\mathbb{E}\left[e^{-\delta\overline{\mathcal{G}}_{\tau_u^+}-\delta\left(\tau_u^+-\overline{\mathcal{G}}_{\tau_u^+}\right)}1_{\{\tau_u^+<\infty,\mathcal{X}^*(\tau_u^+)=u\}}\right] = \mathbb{E}\left[e^{-\delta\tau_u^+}1_{\{\tau_u^+<\infty\}}\right] - H_{\delta,\omega} * W_\delta(u).$$

By Lemma 19 a), the above equality is equivalent to

$$\mathbb{E}\left[e^{-\delta\overline{\mathcal{G}}_{\tau_u^+}-\delta\left(\tau_u^+-\overline{\mathcal{G}}_{\tau_u^+}\right)}1_{\{\tau_u^+<\infty,\mathcal{X}^*(\tau_u^+)=u\}}\right] = \mathbb{P}\left[-I^{\mathcal{X}}_{e_\delta} > u\right] - H_{\delta,\omega} * W_\delta(u)$$

$$= a_\delta \int_u^\infty W_\delta(y)dy - H_{\delta,\omega} * W_\delta(u), \qquad (3.51)$$

where we have used that $a_\delta W_\delta(u)du$ is the probability density of $-I^{\mathcal{X}}_{e_\delta}$. Taking Laplace transforms in both sides of the equality above yields:

$$\int_0^\infty e^{-ru}\mathbb{E}\left[e^{-\delta\overline{\mathcal{G}}_{\tau_u^+}-\delta\left(\tau_u^+-\overline{\mathcal{G}}_{\tau_u^+}\right)}1_{\{\tau_u^+<\infty,\mathcal{X}^*(\tau_u^+)=u\}}\right]du = \frac{1-a_\delta\widehat{W}_\delta(r)}{r} - \widehat{H}_{\delta,\omega}(r)\widehat{W}_\delta(r)$$
$$= \frac{1}{r} - \frac{\widehat{W}_\delta(r)}{r}\left(a_\delta + r\widehat{H}_{\delta,\omega}(r)\right). \qquad (3.52)$$

Now we calculate $\widehat{H}_{\delta,\omega}(r)$ for $\omega(x,y,z)\equiv 1$.

First we note that the term $\mathbb{E}\left[e^{-\delta\overline{\mathcal{G}}_{\tau_u^+}-\delta\left(\tau_u^+-\overline{\mathcal{G}}_{\tau_u^+}\right)}1_{\{\tau_u^+<\infty,\mathcal{X}^*(\tau_u^+)=u\}}\right]$ is not zero only in Case C, and in this case we have $H_{\delta,\omega}(u) = \sum_{j=1}^{R}\sum_{a=0}^{k_j-1} E(j,a,\delta)J_{\omega,\delta,a,j}(u)$. With this in mind, we obtain for $\omega(x,y,z)\equiv 1$:

$$H_{\delta,\omega}(y) = \sum_{j=1}^{R}\sum_{a=0}^{k_j-1} E(j,a,\delta)\int_y^\infty (v-y)^a\, e^{-\rho_{j,\delta}(v-y)}\int_v^\infty \nu_{\mathcal{S}}(dx)dv$$
$$= \sum_{j=1}^{R}\sum_{a=0}^{k_j-1} E(j,a,\delta)\int_y^\infty (v-y)^a\, e^{-\rho_{j,\delta}(v-y)}\overline{\mathcal{V}}_{\mathcal{S}}(v)dv, \qquad (3.53)$$

where $\overline{\mathcal{V}}_{\mathcal{S}}(v)$ is the tail on v of $\nu_{\mathcal{S}}(dx)$.

Using the definition of $\mathcal{T}_{\rho_{j,\delta;a}}\overline{\mathcal{V}}_{\mathcal{S}}(v)$ given in (3.9) and then taking Laplace transforms in the equality above, we obtain

$$\widehat{H}_{\delta,\omega}(r) = \sum_{j=1}^{R}\sum_{a=0}^{k_j-1} E(j,a,\delta)\widehat{\mathcal{T}}_{\rho_{j,\delta;a}}\overline{\mathcal{V}}_{\mathcal{S}}(r) \qquad (3.54)$$

Now we note that

$$\widehat{\mathcal{L}}_\delta(r) - \frac{\Psi_{\mathcal{S}}(r)}{r} = \sum_{l=1}^{R} \frac{(-1)^{1-k_l}}{(k_l-1)!}\frac{\partial^{k_l-1}}{\partial s^{k_l-1}}\left[\frac{\prod_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}} s\widehat{T}_s\overline{\mathcal{V}}_{\mathcal{S}}(r)\right]_{s=\rho_{l,\delta}} - \frac{\Psi_{\mathcal{S}}(r)}{r}. \quad (3.55)$$

Our goal now is to prove that (3.55) equals $C_\delta + r\widehat{H}_{\delta,\omega}(r)$ for some C_δ.

Using Hermite interpolation polynomials it can be proved that:

$$\sum_{j=1}^{R} \frac{(-1)^{1-k_j}}{(k_j-1)!}\frac{\partial^{k_j-1}}{\partial s^{k_j-1}}\left[\frac{\prod_{l=1}^{N}(q_l-s)^{m_l}(\rho_{j,\delta}-s)^{k_j}}{\prod_{l=1}^{R}(\rho^n_{l,\delta}-s)^{k_l}}\right]_{s=\rho_{j,\delta}}$$

$$= (-1)^m \sum_{j=1}^{R} \frac{(-1)^{m+1-k_j}}{(k_j-1)!}\frac{\partial^{k_j-1}}{\partial s^{k_j-1}}\left[\frac{\prod_{l=1}^{N}(q_l-s)^{m_l}(\rho_{j,\delta}-s)^{k_j}}{\prod_{l=1}^{R}(\rho^n_{l,\delta}-s)^{k_l}}\right]_{s=\rho_{j,\delta}} = (-1)^m(-1)^m = 1. \quad (3.56)$$

Hence the expression in the right-hand side of (3.55) can be rewritten as

$$\sum_{l=1}^{R} \frac{(-1)^{1-k_l}}{(k_l-1)!}\frac{\partial^{k_l-1}}{\partial s^{k_l-1}}\left[\frac{\prod_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}}\left(s\widehat{T}_s\overline{\mathcal{V}}_{\mathcal{S}}(r) - \frac{\Psi_{\mathcal{S}}(r)}{r}\right)\right]_{s=\rho_{l,\delta}}.$$

By the second equality in (3.12), the expression above is equivalent to

$$\sum_{l=1}^{R} \frac{(-1)^{1-k_l}}{(k_l-1)!}\frac{\partial^{k_l-1}}{\partial s^{k_l-1}}\left[\frac{\prod_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}}\left(r\widehat{T}_s\overline{\mathcal{V}}_{\mathcal{S}}(r) - \frac{\Psi_{\mathcal{S}}(s)}{s}\right)\right]_{s=\rho_{l,\delta}}$$

$$= r\sum_{l=1}^{R} \frac{(-1)^{1-k_l}}{(k_l-1)!}\frac{\partial^{k_l-1}}{\partial s^{k_l-1}}\left[\frac{\prod_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}}\widehat{T}_s\overline{\mathcal{V}}_{\mathcal{S}}(r)\right]_{s=\rho_{l,\delta}} - C_\delta, \quad (3.57)$$

where $C_\delta = \sum_{l=1}^{R} \frac{(-1)^{1-k_l}}{(k_l-1)!}\frac{\partial^{k_l-1}}{\partial s^{k_l-1}}\left[\frac{\prod_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}}\left(-\frac{\Psi_{\mathcal{S}}(s)}{s}\right)\right]_{s=\rho_{l,\delta}}.$

This gives:

$$\widehat{\mathcal{L}}_\delta(r) - \frac{\Psi_{\mathcal{S}}(r)}{r} = \sum_{l=1}^{R} \frac{(-1)^{1-k_l}}{(k_l-1)!} \frac{\partial^{k_l-1}}{\partial s^{k_l-1}} \left[\frac{\prod_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}} \left(r\widehat{T}_s\overline{\mathcal{V}}_{\mathcal{S}}(r) - \frac{\Psi_{\mathcal{S}}(s)}{s} \right) \right]_{s=\rho_{l,\delta}}$$

$$= r\sum_{l=1}^{R} \frac{(-1)^{1-k_l}}{(k_l-1)!} \frac{\partial^{k_l-1}}{\partial s^{k_l-1}} \left[\frac{\prod_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}} \widehat{T}_s\overline{\mathcal{V}}_{\mathcal{S}}(r) \right]_{s=\rho_{l,\delta}} + C_\delta$$

$$= r\sum_{l=1}^{R}\sum_{a=0}^{k_l-1} \binom{k_l-1}{a} \frac{\partial^{k_l-1-a}}{\partial s^{k_l-1-a}} \frac{(-1)^{1-k_l}}{(k_l-1)!} \frac{\partial^{k_l-1}}{\partial s^{k_l-1}} \left[\frac{\prod_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}} \right]_{s=\rho_{l,\delta}}$$

$$\times \frac{\partial^a}{\partial s^a}\left[\widehat{T}_s\overline{\mathcal{V}}_{\mathcal{S}}(r)\right]_{s=\rho_{j,\delta}} + C_\delta = r\sum_{l=1}^{R}\sum_{a=0}^{k_l-1} E(j,a,\delta)\widehat{\mathcal{T}}_{\rho_{j,\delta};a}\overline{\mathcal{V}}_{\mathcal{S}}(r) + C_\delta = r\widehat{H}_{\delta,\omega}(r) + C_\delta.$$

where in the third equality we used Leibniz rule. Setting $r = 0$ in the equality above, we obtain $\widehat{\mathcal{L}}_\delta(0) = C_\delta$. Therefore $\widehat{\mathcal{L}}_\delta(r) - \frac{\Psi_{\mathcal{S}}(r)}{r} = r\widehat{H}_{\delta,\omega}(r) + \widehat{\mathcal{L}}_\delta(0)$. This implies

$$r\widehat{H}_{\delta,\omega}(r) = -\frac{\Psi_{\mathcal{S}}(r)}{r} - \left[\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r)\right] \tag{3.58}$$

On the other hand, we know from (3.24) that:

$$\widehat{W}_\delta(r) = \frac{1}{a_\delta + \gamma^2 r - \frac{\Psi_{\mathcal{S}}(r)}{r} - \left[\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r)\right]} \tag{3.59}$$

Now we substitute (3.58) and (3.59) in (3.52), and obtain:

$$\int_0^\infty e^{-ru}\mathbb{E}\left[e^{-\delta\overline{\mathcal{G}}_{\tau_u^+}-\delta\left(\tau_u^+-\overline{\mathcal{G}}_{\tau_u^+}\right)} 1_{\{\tau_u^+<\infty,\mathcal{X}^*(\tau_u^+)=u\}}\right] du$$
$$= \frac{1}{r} - \frac{1}{r}\frac{1}{a_\delta + \gamma^2 r - \frac{\Psi_{\mathcal{S}}(r)}{r} - \left[\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r)\right]}\left(a_\delta - \frac{\Psi_{\mathcal{S}}(r)}{r} - \left[\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r)\right]\right)$$

$$= \frac{1}{r} - \frac{1}{r}\frac{1}{a_\delta + \gamma^2 r - \frac{\Psi_{\mathcal{S}}(r)}{r} - \left[\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r)\right]}\left(a_\delta \pm \gamma^2 r - \frac{\Psi_{\mathcal{S}}(r)}{r} - \left[\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r)\right]\right)$$
$$= \frac{1}{r} - \frac{1}{r}\left(1 - \frac{\gamma^2 r}{a_\delta + \gamma^2 r - \frac{\Psi_{\mathcal{S}}(r)}{r} - \left[\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r)\right]}\right)$$
$$= \frac{\gamma^2}{a_\delta + \gamma^2 r - \frac{\Psi_{\mathcal{S}}(r)}{r} - \left[\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r)\right]} = \gamma^2 \widehat{W}_\delta(r),$$

where in the last equality we used (3.59) again. Inverting the equality above yields

$$\mathbb{E}\left[e^{-\delta \overline{\mathcal{G}}_{\tau_u^+} - \delta\left(\tau_u^+ - \overline{\mathcal{G}}_{\tau_u^+}\right)} 1_{\{\tau_u^+ < \infty, \mathcal{X}^*(\tau_u^+) = u\}}\right] = \mathbb{E}\left[e^{-\delta \tau_u^+} 1_{\{\tau_u^+ < \infty, \mathcal{X}^*(\tau_u^+) = u\}}\right] = \gamma^2 W_\delta(u). \tag{3.60}$$

Step 4: We substitute (3.49) and (3.60) in (3.38). This completes the proof. ■

Lemma 19 a) and Theorem 14 give the following corollary.

Corollary 7. *Let*

$$\Phi(u) := 1 - \mathbb{P}\left[\tau_0^- < \infty | \mathcal{X}^u(0) = u\right]$$

and

$$\varphi_\delta(u) = 1 - \mathbb{E}\left[e^{-\tau_0^-} 1_{\{\tau_0^- < \infty\}} \middle| \mathcal{X}^u(0) = u\right], \delta > 0,$$

denote respectively, the survival probability of $\mathcal{X}^u$ and one minus the Laplace transform of the time to ruin of $\mathcal{X}^u$, then $\Phi(u)$ and $\varphi_\delta(u)$ are distribution functions with densities $\Phi'(u)$ and $\varphi'_\delta(u)$.

Moreover, for any penalty function ω we have the equalities:

$$\phi(u; \delta, \omega) = \frac{\omega_0 \gamma^2}{a_\delta} \varphi'_\delta(u) + \frac{1}{a_\delta} H_{\delta,\omega} * \varphi'_\delta(u), \delta > 0,$$

and

$$\phi(u;0,\omega) = \frac{\omega_0\gamma^2}{a_0}\Phi'(u) + \frac{1}{a_0}H_{0,\omega} * \Phi'(u), \delta = 0,$$

where $H_{\delta,\omega}(u)$ is given in Theorem 14.

Proof. We only need to prove that $W_\delta(u) = \frac{1}{a_\delta}\varphi_\delta'(u)$ a.s. for $\delta > 0$ and $W_0(u) = \frac{1}{a_0}\Phi'(u)$ a.s., both for all $u > 0$. Clearly, the second equality follows from the first by taking limits when $\delta \downarrow 0$, which implies we only need to prove the first equality.

For this we recall that $1 - \varphi_\delta(u) = \phi(u;\delta,\omega)$ in the case when $\omega \equiv 1$. Therefore by Lemma 19 a) and the fact that the first passage of $\mathcal{X}^u$ below zero (time to ruin) equals the first passage of $-\mathcal{X}$ above u, we obtain $1 - \varphi_\delta(u) = \mathbb{P}\left[-I_{e_\delta}^{\mathcal{X}} > u\right]$. Now we use the equality $a_\delta \int_u^\infty W_\delta(y)dy = \mathbb{P}\left[-I_{e_\delta}^{\mathcal{X}} > u\right]$, and conclude that $W_\delta(u) = \frac{1}{a_\delta}\varphi_\delta'(u)$ a.s. ∎

3.4 Examples

In this final section we present a few particular examples of the function $W_\delta(u)$ under cases A, B and C. Here we take as case C1 the one considered in Proposition 6.

We make use of the two-parameter Mittag-Leffler function, denoted by $\mathcal{E}_{\alpha,\beta}(x)$ and defined as:

$$\mathcal{E}_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)}. \tag{3.61}$$

For $c > 0$, $\alpha \in (1,2)$ and $\beta = 1$, we recall that the Mittag-Leffler function $\mathcal{E}_{\alpha-1,1}(-cx^{\alpha-1})$ is the tail of the extremal stable distribution, which has a density denoted by $z_{\alpha,c}(u)$ for $u > 0$. The function $z_{\alpha,c}(u)$ has a Laplace transform $\widehat{z}_{\alpha,c}(r)$ given by

$$\widehat{z}_{\alpha,c}(r) = \frac{c}{c + r^{\alpha-1}} \tag{3.62}$$

The proof of the equality above can be found in Furrer [1998]. These properties are used in many of the examples presented below.

Example 1 (Case A): Let us suppose that $G_{\mathcal{S}}(r) = \sqrt{b+r} - b$ (this is a particular case of an Inverse Gaussian subordinator). We focus our attention in the function $\theta_\delta(u)$, whose Laplace transform is, by defitinion, given by $\frac{a_\delta}{a_\delta + G_{\mathcal{S}}(r)}$. In this case we have $\widehat{\theta}_\delta(r) = \frac{a_\delta}{a_\delta - b} \frac{a_\delta - b}{a_\delta - b + (b+r)^{1/2}}$, hence if $a_\delta > b$, by (3.62) it follows that

$$\theta_\delta(u) = \frac{a_\delta}{a_\delta - b} e^{-bu} z_{\alpha, a_\delta - b}(u), u > 0,$$

where in this case $\alpha = 3/2$. Using Theorem 13 we obtain:

$$\begin{aligned} &W_\delta(u) \\ &= \frac{1}{a_\delta - b} e^{-bu} z_{\alpha, a_\delta - b}(u) \\ &+ \frac{1}{a_\delta - b} \int_0^u e^{-b(u-y)} z_{\alpha, a_\delta - b}(u-y) \sum_{n=1}^{\infty} \left(-\frac{1}{a_\delta} \right)^n \left(\widehat{\ell}_\delta(0) \theta_\delta - \ell_\delta * \theta_\delta \right)^{*n} (y) dy, \qquad (3.63) \end{aligned}$$

where the function $\ell_\delta(u)$ is given in (3.10). Let us consider the particular case when $b = 1, \lambda_1 = 2, \delta = 0.5$ and $\widehat{f_1}(r) = \frac{1}{(1+r)^2}$.

We have

$$L_{\mathcal{X}}(r) - \delta = 2 \frac{1}{(1-r)^2} + 1 - (1+r)^{1/2} - 2 - 0.5,$$

In this case the two different roots are 0.1154509835 and 1.794106343, which gives $a_\delta = 2.413927106 > 1 = b$.

In the particular case when $\mathcal{S}$ is a compound Poisson process with Laplace exponent $G_{\mathcal{S}}(r) = \lambda_2 \int_0^\infty (1 - e^{-rx}) f_2(x) dx$, where $f_2(x)$ is a density function and $\lambda_2 > 0$, we have that

$$\widehat{\theta}_\delta(r) = \frac{a_\delta}{a_\delta + \lambda_2 - \lambda_2 \widehat{f}_2(r)} = \frac{a_\delta}{a_\delta + \lambda_2} \sum_{n=0}^{\infty} \left(\frac{\lambda_2}{a_\delta + \lambda_2} \right)^n \left(\widehat{f}_2(r) \right)^n.$$

Inverting the above Laplace transform yields

$$\theta_\delta(u) = \frac{a_\delta}{a_\delta + \lambda_2} \delta_0(u) + \frac{a_\delta}{a_\delta + \lambda_2} \sum_{n=1}^{\infty} \left(\frac{a_\delta}{a_\delta + \lambda_2} \right)^n f_2^{*n}(u), u > 0.$$

Example 2 (Case B): Let us suppose that $G_{\mathcal{S}}(r) = \lambda_2 \widehat{f}_2(r) - \lambda_2$, i.e. $\mathcal{S}$ is a compound Poisson process with Lévy measure $\lambda_2 f_2(x)$ for $\lambda_2 > 0$. In this case the resulting Lévy risk process is the classical two-sided jumps risk process.

Then $\ell_\delta(u) = \lambda_2 \sum\limits_{j=1}^{R} \sum\limits_{a=0}^{k_j-1} E(j,a,\delta) \int\limits_u^\infty (y-u)^a e^{-\rho_{j,\delta}(y-u)} f_2(y)dy$, and using (A.56) we obtain $a_\delta + \widehat{\ell}_\alpha(0) = c$, where c is the drift of the process $\mathcal{X}$ in this case. Hence by Theorem 13 we have

$$W_\delta(u) = \frac{1}{c}\delta_0(u) + \frac{1}{c}\sum_{n=1}^{\infty}\left(\frac{1}{c}\right)^n \ell_\delta^{*n}(u), u > 0.$$

In this case we have the following equivalent representation for ℓ_δ:

Let us consider the probability $\mathbb{P}\left[S_{e_\delta}^{\mathcal{X}} - Y > -u\right]$, where Y has probability density f_2 and it is independent of $S_{e_\delta}^{\mathcal{X}}$. We can think of Y as one of the claims associated to the spectrally negative Lévy process $\mathcal{S}$.

Then $\mathbb{P}\left[S_{e_\delta}^{\mathcal{X}} - Y > -u\right] = \frac{\delta}{a_\delta} \sum\limits_{j=1}^{R} \sum\limits_{a=0}^{k_j-1} E(j,a,\delta) \int\limits_u^\infty (y-u)^a e^{-\rho_{j,\delta}(y-u)} f_2(y)dy$ and it follows that $\ell_\delta(u) = \lambda_2 \delta^{-1} a_\delta \mathbb{P}\left[S_{e_\delta}^{\mathcal{X}} - Y > -u\right]$.

Example 3 (Case C): Let us suppose now that $L_{\mathcal{X}}(r) - \delta = \lambda_1 \frac{Q(-r)}{\prod_{j=1}^N (q_j - r)^{m_j}} + cr + b^\alpha \Gamma(-\alpha) r - b^\alpha \Gamma(-\alpha)(b+r)^\alpha r - \lambda_1 - \delta$, where $\alpha \in (0,1)$, $c, b > 0$ and $\gamma = 0$. This is a particular case of the process considered in Section 6 in Hubalek and Kyprianou [2011]. It follows from Theorem 12 that:

$$\widehat{\mathfrak{W}}_{\mathcal{Y}}(r) = \frac{1}{(a_\delta + b^\alpha \Gamma(-\alpha))r - b^\alpha \Gamma(-\alpha) r (b+r)^\alpha}.$$

Hence, by Theorem 6.2 in Hubalek and Kyprianou [2011] we have

$$\mathfrak{W}_{\mathcal{Y}}(u) = \frac{1}{-\Gamma(-\alpha)} \int\limits_0^u e^{-by} y^{\alpha-1} \mathcal{E}_{\alpha,\alpha}\left(\frac{a_\delta + \Gamma(-\alpha) b^\alpha y^\alpha}{\Gamma(-\alpha)}\right) dy, \tag{3.64}$$

where $\mathcal{E}_{\alpha,\alpha}(x)$ is defined in (3.61). This yields:

$$\mathfrak{W}'_{\mathcal{Y}}(u) = \frac{1}{-\Gamma(-\alpha)} e^{-bu} u^{\alpha-1} \mathcal{E}_{\alpha,\alpha}\left(\frac{a_\delta + \Gamma(-\alpha) b^\alpha u^\alpha}{\Gamma(-\alpha)}\right), u > 0.$$

Hence, in view of Theorem 13, we have

$$\begin{aligned}
&W_\delta(u)\\
&= \frac{1}{-\Gamma(-\alpha)} e^{-bu} u^{\alpha-1} \mathcal{E}_{\alpha,\alpha}\left(\frac{a_\delta + \Gamma(-\alpha) b^\alpha u^\alpha}{\Gamma(-\alpha)}\right)\\
&+ \frac{1}{-\Gamma(-\alpha)} \int_0^u e^{-b(u-y)} (u-y)^{\alpha-1} \mathcal{E}_{\alpha,\alpha}\left(\frac{a_\delta + \Gamma(-\alpha) b^\alpha (y-u)^\alpha}{\Gamma(-\alpha)}\right) \sum_{n=1}^{\infty} \sigma^{*n}_{\delta,\alpha,b}(y) dy,
\end{aligned}$$

for $u > 0$, where

$$\begin{aligned}
\sigma_{\delta,\alpha,b}(u) = {}& \widehat{\mathcal{L}}_\delta(0) e^{-bu} u^{\alpha-1} \mathcal{E}_{\alpha,\alpha}\left(\frac{a_\delta + \Gamma(-\alpha) b^\alpha u^\alpha}{\Gamma(-\alpha)}\right)\\
& - \int_0^u \mathcal{L}_\delta(u-y) e^{-by} y^{\alpha-1} \mathcal{E}_{\alpha,\alpha}\left(\frac{a_\delta + \Gamma(-\alpha) b^\alpha y^\alpha}{\Gamma(-\alpha)}\right) dy
\end{aligned}$$

and

$$\mathcal{L}_\delta(u) = \sum_{j=1}^{R} \sum_{a=0}^{k_j-1} E_*(j,a,\delta) \int_u^\infty (y-u)^a e^{-\rho_{j,\delta}(y-u)} \int_y^\infty e^{-bx} \left(\frac{bx + (\alpha+1)}{x^{\alpha+2}}\right) dx dy.$$

Example 4: As a particular case of Proposition 6, we set $\gamma = 0$ and take $\mathcal{M}$ as an α-stable process with $\alpha \in (1,2)$, only positive jumps and $-\Psi_{\mathcal{M}}(r) = \eta^\alpha r^\alpha$. This gives $-\frac{\Psi_{\mathcal{M}}(r)}{r} = \eta^\alpha r^{\alpha-1}$. Hence, using (3.36) and (3.62), we obtain:

$$\widehat{\mathfrak{W}}'_{\mathcal{M}_-}(r) = \frac{1}{c + \kappa_\delta + \eta^\alpha r^{\alpha-1}} = \frac{1}{c+\kappa_\delta} \frac{\frac{c+\kappa_\delta}{\eta^\alpha}}{\frac{c+\kappa_\delta}{\eta^\alpha} + r^{\alpha-1}} = \frac{1}{c+\kappa_\delta} \widehat{z}_{\alpha,\frac{c+\kappa_\delta}{\eta^\alpha}}(r).$$

The equality above implies that:

$$\mathfrak{W}'_{\mathcal{M}_-}(u) = \frac{1}{c+\kappa_\delta} z_{\alpha,\frac{c+\kappa_\delta}{\eta^\alpha}}(u), u > 0.$$

Recalling that, in this case, we have $\overline{\mathcal{V}}_{\mathcal{M}}(x) = \eta^\alpha \frac{\alpha-1}{\Gamma(2-\alpha)} x^{-\alpha}$ and using the equality above and Proposition 6, we obtain

$$\vartheta_\delta(u) = \frac{1}{c+\kappa_\delta} z_{\alpha,b}(u) + \frac{1}{c+\kappa_\delta} z_{\alpha,b} * \sum_{n=1}^{\infty} \left(-\frac{1}{c+\kappa_\delta}\right)^n \left(\mathcal{L}_{\delta,\mathcal{M}} * z_{\alpha,b}\right)^{*n}(u),$$

where in this case $b = \frac{c+\kappa_\delta}{\eta^\alpha}$ and

$$\mathcal{L}_{\delta,\mathcal{M}}(u) = \eta^\alpha \frac{\alpha-1}{\Gamma(2-\alpha)} \sum_{j=1}^{R} \sum_{a=0}^{k_j-1} E_*(j,a,\delta) \int_u^\infty (y-u)^a e^{-\rho_{j,\delta}(y-u)} y^{-\alpha} dy.$$

Recalling Proposition 6, the density of the negative Wiener-Hopf factor is given by the expression:

$$W_\delta(u) = \vartheta_\delta(u) + \vartheta_\delta * \sum_{n=1}^{\infty} (\kappa_\delta \vartheta_\delta + \ell_\delta * \vartheta_\delta)^{*n} (u), u > 0,$$

where $\ell_\delta(u) = \sum_{j=1}^{R} \sum_{a=0}^{k_j-1} E(j,a,\delta) \int_u^\infty (y-u)^a e^{-\rho_{j,\delta}(y-u)} \nu_{\mathcal{S}}(dy)$. Hence, we can obtain an explicit formula for $W_\delta(u)$ above by substitution of ℓ_δ and ϑ_δ. Since the expression for ϑ_δ also involves an infinite sum of convolutions, the resulting expression for $W_\delta(u)$ is too large, and therefore we omit it.

From the above calculations we note that, in the case when all the roots $\rho_{j,\delta}, j = 1, 2, \ldots, m+1$ are assumed to be different and $\mathcal{S}$ is a compound Poisson process, we obtain the particular case of the process studied in Chapter 2. Moreover, we also obtain the density of the negative Wiener-Hopf factor of the process studied in the aforementioned chapter, in the more general case when the roots $\rho_{j,\delta}$ are allowed to have multiplicities greater than 1 and $\mathcal{S}$ is a subordinator.

Chapter 4

Examples of q-scale functions

In this final chapter we use the techniques developed previously to obtain explicit expressions for some cases of q-scale functions. First we consider the case when $\mathcal{X} = \{\mathcal{X}(t), t \geq 0\}$ is given by

$$\mathcal{X}(t) = ct + \eta \mathcal{W}_\alpha(t) - \mathcal{S}(t), \ \eta \geq 0 \tag{4.1}$$

where $c \geq 0$, $\mathcal{W}_\alpha = \{\mathcal{W}_\alpha(t), t \geq 0\}$ is an α-stable process with $\alpha \in (1, 2)$, only negative jumps and Lévy measure $\nu_{\mathcal{W}}(dx)$, and $\mathcal{S} = \{\mathcal{S}(t), t \geq 0\}$ is an independent subordinator with Laplace exponent $-G_{\mathcal{S}}(r) = -\int\limits_{0+}^{\infty} (1 - e^{-rx}) \nu_{\mathcal{S}}(dx)$. This process is a generalization of the one studied in Kolkovska and Martín-González [2016], where $\mathcal{S}$ is taken as a compound Poisson process.

We let $\overline{\mathcal{V}}_{\mathcal{W}}(x)$ denote the tail of $\nu_{\mathcal{W}}(dx)$ and denote again the Generalized Lundberg function of $\mathcal{X}$ as $L_{\mathcal{X}}(r)$ (which, in this case, coincides with the Laplace exponent of $\mathcal{X}$). We assume that $\mathbb{E}[\mathcal{X}(1)] > 0$. Under this assumption, it can be proved that the equation $L_{\mathcal{X}}(r) - q = 0$ has exactly one nonnegative root (see, for instance, Biffis and Kyprianou [2009]), which equals zero when $q = 0$. We denote this root by ρ. The following two results are needed.

Proposition 8. *For $r \geq \rho$ and $c' = c/\eta^\alpha$, the function*

$$\widehat{V}_{\alpha,\rho}(r) = \frac{\widehat{z}_{c',\alpha}(r)}{1 + \frac{\eta^\alpha \rho}{c} \widehat{T}_\rho \overline{\mathcal{V}}_{\mathcal{W}}(r) \widehat{z}_{c',\alpha}(r)} \tag{4.2}$$

is the Laplace transform of a function $V_{\alpha,\rho}$ which admits the series representation

$$V_{\alpha,\rho}(u) = z_{c',\alpha} * \sum_{n=0}^{\infty} (-1)^n \left(\frac{\eta^\alpha \rho}{c}\right)^n \left(T_\rho \overline{\mathcal{V}}_{\mathcal{W}}\right)^{*n} * z_{c',\alpha}^{*n}(u), \quad u > 0,$$

where $\widehat{z}_{c',\alpha}(r)$ is defined in (3.62).

Proof. We have

$$\frac{\eta^\alpha \rho}{c} \widehat{T}_\rho \overline{\mathcal{V}}_{\mathcal{W}}(\rho) \widehat{z}_{c',\alpha}(\rho) = \frac{(\alpha - 1)\eta^\alpha \rho^{\alpha-1}}{c + \eta^\alpha \rho^{\alpha-1}} < 1. \tag{4.3}$$

Hence, for $r \geq \rho$,

$$\widehat{V}_{\alpha,\rho}(r) = \frac{\widehat{z}_{c',\alpha}(r)}{1 + \frac{\eta^\alpha \rho}{c} \widehat{T}_\rho \overline{\mathcal{V}}_{\mathcal{W}}(r) \widehat{z}_{c',\alpha}(r)} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{\eta^\alpha \rho}{c} \right)^n \left(\widehat{T}_\rho \overline{\mathcal{V}}_{\mathcal{W}}(r) \right)^n \widehat{z}_{c',\alpha}^{n+1}(r). \quad (4.4)$$

Let us define $S_n(u) := z_{c',\alpha} * \sum_{k=0}^{n} \left(\frac{\eta^\alpha \rho}{c}\right)^k \left(T_\rho \overline{\mathcal{V}}_{\mathcal{W}}\right)^{*k} * z_{c',\alpha}^{*k}(u)$. Using the monotone convergence theorem we get, for $r \geq 0$,

$$\begin{aligned}
&\int_0^\infty e^{-(r+\rho)u} z_{c',\alpha} * \sum_{k=0}^{\infty} \left(\frac{\eta^\alpha \rho}{c} \right)^k \left(T_\rho \overline{\mathcal{V}}_{\mathcal{W}}\right)^{*k} * z_{c',\alpha}^{*k}(u)\, du \\
&= \lim_{n\to\infty} \int_0^\infty e^{-(r+\rho)u} S_n(u)\, du \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{\eta^\alpha \rho}{c} \right)^k \int_0^\infty e^{-(r+\rho)u} \left(T_\rho \overline{\mathcal{V}}_{\mathcal{W}}\right)^{*k} * z_{c',\alpha}^{*(k+1)}(u)\, du \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{\eta^\alpha \rho}{c} \right)^k \left(\widehat{T}_\rho \overline{\mathcal{V}}_{\mathcal{W}}(r+\rho) \right)^k \left(\widehat{z}_{c',\alpha}(r+\rho) \right)^{k+1} \\
&= \sum_{k=0}^{\infty} \left(\frac{\eta^\alpha \rho}{c} \right)^k \left(\widehat{T}_\rho \overline{\mathcal{V}}_{\mathcal{W}}(r+\rho) \right)^k \left(\widehat{z}_{c',\alpha}(r+\rho) \right)^{k+1} < \infty,
\end{aligned}$$

where in the last inequality we used (4.3). This implies that the series in the right-hand side of (2.8) is absolutely convergent, and shows that the Laplace transform of that series equals the right-hand side of (4.4), which proves the result. ■

Corollary 8. *If $\rho = 0$, we have the equality $V_{\alpha,0}(u) = z_{c',\alpha}(u)$ for $u > 0$.*

Proof. This follows from Proposition 8 by setting $\rho = 0$ in (4.2). ■

Now we are ready to state and prove our result for the q-scale function of the process $\mathcal{X}$.

Proposition 9. *The q-scale function of the process $\mathcal{X}$ defined in (4.1) is given by*

$$\mathfrak{W}_{\mathcal{X}}^{(q)}(x) = \frac{1}{c} e^{\rho x} * \sum_{n=0}^{\infty} \left(\frac{1}{c}\right)^n (T_\rho \nu_{\mathcal{S}})^{*n} * V_{\alpha,\rho}^{*(n+1)}(x), \quad q > 0, \tag{4.5}$$

and

$$\mathfrak{W}_{\mathcal{X}}(x) = \frac{1}{c} \sum_{n=0}^{\infty} \left(\frac{1}{c}\right)^n Z_{c',\alpha} * \overline{\mathcal{V}}_{\mathcal{S}}^{*n} * z_{c',\alpha}^{*n}(x), \quad q = 0, \tag{4.6}$$

where $Z_{c',\alpha}(x) = \int\limits_0^x z_{c',\alpha}(y)dy$.

Proof. Since $L_{\mathcal{X}}(r) - q = cr + \eta^\alpha r^\alpha - G_{\mathcal{S}}(r) - q$, it follows that $c\rho + \eta^\alpha \rho^\alpha + G_{\mathcal{S}}(\rho) = q$. These equalities imply that:

$$\int\limits_0^\infty e^{-rx} \mathfrak{W}_{\mathcal{X}}^{(q)}(x)dx = \frac{\frac{1}{r-\rho}}{c + \eta^\alpha \left(\frac{\rho^\alpha - r^\alpha}{\rho - r}\right) - \frac{\int\limits_{0+}^\infty (e^{-rx} - e^{-\rho x})\nu_{\mathcal{S}}(dx)}{\rho - r}}.$$

By (3.12) we have $\frac{\rho^\alpha - r^\alpha}{\rho - r} = r^{\alpha-1} + \rho \frac{\rho^{\alpha-1} - r^{\alpha-1}}{\rho - r} = r^{\alpha-1} + \rho \widehat{T}_\rho \overline{\mathcal{V}}_{\mathcal{W}}(r)$. Hence we obtain, for $r > \rho$,

$$\begin{aligned}
\int\limits_0^\infty e^{-rx} \mathfrak{W}_{\mathcal{X}}^{(q)}(x)dx &= \frac{\frac{1}{r-\rho}}{c + \eta^\alpha r^{\alpha-1} + \eta^\alpha \rho \frac{\rho^{\alpha-1} - r^{\alpha-1}}{\rho - r} - \widehat{T}_\rho \nu_{\mathcal{S}}(r)} \\
&= \frac{\frac{1}{r-\rho} \frac{1}{c + \eta^\alpha r^{\alpha-1} + \eta^\alpha \rho \frac{\rho^{\alpha-1} - r^{\alpha-1}}{\rho - r}}}{1 - \widehat{T}_\rho \nu_{\mathcal{S}}(r) \frac{1}{c + \eta^\alpha r^{\alpha-1} + \eta^\alpha \rho \frac{\rho^{\alpha-1} - r^{\alpha-1}}{\rho - r}}} \\
&= \frac{\frac{1}{c} \frac{\widehat{V}_{\alpha,\rho}(r)}{r-\rho}}{1 - \frac{1}{c} \widehat{T}_\rho \nu_{\mathcal{S}}(r) \widehat{V}_{\alpha,\rho}(r)},
\end{aligned}$$

where in the last equality we used (4.2). This gives:

$$\widehat{\mathfrak{W}}_{\mathcal{X}}^{(q)}(r) = \frac{\frac{1}{c}\frac{\widehat{V}_{\alpha,\rho}(r)}{r-\rho}}{1-\frac{1}{c}\widehat{T}_{\rho}\nu_{\mathcal{S}}(r)\widehat{V}_{\alpha,\rho}(r)} = \frac{\widehat{V}_{\alpha,\rho}(r)}{c(r-\rho)}\sum_{n=0}^{\infty}\left(\frac{1}{c}\right)^{n}\left(\widehat{T}_{\rho}\nu_{\mathcal{S}}(r)\widehat{V}_{\alpha,\rho}(r)\right)^{n}. \qquad (4.7)$$

Since for $r > \rho$ we have that $\frac{1}{r-\rho}$ is the Laplace transform of $e^{\rho x}$, with the help of Proposition 8 we can invert (4.7) and obtain (4.5). To obtain (4.6), we set $\rho = 0$ in (4.7) and use Corollary 8 and the equalities $r\widehat{Z}_{c',\alpha}(r) = \widehat{z}_{c',\alpha}(r)$ and $\widehat{T}_{\rho}\nu_{\mathcal{S}}(r)|_{\rho=0} = \widehat{\overline{\mathcal{V}}}_{\mathcal{S}}(r)$. This results to:

$$\widehat{\mathfrak{W}}_{\mathcal{X}}(r) = \frac{\frac{1}{c}\frac{\widehat{z}_{c',\alpha}(r)}{r}}{1-\frac{1}{c}\widehat{\overline{\mathcal{V}}}_{\mathcal{S}}(r)\widehat{z}_{c',\alpha}(r)} = \frac{\widehat{Z}_{c',\alpha}(r)}{c}\sum_{n=0}^{\infty}\left(\frac{1}{c}\right)^{n}\left(\widehat{\overline{\mathcal{V}}}_{\mathcal{S}}(r)\widehat{z}_{c',\alpha}(r)\right)^{n}.$$

Now (4.6) follows by inverting the expression above. ■

From the proposition above we obtain the following corollary.

Corollary 9. *Suppose that $\eta\mathcal{W}_{\alpha}$ in (4.1) is replaced by $\gamma\mathfrak{B}$, where $\mathfrak{B} = \{\mathfrak{B}(t), t \geq 0\}$ is a brownian motion with zero mean and variance 2, then the q-scale function of the resulting spectrally negative Lévy process $\mathcal{X}$ is given by*

$$\mathfrak{W}_{\mathcal{X}}^{(q)}(x) = \frac{1}{c+\gamma^2\rho}e^{\rho x} * \sum_{n=0}^{\infty}\left(\frac{1}{c+\gamma^2\rho}\right)^{n}(T_{\rho}\nu_{\mathcal{S}})^{*n} * e_{c'}^{*(n+1)}(x), \quad q > 0,$$

and

$$\mathfrak{W}_{\mathcal{X}}(x) = \frac{1}{c}\sum_{n=0}^{\infty}\left(\frac{1}{c}\right)^{n} E_{c'} * \overline{\mathcal{V}}_{\mathcal{S}}^{*n} * e_{c'}^{*n}(x), \quad q = 0,$$

where in this case $c' = c/\gamma^2 + \rho$, $e_a(x) = ae^{-ax}, x > 0$ and $E_a(x) = \int\limits_0^x e_a(y)dy$.

Proof. In this case we have

$$\int_0^\infty e^{-rx}\mathfrak{W}_{\mathcal{X}}^{(q)}(x)dx = \frac{\frac{1}{r-\rho}}{c+\gamma^2\left(\frac{\rho^2-r^2}{\rho-r}\right)-\frac{\int_{0+}^{\infty}(e^{-rx}-e^{-\rho x})\nu_{\mathcal{S}}(dx)}{\rho-r}} = \frac{\frac{1}{r-\rho}}{c+\gamma^2(r+\rho)-\widehat{T}_\rho\nu_{\mathcal{S}}(r)}$$
$$= \frac{\frac{1}{r-\rho}\frac{1}{c+\gamma^2\rho+\gamma^2 r}}{1-\widehat{T}_\rho\nu_{\mathcal{S}}(r)\frac{1}{c+\gamma^2\rho+\gamma^2 r}} = \frac{\frac{1}{c+\gamma^2\rho}\frac{1}{r-\rho}\widehat{e}_{c'}(r)}{1-\frac{1}{c+\gamma^2\rho}\widehat{T}_\rho\nu_{\mathcal{S}}(r)\widehat{e}_{c'}(r)}$$

The rest of the proof is analogous to that of Proposition 9. ∎

The second formula in the Corollary above clearly gives the result in Proposition 7 by taking derivatives and considering $c = a_\delta + \mu_{\mathcal{S}}$.

Appendix A

Proofs of technical results

Proof of Lemma 3. The existence of the integral I_n follows from the existence of both, $\widehat{g}_{\alpha,1}$ and the first moment of $g_{\alpha,1}$ for $\alpha \in (1,2)$. Setting $y = -n^{1/\alpha}x$ we obtain

$$I_n = \int_0^\infty n\left(1 - e^{ry/n^{1/\alpha}} + \frac{ry}{n^{1/\alpha}}\right) g_{\alpha,1}(-y)dy,$$

and putting $I_n^*(y,r) = n\left(1 - e^{ry/n^{1/\alpha}} + \frac{ry}{n^{1/\alpha}}\right) g_{\alpha,1}(-y)$ gives

$$|I_n^*(y,r)| = \left|\sum_{k=2}^\infty r^k \frac{y^k}{n^{k/\alpha-1}} g_{\alpha,1}(-y)\right| \le \sum_{k=2}^\infty |r|^k y^k g_{\alpha,1}(-y) = \left(e^{|r|y} - 1 - |r|y\right) g_{\alpha,1}(-y).$$

Since $\int\limits_0^\infty \left(e^{|r|y} - 1 - |r|y\right) g_{\alpha,1}(-y)dy = \int\limits_{-\infty}^0 \left(e^{-|r|y} - 1 + |r|y\right) g_{\alpha,1}(y)dy < \infty$, we use the inequality $|I_n| \le \int_0^\infty |I_n^*(x,r)|\, dx$ and apply the dominated convergence theorem, and the result follows. ■

Proof of Lemma 4. Hypothesis 1 implies that $\phi_{n,k}(u)$ is bounded for all $u \ge 0$, hence $K_0(n,k,r)$ is finite. Using integration by parts it follows that

$$\int_0^x \frac{y^{k-1} q_j^k e^{-q_j y}}{(k-1)!} dy = 1 - \sum_{l=0}^{k-1} \frac{e^{-q_j x}}{l!} q_j^l x^l, \tag{A.1}$$

and performing the change of variables $z = u + x$ we obtain

$$\begin{aligned}
K_0(n,k,r) &= \lambda_1(k)A(k)\int_0^\infty \int_x^\infty e^{-rz}\phi_{n,k}(z) b_n(k) e^{(r-b_n(k))x} dz dx \\
&\quad + \lambda_1 \sum_{i=1}^N \sum_{j=1}^{m_i} \beta_{ij} \int_0^\infty \int_x^\infty e^{-rz}\phi_{n,k}(z) \frac{x^{j-1} q_i^j e^{(r-q_i)x}}{(j-1)!} dz dx \\
&= \lambda_1(k)A(k)\int_0^\infty e^{-rz}\phi_{n,k}(z) \int_0^z b_n(k) e^{-(b_n(k)-r)x} dx dz
\end{aligned}$$

$$+ \lambda_1 \sum_{i=1}^{N} \sum_{j=1}^{m_i} \beta_{ij} \int_0^\infty e^{-rz} \phi_{n,k}(z) \int_0^z \frac{x^{j-1} q_i^j e^{(r-q_i)x}}{(j-1)!} dxdz. \tag{A.2}$$

Using (A.1) we get

$$\begin{aligned}
K_0(n,k,r) &= \lambda_1(k) A(k) \frac{b_n(k)}{b_n(k)-r} \int_0^\infty e^{-rz} \phi_{n,k}(z) dz \\
&- \lambda_1(k)(1-A(k)) \frac{b_n(k)}{b_n(k)-r} \int_0^\infty \phi_{n,k}(z) e^{-b_n(k)z} dz \\
&+ \lambda_1 \sum_{i=1}^{N} \sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} \int_0^\infty e^{-rz} \phi_{n,k}(z) dz \qquad \text{(A.3)} \\
&- \lambda_1 \sum_{i=1}^{N} \sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} \int_0^\infty e^{-rz} \phi_{n,k}(z) \left\{ \sum_{l=0}^{j-1} \frac{e^{-q_i z + rz}}{l!} (q_i - r)^l z^l \right\} dz \\
&= \left[\frac{\lambda_1(k) A(k) b_n(k)}{b_n(k) - r} + \lambda_1 \sum_{i=1}^{N} \sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} \right] \widehat{\phi}_{n,k}(r) \\
&- \frac{\lambda_1(k)(1-A(k)) b_n(k)}{b_n(k) - r} \widehat{\phi}_{n,k}(b_n(k)) - \lambda_1 \sum_{i=1}^{N} \sum_{j=1}^{m_i} \beta_{ij} q_i^j \sum_{l=0}^{j-1} \frac{(q_i - r)^l \gamma_{l,i}(n,k)}{l!(q_i - r)^j},
\end{aligned}$$

and the result follows. ■

Proof of Lemma 5. We set $e(u; r_1, r_2) = e^{-r_1 u} - e^{-r_2 u}$ and recall that $d(\alpha) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}$. Hence

$$\eta^{-\alpha}(M_{\alpha,n}(r_1) - M_{\alpha,n}(r_2)) = n^{1+1/\alpha} \int_0^\infty \int_u^\infty e(u; r_1, r_2) \omega(x-u, u) g_{\alpha,1}(n^{1/\alpha} x) dxdu. \tag{A.4}$$

By formula (14.37), p. 89 in Sato (1999), we get $\lim_{x\to\infty} \frac{g_{\alpha,1}(x)}{d(\alpha) x^{-1-\alpha}} = 1$. Hence, for every $\varepsilon > 0$ there exists a positive number $A_\varepsilon > 1$ such that for all $u > A_\varepsilon$,

$$\frac{g_{\alpha,1}(x)}{d(\alpha)x^{-1-\alpha}} < (1+\varepsilon). \tag{A.5}$$

We take $A = A_\varepsilon$ and $n > A^\alpha$, and split (A.4) as

$$\begin{aligned}\eta^{-\alpha}\left[M_\alpha(r_1) - M_\alpha(r_2)\right] = n^{1+1/\alpha} &\int_1^\infty \int_u^\infty e(u;r_1,r_2)\omega(x-u,u)g_{\alpha,1}(n^{1/\alpha}x)dxdu \\ + n^{1+1/\alpha} &\int_{A/n^{1/\alpha}}^1 \int_u^\infty e(u;r_1,r_2)\omega(x-u,u)g_{\alpha,1}(n^{1/\alpha}x)dxdu \\ + n^{1+1/\alpha} &\int_0^{A/n^{1/\alpha}} \int_u^\infty e(u;r_1,r_2)\omega(x-u,u)g_{\alpha,1}(n^{1/\alpha}x)dxdu.\end{aligned} \tag{A.6}$$

Since $u \geq 1$ and $x \geq u$ imply $n^{1/\alpha}x > A$, from (A.5) we obtain that the first term in (A.6) is bounded above by

$$2d(\alpha)B(1+\varepsilon)\int_1^\infty \int_u^\infty x^{-1-\alpha}dxdu = \frac{2d(\alpha)B(1+\varepsilon)}{\alpha(\alpha-1)}.$$

Hence, using dominated convergence it follows that

$$\begin{aligned}&\lim_{n\to\infty} n^{1+1/\alpha}\int_1^\infty \int_u^\infty e(u;r_1,r_2)\omega(x-u,u)g_{\alpha,1}(n^{1/\alpha}x)\,dx\,du \\ &= d(\alpha)\int_1^\infty \int_u^\infty e(u;r_1,r_2)\omega(x-u,u)x^{-1-\alpha}\,dx\,du.\end{aligned} \tag{A.7}$$

Now we consider the second term in (A.6). In this case $n^{1/\alpha}x \geq A$, hence

$$
\begin{aligned}
&\left| n^{1+1/\alpha} \int_{A/n^{1/\alpha}}^{1} \int_{u}^{\infty} e(u; r_1, r_2)\omega(x-u,u) g_{\alpha,1}(n^{1/\alpha}x) dx du \right| \\
&\leq d(\alpha) B \int_0^1 \int_u^\infty |e(u; r_1, r_2)| \frac{g_{\alpha,1}(n^{1/\alpha}x)}{d(\alpha)\left[n^{1/\alpha}x\right]^{-1-\alpha}} d(\alpha) x^{-1-\alpha} dx du \\
&\leq d(\alpha) B (1+\varepsilon) \int_0^1 \sum_{k=1}^{\infty} \frac{|r_2^k - r_1^k| u^k}{k!} \int_u^\infty \frac{1}{x^{1+\alpha}} dx du \\
&\leq \frac{2 d(\alpha) B}{\alpha} \left(\frac{|r_1 - r_2|}{2-\alpha} + e^{|r_1|} + e^{|r_2|} \right).
\end{aligned}
$$

Using again dominated convergence yields

$$
\begin{aligned}
&\lim_{n\to\infty} n^{1+1/\alpha} \int_{\frac{A}{n^{1/\alpha}}}^{1} \int_{u}^{\infty} e(u; r_1, r_2)\omega(x-u,u) g_{\alpha,1}(n^{1/\alpha}x) dx du \\
&= d(\alpha) \int_0^1 \int_u^\infty e(u; r_1, r_2)\omega(x-u,u) x^{-1-\alpha} dx du. \qquad \text{(A.8)}
\end{aligned}
$$

For the third term in (A.6) we use the change of variables $y = n^{1/\alpha}u$ and $z = n^{1/\alpha}x$. This yields

$$
\begin{aligned}
&\left| n^{1+1/\alpha} \int_0^{A/n^{1/\alpha}} \int_u^\infty e(u; r_1, r_2)\omega(x-u,u) g_{\alpha,1}(n^{1/\alpha}x) dx du \right| \\
&\leq n^{1-1/\alpha} \int_0^A \int_y^\infty \left| e^{-r_1 y/n^{1/\alpha}} - e^{-r_2 y/n^{1/\alpha}} \right| \omega\left(\frac{z-y}{n^{1/\alpha}}, \frac{y}{n^{1/\alpha}} \right) g_{\alpha,1}(z) dz dy \\
&\leq B \int_0^A \int_y^\infty \sum_{k=1}^{\infty} \frac{|r_1^k - r_2^k| y^k}{k! n^{(k+1-\alpha)/\alpha}} g_{\alpha,1}(z) dz dy \leq \frac{B}{n^{(2-\alpha)/\alpha}} \int_0^A \left(e^{|r_1| y} + e^{|r_2| y} \right) \overline{G}_{\alpha,1}(y) dy \\
&\leq \frac{B}{n^{(2-\alpha)/\alpha}} A \left(e^{|r_1| A} + e^{|r_2| A} \right) \overline{G}_{\alpha,1}(0),
\end{aligned}
$$

where the last inequality follows using that $n \geq A^{\alpha} > 1$, which implies $n^{-(2-\alpha)/\alpha} \geq n^{-(k+1-\alpha)/\alpha}$ for all $k \geq 2$. The last inequality renders

$$\lim_{n\to\infty} n^{1+1/\alpha} \int_0^{A/n^{1/\alpha}} \int_u^{\infty} e(u; r_1, r_2)\omega(x-u, u) g_{\alpha,1}(n^{1/\alpha}x) dx du = 0,$$

and using (A.8) and (A.8), we obtain (2.2).

Now we will prove (2.3). First we note that

$$\begin{aligned} \frac{M_\alpha(r_1) - M_\alpha(r_2)}{r_2 - r_1} &= \omega_0^+ \frac{d(\alpha)}{r_2 - r_1} \int_0^\infty \int_u^\infty e(u; r_1, r_2) x^{-1-\alpha} dx du \\ &+ \frac{d(\alpha)}{r_2 - r_1} \int_0^\infty \int_u^\infty e(u; r_1, r_2)\omega^*(x-u, u) x^{-1-\alpha} dx du, \end{aligned} \tag{A.9}$$

where $\omega^*(x, y) = \omega(x, y) - \omega_0^+$. Using (2.6), we have that the first term in (A.9) equals $\omega_0^+ \frac{r_2^{\alpha-1} - r_1^{\alpha-1}}{r_2 - r_1}$, which converges to ω_0^+ when $\alpha \to 2$.

Hypothesis 1 implies that, for any $\varepsilon > 0$, there exists a $\delta < \sqrt{2}$ such that $\left|\omega(x, y) - \omega_0^+\right| < \varepsilon$ for all (x, y) with $|(x, y)| < \delta$. Let us split the second double integral in (A.9) as follows:

$$\frac{d(\alpha)}{r_2 - r_1} \int_1^\infty \int_u^\infty e(u; r_1, r_2)\omega^*(x-u, u) x^{-1-\alpha} dx du \tag{A.10}$$

$$+ \frac{d(\alpha)}{r_2 - r_1} \int_0^1 \int_u^\infty \left[e(u; r_1, r_2) - r_1 u + r_2 u\right] \omega^*(x-u, u) x^{-1-\alpha} dx du \tag{A.11}$$

$$+ d(\alpha) \int_{\frac{\delta}{\sqrt{2}}}^1 \int_u^{\frac{\delta}{\sqrt{2}}} u\omega^*(x-u, u) x^{-1-\alpha} dx du \tag{A.12}$$

$$+ d(\alpha) \int_0^1 \int_{\frac{\delta}{\sqrt{2}}}^\infty u\omega^*(x-u, u) x^{-1-\alpha} dx du \tag{A.13}$$

$$+ d(\alpha) \int_0^{\frac{\delta}{\sqrt{2}}} \int_u^{\frac{\delta}{\sqrt{2}}} u\omega^*(x-u, u) x^{-1-\alpha} dx du. \tag{A.14}$$

For (A.10) we have

$$\left|\frac{d(\alpha)}{r_2 - r_1}\int_1^{\infty}\int_u^{\infty} e(u;r_1,r_2)\omega^*(x-u,u)x^{-1-\alpha}dxdu\right| \leq \frac{2Bd(\alpha)}{r_2 - r_1}\int_1^{\infty}\int_u^{\infty} x^{-1-\alpha}dxdu$$
$$= \frac{2B\Gamma(\alpha+1)\sin\left[\pi(2-\alpha)\right]}{\pi\alpha(\alpha-1)(r_2-r_1)}$$

Using the continuity of the function $\Gamma(x)$ at the positive integers and the equality $\lim_{\alpha\to 2}\sin[\pi(2-\alpha)] = 0$, we obtain that (A.10) tends to zero as $\alpha \to 2$.

We have for (A.11):

$$\left|\frac{d(\alpha)}{r_2 - r_1}\int_0^{1}\int_u^{\infty}\left[e^{-r_1 u} - r_1 u - e^{-r_2 u} + r_2 u\right]\omega(x-u,u)x^{-1-\alpha}dxdu\right|$$
$$\leq \frac{B(1/\pi)\Gamma(\alpha+1)\sin[\pi(2-\alpha)]}{r_2 - r_1}\int_0^1 \left|e^{-r_1 u} - r_1 u - e^{-r_2 u} + r_2 u\right| u^{-\alpha}du$$
$$\leq \frac{B(1/\pi)\Gamma(\alpha+1)\sin[\pi(2-\alpha)]}{r_2 - r_1}\int_0^1 \sum_{k=2}^{\infty}\frac{|r_1^k - r_2^k|u^{2-\alpha}}{k!}du$$
$$\leq \frac{B(1/\pi)\Gamma(\alpha+1)\sin[\pi(2-\alpha)]}{r_2 - r_1}\int_0^1 \sum_{k=2}^{\infty}\frac{|r_1|^k + |r_2|^k}{k!}du$$
$$\leq \frac{B(1/\pi)\Gamma(\alpha+1)\sin[\pi(2-\alpha)]}{r_2 - r_1}\left(e^{|r_1|} + e^{|r_2|}\right)$$

where the second last equality follows using that $k - \alpha > 0$ for $k \geq 2$, which gives $u^{k-\alpha} \leq 1$ for $u \in [0, 1]$. Using the continuity of the function $\Gamma(x)$ at the positive integers and $\lim_{\alpha\to 2}\sin[\pi(2-\alpha)] = 0$, this gives:

$$\lim_{\alpha\to 2}\left|\frac{d(\alpha)}{r_2-r_1}\int_0^1\int_u^\infty\left[e^{-r_1u}-r_1u-e^{-r_2u}+r_2u\right]\omega(x-u,u)x^{-1-\alpha}dxdu\right|=0$$

This implies that (A.11) tends to zero as $\alpha\to 2$.

Since $\lim\limits_{(x,y)\to(0+,0+)}\omega(x,y)=\omega_0^+$, for $\varepsilon>0$ there exists a $\delta<\sqrt{2}$ such that $\left|\omega(x,y)-\omega_0^+\right|<\varepsilon$, for all (x,y) such that $|(x,y)|<\delta$.

We take $|\cdot|$ as the euclidean distance and note that, for the right-hand side of (A.14), we have $(x-u)^2<\frac{\delta^2}{2}$ and $u^2<\frac{\delta^2}{2}$, which implies $\sqrt{(x-u)^2+u^2}<\delta$. Hence, for each $1<\alpha<2$ we obtain:

$$\left|d(\alpha)\int_0^{\frac{\delta}{\sqrt{2}}}\int_u^{\frac{\delta}{\sqrt{2}}}u\omega^*(x-u,u)x^{-1-\alpha}dxdu\right|\leq d(\alpha)\varepsilon\int_0^{\frac{\delta}{\sqrt{2}}}\int_u^{\frac{\delta}{\sqrt{2}}}ux^{-1-\alpha}dxdu \tag{A.15}$$

Now we note that the right-hand side of (A.15) satisfies the equalities:

$$\begin{aligned}
d(\alpha)\varepsilon\int_0^{\frac{\delta}{\sqrt{2}}}\int_u^{\frac{\delta}{\sqrt{2}}}ux^{-1-\alpha}dxdu &= \frac{\varepsilon\Gamma(\alpha+1)\sin[\pi(2-\alpha)]}{\pi\alpha}\int_0^{\frac{\delta}{\sqrt{2}}}u\left[u^{-\alpha}-\left(\frac{\delta}{\sqrt{2}}\right)^{-\alpha}\right]du\\
&= \frac{\varepsilon\Gamma(\alpha+1)\sin[\pi(2-\alpha)]}{\pi\alpha}\left[\frac{1}{2-\alpha}\left(\frac{\delta}{\sqrt{2}}\right)^{2-\alpha}-\frac{1}{2}\left(\frac{\delta}{\sqrt{2}}\right)^{2-\alpha}\right]\\
&= \frac{\varepsilon}{\pi}\Gamma(\alpha+1)\frac{\sin[\pi(2-\alpha)]}{(2-\alpha)}\left(\frac{\delta}{\sqrt{2}}\right)^{2-\alpha}\left(\frac{1}{\alpha}-\frac{2-\alpha}{2\alpha}\right)
\end{aligned}$$

Using the continuity of the gamma function at the positive integers and the equality $\lim\limits_{\alpha\to 2}\frac{\sin[\pi(2-\alpha)]}{2-\alpha}=\pi$, we see that the right-hand side of the equality above tends to $\varepsilon/2$ as $\alpha\to 2$. Applying this to (A.15) we obtain:

$$\limsup_{\alpha\to 2}\left| d(\alpha)\int_0^{\frac{\delta}{\sqrt{2}}}\int_u^{\frac{\delta}{\sqrt{2}}} u\omega^*(x-u,u)x^{-1-\alpha}dxdu\right| < \varepsilon. \tag{A.16}$$

The inequality $|\omega(x,y)-\omega_0^+|\le B$ gives for (A.12) and (A.13) as before:

$$\begin{aligned}
&\left| d(\alpha)\int_{\frac{\delta}{\sqrt{2}}}^{1}\int_u^{\frac{\delta}{\sqrt{2}}} u\omega^*(x-u,u)x^{-1-\alpha}dxdu\right| \\
&\le B\frac{\Gamma(\alpha+1)\sin[\pi(2-\alpha)]}{\pi\alpha}\int_{\frac{\delta}{\sqrt{2}}}^{1} u\left[u^{-\alpha}-\left(\frac{\delta}{\sqrt{2}}\right)^{-\alpha}\right]du \\
&= B\frac{\Gamma(\alpha+1)\sin[\pi(2-\alpha)]}{\pi\alpha}\left[\frac{1-\left(\delta/\sqrt{2}\right)^{2-\alpha}}{2-\alpha}\right. \\
&\left. -\frac{1}{2}\left(\frac{\delta}{\sqrt{2}}\right)^{-\alpha}\left(1-\left(\frac{\delta}{\sqrt{2}}\right)^{2}\right)\right] \\
&= B\frac{\Gamma(\alpha+1)\sin[\pi(2-\alpha)]}{\pi\alpha(2-\alpha)}\left[1-\left(\delta/\sqrt{2}\right)^{2-\alpha}\right] \\
&- B\frac{\Gamma(\alpha+1)\sin[\pi(2-\alpha)]}{2\pi\alpha}\left(\frac{\delta}{\sqrt{2}}\right)^{-\alpha}\left[1-\left(\frac{\delta}{\sqrt{2}}\right)^{2}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left| d(\alpha)\int_0^{1}\int_{\frac{\delta}{\sqrt{2}}}^{\infty} u\omega^*(x-u,u)x^{-1-\alpha}dxdu\right| \\
&\le Bd(\alpha)\int_0^{1}\int_{\frac{\delta}{\sqrt{2}}}^{\infty} ux^{-1-\alpha}dxdu = B\frac{\Gamma(\alpha+1)\sin[\pi(2-\alpha)]}{\pi\alpha}\int_0^{1} u\left(\frac{\delta}{\sqrt{2}}\right)^{-\alpha}du
\end{aligned}$$

$$= B\frac{\Gamma(\alpha+1)\sin[\pi(2-\alpha)]}{2\pi\alpha}\left(\frac{\delta}{\sqrt{2}}\right)^{-\alpha}.$$

Since $\lim_{\alpha\to 2} B\frac{(1/\pi)\Gamma(\alpha+1)\sin[\pi(2-\alpha)]}{\alpha(2-\alpha)}\left[1-\left(\delta/\sqrt{2}\right)^{2-\alpha}\right]=0$, we obtain:

$$\limsup_{\alpha\to 2}\left|d(\alpha)\int_{\frac{\delta}{\sqrt{2}}}^{1}\int_{u}^{\frac{\delta}{\sqrt{2}}} u\omega^*(x-u,u)x^{-1-\alpha}dxdu\right| = 0$$

and

$$\limsup_{\alpha\to 2}\left|d(\alpha)\int_{0}^{1}\int_{\frac{\delta}{\sqrt{2}}}^{\infty} u\omega^*(x-u,u)x^{-1-\alpha}dxdu\right| = 0.$$

Hence using these equalities together with (A.16), the triangle inequality and properties of the limit superior, we obtain:

$$\limsup_{\alpha\to 2}\left|d(\alpha)\int_{0}^{1}\int_{u}^{\infty} u\omega^*(x-u,u)x^{-1-\alpha}dxdu\right| \leq \varepsilon.$$

Since ε is arbitrary, we obtain

$$\lim_{\alpha\to 2}\left|d(\alpha)\int_{0}^{1}\int_{u}^{\infty} u\omega^*(x-u,u)x^{-1-\alpha}dxdu\right| = 0. \tag{A.17}$$

Hence

$$\limsup_{\alpha\to 2}\left|\frac{d(\alpha)}{r_2-r_1}\int_0^\infty\int_u^\infty e(u;r_1,r_2)\omega^*(x-u,u)x^{-1-\alpha}dxdu\right|\le\varepsilon,$$

and (2.3) follows. ■

Proof of Proposition 2.

a) We set

$$J_0(r)=\lambda_1\sum_{j=1}^{m+1}Q_1(\rho_{j,\delta})\frac{1}{\prod\limits_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}-\rho_{j,\delta})}\frac{\frac{Q(-r)}{\prod_{j=1}^{N}(q_j-r)^{m_j}}-\frac{Q(-\rho_{l,\delta})}{\prod_{j=1}^{N}(q_j-\rho_{l,\delta})^{m_j}}}{\rho_{j,\delta}-r}.$$

First we prove that $J_0(r)=0$ for all $r\ge 0$. From (2.6) we get $\frac{Q(r)}{\prod_{j=1}^{N}(q_j+r)^{m_j}}=\sum\limits_{k=1}^{N}\sum\limits_{l=1}^{m_k}\frac{\beta_{kl}q_k^l}{(q_k+r)^l}$. Now, for any fixed $r\ge 0$ we define $\rho^*_{j,\delta}(r)=\rho_{j,\delta}-r$ and $q^*_i(r)=q_i-r$. This gives:

$$\begin{aligned}J_0(r)&=\lambda_1\sum_{j=1}^{m+1}\frac{\prod\limits_{h=1}^{N}(q^*_h(r)-\rho^*_{j,\delta}(r))^{m_h}}{\prod\limits_{i=1,i\neq j}^{m+1}(\rho^*_{i,\delta}(r)-\rho^*_{j,\delta}(r))}\sum_{k=1}^{N}\sum_{l=1}^{m_k}\beta_{kl}q_k^l\frac{(q^*_k(r)-\rho^*_{j,\delta}(r))^l-(q^*_k(r))^l}{(q^*_k(r))^l(q^*_k(r)-\rho^*_{j,\delta}(r))^l\rho^*_{j,\delta}(r)}\\&=\lambda_1\sum_{j=1}^{m+1}\frac{\prod\limits_{h=1}^{N}(q^*_h(r)-\rho^*_{j,\delta}(r))^{m_h}}{\prod\limits_{i=1,i\neq j}^{m+1}(\rho^*_{i,\delta}(r)-\rho^*_{j,\delta}(r))}\sum_{k=1}^{N}\sum_{l=1}^{m_k}\beta_{kl}q_k^l\frac{P^*_l(\rho^*_{j,\delta}(r))}{(q^*_k(r))^l(q^*_k(r)-\rho^*_{j,\delta}(r))^l},\end{aligned}\tag{A.18}$$

where P^*_l is a polynomial of degree $l-1$. We note that for each $j\in\{1,\dots,m+1\}$,

$$\sum_{k=1}^{N}\sum_{l=1}^{m_k}\beta_{kl}q_k^l\frac{P^*_l(\rho^*_{j,\delta}(r))}{(q^*_k(r))^l(q^*_k(r)-\rho^*_{j,\delta}(r))^l}=\frac{P^{**}(\rho^*_{j,\delta}(r))}{\prod\limits_{h=1}^{N}(q^*_h(r)-\rho^*_{j,\delta}(r))^{m_h}},$$

where $P^{**}(\rho^*_{j,\delta}(r))$ is a polynomial on $\rho^*_{j,\delta}(r)$ of degree at most $m-1$. Using the above equality in (A.18) gives

$$J_0(r) = \lambda_1 \sum_{j=1}^{m+1} \frac{P^{**}(\rho^*_{j,\delta}(r))}{\prod\limits_{i=1,i\neq j}^{m+1} (\rho^*_{i,\delta}(r) - \rho^*_{j,\delta}(r))} = 0,$$

due to (2.5). From the equality $\lambda_1 + \lambda_2 + \delta = \lambda_2 \widehat{f}_2(\rho_{j,\delta}) + \lambda_1 \frac{Q(-\rho_{j,\delta})}{\prod_{l=1}^N (q_l - \rho_{j,\delta})^{m_l}} + c\rho_{j,\delta} + \eta^\alpha \rho^\alpha_{j,\delta}$, we obtain that the denominator of the right-hand side of (2.4) is given by

$$\begin{aligned} &\sum_{j=1}^{m+1} \frac{Q_1(\rho_{j,\delta})}{\prod\limits_{l\neq j}(\rho_{l,\delta} - \rho_{j,\delta})(\rho_{j,\delta} - r)} L_\alpha(r) \\ &= \sum_{j=1}^{m+1} \frac{Q_1(\rho_{j,\delta})}{\prod\limits_{i=1,i\neq j}^{m+1} (\rho_{i,\delta} - \rho_{j,\delta})(\rho_{j,\delta} - r)} \left(\lambda_2 (\widehat{f}_2(r) - \widehat{f}_2(\rho_{j,\delta})) - c(\rho_{j,\delta} - r) - \eta^\alpha (\rho^\alpha_{j,\delta} - r^\alpha) \right). \end{aligned}$$

From the equality above, (2.4) and $\frac{\rho^\alpha_{j,\delta} - r^\alpha}{\rho_{j,\delta} - r} = \rho_{j,\delta} \frac{\rho^{\alpha-1}_{j,\delta} - r^{\alpha-1}}{\rho_{j,\delta} - r} + r^{\alpha-1}$, we have

$$\begin{aligned} \widehat{W}_\delta(r) &= \frac{1}{\sum\limits_{j=1}^{m+1} \frac{Q_1(\rho_{j,\delta})}{\prod\limits_{i=1,i\neq j}^{m+1} (\rho_{i,\delta} - \rho_{j,\delta})} \left(-\lambda_2 \frac{(\widehat{f}_2(r) - \widehat{f}_2(\rho_{j,\delta}))}{\rho_{j,\delta} - r} + c + \eta^\alpha \frac{\rho^\alpha_{j,\delta} - r^\alpha}{\rho_{j,\delta} - r} \right)} \\ &= \frac{1}{\sum\limits_{j=1}^{m+1} E(\rho_{j,\delta}) \left(-\lambda_2 \widehat{T}_{\rho_{j,\delta}} f_2(r) + c + \eta^\alpha \rho_{j,\delta} \frac{\rho^{\alpha-1}_{j,\delta} - r^{\alpha-1}}{\rho_{j,\delta} - r} + \eta^\alpha r^{\alpha-1} \right)} \\ &= \frac{1}{c + \eta^\alpha r^{\alpha-1} - \lambda_2 \sum\limits_{j=1}^{m+1} E(\rho_{j,\delta}) \widehat{T}_{\rho_{j,\delta}} f_2(r) + \eta^\alpha \sum\limits_{j=1}^{m+1} E(\rho_{j,\delta}) \rho_{j,\delta} \frac{\rho^{\alpha-1}_{j,\delta} - r^{\alpha-1}}{\rho_{j,\delta} - r}} \\ &= \frac{\frac{1}{\eta^\alpha \theta_\delta} \widehat{\nu}_{\alpha,\delta}(r)}{1 - \frac{1}{\theta_\delta} \left[\eta^{-\alpha} \widehat{g}_\delta(r) + \kappa_\delta \right] \widehat{\nu}_{\alpha,\delta}(r)} \end{aligned} \qquad \text{(A.19)}$$

where the last equality is obtained by dividing the nominator and the denominator by $c + \eta^\alpha \kappa_\delta + \eta^\alpha r^{\alpha-1} + \eta^\alpha \sum_{j=1}^{m+1} E(\rho_{j,\delta}) \rho_{j,\delta} \widehat{T}_{\rho_{j,\delta}} \ell_\alpha(r)$.

b) We define the function $\widehat{\nu}^*_{\alpha,\delta}(r) = \frac{1}{c+\eta^\alpha r^{\alpha-1}+\eta^\alpha \widehat{f}_{\alpha,\delta}(r)}$. From (2.2) and (A.19) we obtain

$$\widehat{\phi}(r) = \frac{\widehat{h}_{\alpha,\delta,\omega}(r)\widehat{\nu}^*_{\alpha,\delta}(r)}{1-\widehat{g}_\delta(r)\widehat{\nu}^*_{\alpha,\delta}(r)}. \tag{A.20}$$

First we consider the case when $\delta > 0$. We will show in this case that if $\omega(x,y) \equiv 1$, then

$$\frac{1}{r}\left[1-\widehat{g}_\delta(r)\widehat{\nu}^*_{\alpha,\delta}(r) - \delta R\widehat{\nu}_{\alpha,\delta}(r)\right] = \widehat{h}_{\alpha,\delta,\omega}(r)\widehat{\nu}^*_{\alpha,\delta}(r), \tag{A.21}$$

where $R = \frac{\prod_{i=1}^{N} q_i^{m_i}}{\prod_{j=1}^{m+1} \rho_{j,\delta}}$. Using that $L_\alpha(\rho_{j,\delta}) = 0$ and Lemma 6 we get

$$\begin{aligned}
\delta R &= \sum_{j=1}^{m+1} E(\rho_{j,\delta})\frac{\delta}{\rho_{j,\delta}} \\
&= \sum_{j=1}^{m+1} E(\rho_{j,\delta})\left[\frac{\lambda_2\left(\widehat{f}_2(\rho_{j,\delta})-1\right)}{\rho_{j,\delta}} + c + \eta^\alpha\rho_{j,\delta}^{\alpha-1} + \frac{\lambda_1\left(\frac{Q(-\rho_{j,\delta})}{\prod_{l=1}^N (q_l-\rho_{j,\delta})^{m_l}}-1\right)}{\rho_{j,\delta}}\right] \\
&= \sum_{j=1}^{m+1} E(\rho_{j,\delta})\left[\frac{\lambda_2\left(\widehat{f}_2(\rho_{j,\delta})-1\right)}{\rho_{j,\delta}} + c + \eta^\alpha\rho_{j,\delta}^{\alpha-1}\right] \\
&+ \lambda_1\sum_{j=1}^{m+1} E(\rho_{j,\delta})\frac{\frac{Q(-\rho_{j,\delta})}{\prod_{l=1}^N (q_l-\rho_{j,\delta})^{m_l}}}{\rho_{j,\delta}} - \lambda_1\sum_{j=1}^{m+1}\frac{E(\rho_{j,\delta})}{\rho_{j,\delta}}
\end{aligned} \tag{A.22}$$

From Lemma 6 it follows that $\lambda_1\sum_{j=1}^{m+1}\frac{E(\rho_{j,\delta})}{\rho_{j,\delta}} = \lambda_1 R$. From the definition of $E(\rho_{j,\delta})$ we obtain:

$$\begin{aligned}
\lambda_1\sum_{j=1}^{m+1} E(\rho_{j,\delta})\frac{\frac{Q(-\rho_{j,\delta})}{\prod_{l=1}^N (q_l-\rho_{j,\delta})^{m_l}}}{\rho_{j,\delta}} &= \lambda_1\sum_{j=1}^{m+1}\frac{\prod_{l=1}^{N}(q_l-\rho_{j,\delta})^{m_l}}{\prod_{k=1,k\neq j}^{m+1}(\rho_{k,\delta}-\rho_{j,\delta})}\frac{Q(-\rho_{j,\delta})}{\prod_{l=1}^{N}(q_l-\rho_{j,\delta})^{m_l}\rho_{j,\delta}} \\
&= \lambda_1\sum_{j=1}^{m+1}\frac{Q(-\rho_{j,\delta})}{\prod_{k=1,k\neq j}^{m+1}(\rho_{k,\delta}-\rho_{j,\delta})\rho_{j,\delta}}.
\end{aligned}$$

Since Q is a polynomial in $\rho_{j,\delta}$ of degree at most $m-1$ and constant term $\prod_{i=1}^{N} q_i^{m_i}$, it follows that $Q(-\rho_{j,\delta})\rho_{j,\delta}^{-1} = (\rho_{j,\delta})^{-1}\prod_{i=1}^{N} q_i^{m_i} + Q_0(-\rho_{j,\delta})$, where $Q_0(r)$ is a polynomial of degree at most $m-2$. Hence, applying (2.5), we obtain $\lambda_1 \sum_{j=1}^{m+1} E(\rho_{j,\delta}) \frac{\frac{Q(-\rho_{j,\delta})}{\prod_{l=1}^{N}(q_l-\rho_{j,\delta})^{m_l}}}{\rho_{j,\delta}} = \lambda_1 R$, and (A.22) simplifies to

$$\delta R = \sum_{j=1}^{m+1} E(\rho_{j,\delta}) \left[\frac{\lambda_2\left(\widehat{f}_2(\rho_{j,\delta})-1\right)}{\rho_{j,\delta}} + c + \eta^{\alpha}\rho_{j,\delta}^{\alpha-1} \right]. \tag{A.23}$$

On the other hand, from the definition of $g_\delta(x)$ and Lemma 6 it follows that

$$\begin{aligned} 1-\widehat{g}_\delta(r)\widehat{\nu}^*_{\alpha,\delta}(r) &= 1-\left(\lambda_2 \sum_{j=1}^{m+1} E(\rho_{j,\delta})\widehat{T}_{\rho_{j,\delta}} f_2(r)\right)\widehat{\nu}^*_{\alpha,\delta}(r) \\ &= \sum_{j=1}^{m+1} E(\rho_{j,\delta})\left(1-\lambda_2 \widehat{T}_{\rho_{j,\delta}} f_2(r)\widehat{\nu}^*_{\alpha,\delta}(r)\right), \end{aligned} \tag{A.24}$$

and due to (A.23) and (A.24),

$$\begin{aligned} \frac{1}{r}\left[1-\widehat{g}_\delta(r)\widehat{\nu}^*_{\alpha,\delta}(r)-\delta R\widehat{\nu}^*_{\alpha,\delta}(r)\right] = \frac{1}{r}\sum_{j=1}^{m+1} E(\rho_{j,\delta}) \Bigg\{ 1-\lambda_2\widehat{T}_{\rho_{j,\delta}} f_2(r)\widehat{\nu}^*_{\alpha,\delta}(r) \\ - \left[\frac{\lambda_2\left(\widehat{f}_2(\rho_{j,\delta})-1\right)}{\rho_{j,\delta}} + c + \eta^{\alpha}\rho_{j,\delta}^{\alpha-1}\right]\widehat{\nu}^*_{\alpha,\delta}(r)\Bigg\}. \end{aligned} \tag{A.25}$$

Using the equality $\widehat{\nu}^*_{\alpha,\delta}(r) = \dfrac{1}{c+\eta^{\alpha}r^{\alpha-1}+\eta^{\alpha}\widehat{f}_{\alpha,\delta}(r)}$ and Lemma 6, we get:

$$\sum_{j=1}^{m+1} E(\rho_{j,\delta}) \left[c+\eta^{\alpha}r^{\alpha-1} + \eta^{\alpha}\rho_{j,\delta}\frac{\rho_{j,\delta}^{\alpha-1}-r^{\alpha-1}}{\rho_{j,\delta}-r} \right]\widehat{\nu}^*_{\alpha,\delta}(r) = 1. \tag{A.26}$$

From (A.26) and (A.25) we obtain, for $r \neq \rho_{j,\delta}$:

$$\begin{aligned}
&\frac{1}{r}\left[1-\widehat{g}_\delta(r)\widehat{\nu}^*_{\alpha,\delta}(r)-\delta R\widehat{\nu}^*_{\alpha,\delta}(r)\right] \\
&=\frac{1}{r}\sum_{j=1}^{m+1}E(\rho_{j,\delta})\left[c+\eta^\alpha r^{\alpha-1}+\eta^\alpha\rho_{j,\delta}\frac{\rho_{j,\delta}^{\alpha-1}-r^{\alpha-1}}{\rho_{j,\delta}-r}\right]\widehat{\nu}^*_{\alpha,\delta}(r) \\
&-\frac{1}{r}\lambda_2\sum_{j=1}^{m+1}E(\rho_{j,\delta})\widehat{T}_{\rho_{j,\delta}}f_2(r)\widehat{\nu}^*_{\alpha,\delta}(r) \\
&-\frac{1}{r}\sum_{j=1}^{m+1}E(\rho_{j,\delta})\left(\frac{\lambda_2(\widehat{f}_2(\rho_{j,\delta})-1)}{\rho_{j,\delta}}+c+\eta^\alpha\rho_{j,\delta}^{\alpha-1}\right)\widehat{\nu}^*_{\alpha,\delta}(r) \\
&=\frac{1}{r}\sum_{j=1}^{m+1}E(\rho_{j,\delta})\left\{\left(\eta^\alpha r\frac{\rho_{j,\delta}^{\alpha-1}-r^{\alpha-1}}{\rho_{j,\delta}-r}+\lambda_2 r\frac{\frac{1-\widehat{f}_2(r)}{r}-\frac{1-\widehat{f}_2(\rho_{j,\delta})}{\rho_{j,\delta}}}{\rho_{j,\delta}-r}\right)\widehat{\nu}^*_{\alpha,\delta}(r)\right\} \\
&=\sum_{j=1}^{m+1}E(\rho_{j,\delta})\left(\eta^\alpha\widehat{T}_{\rho_{j,\delta}}\ell_\alpha(r)+\lambda_2\widehat{T}_{\rho_{j,\delta}}\overline{F}_2(r)\right)\widehat{\nu}^*_{\alpha,\delta}(r).
\end{aligned}$$

Since $\xi_\omega(u)=\overline{F}_2(u)$ when $\omega(x,y)=1$, from (2.3), (2.1) and the above equality we obtain

$$\begin{aligned}
\widehat{h}_{\alpha,\delta,\omega}(r) &= \sum_{j=1}^{m+1}E(\rho_{j,\delta})\left(\eta^\alpha\widehat{T}_{\rho_{j,\delta}}\ell_\alpha(r)+\lambda_2\widehat{T}_{\rho_{j,\delta}}\overline{F}_2(r)\right) \\
&= \frac{1}{r}\frac{\left[1-\widehat{g}_\delta(r)\widehat{\nu}^*_{\alpha,\delta}(r)-\delta R\widehat{\nu}^*_{\alpha,\delta}(r)\right]}{\widehat{\nu}^*_{\alpha,\delta}(r)},
\end{aligned}$$

and (A.21) is proved. From (A.21) and (A.20) it follows that

$$\begin{aligned}
\widehat{\varphi}_\delta(r) &= \frac{1}{r}\frac{1-\widehat{g}_\delta(r)\widehat{\nu}^*_{\alpha,\delta}(r)-\delta R\widehat{\nu}^*_{\alpha,\delta}(r)}{1-\widehat{g}_\delta(r)\widehat{\nu}^*_{\alpha,\delta}(r)} = \frac{1}{r}-\frac{1}{r}\left[\frac{\delta R\widehat{\nu}^*_{\alpha,\delta}(r)}{1-\widehat{g}_\delta(r)\widehat{\nu}^*_{\alpha,\delta}(r)}\right] \\
&= \frac{1}{r}-\frac{1}{r}\left[\frac{\frac{\delta R}{\eta^\alpha}}{\theta_\delta+r^{\alpha-1}+\widehat{f}_{\alpha,\delta}(r)-\eta^{-\alpha}\widehat{g}_\delta(r)-\kappa_\delta}\right]
\end{aligned}$$

$$= \frac{1}{r} - \frac{1}{r}\left[\frac{\frac{\delta R}{\eta^{\alpha}\theta_\delta}\widehat{\nu}_{\alpha,\delta}(r)}{1-\frac{1}{\theta_\delta}\left[\eta^{-\alpha}\widehat{g}_\delta(r)+\kappa_\delta\right]\widehat{\nu}_{\alpha,\delta}(r)}\right] = \frac{1}{r} - \frac{1}{r}\delta R\widehat{W}_\delta(r),$$

where the last equality follows from (2.7). This proves d).

■

Proof of Proposition 4.

a) Let us define $F^*(u) = \int_0^u \overline{F}_\alpha(x)dx$. From (2.22) we obtain $\widehat{F}^*(r) = \frac{1-\frac{1}{\mathcal{C}_F}\widehat{f}_\alpha(r)}{r^2}$.

The idea now is to use Lemma 9 to prove the assertion in a). For this, we consider the limit $\lim_{r\downarrow 0}\frac{r\widehat{F}^*(r)}{r^{\alpha-2}}$, which by the equality above is equivalent to $\lim_{r\downarrow 0}\frac{1-\frac{1}{\mathcal{C}_F}\widehat{f}_{\alpha,0}(r)}{r^{\alpha-1}}$.

By definition of C_F it follows that $1-\frac{1}{C_F}\widehat{f}_\alpha(0)=0$, hence we use LH́opital's rule to obtain:

$$\begin{aligned}
\lim_{r\downarrow 0}\frac{1-\frac{1}{\mathcal{C}_F}\widehat{f}_{\alpha,0}(r)}{r^{\alpha-1}} &= \lim_{r\downarrow 0}\frac{1+\dfrac{1}{\mathcal{C}_F}\sum_{j=2}^{m+1}E(\rho_j)\rho_j\dfrac{\rho_j^{\alpha-1}-r^{\alpha-1}}{\rho_j-r}}{r^{\alpha-1}} \\
&= \lim_{r\downarrow 0}\frac{\dfrac{1}{\mathcal{C}_F}\sum_{j=2}^{m+1}E(\rho_j)\rho_j\left(\dfrac{\rho_j^{\alpha-1}-r^{\alpha-1}}{(\rho_j-r)^2}-\dfrac{(\alpha-1)r^{\alpha-2}}{\rho_j-r}\right)}{(\alpha-1)r^{\alpha-2}} \\
&= \lim_{r\downarrow 0}\frac{\dfrac{1}{\mathcal{C}_F}\sum_{j=2}^{m+1}E(\rho_j)\left(\dfrac{r^{2-\alpha}\rho_j^{\alpha-1}-r}{(\rho_j-r)^2}-\dfrac{(\alpha-1)}{\rho_j-r}\right)}{(\alpha-1)} \\
&= -\frac{1}{\mathcal{C}_F}\sum_{j=2}^{m+1}E(\rho_j) = \frac{1}{\mathcal{C}_F}\left(1-\frac{\prod_{i=1}^{N}q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}\right)
\end{aligned}$$

where the fifth equality follows by taking limits as $\delta\downarrow 0$ in $\sum_{j=1}^{m+1}E(\rho_j(\delta))=1$. By Lemma 9 we obtain (2.23). Since (2.23) implies that $\overline{F}_\alpha(x)$ is regularly varying, and all regularly varying distributions are also subexponential, we also obtain that $F_\alpha(x)\in\mathcal{S}$.

b) For any $\rho_j, j = 2, 3, \ldots, m+1$, we have

$$\lim_{x\to\infty} \left| \frac{\int_x^\infty T_{\rho_j} f_2(y)dy}{\overline{F}_{2,I}(x)} \right| \leq \lim_{x\to\infty} \frac{\int_x^\infty \int_y^\infty e^{-Re(\rho_j)(z-y)} f_2(z)dzdy}{\overline{F}_{2,I}(x)} \tag{A.27}$$

Taking limits when $x \to \infty$ in the right-hand side of (A.27) yields:

$$\begin{aligned}
\lim_{x\to\infty} \frac{\int_x^\infty e^{Re(\rho_j)y} \int_y^\infty e^{-Re(\rho_j)z} f_2(z)dzdy}{\overline{F}_{2,I}(x)} &= \lim_{x\to\infty} \frac{\int_x^\infty e^{-Re(\rho_j)z} f_2(z)dzdy}{e^{-Re(\rho_j)x}\overline{F}_2(x)} \\
&= \lim_{x\to\infty} \frac{e^{-Re(\rho_j)x} f_2(x)}{Re(\rho_j)e^{-Re(\rho_j)x}\overline{F}_2(x) + e^{-Re(\rho_j)x} f_2(x)} \\
&= \lim_{x\to\infty} \frac{\frac{f_2(x)}{\overline{F}_2(x)}}{Re(\rho_j) + \frac{f_2(x)}{\overline{F}_2(x)}}, \qquad \text{(A.28)}
\end{aligned}$$

where the first and second equalities follow by L'Hopital's rule. Using the assumption that $F_2 \in \mathcal{R}_0$, we obtain from (A.28) and (A.27) that

$$\lim_{x\to\infty} \left| \frac{\int_x^\infty T_{\rho_j} f_2(y)dy}{\overline{F}_{2,I}(x)} \right| = 0. \tag{A.29}$$

Since $\int_x^\infty g_0(y)dy = \lambda_2 \frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1} \rho_j} \mu_2 \overline{F}_{2,I}(x) - \lambda_2 \sum_{j=2}^{m+1} E(\rho_j) \int_x^\infty T_{\rho_j} f_2(y)$, the triangle inequality yields:

$$\frac{\lambda_2\mu_2}{\mathcal{C}_G} \frac{\prod_{i=1}^{N} q_i^{m_i}}{\prod_{j=2}^{m+1} \rho_{j,0}} - \left| \frac{-\frac{\lambda_2}{C_G} \sum_{j=2}^{m+1} E(\rho_j) \int_x^\infty T_{\rho_j} f_2(y)dy}{\overline{F}_{2,I}(x)} \right| \leq \left| \frac{\overline{G}_0(x)}{\overline{F}_{2,I}(x)} \right| \tag{A.30}$$

and

$$\left|\frac{\overline{G}_0(x)}{\overline{F}_{2,I}(x)}\right| \le \frac{\lambda_2\mu_2}{\mathcal{C}_G}\frac{\prod_{i=1}^{N} q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}} + \left|\frac{-\frac{\lambda_2}{C_G}\sum_{j=2}^{m+1} E(\rho_j)\int_x^\infty T_{\rho_j} f_2(y)dy}{\overline{F}_{2,I}(x)}\right|. \tag{A.31}$$

Taking limits when $x \to \infty$ in (A.30) and (A.31) and using (A.29), we obtain (2.24). Assuming that $F_{2,I} \in \mathcal{S}$, we obtain $G_0 \in \mathcal{S}$ from Lemma 8 c).

c) Let us assume that $\overline{F}_2(x) = o(x^{-\alpha})$, hence L'Hospital's rule implies that

$$\lim_{x\to\infty}\frac{\overline{F}_{2,I}(x)}{x^{1-\alpha}} = 0 \tag{A.32}$$

This yields

$$\lim_{x\to\infty}\frac{\int_x^\infty\int_y^\infty e^{-Re(\rho_j)(z-y)} f_2(z)dzdy}{x^{1-\alpha}} \le \lim_{x\to\infty}\frac{\int_x^\infty\int_y^\infty f_2(z)dzdy}{x^{1-\alpha}} = \mu_2 \lim_{x\to\infty}\frac{\overline{F}_{2,I}(x)}{x^{1-\alpha}} = 0,$$

so we obtain from (A.32):

$$\lim_{x\to\infty}\left|\frac{\int_x^\infty T_{\rho_j} f_2(y)dy}{x^{1-\alpha}}\right| = 0. \tag{A.33}$$

Replacing $\overline{F}_{2,I}(x)$ by $x^{1-\alpha}$ in (A.30) and (A.31) we obtain the inequalities:

$$\frac{\mu_2}{\mathcal{C}_G}\frac{\prod_{i=1}^{N} q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}\frac{\overline{F}_{2,I}(x)}{x^{1-\alpha}} - \left|\frac{-\frac{1}{C_G}\sum_{j=2}^{m+1} E(\rho_j)\int_x^\infty T_{\rho_j} f_2(y)dy}{x^{1-\alpha}}\right| \le \left|\frac{\overline{G}_0(x)}{x^{1-\alpha}}\right| \tag{A.34}$$

and

$$\left|\frac{\overline{G}_0(x)}{x^{1-\alpha}}\right| \leq \frac{\mu_2}{\mathcal{C}_G}\frac{\prod\limits_{i=1}^{N} q_i^{m_i}}{\prod\limits_{j=2}^{m+1}\rho_{j,0}}\frac{\overline{F}_{2,I}(x)}{x^{1-\alpha}} + \left|\frac{-\frac{1}{C_G}\sum_{j=2}^{m+1} E(\rho_j)\int\limits_x^\infty T_{\rho_j} f_2(y)dy}{x^{1-\alpha}}\right| \tag{A.35}$$

Hence, the result follows by taking limits when $x \to \infty$ in (A.34) and (A.35), and applying (A.32) and (A.33).

d) Putting $r = 0$ in (2.19) we obtain $C_U = \frac{1}{1+\frac{C_F}{\theta}}$. Multiplying both sides in (2.19) by C_U we obtain $\frac{\widehat{\nu}_{\alpha,0}(r)}{\mathcal{C}_U}\left(1+\frac{1}{\theta}\widehat{f}_{\alpha,0}(r)\widehat{z}_{\alpha,\theta_0}(r)\right) = \frac{\widehat{z}_{\alpha,\theta_0}(r)}{\mathcal{C}_U}$

$$\begin{aligned}\left(1-\frac{\widehat{\nu}_{\alpha,0}(r)}{\mathcal{C}_U}\right)\left(1+\frac{1}{\theta}\widehat{f}_{\alpha,0}(r)\widehat{z}_{\alpha,\theta_0}(r)\right) &= 1+\frac{1}{\theta}\widehat{f}_{\alpha,0}(r)\widehat{z}_{\alpha,\theta_0}(r) - \frac{\widehat{z}_{\alpha,\theta_0}(r)}{\mathcal{C}_U}\\ &= 1+\frac{1}{\theta}\widehat{f}_{\alpha,0}(r)\widehat{z}_{\alpha,\theta_0}(r) - \left(1+\frac{\mathcal{C}_F}{\theta}\right)\widehat{z}_{\alpha,\theta_0}(r)\\ &= 1-\widehat{z}_{\alpha,\theta_0}(r) - \frac{\mathcal{C}_F}{\theta}\widehat{z}_{\alpha,\theta_0}(r)\left(1-\frac{1}{\mathcal{C}_F}\widehat{f}_{\alpha,0}(r)\right)\end{aligned} \tag{A.36}$$

We define the function $U_\alpha^*(x) = \int\limits_0^x \overline{U}_\alpha(y)dy, x > 0$. By (2.22), the Laplace transform of this function satisfies the equalities:

$$\widehat{U}_\alpha^*(r) = \frac{\widehat{\overline{U}}_\alpha(r)}{r} = \frac{1-\frac{\widehat{\nu}_{\alpha,0}(r)}{C_U}}{r^2}. \tag{A.37}$$

Hence (A.36) yields:

$$\frac{r\widehat{U}_\alpha^*(r)}{r^{\alpha-2}} = \frac{\frac{1-\widehat{z}_{\alpha,\theta_0}(r)}{r^{\alpha-1}} - \frac{\mathcal{C}_F}{\theta}\frac{\widehat{z}_{\alpha,\theta_0}(r)\left(1-\frac{1}{\mathcal{C}_F}\widehat{f}_{\alpha,0}(r)\right)}{r^{\alpha-1}}}{1+\frac{1}{\theta}\widehat{f}_{\alpha,0}(r)\widehat{z}_{\alpha,\theta_0}(r)} \tag{A.38}$$

Using that $\widehat{z}_{\alpha,\theta}(r) = \frac{\theta}{\theta+r^{\alpha-1}}$ we obtain $\lim_{r\downarrow 0}\frac{1-\widehat{z}_{\alpha,\theta}(r)}{r^{\alpha-1}} = \lim_{r\downarrow 0}\frac{1}{\theta+r^{\alpha-1}} = \frac{1}{\theta}$, and from the proof of a) we have $\lim_{r\downarrow 0}\frac{1-\frac{1}{\mathcal{C}_F}\widehat{f}_{\alpha,0}(r)}{r^{\alpha-1}}=\frac{1}{\mathcal{C}_F}\left(1-\frac{\prod_{i=1}^{N}q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}\right)$.

Letting $r\downarrow 0$ in (A.38) and using the two equalities above, we obtain:

$$\lim_{r\downarrow 0}\frac{r\widehat{U}^*_\alpha(r)}{r^{\alpha-2}}=\frac{\frac{1}{\theta}-\frac{1}{\theta}+\frac{1}{\theta}\frac{\prod_{i=1}^{N}q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}}{1+\frac{\mathcal{C}_F}{\theta}}=\frac{\mathcal{C}_U}{\theta}\frac{\prod_{i=1}^{N}q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}, \tag{A.39}$$

where in the last equality we used that $C_U=\frac{1}{1+\frac{\mathcal{C}_F}{\theta}}$.

Since $U^*_\alpha(x)$ has the monotone density $\overline{U}_\alpha(x)$, Lemma 9, gives (2.25). This implies that the tail of U_α is asymptotically regularly varying with index $1-\alpha$, hence we conclude that $U_\alpha\in\mathcal{S}$.

■

Proof of Lemma 15. First we prove that

$$\int_{0+}^{\infty}\left|1-e^{-rx}\right|\overline{\mathcal{V}}_{\mathcal{S}}(x)dx<\infty \tag{A.40}$$

for any $r\in\mathbb{C}_+$. We have:

$$\begin{aligned}\int_{0+}^{\infty}\left|1-e^{-rx}\right|\overline{\mathcal{V}}_{\mathcal{S}}(x)dx&=\int_{0+}^{1}\left|1-e^{-rx}\right|\overline{\mathcal{V}}_{\mathcal{S}}(x)dx+\int_{1}^{\infty}\left|1-e^{-rx}\right|\overline{\mathcal{V}}_{\mathcal{S}}(x)dx\\&\le\int_{0+}^{1}\left|1-e^{-rx}\right|\overline{\mathcal{V}}_{\mathcal{S}}(x)dx+\int_{1}^{\infty}\overline{\mathcal{V}}_{\mathcal{S}}(x)dx\\&\le\sum_{k=1}^{\infty}\frac{|r|^k}{k!}\int_{0+}^{1}x^k\overline{\mathcal{V}}_{\mathcal{S}}(x)dx+\int_{1}^{\infty}\overline{\mathcal{V}}_{\mathcal{S}}(x)dx\end{aligned}$$

$$\le \sum_{k=1}^{\infty} \frac{|r|^k}{k!} \int_{0+}^{1} x\overline{\mathcal{V}}_{\mathcal{S}}(x)dx + \int_{1}^{\infty} \overline{\mathcal{V}}_{\mathcal{S}}(x)dx$$

$$= \left(e^{|r|} - 1\right) \int_{0+}^{1} x\overline{\mathcal{V}}_{\mathcal{S}}(x)dx + \int_{1}^{\infty} \overline{\mathcal{V}}_{\mathcal{S}}(x)dx \tag{A.41}$$

Since $x\overline{\mathcal{V}}_{\mathcal{S}}(x) \ge 0$, by Fubini's theorem we have

$$\int_{0+}^{1} x\overline{\mathcal{V}}_{\mathcal{S}}(x)dx = \int_{0+}^{1} x \int_{x}^{\infty} \nu_{\mathcal{S}}(dy)dx = \int_{0+}^{1} \int_{0}^{y} xdx\nu_{\mathcal{S}}(dy) + \int_{1}^{\infty} \int_{0+}^{1} xdx\nu_{\mathcal{S}}(dy)$$

$$= \int_{0+}^{1} \frac{y^2}{2}\nu_{\mathcal{S}}(dy) + \frac{1}{2}\int_{1}^{\infty} \nu_{\mathcal{S}}(dy),$$

where both integrals are finite because of condition $\int_{0+}^{\infty} \left(x^2 \wedge x\right) \nu_{\mathcal{S}}(dx)$, which implies $\int_{0+}^{1} x^2\nu_{\mathcal{S}}(dx) < \infty$ and $\int_{1}^{\infty} \nu_{\mathcal{S}}(dx) \le \int_{1}^{\infty} x\nu_{\mathcal{S}}(dx) < \infty$.

Now we consider the integral $\int_{1}^{\infty} \overline{\mathcal{V}}_{\mathcal{S}}(x)dx$. We apply integration by parts and obtain:

$$\int_{1}^{\infty} \overline{\mathcal{V}}_{\mathcal{S}}(x)dx = x\overline{\mathcal{V}}_{\mathcal{S}}(x)\Big|_{x=1}^{\infty} + \int_{1}^{\infty} x\nu_{\mathcal{S}}(dx)$$

We note that $x\overline{\mathcal{V}}_{\mathcal{S}}(x) = \int_{x}^{\infty} x\nu_{\mathcal{S}}(dy) \le \int_{x}^{\infty} y\nu_{\mathcal{S}}(dy)$, and since the right-hand side of this inequality is finite for $x \ge 1$ by the condition $\int_{0+}^{\infty} \left(x^2 \wedge x\right) \nu_{\mathcal{S}}(dx)$ and it tends to zero as $x \to \infty$, it follows that $\lim_{x\to\infty} x\overline{\mathcal{V}}_{\mathcal{S}}(x) = 0$. Hence

$$\int_{1}^{\infty} \overline{\mathcal{V}}_{\mathcal{S}}(x)dx = \int_{1}^{\infty} x\nu_{\mathcal{S}}(dx) - \overline{\mathcal{V}}_{\mathcal{S}}(1) = \int_{1}^{\infty} (x-1)\nu_{\mathcal{S}}(dx) \in [0, \infty).$$

This proves that the right-hand side in (A.41) is finite. Therefore we obtain (3.13) using Fubini's theorem. To prove (3.12) we have

$$\begin{aligned}
\frac{\Psi_{\mathcal{S}}(r_1)}{r_1} - \frac{\Psi_{\mathcal{S}}(r_2)}{r_2} &= \int_{0+}^{\infty} \left[\frac{1-e^{-r_1 x} - r_1 x}{r_1} - \frac{1-e^{-r_2 x} - r_2 x}{r_2}\right] \nu_{\mathcal{S}}(dx) \\
&= \int_{0+}^{\infty} \int_{0}^{x} \left[e^{-r_1 y} - 1 - \left(e^{-r_2 y} - 1\right)\right] dy \nu_{\mathcal{S}}(dx) \\
&= \int_{0+}^{\infty} \int_{y}^{\infty} \nu_{\mathcal{S}}(dx) \left[e^{-r_1 y} - e^{-r_2 y}\right] dy \\
&= \int_{0+}^{\infty} \overline{\mathcal{V}}_{\mathcal{S}}(y) \left[e^{-r_1 y} - e^{-r_2 y}\right] dy = \widehat{\overline{\mathcal{V}}}_{\mathcal{S}}(r_1) - \widehat{\overline{\mathcal{V}}}_{\mathcal{S}}(r_2) \qquad \text{(A.42)}
\end{aligned}$$

where the third equality follows by changing the order of integration, which is possible because of (A.40). Now we note that:

$$\begin{aligned}
\frac{\Psi_{\mathcal{S}}(r_1) - \Psi_{\mathcal{S}}(r_2)}{r_2 - r_1} &= \frac{r_1 \frac{\Psi_{\mathcal{S}}(r_1)}{r_1} - r_2 \frac{\Psi_{\mathcal{S}}(r_2)}{r_2} \pm r_2 \frac{\Psi_{\mathcal{S}}(r_1)}{r_1}}{r_2 - r_1} = \frac{r_1 - r_2}{r_2 - r_1} \frac{\Psi_{\mathcal{S}}(r_1)}{r_1} + r_2 \frac{\frac{\Psi_{\mathcal{S}}(r_1)}{r_1} - \frac{\Psi_{\mathcal{S}}(r_2)}{r_2}}{r_2 - r_1} \\
&= r_2 \frac{\frac{\Psi_{\mathcal{S}}(r_1)}{r_1} - \frac{\Psi_{\mathcal{S}}(r_2)}{r_2}}{r_2 - r_1} - \frac{\Psi_{\mathcal{S}}(r_1)}{r_1}. \qquad \text{(A.43)}
\end{aligned}$$

and analogously:

$$\frac{\Psi_{\mathcal{S}}(r_1) - \Psi_{\mathcal{S}}(r_2)}{r_2 - r_1} = r_1 \frac{\frac{\Psi_{\mathcal{S}}(r_1)}{r_1} - \frac{\Psi_{\mathcal{S}}(r_2)}{r_2}}{r_2 - r_1} - \frac{\Psi_{\mathcal{S}}(r_2)}{r_2}. \qquad \text{(A.44)}$$

Hence the result follows substituting (A.42) in (A.43) and (A.44) and using (1.5). ■

Proof of Lemma 16. We construct $m+1$ different numbers depending on $\varepsilon > 0$ and such that k_1 of these numbers converge to $\rho_{1,\delta}$, k_2 of these numbers, different from the previous k_1, converge to $\rho_{2,\delta}$, $\ldots$, k_R numbers different than the previous $m + 1 - \sum_{j=1}^{R-1} k_j$ converge to $\rho_{R,\delta}$. For this we take $\varepsilon \in (0, E)$, where $E =$

$\min\{|Re\rho_{i,\delta} - Re\rho_{j,\delta}| : Re\rho_{i,\delta} - Re\rho_{j,\delta} \neq 0\}$ and consider the complex numbers

$$\begin{aligned}
&\rho^*_{1,\delta}(\varepsilon) = \rho_{1,\delta},\ \rho^*_{2,\delta}(\varepsilon) = \rho_{1,\delta} + \frac{\varepsilon}{m+1} \dots, \rho^*_{k_1,\delta}(\varepsilon) = \rho_{1,\delta} + \frac{k_1-1}{m+1}\varepsilon,\\
&\rho^*_{k_1+1,\delta}(\varepsilon) = \rho_{2,\delta},\ \rho^*_{k_1+2,\delta}(\varepsilon) = \rho_{2,\delta} + \frac{k_1+1}{m+1}\varepsilon, \dots, \rho^*_{k_1+k_2,\delta}(\varepsilon) = \rho_{\delta} + \frac{k_1+k_2-1}{m+1}\varepsilon, \dots\\
&\vdots\\
&\rho^*_{k_1+\dots+k_{R-1}+1,\delta}(\varepsilon) = \rho_{R,\delta},\ \rho^*_{k_1+\dots+k_{R-1}+2,\delta}(\varepsilon) = \rho_{R,\delta} + \frac{\sum_{j=1}^{R-1} k_j + 1}{m+1}\varepsilon, \dots,\\
&\rho^*_{m+1,\delta}(\varepsilon) = \rho_{R,\delta} + \frac{\sum_{j=1}^{R} k_j - 1}{m+1}\varepsilon,
\end{aligned} \tag{A.45}$$

Clearly we have:

$$\lim_{\varepsilon\to 0} \rho^*_{l_j+a_j,\delta}(\varepsilon) = \rho_{j,\delta}, j = 1, 2, \dots, R,$$

for $l_1 = 0, l_2 = k_1, \dots, l_R = k_{R-1}$ and $a_j = 1, 2, \dots, k_j$.

This gives $m+1$ distinct numbers $\rho^*_{1,\delta}(\varepsilon), \rho^*_{2,\delta}(\varepsilon), \dots, \rho^*_{m+1,\delta}(\varepsilon)$ such that, as $\varepsilon \downarrow 0$, the first k_1 converge to $\rho_{1,\delta}$, the next k_2 converge to $\rho_{2,\delta}$, etc. This construction plays a critical role in the proof of Lemma 16. The following technical lemma is also required. First we obtain the result for case C. Using the complex numbers $\rho^*_{1,\delta}(\varepsilon), \dots \rho^*_{m+1,\delta}(\varepsilon)$ defined in (A.45), first we have the equality

$$\begin{aligned}
L_{\mathcal{X}}\left[\rho^*_{j,\delta}(\varepsilon)\right] - \delta = &- \Psi_{\mathcal{S}}\left[\rho^*_{j,\delta}(\varepsilon)\right] + \lambda_1 \frac{Q(-\rho^*_{j,\delta}(\varepsilon))}{\prod_{j=1}^{N}\left(q_j - \rho^*_{j,\delta}(\varepsilon)\right)^{m_j}}\\
&+ c\rho^*_{j,\delta}(\varepsilon) + \gamma^2\left[\rho^*_{j,\delta}(\varepsilon)\right]^2 - \delta - \lambda_1.
\end{aligned}$$

Therefore, for each $j = 1, 2, \dots, m+1$ we obtain:

$$\begin{aligned}
\lambda_1 + \delta = &- \Psi_{\mathcal{S}}\left[\rho^*_{j,\delta}(\varepsilon)\right] + \lambda_1 \frac{Q(-\rho^*_{j,\delta}(\varepsilon))}{\prod_{j=1}^{N}\left(q_j - \rho^*_{j,\delta}(\varepsilon)\right)^{m_j}}\\
&+ c\rho^*_{j,\delta}(\varepsilon) + \gamma^2\left[\rho^*_{j,\delta}(\varepsilon)\right]^2 - \left(L_{\mathcal{X}}\left[\rho^*_{j,\delta}(\varepsilon)\right] - \delta\right).
\end{aligned}$$

This yields:

$$L_{\mathcal{X}}(r)-\delta = -\Psi_{\mathcal{S}}(r)+\lambda_1\frac{Q(-r)}{\prod_{j=1}^{N}(q_j-r)^{m_j}}+cr+\gamma^2r^2+\Psi_{\mathcal{S}}\left[\rho^*_{j,\delta}(\varepsilon)\right]$$
$$-\lambda_1\frac{Q(-\rho^*_{j,\delta}(\varepsilon))}{\prod\limits_{j=1}^{N}\left(q_j-\rho^*_{j,\delta}(\varepsilon)\right)^{m_j}}-c\rho^*_{j,\delta}(\varepsilon)-\gamma^2\left[\rho^*_{j,\delta}(\varepsilon)\right]^2+L_{\mathcal{X}}\left[\rho^*_{j,\delta}(\varepsilon)\right]-\delta. \tag{A.46}$$

First we prove the result for case C. Let us recall that $Q_1(r)=\prod\limits_{j=1}^{N}(q_j-r)^{m_j}$ for $r\in\mathbb{C}_+$. Since this polynomial has degree m, it admits the equivalent representation

$$Q_1(r)=\sum_{l=1}^{m+1}\frac{\prod\limits_{j=1}^{N}\left[q_j-\rho^*_{l,\delta}(\varepsilon)\right]^{m_j}}{\prod\limits_{j\neq l}\left[\rho^*_{j,\delta}(\varepsilon)-\rho^*_{l,\delta}(\varepsilon)\right]}\prod_{j\neq l}\left[\rho^*_{j,\delta}(\varepsilon)-r\right],$$

which follows from Lagrange interpolation. This and (A.46) give:

$$\left[L_{\mathcal{X}}(r)-\delta\right]Q_1(r)=\sum_{l=1}^{m+1}\frac{Q_1\left[\rho^*_{l,\delta}(\varepsilon)\right]}{\prod\limits_{j\neq l}\left[\rho^*_{j,\delta}(\varepsilon)-\rho^*_{l,\delta}(\varepsilon)\right]}\prod_{j\neq l}\left[\rho^*_{j,\delta}(\varepsilon)-r\right]\Bigg\{\left[-(\rho^*_{l,\delta}(\varepsilon)-r)\right]$$
$$\times\left[\frac{\Psi_{\mathcal{S}}(r)-\Psi_{\mathcal{S}}(\rho^*_{l,\delta}(\varepsilon))}{\rho^*_{l,\delta}(\varepsilon)-r}+\lambda_1\frac{\frac{Q(-r)}{\prod_{j=1}^{N}(q_j-r)^{m_j}}-\frac{Q\left(\rho^*_{l,\delta}(\varepsilon)\right)}{\prod_{j=1}^{N}\left(q_j-\rho^*_{l,\delta}(\varepsilon)\right)^{m_j}}}{\rho^*_{l,\delta}(\varepsilon)-r}\right]$$
$$-c(\rho^*_{l,\delta}(\varepsilon)-r)-\gamma^2(\rho^*_{l,\delta}(\varepsilon)-r)(\rho^*_{l,\delta}(\varepsilon)+r)$$
$$-\eta^\alpha(\rho^*_{l,\delta}(\varepsilon)-r)\frac{\left[\rho^*_{l,\delta}(\varepsilon)\right]^\alpha-r^\alpha}{(\rho^*_{l,\delta}(\varepsilon)-r)}+\left[L_{\mathcal{X}}(\rho^*_{l,\delta}(\varepsilon))-\delta\right]\Bigg\}$$
$$=\prod_{j=1}^{m+1}\left[\rho^*_{j,\delta}(\varepsilon)-r\right]\sum_{l=1}^{m+1}\frac{Q_1\left[\rho^*_{l,\delta}(\varepsilon)\right]}{\prod\limits_{j\neq l}\left[\rho^*_{j,\delta}(\varepsilon)-\rho^*_{l,\delta}(\varepsilon)\right]}\Bigg\{-\frac{\Psi_{\mathcal{S}}(r)-\Psi_{\mathcal{S}}(\rho^*_{l,\delta}(\varepsilon))}{\rho^*_{l,\delta}(\varepsilon)-r}$$
$$+\lambda_1\frac{\frac{Q(-r)}{\prod_{j=1}^{N}(q_j-r)^{m_j}}-\frac{Q\left(\rho^*_{l,\delta}(\varepsilon)\right)}{\prod_{j=1}^{N}\left(q_j-\rho^*_{l,\delta}(\varepsilon)\right)^{m_j}}}{\rho^*_{l,\delta}(\varepsilon)-r}-c-\gamma^2(\rho^*_{l,\delta}(\varepsilon)+r)$$

$$-\eta^{\alpha}\frac{\left[\rho^*_{l,\delta}(\varepsilon)\right]^{\alpha}-r^{\alpha}}{(\rho^*_{l,\delta}(\varepsilon)-r)}+\frac{L_{\mathcal{X}}(\rho^*_{l,\delta}(\varepsilon))-\delta}{\rho^*_{l,\delta}(\varepsilon)-r}\Bigg\}. \tag{A.47}$$

Formula (2.5) and the first part of the proof of Proposition 2 imply, respectively:

$$\begin{gathered}\sum_{l=1}^{m+1}\frac{Q_1\left[\rho^*_{l,\delta}(\varepsilon)\right]}{\prod\limits_{j\neq l}\left[\rho^*_{j,\delta}(\varepsilon)-\rho^*_{l,\delta}(\varepsilon)\right]}=1\\ \text{and}\\ \lambda_1\sum_{l=1}^{m+1}\frac{Q_1\left[\rho^*_{l,\delta}(\varepsilon)\right]}{\prod\limits_{j\neq l}\left[\rho^*_{j,\delta}(\varepsilon)-\rho^*_{l,\delta}(\varepsilon)\right]}\left[\frac{\frac{Q(-r)}{\prod_{j=1}^{N}\left(q_j-r\right)^{m_j}}-\frac{Q\left(\rho^*_{l,\delta}(\varepsilon)\right)}{\prod_{j=1}^{N}\left(q_j-\rho^*_{l,\delta}(\varepsilon)\right)^{m_j}}}{\rho^*_{l,\delta}(\varepsilon)-r}\right]=0,\ r\in\mathbb{C}_+.\end{gathered} \tag{A.48}$$

Hence, substituting these two equalities in (A.47) and applying the first equality in (3.12) to $\frac{\Psi_{\mathcal{S}}(r)-\Psi_{\mathcal{S}}(\rho^*_{j,\delta}(\varepsilon))}{\rho^*_{j,\delta}(\varepsilon)-r}$, it follows that

$$\begin{aligned}\left[L_{\mathcal{X}}(r)-\delta\right]Q_1(r)=\prod_{j=1}^{m+1}\left[\rho^*_{j,\delta}(\varepsilon)-r\right]\sum_{l=1}^{m+1}\frac{Q_1\left[\rho^*_{l,\delta}(\varepsilon)\right]}{\prod\limits_{j\neq l}\left[\rho^*_{j,\delta}(\varepsilon)-\rho^*_{l,\delta}(\varepsilon)\right]}\Bigg\{-\rho^*_{l,\delta}(\varepsilon)\widehat{T}_{\rho^*_{l,\delta}(\varepsilon)}\overline{\mathcal{V}}_{\mathcal{S}}(r)\\ +\frac{\Psi_{\mathcal{S}}(r)}{r}-c-\gamma^2\left[r+\rho^*_{l,\delta}(\varepsilon)\right]+\frac{L_{\mathcal{X}}\left[\rho^*_{l,\delta}(\varepsilon)\right]-\delta}{\rho^*_{l,\delta}(\varepsilon)-r}\Bigg\}.\end{aligned}$$

Dividing both sides in the equality above by $\prod_{j=1}^{m+1}\left[\rho^*_{j,\delta}(\varepsilon)-r\right]$ we obtain

$$\begin{aligned}\frac{\left[L_{\mathcal{X}}(r)-\delta\right]Q_1(r)}{\prod_{j=1}^{m+1}\left[\rho^*_{j,\delta}(\varepsilon)-r\right]}&=\sum_{l=1}^{m+1}\frac{Q_1\left[\rho^*_{l,\delta}(\varepsilon)\right]}{\prod\limits_{j\neq l}\left[\rho^*_{j,\delta}(\varepsilon)-\rho^*_{l,\delta}(\varepsilon)\right]}\Bigg\{-\rho^*_{l,\delta}(\varepsilon)\widehat{T}_{\rho^*_{l,\delta}(\varepsilon)}\overline{\mathcal{V}}_{\mathcal{S}}(r)+\frac{\Psi_{\mathcal{S}}(r)}{r}-c\\ &\quad-\gamma^2\left[r+\rho^*_{l,\delta}(\varepsilon)\right]+\frac{L_{\mathcal{X}}\left[\rho^*_{l,\delta}(\varepsilon)\right]-\delta}{\rho^*_{j,\delta}(\varepsilon)-r}\Bigg\}\\ &=\frac{\Psi_{\mathcal{S}}(r)}{r}-c-\gamma^2 r-\sum_{l=1}^{m+1}\frac{Q_1\left[\rho^*_{l,\delta}(\varepsilon)\right]}{\prod\limits_{j\neq l}\left[\rho^*_{j,\delta}(\varepsilon)-\rho^*_{l,\delta}(\varepsilon)\right]}\Bigg\{\rho^*_{l,\delta}(\varepsilon)\widehat{T}_{\rho^*_{l,\delta}(\varepsilon)}\overline{\mathcal{V}}_{\mathcal{S}}(r)\\ &\quad+\gamma^2\rho^*_{l,\delta}(\varepsilon)-\frac{L_{\mathcal{X}}\left[\rho^*_{l,\delta}(\varepsilon)\right]-\delta}{\rho^*_{j,\delta}(\varepsilon)-r}\Bigg\}\end{aligned} \tag{A.49}$$

Taking limits as $\varepsilon \downarrow 0$ in both sides of the equality above, and using (1.12) yields:

$$\begin{aligned}
&[L_{\mathcal{X}}(r)-\delta]\,\widehat{e}_{\delta}^{+}(r)\\
&=\frac{\Psi_{\mathcal{S}}(r)}{r}-c-\gamma^2 r-\sum_{l=1}^{R}\frac{(-1)^{1-k_l}}{(k_l-1)!}\frac{\partial^{k_l-1}}{\partial s^{k_l-1}}\left[\frac{\prod_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}}s\widehat{T}_s\overline{\mathcal{V}}_{\mathcal{S}}(r)\right]_{s=\rho_{l,\delta}}\\
&-\gamma^2\sum_{l=1}^{R}\frac{(-1)^{1-k_l}}{(k_l-1)!}\frac{\partial^{k_l-1}}{\partial s^{k_j-1}}\left[\frac{\prod_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}}s\right]_{s=\rho_{l,\delta}}\\
&+\sum_{l=1}^{R}\frac{(-1)^{1-k_l}}{(k_l-1)!}\frac{\partial^{k_l-1}}{\partial s^{k_l-1}}\left[\frac{Q_1(s)(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}}\frac{L_{\mathcal{X}}(s)-\delta}{s-r}\right]_{s=\rho_{l,\delta}}
\end{aligned} \tag{A.50}$$

Since $\rho_{j,\delta}$ for $j=1,2,\ldots,R$ are the roots of $L_{\mathcal{X}}(s)-\delta=0$ in $\mathbb{C}_{++}$, and they have multiplicities k_j for $j=1,2,\ldots,m+1$, it follows that:

$$\sum_{l=1}^{R}\frac{(-1)^{1-k_l}}{(k_l-1)!}\frac{\partial^{k_l-1}}{\partial s^{k_l-1}}\left[\frac{Q_1(s)(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}}\frac{L_{\mathcal{X}}(s)-\delta}{s-r}\right]_{s=\rho_{l,\delta}}=0.$$

Hence, substituting this in the equality above, setting

$$D_{\delta}=\sum_{l=1}^{R}\frac{(-1)^{1-k_l}}{(k_l-1)!}\frac{\partial^{k_l-1}}{\partial s^{k_j-1}}\left[\frac{\prod_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}}s\right]_{s=\rho_{l,\delta}},$$

and multiplying both sides by -1 we obtain:

$$[\delta - L_{\mathcal{X}}(r)]\,\widehat{e}_\delta^+(r) = c + \gamma^2 D_\delta + \gamma^2 r - \frac{\Psi_{\mathcal{S}}(r)}{r} + \sum_{l=1}^{R} \frac{(-1)^{1-k_l}}{(k_l-1)!}\frac{\partial^{k_l-1}}{\partial s^{k_l-1}}\left[\frac{\prod_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}} s\widehat{T}_s\overline{\mathcal{V}}_{\mathcal{S}}(r)\right]_{s=\rho_{l,\delta}} \tag{A.51}$$

Now we apply Leibniz rule and Lemma 14 to the last term in the equality above. This gives:

$$\begin{aligned}[\delta - L_{\mathcal{X}}(r)]\,\widehat{e}_\delta^+(r) =& c + \gamma^2 D_\delta + \gamma^2 r - \frac{\Psi_{\mathcal{S}}(r)}{r} + \sum_{l=1}^{R}\sum_{a=0}^{k_l-1}\binom{k_l-1}{a}\frac{(-1)^{1-k_l}}{(k_l-1)!} \\ &\times \frac{\partial^{k_l-1-i}}{\partial s^{k_l-1-i}}\left[\frac{\prod_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}}\right]_{s=\rho_{l,\delta}} \frac{\partial^a}{\partial s^a}\left[\widehat{T}_s\overline{\mathcal{V}}_{\mathcal{S}}(r)\right]_{s=\rho_{l,\delta}} \\ =& c + \gamma^2 D_\delta + \gamma^2 r - \frac{\Psi_{\mathcal{S}}(r)}{r} + \sum_{l=1}^{R}\sum_{a=0}^{k_l-1}\binom{k_l-1}{a}\frac{(-1)^{1-k_l+a}}{(k_l-1)!} \\ &\times \frac{\partial^{k_l-1-a}}{\partial s^{k_l-1-i}}\left[\frac{\prod_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}}\right]_{s=\rho_{l,\delta}} \widehat{\mathcal{T}}_{\rho_{l,\delta};a}\overline{\mathcal{V}}_{\mathcal{S}}(r) \\ =& c + \gamma^2 D_\delta + \gamma^2 r - \frac{\Psi_{\mathcal{S}}(r)}{r} + \sum_{l=1}^{R}\sum_{a=0}^{k_l-1} E_*(l,a,\delta)\widehat{\mathcal{T}}_{\rho_{l,\delta};a}\overline{\mathcal{V}}_{\mathcal{S}}(r) \\ =& c + \gamma^2 D_\delta + \gamma^2 r - \frac{\Psi_{\mathcal{S}}(r)}{r} + \widehat{\mathcal{L}}_\delta(r).\end{aligned}$$

Using L'Hospital's rule we obtain $\lim_{r\downarrow 0}\frac{\Psi_{\mathcal{S}}(r)}{r} = 0$, hence setting $r = 0$ in both sides of the equality above and using $[\delta - L_{\mathcal{X}}(0)]\,\widehat{e}_\delta^+(0) = a_\delta$, we obtain $a_\delta = c + \gamma^2 D_\delta + \widehat{\mathcal{L}}_\delta(0)$.

It follows that:

$$[\delta - L_{\mathcal{X}}(r)]\,\widehat{e}_\delta^{+}(r) = a_\delta - \widehat{\mathcal{L}}_\delta(0) + \gamma^2 r - \frac{\Psi_{\mathcal{S}}(r)}{r} + \widehat{\mathcal{L}}_\delta(r)$$
$$= a_\delta + \gamma^2 r - \frac{\Psi_{\mathcal{S}}(r)}{r} - \left(\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r)\right).$$

This gives the result for case C.

For case B, we use (A.47) replacing $\Psi_{\mathcal{S}}(r)$ by $G_{\mathcal{S}}(r)$ and setting $\gamma = 0$. This gives:

$$\begin{aligned}
&[L_{\mathcal{X}}(r) - \delta]\,Q_1(r) \\
&= \prod_{j=1}^{m+1}\left[\rho_{j,\delta}^*(\varepsilon) - r\right]\sum_{l=1}^{m+1}\frac{Q_1\left[\rho_{l,\delta}^*(\varepsilon)\right]}{\prod_{j\neq l}\left[\rho_{j,\delta}^*(\varepsilon) - \rho_{l,\delta}^*(\varepsilon)\right]}\left\{-\frac{G_{\mathcal{S}}(r) - G_{\mathcal{S}}(\rho_{l,\delta}^*(\varepsilon))}{\rho_{l,\delta}^*(\varepsilon) - r}\right. \\
&\left. +\lambda_1\frac{\frac{Q(-r)}{\prod_{j=1}^{N}(q_j - r)^{m_j}} - \frac{Q\left(\rho_{l,\delta}^*(\varepsilon)\right)}{\prod_{j=1}^{N}\left(q_j - \rho_{l,\delta}^*(\varepsilon)\right)^{m_j}}}{\rho_{l,\delta}^*(\varepsilon) - r} - c + \frac{L_{\mathcal{X}}(\rho_{l,\delta}^*(\varepsilon)) - \delta}{\rho_{l,\delta}^*(\varepsilon) - r}\right\} \\
&= \prod_{j=1}^{m+1}\left[\rho_{j,\delta}^*(\varepsilon) - r\right]\sum_{l=1}^{m+1}\frac{Q_1\left[\rho_{l,\delta}^*(\varepsilon)\right]}{\prod_{j\neq l}\left[\rho_{j,\delta}^*(\varepsilon) - \rho_{l,\delta}^*(\varepsilon)\right]}\left\{\widehat{T}_{\rho_{l,\delta}^*(\varepsilon)}\nu_{\mathcal{S}}(r) - c + \frac{L_{\mathcal{X}}(\rho_{l,\delta}^*(\varepsilon)) - \delta}{\rho_{l,\delta}^*(\varepsilon) - r}\right\}
\end{aligned} \tag{A.52}$$

where in the second equality we used

$$\begin{aligned}
\frac{G_{\mathcal{S}}(r) - G_{\mathcal{S}}(s)}{s - r} &= -\frac{\int_{0+}^{\infty}\left[e^{-rx} - 1 - (e^{-sx} - 1)\right]\nu_{\mathcal{S}}(dx)}{s - r} \\
&= -\frac{\int_{0+}^{\infty}\left[e^{-rx} - e^{-sx}\right]\nu_{\mathcal{S}}(dx)}{s - r} = -\widehat{T}_s\nu_{\mathcal{S}}(r).
\end{aligned} \tag{A.53}$$

and the second equality in (A.48). Using the first equality in (A.48) and dividing both sides in (A.52) by $-\prod_{j=1}^{m+1}\left[\rho^*_{j,\delta}(\varepsilon)-r\right]$, we obtain

$$[\delta - L_{\mathcal{X}}(r)]\,\frac{Q_1(r)}{\prod\limits_{j=1}^{m+1}\left[\rho^*_{j,\delta}(\varepsilon)-r\right]}$$
$$= c - \sum_{l=1}^{m+1}\frac{Q_1\left[\rho^*_{l,\delta}(\varepsilon)\right]}{\prod\limits_{j\neq l}\left[\rho^*_{j,\delta}(\varepsilon)-\rho^*_{l,\delta}(\varepsilon)\right]}\left\{\widehat{T}_{\rho^*_{l,\delta}(\varepsilon)}\nu_{\mathcal{S}}(r)+\frac{L_{\mathcal{X}}(\rho^*_{l,\delta}(\varepsilon))-\delta}{\rho^*_{l,\delta}(\varepsilon)-r}\right\} \tag{A.54}$$

Taking limits as $\varepsilon \downarrow 0$ and using (1.12), we obtain:

$$[\delta - L_{\mathcal{X}}(r)]\,\widehat{e}^{+}_{\delta}(r) = c - \sum_{l=1}^{R}\frac{(-1)^{1-k_l}}{(k_l-1)!}\frac{\partial^{k_l-1}}{\partial s^{k_l-1}}\left[\frac{\prod\limits_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod\limits_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}}\widehat{T}_s\nu(r)\right]_{s=\rho_{l,\delta}}$$
$$= c - \sum_{l=1}^{R}\sum_{a=0}^{k_l-1}\binom{k_l-1}{a}\frac{(-1)^{1-k_l}}{(k_l-1)!}\frac{\partial^{k_l-1-i}}{\partial s^{k_l-1-i}}\left[\frac{\prod\limits_{j=1}^{N}(q_j-s)^{m_j}(\rho_{l,\delta}-s)^{k_l}}{\prod\limits_{j=1}^{R}(\rho_{j,\delta}-s)^{k_j}}\right]_{s=\rho_{l,\delta}}$$
$$\times\frac{\partial^a}{\partial s^a}\left[\widehat{T}_s\nu_{\mathcal{S}}(r)\right]_{s=\rho_{l,\delta}} = c - \sum_{l=1}^{R}\sum_{a=0}^{k_l-1}E(l,a,\delta)\widehat{\mathcal{T}}_{\rho_{l,\delta};a}\nu_{\mathcal{S}}(r)$$
$$= c - \widehat{\ell}_{\delta}(r), \tag{A.55}$$

where in the second equality we used Leibniz rule and in the third equality we used Lemma 14. Setting $r=0$ in the equality above and using $[\delta - L_{\mathcal{X}}(0)]\,\widehat{e}^{+}_{\delta}(0)=a_\delta$, we obtain $c-\widehat{\ell}_{\delta}(0)=a_\delta$, or equivalently

$$c = a_\delta + \widehat{\ell}_{\delta}(0). \tag{A.56}$$

Therefore:

$$[\delta - L_{\mathcal{X}}(r)]\,\widehat{e}^{+}_{\delta}(r) = a_\delta + \left[\widehat{\ell}_{\delta}(0)-\widehat{\ell}_{\delta}(r)\right]$$

This implies the result for case B. To obtain the result for case A, we suppose that $L_{\mathcal{X}}(r) = \lambda_1 \left(\frac{Q(-r)}{\prod_{j=1}^{N}(q_j - r)^{m_j}} - 1 \right) - G_{\mathcal{S}}(r)$, and $\mathcal{S}$ is a subordinator but not a compound Poisson process. Note that we have also assumed that the drift term $c = c_0 + \mu_{\mathcal{S}}$ equals zero.

In this case we know by Lemma 13 that $L_{\mathcal{X}}(r) - \delta = 0$ has m roots in $\mathbb{C}_{++}$. We denote them again as $\rho_{1,\delta}, \ldots, \rho_{R,\delta}$ and assume that they have multiplicities $k_1 \equiv 1, k_2, \ldots, k_R$ such that $\sum_j k_j = m$. We consider the numbers $\rho^*_{1,\delta}(\varepsilon), \ldots, \rho^*_{m,\delta}(\varepsilon)$ as defined before, and instead of $\rho^*_{m+1,\delta}(\varepsilon)$ we take $\rho_\infty(n) = \sqrt{n}$. We also take $c_n = \frac{1}{n}$. Clearly $\lim_{n\to\infty} c_n = 0$ and $\lim_{n\to\infty} c_n \rho_\infty(n) = 0$.

Note that, in this case, the function $L_{\mathcal{X}}(r) + c_n r$ is exponent of the moment generating function of some Lévy process of the form (3.1) with $\gamma = 0$ and drift term c_n. Hence we can use (A.47) with $\gamma = 0$, $G_{\mathcal{S}}$ instead of $\Psi_{\mathcal{S}}$ and c_n instead of c. This yields:

$$
\begin{aligned}
&\left[L_{\mathcal{X}}(r) + c_n r - \delta\right] Q_1(r) \\
&= (\rho_\infty(n) - r) \prod_{j=1}^{m} \left[\rho^*_{j,\delta}(\varepsilon) - r\right] \sum_{l=1}^{m} \frac{Q_1\left[\rho^*_{l,\delta}(\varepsilon)\right]}{\prod_{j\neq l}\left[\rho^*_{j,\delta}(\varepsilon) - \rho^*_{l,\delta}(\varepsilon)\right]} \left\{ -\frac{G_{\mathcal{S}}(r) - G_{\mathcal{S}}(\rho^*_{l,\delta}(\varepsilon))}{\rho^*_{l,\delta}(\varepsilon) - r} \right. \\
&\left. + \lambda_1 \frac{\frac{Q(-r)}{\prod_{j=1}^{N}(q_j - r)^{m_j}} - \frac{Q\left(\rho^*_{l,\delta}(\varepsilon)\right)}{\prod_{j=1}^{N}\left(q_j - \rho^*_{l,\delta}(\varepsilon)\right)^{m_j}}}{\rho^*_{l,\delta}(\varepsilon) - r} - c - \gamma^2(\rho^*_{l,\delta}(\varepsilon) + r) + \frac{L_{\mathcal{X}}(\rho^*_{l,\delta}(\varepsilon)) - \delta}{\rho^*_{l,\delta}(\varepsilon) - r} \right\}. \\
&+ (\rho_\infty(n) - r) \prod_{j=1}^{m} \left[\rho^*_{j,\delta}(\varepsilon) - r\right] \left\{ -\frac{G_{\mathcal{S}}(r) - G_{\mathcal{S}}(\rho^*_{l,\delta}(\varepsilon))}{\rho^*_{l,\delta}(\varepsilon) - r} \right. \\
&\left. + \lambda_1 \frac{\frac{Q(-r)}{\prod_{j=1}^{N}(q_j - r)^{m_j}} - \frac{Q(-\rho_\infty(n))}{\prod_{j=1}^{N}(q_j - \rho_\infty(n))^{m_j}}}{\rho_\infty(n) - r} - c - \gamma^2(\rho_\infty(n) + r) + \frac{L_{\mathcal{X}}(\rho_\infty(n)) - \delta}{\rho_\infty(n) - r} \right\}
\end{aligned}
\tag{A.57}
$$

By (A.48) we have

$$
\sum_{l=1}^{m} \frac{Q_1\left[\rho^*_{l,\delta}(\varepsilon)\right]}{\prod_{j\neq l}\left[\rho^*_{j,\delta}(\varepsilon) - \rho^*_{l,\delta}(\varepsilon)\right]} + \frac{Q_1\left[\rho_\infty(n)\right]}{\prod_{j=1}^{m}\left[\rho^*_{j,\delta}(\varepsilon) - \rho_\infty(n)\right]} = 1
$$

and

$$\sum_{l=1}^{m} \frac{Q_1\left[\rho_{l,\delta}^*(\varepsilon)\right]}{\prod\limits_{j\neq l}\left[\rho_{j,\delta}^*(\varepsilon)-\rho_{l,\delta}^*(\varepsilon)\right]}\left(\frac{\frac{Q(-r)}{\prod_{j=1}^{N}(q_j-r)^{m_j}}-\frac{Q\left(\rho_{l,\delta}^*(\varepsilon)\right)}{\prod_{j=1}^{N}\left(q_j-\rho_{l,\delta}^*(\varepsilon)\right)^{m_j}}}{\rho_{l,\delta}^*(\varepsilon)-r}\right)$$
$$+\frac{Q_1\left[\rho_\infty(n)\right]}{\prod\limits_{j=1}^{m}\left[\rho_{j,\delta}^*(\varepsilon)-\rho_\infty(n)\right]}\left(\frac{\frac{Q(-r)}{\prod_{j=1}^{N}(q_j-r)^{m_j}}-\frac{Q(-\rho_\infty(n))}{\prod\limits_{j=1}^{N}(q_j-\rho_\infty(n))^{m_j}}}{\rho_\infty(n)-r}\right)=0. \quad \text{(A.58)}$$

Applying the equalities above and $\frac{G_{\mathcal{S}}(r)-G_{\mathcal{S}}(s)}{s-r}=-\widehat{T}_s\nu_{\mathcal{S}}(r)$ (which is due to (A.53)), to (A.57), then dividing both sides by $\prod\limits_{j=1}^{m}\left[\rho_{j,\delta}^*(\varepsilon)-r\right]$, we obtain:

$$\left[L_{\mathcal{X}}(r)+c_n r-\delta\right]\frac{Q_1(r)}{\prod_{j=1}^{m}\left[\rho_{j,\delta}^*(\varepsilon)-r\right]}$$
$$=-c_n(\rho_\infty(n)-r)+\sum_{l=1}^{m}\frac{\rho_\infty(n)-r}{\rho_\infty(n)-\rho_{l,\delta}^*(\varepsilon)}\frac{Q_1\left[\rho_{l,\delta}^*(\varepsilon)\right]}{\prod\limits_{j\neq l}\left[\rho_{j,\delta}^*(\varepsilon)-\rho_{l,\delta}^*(\varepsilon)\right]}\left\{\widehat{T}_{\rho_{l,\delta}^*(\varepsilon)}\nu_{\mathcal{S}}(r)\right.$$
$$\left.+c_n\rho_{l,\delta}^*(\varepsilon)+\frac{L_{\mathcal{X}}\left[\rho_{l,\delta}^*(\varepsilon)\right]-\delta}{\rho_{l,\delta}^*(\varepsilon)-r}\right\}$$
$$+\frac{Q_1\left[\rho_\infty(n)\right]}{\prod\limits_{j=1}^{m}\left[\rho_{j,\delta}^*(\varepsilon)-\rho_\infty(n)\right]}\left\{(\rho_\infty(n)-r)\,\widehat{T}_{\rho_\infty(n)}\nu_{\mathcal{S}}(r)+c_n\rho_\infty(n)+(L_{\mathcal{X}}\left[\rho_\infty(n)\right]-\delta)\right\}$$

$$=-c_n(\rho_\infty(n)-r)+\sum_{l=1}^{m}\frac{\rho_\infty(n)-r}{\rho_\infty(n)-\rho_{l,\delta}^*(\varepsilon)}\frac{Q_1\left[\rho_{l,\delta}^*(\varepsilon)\right]}{\prod\limits_{j\neq l}\left[\rho_{j,\delta}^*(\varepsilon)-\rho_{l,\delta}^*(\varepsilon)\right]}\left\{\widehat{T}_{\rho_{l,\delta}^*(\varepsilon)}\nu_{\mathcal{S}}(r)\right.$$
$$\left.+c_n\rho_{l,\delta}^*(\varepsilon)+\frac{L_{\mathcal{X}}\left[\rho_{l,\delta}^*(\varepsilon)\right]-\delta}{\rho_{l,\delta}^*(\varepsilon)-r}\right\}$$
$$+\frac{Q_1\left[\rho_\infty(n)\right]}{\prod\limits_{j=1}^{m}\left[\rho_{j,\delta}^*(\varepsilon)-\rho_\infty(n)\right]}\left\{-G_{\mathcal{S}}(r)+G_{\mathcal{S}}(\rho_\infty(n))+c_n\rho_\infty(n)+(L_{\mathcal{X}}\left[\rho_\infty(n)\right]-\delta)\right\}$$

where in the second equality we have used the equality $(\rho_\infty(n) - r)\widehat{T}_{\rho_\infty(n)}\nu_{\mathcal{S}}(r) = (\rho_\infty(n) - r)\left(\frac{-G_{\mathcal{S}}(r)+G_{\mathcal{S}}(\rho_\infty(n))}{\rho_\infty(n)-r}\right)$.

Now we substitute $L_{\mathcal{X}}(\rho_\infty(n)) = \lambda_1\left(\frac{Q(-\rho_\infty(n))}{\prod\limits_{j=1}^{N}(q_j-\rho_\infty(n))^{m_j}} - 1\right) - G_{\mathcal{S}}(\rho_\infty(n))$ and obtain:

$$\begin{aligned}
&[L_{\mathcal{X}}(r) + c_n r - \delta]\frac{Q_1(r)}{\prod_{j=1}^{m}\left[\rho^*_{j,\delta}(\varepsilon) - r\right]} \\
&= -c_n(\rho_\infty(n) - r) + \sum_{l=1}^{m}\frac{\rho_\infty(n) - r}{\rho_\infty(n) - \rho^*_{l,\delta}(\varepsilon)}\frac{Q_1\left[\rho^*_{l,\delta}(\varepsilon)\right]}{\prod\limits_{j\neq l}\left[\rho^*_{j,\delta}(\varepsilon) - \rho^*_{l,\delta}(\varepsilon)\right]}\Bigg\{\widehat{T}_{\rho^*_{l,\delta}(\varepsilon)}\nu_{\mathcal{S}}(r) \\
&+ c_n\rho^*_{j,\delta}(\varepsilon) + \frac{L_{\mathcal{X}}\left[\rho^*_{l,\delta}(\varepsilon)\right] - \delta}{\rho^*_{l,\delta}(\varepsilon) - r}\Bigg\} + \frac{Q_1\left[\rho_\infty(n)\right]}{\prod\limits_{j=1}^{m}\left[\rho^*_{j,\delta}(\varepsilon) - \rho_\infty(n)\right]}\Bigg\{ - G_{\mathcal{S}}(r) + G_{\mathcal{S}}(\rho_\infty(n)) \\
&+ \left(c_n\rho_\infty(n) + \lambda_1\frac{Q\left(-\rho_\infty(n)\right)}{\prod_{j=1}^{N}\left(q_j - \rho_\infty(n)\right)^{m_j}} - G_{\mathcal{S}}\left(\rho_\infty(n)\right) - \lambda_1 - \delta\right)\Bigg\} \\
&= -c_n(\rho_\infty(n) - r) + \sum_{l=1}^{m}\frac{\rho_\infty(n) - r}{\rho_\infty(n) - \rho^*_{l,\delta}(\varepsilon)}\frac{Q_1\left[\rho^*_{l,\delta}(\varepsilon)\right]}{\prod\limits_{j\neq l}\left[\rho^*_{j,\delta}(\varepsilon) - \rho^*_{l,\delta}(\varepsilon)\right]}\Bigg\{\widehat{T}_{\rho^*_{l,\delta}(\varepsilon)}\nu_{\mathcal{S}}(r) + c_n\rho^*_{l,\delta}(\varepsilon) \\
&+ \frac{L_{\mathcal{X}}\left[\rho^*_{l,\delta}(\varepsilon)\right] - \delta}{\rho^*_{l,\delta}(\varepsilon) - r}\Bigg\} - \frac{Q_1\left[\rho_\infty(n)\right]}{\prod\limits_{j=1}^{m}\left[\rho^*_{j,\delta}(\varepsilon) - \rho_\infty(n)\right]}G_{\mathcal{S}}(r) \\
&+ \frac{Q_1\left[\rho_\infty(n)\right]}{\prod\limits_{j=1}^{m}\left[\rho^*_{j,\delta}(\varepsilon) - \rho_\infty(n)\right]}\left\{c_n\rho_\infty(n) + \lambda_1\frac{Q\left(-\rho_\infty(n)\right)}{\prod_{j=1}^{N}\left(q_j - \rho_\infty(n)\right)^{m_j}} - (\lambda_1 + \delta)\right\} \qquad \text{(A.59)}
\end{aligned}$$

Since $Q_1(r) = \prod_{j=1}^{N}(q_j - r)^{m_j}$ and $\prod_{j=1}^{m}\left(\rho^*_{j,\delta}(\varepsilon) - r\right)$ both have degree m, and since in the quotient $\frac{Q(r)}{Q_1(r)}$, $Q(r)$ is a polynomial of degree at most $m-1$, it follows that for any r, s such that $r \neq \rho_\infty(n)$ and $s \neq \rho_\infty(n)$:

$$\lim_{n\to\infty}\frac{\rho_\infty(n) - r}{\rho_\infty(n) - s} = 1$$

$$\lim_{n\to\infty} \frac{Q_1(\rho_\infty(n))}{\prod_{j=1}^m \left(\rho^*_{j,\delta}(\varepsilon) - \rho_\infty(n)\right)} = 1$$
$$\lim_{n\to\infty} \frac{Q(-\rho_\infty(n))}{\prod\limits_{j=1}^N (q_j - \rho_\infty(n))^{m_j}} = 0.$$

We let $n \to \infty$ in both sides of (A.59) and multiply the resulting equality by -1. This gives:

$$[\delta - L_{\mathcal{X}}(r)] \frac{\prod_{j=1}^N (q_j - r)^{m_j}}{\prod_{j=1}^m \left(\rho^*_{j,\delta}(\varepsilon) - r\right)}$$
$$= -\sum_{l=1}^m \frac{Q_1\left[\rho^*_{l,\delta}(\varepsilon)\right]}{\prod\limits_{j\neq l}\left[\rho^*_{j,\delta}(\varepsilon) - \rho^*_{l,\delta}(\varepsilon)\right]} \left\{ \widehat{T}_{\rho^*_{l,\delta}(\varepsilon)} \nu_{\mathcal{S}}(r) + \frac{L_{\mathcal{X}}\left[\rho^*_{l,\delta}(\varepsilon)\right] - \delta}{\rho^*_{l,\delta}(\varepsilon) - r} \right\} + G_{\mathcal{S}}(r) + \lambda_1 + \delta.$$

Letting $\varepsilon \downarrow 0$ in the equality above and using again (1.12) and that $\rho_{j,\delta}$ for $j = 1, 2, \ldots, R$ are roots of $L_{\mathcal{X}}(r) - \delta$ in $\mathbb{C}_{++}$, with multiplicities k_j, $j = 1, 2, \ldots, R$ we obtain:

$$\begin{aligned}
[\delta - L_{\mathcal{X}}(r)] \frac{\prod_{j=1}^N (q_j - r)^{m_j}}{\prod_{j=1}^m \left(\rho_{j,\delta} - r\right)^{k_j}} &= G_{\mathcal{S}}(r) + \lambda_1 + \delta \\
&\quad - \sum_{l=1}^R \frac{(-1)^{1-k_l}}{(k_l - 1)!} \frac{\partial^{k_l - 1}}{\partial s^{k_l - 1}} \left[\frac{\prod\limits_{j=1}^N (q_j - s)^{m_j} (\rho_{l,\delta} - s)^{k_l}}{\prod\limits_{j=1}^R (\rho_{j,\delta} - s)^{k_j}} \widehat{T}_s \nu_{\mathcal{S}}(r) \right]_{s=\rho_{l,\delta}} \\
&= G_{\mathcal{S}}(r) + \lambda_1 + \delta - \widehat{\ell}_\delta(r)
\end{aligned} \tag{A.60}$$

From this we obtain $a_\delta = \lambda_1 + \delta - \widehat{\ell}(0)$ by taking $r = 0$, or equivalently $\lambda_1 + \delta = a_\delta + \widehat{\ell}(0)$. Substituting this in (A.60) results to:

$$[\delta - L_{\mathcal{X}}(r)] \frac{\prod_{j=1}^N (q_j - r)^{m_j}}{\prod_{j=1}^m \left(\rho_{j,\delta} - r\right)^{k_j}} = a_\delta + G_{\mathcal{S}}(r) + \widehat{\ell}_\delta(0) - \widehat{\ell}_\delta(r). \tag{A.61}$$

■

Proof of Lemma 18. We prove b). First we consider cases B and C, in which the roots in $\mathbb{C}_{++}$ are $m+1$, and hence, $\prod_{j=1}^{R}(\rho_{j,\delta}-r)^{k_j}$ has degree $m+1$. We also recall that $\prod_{l=1}^{N}(q_l-r)^{m_l}$ has degree m for cases A, B and C.

By (A.10) in Labbé et al. [2011] the polynomial $\prod_{l=1}^{N}(q_l-r)^{m_l}$ has the equivalent representation

$$\prod_{l=1}^{N}(q_l-r)^{m_l}=\sum_{j=1}^{R}\sum_{a=0}^{k_j-1}\frac{1}{a!}\frac{\partial^a}{\partial s^a}\left[\frac{\prod_{l=1}^{N}(q_l-s)^{m_l}(s-\rho_{j,\delta})^{k_j}}{\prod_{l=1}^{R}(s-\rho_{l,\delta})^{k_l}}\right]_{s=\rho_{j,\delta}}\frac{\prod_{l=1}^{R}(r-\rho_{l,\delta})^{k_j}}{(r-\rho_{j,\delta})^{k_j-a}}.$$

Hence, dividing both sides in the equality above by $\prod_{j=1}^{R}(\rho_{j,\delta}-r)^{k_j}$ and factorizing $(-1)^{m+1}$, we obtain:

$$\begin{aligned}\widehat{e}_{\delta}^{+}(r)&=(-1)^{m+1}\sum_{j=1}^{R}\sum_{a=0}^{k_j-1}\frac{1}{a!}\frac{\partial^a}{\partial s^a}\left[\frac{\prod_{l=1}^{N}(q_l-s)^{m_l}(-1)^{k_j}(\rho_{j,\delta}-s)^{k_j}}{(-1)^{m+1}\prod_{l=1}^{R}(\rho_{l,\delta}-s)^{k_l}}\right]_{s=\rho_{j,\delta}}\frac{(-1)^{k_j-a}}{(\rho_{j,\delta}-r)^{k_j-a}}\\&=\sum_{j=1}^{R}\sum_{a=0}^{k_j-1}\frac{1}{a!}\frac{\partial^a}{\partial s^a}\left[\frac{\prod_{l=1}^{N}(q_l-s)^{m_l}(\rho_{j,\delta}-s)^{k_j}}{\prod_{l=1}^{R}(\rho_{l,\delta}-s)^{k_l}}\right]_{s=\rho_{j,\delta}}\frac{(-1)^{a}}{(\rho_{j,\delta}-r)^{k_j-a}}\end{aligned}$$

We set $b=k_j-1-a$ and note that b moves from 0 to k_j-a. We also use that $(-1)^{1-k_j+b}=(-1)^{k_j-1-b}$ and rewrite the equality above as

$$\begin{aligned}&\widehat{e}_{\delta}^{+}(r)\\&=\sum_{j=1}^{R}\sum_{b=0}^{k_j-1}\frac{(-1)^{1-k_j+b}}{(k_j-1-b)!}\frac{\partial^{k_j-1-b}}{\partial s^{k_j-1-b}}\left[\frac{\prod_{l=1}^{N}(q_l-s)^{m_l}(\rho_{j,\delta}-s)^{k_j}}{\prod_{l=1}^{R}(\rho_{l,\delta}-s)^{k_l}}\right]_{s=\rho_{j,\delta}}\frac{1}{(\rho_{j,\delta}-r)^{b+1}}\\&=\sum_{j=1}^{R}\sum_{b=0}^{k_j-1}\frac{(k_j-1)!}{(k_j-1)!}\frac{b!}{b!}\frac{(-1)^{1-k_j+b}}{(k_j-1-b)!}\frac{\partial^{k_j-1-b}}{\partial s^{k_j-1-b}}\left[\frac{\prod_{l=1}^{N}(q_l-s)^{m_l}(\rho_{j,\delta}-s)^{k_j}}{\prod_{l=1}^{R}(\rho_{l,\delta}-s)^{k_l}}\right]_{s=\rho_{j,\delta}}\frac{1}{(\rho_{j,\delta}-r)^{b+1}}\\&=\sum_{j=1}^{R}\sum_{b=0}^{k_j-1}\binom{k_j-1}{b}\frac{(-1)^{1-k_j+b}}{(k_j-1)!}\frac{\partial^{k_j-1-b}}{\partial s^{k_j-1-b}}\left[\frac{\prod_{l=1}^{N}(q_l-s)^{m_l}(\rho_{j,\delta}-s)^{k_j}}{\prod_{l=1}^{R}(\rho_{l,\delta}-s)^{k_l}}\right]_{s=\rho_{j,\delta}}\frac{b!}{(\rho_{j,\delta}-r)^{b+1}}\\&=\sum_{j=1}^{R}\sum_{b=0}^{k_j-1}E(j,b)\frac{b!}{(\rho_{j,\delta}-r)^{b+1}}\end{aligned}\tag{A.62}$$

Now we use that $\int_0^{\infty}e^{rx}y^{b}e^{-\rho_{j,\delta}x}dx=\frac{b!}{(\rho_{j,\delta}-r)^{b+1}}$, and the result follows in this case.

In case A there are only m roots in $\mathbb{C}_{++}$, hence $\prod_{j=1}^{R}(\rho_{j,\delta}-r)^{k_j}$ has degree m. We note that the term with degree m in this polynomial is given by $(-1)^m r^m$, and also the term with degree m in $\prod_{l=1}^{N}(q_l-r)^{m_l}$ is given by $(-1)^m r^m$. Therefore

$P(r) = \prod_{l=1}^{N}(q_l - r)^{m_l} - \prod_{j=1}^{R}(\rho_{j,\delta} - r)^{k_j}$ is a polynomial with degree $m-1$. We apply (A.10) in Labbé et al. [2011] to $P(r)$ and obtain:

$$P(r) = (-1)^{m+1} \sum_{j=1}^{R} \sum_{a=0}^{k_j-1} \frac{1}{a!} \frac{\partial^a}{\partial s^a} \left[\frac{P(r)(-1)^{k_j}(\rho_{j,\delta} - s)^{k_j}}{(-1)^{m+1} \prod_{l=1}^{R}(\rho_{l,\delta} - s)^{k_l}} \right]_{s=\rho_{j,\delta}} \frac{(-1)^{k_j-a}}{(\rho_{j,\delta} - r)^{k_j-a}}$$
$$= \sum_{j=1}^{R} \sum_{a=0}^{k_j-1} \frac{1}{a!} \frac{\partial^a}{\partial s^a} \left[\frac{P(r)(\rho_{j,\delta} - s)^{k_j}}{\prod_{l=1}^{R}(\rho_{l,\delta} - s)^{k_l}} \right]_{s=\rho_{j,\delta}} \frac{(-1)^{a}}{(\rho_{j,\delta} - r)^{k_j-a}}$$

$$= \sum_{j=1}^{R} \sum_{b=0}^{k_j-1} \frac{(-1)^{1-k_j+b}}{(k_j-1-b)!} \frac{\partial^{k_j-1-b}}{\partial s^{k_j-1-b}} \left[\frac{P(r)(\rho_{j,\delta} - s)^{k_j}}{\prod_{l=1}^{R}(\rho_{l,\delta} - s)^{k_l}} \right]_{s=\rho_{j,\delta}} \frac{1}{(\rho_{j,\delta} - r)^{b+1}}$$
$$= \sum_{j=1}^{R} \sum_{b=0}^{k_j-1} \frac{(k_j-1)!}{(k_j-1)!} \frac{b!}{b!} \frac{(-1)^{1-k_j+b}}{(k_j-1-b)!} \frac{\partial^{k_j-1-b}}{\partial s^{k_j-1-b}} \left[\frac{P(r)(\rho_{j,\delta} - s)^{k_j}}{\prod_{l=1}^{R}(\rho_{l,\delta} - s)^{k_l}} \right]_{s=\rho_{j,\delta}} \frac{1}{(\rho_{j,\delta} - r)^{b+1}}$$
$$= \sum_{j=1}^{R} \sum_{b=0}^{k_j-1} \binom{k_j-1}{b} \frac{(-1)^{1-k_j+b}}{(k_j-1)!} \frac{\partial^{k_j-1-b}}{\partial s^{k_j-1-b}} \left[\frac{P(r)(\rho_{j,\delta} - s)^{k_j}}{\prod_{l=1}^{R}(\rho_{l,\delta} - s)^{k_l}} \right]_{s=\rho_{j,\delta}} \frac{b!}{(\rho_{j,\delta} - r)^{b+1}}$$
$$= \sum_{j=1}^{R} \sum_{b=0}^{k_j-1} \binom{k_j-1}{b} \frac{(-1)^{1-k_j+b}}{(k_j-1)!} \frac{\partial^{k_j-1-b}}{\partial s^{k_j-1-b}} \left[\frac{\prod_{l=1}^{N}(q_l - s)^{m_l}(\rho_{j,\delta} - s)^{k_j}}{\prod_{l=1}^{R}(\rho_{l,\delta} - s)^{k_l}} \right]_{s=\rho_{j,\delta}} \frac{b!}{(\rho_{j,\delta} - r)^{b+1}}$$
$$- \sum_{j=1}^{R} \sum_{b=0}^{k_j-1} \binom{k_j-1}{b} \frac{(-1)^{1-k_j+b}}{(k_j-1)!} \frac{\partial^{k_j-1-b}}{\partial s^{k_j-1-b}} \left[\frac{\prod_{l=1}^{R}(\rho_{l,\delta} - s)^{k_l}(\rho_{j,\delta} - s)^{k_j}}{\prod_{l=1}^{R}(\rho_{l,\delta} - s)^{k_l}} \right]_{s=\rho_{j,\delta}} \frac{b!}{(\rho_{j,\delta} - r)^{b+1}}$$

where in the sixth equality we used the linearity of the partial derivatives. We note that

$$\sum_{j=1}^{R} \sum_{b=0}^{k_j-1} \binom{k_j-1}{b} \frac{(-1)^{1-k_j+b}}{(k_j-1)!} \frac{\partial^{k_j-1-b}}{\partial s^{k_j-1-b}} \left[\frac{\prod_{l=1}^{R}(\rho_{l,\delta} - s)^{k_l}(\rho_{j,\delta} - s)^{k_j}}{\prod_{l=1}^{R}(\rho_{l,\delta} - s)^{k_l}} \right]_{s=\rho_{j,\delta}} \frac{b!}{(\rho_{j,\delta} - r)^{b+1}}$$
$$= \sum_{j=1}^{R} \sum_{b=0}^{k_j-1} \binom{k_j-1}{b} \frac{(-1)^{1-k_j+b}}{(k_j-1)!} \frac{\partial^{k_j-1-b}}{\partial s^{k_j-1-b}} \left[(\rho_{j,\delta} - s)^{k_j} \right]_{s=\rho_{j,\delta}} \frac{b!}{(\rho_{j,\delta} - r)^{b+1}} = 0,$$

because all the derivatives $\frac{\partial^{k_j-1-b}}{\partial s^{k_j-1-b}} \left[(\rho_{j,\delta} - s)^{k_j} \right]_{s=\rho_{j,\delta}}$ equal zero. Hence:

$$P(r) = \sum_{j=1}^{R} \sum_{b=0}^{k_j-1} \binom{k_j-1}{b} \frac{(-1)^{1-k_j+b}}{(k_j-1)!} \frac{\partial^{k_j-1-b}}{\partial s^{k_j-1-b}} \left[\frac{\prod_{l=1}^{N}(q_l - s)^{m_l}(\rho_{j,\delta} - s)^{k_j}}{\prod_{l=1}^{R}(\rho_{l,\delta} - s)^{k_l}} \right]_{s=\rho_{j,\delta}} \frac{b!}{(\rho_{j,\delta} - r)^{b+1}} \quad \text{(A.63)}$$

We note that $\widehat{e_\delta^+}(r)$ can be rewritten as $\widehat{e_\delta^+}(r) = 1 + \frac{P(r)}{\prod_{l=1}^{R}(\rho_{l,\delta}-r)^{k_l}}$. Therefore (A.63) implies that $\widehat{e_\delta^+}(r)$ equals:

$$1 + \sum_{j=1}^{R} \sum_{b=0}^{k_j-1} \binom{k_j-1}{b} \frac{(-1)^{1-k_j+b}}{(k_j-1)!} \frac{\partial^{k_j-1-b}}{\partial s^{k_j-1-b}} \left[\frac{\prod_{l=1}^{N}(q_l-s)^{m_l}(\rho_{j,\delta}-s)^{k_j}}{\prod_{l=1}^{R}(\rho_{l,\delta}-s)^{k_l}} \right]_{s=\rho_{j,\delta}} \frac{b!}{(\rho_{j,\delta}-r)^{b+1}}.$$

Using $\int\limits_0^\infty e^{rx} y^b e^{-\rho_{j,\delta}x} dx = \frac{b!}{(\rho_{j,\delta}-r)^{b+1}}$ and $\int\limits_0^\infty e^{rx}\delta_0(x)dx = 1$ we obtain the result in this case. ■

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