CIMAT

CENTRO DE INVESTIGACIÓN EN MATEMÁTICAS

GENERALISATIONS OF CONTINUOUS STATE BRANCHING PROCESSES

T H E S I S

In partial fulfilment of the requirements for the degree of:

DOCTOR OF SCIENCES

with orientation in

PROBABILITY AND STATISTIC

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Guanajuato, México. August, 2016

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Summary

This thesis concerns on new developments of continuous-state branching processes. In particular, we focus on two different topics on this subject. The first topic concerns continuous-state branching processes in a Lévy random environment. In order to define this class of processes, we study the existence and uniqueness of strong solutions of a particular class of non-negative stochastic differential equations driven by Brownian motions and Poisson random measures which are mutually independent. The long-term behaviours of absorption and explosion are also studied.

The second topic is related to multi-type continuous-state branching processes with a countable infinite number of types. We define this kind of processes as super Markov chains with both local and non-local branching mechanisms. Special attention is given to extinction events; in particular local and global extinction are studied.

Acknowledgements

Quiero agradecer a mis papás porque siempre me han impulsado para que sea feliz, en particular me han apoyado económica y mentalmente para que siga este camino. A mis hermanos por todo su cariño y sus palabras de amor. A Ro porque ha hecho que este andar sea mucho más agradable. He tenido la suerte de encontrar alguien que me entiende y comparte la misma pasión que yo. También quiero agradecer a mis tios Fili y Heidi porque cuando empecé con la idea de estudiar matemáticas, siempre me apoyaron y durante mi estancia en el DF me hicieron sentir como en casa. Y a todas las pericas por nuestro chat lleno de amor.

Quiero agradecer a Juan Carlos Pardo. Es un asesor impecable, la verdad estuve más que consentida. Me ayudó a crear matemáticas, me consiguió viáticos para todos los viajes que tuve y me impulso a hacer cosas que jamás creí realizar. Además siempre ha sido un gran consejero y una persona que ayuda y se preocupa por los demás.

Agradezco a Victor Rivero por todos los consejos que me dio y porque sin duda, sin él y Maika mi vida en Bath no hubiera sido tan amena como lo fue. Siempre estaré agradecida con Gerónimo Uribe por todos los consejos y pláticas que tuvimos, tanto académicas como no académicas. De hecho, puedo decir que mi decisión de hacer un doctorado en México y en particular en CIMAT fue influenciada por algunas de las pláticas. Y, fue una gran decisión.

A special recognition to the Mexican people who made possible the existence of postgraduate fellowships. Through the Mexican Council of Science, CONACyT, I received the grant number 351643 to do my PhD. I also want to thanks the Department of Mathematical Sciences of the University of Bath where part of this thesis was done.

Andreas Kyprianou was a really good advisor that helped me a lot before and after my time at Bath. I am sure that without him and Juan Carlos, my Ph.D. would have been more difficult. I also want to thanks the reviewers of my thesis. Thank you for took your time and checked the details.

Es una suerte haber conocido a Javier Fernández, de él aprendí el amor por las matemáticas. La formalidad de éstas la aprendí con Guillermo Grabinsky. Agradezco a Airam por estar juntas en el doctorado, fue una suerte compartir cubículo, es una gran compañera de risas y pláticas profundas. Finalmente, quiero agradecer a todos mis amigos, a aquellos que aún frecuento y aquellos que el camino distancio. Gracias por las pláticas, las risas, las salidas y bailes, por formar parte de mi vida y enseñarme a disfrutar los distintos matices de ésta. En particular gracias a Adriana, Alfredo, Amauri, Arturo, Damián, Daniel, Ehyter, Elia, George, Gerónimo, Gris, Harold, Jan, José Luis, Lu, Mario, Octavio, Osvaldo, Toño,... I cannot forget my friends at Bath; Matt D., Sam, Dorka and Marcus, thanks for making my time there more enjoyable.

Contents

Introduction

In many biological systems, when the population size is large enough, many birth and death events occur. Therefore, the dynamics of the population become difficult to describe. Under this scenario, continuous-state models are good approximations of these systems and sometimes they can be simpler and computationally more tractable. Moreover, the qualitative behaviour of the approximate models may be easier to understand.

The simplest branching model in continuous time and space is perhaps the so called continuous-state branching process (or CB-process for short). They have been the subject of intensive study since their introduction by Jiřina [58]. This model arises as the limit of Galton-Watson processes; where individuals behave independently one from each other and each individual gives birth to a random number of offspring, with the same offspring distribution (see for instance Grimvall [49], for a general background see Chapter 12 of [64] or Chapter 3 of [70]). More precisely, a $[0,\infty]$-valued strong Markov process $Y = (Y_t, t \geq 0)$ with probabilities $(\mathbb{P}_x,\, x \geq 0)$ is called a *continuous-state branching process* if it has paths that are right-continuous with left limits and its law observes the branching property: for all $\theta \geq 0$ and $x, y \geq 0$,

$$\mathbb{E}_{x+y}\left[e^{-\theta Y_t}\right] = \mathbb{E}_x\left[e^{-\theta Y_t}\right]\mathbb{E}_y\left[e^{-\theta Y_t}\right], \qquad t \geq 0.$$

Moreover, its law is completely characterized by the latter identity, i.e.

$$\mathbb{E}_x\left[e^{-\lambda Y_t}\right] = e^{-x u_t(\lambda)}, \qquad t, \lambda \geq 0, \tag{1}$$

where u is a differentiable function in t satisfying

$$\frac{\partial u_t(\lambda)}{\partial t} = -\psi(u_t(\lambda)), \qquad u_0(\lambda) = \lambda, \tag{2}$$

and ψ satisfies the celebrated Lévy-Khintchine formula, i.e.

$$\psi(\lambda) = -q - a\lambda + \gamma^2\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}}\right)\mu(\mathrm{d}x), \qquad \lambda \geq 0,$$

where $a \in \mathbb{R}$, $q, \gamma \geq 0$ and μ is a measure concentrated on $(0,\infty)$ such that $\int_{(0,\infty)} \left(1 \wedge x^2\right)\mu(\mathrm{d}x)$ is finite. The function ψ is convex and is known as the branching mechanism of Y.

Let

$$T_0 = \inf\{t \geq 0 : Y_t = 0\} \qquad \text{and} \qquad T_\infty = \inf\{t \geq 0 : Y_t = \infty\}$$

denote the absorption and explosion times, respectively. Then $Y_t = 0$ for every $t \geq T_0$ and $Y_t = \infty$ for every $t \geq T_\infty$. Plainly, equation (2) can be solved in terms of ψ, and this readily yields the law of the absorption and extinction times (see Grey [48]). More precisely, let η be the largest root of the branching mechanism ψ, i.e. $\eta = \sup\{\theta \geq 0 : \psi(\theta) = 0\}$, (with $\infty = \sup\{\emptyset\}$). Then for every $x > 0$:

i) if $\eta = 0$ or if $\int_{0+} \mathrm{d}\theta/|\psi(\theta)| = \infty$, we have $\mathbb{P}_x(T_\infty < \infty) = 0$,

ii) if $\eta > 0$ and $\int_{0+} \mathrm{d}\theta/|\psi(\theta)| < \infty$, we define

$$g(t) = -\int_0^t \frac{\mathrm{d}\theta}{\psi(\theta)}, \qquad t \in (0, \eta).$$

The mapping $g : (0, \eta) \to (0, \infty)$ is bijective, and we write $\gamma : (0, \infty) \to (0, \eta)$ for its right-continuous inverse. Thus

$$\mathbb{P}_x(T_\infty > t) = \exp\{-x\gamma(t)\}, \qquad x, t > 0.$$

iii) if $\psi(\infty) < 0$ or if $\int^\infty \mathrm{d}\theta/\psi(\theta) = \infty$, we have $\mathbb{P}_x(T_0 < \infty) = 0$,

iv) if $\psi(\infty) = \infty$ and $\int^\infty \mathrm{d}\theta/\psi(\theta) < \infty$, we define

$$\phi(t) = \int_t^\infty \frac{\mathrm{d}\theta}{\psi(\theta)}, \qquad t \in (\eta, \infty).$$

The mapping $\phi : (\eta, \infty) \to (0, \infty)$ is bijective, we write $\varphi : (0, \infty) \to (\eta, \infty)$ for its right-continuous inverse. Thus

$$\mathbb{P}_x(T_0 < t) = \exp\{-x\varphi(t)\}, \qquad x, t > 0.$$

From (ii), we get that $\mathbb{P}_x(T_\infty < \infty) = 1 - \exp\{-x\eta\}$. Hence from the latter and (i), we deduce that a CB-process has a finite explosion time with positive probability if and only if

$$\int_{0+} \frac{\mathrm{d}u}{|\psi(u)|} < \infty \qquad \text{and} \qquad \eta > 0,$$

When $\eta < \infty$, the condition $\eta > 0$ is equivalent to $\psi'(0+) < 0$.

Similarly from (iv), we deduce that $\mathbb{P}_x(T_0 < \infty) = \exp\{-x\eta\}$. Hence, the latter identity and (iii) imply that a CB-process has a finite absorption time a.s. if and only if

$$\psi(\infty) = \infty, \qquad \int^\infty \frac{\mathrm{d}u}{\psi(u)} < \infty \qquad \text{and} \qquad \psi'(0+) \geq 0.$$

We define the extinction event as $\{\lim_{t\to\infty} Y_t = 0\}$. When $\psi(\infty) = \infty$, we have that for all $x \geq 0$,

$$\mathbb{P}_x\left(\lim_{t\to\infty} Y_t = 0\right) = \exp\{-x\eta\}.$$

The value of $\psi'(0+)$ also determines whether its associated CB-process will, on average, decrease, remain constant or increase. More precisely, under the assumption that $q = 0$, we observe that the first moment of a CB-process can be obtained by differentiating (1) with respect to λ. In particular, we may deduce

$$\mathbb{E}_x[Y_t] = xe^{-\psi'(0+)t}, \qquad \text{for} \qquad x, t \geq 0.$$

Hence using the same terminology as for Bienaymé-Galton-Watson processes, in respective order, a CB-process is called *supercritical, critical* or *subcritical* depending on the behaviour of its mean, in other words on whether $\psi'(0+) < 0$, $\psi'(0+) = 0$ or $\psi'(0+) > 0$.

A process in this class can also be defined as the unique non-negative strong solution of the following stochastic differential equation (SDE for short)

$$
\begin{aligned}
Y_t =& Y_0 + a\int_0^t Y_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s \\
&+ \int_0^t \int_{(0,1)} \int_0^{Y_{s-}} z\widetilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t \int_{[1,\infty]} \int_0^{Y_{s-}} zN(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u),
\end{aligned}
$$

where $B=(B_t, t\geq 0)$ is a standard Brownian motion, $N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ is a Poisson random measure independent of B, with intensity $\mathrm{d}s\Lambda(\mathrm{d}z)\mathrm{d}u$ where Λ is a measure on $(0,\infty]$ defined as $\Lambda(\mathrm{d}z)=\mathbf{1}_{(0,\infty)}(z)\mu(\mathrm{d}z)+q\delta_\infty(\mathrm{d}z)$, and $\widetilde{N}$ is the compensated measure of N, see for instance [45].

There has been some interest in extending CB-processes to other population models. By analogy with multi-type Galton-Watson processes, a natural extension would be to consider a multi-type Markov population model in continuous time which exhibits a branching property. Multi-type CB-processes (MCBPs) should have the property that the continuum mass of each type reproduces within its own population type in a way that is familiar to a CB-process, but also allows for the migration and/or seeding of mass into other population types.
Recently in [12], the notion of a multi-type continuous-state branching process (with immigration) having d-types was introduced as a solution to an d-dimensional vector-valued SDE with both Gaussian and Poisson driving noises. Simultaneously, in [23], the pathwise construction of these d-dimensional processes was given in terms of a multiparameter time change of Lévy processes (see also [46]). Preceding that, work on affine processes, originally motivated by mathematical finance, in [33] also showed the existence of such processes. Older work on multi-type continuous-state branching processes is more sparse but includes [73] and [88] , where only two types are considered.

In this thesis, we introduce multi-type continuous-state branching processes (MCBPs) as super Markov chains with both local and non-local branching mechanisms. That is to say we defined MCBPs as superprocesses whose associated underlying Markov movement generator is that of a Markov chain. This allows us the possibility of working with a countably infinite number of types. We are interested in particular in the event of extinction and growth rates. Lessons learnt from the setting of super diffusions tells us that, in the case that the number of types is infinite, we should expect to see the possibility that the total mass may grow arbitrarily large whilst the population of each type dies out; see for example the summary in Chapter 2 of [39]. This type of behaviour can be attributed to the notion of transient 'mass transfer' through the different types and is only possible with an infinite number of types. In the case that the number of types is finite, we know from the setting of multi-type Bienaymé–Galton–Watson processes (MBGW) that all types grow at the same rate. Moreover, this rate is determined by a special eigenvalue associated to the linear semigroup of the process. In our case, the so-called spectral radius of the linear semigroup will have an important roll in the asymptotic behaviour of our process, in particular, it will determine the phenomenon of local extinction. In order to study this phenomenon, we develop some standard tools based around a spine decomposition.

Another natural extension of CB-processes is to include immigration, competition or dependence on the environment. The interest in these new models comes from the fact that they arise as limits of discrete population models where there are interactions between individuals or where the offspring distribution depends on the environment (see for instance Lambert [68], Kawasu and Watanabe [60], Bansaye and Simatos [10]).

Recall that a CB-process with immigration (or CBI-process) is a strong Markov process taking values in $[0,\infty]$, where 0 is no longer an absorbing state. It is characterized by a branching mechanism ψ and an immigration mechanism,

$$\phi(u) := \mathtt{d}u + \int_0^\infty (1-e^{-ut})\nu(\mathrm{d}t), \qquad u \geq 0$$

where $\mathtt{d} \geq 0$ and

$$\int_0^\infty (1 \wedge x)\nu(\mathrm{d}x) < \infty.$$

It is well-known that if $(Y_t, t \geq 0)$ is a process in this class, then its semi-group is characterized by

$$\mathbb{E}_x\left[e^{-\lambda Y_t}\right] = \exp\left\{-xu_t(\lambda) - \int_0^t \phi(u_s)\mathrm{d}s\right\}, \qquad \text{for} \quad \lambda, x, t \geq 0,$$

where u_t solves (2).

According to Fu and Li [45], under the condition that $\int_{(0,\infty)}(x \wedge x^2)\mu(\mathrm{d}x)$ is finite, a CBI-process can be defined as the unique non-negative strong solution of the SDE

$$\begin{aligned} Y_t = Y_0 + &\int_0^t (\mathtt{d} + aY_s)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s \\ &+ \int_0^t \int_{(0,\infty)} \int_0^{Y_{s-}} z\widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{(0,\infty)} zM^{(im)}(\mathrm{d}s, \mathrm{d}z), \end{aligned}$$

where $M^{(im)}(\mathrm{d}s, \mathrm{d}z)$ is a Poisson random measure with intensity $\mathrm{d}s\nu(\mathrm{d}z)$, independent of B and N.

CB-processes with competition were first studied by Lambert [68], under the name of logistic branching processes, and more recently studied by Ma [74] and Beresticky et al. [13]. Under the assumptions that $q = 0$ and $\int_{(0,\infty)}\left(x \wedge x^2\right)\mu(\mathrm{d}x) < \infty$, the CB-process with competition is defined as the unique strong solution of the following SDE

$$Y_t = Y_0 + a\int_0^t Y_s\mathrm{d}s - \int_0^t \beta(Y_s)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s + \int_0^t \int_{(0,\infty)} \int_0^{Y_{s-}} z\widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

where β is a continuous non-decreasing function on $[0,\infty)$ with $\beta(0) = 0$, which is called the competition mechanism. The interpretation of the function β is the following: in a given population of size z, an additional individual would be killed a rate $\beta(z)$.

Smith and Wilkinson introduced and studied branching processes in random environment (BPREs). Roughly speaking, BPREs are a generalization of Galton-Watson processes, where at each generation the offspring distribution is picked randomly in an i.i.d manner. This type of process has attracted considerable interest in the last decade, see for instance [2, 3, 8] and the references therein. One of the reason is that BPREs are more realistic models than classical branching processes. And, from the mathematical point of view, they have more interesting features such as a phase transition in the subcritical regime. Scaling limits for BPREs have been studied by Kurtz [62] in the continuous case and more recently by Bansaye and Simatos [10] in a more general setting.

CB-processes in random environment, the continuous analogue in time and state space of BPREs, can be defined as a strong solution of a particular stochastic differential equation. They have been studied recently by several authors in different settings. More precisely, Böinghoff and Hutzenthaler [20] studied the case when the process possesses continuous paths. This process is the strong solution of the following SDE

$$Z_t = Z_0 + a\int_0^t Z_s ds + \int_0^t \sqrt{2\gamma^2 Z_s} dB_s + \int_0^t Z_s dS_s, \tag{3}$$

where the process $S = (S_t, t \geq 0)$ is a Brownian motion with drift which is independent of B. Bansaye and Tran [11] studied a cell division model, where the cells are infected by parasites. Informally, the quantity of parasites in a cell evolves as a Feller diffusion. The cells divide in continuous time at rate $r(x)$, which may depend on the quantity of parasites x that they contain. When a cell divides, a random fraction θ of parasites goes in the first daughter cell and the rest in the second one. In each division, they only keep one cell and consider the quantity of parasites inside. Assuming that the rate r is constant and θ is a r.v. in $(0,1)$ with distribution F, the model follows a Feller diffusion with multiplicative jumps of independent sizes distributed as F and which occurs at rate r. In particular, the model can be described as in (3) with S satisfying

$$S_t = -r\int_0^t \int_{(0,1)} (1-\theta) M(\mathrm{d}s, \mathrm{d}\theta)$$

where M is a Poisson random measure with intensity $\mathrm{d}sF(\mathrm{d}\theta)$. Inspired in this model, Bansaye et al. [9] studied more general CB-processes in random environment which are driven by Lévy processes whose paths are of bounded variation and their associated Lévy measure satisfies $\int_{(1,\infty)} x\mu(\mathrm{d}x) < \infty$. They are know as CB-processes with catastrophes motivated by the fact that the presence of a negative jump in the random environment represents that a proportion of a population, following the dynamics of the CB-process, is killed. The process is defined as the unique non negative strong solution of the SDE

$$Z_t = Z_0 + a\int_0^t Z_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s} \mathrm{d}B_s + \int_0^t \int_{(0,\infty)} \int_0^{Z_{s-}} z\widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t Z_{s-} \mathrm{d}S_s,$$

where

$$S_t = \int_0^t \int_{(0,\infty)} (m-1) M(\mathrm{d}s, \mathrm{d}m),$$

M is a Poisson random measure independent of N and B, with intensity $\mathrm{d}s\nu(\mathrm{d}m)$ such that

$$\nu(\{0\}) = 0 \qquad \text{and} \qquad 0 < \int_{(0,\infty)} (1 \wedge |m-1|)\nu(\mathrm{d}m) < \infty.$$

In all those works, the existence of such processes is given via a stochastic differential equation. Böinghoff and Hutzenthaler computed the exact asymptotic behaviour of the survival probability using a time change method and in consequence, they described the so called Q-process. This is the process conditioned to be never absorbed. Similarly to the discrete case, the authors in [20] found a phase transition in the subcritical regime that depends on the parameters of the random environment. Bansaye et al. also studied the survival probability but unlike the case studied in [20], they used a martingale technique since the time change technique does not hold in general. In the particular case where the branching mechanism is stable, the authors in [9] computed the exact asymptotic behaviour of the survival probability and obtained similar results to those found in [20].

In this thesis, one of our aims is to construct a continuous state branching processes with immigration in a Lévy random environment as a strong solution of a stochastic differential equation. In order to do so, we study a particular class of non-negative stochastic differential equations driven by Brownian motions and Poisson random measures which are mutually independent. The existence and uniqueness of strong solutions are established under some general conditions that allows us to consider the case when the strong solution explodes at a finite time. It is important to note that this result is of particular interest on its own. We also study the long-term behaviour of these processes. And, in the particular case where the branching mechanism is stable, i.e. $\psi(\lambda) = \lambda^{\beta+1}$, with $\beta \in (-1, 0) \cup (0, 1]$, we study the asymptotic behaviour of the absorption and explosion probability. Up to our knowledge, the explosion case has never been studied before even in the discrete setting.

A key tool is a fine development in the asymptotic behaviour of exponential functionals of general Lévy processes. Recall that a one-dimensional Lévy process, $\xi = (\xi_t : t \geq 0)$, is a stochastic process issued from the origin with stationary and independent increments and a.s. cádlág paths. Its exponential functional is defined by

$$I_t(\xi) := \int_0^t e^{-\xi_s} \mathrm{d}s, \qquad t \geq 0. \tag{4}$$

In recent years there has been a general recognition that exponential functionals of Lévy processes play an important role in various domains of probability theory such as self-similar Markov processes, generalized Ornstein-Uhlenbeck processes, random processes in random environment, fragmentation processes, branching processes, mathematical finance, Brownian motion on hyperbolic spaces, insurance risk, queueing theory, to name but a few (see [19, 24, 67] and references therein). There is a vast literature about exponential functionals of Lévy processes drifting to $+\infty$ or killed at an independent exponential time $\mathbf{e}_q$ with parameter $q \geq 0$, see for instance [5, 19]. Most of the known results on $I_\infty(\xi)$ and $I_{\mathbf{e}_q}(\xi)$ are related to the knowledge of their densities or the behaviour of their tail distributions. In particular, it is know from Theorem 1 in Bertoin and Yor [19] that

$$\text{a Lévy process } \xi \text{ drifts to } \infty \quad \text{if and only if} \quad I_\infty(\xi) < \infty \text{ a.s.} \tag{5}$$

According to Theorem 3.9 in Bertoin et al. [16], there exists a density for $I_\infty(\xi)$, here denoted by h. In the case when $q > 0$, the existence of the density of $I_{\mathbf{e}_q}(\xi)$ appears in Pardo et al. [82]. Moreover, according to Theorem 2.2. in Kuznetsov et al. [63], under the assumption that $\mathbb{E}[|\xi_1|] < \infty$, the density h is completely determined by the following integral equation: for $v > 0$,

$$\begin{aligned} \mu \int_v^\infty h(x)\mathrm{d}x + \frac{\rho^2}{2} v h(v) + \int_v^\infty \overline{\overline{\Pi}}^{(-)} \left(\ln \frac{x}{v} \right) h(x)\mathrm{d}x \\ + \int_0^v \overline{\overline{\Pi}}^{(+)} \left(\ln \frac{x}{v} \right) h(x)\mathrm{d}x + \int_v^\infty \frac{h(x)}{x}\mathrm{d}x = 0, \end{aligned} \tag{6}$$

where

$$\overline{\overline{\Pi}}^{(+)}(x) = \int_x^\infty \int_y^\infty \Pi(\mathrm{d}z)\mathrm{d}y \qquad \text{and} \qquad \overline{\overline{\Pi}}^{(-)}(x) = \int_x^\infty \int_{-\infty}^{-y} \Pi(\mathrm{d}z)\mathrm{d}y.$$

We refer to [16, 63, 82], and the references therein, for more details about these facts. The case when the exponential functional of a Lévy process does not converge has only been studied in a few papers and not in its most general form, see for instance [9, 20].

One of our aims in this thesis is to study the asymptotic behaviour of

$$\mathbb{E}\Big[F\big(I_t(\xi)\big)\Big] \qquad \text{as} \qquad t \to \infty,$$

where F is a non-increasing function with polynomial decay at infinity and under some exponential moment conditions on ξ, and $I_t(\xi)$ does not converge a.s. to a finite random variable, as t goes to ∞. In particular, we find five different regimes that depend on the shape of the Laplace exponent of ξ. These results will be applied for the particular functions such that

$$F(x) = x^{-p}, \qquad F(x) = 1 - e^{x^{-p}}, \quad F(x) = e^{-x}, \quad \text{or} \quad F(x) = \frac{a}{b+x} \quad \text{for } a, b, p, x > 0.$$

Exponential functionals also appear in the study of diffusions in random potential, which we now describe informally. Associated with a stochastic process $V = (V(x), x \in \mathbb{R})$ such that $V(0) = 0$, a diffusion $X_V = (X_V(t), t \geq 0)$ in the random potential V is, loosely speaking, a solution of the stochastic differential equation

$$\mathrm{d}X_V(t) = \mathrm{d}\beta_t - \frac{1}{2}V'(X_V(t))\mathrm{d}t, \qquad X_V(0) = 0,$$

where $(\beta_t, t \geq 0)$ is a standard Brownian motion independent of V. More rigorously, the process X_V should be considered as a diffusion whose conditional generator, given V, is:

$$\frac{1}{2}\exp(V(x))\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-V(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

Observe that from Feller's construction of such diffusions, the potential V does not need to be differentiable. Kawazu and Tanaka [59] studied the asymptotic behaviour of the tail of the distribution of the maximum of a diffusion in a drifted Brownian potential. Carmona et al. [24] considered the case when the potential is a Lévy process whose jump structure is of bounded variation. More precisely, they studied the following question: How fast does $\mathbb{P}(\max_{t\geq 0} X_V(t) > x)$ decay as $x \to \infty$? From these works, we know that

$$\mathbb{P}\left(\max_{t\geq 0} X_V(t) > x\right) = \mathbb{E}\left[\frac{A}{A + B_x}\right], \qquad x > 0$$

where

$$A = \int_{-\infty}^0 e^{V(t)}\mathrm{d}t \quad \text{and} \quad B_x = \int_0^x e^{V(t)}\mathrm{d}t, \quad x > 0$$

are independent. As a consequence, exponential functionals play an essential role in this domain.

Outline

We now give a detailed summary of the main body of this thesis.
Chapters 1-4 are dedicated to study Continuous-state branching processes in a Lévy random environment with immigration and competition. They are based on the papers:

[79] S. PALAU and J.C. PARDO. Branching processes in a Lévy random environment. *Preprint arXiv:1512.07691, submitted* (2015).
[80] S. PALAU and J.C. PARDO. Continuous state branching processes in random environment: The Brownian case. *Stochastic Processes and their Applications.* (2016) 10.1016/j.spa.2016.07.006
[81] S. PALAU, J.C. PARDO and C.SMADI. Asymptotic behaviour of exponential functionals of Lévy processes with applications to random processes in random environment. *Preprint arXiv:1601.03463, submitted* (2016).

We want to remark that whilst writing [79] and [81], Hui He, Zenghu Li and Wei Xu independently developed similar results in the following papers.

[51] H. HE, Z. LI and W. XU. Continuous-state branching processes in Lévy random environments. *Preprint arXiv:1601.04808,* (2016).
[72] Z. LI and W. XU. Asymptotic results for exponential functionals of Lévy processes. *Preprint arXiv:1601.02363,* (2016).

One of our aims is to construct a continuous state branching processes with immigration in a Lévy random environment as a strong solution of a stochastic differential equation (SDE for short). In order to do so, in Section 1.1, we study a particular class of non-negative stochastic differential equations driven by Brownian motions and Poisson random measures which are mutually independent. The existence and uniqueness of strong solutions are established under some general conditions that allows us to consider the case when the strong solution explodes at a finite time. This result is of particular interest on its own. Section 1.2 is devoted to the construction of CBI-processes with competition in a Lévy random environment. We end the section with some examples.

The long-term behaviour of CB-processes in Lévy random environment is studied in Chapter 2. In particular, we discuss when the process is conservative and the explosion and extinction events. We provide some examples where we can found explicitly the probability of such events. In the second section, we study a competition model in a Lévy random environment. This process can be seen as a population model that extends the competition model given in Evans et al. [44]. We provide the long term behavior of the process. When the random environment has no negative jumps, we compute the Laplace transform of the first passage time from below a level.

In Chapter 3, we study the exponential functional of a Lévy process. In Section 3.1 we expose the asymptotic behaviour of

$$\mathbb{E}\Big[F\big(I_t(\xi)\big)\Big] \qquad \text{as} \qquad t \to \infty,$$

where $I_t(\xi)$ is given by (4) and F is a non-increasing function with polynomial decay at infinity and under some exponential moment conditions on ξ. In particular, we find five different regimes that depend on the shape of the Laplace exponent of ξ. If the exponential moment conditions are not satisfied, we still can find the asymptotic behaviour of $\mathbb{E}\Big[\big(I_t(\xi)\big)^{-p}\Big]$, for $p \in (0,1]$, under the so-called Spitzer's condition. i.e. if there exists $\delta \in (0,1]$ such that

$$\lim_{t\to\infty} t^{-1}\int_0^t \mathbb{P}\left(\xi_s \geq 0\right) \mathrm{d}s = \delta.$$

In Section 3.2 we apply the results to the following classes of processes in random environment: the competition model given en Section 2.2 and diffusion processes whose dynamics are perturbed by a Lévy random environment. For the competition model, we describe the asymptotic behaviour of its mean. For the diffusion processes, we provide the asymptotic behaviour of the tail probability of its global maximum. Finally, Section 3.3 is devoted to the proofs of the main results of the Chapter. The proof under the exponential moment conditions on ξ, relies on a discretisation of the exponential functional $I_t(\xi)$ and on the asymptotic behaviour of functionals of semi-direct products of random variables which was described by Guivarc'h and Liu [50]. Li and Xu in [72] obtained similar results by using fluctuation theory for Lévy processes and the knowledge of Lévy processes conditioned to stay positive. The proof under Spitzer's condition relies in a factorisation of $I_t(\xi)$ given by Arista and Rivero [5].

These results allow us to find the asymptotic behaviour of absorption and explosion probabilities for stable continuous state branching processes in a Lévy random environment. The speed of explosion is studied in Section 4.2. We find 3 different regimes: subcritical-explosion, critical-explosion and supercritical explosion. The speed of explosion is studied in Section 4.2. We find 3 different regimes: subcritical-explosion, critical-explosion and supercritical explosion. The speed of absorption is studied in Section 4.3. As in the discrete case (time and space), we find five different regimes: supercritical, critical, weakly subcritical, intermediately subcritical and strongly subcritical. When the random environment is driven by a Brownian motion with drift, the limiting coefficients of the asymptotic behaviour of the absorption probability are explicit and written in terms of the initial population. In a general Lévy environment, the latter coefficients are also explicit in 3 out of the 5 regimes (supercritical, intermediate subcritical and strongly subcritical cases). This allows us to study two conditioned versions of the process: the process conditioned to be never absorbed (or Q-process) and the process conditioned on eventual absorption. Both processes are studied in Section 4.4.

Finally, Section 5 is devoted to Multi-type continuous-state branching processes (MCBPs). It is based on the paper

[66] A. KYPRIANOU and S. PALAU. Extinction properties of multi-type continuous-state branching processes. *Preprint arXiv:1604.04129, submitted*, (2016).

The main results and some open questions are presented in Section 5.1. We defined a multi-type continuous-state branching process as a super Markov chain with both a local and a non-local branching mechanism. This allows us the possibility of working with a countably infinite number of types. In Section 5.2 we give the construction of MCBPs as a scaling limit of MBGW processes; that is to say, in terms of branching Markov chains. The spectral radius of the associated linear semigroup will have an important roll in the asymptotic behaviour of our process, in particular, it will determine the phenomenon of local extinction. The properties of this semigroup are studied in Section 5.3. In Sections 5.4 and 5.5 we develop some standard tools based around a spine decomposition. In this setting, the spine is a Markov chain and we note in particular that the non-local nature of the branching mechanism induces a new additional phenomenon in which a positive, random amount of mass immigrates off the spine each time it jumps from one state to another. Moreover, the distribution of the immigrating mass depends on where the spine jumped from and where it jumped to. In Section 5.6, we give the proof of the main results. Finally in Section 5.7, we provide examples to illustrate the local phenomenon property.

Chapter 1

Branching processes in a Lévy random environment

This chapter is based in paper [79] elaborated in collaboration with Juan Carlos Pardo. Here, we introduce branching processes in a Lévy random environment. In order to define this class of processes, we study a particular class of non-negative stochastic differential equations driven by Brownian motions and Poisson random measures which are mutually independent. The existence and uniqueness of strong solutions are established under some general conditions that allows us to consider the case when the strong solution explodes at a finite time. Section 1 is devoted to prove the existence and uniqueness of a non-negative strong solution of a particular class of SDE. In Section 2, we construct a branching model in continuous time and space that is affected by a random environment as the unique strong solution of a SDE that satisfies the conditions of Section 1. This model has a branching part, an immigration part, a competition part, and the random environment is driven by a general Lévy process. We provide the Laplace exponent of the process given the environment. The chapter finishes with some examples where the Laplace exponent can be computed explicitly.

1.1 Stochastic differential equations

Stochastic differential equations with jumps have been playing an ever more important role in various domains of applied probability theory such as financial mathematics or mathematical biology. Under Lipschitz conditions, the existence and uniqueness of strong solutions of SDEs with jumps can be established by arguments based on Gronwall's inequality and the results on continuous-type equation, see for instance the monograph of Ikeda and Watanabe [57]. In view of the results of Fu and Li [45], Dawson and Li [28] and Li and Pu [71] weaker conditions would be sufficient for the existence and uniqueness of strong solutions for one-dimensional equations.

Fu and Li [45], motivated by describing CBI processes via SDEs, studied general SDEs that describes non-negative processes with jumps under general conditions. The authors in [45] (see also [28, 71]) provided criteria for the existence and uniqueness of strong solutions of those equations. The main idea of their criteria is to assume a monotonicity condition on the kernel associated with the compensated noise so that the continuity conditions can be weaken. Nonetheless, their criteria do not include the case where the branching mechanism of a CBI process has infinite mean and also the possibility of including a general random environment. This excludes some interesting models that can be of particular interest for applications.

Our goal in this section is to describe a general one-dimensional SDE that may relax the moment condition of Fu and Li [45] (see also [28] and [71]) and also include some extra randomness that can help us to define branching processes in more general random environment that those considered by Böinghoff and Hutzenthaler [20] and Bansaye et al. [9].

For $m, d, l \geq 1$, we define the index sets $I = \{1, \dots, m\}$, $J = \{1, \dots, l\}$ and $K = \{1, \dots, d\}$, and take $(U_i)_{i\in I}$ and $(V_j)_{j\in J}$ separable topological spaces whose topologies can be defined by complete metrics. Suppose that $(\mu_i)_{i\in I}$ and $(\nu_j)_{j\in J}$ are σ-finite Borel measures such that each μ_i and ν_j are defined on U_i and V_j, respectively. We say that the parameters $(b, (\sigma_k)_{k\in K}, (h_i)_{i\in I}, (g_j)_{j\in J})$ are *admissible* if

i) $b : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function such that $b(0) \geq 0$,

ii) for $k \in K$, $\sigma_k : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $\sigma_k(0) = 0$,

iii) for $i \in I$, let $g_i : \mathbb{R}_+ \times U_i \to \mathbb{R}$ be Borel functions such that $g_i(x, u_i) + x \geq 0$ for $x \geq 0$, $u_i \in U_i$ and $i \in I$,

iv) for $j \in J$, let $h_j : \mathbb{R}_+ \times V_j \to \mathbb{R}$ be Borel functions such that $h_j(0, v_j) = 0$ and $h_j(x, v_j) + x \geq 0$ for $x > 0$, $v_j \in V_j$ and $j \in J$.

For each $k \in K$, let $B^{(k)} = (B^{(k)}_t, t \geq 0)$ be a standard Brownian motion. We also let $(M_i)_{i\in I}$ and $(N_j)_{j\in J}$ be two sequences of Poisson random measures such that each $M_i(\mathrm{d}s, \mathrm{d}u)$ and $N_j(\mathrm{d}s, \mathrm{d}u)$ are defined on $\mathbb{R}_+ \times U_i$ and $\mathbb{R}_+ \times V_j$, respectively, and with intensities given by $\mathrm{d}s\mu_i(\mathrm{d}u)$ and $\mathrm{d}s\nu_j(\mathrm{d}v)$. We also suppose that $(B^{(k)})_{k\in K}$, $(M_i)_{i\in I}$ and $(N_j)_{j\in J}$ are independent of each other. The compensated measure of N_j is denoted by $\widetilde{N}_j$.

For each $i \in I$, let W_i be a subset in U_i such that $\mu_i(U_i \setminus W_i) < \infty$. For our purposes, we consider the following conditions on the parameters $(b, (\sigma_k)_{k\in K}, (h_i)_{i\in I}, (g_j)_{j\in J})$:

a) For each n, there is a positive constant A_n such that

$$\sum_{i\in I} \int_{W_i} |g_i(x, u_i) \wedge 1| \mu_i(\mathrm{d}u_i) \leq A_n(1 + x), \qquad \text{for every} \quad x \in [0, n].$$

b) Let $b(x) = b_1(x) - b_2(x)$, where b_2 is a non-decreasing continuous function. For each $n \geq 0$, there is a non-decreasing concave function $z \mapsto r_n(z)$ on $\mathbb{R}_+$ satisfying $\int_{0+} r_n(z)^{-1}\mathrm{d}z = \infty$ and

$$|b_1(x) - b_1(y)| + \sum_{i\in I} \int_{W_i} |g_i(x, u_i) \wedge n - g_i(y, u_i) \wedge n| \mu_i(\mathrm{d}u_i) \leq r_n(|x - y|)$$

for every $0 \leq x, y \leq n$.

c) For each $n \geq 0$ and $(v_1, \cdots, v_l) \in \mathcal{V}$, the function $x \mapsto x + h_j(x, v_j)$ is non-decreasing and there is a positive constant B_n such that for every $0 \leq x, y \leq n$,

$$\sum_{k\in K} |\sigma_k(x) - \sigma_k(y)|^2 + \sum_{j\in J} \int_{V_j} l_j^2(x, y, v_j)\nu_j(\mathrm{d}v_j) \leq B_n|x - y|$$

where $l_j(x, y, v_j) = h_j(x, v_j) \wedge n - h_j(y, v_j) \wedge n$.

A $[0,\infty]$-valued process $Z=(Z_t, t\geq 0)$ with càdlàg paths is called a *solution* of

$$\begin{aligned} Z_t =& Z_0 + \int_0^t b(Z_s)\mathrm{d}s + \sum_{k\in K}\int_0^t \sigma_k(Z_s)\mathrm{d}B_s^{(k)} \\ &+ \sum_{i\in I}\int_0^t\int_{U_i} g_i(Z_{s-},u_i)M_i(\mathrm{d}s,\mathrm{d}u_i) + \sum_{j\in J}\int_0^t\int_{V_j} h_j(Z_{s-},v_j)\widetilde{N}_j(\mathrm{d}s,\mathrm{d}v_j), \end{aligned} \tag{1.1}$$

if it satisfies the stochastic differential equation (1.1) up to the time $\tau_n := \inf\{t\geq 0 : Z_t \geq n\}$ for all $n\geq 1$, and $Z_t=\infty$ for all $t\geq \tau := \lim_{n\to\infty}\tau_n$. We say that Z is a *strong solution* if, in addition, it is adapted to the augmented natural filtration generated by $(B^{(k)})_{k\in K}$, $(M_i)_{i\in I}$ and $(N_j)_{j\in J}$.

Theorem 1. *Suppose that $(b,(\sigma_k)_{k\in K},(h_i)_{i\in I},(g_j)_{j\in J})$ are admissible parameters satisfying conditions a), b) and c). Then, the stochastic differential equation (1.1) has a unique non-negative strong solution. The process $Z=(Z_t,t\geq 0)$ is a Markov process and its infinitesimal generator $\mathcal{L}$ satisfies, for every $f\in C_b^2(\overline{\mathbb{R}}_+)$,*[1]

$$\begin{aligned} \mathcal{L}f(x) = b(x)f'(x) + \frac{1}{2}f''(x)\sum_{k\in K}\sigma_k^2(x) + \sum_{i\in I}\int_{U_i}\Big(f(x+g_i(x,u_i)) - f(x)\Big)\mu_i(\mathrm{d}u_i) \\ + \sum_{j\in J}\int_{V_j}\Big(f(x+h_j(x,v_j)) - f(x) - f'(x)h_j(x,v_j)\Big)\nu_j(\mathrm{d}v_j). \end{aligned} \tag{1.2}$$

Proof. The proof of this theorem uses several Lemmas that can be found in the Appendix.

We can extend the functions b,σ_k,g_i,h_j to $\mathbb{R}$ in the way that b is continuous with $b(x)\geq 0$ for all $x\leq 0$, and $\sigma_k(x)=g_i(x,u_i)=h_j(x,v_j)=0$ for all $x\leq 0$ and $u_i\in U_i$, $v_j\in V_j$. As in the proof of Proposition 2.1 in [45], if there exists $\epsilon>0$ such that $\mathbb{P}(\tau := \inf\{t\geq 0 : Z_t\leq -\epsilon\}<\infty)>0$. Then, by the assumptions iii) and iv), the process doesn't jump to $(-\infty,-\epsilon]$. Therefore on the event $\{\tau<\infty\}$, $Z_\tau = Z_{-\tau} = -\epsilon$ and $\tau > \varsigma := \sup\{s>\tau : Z_t\geq 0 \text{ for all } s\leq t\leq \tau\}$. Let $r\geq 0$ such that $\mathbb{P}(\tau>r>\varsigma)>0$, the contradiction occurs by observing that $Z_{t\wedge\tau}$ is non-decreasing in (r,∞) and $Z_r>-\epsilon$. Then any solution of (1.1) is non-negative.

For each $i\in I$ and $j\in J$, let $\{W_i^m : m\in\mathbb{N}\}$ and $\{V_j^m : m\in\mathbb{N}\}$ be non-increasing sequences of Borel subsets of W_i and V_j, such that $\bigcup_{m\in\mathbb{N}} W_i^m = W_i$ and $\mu_i(W_i^m)<\infty$; $\bigcup_{m\in\mathbb{N}} V_j^m = V_j$ and $\nu_j(V_j^m)<\infty$, respectively.

By the results for continuous-type stochastic equation (see for example Ikeda and Watanabe [57] Theorem IV.2.3), for each $n,m\in\mathbb{N}$, there is a non-negative weak solution to

$$\begin{aligned} Z_t = Z_0 + \int_0^t b(Z_s\wedge n)\mathrm{d}s - \sum_{j\in J}\int_0^t\int_{V_j^m}\big(h_j(Z_s\wedge n, v_j)\wedge n\big)\mu_j(\mathrm{d}v_j)\mathrm{d}s \\ + \sum_{k\in K}\int_0^t \sigma_k(Z_s^{(n)}\wedge n)\mathrm{d}B_s^{(k)}. \end{aligned} \tag{1.3}$$

By Hölder inequality and hypothesis c), the functions $x\mapsto \int_{V_j^m}\big(h_j(x\wedge n,v_j)\wedge n\big)\nu_j(\mathrm{d}v_j)$ are continuous for each $j\in J$ and $m\in\mathbb{N}$. Moreover $b(x\wedge n) - \sum_{j\in J}\int_{V_j^m}\big(h_j(x\wedge n,v_j)\wedge n\big)\nu_j(\mathrm{d}v_j)$

[1] $\overline{\mathbb{R}}_+ = [0,\infty]$ and $C_b^2(\overline{\mathbb{R}}_+) = \{$twice differentiable functions such that $f(\infty)=0\}$

is the difference between the continuous function $b_1(x \wedge n) + \sum_{j\in J}(x \wedge n)\nu(V_j^m)$ and the non-decreasing continuous function $b_2(x \wedge n) + \sum_{j\in J} \int_{V_j^m}[(x \wedge n) + \big(h_j(x \wedge n, v_j) \wedge n\big)]\nu_j(\mathrm{d}v_j)$. Then, by Lemma 10 (see the Appendix) the pathwise uniqueness holds for (1.3), so the equation has a unique non-negative strong solution. (see Situ [87], p. 104).

Now, by applying lemma 11 (see the Appendix), we deduce that for each $n, m \in \mathbb{N}$, there is a unique non-negative strong solution to

$$\begin{aligned} Z_t^{(n,m)} = Z_0 &+ \int_0^t b(Z_s^{(n,m)} \wedge n)\mathrm{d}s + \sum_{k\in K} \int_0^t \sigma_k(Z_s^{(n,m)} \wedge n)\mathrm{d}B_s^{(k)} \\ &+ \sum_{i\in I} \int_0^t \int_{W_i^m} \big(g_i(Z_{s-}^{(n,m)} \wedge n, u_i) \wedge n\big) M_i(\mathrm{d}s, \mathrm{d}u_i) \\ &+ \sum_{j\in J} \int_0^t \int_{V_j^m} \big(h_j(Z_{s-}^{(n,m)} \wedge n, v_j) \wedge n\big) \widetilde{N}_j(\mathrm{d}s, \mathrm{d}v_j). \end{aligned} \tag{1.4}$$

By Lemma 12 (in the Appendix), for each $n \in \mathbb{N}$, the sequence $\{Z_t^{(n,m)} : m \in \mathbb{N}\}$ is tight in $D([0,\infty), \mathbb{R}_+)$. Moreover, by Lemma 14 (see the Appendix) the weak limit point of the sequence is a non-negative weak solution to

$$\begin{aligned} Z_t^{(n)} = Z_0 &+ \int_0^t b(Z_s^{(n)} \wedge n)\mathrm{d}s + \sum_{k\in K} \int_0^t \sigma_k(Z_s^{(n)} \wedge n)\mathrm{d}B_s^{(k)} \\ &+ \sum_{i\in I} \int_0^t \int_{W_i} \big(g_i(Z_{s-}^{(n)} \wedge n, u_i) \wedge n\big) M_i(\mathrm{d}s, \mathrm{d}u_i) \\ &+ \sum_{j\in J} \int_0^t \int_{V_j} \big(h_j(Z_{s-}^{(n)} \wedge n, v_j) \wedge n\big) \widetilde{N}_j(\mathrm{d}s, \mathrm{d}v_j). \end{aligned} \tag{1.5}$$

By lemma 10 (in the Appendix) the pathwise uniqueness holds for (1.5). This guaranties that there is a unique non-negative strong solution of (1.5). Next, we apply Lemma 11 (see the Appendix) that allows us to replace the space W_i by U_i in the SDE (1.5). In other words we deduce that for $n \geq 0$ there is a unique non-negative strong solution of

$$\begin{aligned} Z_t^{(n)} = Z_0 &+ \int_0^t b(Z_s^{(n)} \wedge n)\mathrm{d}s + \sum_{k\in K} \int_0^t \sigma_k(Z_s^{(n)} \wedge n)\mathrm{d}B_s^{(k)} \\ &+ \sum_{i\in I} \int_0^t \int_{U_i} \big(g_i(Z_{s-}^{(n)} \wedge n, u_i) \wedge n\big) M_i(\mathrm{d}s, \mathrm{d}u_i) \\ &+ \sum_{j\in J} \int_0^t \int_{V_j} \big(h_j(Z_{s-}^{(n)} \wedge n, v_j) \wedge n\big) \widetilde{N}_j(\mathrm{d}s, \mathrm{d}v_j). \end{aligned} \tag{1.6}$$

Finally, we proceed to show that there is a unique non-negative strong solution to the SDE (1.1). In order to do so, we first define $\tau_m = \inf\{t \geq 0 : Z_t^{(m)} \geq m\}$, for $m \geq 0$, and then we prove that the sequence $(\tau_m, m \geq 0)$ is non-decreasing and that $Z_t^{(m)} = Z_t^{(n)}$ for $m \leq n$ and $t < \tau_m$. Since the Poisson random measures are independent, they do not jump simultaneously. Therefore, by using the fact that the trajectory $t \mapsto Z_t^{(m)}$ has no jumps larger than m on the interval $[0, \tau_m)$, we obtain that for each $i \in I$, $j \in J$, and $0 \le t < \tau_m$,

$$g_i(Z_t^{(m)}, u) \le m \qquad \text{and} \qquad h_j(Z_t^{(m)}, v) \le m, \quad u \in U_i, v \in V_j.$$

This implies that $Z_t^{(m)}$ satisfies (1.1) on the interval $[0, \tau_m)$. For $0 \le m \le n$, let $(Y_t^{(n)}, t \ge 0)$ be the strong solution to

$$\begin{aligned} Y_t^{(n)} = Z_{\tau_m-}^{(m)} &+ \int_0^t b(Y_s^{(n)} \wedge n)\mathrm{d}s + \sum_{k\in K}\int_0^t \sigma_k(Y_s^{(n)} \wedge n)\mathrm{d}B_{\tau_m+s}^{(k)} \\ &+ \sum_{i\in I}\int_0^t\int_{U_i}\big(g_i(Y_{s-}^{(n)} \wedge n, u)\wedge n\big)M_i(\tau_m + \mathrm{d}s, \mathrm{d}u) \\ &\qquad + \sum_{j\in J}\int_0^t\int_{V_j}\big(h_j(Y_{s-}^{(n)} \wedge n, v)\wedge n\big)\widetilde{N}_j(\tau_m + \mathrm{d}s, \mathrm{d}v). \end{aligned}$$

We define $\tilde{Y}_t^{(n)} = Z_t^{(m)}$ for $0 \le t \le \tau_m$ and $\tilde{Y}_t^{(n)} = Y_{t-\tau_m}^{(n)}$ for $t \ge \tau_m$. Note that $(\widetilde{Y}_t^{(n)}, t \ge 0)$ is solution to (1.6). From the uniqueness, we deduce that $Z_t^{(n)} = \widetilde{Y}_t^{(n)}$ for all $t \ge 0$. In particular, we have that $Z_t^{(n)} = Z_t^{(m)} < m$ for $0 \le t < \tau_m$. Consequently, the sequence $(\tau_m, m \ge 0)$ is non-decreasing.

Next, we define the process $Z = (Z_t, t \ge 0)$ as

$$Z_t = \begin{cases} Z_t^{(m)} & \text{if} \quad t < \tau_m, \\ \infty & \text{if} \quad t \ge \lim_{m\to\infty} \tau_m. \end{cases}$$

It is not difficult to see that Z is a weak solution to (1.1). In order to prove our result, we consider two solutions to (1.1), Z' and Z'', and consider $\tau'_m = \inf\{t \ge 0 : Z'_t \ge m\}$, $\tau''_m = \inf\{t \ge 0 : Z''_t \ge m\}$ and $\tau_m = \tau'_m \wedge \tau''_m$. Therefore Z' and Z'' satisfy (1.6) on $[0, \tau_m)$, implying that they are indistinguishable on $[0, \tau_m)$. If $\tau_\infty = \lim_{m\to\infty} \tau_m < \infty$, we have two possibilities either $Z'_{\tau_\infty-} = Z''_{\tau_\infty-} = \infty$ or one of them has a jump of infinity size at τ_∞. In the latter case, this jump comes from an atom of one of the Poisson random measures $(M_i)_{i\in I}$ or $(N_j)_{j\in J}$, so both processes have it. Since after this time both processes are equal to ∞, we obtain that Z' and Z'' are indistinguishable. In other words, there is a unique strong solution to (1.1). The strong Markov property is due to the fact that we have a strong solution, the integrators are Lévy processes and the integrand functions doesn't depend on the time. (See Theorem V.32 in Protter [83], where the Lipschitz continuity is just to guarantee the existence and uniqueness of the solution) and by Itô's formula it is easy to show that the infinitesimal generator of $(Z_t, t \ge 0)$ is given by (1.2). □

1.2 CBI-processes with competition in a Lévy random environment

In this section, we construct a branching model in continuous time and space that is affected by a random environment as the unique strong solution of a SDE that satisfies the conditions of Theorem 1. In this model, the random environment is driven by a general Lévy process. In order to define CBI-processes in a Lévy random environment (CBILRE for short), we first introduce the objects that are involved in the *branching, immigration* and *environmental* parts. For the *branching part*, we introduce $B^{(b)} = (B^{(b)}_t, t \geq 0)$ a standard Brownian motion and $N^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ a Poisson random measure independent of $B^{(b)}$, with intensity $\mathrm{d}s\Lambda(\mathrm{d}z)\mathrm{d}u$ where $\Lambda(\mathrm{d}z) = \mu(\mathrm{d}z) + q\delta_\infty(\mathrm{d}z)$, for $q \geq 0$. We denote by $\widetilde{N}^{(b)}$ for the compensated measure of $N^{(b)}$ and recall that the measure μ is concentrated on $(0,\infty)$ and satisfies

$$\int_{(0,\infty)} (1 \wedge z^2)\mu(\mathrm{d}z) < \infty.$$

The *immigration* term is given by a Poisson random measure $M(\mathrm{d}s, \mathrm{d}z)$ with intensity $\mathrm{d}s\nu(\mathrm{d}z)$ where the measure ν is supported in $(0,\infty)$ and satisfies

$$\int_{(0,\infty)} (1 \wedge z)\nu(\mathrm{d}z) < \infty.$$

Finally, for the *environmental* term, we introduce $B^{(e)} = (B^{(e)}_t, t \geq 0)$ a standard Brownian motion and $N^{(e)}(\mathrm{d}s, \mathrm{d}z)$ a Poisson random measure in $\mathbb{R}_+ \times \mathbb{R}$ independent of $B^{(e)}$ with intensity $\mathrm{d}s\pi(\mathrm{d}y)$, $\widetilde{N}^{(e)}$ its compensated version and π is a measure concentrated on $\mathbb{R} \setminus \{0\}$ such that

$$\int_{\mathbb{R}} (1 \wedge z^2)\pi(\mathrm{d}z) < \infty.$$

We will assume that all the objects involve in the branching, immigration and environmental terms are mutually independent.

A CB-processes in a Lévy random environment with immigration and competition is defined as the solution of the stochastic differential equation

$$\begin{aligned} Z_t =& Z_0 + \int_0^t (\mathbf{d} + aZ_s)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B^{(b)}_s \\ &- \int_0^t \beta(Z_s)\mathrm{d}s + \int_0^t \int_{(0,\infty)} zM^{(im)}(\mathrm{d}s, \mathrm{d}z) + \int_0^t Z_{s-}\mathrm{d}S_s \\ &+ \int_0^t \int_{(0,1)} \int_0^{Z_{s-}} z\widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[1,\infty)} \int_0^{Z_{s-}} zN^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \end{aligned} \tag{1.7}$$

where $a \in \mathbb{R}$, $\mathbf{d}, \gamma \geq 0$, β is a continuous non-decreasing function on $[0,\infty)$ with $\beta(0) = 0$,

$$S_t = \alpha t + \sigma B^{(e)}_t + \int_0^t \int_{(-1,1)} (e^z - 1)\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{\mathbb{R}\setminus(-1,1)} (e^z - 1)N^{(e)}(\mathrm{d}s, \mathrm{d}z), \tag{1.8}$$

with $\alpha \in \mathbb{R}$ and $\sigma \geq 0$.

Corollary 1. *The stochastic differential equation (1.7) has a unique non-negative strong solution. The CBLRE $Z = (Z_t, t \geq 0)$ is a Markov process and its infinitesimal generator $\mathcal{A}$ satisfies, for every $f \in C^2_b(\overline{\mathbb{R}}_+)$,*

$$\begin{aligned} \mathcal{A}f(x) =& \Big(ax + \alpha x - \beta(x) + \mathbf{d}\Big)f'(x) + \int_{(0,\infty)} \Big(f(x+z) - f(x)\Big)\nu(\mathrm{d}z) \\ &+ \left(\gamma^2 x + \frac{\sigma^2}{2}x^2\right) f''(x) + x\int_{(0,\infty)} \Big(f(x+z) - f(x) - zf'(x)\mathbf{1}_{\{z<1\}}\Big)\Lambda(\mathrm{d}z) \\ &+ \int_{\mathbb{R}} \Big(f(xe^z) - f(x) - x(e^z - 1)f'(x)\mathbf{1}_{\{|z|<1\}}\Big)\pi(\mathrm{d}z). \end{aligned} \tag{1.9}$$

Proof. The proof of this result is a straightforward application of Theorem 1. Take the set of index $K = J = \{1, 2\}$ and $I = \{1, 2, 3\}$; the spaces

$$
\begin{aligned}
&U_1 = W_1 = [1, \infty) \times \mathbb{R}_+, && U_2 = \mathbb{R} \setminus (-1, 1), && W_2 = (-\infty, -1], \\
&U_3 = W_3 = \mathbb{R}_+, && V_1 = (0, 1) \times \mathbb{R}_+, && V_2 = (-1, 1),
\end{aligned}
$$

with associated Poisson random measures $M_1 = N^{(b)}, M_2 = N^{(e)}, M_3 = M^{(im)}, N_1 = N^{(b)}$ and $N_2 = N^{(e)}$, respectively; and standard Brownian motions $B^{(1)} = B^{(b)}$ and $B^{(2)} = B^{(e)}$. We also take the functions

$$
\begin{aligned}
&b(x) = ax - \beta(x) + \mathbf{d}, && \sigma_1(x) = \sqrt{2\gamma^2 x}, && \sigma_2(x) = \sigma x, \\
&g_1(x, z, u) = z\mathbf{1}_{\{u \le x\}}, && g_2(x, z) = x(e^z - 1), && g_3(x, z) = z, \\
&h_1(x, z, u) = z\mathbf{1}_{\{u \le x\}}, && h_2(x, z) = x(e^z - 1), &&
\end{aligned}
$$

which are admissible and verify conditions a), b) and c). □

Similarly to the results of Bansaye et al. [9], we can compute the Laplace transform of a reweighted version of Z given the environment and under the assumption that $q = 0$ and $\beta \equiv 0$. In order to do so, we define the following hypothesis

$$
\int_{[1,\infty)} x\mu(\mathrm{d}x) < \infty. \tag{\textbf{H1}}
$$

It is important to note that conditionally on the environment K, the process Z satisfies the branching property. This property is inherited from the branching property of the original CBI process and the fact that the additional jumps are multiplicative.

Recall that the associated branching mechanism ψ satisfies the celebrated Lévy-Khintchine formula, i.e.

$$
\psi(\lambda) = -a\lambda + \gamma^2\lambda^2 + \int_{(0,\infty)} \big(e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}}\big)\mu(\mathrm{d}x), \qquad \lambda \ge 0
$$

and observe that from our assumption, $|\psi'(0+)| < \infty$ and

$$
\psi'(0+) = -a - \int_{[1,\infty)} x\mu(\mathrm{d}x).
$$

We also recall that the immigration mechanism is given by

$$
\phi(u) = \mathtt{d}u + \int_0^\infty (1 - e^{-ut})\nu(\mathrm{d}t), \qquad u \ge 0.
$$

When (**H1**) holds, we define the auxiliary process

$$
K_t^{(0)} = \mathbf{m}t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} v\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}v) + \int_0^t \int_{\mathbb{R}\setminus(-1,1)} vN^{(e)}(\mathrm{d}s, \mathrm{d}v), \tag{1.10}
$$

where

$$
\mathbf{m} = \alpha - \psi'(0+) - \frac{\sigma^2}{2} - \int_{(-1,1)} (e^v - 1 - v)\pi(\mathrm{d}v).
$$

Proposition 1. *Suppose that* (**H1**) *holds. Then for every* $z, \lambda, t > 0$,

$$\mathbb{E}_z\left[\exp\left\{-\lambda Z_t e^{-K_t^{(0)}}\right\}\big|K^{(0)}\right] = \exp\left\{-zv_t(0,\lambda,K^{(0)}) - \int_0^t \phi\Big(v_t(r,\lambda,K^{(0)})e^{-K_r^{(0)}}\Big)\mathrm{d}r\right\} \quad a.s., \tag{1.11}$$

where for every $t, \lambda \geq 0$, *the function* $(v_t(s,\lambda,K^{(0)}), s \leq t)$ *is the a.s. unique solution of the backward differential equation*

$$\frac{\partial}{\partial s}v_t(s,\lambda,K^{(0)}) = e^{K_s^{(0)}}\psi_0(v_t(s,\lambda,K^{(0)})e^{-K_s^{(0)}}), \qquad v_t(t,\lambda,K^{(0)}) = \lambda, \tag{1.12}$$

and

$$\psi_0(\lambda) = \psi(\lambda) - \lambda\psi'(0) = \gamma^2\lambda^2 + \int_{(0,\infty)}\big(e^{-\lambda x} - 1 + \lambda x\big)\mu(\mathrm{d}x) \qquad \lambda \geq 0.$$

Proof. The first part of the proof follows similar arguments as those used in Bansaye et al. [9]. The main problem in proving our result is finding the a.s. unique solution of the backward differential equation (1.12) in the general case. In order to do so, we need an approximation technique based on the Lévy-Itô decomposition of the Lévy process $K^{(0)}$. The proof of the latter can be found in the appendix in Lemma 15.

For sake of completeness, we provide the main steps of the proof which are similar as those used in [9]. We first define $\widetilde{Z}_t = Z_t e^{-K_t^{(0)}}$, for $t \geq 0$, and choose

$$F(s,x) = \exp\left\{-xv_t(s,\lambda,K^{(0)}) - \int_s^t \phi(v_t(r,\lambda,K^{(0)})e^{-K_r^{(0)}})\mathrm{d}r\right\}, \qquad s \leq t,\ 0 \leq x$$

where $v_t(s,\lambda,K^{(0)})$ is differentiable with respect to the variable s, non-negative and such that $v_t(t,\lambda,K^{(0)}) = \lambda$ for all $\lambda \geq 0$. We observe that conditionally on $K^{(0)}$, that $(F(s,\widetilde{Z}_s), s \in [0,t])$ is a martingale (using Itô's formula) if and only if

$$\frac{\partial}{\partial s}v_t(s,\lambda,K^{(0)}) = \gamma^2 v_t(s,\lambda,K^{(0)})^2 e^{-K_s^{(0)}} + e^{K_s^{(0)}}\int_0^\infty\left(e^{-e^{-K_s^{(0)}}v_t(s,\lambda,K^{(0)})z} - 1 + e^{-K_s^{(0)}}v_t(s,\lambda,K^{(0)})z\right)\mu(\mathrm{d}z).$$

That is $v_t(s,\lambda,K^{(0)})$ solves (1.12). Provided that $v_t(s,\lambda,K^{(0)})$ exist a.s., we get that the process $\left(\exp\left\{-\widetilde{Z}_s v_t(s,\lambda,K^{(0)})\right\}, 0 \leq s \leq t\right)$ conditionally on K is a martingale, and hence

$$\mathbb{E}_z\left[\exp\left\{-\lambda\widetilde{Z_t}\right\}\big|K^{(0)}\right] = \exp\left\{-zv_t(0,\lambda,K^{(0)}) - \int_0^t \phi(v_t(r,\lambda,K^{(0)})e^{-K_r^{(0)}})\mathrm{d}r\right\}.$$

□

Remark 1. *When* $|\psi'(0+)| = \infty$, *the auxiliary process can be taken as follows*

$$K_t = \mathbf{n}t + \sigma B_t^{(e)} + \int_0^t\int_{(-1,1)} v\widetilde{N}^{(e)}(\mathrm{d}s,\mathrm{d}v) + \int_0^t\int_{\mathbb{R}\setminus(-1,1)} vN^{(e)}(\mathrm{d}s,\mathrm{d}v), \tag{1.13}$$

where

$$\mathbf{n} = \alpha - \frac{\sigma^2}{2} - \int\limits_{(-1,1)} (e^v - 1 - v)\pi(\mathrm{d}v).$$

Suppose that there is a unique a.s. solution $v_t(s, \lambda, K)$ to the backward differential equation

$$\frac{\partial}{\partial s} v_t(s, \lambda, K) = e^{K_s}\psi(v_t(s, \lambda, K)e^{-K_s}), \qquad v_t(t, \lambda, K) = \lambda, \tag{1.14}$$

In this case, by following the same arguments as in the last part of the proof of the previous proposition, the process Z conditioned on K satisfies that for every $z, \lambda, t > 0$,

$$\begin{aligned}\mathbb{E}_z &\left[\exp\left\{-\lambda Z_t e^{-K_t}\right\} \big| K\right] \\ &= \exp\left\{-z v_t(0, \lambda, K) - \int_0^t \phi\Big(v_t(r, \lambda, K)e^{-K_r}\Big)\mathrm{d}r\right\} \quad a.s.\end{aligned} \tag{1.15}$$

Before we continue with the exposition of this manuscript, we would like to provide some examples where we can compute explicitly the Laplace exponent of the CB-process in a Lévy random environment without immigration ($\phi \equiv 0$).

Example 1 (**Neveu case).** The Neveu branching process in a Lévy random environment has branching mechanism given by

$$\psi(u) = u\log(u) = cu + \int_{(0,\infty)} \big(e^{-ux} - 1 + ux\mathbf{1}_{\{x<1\}}\big)x^{-2}\mathrm{d}x, \qquad u > 0,$$

where $c \in \mathbb{R}$ is a suitable constant. In this particular case the backward differential equation (1.14) satisfies

$$\frac{\partial}{\partial s} v_t(s, \lambda, K) = v_t(s, \lambda, \delta)\log(e^{-K_s}v_t(s, \lambda, K)), \qquad v_t(t, \lambda, K) = \lambda.$$

One can solve the above equation and deduce

$$v_t(s, \lambda, K) = \exp\left\{e^s\left(\int_s^t e^{-u}K_u\mathrm{d}u + \log(\lambda)e^{-t}\right)\right\}, \qquad \text{for } s \le t.$$

Hence, from identity (1.15) for all $z, \lambda, t > 0$

$$\mathbb{E}_z\Big[\exp\Big\{-\lambda Z_t e^{-K_t}\Big\}\Big|K\Big] = \exp\left\{-z\lambda^{e^{-t}}\exp\left\{\int_0^t e^{-s}K_s\mathrm{d}s\right\}\right\} \qquad \text{a.s.} \tag{1.16}$$

Observe that

$$\int_0^t e^{-s}K_s\mathrm{d}s = -e^{-t}K_t + \int_0^t e^{-s}\mathrm{d}K_s.$$

According to Sato ([84], Chapter 17), $\int_0^t e^{-s}\mathrm{d}K_s$ is an infinitely divisible random variable with characteristic exponent

$$\psi(\lambda) = \int_0^t \psi_K(\lambda e^{-s})\mathrm{d}s, \qquad \lambda \ge 0,$$

where ψ_K is the characteristic exponent of K.
In particular, when K has continuous paths, the r.v. $\int_0^t e^{-s}K_s\mathrm{d}s$, is normal distributed with mean $(\alpha - \frac{\sigma^2}{2})(1 - e^{-t} - te^{-t})$ and variance $\frac{\sigma^2}{2}(1 + 4e^{-t} - 3e^{-2t})$, for $t \ge 0$. In other words, the Laplace transform of $Z_t e^{-K_t}$ can be determined by the Laplace transform of a log-normal distribution which we know exists but there is not an explicit form of it.

Example 2 (Feller case). Assume that $a = \mu(0,\infty) = 0$, and the environment is a Brownian motion with drift. Thus the CB-process in a Brownian random environment (1.7) is reduced to the following SDE

$$Z_t = Z_0 + \alpha \int_0^t Z_s \mathrm{d}s + \sigma \int_0^t Z_s \mathrm{d}B_s^{(e)} + \int_0^t \sqrt{2\gamma^2 Z_s} \mathrm{d}B_s,$$

where the random environment is given by $S_t = \alpha t + \sigma B_t^{(e)}$. This SDE is equivalent to the strong solution of the SDE

$$\begin{aligned} dZ_t =& \frac{\sigma^2}{2} Z_t \mathrm{d}t + Z_t \mathrm{d}K_t + \sqrt{2\gamma^2 Z_s} \mathrm{d}B_s, \\ dK_t =& a_0 \mathrm{d}t + \sigma \mathrm{d}B_t^{(e)}, \end{aligned}$$

where $a_0 = \alpha - \sigma^2/2$, which is the branching diffusion in random environment studied by Böinghoff and Hutzenthaler [20].

Observe that in this case $\psi'(0+) = 0$. Hence $K = K^{(0)}$ and the backward differential equation (1.14) satisfies

$$\frac{\partial}{\partial s} v_t(s,\lambda,K) = \gamma^2 v_t^2(s,\lambda,K) e^{-K_s}, \qquad v_t(t,\lambda,K) = \lambda.$$

The above equation can be solved and after some computations one can deduce

$$v_t(s,\lambda,K) = \left(\lambda^{-1} + \gamma^2 \int_s^t e^{-K_u} \mathrm{d}u\right)^{-1} \qquad \text{for } s \le t.$$

Hence, from identity (1.15) we get

$$\mathbb{E}_z\Big[\exp\Big\{-\lambda Z_t e^{-K_t}\Big\}\Big|K\Big] = \exp\left\{-z\left(\lambda^{-1} + \gamma^2 \int_0^t e^{-K_u} \mathrm{d}u\right)^{-1}\right\} \qquad \text{a.s.} \tag{1.17}$$

Example 3 (Stable case). Now, we assume that the branching mechanism is of the form

$$\psi(\lambda) = c\lambda^{\beta+1}, \qquad \lambda \ge 0,$$

for some $\beta \in (-1,0) \cup (0,1]$ and c is such that

$$\begin{cases} c < 0 & \text{if } \beta \in (-1,0), \\ c > 0 & \text{if } \beta \in (0,1]. \end{cases}$$

Under this assumption, the process Z satisfies the following stochastic differential equation

$$Z_t = Z_0 + \mathbf{1}_{\{\beta=1\}} \int_0^t \sqrt{2cZ_s} \mathrm{d}B_s + \mathbf{1}_{\{\beta\neq1\}} \int_0^t \int_0^\infty \int_0^{Z_{s-}} z\widehat{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t Z_{s-} \mathrm{d}S_s, \tag{1.18}$$

where the process S is defined as in (1.8), $B = (B_t, t \ge 0)$ is a standard Brownian motion, N is a Poisson random measure with intensity

$$\frac{c\beta(\beta+1)}{\Gamma(1-\beta)} \frac{1}{z^{2+\beta}} \mathrm{d}s\mathrm{d}z\mathrm{d}u,$$

$\widetilde{N}$ is its compensated version, and

$$\widehat{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = \begin{cases} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) & \text{if } \beta \in (-1, 0), \\ \widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) & \text{if } \beta \in (0, 1), \end{cases}$$

Note that

$$\psi'(0+) = \begin{cases} -\infty & \text{if } \beta \in (-1, 0), \\ 0 & \text{if } \beta \in (0, 1]. \end{cases}$$

Hence, when $\beta \in (0, 1]$, we have $K_t^{(0)} = K_t$, for $t \geq 0$. In both cases, we use the backward differential equation (1.14),

$$\frac{\partial}{\partial s} v_t(s, \lambda, K) = -c v_t^{\beta+1}(s, \lambda, K) e^{-\beta K_s} \qquad v_t(t, \lambda, K) = \lambda.$$

Similarly to the Feller case, we can solve the above equation and get

$$v_t(s, \lambda, K) = \left(\lambda^{-\beta} + \beta c \int_s^t e^{-\beta K_u} \mathrm{d}u \right)^{-1/\beta} \qquad \text{for } s \leq t.$$

Hence, from (1.15) we get the following a.s. identity

$$\mathbb{E}_z \Big[\exp \Big\{ -\lambda Z_t e^{-K_t} \Big\} \Big| K \Big] = \exp \left\{ -z \left(\lambda^{-\beta} + \beta c \int_0^t e^{-\beta K_u} \mathrm{d}u \right)^{-1/\beta} \right\}, \tag{1.19}$$

Our last example is a stable CBI in a Lévy random environment. In this case, the branching and immigration mechanisms are β-stable.

Example 4 (**Stable case with immigration).** Here we assume that the branching and immigration mechanisms are of the form $\psi(\lambda) = c\lambda^{\beta+1}$ and $\phi(\lambda) = d\lambda^{\beta}$, where $\beta \in (0, 1]$, $c, d > 0$. Hence, the stable CBILRE-process is given as the unique non-negative strong solution of the stochastic differential equation

$$\begin{aligned} Z_t = & Z_0 + \mathbf{1}_{\{\beta=1\}} \left(\int_0^t \sqrt{2cZ_s} \mathrm{d}B_s + dt \right) + \int_0^t Z_{s-} \mathrm{d}S_s \\ & + \mathbf{1}_{\{\beta\neq1\}} \left(\int_0^t \int_0^\infty \int_0^{Z_{s-}} z\widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_0^\infty zM(\mathrm{d}s, \mathrm{d}z) \right), \end{aligned}$$

where the process S is defined as in (1.8), $B = (B_t, t \geq 0)$ is a standard Brownian motion, and N and M are two independent Poisson random measures with intensities

$$\frac{c\beta(\beta+1)}{\Gamma(1-\beta)} \frac{1}{z^{2+\beta}} \mathrm{d}s\mathrm{d}z\mathrm{d}u \qquad \text{and} \qquad \frac{d\beta}{\Gamma(1-\beta)} \frac{1}{z^{1+\beta}} \mathrm{d}s\mathrm{d}z.$$

Observe that $\psi'(0+) = 0$ and $K = K^{(0)}$. From (1.12) we get the following a.s. identity

$$\begin{aligned} \mathbb{E}_z \left[\exp\left\{-\lambda Z_t e^{-K_t}\right\} \middle| K \right] = & \exp\left\{ -z \left(\beta c \int_0^t e^{-\beta K_s} \mathrm{d}s + \lambda^{-\beta} \right)^{-1/\beta} \right\} \\ & \times \exp\left\{ -\frac{d}{\beta c} \ln \left(\beta c \lambda^{\beta} \int_0^t e^{-\beta K_s} \mathrm{d}s + 1 \right) \right\}. \end{aligned} \tag{1.20}$$

If we take limits as $z \downarrow 0$, we deduce that the entrance law at 0 of the process $(Z_t e^{-K_t}, t \geq 0)$ satisfies

$$\mathbb{E}_0 \left[\exp\left\{-\lambda Z_t e^{-K_t}\right\} \middle| K \right] = \exp\left\{ -\frac{d}{\beta c} \ln \left(\beta c \lambda^{\beta} \int_0^t e^{-\beta K_s} \mathrm{d}s + 1 \right) \right\}.$$

Chapter 2

Long term behavior

This chapter is based in paper [79] elaborated in collaboration with Juan Carlos Pardo. We study the long-term behavior of the two classes of processes; CB-processes in a Lévy random environment and a competition model in a Lévy random environment. In the first section we study general CB-processes in a Lévy random environment. In particular, we discuss when the process is conservative and explosion and extinction events. We provide some examples where we can find explicitly the probability of such events. In the second section, we study the competition model in a Lévy random environment. This process can be seen as a population model that extends the competition model given in Evans et al. [44]. We provide the long term behavior of the process and in the case when the random environment has no negative jumps, we compute the Laplace transform of the first passage time below a level.

2.1 CB-processes in a Lévy random environment

In the sequel, we exclude from the model defined by (1.7), the competition mechanism β and the immigration term $M^{(im)}$. Let ψ be a branching mechanism, i.e.

$$\psi(\lambda) = -q - a\lambda + \gamma^2\lambda^2 + \int_{(0,\infty)} \big(e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}}\big)\mu(\mathrm{d}x), \qquad \lambda \geq 0,$$

where $a \in \mathbb{R}$, $q, \gamma \geq 0$ and μ is a measure concentrated on $(0,\infty)$ such that $\int_{(0,\infty)} \big(1 \wedge x^2\big)\mu(\mathrm{d}x)$ is finite. Recall that a CB-processes in a Lévy random environment (CBLRE) with branching mechanism ψ, is defined as the solution of the stochastic differential equation

$$\begin{aligned} Z_t =& Z_0 + \int_0^t aZ_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s^{(b)} + \int_0^t Z_{s-}\mathrm{d}S_s \\ &+ \int_0^t \int_{(0,1)} \int_0^{Z_{s-}} z\widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[1,\infty)} \int_0^{Z_{s-}} zN^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \end{aligned} \tag{2.1}$$

where $B^{(b)} = (B_t^{(b)}, t \geq 0)$ is a standard Brownian motion, $N^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ is a Poisson random measure with intensity $\mathrm{d}s\Lambda(\mathrm{d}z)\mathrm{d}u$ where $\Lambda(\mathrm{d}z) = \mu(\mathrm{d}z) + q\delta_\infty(\mathrm{d}z)$, and S_t is the environment given by (1.8), i.e.

$$S_t = \alpha t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} (e^z - 1)\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{\mathbb{R}\setminus(-1,1)} (e^z - 1)N^{(e)}(\mathrm{d}s, \mathrm{d}z),$$

with $\alpha \in \mathbb{R}$, $\sigma \geq 0$, $B^{(e)} = (B_t^{(e)}, t \geq 0)$ a standard Brownian motion and $N^{(e)}(\mathrm{d}s, \mathrm{d}z)$ a Poisson random measure in $\mathbb{R}_+ \times \mathbb{R}$ with intensity $\mathrm{d}s\pi(\mathrm{d}y)$. Additionally, all the process are independent of each other.

In this section, we are interested in determining the long term behaviour of CB-processes in a Lévy random environment. Similarly to the CB-processes case, there are three events which are of immediate concern for the process Z, *explosion*, *absorption* and *extinction*. Recall that the event of explosion at fixed time t, is given by $\{Z_t = \infty\}$. When $\mathbb{P}_z(Z_t < \infty) = 1$, for all $t > 0$ and $z > 0$, we say the process is *conservative*. In the second event, we observe from the definition of Z that if $Z_t = 0$ for some $t > 0$, then $Z_{t+s} = 0$ for all $s \geq 0$, which makes 0 an absorbing state. As Z_t is to be thought of as the size of a given population at time t, the event $\{\lim_{t\to\infty} Z_t = 0\}$ is referred as *extinction*.

To the best knowledge of the author, *explosion* has never been studied before for branching processes in random environment even in the discrete setting. Most of the results that appear in the literature are related to *extinction*. In this section, we first provide a sufficient condition under which the process Z is conservative and an example where we can determine explicitly the probability of explosion. Under the condition that the process is conservative, we study the probability of extinction under the influence of the random environment.

Without presence of environment, it is completely characterized when a CB-process presents explosion or extinction a.s. Moreover, the explosion and extinction probabilities are known. If the environment is affecting the process, it is not easy to deduce these probabilities. In the next chapter, we provide an example under which both events can be computed explicitly, as well as their asymptotic behaviour when time increases. In order to find this behaviour, first, we study the exponential functional of a Lévy processes.

Recall that $\psi'(0+) \in [-\infty, \infty)$, and that whenever $|\psi'(0+)| < \infty$, we write

$$\mathbf{m} = \alpha - \psi'(0+) - \frac{\sigma^2}{2} - \int\limits_{(-1,1)} (e^v - 1 - v)\pi(\mathrm{d}v),$$

and

$$\psi_0(\lambda) = \psi(\lambda) - \lambda\psi'(0+), \qquad \lambda \geq 0.$$

The following proposition provides necessary conditions under which the process Z is conservative.

Proposition 2. *Assume that $q = 0$ and $|\psi'(0+)| < \infty$, then a CBLRE with branching mechanism ψ is conservative.*

Proof. Under our assumption, the auxiliary process (1.10) takes the form

$$K_t^{(0)} = \mathbf{m}t + \sigma B_t^{(e)} + \int_0^t \int\limits_{(-1,1)} v\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}v) + \int_0^t \int_{\mathbb{R}\setminus(-1,1)} vN^{(e)}(\mathrm{d}s, \mathrm{d}v),$$

and $v_t(s, \lambda, K^{(0)})$ is the unique solution to the backward differential equation (1.12). From identity (1.11), we know that for $z, \lambda, t > 0$

$$\mathbb{E}_z\Big[\exp\Big\{ - \lambda Z_t e^{-K_t^{(0)}}\Big\}\Big] = \mathbb{E}\Big[\exp\Big\{ - zv_t(0, \lambda, K^{(0)})\Big\}\Big].$$

Thus if we take limits as $\lambda \downarrow 0$, we deduce

$$\mathbb{P}_z\big(Z_t < \infty\big) = \lim_{\lambda\downarrow 0} \mathbb{E}_z\Big[\exp\Big\{ -\lambda Z_t e^{-K_t^{(0)}}\Big\}\Big] = \mathbb{E}\left[\exp\left\{-z\lim_{\lambda\downarrow 0} v_t(0,\lambda,K^{(0)})\right\}\right], \qquad z,t>0,$$

where the limits are justified by monotonicity and dominated convergence. This implies that a CBLRE is conservative if and only if

$$\lim_{\lambda\downarrow 0} v_t(0,\lambda,K^{(0)}) = 0.$$

Let us introduce the function $\Phi(\lambda) = \lambda^{-1}\psi_0(\lambda)$, $\lambda > 0$ and observe that $\Phi(0) = \psi_0'(0+) = 0$. Since ψ_0 is convex, we deduce that Φ is increasing. Finally, solving the equation (1.12) with $\psi_0(\lambda) = \lambda\Phi(\lambda)$, we get

$$v_t(s,\lambda,K^{(0)}) = \lambda \exp\left\{-\int_s^t \Phi(e^{-K_r^{(0)}} v_t(r,\lambda,K^{(0)}))\mathrm{d}r\right\}.$$

Therefore, since Φ is increasing and $\Phi(0) = 0$, we have

$$0 \le \lim_{\lambda\to 0} v_t(0,\lambda,K^{(0)}) = \lim_{\lambda\to 0} \lambda \exp\left\{-\int_0^t \Phi(e^{-K_r} v_t(r,\lambda,K^{(0)}))\mathrm{d}r\right\} \le \lim_{\lambda\to 0}\lambda = 0,$$

implying that Z is conservative.

□

Recall that in the case when there is no random environment, i.e. $S = 0$, we know that a CB-process with branching mechanism ψ is conservative if and only if

$$\int_{0+} \frac{\mathrm{d}u}{|\psi(u)|} = \infty. \tag{2.2}$$

The key part of the proof was observing that for all $\lambda > 0$, the solution to (1.14) can be uniquely identified by the relation

$$t = \int_0^t \frac{\frac{\partial}{\partial s} v_t(s,\lambda,0)}{\psi(v_t(s,\lambda,0))}\mathrm{d}s = \int_{v_t(0,\lambda,0)}^{\lambda} \frac{\mathrm{d}\epsilon}{\psi(\epsilon)}.$$

And therefore, it is easy to see that $\lim_{\lambda\downarrow 0} v_t(0,\lambda,K^{(0)}) = 0$ if and only if (2.2) holds. In the case when the random environment is present, it is not so clear how to get a necessary and sufficient condition in terms of the branching mechanism. The reason is that it is not enough to do a change of variable in the integral

$$t = \int_0^t \frac{e^{-K_s}\frac{\partial}{\partial s} v_t(s,\lambda,K)}{\psi(e^{-K_s} v_t(s,\lambda,K))}\mathrm{d}s.$$

To expose the diverse behaviours that may arise, we now provide two interesting examples in the case when $\psi'(0+) = -\infty$.

Example 5 (Neveu case). In this case, recall that $\psi(u) = u \log u$. In particular

$$\psi'(0+) = -\infty \qquad \text{and} \qquad \int_{0+} \frac{\mathrm{d}u}{|u \log u|} = \infty.$$

By taking limits as $\lambda \downarrow 0$ in (1.16), one can see that the process is conservative conditionally on the environment, i.e.

$$\mathbb{P}_z \left(Z_t < \infty \middle| K \right) = 1,$$

for all $t \in (0, \infty)$ and $z \in [0, \infty)$.

Example 6 (Stable case with $\beta \in (-1, 0)$). Here $\psi(u) = cu^{\beta+1}$, where $a \in \mathbb{R}$ and c is a negative constant. From straightforward computations, we get

$$\psi'(0+) = -\infty, \qquad \text{and} \qquad \int_{0+} \frac{\mathrm{d}u}{|\psi(u)|} < \infty,$$

Moreover, by taking limits as $\lambda \downarrow 0$ in (1.19), deduce that for $z, t > 0$

$$\mathbb{P}_z\Big(Z_t < \infty \Big| K \Big) = \exp\left\{ -z \left(\beta c \int_0^t e^{-\beta K_u} \mathrm{d}u \right)^{-1/\beta} \right\} \qquad \text{a.s.,}$$

implying

$$\mathbb{P}_z\Big(Z_t = \infty \Big| K \Big) = 1 - \exp\left\{ -z \left(\beta c \int_0^t e^{-\beta K_u} \mathrm{d}u \right)^{-1/\beta} \right\} > 0.$$

In other words the stable CBLRE with $\beta \in (-1, 0)$ explodes with positive probability for any $t > 0$. Moreover, if the process $(K_u, u \geq 0)$ does not drift to $-\infty$, from (5) we deduce that

$$\lim_{t \to \infty} Z_t = \infty, \qquad \text{a.s.}$$

On the other hand, if the process $(K_u, u \geq 0)$ drifts to $-\infty$, we have an interesting long-term behaviour of the process Z. In fact, we deduce from the Dominated Convergence Theorem

$$\mathbb{P}_z\Big(Z_\infty = \infty \Big) = 1 - \mathbb{E}\left[\exp\left\{ -z \left(\beta c \int_0^\infty e^{-\beta K_u} \mathrm{d}u \right)^{-1/\beta} \right\} \right], \qquad z > 0.$$

By (5), the above probability is positive. In this particular case, we will discuss the asymptotic behaviour of the probability of explosion in Chapter 3.

As before we denote by Φ the function $\Phi(\lambda) = \lambda^{-1} \psi_0(\lambda)$, for $\lambda \geq 0$, and we introduce

$$A(x) = \mathbf{m} + \pi((1, \infty)) + \int_1^x \pi((y, \infty)) \mathrm{d}y, \qquad \text{for} \quad x > 0.$$

Proposition 3. *Assume that $\int_{[1,\infty)} x \mu(\mathrm{d}x) < \infty$. Let $(Z_t, t \geq 0)$ be a CBLRE with branching mechanism given by ψ and $z \geq 0$.*

i) If the process $K^{(0)}$ drifts to $-\infty$, then $\mathbb{P}_z \Big(\lim_{t \to \infty} Z_t = 0 \Big| K^{(0)} \Big) = 1$, a.s.

ii) If the process $K^{(0)}$ oscillates, then $\mathbb{P}_z\Big(\liminf_{t\to\infty} Z_t = 0\Big|K^{(0)}\Big) = 1$, a.s. Moreover if $\gamma > 0$ then

$$\mathbb{P}_z\Big(\lim_{t\to\infty} Z_t = 0\Big|K^{(0)}\Big) = 1, \ a.s.$$

iii) If the process $K^{(0)}$ drifts to $+\infty$, so that $A(x) > 0$ for all x large enough. Then if

$$\int_{(a,\infty)} \frac{x}{A(x)}\big|\mathrm{d}\Phi(e^{-x})\big| < \infty \qquad \textit{for some } a > 0, \tag{2.3}$$

we have $\mathbb{P}_z\Big(\liminf_{t\to\infty} Z_t > 0\Big|K^{(0)}\Big) > 0$ a.s., for all $z > 0$, and there exists a non-negative finite r.v. W such that

$$Z_t e^{-K_t^{(0)}} \underset{t\to\infty}{\longrightarrow} W, \ a.s \qquad \textit{and} \qquad \big\{W = 0\big\} = \Big\{\lim_{t\to\infty} Z_t = 0\Big\}.$$

In particular, if $0 < \mathbb{E}[K_1^{(0)}] < \infty$ then the above integral condition is equivalent to

$$\int^{\infty} x\log(x)\,\mu(\mathrm{d}x) < \infty.$$

iv) Assume that $K^{(0)}$ has continuous paths and drifts to infinite, i.e. $K_t^{(0)} = \mathbf{m}t + \sigma B_t^{(e)}$ with $\mathbf{m} = \alpha + \psi'(0+) - \sigma^2/2 > 0$, and that

$$\gamma > 0 \qquad \textit{and} \qquad \kappa := \int_1^{\infty} x^2\mu(\mathrm{d}x) < \infty.$$

Then, for $z > 0$,

$$\left(1 + \frac{z\sigma^2}{2\gamma^2}\right)^{-\frac{2\mathbf{m}}{\sigma^2}} \le \mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0\right) \le \left(1 + \frac{1}{2}\frac{z\sigma^2}{\gamma^2 + \kappa}\right)^{-\frac{2\mathbf{m}}{\sigma^2}}.$$

Proof. Recall that under our assumption, the function $v_t(s, \lambda, K^{(0)})$ satisfies the backward differential equation (1.12). Similarly to the last part of the proof of Proposition 2, one can prove that $Z_t e^{-K_t^{(0)}}$ is a non-negative local martingale. Therefore $Z_t e^{-K_t^{(0)}}$ is a non-negative supermartingale and it converges a.s. to a non-negative finite random variable, here denoted by W. This implies the statement of part (i) and the first statement of part (ii).

In order to prove the second statement of part (ii), we observe that if $\gamma > 0$, then the solution to (1.12) also satisfies

$$\frac{\partial}{\partial s} v_t(s, \lambda, K^{(0)}) \ge \gamma^2 v_t(s, \lambda, K^{(0)})^2 e^{-K_s^{(0)}}.$$

Therefore

$$v_t(s, \lambda, K^{(0)}) \le \left(\frac{1}{\lambda} + \gamma^2 \int_s^t e^{-K_s^{(0)}}\mathrm{d}s\right)^{-1},$$

which implies the following inequality,

$$\mathbb{P}_z(Z_t = 0|K^{(0)}) \ge \exp\left\{-z\left(\gamma^2 \int_0^t e^{-K_s^{(0)}}\mathrm{d}s\right)^{-1}\right\}. \tag{2.4}$$

By (5), we have

$$\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0 \Big| K^{(0)}\right) = 1 \quad \text{a.s.}$$

Now, we prove part (iii). From the non-negative property of ψ_0, it follows that $v_t(\cdot, \lambda, K^{(0)})$, the a.s. solution to (1.12), is non-decreasing on $[0, t]$. Thus for all $s \in [0, t]$, $v_t(s, \lambda, K^{(0)}) \leq \lambda$. Since ψ_0 is convex and $\Phi(0) = \psi_0'(0+) = 0$, we dedude that Φ is increasing. Hence

$$\frac{\partial}{\partial s} v_t(s, \lambda, K^{(0)}) = v_t(s, \lambda, K^{(0)})\Phi(v_t(s, \lambda, K^{(0)})e^{-K_s^{(0)}}) \leq v_t(s, \lambda, K^{(0)})\Phi(\lambda e^{-K_s^{(0)}}).$$

Therefore, for every $s \leq t$, we have

$$v_t(s, \lambda, K^{(0)}) \geq \lambda \exp\left\{-\int_s^t \Phi(\lambda e^{-K_s^{(0)}})\mathrm{d}s\right\}.$$

In particular,

$$\liminf_{t\to\infty} v_t(0, \lambda, K^{(0)}) \geq \lambda \exp\left\{-\int_0^\infty \Phi(\lambda e^{-K_s^{(0)}})\mathrm{d}s\right\}.$$

If the integral on the right-hand side is a.s. finite, then

$$\liminf_{t\to\infty} v_t(0, \lambda, K^{(0)}) \geq \lambda \exp\left\{-\int_0^\infty \Phi(\lambda e^{-K_s^{(0)}})\mathrm{d}s\right\} > 0, \qquad \text{a.s.},$$

implying that for $z > 0$

$$\mathbb{E}_z\left[e^{-\lambda W}\Big| K^{(0)}\right] \leq \exp\left\{-z\ \lambda \exp\left\{-\int_0^\infty \Phi(\lambda e^{-K_s^{(0)}})\mathrm{d}s\right\}\right\} < 1, \qquad \text{a.s.}$$

and in particular $\mathbb{P}_z\left(\liminf_{t\to\infty} Z_t > 0 \Big| K^{(0)}\right) > 0$ a.s. Next, we use Lemma 20 in [9] and the branching property of Z, to deduce

$$\{W = 0\} = \left\{\lim_{t\to\infty} Z_t = 0\right\}.$$

In order to finish our proof, we show that the integral condition (2.3) implies

$$\int_0^\infty \Phi(\lambda e^{-K_s^{(0)}})\mathrm{d}s < \infty \quad \text{a.s.}$$

We first introduce $\varsigma = \sup\{t \geq 0 : K_t^{(0)} \leq 0\}$ and observe

$$\int_0^\infty \Phi(\lambda e^{-K_s^{(0)}})\mathrm{d}s = \int_0^\varsigma \Phi(\lambda e^{-K_s^{(0)}})\mathrm{d}s + \int_\varsigma^\infty \Phi(\lambda e^{-K_s^{(0)}})\mathrm{d}s \tag{2.5}$$

Since $\varsigma < \infty$ a.s., the first integral of the right-hand side is a.s. finite. For the second integral, we use Theorem 1 in Erickson and Maller [42] which ensures us that

$$\int_\varsigma^\infty \Phi(\lambda e^{-K_s^{(0)}})\mathrm{d}s < \infty, \qquad a.s.,$$

if the integral condition (2.3) holds.

Now, we assume that $0 \leq \mathbb{E}[K_1^{(0)}] < \infty$ and observe that $\lim_{x\to\infty} A(x)$ is finite. In particular, this implies that the integral condition (2.3) is equivalent to

$$\int_0^\infty \Phi(\lambda e^{-y})\mathrm{d}y < \infty.$$

Moreover, we have

$$\begin{aligned}\int_0^\infty \Phi(\lambda e^{-y})\mathrm{d}y &= \int_0^\lambda \frac{\Phi(\theta)}{\theta}\mathrm{d}\theta \\ &= \gamma^2\lambda + \int_0^\lambda \frac{\mathrm{d}\theta}{\theta^2}\int_{(0,\infty)} (e^{-\theta x} - 1 + \theta x)\mu(\mathrm{d}x) \\ &= \gamma^2\lambda + \int_{(0,\infty)} \mu(\mathrm{d}x)\int_0^\lambda (e^{-\theta x} - 1 + \theta x)\frac{\mathrm{d}\theta}{\theta^2} \\ &= \gamma^2\lambda + \int_{(0,\infty)} x\left(\int_0^{\lambda x} (e^{-y} - 1 + y)\frac{\mathrm{d}y}{y^2}\right)\mu(\mathrm{d}x).\end{aligned}$$

Since the function

$$g_\lambda(x) = \int_0^{\lambda x} (e^{-y} - 1 + y)\frac{\mathrm{d}y}{y^2},$$

is equivalent to $\lambda x/2$ as $x \to 0$ and equivalent to $\ln x$ as $x \to \infty$. Using the integrability condition $\int_{(0,\infty)}(x \wedge x^2)\mu(\mathrm{d}x) < \infty$, we deduce that

$$\int_0^\infty \Phi(\lambda e^{-y})\mathrm{d}y < \infty \qquad \text{if and only if} \qquad \int^\infty x\log(x)\mu(\mathrm{d}x) < \infty.$$

Finally, we prove part (iv). From inequality (2.4) and the Dominate Convergence Theorem, we deduce

$$\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0\Big|K^{(0)}\right) \geq \exp\left\{-z\left(\gamma^2\int_0^\infty e^{-K_s^{(0)}}\mathrm{d}s\right)^{-1}\right\} \qquad \text{a.s.}$$

According to Dufresne [34], when $K_t^{(0)} = \mathbf{m}t + \sigma B_t^{(e)}$ and $\mathbf{m} > 0$,

$$\int_0^\infty e^{-K_s^{(0)}}\mathrm{d}s \quad \text{has the same law as} \quad \left(2\Gamma_{\frac{2\mathbf{m}}{\sigma^2}}\right)^{-1}, \tag{2.6}$$

where Γ_v is Gamma variable with shape parameter v, i.e. of density:

$$\mathbb{P}(\Gamma_v \in \mathrm{d}x) = \frac{x^{v-1}}{\Gamma(v)}e^{-x}\mathbf{1}_{\{x>0\}}.$$

After straightforward computations, we deduce that for $z > 0$

$$\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0\right) \geq \left(1 + \frac{z\sigma^2}{\gamma^2}\right)^{-\frac{2\mathbf{m}}{\sigma^2}}.$$

In a similar way, the upper bound follows from the a.s. inequality

$$\frac{\partial}{\partial s}v_t(s,\lambda,K^{(0)}) \leq (\gamma^2 + \kappa)v_t(s,\lambda,K^{(0)})^2 e^{-K_s^{(0)}}.$$

□

Now, we derive a central limit theorem in the supercritical regime which follows from Theorem 3.5 in Doney and Maller [31] and similar arguments as those used in Corollary 3 in [9], so we skip its proof.

For $x > 0$ let

$$T(x) = \pi((x, \infty)) + \pi((-\infty, -x)) \qquad \text{and} \qquad U(x) = \sigma^2 + \int_0^x yT(y)\mathrm{d}y$$

Corollary 2. *Assume that $K^{(0)}$ drifts to $+\infty$, $T(x) > 0$ for all $x > 0$, and (2.3) is satisfied. There are two measurable functions $a(t), b(t) > 0$ such that , conditionally on $\{W > 0\}$,*

$$\frac{\log(Z_t) - a(t)}{b(t)} \xrightarrow[t\to\infty]{d} \mathcal{N}(0, 1),$$

if and only if

$$\frac{U(x)}{x^2 T(x)} \to \infty \qquad as \quad x \to \infty,$$

where $\stackrel{d}{\to}$ means convergence in distribution and $\mathcal{N}(0, 1)$ denotes a centred Gaussian random variable with variance equals 1.

It is important to note that if $\int_{\{|x|>1\}} x^2\pi(\mathrm{d}x) < \infty$, then for $t > 0$,

$$a(t) := \left(\mathbf{m} + \int_{\{|x|\geq 1\}} x\pi(\mathrm{d}x)\right) t \qquad \text{and} \qquad b^2(t) := \left(\sigma^2 + \int_{\mathbb{R}} x^2\pi(\mathrm{d}x)\right) t,$$

which is similar to the result obtained in Corollary 3 in [9].

We finish this section with 3 examples where we find explicitly the probabilities that we studied before. In the Neveu example, we show that the process survives a.s. and it has positive extinction probability.

Example 7 (Stable case with $\beta \in (0, 1]$). When the branching mechanism is of the form $\psi(u) = cu^{\beta+1}$ for $\beta \in (0, 1]$, $\phi \equiv 0$ and $\mathbf{m} > 0$, one can deduce directly from (1.19) by taking λ and t to ∞, and (5) that for $z \geq 0$

$$\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0 \big| K\right) = \exp\left\{-z\left(\beta c \int_0^\infty e^{-\beta K_u}\mathrm{d}u\right)^{-1/\beta}\right\}, \qquad \text{a.s.,}$$

and in particular

$$\mathbb{P}\Big(W = 0\Big) = \mathbb{P}_z\Big(\lim_{t\to\infty} Z_t = 0\Big) = \mathbb{E}_z\left[\exp\left\{-z\left(\beta c \int_0^\infty e^{-\beta K_u}\mathrm{d}u\right)^{-1/\beta}\right\}\right].$$

Example 8 (Stable case with immigration). Now, we assume that the branching and immigration mechanisms are of the form $\psi(\lambda) = c\lambda^{\beta+1}$ and $\phi(\lambda) = d\lambda^\beta$, where $\beta \in (0, 1]$, $c, d > 0$ and $a \in \mathbb{R}$. If we take limits as $\lambda \uparrow \infty$ in (1.20), we obtain

$$\mathbb{P}_z\Big(Z_t > 0 \Big| K\Big) = 1, \qquad \text{for} \quad z > 0.$$

Similarly if we take limits as $\lambda \downarrow 0$ in (1.20), we deduce

$$\mathbb{P}_z\Big(Z_t < \infty \Big| K\Big) = 1, \qquad \text{for} \quad z \geq 0.$$

In other words, the stable CBLRE with immigration is conservative and positive at finite time a.s.

An interesting question is to study the long-term behaviour of the stable CBILRE. Now, if we take limits as $t \uparrow \infty$ in (1.20), we deduce that when $\mathbf{m} > 0$,

$$\begin{aligned}\mathbb{E}_z\left[\exp\left\{-\lambda \lim_{t\to\infty} Z_t e^{-K_t}\right\}\right] =& \mathbb{E}\left[\exp\left\{-z\left(\beta c \int_0^\infty e^{-\beta K_s}\mathrm{d}s + \lambda^{-\beta}\right)^{-1/\beta}\right\}\right. \\ &\left.\times \exp\left\{-\frac{d}{\beta c}\ln\left(\beta c \lambda^\beta \int_0^\infty e^{-\beta K_s}\mathrm{d}s + 1\right)\right\}\right],\end{aligned}$$

where we recall that by (5), $\int_0^\infty e^{-\beta K_s}\mathrm{d}s < \infty$ a.s. In other words, $Z_t e^{-K_t}$ converges in distribution to a r.v. whose Laplace transform is given by the previous identity.

If $\mathbf{m} \leq 0$, we deduce

$$\lim_{t\to\infty} Z_t e^{-K_t} = \infty, \qquad \mathbb{P}_z - \text{a.s.} \qquad z > 0$$

We observe that when $\mathbf{m} = 0$, the process K oscillates implying that

$$\lim_{t\to\infty} Z_t = \infty, \qquad \mathbb{P}_z - \text{a.s.}, \qquad z > 0$$

Example 9 (Neveu case). According to (1.16), for $z, \lambda, t > 0$

$$\mathbb{E}_z\Big[\exp\Big\{-\lambda Z_t e^{-K_t}\Big\}\Big| K\Big] = \exp\left\{-z\lambda^{e^{-t}}\exp\left\{\int_0^t e^{-s}K_s\mathrm{d}s\right\}\right\} \qquad \text{a.s.} \tag{2.7}$$

If we take limits as $\lambda \uparrow \infty$, in (2.7), we obtain that the Neveu CBLRE survives conditionally on the environment, in other words

$$\mathbb{P}_z\left(Z_t > 0 \big| K\right) = 1,$$

for all $t \in (0, \infty)$ and $z \in (0, \infty)$. Moreover since the process has càdlàg paths, we deduce the Neveu CBLRE survives a.s., i.e.

$$\mathbb{P}_z\Big(Z_t > 0, \text{ for all } t \geq 0\Big) = 1, \qquad z > 0.$$

On the one hand, using integration by parts we obtain

$$\int_0^t e^{-s}K_s\mathrm{d}s = -e^{-t}K_t + \int_0^t e^{-s}\mathrm{d}K_s.$$

Let ψ_K be the characteristic exponent of K and ρ its Lévy measure. According to Theorem 17.5 in Sato [84], if ρ satisfies

$$\int_{|x|>2} \log|x|\rho(\mathrm{d}x) < \infty,$$

the law of $\int_0^t e^{-s}\mathrm{d}K_s$ converges to a self-decomposable law, denoted by $\int_0^\infty e^{-s}\mathrm{d}K_s$, as t goes to ∞. Moreover, the characteristic exponent of $\int_0^\infty e^{-s}\mathrm{d}K_s$ is given by

$$\Psi_K(\lambda) = \int_0^\infty \psi_K(\lambda e^{-s})\mathrm{d}s \qquad \lambda \geq 0.$$

In particular, if $\mathbb{E}[|K_1|] < \infty$, then $\int_{|x|>2} \log|x|\rho(\mathrm{d}x) < \infty$ and by the Strong Law of Large Numbers, $e^{-t}K_t \to 0$ as t goes to ∞. Hence, if we take limits as $t \uparrow \infty$ in (2.7), we observe

$$\mathbb{E}_z\left[\exp\left\{ -\lambda \lim_{t\to\infty} Z_t e^{-K_t}\right\}\Big| K\right] \stackrel{\mathcal{L}}{=} \exp\left\{-z\exp\left\{\int_0^\infty e^{-s}\mathrm{d}K_s\right\}\right\}, \qquad z,\lambda > 0.$$

Since the right-hand side of the above identity does not depend on λ, this implies that

$$\mathbb{P}_z\Big(\lim_{t\to\infty} Z_t e^{-K_t} = 0\Big| K\Big) \stackrel{\mathcal{L}}{=} \exp\left\{-z\exp\left\{\int_0^\infty e^{-s}\mathrm{d}K_s\right\}\right\}, \qquad z > 0$$

and taking expectations in the above identity, we deduce

$$\mathbb{P}_z\Big(\lim_{t\to\infty} Z_t e^{-K_t} = 0\Big) = \mathbb{E}\left[\exp\left\{-z\exp\left\{\int_0^\infty e^{-s}\mathrm{d}K_s\right\}\right\}\right], \qquad z > 0.$$

In conclusion, the Neveu process in Lévy random environment is conservative and survives a.s. But when $\mathbb{E}[K_1] < 0$, the extinction probability is given by the previous expression. In addition, when the random environment is continuous and $\alpha < \sigma^2/2$, the extinction probability is given by the Laplace transform of a log-normal distribution with mean $\alpha - \sigma^2/2$ and variance $\sigma^2/2$.

2.2 Competition model in a Lévy random environment

We now study an extension of the competition model given in Evans et al. [44]. In this model, we exclude the immigration term and take the branching and competition mechanisms as follows

$$\beta(x) = kx^2 \qquad \text{and} \qquad \psi(\lambda) = a\lambda, \qquad \text{for } x, \lambda \geq 0$$

where k is a positive constant. Hence, we define a competition model in a Lévy random environment process $(Z_t, t \geq 0)$ as the solution of the SDE

$$Z_t = Z_0 + \int_0^t Z_s(a - kZ_s)\mathrm{d}s + \int_0^t Z_{s-}\mathrm{d}S_s \tag{2.8}$$

where the environment is given by the Lévy process described in (1.8).

From Corollary 1, there is a unique non negative strong solution of (2.8) satisfying the Markov property. Moreover, we have the following result, which that in particular tells us that the process Z is the inverse of a generalised Ornstein-Uhlenbeck process.

Proposition 4. *Suposse that $(Z_t, t \geq 0)$ is the unique strong solution of (2.8). Then, it satisfies*

$$Z_t = \frac{Z_0 e^{K_t}}{1 + kZ_0 \int_0^t e^{K_s}\mathrm{d}s}, \qquad t \geq 0, \tag{2.9}$$

where K is the Lévy process defined in (1.10). Moreover, if $Z_0 = z > 0$ then, $Z_t > 0$ for all $t \geq 0$ a.s. and it has the following asymptotic behaviour:

i) If the process K drifts to $-\infty$, then $\lim_{t\to\infty} Z_t = 0$ *a.s.*

ii) If the process K oscillates, then $\liminf_{t\to\infty} Z_t = 0$ *a.s.*

iii) If the process K drifts to ∞, then $(Z_t, t \geq 0)$ has a stationary distribution whose density satisfies for $z > 0$,

$$\mathbb{P}_z(Z_\infty \in \mathrm{d}x) = h\left(\frac{1}{kx}\right)\frac{\mathrm{d}x}{x^2}, \qquad x > 0,$$

where

$$\int_t^\infty h(x)\mathrm{d}x = \int_{\mathbb{R}} h(te^{-y})U(\mathrm{d}y), \qquad a.e.\ t\ on\ (0,\infty),$$

and U denotes the potential measure associated to K, i.e.

$$U(\mathrm{d}x) = \int_0^\infty \mathbb{P}(K_s \in \mathrm{d}x)\mathrm{d}s \qquad x \in \mathbb{R}.$$

Moreover, if $0 < \mathbb{E}[K_1] < \infty$, *then*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t Z_s \mathrm{d}s = \frac{1}{k}\mathbb{E}[K_1], \qquad a.s.$$

and for every measure function $f : \mathbb{R}_+ \to \mathbb{R}_+$ we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(Z_s)\mathrm{d}s = \mathbb{E}\left[f\left(\frac{1}{kI_\infty(-K)}\right)\right], \qquad a.s.$$

where $I_\infty(-K) = \int_0^\infty e^{K_s}\mathrm{d}s$.

Proof. By Itô's formula, we see that the process Z satisfies (2.9). Moreover, since the Lévy process K has infinite lifetime, then we necessarily have $Z_t > 0$ a.s. Part (i) follows directly from (5) and (2.9). Next, we prove part (ii). Assume that the process K oscillates. On the one hand, we have

$$Z_t = \frac{Z_0}{e^{-K_t} + kZ_0 e^{-K_t}\displaystyle\int_0^t e^{K_s}\mathrm{d}s} \leq \frac{1}{ke^{-K_t}\displaystyle\int_0^t e^{K_s}\mathrm{d}s}.$$

On the other hand, the Duality Lemma (see for instance Lemma 3.4 in [64]) tells us that $\{K_{(t-s)-} - K_t : 0 \leq s \leq t\}$ and $\{-K_s : 0 \leq s \leq t\}$ have the same law under $\mathbb{P}$. Then, we deduce

$$\left(K_t, e^{-K_t}\int_0^t e^{K_s}\mathrm{d}s\right) \qquad \text{is equal in law to} \qquad \left(K_t, \int_0^t e^{-K_s}\mathrm{d}s\right).$$

From (5) and our assumption, we have that the exponential functional of K goes to ∞ as $t \to \infty$. This implies that $\lim_{t\to\infty} Z_t = 0$ in distribution and therefore,

$$\liminf_{t\to\infty} Z_t = 0, \qquad \text{a.s.}$$

Finally, we assume that the process K drifts to ∞. Then, from the previous observation, we have the following identity in law

$$Z_t \overset{\mathcal{L}}{=} \frac{Z_0}{e^{-K_t} + kZ_0\displaystyle\int_0^t e^{-K_s}\mathrm{d}s}.$$

Using (5), we have that Z_t converges in distribution to

$$\left(k\int_0^\infty e^{-K_s}\mathrm{d}s\right)^{-1}.$$

The form of the density follows from Theorem 1 of Arista and Rivero [5].

Now, observe that

$$\int_0^t Z_s\mathrm{d}s = \frac{1}{k}\ln\left(1 + kZ_0\int_0^t e^{K_s}\mathrm{d}s\right). \tag{2.10}$$

Therefore if $0 < \mathbb{E}[K_1] < \infty$, a simple application of the Law of Large Numbers allow us to deduce (see also Proposition 4.1 in Carmona et al. [24])

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t Z_s\mathrm{d}s = \lim_{t\to\infty}\frac{1}{kt}\ln\left(\int_0^t e^{K_s}\mathrm{d}s\right) = \frac{1}{k}\mathbb{E}[K_1], \quad \text{a.s..} \tag{2.11}$$

In order to prove the last assertion of our proposition, let us introduce $X^{(x)} = (X_t^{(x)}, t \geq 0)$ the positive self-similar Markov process associated to K via the Lamperti transform with scaling index 1. That is to say, for all $x > 0$

$$X_t^{(x)} = xe^{K_{\tau(tx^{-1})}}, \qquad t \geq 0,$$

where the time change τ is defined as follows

$$\tau(t) = \inf\left\{s \geq 0 : \int_0^s e^{K_r}dr > t\right\}.$$

This process satisfies the scaling property, i.e. for $a > 0$, the following identity in law follows

$$\left(aX_t^{(x)}, t \geq 0\right) \stackrel{\mathcal{L}}{=} \left(X_{at}^{(ax)}, t \geq 0\right).$$

Next, we define the process Y as follows

$$Y_t^{(x)} := e^{-kt}X^{(x)}_{(e^{kt}-1)/k}, \qquad t \geq 0,$$

By the scaling property, it turns out to have the same law as

$$\left(X^{(xe^{-kt})}_{(1-e^{-kt})/k}, t \geq 0\right).$$

Since $0 < \mathbb{E}[K_1] < \infty$, Theorem 1 of Bertoin and Yor in [18] tells us that for all $x > 0$, $Y_t^{(x)} \to X_{1/k}^{(0)}$ as $t \to \infty$, and for any measurable function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$

$$\mathbb{E}\left[f(X_{1/k}^{(0)})\right] := \int_{(0,\infty)} f(x)\mu(\mathrm{d}x) = \frac{1}{\mathbb{E}[K_1]}\mathbb{E}\left[f\left(\frac{1}{kI_\infty(-K)}\right)\frac{1}{I_\infty(-K)}\right], \tag{2.12}$$

where $I_\infty(-K) = \int_0^\infty e^{-K_s}\mathrm{d}s$. Then, Y is a Markov process with invariant distribution μ. Moreover, by the Ergodic Theorem

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(Y_s^{(x)})\mathrm{d}s = \int_{(0,\infty)} f(x)\mu(\mathrm{d}x). \tag{2.13}$$

In one hand, observe that by the definition of Y and τ,

$$Y_t^{(Z_0)} = Z_0 e^{-kt} e^{K_{\tau((e^{kt}-1)/(kZ_0))}} = \left. \frac{Z_0 e^{K_v}}{1 + kZ_0 \int_0^v e^{K_s} \mathrm{d}s} \right|_{v=\tau((e^{kt}-1)/(kZ_0))} = Z_{\tau((e^{kt}-1)/(kZ_0))}.$$

On the other hand, by identity (2.10), we deduce for all $t \geq 0$

$$\tau\left(\frac{e^{kt}-1}{kZ_0}\right) = \inf\left\{s > 0 : \int_0^s Z_r \mathrm{d}r > t\right\}.$$

This implies

$$\int_0^t f(Z_s)\mathrm{d}s = \int_0^{\tau\left(\frac{e^{kt}-1}{kZ_0}\right)} f\left(Z_{\tau\left(\frac{e^{ks}-1}{kZ_0}\right)}\right) \frac{1}{Z_{\tau\left(\frac{e^{ks}-1}{kZ_0}\right)}} \mathrm{d}s = \int_0^{\tau\left(\frac{e^{kt}-1}{kZ_0}\right)} f\left(Y_s^{(Z_0)}\right) \frac{1}{Y_s^{(Z_0)}} \mathrm{d}s. \quad (2.14)$$

By taking $f(x) = x$ in the previous equality, we get

$$\lim_{t\to\infty} \frac{\tau\left(\frac{e^{kt}-1}{kZ_0}\right)}{t} = \lim_{t\to\infty} \frac{1}{t}\int_0^t Z_s \mathrm{d}s.$$

Putting all the pieces together, i.e. by (2.11), (2.12), (2.13) and (2.14), we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(Z_s)\mathrm{d}s = \mathbb{E}\left[f\left(\frac{1}{kI_\infty(-K)}\right)\right], \qquad \text{a.s.}$$

This completes the proof. □

We finish this section with two important observations in two particular cases. We first assume that the process K drifts to $+\infty$ and that satisfies

$$\int_{[1,\infty)} e^{qx}\pi(\mathrm{d}x) < \infty \qquad \text{for every} \quad q > 0,$$

i.e. that has exponential moments of all positive orders. Let us denote by Ψ_K the characteristic exponent of the Lévy process K, i.e.

$$\Psi_K(\theta) = -\log \mathbb{E}[e^{i\theta K_1}] \qquad \text{for} \quad \theta \in \mathbb{R}.$$

In this situation, the characteristic exponent $\Psi_k(\theta)$ has an analytic extension to the half-plane with negative imaginary part, and one has

$$\mathbb{E}[e^{qK_t}] = e^{t\psi_K(q)} < \infty, \qquad t, q \geq 0,$$

where $\psi_K(q) = -\Psi_K(-iq)$ for $q \geq 0$. Hence, according to Theorem 3 in Bertoin and Yor [19] the stationary distribution has positive moments and satisfies, for $z > 0$ and $n \geq 1$,

$$\mathbb{E}_z\left[Z_\infty^n\right] = \psi_K'(0+)\frac{\psi_K(1)\cdots\psi_K(n-1)}{(n-1)!}.$$

Finally, we assume that the process K drifts to $-\infty$ and has no negative jumps. Observe that the process Z inherited the latter property and we let $Z_0 = z > 0$. Under this assumption, we can compute the Laplace transform of the first passage time from below a level $z > b > 0$ of the process Z, i.e.

$$\sigma_b = \inf\{s \geq 0 : Z_s \leq b\}.$$

In this case, Ψ_k has an analytic extension to the half-plane with positive imaginary part, and

$$\mathbb{E}[e^{-qK_t}] = e^{t\hat{\psi}_K(q)} < \infty, \qquad t, q \geq 0,$$

where $\hat{\psi}_K(q) = -\Psi_K(iq)$ for $q \geq 0$. Define, for all $t \geq 0$, $\mathcal{F}_t = \sigma(K_s : s \leq t)$ and consider the exponential change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}^{\kappa(\lambda)}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = e^{-\kappa(\lambda)K_t - \lambda t}, \qquad \text{for } \lambda \geq 0, \tag{2.15}$$

where $\kappa(\lambda)$ is the largest solution to $\hat{\psi}_K(u) = \lambda$. Under $\mathbb{P}^{\kappa(\lambda)}$, the process K is still a spectrally positive and its Laplace exponent, $\hat{\psi}_{\kappa(\lambda)}$ satisfies the relation

$$\hat{\psi}_{\kappa(\lambda)}(u) = \hat{\psi}_K(\kappa(\lambda) + u) - \lambda, \qquad \text{for} \quad u \geq 0.$$

See for example Chapter 8 of [64] for further details on the above remarks. Note in particular that it is easy to verify that $\hat{\psi}'_{\kappa(\lambda)}(0+) > 0$ and hence the process K under $\mathbb{P}^{\kappa(\lambda)}$ drifts to $-\infty$. According to earlier discussion, this guarantees that also under $\mathbb{P}^{\kappa(\lambda)}$, the process Z goes to 0 as $t \to \infty$.

Lemma 1. *Suppose that $\lambda \geq 0$ and that $\kappa(\lambda) > 1$, then for all $0 < b \leq z$,*

$$\mathbb{E}_z\left[e^{-\lambda\sigma_b}\right] = \frac{\mathbb{E}^{\kappa(\lambda)}\left[(1 + kzI_\infty(K))^{\kappa(\lambda)}\right]}{\mathbb{E}^{\kappa(\lambda)}\left[(zb^{-1} + kzI_\infty(K))^{\kappa(\lambda)}\right]},$$

where

$$I_\infty(K) = \int_0^\infty e^{K_s}\mathrm{d}s.$$

Proof. From the absence of negative jumps we have $Z_{\sigma_b} = b$ on the event $\{\sigma_b < \infty\}$ and in particular

$$b = \frac{ze^{K_{\sigma_b}}}{1 + kz\displaystyle\int_0^{\sigma_b} e^{K_s}\mathrm{d}s}.$$

On the other hand, from the Markov property and the above identity, we have

$$1 + kzI_\infty(K) = 1 + kz\int_0^{\sigma_b} e^{K_s}\mathrm{d}s + kze^{K_{\sigma_b}}\int_0^\infty e^{K_{\sigma_b+s} - K_{\sigma_b}}\mathrm{d}s = e^{K_{\sigma_b}}\left(\frac{z}{b} + zkI'_\infty\right),$$

where I'_∞ is an independent copy of $I_\infty(K)$.

The latter identity and the Escheer transform imply that for $\lambda \geq 0$

$$\mathbb{E}_z\left[e^{-\lambda\sigma_b}\right] = \mathbb{E}^{\kappa(\lambda)}\left[e^{\kappa(\lambda)K_{\sigma_b}}\right] = \frac{\mathbb{E}^{\kappa(\lambda)}\left[(1+kzI_\infty(K))^{\kappa(\lambda)}\right]}{\mathbb{E}^{\kappa(\lambda)}\left[\left(\frac{z}{b}+zkI_\infty(K)\right)^{\kappa(\lambda)}\right]},$$

provided the quantity $\mathbb{E}^{\kappa(\lambda)}[(a+kzI_\infty(K))^{\kappa(\lambda)}]$ is finite, for $a>0$.

Observe that for $s \geq 1$,

$$\mathbb{E}^{\kappa(\lambda)}\left[(a+I_\infty(K))^s\right] \leq 2^{s-1}\Big(a^s + \mathbb{E}^{\kappa(\lambda)}[I_\infty(K)^s]\Big),$$

hence it suffices to investigate the finiteness of $\mathbb{E}^{\kappa(\lambda)}[I_\infty^s]$. According to Lemma 2.1 in Maulik and Zwart [76] the expectation $\mathbb{E}^{\kappa(\lambda)}[I_\infty(K)^s]$ is finite for all $s \geq 0$ such that $-\hat{\psi}_{\kappa(\lambda)}(-s) > 0$. Since $\hat{\psi}_{\kappa(\lambda)}(-s)$ is well defined for $\kappa(\lambda) - s \geq 0$, then a straightforward computation gives us that $\mathbb{E}^{\kappa(\lambda)}[I_\infty(K)^s] < \infty$ for $s \in [0, \kappa(\lambda)]$. □

Chapter 3

Asymptotic behaviour of exponential functionals of Lévy processes

This chapter is based in paper [81] elaborated in collaboration with Juan Carlos Pardo and Charline Smadi. Here, we study The exponential functional of a Lévy process is the main topic of this chapter. We study the asymptotic behaviour of

$$\mathbb{E}\Big[F\big(I_t(\xi)\big)\Big] \qquad \text{as} \qquad t \to \infty,$$

where $I_t(\xi)$ is given by (4) and F is a non-increasing function with polynomial decay at infinity and under some exponential moment conditions on ξ. If the exponential moment conditions are not satisfied, we still can find the asymptotic behaviour of $\mathbb{E}\Big[\big(I_t(\xi)\big)^{-p}\Big]$, for $p \in (0,1]$, under Spitzer's condition. We describe the main results of the chapter in Section 3.1. In the next section we apply the results to the following classes of processes in random environment: the competition model given in Section 2.2 and diffusion processes whose dynamics are perturbed by a Lévy random environment. For the competition model, we describe the asymptotic behaviour of its mean. For the diffusion processes, we provide the asymptotic behaviour of the tail probability of its global maximum. Finally, Section 3.3 is devoted to the proofs of the main results of the Chapter. The proof under the exponential moment conditions on ξ relies on a discretisation of the exponential functional $I_t(\xi)$ and is closely related to the behaviour of functionals of semi-direct products of random variables. The proof under Spitzer's condition relies in a factorisation of $I_t(\xi)$ given by Arista and Rivero [5].

3.1 Introduction and main results

Let $\xi = (\xi_t : t \geq 0)$ be a Lévy process with characteristic triplet (α, σ, Π) where $\alpha \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$ and Π is a measure on $\mathbb{R}\backslash\{0\}$ satisfying the integrability condition $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$. Recall that for all $z \in \mathbb{R}$

$$\mathbb{E}[e^{iz\xi_t}] = e^{t\psi(iz)},$$

where the Laplace exponent $\psi(z)$ is given by the Lévy-Khintchine formula

$$\psi(z) = \alpha z + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \left(e^{zx} - 1 - zxh(x)\right)\Pi(\mathrm{d}x), \qquad z \in \mathbb{R}.$$

Here, $h(x)$ is the cutoff function which is usually taken to be $h(x) \equiv \mathbf{1}_{\{|x|<1\}}$. Whenever the process ξ has finite mean, we will take $h(x) \equiv 1$.

In this chapter, we are interested in studying the exponential functional of ξ, defined by

$$I_t(\xi) := \int_0^t e^{-\xi_s} \mathrm{d}s, \qquad t \geq 0.$$

More precisely, one of our aims is to study the asymptotic behaviour of

$$\mathbb{E}\Big[F\big(I_t(\xi)\big)\Big] \qquad \text{as} \qquad t \to \infty,$$

where F is a non-increasing function with polynomial decay at infinity and under some exponential moment conditions on ξ. In particular, we find five different regimes that depend on the shape of $\psi(z)$, whenever it is well-defined. Let us now state our main results. Assume that

$$\theta^+ = \sup\{\lambda > 0 : \psi(\lambda) < \infty\} \tag{3.1}$$

exists and is positive. In other words, the Laplace exponent of the Lévy process ξ can be defined on $[0, \theta^+)$, see for instance Lemma 26.4 in Sato [84]. Besides, ψ satisfies

$$\psi(\lambda) = \log \mathbb{E}\left[e^{\lambda \xi_1}\right], \qquad \lambda \in [0, \theta^+).$$

From Theorem 25.3 in [84], $\psi(\lambda) < \infty$ is equivalent to

$$\int_{\{|x|>1\}} e^{\lambda x}\, \Pi(\mathrm{d}x) < \infty. \tag{3.2}$$

Moreover ψ belongs to $C^\infty((0, \theta^+))$ with $\psi(0) = 0$, $\psi'(0+) \in [-\infty, \infty)$ and $\psi''(\lambda) > 0$, for $\lambda \in (0, \theta^+)$ (see Lemma 26.4 in [84]). Hence, the Laplace exponent ψ is a convex function on $[0, \theta^+)$ implying that either it is positive or it may have another root in $(0, \theta^+)$. In the latter scenario, ψ has at most one global minimum on $(0, \theta^+)$. Whenever such a global minimum exists, we denote by τ the position where it is reached. As we will see below, this parameter is relevant to determine the asymptotic behaviour of $\mathbb{E}\left[I_t(\xi)^{-p}\right]$, for $0 < p < \theta^+$.

Let us introduce the exponential change of measure known as the Esscher transform. According to Theorem 3.9 in Kyprianou [64], for any λ such that (3.2) is satisfied, we can perform the following change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}^{(\lambda)}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = e^{\lambda \xi_t - \psi(\lambda) t}, \qquad t \geq 0 \tag{3.3}$$

where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration generated by ξ which is naturally completed. Moreover, under $\mathbb{P}^{(\lambda)}$ the process ξ is still a Lévy process with Laplace exponent given by

$$\psi_\lambda(z) = \psi(\lambda + z) - \psi(\lambda), \qquad z \in \mathbb{R}.$$

Theorem 2. *Assume that* $0 < p < \theta^+$.

i) If $\psi'(0+) > 0$, *then*

$$\lim_{t\to\infty} \mathbb{E}\left[I_t(\xi)^{-p}\right] = \mathbb{E}\left[I_\infty(\xi)^{-p}\right] > 0.$$

ii) If $\psi'(0+) = 0$ and $\psi''(0+) < \infty$, then there exists a positive constant c_1 such that

$$\lim_{t\to\infty} \sqrt{t}\mathbb{E}\left[I_t(\xi)^{-p}\right] = c_1.$$

iii) Assume that $\psi'(0+) < 0$

a) if $\psi'(p) < 0$, then

$$\lim_{t\to\infty} e^{-t\psi(p)}\mathbb{E}\left[I_t(\xi)^{-p}\right] = \mathbb{E}^{(p)}[I_\infty(-\xi)^{-p}] > 0.$$

b) if $\psi'(p) = 0$, then there exists a positive constant c_2 such that

$$\lim_{t\to\infty} \sqrt{t}e^{-t\psi(p)}\mathbb{E}\left[I_t(\xi)^{-p}\right] = c_2.$$

c) $\psi'(p) > 0$, then

$$\mathbb{E}\left[I_t(\xi)^{-p}\right] = o(t^{-1/2}e^{t\psi(\tau)}), \qquad \textit{as} \quad t\to\infty,$$

where τ is the position where the global minimum is reached. Moreover if we also assume that ξ is non-arithmetic (or non-lattice) then

$$\mathbb{E}\left[I_t(\xi)^{-p}\right] = O(t^{-3/2}e^{t\psi(\tau)}), \qquad \textit{as} \quad t\to\infty.$$

It is important to note that for any $q > 0$ satisfying (3.2), we necessarily have that $\mathbb{E}\left[I_t(\xi)^{-q}\right]$ is finite for all $t > 0$. Indeed, since $(e^{q\xi_t - t\psi(q)}, t \geq 0)$ is a positive martingale, we deduce from L_1-Doob's inequality (see for instance [1]) and the Esscher transform (3.3), that the following series of inequalities hold: for $t \leq 1$,

$$\begin{aligned}\mathbb{E}\left[I_t(\xi)^{-q}\right] &\leq t^{-q}\mathbb{E}\left[\sup_{0\leq u\leq 1} e^{q\xi_u}\right] \leq t^{-q}e^{\psi(q)\vee 0}\mathbb{E}\left[\sup_{0\leq u\leq 1} e^{q\xi_u - u\psi(q)}\right] \\ &\leq t^{-q}\frac{e^{1+\psi(q)\vee 0}}{e-1}\left(1 + \mathbb{E}^{(q)}\left[q\xi_1 - \psi(q)\right]\right) = t^{-q}\frac{e^{1+\psi(q)\vee 0}}{e-1}[1 + q\psi'(q) - \psi(q)],\end{aligned} \tag{3.4}$$

which is finite. The finiteness for $t > 1$ follows from the fact that $I_t(\xi)$ is non-decreasing.

We are now interested in extending the above result for a class of functions which have polynomial decay and are non-increasing at ∞. As we will see below such extension is not straightforward and need more conditions on the exponential moments of the Lévy process ξ.

For simplicity, we write

$$\mathcal{E}_F(t) := \mathbb{E}\left[F(I_t(\xi))\right],$$

where F belongs to a particular class of continuous functions on $\mathbb{R}_+$ that we will introduce below. We assume that the Laplace exponent ψ of ξ is well defined on the interval (θ^-, θ^+), where

$$\theta^- := \inf\{\lambda < 0 : \psi(\lambda) < \infty\},$$

and θ^+ is defined as in (3.1). Recall that ψ is a convex function that belongs to $C^\infty((\theta^-, \theta^+))$ with $\psi(0) = 0$, $\psi'(0+) \in [-\infty, \infty)$ and $\psi''(\lambda) > 0$, for $\lambda \in (\theta^-, \theta^+)$. Also recall that $\tau \in [0, \theta^+)$ is the position where the minimum of ψ is reached.

Let $\mathbf{k}$ be a positive constant. We will consider functions F satisfying one of the following conditions: There exists $x_0 \geq 0$ such that $F(x)$ is non-increasing for $x \geq x_0$, and

(**A1**) F satisfies

$$F(x) = \mathbf{k}(x+1)^{-p}\Big[1+(1+x)^{-\varsigma}h(x)\Big], \qquad \text{for all } x>0,$$

where $0<p\leq\tau$, $\varsigma\geq 1$ and h is a Lipschitz function which is bounded.

(**A2**) F is an Hölder function with index $\alpha>0$ satisfying

$$F(x)\leq \mathbf{k}(x+1)^{-p}, \qquad \text{for all } x>0,$$

with $p>\tau$.

Theorem 3. *Assume that $0<p<\theta^+$. We have the following five regimes for the asymptotic behaviour of $\mathcal{E}_F(t)$ for large t.*

i) If $\psi'(0+)>0$ and F is a positive and continuous function which is bounded, then

$$\lim_{t\to\infty}\mathcal{E}_F(t)=\mathcal{E}_F(\infty).$$

*ii) If $\psi'(0+)=0$, F satisfies (**A2**) and $\theta^-<0$, then there exists a positive constant c_3 such that*

$$\lim_{t\to\infty}\sqrt{t}\mathcal{E}_F(t)=c_3.$$

iii) Suppose that $\psi'(0+)<0$:

*a) If F satisfies (**A1**) and $\psi'(p)<0$, then,*

$$\lim_{t\to\infty}e^{-t\psi(p)}\mathcal{E}_F(t)=\lim_{t\to\infty}e^{-t\psi(p)}\mathbf{k}\mathbb{E}\left[I_t(\xi)^{-p}\right]=\mathbf{k}\mathbb{E}^{(p)}\left[I_\infty(-\xi)^{-p}\right].$$

*b) If F satisfies (**A1**) and $\psi'(p)=0$, then,*

$$\lim_{t\to\infty}\sqrt{t}e^{-t\psi(p)}\mathcal{E}_F(t)=\lim_{t\to\infty}\sqrt{t}e^{-t\psi(p)}\mathbf{k}\mathbb{E}\left[I_t(\xi)^{-p}\right]=\mathbf{k}c_2,$$

where c_2 has been defined in point iii) b) of Theorem 2.

*c) If F satisfies (**A2**), $\psi'(p)>0$ and $\tau+p<\theta^+$, then there exists a positive constant c_4 such that*

$$\lim_{t\to\infty}t^{3/2}e^{-t\psi(\tau)}\mathcal{E}_F(t)=c_4.$$

If θ^+ does not exits, we can still provide the asymptotic behaviour of $\mathcal{E}_F(t)$, for $F(x)=x^{-p}$ with $p\in(0,1]$, under the so-called *Spitzer's condition*; i.e. if there exists $\delta\in(0,1]$ such that

$$\lim_{t\to\infty}t^{-1}\int_0^t\mathbb{P}(\xi_s\geq 0)\,\mathrm{d}s=\delta. \tag{3.5}$$

Theorem 4. *Let $p\in(0,1]$ and suppose that ξ satisfies Spitzer's condition (3.5) with $\delta\in(0,1]$. Then, there exists a constant $c(p)$ that depends on p such that*

$$\lim_{t\to\infty}t^{1-\delta}\mathbb{E}\left[\left(\int_0^t e^{-\xi_s}ds\right)^{-p}\right]=c(p).$$

In particular if ξ satisfies Spitzer's condition with $\delta\in(0,1)$ and $0<\theta^+$, we necessary have $\psi'(0+)=\mathbb{E}[\xi_1]=0$ and $\delta=1/2$. Therefore, ξ is under the regimen of Theorem 2 part (ii).

3.2 Applications

Now, we provide two examples where we can apply the main results of this chapter. Both are processes perturbed by Lévy random environments.

3.2.1 Competition model in a Lévy random environment

We now study the asymptotic behaviour of the population model given in Section 2.2. Recall that the competition model in a Lévy random environment, $(Z_t, t \geq 0)$, is the unique strong solution of the SDE

$$Z_t = Z_0 + \int_0^t Z_s(a - kZ_s)\mathrm{d}s + \int_0^t Z_{s-}\mathrm{d}S_s$$

where $a > 0$ is the drift, $k > 0$ is the competition, and the environment is given by the Lévy process defined in (1.8). Moreover, the process Z satisfies the Markov property and we have

$$Z_t = \frac{Z_0 e^{K_t}}{1 + kZ_0 \int_0^t e^{K_s}\mathrm{d}s}, \qquad t \geq 0,$$

where K is the Lévy process defined in (1.10).

The following result studies the asymptotic behaviour of $\mathbb{E}_z[Z_t]$, where $\mathbb{P}_z$ denotes the law of Z starting from z. Before stating our result, let us introduce the Laplace transform of the Lévy process K by

$$e^{\kappa(\theta)} = \mathbb{E}[e^{\theta K_1}], \qquad \theta \in \mathbb{R},$$

when it exists (see discussion on page 30). We assume that the Laplace exponent κ of K is well defined on the interval (θ_K^-, θ_K^+), where

$$\theta_K^- := \inf\{\lambda < 0 : \kappa(\lambda) < \infty\} \quad \text{and} \quad \theta_K^+ := \sup\{\lambda > 0 : \kappa(\lambda) < \infty\}.$$

Let τ be the position of the global minimum in $(0, \theta_K^+)$, and denote by $\mathbf{m} = \kappa'(0)$ and $\mathbf{m}_1 = \kappa'(1)$.

Proposition 5. *Assume that $1 < \theta_K^+$. For $z > 0$, we have the following five regimes for the asymptotic behaviour of $\mathbb{E}_z[Z_t]$.*

i) If $\mathbf{m} > 0$, then for every $z > 0$

$$\lim_{t\to\infty} \mathbb{E}_z[Z_t] = \frac{1}{k}\mathbb{E}\left[\frac{1}{I_\infty(K)}\right] > 0.$$

ii) If $\mathbf{m} = 0$, then

$$\mathbb{E}_z[Z_t] = O(t^{-1/2}).$$

iii) Suppose that $\mathbf{m} < 0$:

a) If $\mathbf{m}_1 < 0$, then,

$$\lim_{t\to\infty} e^{-t\kappa(1)}\mathbb{E}_z[Z_t] = \mathbb{E}^{(1)}\left[\frac{z}{1 + zkI_\infty(-K)}\right] > 0,$$

where $\mathbb{E}^{(1)}$ denotes the Esscher transform (3.3) of K with $\lambda = 1$.

b) If $\mathbf{m}_1 = 0$, then there exists a positive constant $c(z,k)$ that depends on z and k such that

$$\lim_{t\to\infty} \sqrt{t}e^{-t\kappa(1)}\mathbb{E}_z[Z_t] = c(z,k).$$

c) If $\mathbf{m}_1 > 0$ and $\tau + 1 < \theta^+$ then there exists a positive constant $c_1(z,k)$ that depends on z and k such that

$$\lim_{t\to\infty} t^{3/2}e^{-t\kappa(\tau)}\mathbb{E}_z[Z_t] = c_1(z,k).$$

Proof. We first recall that the time reversal process $(K_t - K_{(t-s)^-}, 0 \le s \le t)$ has the same law as $(K_s, 0 \le s \le t)$, (see Lemma II.2 in [14]). Then, for all $t \ge 0$

$$e^{-K_t}I_t(-K) = e^{-K_t}\int_0^t e^{K_{t-s}}\mathrm{d}s = \int_0^t e^{-(K_t-K_{t-s})}\mathrm{d}s \overset{\mathcal{L}}{=} \int_0^t e^{-K_s}\mathrm{d}s = I_t(K), \tag{3.6}$$

implying

$$(e^{-K_t}, e^{-K_t}I_t(-K)) \overset{\mathcal{L}}{=} (e^{-K_t}, I_t(K)).$$

The above implies that

$$\mathbb{E}_z[Z_t] = z\mathbb{E}\left[\left(e^{-K_t} + kze^{-K_t}\int_0^t e^{K_s}\mathrm{d}s\right)^{-1}\right] = z\mathbb{E}\left[\left(e^{-K_t} + kzI_t(K)\right)^{-1}\right]. \tag{3.7}$$

Let us now prove part i). Assume that $\mathbf{m} > 0$, then K drifts to ∞ and e^{-K_t} converges to 0 as t goes to ∞. By Theorem 1 in [19], $I_t(K)$ converges a.s. to $I_\infty(K)$, a non-negative and finite limit as t goes to ∞. We observe that the result follows from identity (2.8) and the Monotone Convergence Theorem.

Part ii) follows from the inequality

$$\mathbb{E}_z[Z_t] = z\mathbb{E}\left[\left(e^{-K_t} + kzI_t(K)\right)^{-1}\right] \le \mathbb{E}\left[(kI_t(K))^{-1}\right],$$

and Theorem 2 part (ii).

Finally, we prove part iii). Observe by applying the Esscher transform (3.3) with $\lambda = 1$ that

$$\mathbb{E}_z[Z_t] = ze^{\kappa(1)t}\mathbb{E}^{(1)}\left[\left(1 + kz\int_0^t e^{K_s}\mathrm{d}s\right)^{-1}\right].$$

Part iii)-a) follows by observing that under the probability measure $\mathbb{P}^{(1)}$, the process K is a Lévy process with mean $\mathbb{E}^{(1)}[K_1] = \kappa'(1) \in (-\infty, 0)$. We then conclude as in the proof of part i) by showing that $\mathbb{E}^{(1)}[(1 + kzI_t(-K))^{-1}]$, converges to $\mathbb{E}^{(1)}[(1 + kzI_\infty(-K))^{-1}]$, as t increases.

Finally parts iii)-b) and c) follows from a direct application of Theorem (3) parts iii)-b) and c), respectively, with the function $F : x \in \mathbb{R}_+ \mapsto z(1 + kzx)^{-1}$. □

3.2.2 Diffusion processes in a Lévy random environment

Let $(V(x), x \in \mathbb{R})$ be a stochastic process defined on $\mathbb{R}$ such that $V(0) = 0$. As presented in the introduction of this thesis (on page xv), a diffusion process $X = (X(t), t \ge 0)$ in a random potential V is a diffusion whose conditional generator given V is

$$\frac{1}{2}e^{V(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-V(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

It is well known that X may be constructed from a Brownian motion through suitable changes of scale and time, see Brox [22].

Kawazu and Tanaka [59] studied the asymptotic behaviour of the tail of the distribution of the maximum of a diffusion in a drifted Brownian potential. Carmona et al. [24] considered the case when the potential is a Lévy process whose discontinuous part is of bounded variation. The problem is the following: How fast does $\mathbb{P}(\max_{t\geq 0} X(t) > x)$ decay as $x \to \infty$? From these works, we know that

$$\mathbb{P}\left(\max_{t\geq 0} X(t) > x\right) = \mathbb{E}\left[\frac{A}{A + B_x}\right]$$

where

$$A = \int_{-\infty}^{0} e^{V(t)} dt \quad \text{and} \quad B_x = \int_{0}^{x} e^{V(t)} dt$$

are independent. In order to make our analysis more tractable, we consider $(\xi_t, t \geq 0)$ and $(\eta_t, t \geq 0)$ two independent Lévy processes, and we define

$$V(x) = \begin{cases} -\xi_x & \text{if} \quad x \geq 0 \\ -\eta_{-x} & \text{if} \quad x \leq 0. \end{cases}$$

We want to determine the asymptotic behaviour of

$$\mathbb{P}\left(\max_{s\geq 0} X(s) > t\right) = \mathbb{E}\left[\frac{I_\infty(\eta)}{I_\infty(\eta) + I_t(\xi)}\right].$$

We assume that η drifts to ∞, and recall the notations in the introduction of this chapter for the Laplace exponent ψ of ξ, and for θ^-, θ^+ and τ.

Proposition 6. *Assume that* $1 < \theta^+$.

i) If $\psi'(0+) > 0$, *then*

$$\lim_{t\to\infty} \mathbb{P}\left(\max_{s\geq 0} X(s) > t\right) = \mathbb{E}\left[\frac{I_\infty(\eta)}{I_\infty(\eta) + I_\infty(\xi)}\right] > 0.$$

ii) If $\psi'(0+) = 0$, *then there exists a positive constant* C_1 *that depends on the law of* $I_\infty(\eta)$ *such that*

$$\lim_{t\to\infty} \sqrt{t}\,\mathbb{P}\left(\max_{s\geq 0} X(s) > t\right) = C_1.$$

iii) Suppose that $\psi'(0+) < 0$:

a) If $\psi'(1) < 0$, *then there exists a positive constant* C_2 *that depends on the law of* $I_\infty(\eta)$ *such that,*

$$\lim_{t\to\infty} e^{-t\psi(1)}\mathbb{P}\left(\max_{s\geq 0} X(s) > t\right) = C_2.$$

b) If $\psi'(1) = 0$, *then there exists a positive constant* C_3 *that depends on the law of* $I_\infty(\eta)$ *such that*

$$\lim_{t\to\infty} \sqrt{t}e^{-t\psi(1)}\mathbb{P}\left(\max_{s\geq 0} X(s) > t\right) = C_3.$$

c) If $\psi'(1) > 0$, and $\tau + 1 < \theta^+$, then

$$\lim_{t\to\infty} \mathbb{P}\left(\max_{s\geq 0} X(s) > t\right) = o(t^{-1/2}e^{-t\psi(\tau)}).$$

Moreover, if the process ξ is non-arithmetic (or non-lattice) then there exists a positive constant C_4 that depends on the law of $I_\infty(\eta)$ such that

$$\lim_{t\to\infty} t^{3/2}e^{-t\psi(\tau)}\mathbb{P}\left(\max_{s\geq 0} X(s) > t\right) = C_4.$$

Furthermore, if there exists a positive ε such that

$$\mathbb{E}[I_\infty(\eta)^{1+\varepsilon}] < \infty,$$

then

$$C_i = c_i\mathbb{E}[I_\infty(\eta)], \quad i \in \{2,3\},$$

where $(c_i, i \in \{2,3\})$ do not depend on the law of $I_\infty(\eta)$.

Proof. Since η and ξ are independent, we have

$$\mathbb{P}\left(\max_{s\geq 0} X(s) > t\right) = \mathbb{E}\left[I_\infty(\eta) f(I_\infty(\eta), t)\right], \qquad t > 0$$

where

$$f(a,t) = \mathbb{E}\left[(a + I_t(\xi))^{-1}\right], \qquad a, t > 0.$$

The result follows from an application of Theorems 2 and 3 with the function

$$F : x \in \mathbb{R}_+ \mapsto z(a+x)^{-1}.$$

We only prove case ii), as the others are analogous. By Theorem 3 there exists $c_1(a) > 0$ such that

$$\lim_{t\to\infty} t^{1/2} f(a,t) = c_1(a).$$

Moreover, by Theorem 2, there exists c_1 such that

$$\lim_{t\to\infty} t^{1/2} f(0,t) = c_1.$$

Let us define $G_t(a) = at^{1/2}f(a,t)$, and $G_t^0(a) = at^{1/2}f(0,t)$. Observe that

$$G_t(a) \leq G_t^0(a), \qquad \text{for all } t, a \geq 0$$

and

$$\lim_{t\to\infty} \mathbb{E}\left[G_t^0(I_\infty(\eta))\right] = c_1\mathbb{E}\left[I_\infty(\eta)\right].$$

Then, by the Dominated Convergence Theorem (see for instance [32] problem 12 p. 145),

$$\lim_{t\to\infty} \sqrt{t}\mathbb{P}\left(\max_{s\geq 0} X(s) > t\right) = \lim_{t\to\infty} \mathbb{E}\left[G_t(I_\infty(\eta))\right] = \mathbb{E}\left[I_\infty(\eta)c_1(I_\infty(\eta))\right].$$

We complete the proof for the existence of the limits by observing that

$$0 < C_1 = \mathbb{E}\left[I_\infty(\eta)c_1(I_\infty(\eta))\right] \le c_1\mathbb{E}\left[I_\infty(\eta)\right] < \infty.$$

The last part of the proof consists in justifying the form of the constants C_2 and C_3 under the additional condition $\mathbb{E}[I_\infty(\eta)^{1+\varepsilon}] < \infty$ for a positive ε. For every $0 \le \varepsilon \le 1$, we have

$$\frac{I_\infty(\eta)}{I_t(\xi)} - \frac{I_\infty(\eta)}{I_\infty(\eta)+I_t(\xi)} = \frac{I_\infty(\eta)}{I_t(\xi)}\frac{I_\infty(\eta)}{I_\infty(\eta)+I_t(\xi)} \le \frac{I_\infty(\eta)}{I_t(\xi)}\left(\frac{I_\infty(\eta)}{I_\infty(\eta)+I_t(\xi)}\right)^{\varepsilon} \le \left(\frac{I_\infty(\eta)}{I_t(\xi)}\right)^{1+\varepsilon}$$

Hence

$$0 \le \mathbb{E}\left[\frac{I_\infty(\eta)}{I_t(\xi)} - \frac{I_\infty(\eta)}{I_\infty(\eta)+I_t(\xi)}\right] \le \mathbb{E}[(I_\infty(\eta))^{1+\varepsilon}]\mathbb{E}\left[\frac{1}{(I_t(\xi))^{1+\varepsilon}}\right].$$

But from point iii)-c) of Theorem 2 and Equation (3.17) in the proof of Theorem 3, we know that in the cases iii)-a) and iii)-b),

$$\mathbb{E}\left[I_t(\xi)^{-(1+\varepsilon)}\right] = o\left(\mathbb{E}\left[I_t(\xi)^{-1}\right]\right).$$

This ends the proof. □

3.3 Proofs of Theorems 2, 3 and 4.

This section is dedicated to the proofs of the main results of this chapter. We first prove Theorem 2. The proof of part ii) is based on the following approximation technique.

Let $(N_t^{(q)}, t \ge 0)$ be a Poisson process with intensity $q > 0$, which is independent of the Lévy process ξ, and denote by $(\tau_n^q)_{n\ge 0}$ its sequence of jump times with the convention that $\tau_0^q = 0$. For simplicity, we also introduce for $n \ge 0$,

$$\xi_t^{(n)} = \xi_{\tau_n^q + t} - \xi_{\tau_n^q}, \qquad t \ge 0.$$

For $n \ge 0$, we define the following random variables

$$S_n^{(q)} := \xi_{\tau_n^q}, \qquad M_n^{(q)} := \sup_{\tau_n^q \le t < \tau_{n+1}^q} \xi_t \qquad \text{and} \qquad I_n^{(q)} := \inf_{\tau_n^q \le t < \tau_{n+1}^q} \xi_t.$$

Observe that $(S_n^{(q)}, n \ge 0)$ is a random walk with step distribution given by $\xi_{\tau_1^q}$ and that τ_1^q is an exponential r.v. with parameter q which is independent of ξ.

Similarly for the process $\xi^{(n)}$, we also introduce

$$m_n^{(q)} := \sup_{t < \tau_{n+1}^q - \tau_n^q} \xi_t^{(n)} \quad \text{and} \quad i_n^{(q)} := \inf_{t < \tau_{n+1}^q - \tau_n^q} \xi_t^{(n)}.$$

Lemma 2. *Using the above notation we have,*

$$M_n^{(q)} = S_n^{(+,q)} + m_0^{(q)}, \qquad I_n^{(q)} = S_n^{(-,q)} + i_0^{(q)}$$

where each of the processes $S^{(+,q)} = (S_n^{(+,q)}, n \ge 0)$ and $S^{(-,q)} = (S_n^{(-,q)}, n \ge 0)$ are random walks with the same distribution as $S^{(q)}$. Moreover $S^{(+,q)}$ and $m_0^{(q)}$ are independent, as are $S^{(-,q)}$ and $i_0^{(q)}$.

The proof of this lemma is based on the Wiener-Hopf factorisation (see Equations (4.3.3) and (4.3.4) in [30]). It follows from similar arguments as those used in the proof of Theorem IV.13 in [30], which considers the case when the exponential random variables are jump times of the process ξ restricted to $\mathbb{R} \setminus [-\eta, \eta]$, for $\eta > 0$. So, we omit it for the sake of brevity.

Recall that τ_1^q goes to 0, in probability, as q increases and that ξ has càdlàg paths. Hence, there exists an increasing sequence $(q_n)_{n\geq 0}$ such that $q_n \to \infty$ and

$$e^{\lambda i_0^{(q_n)}} \underset{n\to\infty}{\longrightarrow} 1, \qquad \text{a.s.} \tag{3.8}$$

We also recall the following form of the Wiener-Hopf factorisation, for $q > \psi(\lambda)$

$$\frac{q}{q-\psi(\lambda)} = \mathbb{E}\left[e^{\lambda i_0^{(q)}}\right] \mathbb{E}\left[e^{\lambda m_0^{(q)}}\right]. \tag{3.9}$$

From the Dominated Convergence Theorem and identity (3.9), it follows that for $\varepsilon \in (0,1)$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$1 - \varepsilon \leq \mathbb{E}\left[e^{\lambda i_0^{(q_n)}}\right] \leq \mathbb{E}\left[e^{\lambda m_0^{(q_n)}}\right] \leq 1 + \varepsilon. \tag{3.10}$$

Next, we introduce the compound Poisson process

$$Y_t^{(q)} := S^{(q)}_{N_t^{(q)}}, \qquad t \geq 0,$$

whose Laplace exponent satisfies

$$\psi^{(q)}(\lambda) := \log \mathbb{E}\left[e^{\lambda Y_1^{(q)}}\right] = \frac{q\psi(\lambda)}{q - \psi(\lambda)},$$

which is well defined for λ such that $q > \psi(\lambda)$. Similarly, we define

$$\widetilde{I}_t^{(q)} = I^{(q)}_{N_t^{(q)}}, \qquad \widetilde{M}_t^{(q)} = M^{(q)}_{N_t^{(q)}}, \qquad Y_t^{(+,q)} = S^{(+,q)}_{N_t^{(q)}}, \qquad \text{and} \qquad Y_t^{(-,q)} = S^{(-,q)}_{N_t^{(q)}}.$$

We observe from the definitions of $\widetilde{M}^{(q)}$ and $\widetilde{I}^{(q)}$, and Lemma 2, that for all $t \geq 0$, the following inequalities are satisfied

$$e^{-m_0^{(q)}} \int_0^t e^{-Y_s^{(+,q)}} \mathrm{d}s \leq \int_0^t e^{-\xi_s} \mathrm{d}s \leq e^{-i_0^{(q)}} \int_0^t e^{-Y_s^{(-,q)}} \mathrm{d}s. \tag{3.11}$$

We have now all the tools needed to prove Theorem 2.

Proof of Theorem 2. i) Assume that $\psi'(0+) > 0$. According to Theorem 1 in [19], $I_t(\xi)$ converges a.s. to $I_\infty(\xi)$, a non-negative and finite limit as t goes to ∞. Then, we observe that the result follows from the Monotone Convergence Theorem.

We now prove part ii). In order to do so, we use the approximation and notation that we introduced at the beginning of this section. Let $(q_n)_{n\geq 1}$ be a sequence defined as in (3.8) and observe that for $n \geq 1$, we have $\psi^{(q_n)}(0) = 0$, $\psi'^{(q_n)}(0+) = 0$ and $\psi''^{(q_n)}(0+) < \infty$. We also observe that the processes $Y^{(+,q_n)}$ and $Y^{(-,q_n)}$ have bounded variation paths.

We take $\ell \geq N$ and $0 < \varepsilon < 1$. Hence from Lemmas 13 and 14 in Bansaye et al. [9], we observe that there exists a positive constant $c_1(\ell)$ such that

$$(1-\varepsilon)c_1(\ell)t^{-1/2} \leq \mathbb{E}\left[\left(\int_0^t e^{-Y_s^{(\pm,q_\ell)}} \mathrm{d}s\right)^{-p}\right] \leq (1+\varepsilon)c_1(\ell)t^{-1/2}, \quad \text{as } t \to \infty.$$

Therefore using (3.10) and (3.11) in the previous inequality, we obtain

$$(1-\varepsilon)^2 c_1(\ell)t^{-1/2} \leq \mathbb{E}[I_t(\xi)^{-p}] \leq (1+\varepsilon)^2 c_1(\ell)t^{-1/2}, \quad \text{as } t \to \infty. \tag{3.12}$$

Next, we take $n, m \geq N$ and observe that the previous inequalities imply

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2 c_1(n) \leq c_1(m) \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2 c_1(n), \qquad \text{for all } n, m \geq N.$$

Thus, we deduce that $(c_1(n))_{n\geq 1}$ is a Cauchy sequence. Let us denote c_1 its limit which, by the previous inequalities is positive. Let $k \geq N$ such that

$$(1-\varepsilon)c_1 \leq c_1(k) \leq (1+\varepsilon)c_1.$$

Using this inequality and (3.12), we observe

$$(1-\varepsilon)^3 c_1 t^{-1/2} \leq \mathbb{E}[I_t(\xi)^{-p}] \leq (1+\varepsilon)^3 c_1 t^{-1/2}, \quad \text{as } t \to \infty.$$

This completes the proof of part ii).

Now, we prove part iii)-a). Recalling (3.6) yields that

$$I_t(\xi) \stackrel{\mathcal{L}}{=} e^{-\xi_t} I_t(-\xi), \qquad t \geq 0. \tag{3.13}$$

Hence using the Esscher transform (3.3), with $\lambda = p$, we have

$$\mathbb{E}\left[I_t(\xi)^{-p}\right] = \mathbb{E}\left[e^{p\xi_t} I_t(-\xi)^{-p}\right] = e^{t\psi(p)}\mathbb{E}^{(p)}\left[I_t(-\xi)^{-p}\right], \qquad t \geq 0. \tag{3.14}$$

The inequality (3.4) with $q = p$ and the previous identity imply that the decreasing function $t \mapsto \mathbb{E}^{(p)}[I_t(-\xi)^{-p}]$ is finite for all $t > 0$. Recall that under the probability measure $\mathbb{P}^{(p)}$, the process ξ is a Lévy process with mean $\mathbb{E}^{(p)}[\xi_1] = \psi'(p) \in (-\infty, 0)$. Then, as in the proof of part i), $\mathbb{E}^{(p)}[I_t(-\xi)^{-p}]$ converges to $\mathbb{E}^{(p)}[I_\infty(-\xi)^{-p}]$, as t increases.

Part iii)-b) follows from part ii) and the Esscher transform (3.3). More precisely, we apply the Esscher transform with $\lambda = p$ and observe that the Laplace transform of the process ξ under the probability measure $\mathbb{P}^{(p)}$, satisfies $\psi_p'(0+) = \psi'(p) = 0$ and $\psi_p''(0+) = \psi''(p) < \infty$. Therefore by applying part ii) and identity (3.14), we get the existence of a constant $c_2 > 0$ such that

$$\mathbb{E}\left[I_t(\xi)^{-p}\right] = e^{t\psi(p)}\mathbb{E}^{(p)}[I_t(-\xi)^{-p}] \sim c_2 t^{-1/2} e^{t\phi(p)}.$$

Finally we prove part iii)-c). Again from the Esscher transform with $\lambda = \tau$, we see

$$\mathbb{E}\left[I_t(\xi)^{-p}\right] = e^{t\psi(\tau)}\mathbb{E}^{(\tau)}[e^{(p-\tau)\xi_t} I_t(-\xi)^{-p}], \qquad t > 0.$$

On the one hand, for $t > 0$,

$$\begin{aligned}
\mathbb{E}^{(\tau)}[e^{(p-\tau)\xi_t} I_t(-\xi)^{-p}] &= \mathbb{E}^{(\tau)}\left[e^{(p-\tau)(\xi_t-\xi_{t/2})}\frac{(e^{-\xi_{t/2}}I_{t/2}(-\xi)+\int_{t/2}^t e^{\xi_u-\xi_{t/2}}\mathrm{d}u)^{-(p-\tau)}}{(I_{t/2}(-\xi)+e^{\xi_{t/2}}\int_{t/2}^t e^{\xi_u-\xi_{t/2}}\mathrm{d}u)^{\tau}}\right]\\
&\le \mathbb{E}^{(\tau)}\left[e^{(p-\tau)(\xi_t-\xi_{t/2})}\frac{(\int_0^{t/2} e^{\xi_{s+t/2}-\xi_{t/2}}\mathrm{d}s)^{-(p-\tau)}}{I_{t/2}(-\xi)^{\tau}}\right]\\
&= \mathbb{E}^{(\tau)}\left[e^{(p-\tau)(\xi_{t/2})}I_{t/2}(-\xi)^{-(p-\tau)}\right]\mathbb{E}^{(\tau)}\left[I_{t/2}(-\xi)^{-\tau}\right],
\end{aligned}$$

where we have used in the last identity the fact that $(\xi_{u+t/2}-\xi_{t/2}, u \ge 0)$ is independent of $(\xi_u, 0 \le u \le t/2)$ and with the same law as $(\xi_u, u \ge 0)$.

On the other hand, from (3.13) we deduce

$$\mathbb{E}^{(\tau)}\left[e^{(p-\tau)(\xi_{t/2})}I_{t/2}(-\xi)^{-(p-\tau)}\right] = \mathbb{E}^{(\tau)}\left[I_{t/2}(\xi)^{-(p-\tau)}\right], \qquad t > 0.$$

Putting all the pieces together, we get

$$\mathbb{E}^{(\tau)}[e^{(p-\tau)\xi_t}I_t(-\xi)^{-p}] \le \mathbb{E}^{(\tau)}\left[I_{t/2}(\xi)^{-(p-\tau)}\right]\mathbb{E}^{(\tau)}\left[I_{t/2}(-\xi)^{-\tau}\right], \qquad t > 0$$

implying

$$\mathbb{E}\left[I_t(\xi)^{-p}\right] \le e^{t\psi(\tau)}\mathbb{E}^{(\tau)}\left[I_{t/2}(\xi)^{-(p-\tau)}\right]\mathbb{E}^{(\tau)}\left[I_{t/2}(-\xi)^{-\tau}\right], \qquad t > 0.$$

Since $\psi'(\tau) = 0$, we have $\mathbb{E}^{(\tau)}[\xi_1] = 0$ and the process ξ oscillates under $\mathbb{P}^{(\tau)}$. Moreover since $\psi''(\tau) < \infty$, we deduce that $\psi''_\tau(0+) < \infty$. The latter condition implies from part ii) that there exists a constant $c_1(\tau) > 0$ such that

$$\mathbb{E}^{(\tau)}[I_t(\xi)^{-(p-\tau)}] \sim c_1(\tau)t^{-1/2} \qquad \text{as} \quad t \to \infty.$$

Since the process ξ oscillates under $\mathbb{P}^{(\tau)}$, the dual $-\xi$ also oscillates. This implies that $I_t(-\xi)$ goes to ∞ and therefore $\mathbb{E}^{(\tau)}[I_t(-\xi)^{-(p-\tau)}]$ goes to 0, as t increases. In other words, we have

$$\mathbb{E}\left[I_t(\xi)^{-p}\right] = o(t^{-1/2}e^{t\psi(\tau)}), \qquad \text{as} \quad t \to \infty,$$

as expected.

We now assume that ξ is non-arithmetic, our arguments are similar to those used in [9]. We will prove

$$\limsup_{t\to\infty} t^{3/2}e^{-t\psi(\tau)}\mathbb{E}\left[I_t(\xi)^{-p}\right] < \infty.$$

In order to prove it, we take $t > 0$ and observe

$$I_{\lfloor t\rfloor}(\xi) = \sum_{k=0}^{\lfloor t\rfloor-1} e^{-\xi_k}\int_0^1 e^{-(\xi_{k+u}-\xi_k)}\mathrm{d}u.$$

Therefore

$$\mathbb{E}\left[I_{\lfloor t\rfloor}(\xi)^{-p}\right] \le \mathbb{E}\left[\min_{k\le\lfloor t\rfloor-1} e^{p\xi_k}\left(\int_0^1 e^{-(\xi_{k+u}-\xi_k)}\mathrm{d}u\right)^{-p}\right].$$

Conditioning on the value when the minimum is attained, let say k', and observing that $e^{p\xi_{k'}}$ is independent of $\left(\int_0^1 e^{-(\xi_{k'+u}-\xi_{k'})}\mathrm{d}u\right)^{-p}$ and the latter has the same law as $\left(\int_0^1 e^{-\xi_u}\mathrm{d}u\right)^{-p}$, we deduce

$$\mathbb{E}\left[I_{\lfloor t\rfloor}(\xi)^{-p}\right] \le \mathbb{E}\left[\min_{k\le\lfloor t\rfloor-1} e^{p\xi_k}\right]\mathbb{E}\left[\left(\int_0^1 e^{-\xi_u}\mathrm{d}u\right)^{-p}\right].$$

Finally, by Lemma 7 in [52], there exists a $C>0$ such that

$$\mathbb{E}\left[\min_{k\le\lfloor t\rfloor-1} e^{p\xi_k}\right] \sim C\lfloor t\rfloor^{-3/2}e^{\lfloor t\rfloor\psi(\tau)}, \qquad \text{for } t \text{ large.}$$

The claim follows from the monotonicity of $\mathbb{E}\left[I_{\lfloor t\rfloor}(\xi)^{-p}\right]$ and the fact that $t\in(\lfloor t\rfloor,\lfloor t\rfloor+1)$. $\square$

The idea of the proof of Theorem 3 is to study the asymptotic behaviour of $\mathcal{E}_F(n/q)$ for q fixed and large n, and then to use the monotonicity of F to deduce the asymptotic behaviour of $\mathcal{E}_F(t)$ when t goes to infinity. In order to do so, we use a key result due to Guivarc'h and Liu (see Theorem 2.1 in [50]) that we state here for the sake of completeness.

Theorem 5 (Giuvarc'h, Liu 01)**.** *Let $(a_n,b_n)_{n\ge0}$ be a $\mathbb{R}_+^2$-valued sequence of i.i.d. random variables such that $\mathbb{E}[\ln a_0]=0$. Assume that $b_0/(1-a_0)$ is not constant a.s. and define*

$$A_0:=1,\quad A_n:=\prod_{k=0}^{n-1}a_k \quad \textit{and} \quad B_n:=\sum_{k=0}^{n-1}A_kb_k, \qquad \textit{for} \quad n\ge1.$$

Let η,κ,ϑ be three positive numbers such that $\kappa<\vartheta$, and $\tilde{\phi}$ and $\tilde{\psi}$ be two positive continuous functions on $\mathbb{R}_+$ such that they do not vanish and for a constant $C>0$ and for every $a>0$, $b\ge0$, $b'\ge0$, we have

$$\tilde{\phi}(a)\le Ca^\kappa, \quad \tilde{\psi}(b)\le\frac{C}{(1+b)^\vartheta}, \qquad \textit{and} \quad |\tilde{\psi}(b)-\tilde{\psi}(b')|\le C|b-b'|^\eta.$$

Moreover, assume that

$$\mathbb{E}\big[a_0^\kappa\big]<\infty, \quad \mathbb{E}\big[a_0^{-\eta}\big]<\infty, \quad \mathbb{E}\big[b_0^\eta\big]<\infty \qquad \textit{and} \quad \mathbb{E}\big[a_0^{-\eta}b_0^{-\vartheta}\big]<\infty.$$

Then, there exist two positive constants $c(\tilde{\phi},\tilde{\psi})$ and $c(\tilde{\psi})$ such that

$$\lim_{n\to\infty}n^{3/2}\mathbb{E}\left[\tilde{\phi}(A_n)\tilde{\psi}(B_n)\right]=c(\tilde{\phi},\tilde{\psi}) \qquad \textit{and} \qquad \lim_{n\to\infty}n^{1/2}\mathbb{E}\left[\tilde{\psi}(B_n)\right]=c(\tilde{\psi}).$$

Let $q>0$ and define the sequence $q_n=n/q$, for $n\ge0$. For $k\ge0$, we also define

$$\widetilde{\xi}^{(k)}_u=\xi_{q_k+u}-\xi_{q_k}, \qquad \text{for} \quad u\ge0,$$

and

$$a_k=e^{-\widetilde{\xi}^{(k)}_{q_{k+1}-q_k}} \qquad \text{and} \qquad b_k=\int_0^{q_{k+1}-q_k}e^{-\widetilde{\xi}^{(k)}_u}\mathrm{d}u. \tag{3.15}$$

Hence, (a_k,b_k) is a $\mathbb{R}_+^2$-valued sequence of i.i.d. random variables. Also observe that

$$a_0=e^{-\xi_{\frac{1}{q}}} \qquad \text{and} \qquad \frac{b_0}{1-a_0}=\frac{I_{\frac{1}{q}}(\xi)}{1-e^{-\xi_{\frac{1}{q}}}},$$

which are not constant a.s. as required by Theorem 5. Moreover, we have

$$\int_{q_i}^{q_{i+1}} e^{-\xi_u}\mathrm{d}u = e^{-\xi_{q_i}} b_i = \prod_{k=0}^{i-1} a_k b_i = A_i b_i,$$

where A_k is defined as in Theorem 5. The latter identity implies

$$I_{q_n}(\xi) = \sum_{i=0}^{n-1} \int_{q_i}^{q_{i+1}} e^{-\xi_u}\mathrm{d}u = \sum_{i=0}^{n-1} A_i b_i := B_n.$$

In other words, we have all the objects required to apply Theorem 5.

Proof of Theorem 3. i) The proof uses similar arguments as those used in the proof of Theorem 2-i).

ii) We now assume that $\psi'(0+) = 0$. We define the sequence $(a_k, b_k)_{k\geq 0}$ as in (3.15) and follow the same notation as in Theorem 5. We take $0 < \eta < \alpha$ and $d_p > 1$ such that $-\theta^-/d_p < p$ and $\theta^- < -\eta < \eta + p < \theta^+$, and let

$$(\eta, \kappa, \vartheta) = \left(\eta, \frac{-\theta^-}{d_p}, p\right).$$

Next, we verify the moment conditions of Theorem 5 for the couple (a_0, b_0). From the definition of (a_0, b_0), it is clear

$$\mathbb{E}\left[\ln a_0\right] = \frac{\psi'(0+)}{q} = 0, \qquad \mathbb{E}\left[a_0^{\kappa}\right] = e^{\psi(-\kappa)/q} \quad \text{and} \quad \mathbb{E}\left[a_0^{-\eta}\right] = e^{\psi(\eta)/q},$$

which are well defined. Similarly as in (3.4), by L_1-Doob's inequality (see [1]) and the Esscher transform (3.3)

$$\mathbb{E}\left[b_0^{\eta}\right] \leq q^{\eta}\mathbb{E}\left[\sup_{0\leq u\leq 1/q} e^{-\eta\xi_u}\right] \leq \frac{e}{e-1} q^{\eta} e^{\frac{\psi(-\eta)}{q}} \left(1 - \eta\psi'(-\eta) - \psi(-\eta)\right) < \infty,$$

and

$$\mathbb{E}\left[a_0^{-\eta} b_0^{-\vartheta}\right] \leq q^{\vartheta}\mathbb{E}\left[e^{\eta\xi_{\frac{1}{q}}} \sup_{0\leq u\leq 1/q} e^{\vartheta\xi_u}\right] \leq q^{\vartheta}\mathbb{E}\left[\sup_{0\leq u\leq 1/q} e^{(\eta+\vartheta)\xi_u}\right] < \infty.$$

Therefore the asymptotic behaviour of $\mathcal{E}_F(q_n)$ for large n, follows from a direct application of Theorem 5. In other words, there exists a positive constant $c(q)$ such that

$$\sqrt{n}\mathcal{E}_F(q_n) \sim c(q), \qquad \text{as} \qquad n \to \infty.$$

In order to get our result, we take t to be a positive real number. Since the mapping $s \mapsto \mathcal{E}_F(s)$ is non-increasing, we get

$$\sqrt{t}\mathcal{E}_F(t) \leq \sqrt{t}\mathcal{E}_F(\lfloor qt\rfloor/q) = \sqrt{\frac{t}{\lfloor qt\rfloor}}\sqrt{\lfloor qt\rfloor}\mathcal{E}_F(\lfloor qt\rfloor/q).$$

Similarly

$$\sqrt{t}\mathcal{E}_F(t) \geq \sqrt{t}\mathcal{E}_F((\lfloor qt \rfloor + 1)/q) = \sqrt{\frac{t}{\lfloor qt \rfloor + 1}}\sqrt{\lfloor qt \rfloor + 1}\mathcal{E}_F((\lfloor qt \rfloor + 1)/q).$$

Therefore

$$\sqrt{t}\mathcal{E}_F(t) \sim c(q)q^{-1/2}, \qquad \text{as } t \to \infty.$$

Moreover, we deduce that $c(q)q^{-1/2}$ is positive and does not depend on q. Hence we denote this constant by c_1. This concludes the proof of point ii).

iii) For the rest of the proof, we assume that $\psi'(0) < 0$. We first prove part a). Since $\psi'(p) < 0$, from Theorem 2 part iii)-a) we know that

$$\mathbb{E}\left[I_t(\xi)^{-p}\right] \sim e^{t\psi(p)}\mathbb{E}^{(p)}[I_\infty(-\xi)^{-p}], \qquad as \quad t \to \infty.$$

Hence the asymptotic behaviour is proven if we show that

$$\mathcal{E}_F(t) \sim \mathbf{k}\mathbb{E}\left[I_t(\xi)^{-p}\right], \quad \text{as} \quad t \to \infty.$$

Since $\psi'(p) < 0$, there is $\varepsilon > 0$ such that $p(1+\varepsilon) < \theta^+$, $\psi(p(1+\varepsilon)) < \psi(p)$ and $\psi'((1+\varepsilon)p) < 0$. Hence, from Lemma 2 (see the Appendix), we deduce that there is a constant M such that

$$\left|F\left(I_t(\xi)\right) - \mathbf{k}I_t(\xi)^{-p}\right| \leq MI_t(\xi)^{-(1+\varepsilon)p}. \tag{3.16}$$

In other words, it is enough to prove

$$\mathbb{E}\left[I_t(\xi)^{-(1+\varepsilon)p}\right] = o(e^{t\psi(p)}), \quad \text{as} \quad t \to \infty. \tag{3.17}$$

From the Esscher transform (3.3) with $\lambda = (1+\varepsilon)p$, we deduce

$$\mathbb{E}\left[I_t(\xi)^{-(1+\varepsilon)p}\right] = \mathbb{E}\left[e^{p(1+\varepsilon)\xi_s}I_t(-\xi)^{-(1+\varepsilon)p}\right] = e^{t\psi(p)}e^{t\psi_p(\varepsilon p)}\mathbb{E}^{((1+\varepsilon)p)}\left[I_t(-\xi)^{-(1+\varepsilon)p}\right].$$

This and Equation (3.4) with $\lambda = (1+\varepsilon)p$ imply that $\mathbb{E}^{((1+\varepsilon)p)}[I_t(-\xi)^{-(1+\varepsilon)p}]$ is finite for all $t > 0$. Similarly as in the proof of Theorem 2 iii)-a), we can deduce that $\mathbb{E}^{((1+\varepsilon)p)}[I_t(-\xi)^{-(1+\varepsilon)p}]$ has a finite limit, as t goes to ∞. We conclude by observing that $\psi_p(\varepsilon p)$ is negative implying that (3.17) holds. We complete the proof of point iii)-a) by observing that (3.16) and (3.17) yield

$$\mathbb{E}[F\left(I_t(\xi)\right)] \sim \mathbf{k}\mathbb{E}[I_t(\xi)^{-p}], \quad t \to \infty.$$

We now prove part b). Since $\psi'(p) = 0$ and $\psi''(p) < \infty$, from Theorem 2 part iii)-b) we know that there exists a positive constant c_2 such that

$$\mathbb{E}\left[I_t(\xi)^{-p}\right] \sim c_2 t^{-1/2}e^{t\psi(p)}, \qquad as \quad t \to \infty.$$

Similarly as in the proof of part a), the asymptotic behaviour is proven if we show that

$$\mathcal{E}_F(t) \sim \mathbf{k}\mathbb{E}\left[I_t(\xi)^{-p}\right], \quad \text{as} \quad t \to \infty,$$

which amounts to showing that

$$\mathbb{E}\left[I_t(\xi)^{-(1+\varepsilon)p}\right] = o(t^{-1/2}e^{t\psi(p)}), \quad \text{as} \quad t\to\infty$$

for ε small enough. The latter follows from of Theorem 2 iii)-c).

Finally, we prove part c). Similarly as in the proof of part ii), we define the sequence $(a_k, b_k)_{k\geq 0}$ as in (3.15) and follow the same notation as in Theorem 5. Let us choose $0<\eta<\alpha$ such that $0<\tau-\eta<\tau+p+\eta<\theta^+$ and take

$$(\eta, \kappa, \vartheta) = (\eta, \tau, p)\,.$$

Next, we apply the Esscher transform (3.3) with $\lambda=\tau$ and observe

$$\mathbb{E}[F(I(q_n))]e^{-q_n\psi(\tau)} = \mathbb{E}^{(\tau)}[e^{-\tau\xi_{q_n}}F(I(q_n))] = \mathbb{E}^{(\tau)}[A_n^\tau F(B_n)]. \tag{3.18}$$

Hence in order to apply Theorem 5, we need the moment conditions on (a_0, b_0) to be satisfied under the probability measure $\mathbb{P}^{(\tau)}$. We first observe,

$$\mathbb{E}^{(\tau)}[\ln a_0] = \mathbb{E}^{(\tau)}[\xi_{1/q}] = e^{-\psi(\tau)/q}\mathbb{E}[\xi_{1/q}e^{\tau\xi_{1/q}}] = \frac{\psi'(\tau)}{q} = 0.$$

Similarly, we get

$$\mathbb{E}^{(\tau)}\left[a_0^{\kappa}\right] = \mathbb{E}^{(\tau)}[e^{-\kappa\xi_{1/q}}] = e^{-\psi(\tau)/q} \quad \text{and} \quad \mathbb{E}^{(\tau)}\left[a_0^{-\eta}\right] = \mathbb{E}^{(\tau)}[e^{\eta\xi_{1/q}}] = e^{\psi_\tau(\eta)/q},$$

where $\psi_\tau(\lambda) = \psi(\tau+\lambda)-\psi(\tau)$. From our assumptions both expectations are finite.

Again, we use similar arguments as those used in (3.4) to deduce

$$\mathbb{E}^{(\tau)}\left[b_0^{\eta}\right] \leq q^{-\eta}\mathbb{E}^{(\tau)}\left[\sup_{0\leq u\leq 1/q} e^{-\eta\xi_u}\right] \leq q^{-\eta}e^{-\psi(\tau)/q}\mathbb{E}\left[\sup_{0\leq u\leq 1} e^{(\tau-\eta)\xi_u}\right] < \infty,$$

and

$$\mathbb{E}^{(\tau)}\left[a_0^{-\eta}b_0^{-p}\right] \leq q^p\mathbb{E}^{(\tau)}\left[e^{\eta\xi_{\frac{1}{q}}}\sup_{0\leq u\leq 1/q} e^{p\xi_u}\right] \leq q^p e^{-\psi(\tau)/q}\mathbb{E}\left[\sup_{0\leq u\leq 1} e^{(\tau+\eta+p)\xi_u}\right] < \infty.$$

Therefore the asymptotic behaviour of $\mathbb{E}^{(\tau)}[A_n^\tau F(B_n)]$ follows from a direct application of Theorem 5 with the functions $\tilde{\psi}(x) = F(x)$ and $\tilde{\phi}(x) = x^\tau$. In other words, we conclude that there exists a positive constant $c(q)$ such that

$$n^{3/2}\mathbb{E}^{(\tau)}[A_n^\tau F(B_n)] \sim c(q), \quad n\to\infty.$$

In particular from (3.18), we deduce

$$\mathcal{E}_F(q_n) \sim c(q)e^{-n\psi(\tau)/q}n^{-3/2}, \quad n\to\infty.$$

Then using the monotonicity of F as in the proof of part ii), we get that for n large enough,

$$c(q)q^{-3/2}e^{-\psi(\tau)/q} \leq n^{3/2}e^{n\psi(\tau)}\mathcal{E}_F(n) \leq c(q)q^{-3/2}. \tag{3.19}$$

A direct application of Lemma 17 then yields the existence of a nonnegative constant c_4 such that

$$\lim_{q\to\infty} c(q)q^{-3/2} = c_4.$$

Moreover, (3.19) yields that c_4 is positive. This ends the proof. $\square$

Now, we proceed with the proof of Theorem 4.

Proof of Theorem 4. We first define the functions

$$f(t) := \mathbb{E}\left[\left(\int_0^t e^{-\xi_s} ds\right)^{-p}\right], \qquad t \geq 0,$$

and

$$g(q) = \int_0^\infty e^{-qt} f(t) \mathrm{d}t, \qquad q \geq 0.$$

According to the Tauberian Theorem and the monotone density Theorem (see for instance [14]), f is regularly varying at ∞ with index $\delta - 1$ if and only if g is regularly varying at $0+$ with index $-\delta$. Therefore, a natural way to analyse the asymptotic behaviour of f is via the asymptotic behaviour of its Laplace transform. In order to do so, we observe from Fubini's Theorem

$$g(q) = \int_0^\infty e^{-qt}\mathbb{E}\left[\left(\int_0^t e^{-\xi_s} ds\right)^{-p}\right] \mathrm{d}t = \frac{1}{q}\mathbb{E}\left[\left(\int_0^{\mathbf{e}_q} e^{-\xi_s} ds\right)^{-p}\right] = \frac{1}{q}\mathbb{E}\left[\left(I_{\mathbf{e}_q}(\xi)\right)^{-p}\right],$$

where $\mathbf{e}_q$ is an independent exponential random variable of parameter q. Note that we can identified $I_{\mathbf{e}_q}(\xi)$ as the exponential functional at ∞ of a Lévy process ξ killed at an independent exponential time with parameter $q > 0$.

Let denote by $\{(L_t^{-1}, H_t), t \geq 0\}$ and $\{(\widehat{L}_t^{-1}, \widehat{H}_t), t \geq 0\}$, for the ascending and descending ladder processes associated to ξ (see [64], Chapter 6 for a proper definition of these processes). The Laplace exponent of both of them will be denoted by $k(\alpha, \beta)$ and $\widehat{k}(\alpha, \beta)$, respectively. In other words,

$$k(\alpha, \beta) = -\log \mathbb{E}\left[e^{-\alpha L_1^{-1} - \beta H_1}\right], \qquad \text{and} \qquad \widehat{k}(\alpha, \beta) = -\log \mathbb{E}\left[e^{-\alpha \widehat{L}_1^{-1} - \beta \widehat{H}_1}\right].$$

If $H^{(q)}$ is the ascending ladder height process associated with the Lévy process ξ killed at an independent exponential time of parameter q, then, its Laplace exponent is $k(q, \cdot)$. This follow from the following identity,

$$\mathbb{E}\left[e^{i\lambda\xi_s}\mathbf{1}_{\{s<e_q\}}\right] = \mathbb{E}\left[e^{qs}e^{i\lambda\xi_s}\right],$$

and by evaluating in $s = L_1^{-1}$ and $\lambda = -i\beta$, i.e.

$$\mathbb{E}\left[e^{-\beta H_1^{(q)}}\right] = \mathbb{E}\left[e^{qL_1^{-1}}e^{-\beta H_1}\right] = e^{-k(q,\beta)}.$$

A similar identity holds for the descending ladder height process $\widehat{H}^{(q)}$, which is associated with the killed Lévy process. From Proposition 1 in [17], there exists a random variable R_q independent of $I_\infty(-\widehat{H}^{(q)})$, whose law is determined by its entire moments and satisfies the recurrent relation

$$\mathbb{E}\left[R_q^\lambda\right] = \widehat{k}(q, \lambda)\mathbb{E}\left[R_q^{\lambda-1}\right], \qquad \lambda > 0.$$

Moreover, we have

$$R_q I_\infty(-\widehat{H}^{(q)}) \stackrel{\mathcal{L}}{=} \mathbf{e}_1, \tag{3.20}$$

where $\mathbf{e}_1$ is an exponential random variable with parameter 1. According to Arista and Rivero ([5], Theorem 2), there exist a random variable J_q whose law is defined by

$$\mathbb{P}\left(J_q \in \mathrm{d}y\right) = \widehat{k}(q,0)y\mathbb{P}\left(\frac{1}{R_q} \in \mathrm{d}y\right), \qquad y \geq 0,$$

and such that

$$I_{\mathbf{e}_q}(\xi) \stackrel{\mathcal{L}}{=} J_q I_\infty(-H^{(q)}). \tag{3.21}$$

Therefore, by identities (3.20) and (3.21), we deduce

$$g(q) = \frac{\widehat{k}(q,0)}{q}\mathbb{E}\left[I_\infty(-H^{(q)})^{-p}\right]\mathbb{E}\left[\mathbf{e}_1^{p-1}\right]\mathbb{E}\left[I_\infty(-\widehat{H}^{(q)})^{p-1}\right]^{-1}. \tag{3.22}$$

On the other hand, we observe

$$\mathbb{E}\left[I_\infty(-H^{(q)})^{-p}\right] \stackrel{q\to\infty}{\longrightarrow} \mathbb{E}\left[I_\infty(-H)^{-p}\right] \qquad \text{and} \qquad \mathbb{E}\left[I_\infty(-\widehat{H}^{(q)})^{p-1}\right] \stackrel{q\to\infty}{\longrightarrow} \mathbb{E}\left[I_\infty(-\widehat{H})^{p-1}\right].$$

By Lemma 2.1 in [76], we also observe

$$0 < \mathbb{E}\left[I_\infty(-H)^{-p}\right]\mathbb{E}\left[I_\infty(-\widehat{H})^{p-1}\right] < \infty.$$

Now, we use Theorem VI.3.14 in [14], that assure that Spitzer's condition (3.5) is equivalent to $\widehat{k}(\cdot,0)$ being a regularly varying function at $0+$ with index $1-\delta$. Therefore, by identity (3.22), g is regularly varying at $0+$ with index $-\delta$ implying that f is regularly varying at ∞ with index $\delta - 1$ as expected, in other words

$$\lim_{t\to\infty} t^{1-\delta}\mathbb{E}\left[\left(\int_0^t e^{-\xi_s} ds\right)^{-p}\right] = c(p),$$

where $c(p)$ is a constant that depends on p.

□

Chapter 4

Stable CBLRE

This chapter is based in papers [80] and [81] elaborated in collaboration with Juan Carlos Pardo and Charline Smadi. We study the asymptotic behaviour of the absorption and explosion probabilities for stable continuous state branching processes in a Lévy random environment. The speed of explosion is studied in Section 4.2. We find 3 different regimes: subcritical-explosion, critical-explosion and supercritical explosion. The speed of absorption is studied in Section 4.3. As in the discrete case (time and space), we find five different regimes: supercritical, critical, weakly subcritical, intermediately subcritical and strongly subcritical. When the random environment is driven by a Brownian motion with drift, the limiting coefficients of the asymptotic behaviour of the absorption probability are explicit and written in terms of the initial population. In a general Lévy environment, the latter coefficients are also explicit in 3 out of the 5 regimes (supercritical, intermediate subcritical and strongly subcritical cases). This allows us to study two conditioned versions of the process: the process conditioned to be never absorbed (or Q-process) and the process conditioned on eventual absorption. Both processes are studied in Section 4.4.

4.1 Introduction

The stable case is perhaps one of the most interesting examples of CB-processes. One of the advantages of this class of CB-processes is that we can perform explicit computations of many functionals, see for instance [15, 67, 69], and that they appear in many other areas of probability such as coalescent theory, fragmentation theory, Lévy trees, self-similar Markov process to name but a few. As we will see below, we can also perform many explicit computations when the stable CB-process is affected by a Lévy random environment. For example, by using Theorem 3, we can find the precise asymptotic behaviour of the explosion and survival probabilities. When the environment is driven by a Brownian motion, by another technique, we can even obtain the constants in a closed form. When the limiting coefficients are explicit and in terms of the initial position, we find two conditioned version of the process: the process conditioned to be never absorbed (or Q-process) and the process conditioned on eventual absorption. We deduce them by using Doob h-transformation.

Let

$$\psi(\lambda) = c\lambda^{\beta+1}, \qquad \lambda \geq 0,$$

for some $\beta \in (-1, 0) \cup (0, 1]$ and c such that $c\beta > 0$. As we said in example 3, a stable continuous state branching processes in a Lévy random environment (SCBLRE) with branching mechanism

ψ is the process Z that satisfies the following stochastic differential equation

$$Z_t = Z_0 + \mathbf{1}_{\{\beta=1\}} \int_0^t \sqrt{2cZ_s}\mathrm{d}B_s + \mathbf{1}_{\{\beta\neq1\}} \int_0^t \int_0^\infty \int_0^{Z_{s-}} z\widehat{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t Z_{s-}\mathrm{d}S_s, \qquad (4.1)$$

where $B = (B_t, t \geq 0)$ is a standard Brownian motion, N is a Poisson random measure with intensity

$$\frac{c\beta(\beta+1)}{\Gamma(1-\beta)} \frac{1}{z^{2+\beta}} \mathrm{d}s\mathrm{d}z\mathrm{d}u,$$

$\widetilde{N}$ is its compensated version,

$$\widehat{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = \begin{cases} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) & \text{if } \beta \in (-1, 0), \\ \widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) & \text{if } \beta \in (0, 1), \end{cases}$$

and the process S is defined as in (1.8), i.e.

$$S_t = \alpha t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} (e^z - 1)\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{\mathbb{R}\setminus(-1,1)} (e^z - 1)N^{(e)}(\mathrm{d}s, \mathrm{d}z). \qquad (4.2)$$

Recall that the random environment was defined as

$$K_t = \mathbf{n}t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} v\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}v) + \int_0^t \int_{\mathbb{R}\setminus(-1,1)} vN^{(e)}(\mathrm{d}s, \mathrm{d}v),$$

where

$$\mathbf{n} = \alpha - \frac{\sigma^2}{2} - \int_{(-1,1)} (e^v - 1 - v)\pi(\mathrm{d}v).$$

According to example 3,

$$\mathbb{E}_z\Big[\exp\Big\{ - \lambda Z_t e^{-K_t}\Big\}\Big|K\Big] = \exp\left\{-z\left(\lambda^{-\beta} + \beta c\int_0^t e^{-\beta K_u}\mathrm{d}u\right)^{-1/\beta}\right\}, \quad \text{a.s.}$$

If we take limits as $\lambda \uparrow \infty$, in the above identity we obtain that for all $z, t > 0$, the *non-explosion probability* is given by

$$\mathbb{P}_z\Big(Z_t < \infty\Big|K\Big) = \mathbf{1}_{\{\beta>0\}} + \mathbf{1}_{\{\beta<0\}} \exp\left\{-z\left(\beta c\int_0^t e^{-\beta K_u}\mathrm{d}u\right)^{-1/\beta}\right\}, \qquad \text{a.s.} \qquad z \geq 0 \quad (4.3)$$

On the other hand, if we take limits as $\lambda \downarrow 0$, we deduce that the *survival probability* satisfies

$$\mathbb{P}_z\Big(Z_t > 0\Big|K\Big) = 1 - \mathbf{1}_{\{\beta>0\}} \exp\left\{-z\left(\beta c\int_0^t e^{-\beta K_u}\mathrm{d}u\right)^{-1/\beta}\right\}, \qquad \text{a.s.} \qquad z \geq 0 \qquad (4.4)$$

It is then clear that if $\beta \in (-1, 0)$, then the survival probability is equal to 1, for all $t \geq 0$. If $\beta \in (0, 1]$, then the process is conservative. In this chapter, we determine the asymptotic behaviour of the missing cases. First, we work with a Brownian environment. We use a time change method similar to the work of Böinghoff and Hutzenthaler [20]. And in consequence, we obtain explicitly the limiting coefficients of the asymptotic behaviour of the absorption and explosion probabilities. These coefficients are written in terms of the initial population. The main results in Chapter 3 allows us to find the latter coefficients when the environment is driven by a Lévy process. Unfortunately, with this technique we know that the constants are in terms of the initial population but we don't know the explicit relation. In their recent work, Li and Xu [72] obtained the same behaviour for the absorption probability. The key of their results was the observation that the asymptotics only depends on the sample paths of the Lévy process with local infimum decreasing slowly. Their coefficients are represented in terms of some transformations based on the renewal functions associated with the ladder process of K and its dual. However, despite the fact that the constants given in [72] are written in terms of the initial population, their explicit form still is hard to compute.

4.2 Speed of explosion of SCBLRE

Let us first study the event of *explosion* for stable branching processes in a Lévy random environment. Let us focus on $\beta \in (-1, 0)$. We recall that when the environment is constant, a stable CB-process explodes at time t with probability $1 - \exp\{-z\beta ct\}$. When a random environment affects the stable CB-process, it also explodes with positive probability, since

$$\mathbb{P}_z\Big(Z_t = \infty \Big| K\Big) = 1 - \exp\left\{-z\left(\beta c \int_0^t e^{-\beta(K_u + au)}\mathrm{d}u\right)^{-1/\beta}\right\} > 0,$$

but three different regimes appear for the asymptotic behaviour of the non-explosion probability that depend on the parameters of the random environment. Up to our knowledge, this behaviour was never observed or studied before, even in the discrete case. We call these regimes *subcritical-explosion*, *critical-explosion* or *supercritical-explosion* depending on whether this probability stays positive, converges to zero polynomially fast or converges to zero exponentially fast.
Before stating this result, let us introduce the Laplace transform of the Lévy process K by

$$e^{\psi_K(\theta)} = \mathbb{E}[e^{\theta K_1}], \tag{4.5}$$

when it exists (see discussion in page 30). We assume that the Laplace exponent ψ_k of K is well defined on an interval (θ_K^-, θ_K^+), where

$$\theta_K^- := \inf\{\lambda < 0 : \psi_K(\lambda) < \infty\} \quad \text{and} \quad \theta_K^+ := \sup\{\lambda > 0 : \psi_K(\lambda) < \infty\}.$$

As we will see in Propositions 7 and 8, the asymptotic behaviour of the probability of explosion depends on the sign of

$$\mathbf{m} = \psi_K'(0+).$$

First, we work in the Brownian environment case; i.e. when $S_t = \alpha t + \sigma B_t^{(e)}$. Then, the auxiliary process is $K_t = (\alpha - \frac{\sigma^2}{2})t + \sigma B_t^{(e)}$ and $\mathbf{m} = \alpha - \frac{\sigma^2}{2}$. In order to simplify notation, we will denote by $I_t^{(\eta)}$ for the exponential functional of a Brownian motion with drift $\eta \in \mathbb{R}$, in other words

$$I_t^{(\eta)} := \int_0^t \exp\Big\{2(\eta s + B_s)\Big\}\mathrm{d}s, \qquad t \in [0, \infty).$$

Let

$$\eta := -\frac{2}{\beta\sigma^2}\mathbf{m} \qquad \text{and} \qquad \mathbf{k} = \left(\frac{\beta\sigma^2}{2c}\right)^{1/\beta},$$

and define

$$g(x) := \exp\left\{-\mathbf{k}x^{1/\beta}\right\}, \qquad \text{for} \qquad x \geq 0.$$

From identity (4.3) and the scaling property, we deduce for $\beta \in (-1, 0)$,

$$\mathbb{P}_z\left(Z_t < \infty\right) = \mathbb{E}\left[g\left(\frac{z^\beta}{2I^{(\eta)}_{\beta^2\sigma^2 t/4}}\right)\right] = \int_0^\infty g(z^\beta v) p_{t\sigma_e^2/4,\eta}(v)\mathrm{d}v, \tag{4.6}$$

where $p_{\nu,\eta}$ denotes the density function of $1/2I_\nu^{(\eta)}$ which according to Matsumoto and Yor [75], satisfies

$$p_{\nu,\eta}(x) = \frac{e^{-\eta^2\nu/2}e^{\pi^2/2\nu}}{\sqrt{2}\pi^2\sqrt{\nu}}\Gamma\left(\frac{\eta+2}{2}\right)e^{-x}x^{-(\eta+1)/2}\int_0^\infty\int_0^\infty e^{\xi^2/2\nu}s^{(\eta-1)/2}e^{-xs} \times \frac{\sinh(\xi)\cosh(\xi)\sin(\pi\xi/\nu)}{(s+\cosh(\xi)^2)^{\frac{\eta+2}{2}}}\mathrm{d}\xi\mathrm{d}s. \tag{4.7}$$

We also denote

$$\mathcal{L}_{\eta,\beta}(\theta) = \mathbb{E}\Big[e^{-\theta\Gamma_{-\eta}^{1/\beta}}\Big], \qquad \text{for} \quad \theta \geq 0.$$

Note, that in the Brownian environment case, we can find explicitly the limiting coefficients.

Proposition 7. *Let $(Z_t, t \geq 0)$ be a SCBBRE with index $\beta \in (-1, 0)$ defined by the SDE (4.1) with $Z_0 = z > 0$ and environment $S_t = \alpha t + \sigma B_t^{(e)}$.*

i) Subcritical-explosion. If $\mathbf{m} < 0$, then

$$\lim_{t\to\infty}\mathbb{P}_z\left(Z_t < \infty\right) = \mathcal{L}_{\eta,\beta}\left(z\mathbf{k}\right). \tag{4.8}$$

ii) Critical-explosion. If $\mathbf{m} = 0$, then

$$\lim_{t\to\infty}\sqrt{t}\ \mathbb{P}_z\left(Z_t < \infty\right) = -\frac{\sqrt{2}}{\sqrt{\pi}\beta\sigma}\int_0^\infty e^{-z\mathbf{k}x^{1/\beta}-x}\frac{\mathrm{d}x}{x}. \tag{4.9}$$

iii) Supercritical-explosion. If $\mathbf{m} > 0$, then

$$\lim_{t\to\infty} t^{\frac{3}{2}} e^{\frac{\mathbf{m}^2 t}{2\sigma^2}}\mathbb{P}_z\left(Z_t < \infty\right) = -\frac{8}{\beta^3\sigma^3}\int_0^\infty g(z^\beta v)\phi_\eta(v)\mathrm{d}v, \tag{4.10}$$

where

$$\phi_\eta(v) = \int_0^\infty\int_0^\infty \frac{1}{\sqrt{2}\pi}\Gamma\left(\frac{\eta+2}{2}\right)e^{-v}v^{-\eta/2}u^{(\eta-1)/2}e^{-u}\frac{\sinh(\xi)\cosh(\xi)\xi}{(u+v\cosh(\xi)^2)^{\frac{\eta+2}{2}}}\mathrm{d}\xi\mathrm{d}u.$$

Proof. Our arguments follow similar reasoning as in the proof of Theorem 1.1 in Böinghoff and Hutzenthaler [20]. For this reason, following the same notation as in [20], we just provide the fundamental ideas of the proof.

The subcritical-explosion case (i) follows from the identity in law by Dufresne (2.6). More precisely, from (2.6), (4.6) and the Dominated Convergence Theorem, we deduce

$$\lim_{t\to\infty} \mathbb{P}_z\left(Z_t < \infty\right) = \mathbb{E}\left[\exp\left\{-z\mathbf{k}\Gamma_{-\eta}^{1/\beta}\right\}\right] = \mathcal{L}_{\eta,\beta}\left(z\mathbf{k}\right).$$

In order to prove the critical-explosion case (ii), we use Lemma 4.4 in [20]. From identity (4.6) and applying Lemma 4.4 in [20] to

$$g(z^\beta x) = \exp\left\{-z\mathbf{k}x^{1/\beta}\right\} \leq \frac{x^{-1/\beta}}{z\mathbf{k}}, \qquad \text{for} \quad x \geq 0, \tag{4.11}$$

we get

$$\begin{aligned}\lim_{t\to\infty} \sqrt{t}\; \mathbb{P}_z\left(Z_t < \infty\right) &= -\frac{2}{\beta\sigma} \lim_{t\to\infty} \sqrt{\frac{t\sigma^2\beta^2}{4}}\; \mathbb{E}\left[g\left(\frac{z^\beta}{2I^{(\eta)}_{\beta^2\sigma^2 t/4}}\right)\right] \\ &= -\frac{\sqrt{2}}{\sqrt{\pi}\beta\sigma} \int_0^\infty e^{-z\mathbf{k}x^{1/\beta}-x} \frac{\mathrm{d}x}{x},\end{aligned}$$

which is finite since the inequality (4.11) holds.

We now consider the supercritical-explosion case (iii). Observe that for all $n \geq 0$,

$$g(z^\beta x) = \exp\left\{-z\mathbf{k}x^{1/\beta}\right\} \leq \frac{x^{-n/\beta}}{n!(z\mathbf{k})^n}, \qquad \text{for} \quad x \geq 0.$$

Therefore using the above inequality for a fixed n, Lemma 4.5 in [20] and identity (4.6), we obtain that for $0 < \mathbf{m} < n\sigma^2/2$, the following limit holds

$$\begin{aligned}\lim_{t\to\infty} t^{3/2}\; e^{\mathbf{m}^2 t/2\sigma^2}\mathbb{P}_z\left(Z_t > 0\right) &= \lim_{t\to\infty} t^{3/2}\; e^{\eta^2\beta^2\sigma^2 t/8}\mathbb{E}\left[g\left(\frac{z^\beta}{2I^{(\eta)}_{\beta^2\sigma^2 t/4}}\right)\right] \\ &= -\frac{8}{\beta^3\sigma^3}\int_0^\infty e^{-z\mathbf{k}v^{1/\beta}}\phi_\eta(v)\mathrm{d}v,\end{aligned}$$

where ϕ_η is defined as in the statement of the proposition. Since this limit holds for any $n \geq 1$, we deduce that it must hold for $\mathbf{m} > 0$. This completes the proof.

□

Now, we study the asymptotic behaviour for a general Lévy environment. The proof of the following proposition is based in the work developed in Chapter 3. We know that the limiting coefficient are in terms of the initial position but in contrast with the continuous environment, in the critical and supercritical explosion cases we don't know the explicit relation. Here, τ is the position where $\psi'_K(\tau) = 0$, and it satisfies $\tau \in (\theta_K^-, 0]$.

Proposition 8. *Let $(Z_t, t \geq 0)$ be the SCBLRE with index $\beta \in (-1, 0)$ defined by the SDE (4.1) with $Z_0 = z > 0$, and recall the definition of the random environment K in (4.2).*

i) Subcritical-explosion. If $\mathbf{m} < 0$*, then, for every* $z > 0$

$$\lim_{t\to\infty} \mathbb{P}_z\Big(Z_t < \infty\Big) = \mathbb{E}\left[\exp\left\{-z\left(\beta c\int_0^\infty e^{-\beta K_u}\mathrm{d}u\right)^{-1/\beta}\right\}\right] > 0.$$

ii) Critical-explosion. If $\mathbf{m} = 0$*, then for every* $z > 0$ *there exists* $c_1(z) > 0$ *such that*

$$\lim_{t\to\infty} \sqrt{t}\mathbb{P}_z\Big(Z_t < \infty\Big) = c_1(z).$$

iii) Supercritical-explosion. If $\mathbf{m} > 0$*. Then for every* $z > 0$ *there exists* $c_2(z) > 0$ *such that*

$$\lim_{t\to\infty} t^{\frac{3}{2}} e^{-t\psi_K(\tau)}\mathbb{P}_z\Big(Z_t < \infty\Big) = c_2(z).$$

Proof. Observe that for a fixed $z > 0$, the function

$$F : x \in \mathbb{R}_+ \mapsto \exp(-z(\beta c x)^{-1/\beta})$$

is non-increasing, continuous, bounded, and satisfies the hypothesis from Theorem 3. Hence Proposition 8 is a direct application of Theorem 3 with $(\xi_t, t \geq 0) = (\beta K_t, t \geq 0)$. (recall that $\beta < 0$). □

4.3 Speed of absorption of SCBLRE

Throughout this section, we assume that $\beta \in (0, 1]$. One of the aims of this section is to compute the asymptotic behaviour of the survival probability and we will see that it depends on the value of $\mathbf{m}$. We find five different regimes as in the discrete case (time and space) [see e.g. [3, 47, 50]]; in the Feller case (see for instance Theorem 1.1 in [20]); and CB processes with catastrophes (see for instance Proposition 5 in [9]). Recall that in the classical theory of branching processes, the survival probability stays positive, converges to zero polynomially fast or converges to zero exponentially fast, depending of whether the process is *supercritical* ($\mathbf{m} > 0$), *critical* ($\mathbf{m} = 0$) or *subcritical* ($\mathbf{m} < 0$), respectively. When a random environment is acting in the process, there is another phase transition in the subcritical regime. This phase transition depends on the second parameter $\mathbf{m}_1 := \psi'_K(1)$. Since ψ_K is a convex function, we recall that $\mathbf{m} \leq \mathbf{m}_1$. We say that the SCBLRE is *strongly subcritical* if $\mathbf{m}_1 < 0$, *intermediately subcritical* if $\mathbf{m}_1 = 0$ and *weakly subcritical* if $\mathbf{m}_1 > 0$.

We start this section with the case where the environment is driven by a Brownian motion. Here,

$$\mathbf{m} = \alpha - \frac{\sigma^2}{2} \qquad \text{and} \qquad \mathbf{m}_1 = \alpha + \frac{\sigma^2}{2}.$$

Recall that

$$\eta := -\frac{2}{\beta\sigma^2}\mathbf{m} \qquad \text{and} \qquad \mathbf{k} = \left(\frac{\beta\sigma^2}{2c}\right)^{1/\beta},$$

and define

$$f(x) := 1 - \exp\left\{-\mathbf{k}x^{1/\beta}\right\}, \qquad \text{for} \qquad x \geq 0.$$

From identity (4.4) and the scaling property, we deduce for $\beta > 0$,

$$\mathbb{P}_z(Z_t > 0) = \mathbb{E}\left[f\left(\frac{z^\beta}{2I^{(\eta)}_{\beta^2\sigma^2 t/4}}\right)\right] = \int_0^\infty f(z^\beta v) p_{\beta^2\sigma^2 t/4,\eta}(v)\mathrm{d}v, \tag{4.12}$$

where $p_{\nu,\eta}$ denotes the density function of $1/2I_\nu^{(\eta)}$ and is given in (4.7). As in the proof of Proposition 7, and following the same notation as in [20], we just provide the fundamental ideas of the proof.

Proposition 9. *Let $(Z_t, t \geq 0)$ be the SCBBRE with index $\beta \in (0, 1]$, Brownian environment and $Z_0 = z > 0$.*

i) Supercritical. If $\mathbf{m} > 0$, then

$$\lim_{t\to\infty} \mathbb{P}_z(Z_t > 0) = 1 - \sum_{n=0}^{\infty} \frac{(-z\mathbf{k})^n}{n!} \frac{\Gamma(\frac{n}{\beta} - \eta)}{\Gamma(-\eta)}. \tag{4.13}$$

ii) Critical. If $\mathbf{m} = 0$, then

$$\lim_{t\to\infty} \sqrt{t}\, \mathbb{P}_z(Z_t > 0) = -\frac{\sqrt{2}}{\sqrt{\pi}\beta\sigma} \sum_{n=1}^{\infty} \frac{(-z\mathbf{k})^n}{n!} \Gamma\left(\frac{n}{\beta}\right). \tag{4.14}$$

iii) Weakly subcritical. If $\mathbf{m} \in (-\sigma^2, 0)$, then

$$\lim_{t\to\infty} t^{\frac{3}{2}} e^{\frac{\mathbf{m}^2 t}{2\sigma^2}} \mathbb{P}_z(Z_t > 0) = \frac{8}{\beta^3\sigma^3} \int_0^\infty f(z^\beta v)\phi_\eta(v)\mathrm{d}v, \tag{4.15}$$

where

$$\phi_\eta(v) = \int_0^\infty \int_0^\infty \frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{\eta + 2}{2}\right) e^{-v} v^{-\eta/2} u^{(\eta-1)/2} e^{-u} \frac{\sinh(\xi)\cosh(\xi)\xi}{(u + v\cosh(\xi)^2)^{\frac{\eta+2}{2}}} \mathrm{d}\xi \mathrm{d}u.$$

iv) Intermediately subcritical. If $\mathbf{m} = -\sigma^2$, then

$$\lim_{t\to\infty} \sqrt{t} e^{\sigma^2 t/2} \mathbb{P}_z(Z_t > 0) = z \frac{\sqrt{2}}{\sqrt{\pi}\beta\sigma} \mathbf{k}\Gamma\left(\frac{1}{\beta}\right). \tag{4.16}$$

v) Strongly subcritical. If $\mathbf{m} < -\sigma^2$, then

$$\lim_{t\to\infty} e^{-\frac{1}{2}(2\mathbf{m}+\sigma^2)t} \mathbb{P}_z(Z_t > 0) = z\mathbf{k} \frac{\Gamma(\eta - 1/\beta)}{\Gamma(\eta - 2/\beta)}. \tag{4.17}$$

Proof. The supercritical case (i) follows from the identity in law by Dufresne (2.6). More precisely, from (2.6), (4.12) and the Dominated Convergence Theorem, we deduce

$$\lim_{t\to\infty} \mathbb{P}_z(Z_t > 0) = \mathbb{E}\left[1 - \exp\left\{-z\mathbf{k}\Gamma_{-\eta}^{1/\beta}\right\}\right] = 1 - \sum_{n=0}^{\infty} (-z\mathbf{k})^n \frac{\Gamma(\frac{n}{\beta} - \eta)}{n!\Gamma(-\eta)}.$$

In order to prove the critical case (ii), we use Lemma 4.4 in [20]. From identity (4.12) and applying Lemma 4.4 in [20] to

$$f(x) = 1 - \exp\left\{-z\mathbf{k}x^{1/\beta}\right\} \leq z\mathbf{k}x^{1/\beta}, \quad x \geq 0, \tag{4.18}$$

we get

$$\begin{aligned}\lim_{t\to\infty}\sqrt{t}\,\mathbb{P}_z\left(Z_t>0\right) &= \frac{2}{\beta\sigma}\lim_{t\to\infty}\sqrt{\frac{t\sigma^2\beta^2}{4}}\;\mathbb{E}\left[1-\exp\left\{-z\mathbf{k}\left(2I^{(\eta)}_{\beta^2\sigma^2t/4}\right)^{-1/\beta}\right\}\right]\\ &= \frac{\sqrt{2}}{\sqrt{\pi}\beta\sigma}\int_0^\infty\left(1-\exp\left\{-z\mathbf{k}x^{1/\beta}\right\}\right)\frac{e^{-x}}{x}\mathrm{d}x.\end{aligned}$$

By Fubini's theorem, it is easy to show that, for all $q\geq 0$

$$\int_0^\infty\left(1-e^{-qx^{1/\beta}}\right)\frac{e^{-x}}{x}\mathrm{d}x = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\Gamma\left(\frac{n}{\beta}\right)q^n,$$

which implies (4.14).

We now consider the weakly subcritical case (iii). Recall that inequality (4.18) still holds, then using Lemma 4.5 in [20] and identity (4.12), we obtain

$$\begin{aligned}\lim_{t\to\infty}t^{3/2}\;e^{\mathbf{m}^2t/2\sigma^2}\mathbb{P}_z\left(Z_t>0\right) &= \lim_{t\to\infty}t^{3/2}\;e^{\eta^2\beta^2\sigma^2t/8}\mathbb{E}\left[f\left(\frac{z^\beta}{2I^{(\eta)}_{\beta^2\sigma^2t/4}}\right)\right]\\ &= \frac{8}{\beta^3\sigma^3}\int_0^\infty\left(1-\exp\left\{-z\mathbf{k}v^{1/\beta}\right\}\right)\phi_\eta(v)\mathrm{d}v,\end{aligned}$$

where ϕ_η is defined as in the statement of the Theorem.

In the remaining two cases we will use Lemma 4.1 in [20] and Lemma 18 in the Appendix. For the intermediately subcritical case (iv), we observe that $\eta=2/\beta$. Hence, applying Lemma 18 in the Appendix with $p=1/\beta$, we get

$$\mathbb{E}\left[\left(I^{(\eta)}_t\right)^{-1/\beta}\right] = e^{-\frac{2}{\beta^2}t}\mathbb{E}\left[\left(I^{(0)}_t\right)^{-1/\beta}\right] \quad\text{and}\quad \mathbb{E}\left[\left(I^{(\eta)}_t\right)^{-2/\beta}\right] \leq e^{-\frac{2}{\beta^2}t}\mathbb{E}\left[\left(I^{(0)}_{t/2}\right)^{-1/\beta}\right]^2.$$

Now, applying Lemma 4.4 in [20], we deduce

$$\lim_{t\to\infty}\sqrt{t}\;\mathbb{E}\left[\left(2I^{(\eta)}_t\right)^{-1/\beta}\right] = \int_0^\infty\frac{1}{\sqrt{2\pi}}\frac{e^{-a}}{a}a^{1/\beta}da = \frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{1}{\beta}\right),$$

and

$$\lim_{t\to\infty}\sqrt{t}\;\mathbb{E}\left[\left(2I^{(\eta)}_t\right)^{-2/\beta}\right] = 0.$$

Therefore, we can apply Lemma 4.1 in [20] with $c_t=\sqrt{t}\;e^{2t}$ and $Y_t=\left(2I^{(\eta)}_t\right)^{-1/\beta}$, and obtain

$$\begin{aligned}\lim_{t\to\infty}\sqrt{t}e^{\sigma^2t/2}\mathbb{P}_z\left(Z_t>0\right) &= \lim_{t\to\infty}\sqrt{t}e^{\sigma^2t/2}\mathbb{E}\left[1-\exp\left\{-z\mathbf{k}\left(2I^{(\eta)}_{\beta^2\sigma^2t/4}\right)^{-1/\beta}\right\}\right]\\ &= \lim_{t\to\infty}\sqrt{t}e^{\sigma^2t/2}z\mathbf{k}\mathbb{E}\left[\left(2I^{(\eta)}_{\beta^2\sigma^2t/4}\right)^{-1/\beta}\right]\\ &= z\frac{\sqrt{2}}{\sqrt{\pi}\beta\sigma}\mathbf{k}\Gamma\left(\frac{1}{\beta}\right).\end{aligned}$$

Finally for the strongly subcritical case, we use again Lemma 18 in the Appendix with $p = 1/\beta$. First observe that $\eta - 2/\beta > 0$. Thus, the Monotone Convergence Theorem and the identity of Dufresne (2.6) yield

$$\lim_{t\to\infty} \mathbb{E}\left[\left(2I_{t/2}^{(-(\eta-2/\beta))}\right)^{-1/\beta}\right] = \mathbb{E}\left[\left(2I_{\infty}^{(-(\eta-2/\beta))}\right)^{-1/\beta}\right] = \mathbb{E}\left[(\Gamma_{\eta-2/\beta})^{1/\beta}\right] = \frac{\Gamma(\eta-1/\beta)}{\Gamma(\eta-2/\beta)}.$$

Since $I_t^{(\eta-2/\beta)} \uparrow \infty$ as t increases, from the Monotone Convergence Theorem, we get

$$\lim_{t\to\infty} \mathbb{E}\left[\left(2I_{t/2}^{(\eta-2/\beta)}\right)^{-1/\beta}\right] = 0.$$

Hence by applying Lemma 4.1 in [20] with $c_t = e^{-(2/\beta^2-2\eta/\beta)t}$ and $Y_t = \left(2I_t^{(\eta)}\right)^{-1/\beta}$ we obtain that

$$\begin{aligned}\lim_{t\to\infty} e^{-\frac{1}{2}(2\mathbf{m}+\sigma^2)t}\mathbb{P}_z\left(Z_t > 0\right) &= \lim_{t\to\infty} c_{\beta^2\sigma^2 t/4}\mathbb{E}\left[1 - \exp\left\{-\mathbf{k}\left(2I^{(\eta)}_{\beta^2\sigma^2 t/4}\right)^{-1/\beta}\right\}\right] \\ &= z\mathbf{k}\frac{\Gamma(\eta-1/\beta)}{\Gamma(\eta-2/\beta)}.\end{aligned}$$

This completes the proof. □

We finish this section with the asymptotic behaviour in the general case. We recall that as in the previous section, some of the limiting constants depends on the initial population but we don't provide the explicitly relation. (See [72] for the "explicit" relation). Here, $\tau \in (0, \theta_K^+)$ is the position where $\psi_K'(\tau) = 0$.

Proposition 10. *Let $(Z_t, t \geq 0)$ be a SCBLRE with index $\beta \in (0,1]$ defined by the SDE (4.1) with $Z_0 = z > 0$, and recall the definition of the random environment K in (4.2). Assume that $1 < \theta_K^+$.*

i) Supercritical. If $\mathbf{m} > 0$, then for every $z > 0$

$$\lim_{t\to\infty} \mathbb{P}_z\Big(Z_t > 0\Big) = \mathbb{E}\left[1 - \exp\left\{-z\left(\beta c\int_0^\infty e^{-\beta K_u}\mathrm{d}u\right)^{-1/\beta}\right\}\right] > 0.$$

ii) Critical. If $\mathbf{m} = 0$, then for every $z > 0$, there exists $c_3(z) > 0$ such that

$$\lim_{t\to\infty} \sqrt{t}\mathbb{P}_z(Z_t > 0) = c_3(z).$$

iii) Subcritical. Assume that $\mathbf{m} < 0$, then

a) Weakly subcritical. If $\mathbf{m}_1 < 0$, then there exists $c_1 > 0$ such that for every $z > 0$,

$$\lim_{t\to\infty} e^{-t\psi_K(1)}\mathbb{P}_z(Z_t > 0) = c_1 z,$$

b) Intermediate subcritical. If $\mathbf{m}_1 = 0$, then there exists $c_2 > 0$ such that for every $z > 0$,

$$\lim_{t\to\infty} \sqrt{t}e^{-t\psi_K(1)}\mathbb{P}_z(Z_t > 0) = c_2 z,$$

c) Strongly subcritical. If $\mathbf{m}_1 > 0$*, then for every* $z > 0$*, there exists* $c_4(z) > 0$ *such that*

$$\lim_{t\to\infty} t^{3/2} e^{-t\psi_K(\tau)} \mathbb{P}_z(Z_t > 0) = c_4(z).$$

Proof. This is a direct application of Theorem 3, with $(\xi_t, t \geq 0) = (\beta K_t, t \geq 0)$ and $F(x) = 1 - \exp(-z(\beta c x)^{-1/\beta})$. □

In the intermediate and strongly subcritical cases *b*) and *c*), $\mathbb{E}[Z_t]$ provides the exponential decay factor of the survival probability which is given by $\psi_K(1)$, and the probability of survival is proportional to the initial state z of the population. In the weakly subcritical case *a*), the survival probability decays exponentially with rate $\psi_K(\tau)$, which is strictly smaller than $\psi_K(1)$, and c_2 may not be proportional to z (it is also the case for c_1). We refer to [7] for a result in this vein for discrete branching processes in random environment.

4.4 Conditioned processes

Here, we are interested in studying two conditioned versions of the processes: the process conditioned to be never absorbed (or Q-process) and the process conditioned on eventual absorption. Our methodology follows similar arguments as those used in Lambert [69] and extend the results obtained by Böinghoff and Hutzenthaler [20] and Hutzenthaler [53]. In order to apply it, we need to know the explicit relation between the limiting constants and the initial value. This means that we can obtain both processes when the environment is driven by a Brownian motion with drift. But, when the environment is a Lévy process, we just know the conditional processes in the supercritical, intermediate subcritical and strongly subcritical cases.

4.4.1 The process conditioned to be never absorbed

In order to study the SCBLRE conditioned to be never absorbed, we need the following lemma.

Lemma 3. *For every* $t \geq 0$*,* Z_t *is integrable.*

Proof. Differentiating the Laplace transform (1.19) of Z_t in λ and taking limits as $\lambda \downarrow 0$, on both sides, we deduce

$$\mathbb{E}_z\left[Z_t \middle| K\right] = z e^{K_t},$$

which is an integrable random variable since by hypothesis $1 < \theta_K^+$. □

Now, we will focus in the Brownian environment case. Recall that

$$\eta := -\frac{2}{\beta\sigma^2}\mathbf{m} \qquad \text{and} \qquad \mathbf{k} = \left(\frac{\beta\sigma^2}{2c}\right)^{1/\beta}.$$

We now define the function $U : [0,\infty) \to (0,\infty)$ as follows

$$U(z) = \begin{cases} -\frac{\sqrt{2}}{\sqrt{\pi}\beta\sigma}\displaystyle\sum_{n=1}^{\infty}\frac{(-z\mathbf{k})^n}{n!}\Gamma\left(\frac{n}{\beta}\right) & \text{if } \mathbf{m}=0,\\ \frac{8}{\beta^3\sigma^3}\displaystyle\int_0^{\infty}\left(1-e^{-z\mathbf{k}v^{1/\beta}}\right)\phi_\eta(v)\mathrm{d}v & \text{if } \mathbf{m}\in(-\sigma^2,0),\\ z\dfrac{\sqrt{2}}{\sqrt{\pi}\beta\sigma}\mathbf{k}\Gamma\left(\dfrac{1}{\beta}\right) & \text{if } \mathbf{m}=-\sigma^2,\\ z\mathbf{k}\dfrac{\Gamma(\eta-1/\beta)}{\Gamma(\eta-2/\beta)} & \text{if } \mathbf{m}<-\sigma^2, \end{cases}$$

where the function ϕ_η is given as in Proposition 9. We also introduce

$$\theta := \theta(\mathbf{m},\sigma) = \begin{cases} 0 & \text{if } \mathbf{m}=0,\\ \dfrac{\mathbf{m}^2}{2\sigma^2} & \text{if } \mathbf{m}\in(-\sigma^2,0),\\ -\dfrac{2\mathbf{m}+\sigma^2}{2} & \text{if } \mathbf{m}\le-\sigma^2. \end{cases}$$

Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration generated by Z and $T_0 = \inf\{t \geq 0 : Z_t = 0\}$ be the absorption time of the process Z. The next proposition states, in the critical and subcritical cases, the existence of the Q-process.

Proposition 11. *Let $(Z_t, t \geq 0)$ be the SCBBRE (Brownian environment) with index $\beta \in (0,1]$ and $Z_0 = z > 0$. Then for $\mathbf{m} \leq 0$:*

i) The conditional laws $\mathbb{P}_z(\cdot \mid T_0 > t+s)$ converge as $s \to \infty$ to a limit denoted by $\mathbb{P}_z^\natural$, in the sense that for any $t \geq 0$ and $\Lambda \in \mathcal{F}_t$,

$$\lim_{s\to\infty}\mathbb{P}_z(\Lambda \mid T_0 > t+s) = \mathbb{P}_z^\natural(\Lambda).$$

ii) The probability measure $\mathbb{P}^\natural$ can be expressed as an h-transform of $\mathbb{P}$ based on the martingale

$$D_t = e^{\theta t}U(Z_t),$$

in the sense that

$$\mathrm{d}\mathbb{P}_z^\natural\big|_{\mathcal{F}_t} = \frac{D_t}{U(z)}\mathrm{d}\mathbb{P}_z\big|_{\mathcal{F}_t}.$$

Proof. We first prove part (i). Let $z, s, t > 0$, and $\Lambda \in \mathcal{F}_t$. From the Markov property, we observe

$$\mathbb{P}_z(\Lambda \mid T_0 > t+s) = \frac{\mathbb{P}_z\Big(\Lambda; T_0 > t+s\Big)}{\mathbb{P}_z(T_0 > t+s)} = \mathbb{E}_z\left[\frac{\mathbb{P}_{Z_t}(Z_s > 0)}{\mathbb{P}_z(Z_{t+s} > 0)}; \Lambda, T_0 > t\right]. \tag{4.19}$$

Since the mapping $t \mapsto I_t^{(\eta)}$ is increasing and the function $f(x) = 1 - \exp\left\{-\mathbf{k}x^{1/\beta}\right\}$ is decreasing, we deduce from (4.12) and the Markov property that for any $z, y > 0$,

$$0 \leq \frac{\mathbb{P}_y\left(Z_s > 0\right)}{\mathbb{P}_z\left(Z_{t+s} > 0\right)} = \frac{\mathbb{E}\left[1 - \exp\left\{-y\mathbf{k}\left(2I^{(\eta)}_{\sigma^2\beta^2 s/4}\right)^{-1/\beta}\right\}\right]}{\mathbb{E}\left[1 - \exp\left\{-z\mathbf{k}\left(2I^{(\eta)}_{\sigma^2\beta^2 (t+s)/4}\right)^{-1/\beta}\right\}\right]}$$
$$\leq \frac{y\mathbf{k}\mathbb{E}\left[\left(2I^{(\eta)}_{\sigma^2\beta^2 s/4}\right)^{-1/\beta}\right]}{\mathbb{E}\left[1 - \exp\left\{-z\mathbf{k}\left(2I^{(\eta)}_{\sigma^2\beta^2 s/4}\right)^{-1/\beta}\right\}\right]}.$$

Moreover, since $I_t^{(\eta)}$ diverge as $t \uparrow \infty$, we have

$$\frac{1}{2}z\mathbf{k}\mathbb{E}\left[\left(2I^{(\eta)}_{\sigma^2\beta^2 s/4}\right)^{-1/\beta}\right] \leq \mathbb{E}\left[1 - \exp\left\{-z\mathbf{k}\left(2I^{(\eta)}_{\sigma^2\beta^2 s/4}\right)^{-1/\beta}\right\}\right] \leq z\mathbf{k}\mathbb{E}\left[\left(2I^{(\eta)}_{\sigma^2\beta^2 s/4}\right)^{-1/\beta}\right],$$

for s sufficiently large. Then for any s greater than some bound chosen independently of Z_t, we necessarily have

$$0 \leq \frac{\mathbb{P}_{Z_t}\left(Z_s > 0\right)}{\mathbb{P}_z\left(Z_{t+s} > 0\right)} \leq \frac{2}{z}Z_t.$$

Now, from the asymptotic behaviour (4.14), (4.15), (4.16) and (4.17), we get

$$\lim_{s\to\infty} \frac{\mathbb{P}_{Z_t}\left(Z_s > 0\right)}{\mathbb{P}_z\left(Z_{t+s} > 0\right)} = \frac{e^{\theta t}U(Z_t)}{U(z)}.$$

Hence, Dominated Convergence and identity (4.19) imply

$$\lim_{s\to\infty} \mathbb{P}_z\left(\Lambda \mid T_0 > t + s\right) = \mathbb{E}_z\left[\frac{e^{\theta t}U(Z_t)}{U(z)}, \Lambda\right]. \tag{4.20}$$

Next, we prove part (ii). In order to do so, we use (4.20) with $\Lambda = \Omega$ to deduce

$$\mathbb{E}_z\left[e^{\theta t}U(Z_t)\right] = U(z).$$

Therefore from the Markov property, we obtain

$$\mathbb{E}_z\left[e^{\theta(t+s)}U(Z_{t+s})\middle|\mathcal{F}_s\right] = e^{\theta s}\mathbb{E}_{Z_s}\left[e^{\theta t}U(Z_t)\right] = e^{\theta s}U(Z_s),$$

implying that D is a martingale. □

Example 10 (SCBBRE Q-process)**.** Here, we assume $\mathbf{m} \leq -\sigma^2$. Recall that

$$\eta := -\frac{2}{\beta\sigma^2}\mathbf{m} \qquad \text{and} \qquad \mathbf{k} = \left(\frac{\beta\sigma^2}{2c}\right)^{1/\beta}.$$

Let $\mathcal{L}$ be the infinitesimal generator of the SCBBRE process. From Proposition 11, we deduce that the form of the infinitesimal generator of the SCBBRE Q-process, here denoted by $\mathcal{L}^\natural$, satisfies for $f \in \mathrm{Dom}(\mathcal{L})$

$$\mathcal{L}^\natural f(x) = \mathcal{L}f(x) + (\mathbf{1}_{\{\beta=1\}}cx + x^2\sigma^2)f'(x)\frac{U'(x)}{U(x)} + \mathbf{1}_{\{\beta\neq1\}}\frac{c\beta(\beta+1)}{\Gamma(1-\beta)}\frac{x}{U(x)}\int_0^\infty \Big(f(x+y)-f(x)\Big)\Big(U(x+y)-U(x)\Big)\frac{\mathrm{d}y}{y^{2+\beta}},$$

with

$$U(z) = \begin{cases} z\dfrac{\sqrt{2}}{\sqrt{\pi}\beta\sigma}\mathbf{k}\Gamma\left(\dfrac{1}{\beta}\right) & \text{if } \mathbf{m} = -\sigma^2, \\ z\mathbf{k}\dfrac{\Gamma(\eta-1/\beta)}{\Gamma(\eta-2/\beta)} & \text{if } \mathbf{m} < -\sigma^2. \end{cases}$$

Replacing the form of U in the infinitesimal generator $\mathcal{L}^\natural$ in both cases, we get

$$\mathcal{L}^\natural f(x) = \frac{1}{2}\sigma^2x^2f''(x) + (a+\alpha+\sigma^2)xf'(x) + \mathbf{1}_{\{\beta=1\}}c\left(2f'(x)+xf''(x)\right) + \mathbf{1}_{\{\beta\neq1\}}\frac{c\beta(\beta+1)}{\Gamma(1-\beta)}\left(\int_0^\infty (f(x+y)-f(x))\frac{\mathrm{d}y}{y^{1+\beta}} + x\int_0^\infty (f(x+y)-f(x)-yf'(x))\frac{\mathrm{d}y}{y^{2+\beta}}\right)$$

From the form of the infinitesimal generator of the stable CBIBRE (4), we deduce that the stable CBBRE Q-process is a stable CBIBRE with branching and immigration mechanisms given by

$$\psi^\natural(\lambda) = c\lambda^{1+\beta} \qquad \text{and} \qquad \phi^\natural(\lambda) = c(\beta+1)\lambda^\beta,$$

and the random environment $S_t^\natural = (\alpha+\sigma^2)t + \sigma B_t^{(e)}$.

Now, we will focus in a general Lévy environment. Recall that in the intermediate and strongly subcritical regimes, the limiting coefficients are written in terms of the initial position. In this case, by following the previous steps, we can obtain the CBLRE conditioned to be never extinct. Let introduce $\theta = -\psi_K(1)$. According to Proposition 10, there exist two constants $c_3, c_4 > 0$ such that c_3z and c_4z are the limiting coefficients in the intermediate and strongly subcritical regimes. We define the function $U : [0,\infty) \to (0,\infty)$ as follows

$$U(z) = \begin{cases} c_3z & \text{if } \mathbf{m} < 0 \text{ and } \mathbf{m}_1 = 0, \\ c_4z & \text{if } \mathbf{m} < 0 \text{ and } \mathbf{m}_1 < 0. \end{cases}$$

Proposition 12. *Let $(Z_t, t \geq 0)$ be the SCBLRE with index $\beta \in (0,1]$ and $Z_0 = z > 0$. Then for $\mathbf{m} < 0$ and $\mathbf{m}_1 \leq 0$:*

i) The conditional laws $\mathbb{P}_z(\cdot \mid T_0 > t+s)$ converge as $s \to \infty$ to a limit denoted by $\mathbb{P}_z^\natural$, in the sense that for any $t \geq 0$ and $\Lambda \in \mathcal{F}_t$,

$$\lim_{s\to\infty}\mathbb{P}_z(\Lambda \mid T_0 > t+s) = \mathbb{P}_z^\natural(\Lambda).$$

ii) The probability measure $\mathbb{P}^\natural$ can be expressed as an h-transform of $\mathbb{P}$ based on the martingale

$$D_t = e^{\theta t} U(Z_t),$$

in the sense that

$$\mathrm{d}\mathbb{P}^\natural_z\big|_{\mathcal{F}_t} = \frac{D_t}{U(z)} \mathrm{d}\mathbb{P}_z\big|_{\mathcal{F}_t}.$$

iii) Its infinitesimal generator satisfies, for every $f \in Dom(\mathcal{A})$.

$$\begin{aligned}\mathcal{L}^\natural f(x) = & \frac{\sigma^2}{2} x^2 f''(x) + (\alpha + \sigma^2) x f'(x) + \mathbf{1}_{\{\beta=1\}} c \left(2f'(x) + x f''(x)\right) \\ & + \mathbf{1}_{\{\beta\neq1\}} \frac{c\beta(\beta+1)}{\Gamma(1-\beta)} \left(\int_0^\infty (f(x+y) - f(x)) \frac{\mathrm{d}y}{y^{1+\beta}} + x \int_0^\infty \big(f(x+y) - f(x) - y f'(x)\big) \frac{\mathrm{d}y}{y^{2+\beta}} \right) \\ & + \int_{\mathbb{R}} \left(f(xe^z) - f(x) - x(e^z - 1) f'(x) \mathbf{1}_{\{|z|<1\}} \right) e^z \pi(\mathrm{d}z) + x f'(x) \int_{(-1,1)} (e^z - 1)^2 \pi(\mathrm{d}z).\end{aligned}$$

iv) Moreover, the stable CBLRE Q-process is a stable CBILRE with branching and immigration mechanisms given by

$$\psi^\natural(\lambda) = c\lambda^{1+\beta} \qquad \text{and} \qquad \phi^\natural(\lambda) = c(\beta+1)\lambda^\beta,$$

and random environment

$$\begin{aligned} S^\natural_t = & \left(\alpha + \sigma^2 + \int_{(-1,1)} (e^z - 1)^2 \pi(\mathrm{d}z) \right) t + \sigma B^{(e)}_t \\ & + \int_0^t \int_{(-1,1)} (e^z - 1) \widetilde{N}^{(e),\natural}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{\mathbb{R}\setminus(-1,1)} (e^z - 1) N^{(e),\natural}(\mathrm{d}s, \mathrm{d}z), \end{aligned}$$

where $N^{(e),\natural}(\mathrm{d}s, \mathrm{d}z)$ is a Poisson random measure in $\mathbb{R}_+ \times \mathbb{R}$ with intensity $e^z \mathrm{d}s \pi(\mathrm{d}z)$. (By hypothesis $1 < \theta^+_K$, then $\int_{\mathbb{R}} (1 \wedge z^2) e^z \pi(\mathrm{d}z) < \infty$ and therefore, the random environment is well defined.)

Proof. The proof of i) and ii) is analogous to the proof in Proposition 11, so we omitted. By ii)

$$\mathcal{L}^\natural f(x) = \frac{\mathcal{L}(fU)(x)}{U(x)} + \theta f(x). \tag{4.21}$$

Since D_t is a martingale,

$$\mathcal{L}U(x) + \theta U(x) = 0.$$

By replacing the value of U, the form of $\mathcal{L}$ and the previous observation in (4.21), we get iii). iv) follows from iii). □

4.4.2 The process conditioned on eventual absorption

In the supercritical case we are interested in the process conditioned on eventual absorption. Assume that $\mathbf{m} > 0$ and define for $z > 0$

$$U_*(z) := \mathbb{E}\left[\exp\left\{-z\left(\beta c\int_0^\infty e^{-\beta K_u}\mathrm{d}u\right)^{-1/\beta}\right\}\right].$$

Observe that if the random environment is a Brownian motion, $U_*(z)$ has the form

$$U_*(z) = \sum_{n=0}^{\infty}\frac{(-z\mathbf{k})^n}{n!}\frac{\Gamma(n/\beta-\eta)}{\Gamma(-\eta)}.$$

Proposition 13. *Let $(Z_t, t\geq 0)$ be the SCBLRE with index $\beta\in(0,1]$ and $Z_0 = z > 0$. Then for $\mathbf{m} > 0$, the conditional law*

$$\mathbb{P}_z^*(\cdot) = \mathbb{P}_z\left(\cdot \mid T_0 < \infty\right),$$

satisfies for any $t\geq 0$,

$$\mathrm{d}\mathbb{P}_z^*\big|_{\mathcal{F}_t} = \frac{U_*(Z_t)}{U_*(z)}\mathrm{d}\mathbb{P}_z\big|_{\mathcal{F}_t}.$$

Moreover, $(U_*(Z_t), t\geq 0)$ is a martingale.

Proof. Let z, $t\geq 0$ and $\Lambda\in\mathcal{F}_t$, then

$$\mathbb{P}_z\left(\Lambda\Big|T_0<\infty\right) = \frac{\mathbb{P}_z\left(\Lambda, T_0<\infty\right)}{\mathbb{P}_z\left(T_0<\infty\right)} = \lim_{s\to\infty}\frac{\mathbb{P}_z\left(\Lambda, Z_{t+s}=0\right)}{U_*(z)}.$$

On the other hand, the Markov property implies

$$\mathbb{P}_z\left(\Lambda, Z_{t+s}=0\right) = \mathbb{E}_z\left[\mathbb{P}_z\Big(Z_{t+s}=0\Big|\mathcal{F}_t\Big),\Lambda\right] = \mathbb{E}_z\left[\mathbb{P}_{Z_t}\Big(Z_s=0\Big),\Lambda\right].$$

Therefore using the Dominated Convergence Theorem, we deduce

$$\mathbb{P}_z\left(\Lambda\Big|T_0<\infty\right) = \lim_{s\to\infty}\frac{\mathbb{E}_z\left[\mathbb{P}_{Z_t}\Big(Z_s=0\Big),\Lambda\right]}{U_*(z)} = \frac{\mathbb{E}_z\Big[U_*(Z_t),\Lambda\Big]}{U_*(z)}.$$

The proof that $(U_*(Z_t), t\geq 0)$ is a martingale follows from the same argument as in the proof of part (ii) of Proposition 11. □

Observe that $\mathbb{P}_z^*(Z_t>0)$ goes to 0 as $t\to\infty$. Hence a natural problem to study is the rates of convergence of the survival probability of the SCBLRE conditioned on eventual extinction. We were able to study them only in the Brownian environment case, so in the last part of the chapter we will assume that K has continuous paths. Here, we obtain a phase transition which is similar to the subcritical regime.

It is important to note that the arguments that we will use below also provides the rate of convergence of the inverse of exponential functionals of a Brownian motion with drift towards its limit, the Gamma random variable. The latter comes from the following observation. Since $U_*(Z_t)$ is a martingale, we deduce

$$\begin{aligned}\mathbb{P}_z^*(Z_t>0) =& \mathbb{P}_z\left(Z_t>0 \mid T_0<\infty\right) = \frac{1}{U_*(z)}\left(U_*(z) - \mathbb{P}_z\left(Z_t=0\right)\right)\\ =& \frac{1}{U_*(z)}\left(\mathbb{E}\left[\exp\left\{-z\mathbf{k}\Gamma_{-\eta}^{1/\beta}\right\}\right] - \mathbb{E}\left[\exp\left\{-z\mathbf{k}\left(2I^{(\eta)}_{\beta^2\sigma^2t/4}\right)^{-1/\beta}\right\}\right]\right).\end{aligned}$$

Another important identity that we will use in our arguments is the following identity in law

$$\left\{ \frac{1}{I_t^{(\eta)}}, t > 0 \right\} \stackrel{\mathcal{L}}{=} \left\{ \frac{1}{I_t^{(-\eta)}} + 2\Gamma_{-\eta}, t > 0 \right\},$$

where $\Gamma_{-\eta}$ and $I_t^{(-\eta)}$ are independent (see for instance identity (1.1) in Matsumoto and Yor [75]). We also introduce

$$h(x, y) = \exp\left\{ -\mathbf{k} x^{1/\beta} \right\} - \exp\left\{ -\mathbf{k} \left(x + y \right)^{1/\beta} \right\}, \qquad x, y \geq 0.$$

Then

$$\mathbb{P}_z^*(Z_t > 0) = \frac{1}{U_*(z)} \mathbb{E}\left[h\left(z^\beta \Gamma_{-\eta}, \frac{z^\beta}{2 I^{(-\eta)}_{\beta^2\sigma^2 t/4}} \right) \right], \tag{4.22}$$

where $\Gamma_{-\eta}$ and $I_t^{(-\eta)}$ are independent.

Proposition 14. *Let $(Z_t, t \geq 0)$ be a supercritical stable CBBRE (Brownian environment) with index $\beta \in (0, 1)$ and $Z_0 = z > 0$.*

i) Weakly supercritical. If $\mathbf{m} \in (0, \beta\sigma^2)$, then

$$\lim_{t\to\infty} t^{\frac{3}{2}} e^{\frac{\mathbf{m}^2 t}{2\sigma^2}} \mathbb{P}_z^*(Z_t > 0) = \frac{8}{\beta^3\sigma^3\Gamma(|\eta|)U_*(z)} \int_0^\infty \int_0^\infty h(z^\beta x, z^\beta y)\phi_{|\eta|}(y) x^{-\eta-1} e^{-x} \mathrm{d}x\mathrm{d}y,$$

where

$$\phi_\eta(y) = \int_0^\infty \int_0^\infty \frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{\eta+2}{2}\right) e^{-y} y^{-\eta/2} u^{(\eta-1)/2} e^{-u} \frac{\sinh(\xi)\cosh(\xi)\xi}{(u + y\cosh(\xi)^2)^{\frac{\eta+2}{2}}} \mathrm{d}\xi \mathrm{d}u.$$

ii) Intermediately supercritical. If $\mathbf{m} = \beta\sigma^2$, then

$$\lim_{t\to\infty} \sqrt{t} e^{\beta^2\sigma^2 t/2} \mathbb{P}_z^*(Z_t > 0) = \frac{z\mathbf{k}\sqrt{2}}{\beta^2\sigma\sqrt{\pi}U_*(z)} \sum_{n=0}^\infty \frac{(-z\mathbf{k})^n}{n!} \Gamma\left(\frac{n+1}{\beta} + 1\right)$$

iii) Strongly supercritical. If $\mathbf{m} > \beta\sigma^2$, then

$$\lim_{t\to\infty} e^{\frac{\beta}{2}(2\mathbf{m}-\beta\sigma^2)t} \mathbb{P}_z^*(Z_t > 0) = \frac{-z\mathbf{k}(\eta+2)}{\beta U_*(z)\Gamma(-\eta)} \sum_{n=0}^\infty \frac{(-z\mathbf{k})^n}{n!} \Gamma\left(\frac{n+1}{\beta} - \eta - 1\right)$$

Proof. Similarly as in the proof of Propositions 7 and 9, and following the same notation as in [20], we just provide the fundamental ideas of the proof.

We first consider the weakly supercritical case (i). Note that for each $x, y > 0$

$$h(x, y) \leq \frac{\mathbf{k}}{\beta} x^{1/\beta - 1}(y \vee y^{1/\beta}).$$

Since $\Gamma_{-\eta}$ and $I_t^{(-\eta)}$ are independent, we deduce

$$\mathbb{E}\left[h\left(z^\beta\Gamma_{-\eta}, \frac{z^\beta}{2I^{(-\eta)}_{\beta^2\sigma^2t/4}}\right)\right] = \mathbb{E}\left[g\left(\frac{z^\beta}{2I^{(-\eta)}_{\beta^2\sigma^2t/4}}\right)\right],$$

where $g(u) := \mathbb{E}\left[h\left(z^\beta\Gamma_{-\eta}, u\right)\right]$. From the inequality of above, we get $g(u) \leq C(u \vee u^{1/\beta})$ for $C > 0$ that depends on $\mathbf{k}, \beta$ and η.

Following step by step the proof of Lemma 4.5 in [20], we can deduce that the statement also holds for our function g with $b = 1$. Actually in the proof of Lemma 4.5 in [20], the authors use the inequality on their statement in order to apply the Dominated Convergence Theorem and they split the integral in (4.24) in [20] into two integrals, one over $[0, 1]$ and another over $(1, \infty)$. In our case, we can take on the integral over $[0, 1]$ the function Cu and on the integral over $(1, \infty)$ the function $Cu^{1/\beta}$ and the result will not change. Therefore

$$\begin{aligned}\lim_{t\to\infty} t^{3/2}\ e^{\mathbf{m}^2t/2\sigma^2}\mathbb{P}^*_z(Z_t > 0) &= \lim_{t\to\infty}\frac{t^{3/2}\ e^{\eta^2\beta^2\sigma^2t/8}}{U_*(z)}\mathbb{E}\left[g\left(\frac{z^\beta}{2I^{(-\eta)}_{\beta^2\sigma^2t/4}}\right)\right]\\ &= \frac{8}{\beta^3\sigma^3\Gamma(|\eta|)U_*(z)}\int_0^\infty\int_0^\infty h(z^\beta x, z^\beta y)\phi_{|\eta|}(y)x^{-\eta-1}e^{-x}\mathrm{d}x\mathrm{d}y,\end{aligned}$$

where $\phi_{|\eta|}$ is defined as in the statement of the Proposition.

In the remaining two cases we use the following inequalities, which hold by the Mean Value Theorem. Let $\epsilon > 0$ then, for each $x, y \geq 0$

$$\frac{\mathbf{k}}{\beta}e^{-\mathbf{k}x^{1/\beta}}x^{1/\beta-1}y \leq h(x,y) \leq \frac{\mathbf{k}}{\beta}e^{-\mathbf{k}x^{1/\beta}}\left((x+\epsilon)^{1/\beta-1}y + \left(\frac{x}{\epsilon}+1\right)^{1/\beta-1}y^{1/\beta}\right). \tag{4.23}$$

For the intermediately supercritical case (ii), we note that $-\eta = 2$. From Lemma 18 (with $p = 1$) and Lemma 4.4 in [20] we deduce

$$\lim_{t\to\infty}\sqrt{t}e^{2t}\mathbb{E}\left[\frac{1}{2I_t^{(2)}}\right] = \frac{1}{\sqrt{2\pi}}.$$

On the other hand, from Lemma 4.5 in [20] with $g(u) = u^{1/\beta}$, we have

$$\lim_{t\to\infty} t^{3/2}e^{2t}\mathbb{E}\left[\frac{1}{(2I_t^{(2)})^{1/\beta}}\right] = \int_0^t g(u)\phi_2(u)\mathrm{d}u,$$

where ϕ_2 is defined as in the statement of the Theorem. Therefore by the previous limits, the independence between Γ_2 and $I_t^{(2)}$, identity (4.22) and inequalities (4.23), we have that for $\epsilon > 0$ the following inequalities hold

$$\begin{aligned}\frac{z\mathbf{k}\sqrt{2}}{\beta^2\sigma\sqrt{\pi}U_*(z)}\mathbb{E}\left[e^{-\mathbf{k}z\Gamma_2^{1/\beta}}\Gamma_2^{1/\beta-1}\right] &\leq \lim_{t\to\infty}\sqrt{t}e^{\beta^2\sigma^2t/2}\mathbb{P}^*_z(Z_t > 0)\\ &\leq \frac{z^\beta\mathbf{k}\sqrt{2}}{\beta^2\sigma\sqrt{\pi}U_*(z)}\mathbb{E}\left[e^{-\mathbf{k}z\Gamma_2^{1/\beta}}(z^\beta\Gamma_2+\epsilon)^{1/\beta-1}\right].\end{aligned}$$

Thus our claim holds true by taking limits as ϵ goes to 0.

Finally, we use similar arguments for the strongly supercritical case (iii). Observe from Lemma 18 and the identity in law by Dufresne (2.6) that

$$\lim_{t\to\infty} e^{-2(1+\eta)t}\mathbb{E}\left[\frac{1}{2I_t^{(-\eta)}}\right] = \mathbb{E}\left[\Gamma_{-(\eta+2)}\right], \tag{4.24}$$

where $\Gamma_{-(\eta+2)}$ is a Gamma r.v. with parameter $-(\eta+2)$. If $-\eta < 2/\beta$, Lemma 4.5 in [20] imply

$$\lim_{t\to\infty} t^{3/2}e^{\eta^2 t/2}\mathbb{E}\left[\frac{1}{(2I_t^{(-\eta)})^{1/\beta}}\right] = \int_0^\infty y^{1/\beta}\phi_{|\eta|}(y)\mathrm{d}y, \tag{4.25}$$

where $\phi_{|\eta|}$ is defined as in the statement of the Proposition. If $-\eta = 2/\beta$, from Lemma 18 and Lemma 4.4 in [20], we get

$$\lim_{t\to\infty} \sqrt{t}e^{2t/\beta^2}\mathbb{E}\left[\frac{1}{(2I_t^{(-\eta)})^{1/\beta}}\right] = \frac{\Gamma(1/\beta)}{\sqrt{2\pi}}. \tag{4.26}$$

Next, if $-\eta > 2/\beta$, from Lemma 18 and the identity in law by Dufresne (**??**) we get

$$\lim_{t\to\infty} e^{-2t(1/\beta+\eta)/\beta}\mathbb{E}\left[\frac{1}{(2I_t^{(-\eta)})^{1/\beta}}\right] = \mathbb{E}\left[\Gamma^{1/\beta}_{-(\eta+2/\beta)}\right], \tag{4.27}$$

where $\Gamma_{-(\eta+2/\beta)}$ is a Gamma r.v. with parameter $-(\eta+2/\beta)$. Therefore, from the independence between $\Gamma_{-\eta}$ and $I_t^{(-\eta)}$, inequalities (4.23) and the limits in (4.24), (4.25),(4.26) and (4.27), we deduce that for $\epsilon > 0$, we have

$$\begin{aligned}\frac{z\mathbf{k}}{\beta U_*(z)}\mathbb{E}\left[e^{-\mathbf{k}z\Gamma^{1/\beta}_{-\eta}}\Gamma^{1/\beta-1}_{-\eta}\right]\mathbb{E}\left[\Gamma_{-(\eta+2)}\right] &\le \lim_{t\to\infty} e^{-\frac{\beta}{2}(2\mathbf{m}-\beta\sigma^2)t}\mathbb{P}^*_z(Z_t>0)\\ &\le \frac{z^\beta\mathbf{k}}{\beta U_*(z)}\mathbb{E}\left[e^{-\mathbf{k}z\Gamma^{1/\beta}_{-\eta}}(z^\beta\Gamma_{-\eta}+\epsilon)^{1/\beta-1}\right]\mathbb{E}\left[\Gamma_{-(\eta+2)}\right].\end{aligned}$$

The proof is completed once we take limits as ϵ goes to 0. □

Chapter 5

Multi-type continuous-state branching processes

This chapter is based in paper [66] elaborated in collaboration with Andreas Kyprianou. We define a multi-type continuous-state branching process (MCBP) as a super Markov chain with both local and non-local branching mechanism. This allows us the possibility of working with a countably infinite number of types. The main results and some open questions are presented in Section 5.1. In Section 5.2 we give the construction of MCBPs as a scaling limit of multi-type Bienayme-Galton Watson processes. The spectral radius of the associated linear semigroup will have an important roll in the asymptotic behaviour of our process, in particular, it will determine the phenomenon of local extinction. The properties of this semigroup are studied in Section 5.3. In Sections 5.4 and 5.5 we develop some standard tools based around a spine decomposition. In Section 5.6, we give the proof of the main results. Finally in Section 5.7, we provide examples to illustrate the local phenomenon property.

5.1 Introduction and main results

Continuous state branching processes (CB-processes) can be seen as high density limits of Bienaymé-Galton-Watson (BGW) processes. By analogy with multi-type BGW processes, a natural extension would be to consider a multi-type Markov population model in continuous time which exhibits a branching property. Indeed, in whatever sense they can be defined, multi-type CB-processes (MCBPs) should have the property that the continuum mass of each type reproduces within its own population type in a way that is familiar to a CB-process, but also allows for the migration and/or seeding of mass into other population types.

Recently in [12], the notion of a multi-type continuous-state branching process (with immigration) having d-types was introduced as a solution to an d-dimensional vector-valued SDE with both Gaussian and Poisson driving noises. Simultaneously, in [23], the pathwise construction of these d-dimensional processes was given in terms of a multiparameter time change of Lévy processes (see also [33, 46, 73, 88], all with a finite number of types).

Our first main result is to identify the existence of multi-type continuous-state branching processes, allowing for up to a countable infinity of types. Denote by $\mathbb{N} = \{1, 2, \cdots\}$ the natural numbers. Let $\mathcal{B}(\mathbb{N})$ be the space of bounded measurable functions on $\mathbb{N}$. Thinking of a member of $\mathcal{B}(\mathbb{N})$, say f, as a vector we will write its entries by $f(i)$, $i \in \mathbb{N}$. Write $\mathcal{M}(\mathbb{N})$ the space of finite

Borel measures on $\mathbb{N}$, let $\mathcal{B}^+(\mathbb{N})$ the subset of bounded positive functions. The next theorem shows the existence of our process, a $[0,\infty)^{\mathbb{N}}$-valued strong Markov process that satisfies the branching property with local mechanism given by ψ and non-local mechanism ϕ.

Theorem 6. *Suppose that*

$$\psi(i,z) = b(i)z + c(i)z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu)\ell(i,\mathrm{d}u), \qquad i \in \mathbb{N}, \quad z \geq 0, \tag{5.1}$$

where $b \in \mathcal{B}(\mathbb{N})$, $c \in \mathcal{B}^+(\mathbb{N})$ and, for each $i \in \mathbb{N}$, $(u \wedge u^2)\ell(i,\mathrm{d}u)$ is a bounded kernel from $\mathbb{N}$ to $(0,\infty)$. Suppose further that

$$\phi(i,f) = -\beta(i)\left[d(i)\langle f, \pi_i\rangle + \int_0^\infty (1 - \mathrm{e}^{-u\langle f,\pi_i\rangle})\mathrm{n}(i,\mathrm{d}u)\right], \qquad i \in \mathbb{N}, f \in \mathcal{B}^+(\mathbb{N}), \tag{5.2}$$

where $d, \beta \in \mathcal{B}^+(\mathbb{N})$, π_i is a probability distribution on $\mathbb{N}\backslash\{i\}$ (specifically $\pi_i(i) = 0$, $i \in \mathbb{N}$) and, for $i \in \mathbb{N}$, $u\mathrm{n}(i,\mathrm{d}u)$ is a bounded kernel from $\mathbb{N}$ to $(0,\infty)$ with

$$d(i) + \int_0^\infty u\mathrm{n}(i,\mathrm{d}u) \leq 1.$$

Then there exists an $[0,\infty)^{\mathbb{N}}$-valued strong Markov process $X := (X_t, t \geq 0)$, where $X_t = (X_t(1), X_t(2), \cdots)$, $t \geq 0$, with probabilities $\{\mathbf{P}_\mu, \mu \in \mathcal{M}(\mathbb{N})\}$ such that

$$\mathbf{E}_\mu[\mathrm{e}^{-\langle f, X_t\rangle}] = \exp\left\{-\langle V_t f, \mu\rangle\right\}, \quad \mu \in \mathcal{M}(\mathbb{N}),\ f \in \mathcal{B}^+(\mathbb{N}), \tag{5.3}$$

where, for $i \in \mathbb{N}$,

$$V_t f(i) = f(i) - \int_0^t \left[\psi(i, V_s f(i)) + \phi(i, V_s f)\right]\mathrm{d}s, \qquad t \geq 0. \tag{5.4}$$

In the above theorem, for $f \in \mathcal{B}^+(\mathbb{N})$ and $\mu \in \mathcal{M}(\mathbb{N})$, we us the notation

$$\langle f, \mu\rangle := \sum_{i \geq 1} f(i)\mu(i).$$

Equation (5.3) tells us that X satisfies the branching property: for $\mu_1, \mu_2 \in \mathcal{M}(\mathbb{N})$,

$$\mathbf{E}_{\mu_1+\mu_2}[\mathrm{e}^{-\langle f, X_t\rangle}] = \mathbf{E}_{\mu_1}[\mathrm{e}^{-\langle f, X_t\rangle}]\mathbf{E}_{\mu_2}[\mathrm{e}^{-\langle f, X_t\rangle}], \qquad t \geq 0.$$

That is to say, $(X, \mathbf{P}_{\mu_1+\mu_2})$ is equal in law to the sum of independent copies of $(X, \mathbf{P}_{\mu_1})$ and $(X, \mathbf{P}_{\mu_2})$. We can also understand the process X to be the natural multi-type generalisation of a CB-process as, for each type $i \in \mathbb{N}$, $X(i)$ evolves, in part from a local contribution which is that of a CB-process with mechanism $\psi(i,z)$, but also from a non-local contribution from other types. The mechanism $\phi(i,\cdot)$ dictates how this occurs. Roughly speaking, each type $i \in \mathbb{N}$ seeds an infinitesimally small mass continuously at rate $\beta(i)d(i)\pi_i(j)$ on to sites $j \neq i$ (recall $\pi_i(i) = 0$, $i \in \mathbb{N}$). Moreover, it seeds an amount of mass $u > 0$ at rate $\beta(i)\mathrm{n}(i,\mathrm{d}u)$ to sites $j \neq i$ in proportion given by $\pi_i(j)$. We refer to the processes described in the above theorem as (ψ,ϕ) multi-type continuous-state branching processes, or (ψ,ϕ)-MCBPs for short.

Our main results concern how the different types of extinction occur for a MCBP X as defined above. As alluded to in the introduction, we must distinguish *local extinction* at a finite number of sites $A \subset \mathbb{N}$, that is,

$$\mathcal{L}_A := \{\lim_{t\to\infty} \langle \mathbf{1}_A, X_t\rangle = 0\},$$

from global extinction of the process X, i.e. the event

$$\mathcal{E} := \{\lim_{t\to\infty} \langle 1, X_t\rangle = 0\}.$$

The distinction between these two has been dealt with in the setting of super diffusions by [40]. In this article, we use techniques adapted from that paper to understand local extinction in the setting here. The case of global extinction can be dealt with in a familiar way. To this end, denote by δ_i the atomic measure consisting of a unit mass concentrated at point $i \in \mathbb{N}$.

Lemma 4. *For each $i \in \mathbb{N}$, let w be the vector with entries $w(i) := -\log \mathbf{P}_{\delta_i}(\mathcal{E})$, $i \in \mathbb{N}$. Then w is a non-negative solution to*

$$\psi(i, w(i)) + \phi(i, w) = 0, \qquad i \in \mathbb{N}. \tag{5.5}$$

For the case of local extinction, a more sophisticated notation is needed. First we must introduce the notion of the linear semigroup. For each $f \in \mathcal{B}(f)$, define the linear semigroup $(\mathcal{M}_t, t \geq 0)$ by

$$\mathcal{M}_t f(i) := \mathbf{E}_{\delta_i}[\langle f, X_t\rangle], \qquad t \geq 0.$$

Define the matrix $M(t)$ by

$$M(t)_{ij} := \mathbf{E}_{\delta_i}[X_t(j)], \qquad t \geq 0,$$

and observe that $\mathcal{M}_t[f](i) = [M(t)f](i)$, for $t \geq 0$ and $f \in \mathcal{B}(\mathbb{N})$. The linear semigroup and its spectral properties play a crucial role in determining the limit behavior of the MCBP. In what follows, we need to assume that $M(t)$ is *irreducible* in the sense that, for any $i, j \in \mathbb{N}$, there exists $t > 0$ such that $M(t)_{ij} > 0$. To this end, we make the following global assumption throughout the paper, which ensures irreducibility of $M(t)$, $t \geq 0$.

(A): *The matrix $\pi_i(j)$, $i, j \in \mathbb{N}$, is the transition matrix of an irreducible Markov chain.*

For each $i, j \in \mathbb{N}$, and $\lambda \in \mathbb{R}$ we define the matrix $H(\lambda)$ by

$$H_{ij}(\lambda) := \int_0^\infty e^{\lambda t} M(t)_{ij} dt.$$

The following result is the analogue of a result proved for linear semigroups of MBGW processes; see e.g. Niemi and Nummelin ([78], Proposition 2.1) or Lemma 1 of [77]. We provide a proof in the appendix.

Lemma 5. *If, for some λ, $H_{ij}(\lambda) < \infty$ for a pair i, j, then $H_{ij}(\lambda) < \infty$ for all $i, j \in \mathbb{N}$. In particular, the parameter*

$$\Lambda_{ij} = \sup\{\lambda \geq -\infty : H_{ij}(\lambda) < \infty\},$$

does not depend on i and j. The common value, $\Lambda = \Lambda_{ij}$, is called the spectral radius of M.

In contrast to Lemma 4, which shows that global extinction depends on the initial configuration of the MCBP through the non-linear functional fixed point equation (5.5), case of local extinction on any finite number of states depends only on the spectral radius Λ. In particular local extinction for finite sets is not a phenomenon that is set-dependent.

Theorem 7 (Local extinction dichotomy)**.** *Fix* $\mu \in \mathcal{M}(\mathbb{N})$ *such that* $\sup\{n : \mu(n) > 0\} < \infty$. *Moreover suppose that*

$$\int_1^\infty (x \log x)\ell(i, \mathrm{d}x) + \int_1^\infty (x \log x)\mathrm{n}(i, \mathrm{d}x) < \infty, \qquad \textit{for all } i \in \mathbb{N}, \tag{5.6}$$

holds.

(i) *For any finite set of states* $A \subseteq \mathbb{N}$, $\mathbf{P}_\mu(\mathcal{L}_A) = 1$ *if and only if* $\Lambda \geq 0$.

(ii) *For any finite set of states* $A \subseteq \mathbb{N}$, *let* v_A *be the vector with entries* $v_A(i) = -\log \mathbf{P}_{\delta_i}(\mathcal{L}_A)$, $i \in \mathbb{N}$, *Then* v_A *is a solution to (5.5), and* $v_A(i) \leq w(i)$ *for all* $i \in \mathbb{N}$.

As we will see in the proof, if $\Lambda \geq 0$, then the process has local extinction a.s. even if (5.6) is not satisfied.

This results open up a number of questions for the MCBPs which are motivated by similar issues that emerge in the setting of CB-processes and super diffusions. For example, by analogy with the setting for super diffusions, under the assumption (5.6), we would expect that when $\Lambda < 0$, the quantity $-\Lambda$ characterises the growth rate of individual types. Specifically we conjecture that, when local extinction fails, $\exp\{\Lambda t\}X_i(t)$ converges almost surely to a non-trivial limit as $t \to \infty$, for each $i \in \mathbb{N}$. Moreover, if the number of types is finite, then $-\Lambda$ is also the growth rate of the total mass. That is to say $\exp\{\Lambda t\}\langle 1, X_t\rangle$ converges almost surely to a non-trivial limit as $t \to \infty$. If the total number of types is infinite then one may look for a discrepancy between the global growth rate and local growth rate. In the setting of super diffusions, [41] have made some progress in this direction. Further still, referring back to classical theory for CB-processes. It is unclear how the event of extinction occurs, both locally and globally. Does extinction occur as a result of mass limiting to zero but remaining positive for all time, or does mass finally disappear after an almost surely finite amount of time? Moreover, how does the way that extinction occur for one type relate to that of another type? An irreducibility property of the type space, e.g. assumption (A), is likely to ensure that mass in all states will experience extinction in a similar way with regard to the two types of extinction described before, but this will not necessarily guarantee that global extinction behaves in the same way as local extinction. We hope to address some of these questions in future work.

We complete this section by giving an overview of the remainder of the chapter. In the next section we give the construction of MCBPs as a scaling limit of MBGW processes; that is to say, in terms of branching Markov chains. We define the linear semigroup associated to the MCBP. The so-called spectral radius of this linear semigroup will have an important roll in the asymptotic behaviour of our process, in particular, it will determine the phenomenon of local extinction. The properties of this semigroup are studied in Section 5.3. In Sections 5.4 and 5.5 we develop some standard tools based around a spine decomposition. In this setting, the spine is a Markov chain and we note in particular that the non-local nature of the branching mechanism induces a new additional phenomenon in which a positive, random amount of mass immigrates off the spine each time it jumps from one state to another. Moreover, the distribution of the immigrating mass depends on where the spine jumped from and where it jumped to. Concurrently to our work we learnt that this phenomenon was also observed recently by Chen, Ren and Song [25]. In Section 5.6, we give the proof of the main results. We note that the main agenda for the proof has heavily influenced by the proof of local extinction in [40] for super diffusions. Finally in Section 5.7, we provide examples to illustrate the local phenomenon property.

5.2 MCBPs as a superprocess

Our objective in this section is to prove Theorem 6. The proof is not novel as we do this by showing that MCBPs can be seen in, in the spirit of the theory of superprocesses, as the scaling limits of MBGW processes with type space $\mathbb{N}$ (or just $\{1, \cdots, n\}$ for some $n \in \mathbb{N}$ in the case of finite types).

To this end, let $\gamma \in \mathcal{B}^+(\mathbb{N})$ and let $F(i, \mathrm{d}\nu)$ be a Markov kernel from $\mathbb{N}$ to $\mathcal{I}(\mathbb{N})$, the space of finite integer-valued measures, such that

$$\sup_{i\in\mathbb{N}} \int_{\mathcal{I}(\mathbb{N})} \nu(1) F(i, \mathrm{d}\nu) < \infty.$$

A branching particle system is described by the following properties:

1. For a particle of type $i \in \mathbb{N}$, which is alive at time $r \geq 0$, the conditional probability of survival during the time interval $[r, t)$ is $\rho_i(r, t) := \exp\{-\gamma(i)(t - r)\}$, $t \geq r$.
2. When a particle of type i dies, it gives birth to a random number of offspring in $\mathbb{N}$ according to the probability kernel $F(i, \mathrm{d}\nu)$.

We also assume that the lifetime and the branching of different particles are independent. Let $X_t(B)$ denote the number of particles in $B \in \mathcal{B}(\mathbb{N})$ that are alive at time $t \geq 0$ and assume $X_0(\mathbb{N}) < \infty$. With a slight abuse of notation, we take $X_0 := \mu$, where $\mu \in \mathcal{I}(\mathbb{N})$. Then $\{X_t : t \geq 0\}$ is a Markov process with state space $\mathcal{I}(\mathbb{N})$, which will be referred as a *branching Markov chain* or *multi-type BGW* with parameters (γ, F). For $\mu \in \mathcal{I}(\mathbb{N})$, let $\mathbf{P}_\mu$ denote the law of $\{X_t : t \geq 0\}$ given $X_0 = \mu$. In the special case that X is issued with a single particle of type i, we write its law by $\mathbf{P}_{\delta_i}$. For $f \in B^+(\mathbb{N})$, $t \geq 0$, $i \in \mathbb{N}$, put

$$u_t(i) := u_t(i, f) = -\log \mathbf{E}_{\delta_i}[\exp\{-\langle f, X_t\rangle\}].$$

The independence hypothesis implies that

$$\mathbf{E}_\mu[\exp\{-\langle f, X_t\rangle\}] = \exp\{-\langle u_t, \mu\rangle\}, \qquad \mu \in \mathcal{I}(\mathbb{N}),\ f \in \mathcal{B}^+(\mathbb{N}),\ t \geq 0. \tag{5.7}$$

Moreover, by conditioning on the first branching event, u_t is determined by the renewal equation

$$\mathrm{e}^{-u_t(i)} = \rho_i(0, t)\mathrm{e}^{-f(i)} + \int_0^t \rho_i(0, s)\gamma(i) \int_{\mathcal{I}(\mathbb{N})} \mathrm{e}^{-\langle u_{t-s},\ \nu\rangle} F(i, \mathrm{d}\nu)\mathrm{d}s.$$

By a standard argument (see for example Lemma 1.2 in Chapter 4 of in [37]) one sees that the last equation is equivalent to

$$\mathrm{e}^{-u_t(i)} = \mathrm{e}^{-f(i)} - \int_0^t \gamma(i)\mathrm{e}^{-u_{t-s}(i)}\mathrm{d}s + \int_0^t \gamma(i) \int_{\mathcal{I}(\mathbb{N})} \mathrm{e}^{-\langle u_{t-s},\ \nu\rangle} F(i, \mathrm{d}\nu)\mathrm{d}s. \tag{5.8}$$

See, for example, Asmussen and Hering [6] or Ikeda et al. [54, 55, 56] for similar constructions.

In preparation for our scaling limit, it is convenient to treat the offspring that start their motion from the death sites of their parents separately from others. To this end, we introduce some additional parameters. Let α and $\beta \in \mathcal{B}^+(\mathbb{N})$ such that $\gamma = \alpha + \beta$. For each $i \in \mathbb{N}$, let π_i be a probability distribution in $\mathbb{N} \setminus \{i\}$ and let g, h be two positive measurable functions from $\mathbb{N} \times [-1, 1]$ to $\mathbb{R}$ such that, for each $i \in \mathbb{N}$,

$$g(i, z) = \sum_{n=0}^{\infty} p_n(i) z^n, \qquad h(i, z) = \sum_{n=0}^{\infty} q_n(i) z^n \qquad |z| \leq 1,$$

are probability generating functions with $\sup_i g'_z(i, 1-) < \infty$ and $\sup_i h'_z(i, 1-) < \infty$. Next, define the probability kernels $F_0(i, d\nu)$ and $F_1(i, d\nu)$ from $\mathbb{N}$ to $\mathcal{I}(\mathbb{N})$ by

$$\int_{\mathcal{I}(\mathbb{N})} \mathrm{e}^{-\langle f,\, \nu \rangle} F_0(i, \mathrm{d}\nu) = g(i, \mathrm{e}^{-f(i)})$$

and

$$\int_{\mathcal{I}(\mathbb{N})} \mathrm{e}^{-\langle f,\, \nu \rangle} F_1(i, \mathrm{d}\nu) = h(i, \langle \mathrm{e}^{-f}, \pi_i \rangle).$$

We replace the role of $F(i, d\nu)$ by

$$\gamma^{-1}(i) \left[\alpha(i) F_0(i, \mathrm{d}\nu) + \beta(i) F_1(i, \mathrm{d}\nu) \right], \qquad i \in \mathbb{N}, \nu \in \mathcal{I}(\mathbb{N}).$$

Intuitively, when a particle of type $i \in \mathbb{N}$ splits, the branching is of local type with probability $\alpha(i)/\gamma(i)$ and is of non-local type with probability $\beta(i)/\gamma(i)$. If branching is of a local type, the distribution of the offspring number is $\{p_n(i)\}$. If branching is of a non-local type, the particle gives birth to a random number of offspring according to the distribution $\{q_n(i)\}$, and those offspring choose their locations in $\mathbb{N} \setminus \{i\}$ independently of each other according to the distribution $\pi_i(\cdot)$. Therefore, u_t is determined by the renewal equation

$$\begin{aligned} \mathrm{e}^{-u_t(i)} = \mathrm{e}^{-f(i)} + \int_0^t \alpha(i) \left[g(i, \mathrm{e}^{-u_{t-s}(i)}) - \mathrm{e}^{-u_{t-s}(i)} \right] \mathrm{d}s \\ + \int_0^t \beta(i) \left[h(i, \langle \mathrm{e}^{-u_{t-s}}, \pi_i \rangle) - \mathrm{e}^{-u_{t-s}(i)} \right] \mathrm{d}s. \end{aligned} \tag{5.9}$$

For the forthcoming analysis, it is more convenient to work with

$$v_t(i) := 1 - \exp\{-u_t(i)\}, \qquad t \geq 0, i \in \mathbb{N}.$$

In that case,

$$v_t(i) = \mathbb{E}_i \left[1 - \mathrm{e}^{f(i)} \right] - \int_0^t \left[\psi(i, v_{t-s}(i)) + \phi(i, v_{t-s}) \right] \mathrm{d}s,$$

where

$$\psi(i, z) = \alpha(i)[g(i, 1 - z) - (1 - z)] + \beta(i) z$$

and

$$\phi(i, f) = \beta(i) \left[h(i, 1 - \langle f, \pi_i \rangle) - 1 \right].$$

Next, we take a scaling limit of the MBGW process. We treat the limit as a superprocess with local and non-local branching mechanism. For each $k \in \mathbb{N}$, let $\{Y^{(k)}(t), t \geq 0\}$ be a sequence of branching particle system determined by $(\alpha_k(\cdot), \beta_k(\cdot), g_k(\cdot), h_k(\cdot), \pi_\cdot)$. Then, for each k,

$$\{X^{(k)}(t) = k^{-1}Y^{(k)}(t), \quad t \geq 0\}$$

defines a Markov process in $N_k(\mathbb{N}) := \{k^{-1}\sigma, \sigma \in N(\mathbb{N})\}$. For $0 \leq z \leq k$ and $f \in \mathcal{B}(\mathbb{N})$, let

$$\psi_k(i, z) = k\alpha_k(i)[g_k(i, 1 - z/k) - (1 - z/k)] + \beta_k(i)z$$

and

$$\phi_k(i, f) = \beta_k(i)k[h_k(i, 1 - k^{-1}\langle f, \pi_i\rangle) - 1].$$

Under certain conditions, Dawson et. al [27] obtained the convergence of $\{X^{(k)}(t), t \geq 0\}$ to some process $\{X(t), t \geq 0\}$. Let $\overline{\mathcal{B}}(\mathbb{N})$ the subset $\mathcal{B}(\mathbb{N})$ with entries uniformly bounded from above and below. We re-word their result for our particular setting here.

Theorem 8. *Suppose that*

$$\sum_{n=0}^{\infty} n q_n^k(i) \leq 1,$$

that $\beta_k \to \beta \in \mathcal{B}^+(\mathbb{N})$ uniformly, $\phi_k(i, f) \to \phi(i, f)$ uniformly on $\mathbb{N} \times \overline{\mathcal{B}}(\mathbb{N})$, and $\psi(i, z) \to \psi(i, z)$ locally uniformly. Then

i) The function $\psi(i, z)$ has representation

$$\psi(i, z) = b(i)z + c(i)z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu)\ell(i, \mathrm{d}u), \qquad i \in \mathbb{N}, \quad z \geq 0, \tag{5.10}$$

where $b \in \mathcal{B}(\mathbb{N})$, $c \in \mathcal{B}^+(\mathbb{N})$ and $(u \wedge u^2)\ell(i, \mathrm{d}u)$ is a bounded kernel from $\mathbb{N}$ to $(0, \infty)$.

ii) The function $\phi(i, f)$ can be represented as

$$\phi(i, f) = -\beta(i)\left[d(i)\langle f, \pi_i\rangle + \int_0^\infty (1 - \mathrm{e}^{-u\langle f, \pi_i\rangle})\mathrm{n}(i, \mathrm{d}u)\right], \tag{5.11}$$

where $d \in \mathcal{B}^+(\mathbb{N})$, and $u\mathrm{n}(i, \mathrm{d}u)$ is a bounded kernel from $\mathbb{N}$ to $(0, \infty)$ with

$$d(i) + \int_0^\infty u\mathrm{n}(i, \mathrm{d}u) \leq 1.$$

iii) To each function ψ and ϕ satisfying (5.10) and (5.11) there correspond a sequence of β_k, ψ_k and ϕ_k.

v) For each $a \geq 0$, the functions $v_t^k(i, f)$ and $ku_t^{(k)}(i, f)$ converge boundedly and uniformly on $[0, a] \times \mathbb{N} \times \overline{\mathcal{B}}(\mathbb{N})$, to the unique bounded positive solution $V_t f(i)$ to the evolution equation

$$V_t f(i) = f(i) - \int_0^t \left[\psi(i, V_{t-s}f(i)) + \phi(i, V_{t-s}f)\right]\mathrm{d}s, \qquad t \geq 0. \tag{5.12}$$

Moreover, there exists a Markov process $\{X_t : t \geq 0\}$ with probabilities $\{\mathbf{P}_\mu, \mu \in \mathcal{M}(\mathbb{N})\}$ such that

$$\mathbf{E}_\mu[\mathrm{e}^{-\langle f, X_t\rangle}] = \exp\left\{-\langle V_t f, \mu\rangle\right\}, \quad \mu \in \mathcal{M}(\mathbb{N}),\ f \in \mathcal{B}^+(\mathbb{N}),$$

and the cumulant semigroup $V_t f$ is given by (5.12).

Theorem 6 now follows directly as a corollary of the above result. Intuitively, $\psi(i, \cdot)$ describes the rate at which a branching event amongst current mass of type $i \in \mathbb{N}$, produces further mass of type i. Moreover, $\phi(i, \cdot)$ describes the rate at which a branching event amongst current mass of type $i \in \mathbb{N}$, produces further mass of other types $\mathbb{N}\backslash\{i\}$.

Remark 1. The non-local branching mechanism is not the most general form that can be assumed in the limit. Indeed, taking account of the class of non-local branching mechanisms that can be developed in [27], [36] and [70], we may do the same here. Nonetheless, we keep to this less-general class for the sake of mathematical convenience.

5.3 Spectral properties of the moment semigroup

Let $(X_t, \mathbf{P}_\mu)$ be a MCBP and define its linear semigroup $(\mathcal{M}_t, t \geq 0)$ by

$$\mathcal{M}_t[f](i) := \mathbf{E}_{\delta_i}[\langle f, X_t\rangle], \qquad i \in \mathbb{N}, f \in \mathcal{B}^+(\mathbb{N}), t \geq 0. \tag{5.13}$$

Replacing f in (5.3) and (5.4), it is easily verified that

$$\mathcal{M}_t[f](i) = f(i) + \int_0^t \mathcal{K}[\mathcal{M}_s[f]](i)\mathrm{d}s - \int_0^t b(i)\mathcal{M}_s[f](i)\mathrm{d}s, \qquad i \in \mathbb{N}, f \in \mathcal{B}^+(\mathbb{N}), t \geq 0,$$

where

$$\mathcal{K}[g](i) = \beta(i)\left(d(i) + \int_0^\infty un(i, \mathrm{d}u)\right)\langle g,\, \pi_i\rangle.$$

In addition, $\mathcal{M}_t$ has formal matrix generator L given by

$$L = \Delta_{-b} + K, \tag{5.14}$$

where the matrices Δ_{-b} and K are given by

$$(\Delta_{-b})_{ij} = -b(i)\mathbf{1}_{i=j}, \qquad \text{and} \qquad K_{ij} = \beta(i)\left(d(i) + \int_0^\infty un(i, \mathrm{d}u)\right)\pi_i(j).$$

Define the matrix $M(t)$ by

$$M(t)_{ij} := \mathbf{E}_{\delta_i}[X_t(j)],$$

and observe that

$$\mathcal{M}_t[f](i) = [M(t)f](i). \tag{5.15}$$

The linear semigroup will play an important role in the proof of Theorem 7, in particular, its spectral properties are of concern to us. Thanks to (5.15), it suffices to study the spectral properties of the matrix $M(t)$. In the forthcoming theory, we will need to assume that $M := \{M(t) : t \geq 0\}$, is *irreducible* in the sense that for any $i, j \in \mathbb{N}$ there exists $t > 0$ such that $M(t)_{ij} > 0$. The following lemma ensures this is the case.

Lemma 6. *Suppose that $\pi_i(j)$, $i, j \in \mathbb{N}$ is the transition matrix of an irreducible Markov chain, then M is irreducible.*

Proof. Let $a(i) = \beta(i)\left(d(i) + \int_0^\infty u n(i, du)\right)$, for i$\in \mathbb{N}$. Define the matrices Q and Δ_{a-b}

$$Q_{ij} = a(i)(\pi_i(j) - \mathbf{1}_{\{i=j\}}) \qquad \text{and} \qquad (\Delta_{a-b})_{ij} = (a(i) - b(i))\mathbf{1}_{\{i=j\}}.$$

By hypothesis, Q is the Q-matrix of an irreducible Markov chain $(\xi_t, \mathbb{P}_i)$. In particular, for each $i, j \in \mathbb{N}$ and $t > 0$, $\mathbb{P}_i(\xi_t = j) > 0$. Observe in (5.14) that $L = Q + \Delta_{a-b}$ which is the formal generator of the semigroup given by

$$\mathcal{T}_t[f](i) = \mathbb{E}_i\left[f(\xi_t)\exp\left\{\int_0^t (a-b)(\xi_s)\mathrm{d}s\right\}\right] \qquad i \in \mathbb{N},\ f \in \mathcal{B}^+(\mathbb{N}),\ t \geq 0. \tag{5.16}$$

By uniqueness of the semigroups, $\mathcal{M}_t f(i) = \mathcal{T}_t[f](i)$, $t \geq 0$, $i \in \mathbb{N}$. In particular, for δ the Dirac function, we have $M(t)_{ij} = \mathcal{T}_t[\delta_j](i) > 0$. And therefore M is irreducible. $\square$

Recall that, for each $i, j \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, we defined the matrix $H(\lambda)$ by

$$H_{ij}(\lambda) := \int_0^\infty \mathrm{e}^{\lambda t} M(t)_{ij}\mathrm{d}t.$$

and that the spectral radius

$$\Lambda := \sup\{\lambda \geq -\infty : H_{ij}(\lambda) < \infty\},$$

does not depend on i and j.

Definition 1. *A non-negative vector x with entries $x(i)$, $i \in \mathbb{N}$, is called right (resp. left) subinvariant λ-vector, if for all $t \geq 0$,*

$$M(t)x \leq \mathrm{e}^{-\lambda t}x, \qquad (\textit{resp. } x^T M(t) \leq \mathrm{e}^{-\lambda t}x).$$

If the equality holds, the vector is call a right (resp. left) invariant λ-vector.

In the next proposition, we appeal to standard techniques (cf. [77] or [86]) and provide sufficient conditions for the existence of subinvariant λ-vectors.

Proposition 15. *If $H(\lambda) < \infty$, then there exists a positive*[1] *right subinvariant λ-vector, x, and a positive left subinvariant λ-vector, y. There exists no left or right subinvariant β-vector for $\beta > \Lambda$.*

Proof. Fix $j \in \mathbb{N}$ and define x and y as follows

$$x(i) = H_{ij}(\lambda) \qquad \text{and} \qquad y(i) = H_{ji}(\lambda).$$

Since the function $t \mapsto M(t)$ is continuous and M is irreducible, $x(i)y(k) > 0$ for all $i, k \in \mathbb{N}$. Let $s \geq 0$, by Fubini's Theorem,

$$[y^T M(s)](i) = \sum_{k\in\mathbb{N}} \int_0^\infty \mathrm{e}^{\lambda t} M(t)_{jk}\mathrm{d}t\ M(s)_{ki} = \int_0^\infty \mathrm{e}^{\lambda t} \sum_{k\in\mathbb{N}} M(t)_{jk} M(s)_{ki}\mathrm{d}t.$$

[1]Recall that a vector x is positive if its entries, $x(i)$, are strictly positive for all i.

The semigroup property implies that

$$[y^T M(s)](i) = \int_0^\infty \mathrm{e}^{\lambda t} M(s+t)_{ji} \mathrm{d}t = \mathrm{e}^{-\lambda s} \int_s^\infty \mathrm{e}^{\lambda t} M(t)_{ji} \mathrm{d}t \le \mathrm{e}^{-\lambda s} y(i).$$

Therefore, y is a left subinvariant λ-vector. A similar computation shows that x is a right subinvariant λ-vector.

Suppose x is a right subinvariant β-vector. Let $\alpha \in (\Lambda, \beta)$, then, for each $i \in \mathbb{N}$,

$$\int_0^\infty \mathrm{e}^{\alpha t} [M(t)x](i) \mathrm{d}t \le \int_0^\infty \mathrm{e}^{\alpha t} \mathrm{e}^{-\beta t} x(i) \mathrm{d}t = x(i)(\beta - \alpha)^{-1}.$$

Let $j \in \mathbb{N}$ such that $x(j) > 0$, then

$$\int_0^\infty \mathrm{e}^{\alpha t} M(t)_{ij} \mathrm{d}t \le \frac{x(i)}{x(j)} (\beta - \alpha)^{-1} < \infty,$$

which is a contradiction with the definition of Λ. In an analogous way, there is no left subinvariant β-vector.

□

When $H_{ij}(\Lambda) = \infty$, Niemi and Nummelin ([78],Theorem 4) proved that there exists unique left and right invariant Λ-vectors as follows.

Proposition 16. *Assume that $H_{ij}(\Lambda) = \infty$ for some $i, j \in \mathbb{N}$. Then,*

i) There exists a unique (up to scalar multiplication) positive left invariant Λ-vector.

ii) There exists a unique (up to scalar multiplication) positive right invariant Λ-vector. Moreover, any right subinvariant Λ-vector is a right invariant vector.

From the previous propositions, there exists at least a positive left (right) subinvariant Λ-vector. One of the reasons we are interested in right (sub)invariant vector, is that we can associate to it a (super)martingale, which will be of use later on in our analysis.

Proposition 17. *Let x be a right subinvariant λ-vector. Then*

$$W_t := \mathrm{e}^{\lambda t} \langle x, X_t \rangle, \qquad t \ge 0,$$

is a supermartingale. If x is also an invariant vector, then $(W_t, t \ge 0)$ is a martingale.

Proof. Let $t, s \ge 0$. By the Markov property and the branching property

$$\mathbf{E}\left[\mathrm{e}^{\lambda(t+s)} \langle x, X_{t+s} \rangle \Big| \mathcal{F}_s \right] = \mathrm{e}^{\lambda(t+s)} \mathbf{E}_{X_s} \left[\langle x, X_t \rangle \right] = \mathrm{e}^{\lambda(t+s)} \sum_{i \in \mathbb{N}} X_s(i) \mathbf{E}_{\delta_i} \left[\langle x, X_t \rangle \right].$$

Since x is a right subinvariant λ-vector,

$$\mathbf{E}_{\delta_i} \left[\langle x, X_t \rangle \right] = [M(t)x](i) \le \mathrm{e}^{-\lambda t} x(i),$$

therefore, we have that

$$\mathbf{E}\left[W_{t+s} | \mathcal{F}_s \right] = \mathrm{e}^{\lambda(t+s)} \sum_{i \in \mathbb{N}} X_s(i) [M(t)x](i) \le \mathrm{e}^{\lambda s} \sum_{i \in \mathbb{N}} x(i) X_s(i) = W_s.$$

In the invariant case, inequalities become equalities. □

Let $[n] = \{1, \cdots, n\}$ and let $X^{[n]} := \{X_t^{[n]} : t \geq 0\}$ be a branching process with the same mechanism as X_t but we kill mass that is is created outside of $[n]$. To be more precise, $X^{[n]}$ has the same local branching mechanisms $\psi(i, \cdot)$ and $\phi(i, \cdot)$, for $i = 1, \cdots, n$, albeit that, now, $\pi_i(j)$, $j \in \mathbb{N}\backslash\{i\}$ is replaced by $\pi_i(j)\mathbf{1}_{(j\leq n)}$, $j \in \mathbb{N}\backslash\{i\}$. Finally $\psi(i, \cdot)$ and $\phi(i, \cdot)$ are set to be zero for $i \geq n$.
Let $M^{[n]}(t)$ be the matrix associated to the linear semigroup of $X^{[n]}$. Then the infinitesimal generator of $M^{[n]}(t)$ is given by

$$L^{[n]} = [\Delta_{-b} + K]\Big|_{[n]}.$$

In order to apply Perron-Froebenius theory to the matrix $M^{[n]}(t)$, we need irreducibility. By Lemma 6, it is enough that $\pi_i(j), i, j \leq n$ is irreducible. There exist simple examples of infinite irreducible matrices such that their upper left square n-corner truncations are not irreducible for all $n \geq 1$. However, according to Seneta ([85], Theorem 3), the irreducibility of π implies that there exists a simultaneous rearrangement of the rows and columns of π, denoted by $\tilde{\pi}$, and a sequence of integers k_n tending to infinity, such that the truncation of $\tilde{\pi}$ to $[k_n]$ is irreducible for all n. Observe that the type space, $\mathbb{N}$, is used as a labelled set and not as an ordered set. It therefore follows that we can assume without loss of generality, that we start with $\tilde{\pi}$ (The vectors b, c, d, β, ℓ and n will require the same rearrangement). In the rest of the paper, when requiring finite truncations to the state space, whilst preserving irreducibility, it is enough to work with the truncations on $[k_n]$. In order to simplify the notation, we will assume without loss of generality that $k_n = n$ for all n.

Perron-Froebenius theory tells us there exist two positive vectors $x^{[n]} = \{x^{[n]}(i) : i = 1, \cdots, n\}$ and $y^{[n]} = \{y^{[n]}(i) : i = 1, \cdots, n\}$, and a real number $\Lambda^{[n]} = \sup\{\lambda \geq -\infty : H_{ij}^{[n]}(\lambda) < \infty\}$, such that

$$M^{[n]}(t)x^{[n]} = \mathrm{e}^{-\Lambda^{[n]}t}x^{[n]} \qquad \text{and} \qquad (y^{[n]})^T M^{[n]}(t) = \mathrm{e}^{-\Lambda^{[n]}t}y^{[n]}.$$

By construction of $X_t^{[n]}$, we have the inequalities

$$M_{ij}^{[n]}(t) \leq M_{ij}^{[n+1]}(t) \leq M_{ij}(t),$$

which naturally leads to the hierarchy of eigenvalues

$$\Lambda \leq \Lambda^{[n+1]} \leq \Lambda^{[n]}. \tag{5.17}$$

Lemma 7.

i) $\Lambda^\infty := \lim_{n\to\infty} \Lambda^{[n]} = \Lambda$.

ii) Let $x^{[n]}$ be a right invariant $\Lambda^{[n]}$-vector for $M^{[n]}$, such that $x^{[n]}(1) = 1$. Then, the vector $\{x^*(j) : j \in \mathbb{N}\}$ given by $x^*(j) = \liminf_{n\to\infty} x^{[n]}(j)$ is a positive right Λ-subinvariant vector. Moreover, it $H_{ij}(\Lambda) = \infty$, then $\{x^*(j) : j \in \mathbb{N}\}$ is the unique positive right invariant Λ-vector of M with $x^*(1) = 1$.

Proof. By inequality (5.17),

$$\Lambda \leq \Lambda^\infty = \lim_{n\to\infty} \Lambda^{[n]}.$$

For any $n \in \mathbb{N}$, let $x^{[n]}$ be a $M^{[n]}$ right invariant vector, such that $x^{[n]}(1) = 1$ for all $n \in \mathbb{N}$, this implies

$$L^{[n]}x^{[n]} = -\Lambda^{[n]}x^{[n]}.$$

Let $x^*(j) = \liminf_{n\to\infty} x^{[n]}(j)$, by Fatou's Lemma

$$Lx^* \le -\Lambda^\infty x^*.$$

Using the fact that $M(t)$ is a non negative matrix and

$$\frac{\mathrm{d}}{\mathrm{d}t}[M(t)x^*](i) = [M(t)Lx^*](i), \qquad i \in \mathbb{N},$$

we find that

$$[M(t)x^*](i) \le \mathrm{e}^{-\Lambda^\infty t} x^*(i), \qquad i \in \mathbb{N}.$$

Since $x^*(1) = 1$, x^* is a right Λ_∞-subinvariant vector. By applying Proposition 15 we have that $\Lambda^\infty \le \Lambda$ and therefore x^* is a right Λ-subinvariant vector. The last part of the claim is true due to Proposition 16. □

Any vector $x \in \mathbb{R}^n$ can be extended to a vector $u \in \mathbb{R}^{\mathbb{N}}$ by the natural inclusion map $u(i) = x(i)\mathbf{1}_{\{i\le n\}}$. Since it will be clear in which space we intend to use the vector, we make an abuse of notation, and in the future we will denote both with x.

5.4 Spine decomposition

According to Dynkin's theory of exit measures [35] it is possible to describe the mass of X as it first exits the growing family of domains $[0,t) \times [n]$ as a sequence of random measures, known as *branching Markov exit measures*, which we denote by $\{X^{[n],t} : t \ge 0\}$. We recover here some of its basic properties. First, $X^{[n],t}$ has support on $(\{t\} \times [n]) \cup ([0,t] \times [n]^c)$. Moreover, under $\{t\} \times [n]$,

$$X^{[n],t}(\{t\} \times B) = X_t^{[n]}(B),$$

for each $B \subset [n]$. We use the obvious notation that for all $f \in \mathcal{B}^+([0,t] \times \mathbb{N})$,

$$\langle f, X^{[n],t}\rangle = \sum_{i\in[n]} f(t,i)X^{[n],t}(\{t\},i) + \sum_{i\in[n]^c} \int_0^t f(s,i)X^{[n],t}(\mathrm{d}s,i).$$

We have that for all $\mu \in M([0,t] \times \mathbb{N})$, and $f \in \mathcal{B}^+([0,t] \times \mathbb{N})$

$$\mathbf{E}_\mu[\mathrm{e}^{-\langle f, X^{[n],t}\rangle}] = \exp\{-\langle V_0^{[n],t} f, \mu\rangle\}, \tag{5.18}$$

where, for $t \ge r \ge 0$, $V_r^{[n],t} f : [n] \to [0,\infty)$ is the unique non-negative solution to

$$V_r^{[n],t} f(i) = \begin{cases} f(t,i) - \int_r^t \left[\psi(i, V_s^{[n],t} f(i)) + \phi(i, V_s^{[n],t} f)\right]\mathrm{d}s & \text{if} \quad i \le n \\ f(r,i) & \text{if} \quad i > n. \end{cases} \tag{5.19}$$

An important observation for later is that temporal homogeneity implies that

$$V_r^{[n],t} f = V_0^{[n],t-r} f, \tag{5.20}$$

for all $f \in \mathcal{B}^+([0,t] \times \mathbb{N})$. Moreover, as a process in time, $X^{[n],\cdot} = \{X^{[n],t} : t \ge 0\}$ is a MCBP with local mechanism $\psi^{[n]} = \psi(i,z)\mathbf{1}_{\{i\le n\}}$ and non-local mechanism $\phi^{[n]} = \phi(i,f)\mathbf{1}_{\{i\le n\}}$.

Any function $g:\mathbb{N}\to[0,\infty)$ can be extended to a function $\bar{g}:[0,\infty)\times\mathbb{N}\to[0,\infty)$ such that $\bar{g}(s,i)=g(i)$. Let x be a $\Lambda^{[n]}$ right invariant vector of $M^{[n]}$. (Note, in order to keep notation to a minimum, we prefer x in place of the more appropriate notation $x^{[n]}$.) By splitting the integral between $\{t\}\times[n]$ and $[0,t]\times[n]^c$, it is easy to show that

$$\langle \bar{x}, X^{[n],t}\rangle = \langle x, X_t^{[n]}\rangle.$$

Using the Markov property of exit measures, the last equality, and Proposition 17, standard computations tell us that

$$Y_t^{[n]} := \mathrm{e}^{\Lambda^{[n]}t}\frac{\langle \bar{x}, X^{[n],t}\rangle}{\langle x,\mu\rangle} = \mathrm{e}^{\Lambda^{[n]}t}\frac{\langle x, X_t^{[n]}\rangle}{\langle x,\mu\rangle}, \qquad t\geq 0,$$

is a mean one $\mathbf{P}_\mu$-martingale. For $\mu\in\mathcal{I}(\mathbb{N})$ such that $\mu(\mathbb{N}\backslash[n])=0$, define $\widetilde{\mathbf{P}}_\mu^{[n]}$ by the martingale change of measure

$$\left.\frac{d\widetilde{\mathbf{P}}_\mu^{[n]}}{d\mathbf{P}_\mu}\right|_{\mathcal{F}_t} = Y_t^{[n]}.$$

Theorem 9. *Let μ a finite measure with support in $[n]$ and $g\in\mathcal{B}^+(\mathbb{N})$. Introduce the Markov chain $(\eta,\mathbb{P}^x_\cdot)$ on $[n]$ with infinitesimal matrix, $\tilde{L}^{[n]}\in M_{n\times n}$, given by*

$$\tilde{L}_{ij}^{[n]} = \frac{1}{x(i)}\left(\Delta_{-b}+K_{ij}+\mathbf{1}_{\{i=j\}}\Lambda^{[n]}\right)x(j).$$

If X is a MCBP, then

$$\widetilde{\mathbf{E}}_\mu^{[n]}\left[\mathrm{e}^{-\langle f, X^{[n],t}\rangle}\frac{\langle \bar{x}\circ\bar{g}, X^{[n],t}\rangle}{\langle \bar{x}, X^{[n],t}\rangle}\right] = \mathbf{E}_\mu\left[\mathrm{e}^{-\langle f, X^{[n],t}\rangle}\right]\times$$
$$\mathbb{E}^x_{x\mu}\left[\exp\left\{-\int_0^t\left(2c(\eta_s)V_0^{[n],t-s}f(\eta_s)+\int_0^\infty u(1-\mathrm{e}^{-uV_0^{[n],t-s}f(\eta_s)})\ell(\eta_s,\mathrm{d}u)\right)\mathrm{d}s\right\}\times\right.$$
$$\left. g(\eta_t)\prod_{s\leq t}\Theta_{\eta_{s-},\eta_s}^{[n],t-s}\right], \tag{5.21}$$

where the matrices $\{\Theta^{[n],s}:s\geq 0\}$, are given by

$$\Theta_{i,j}^{[n],t} = \frac{\pi_i(j)\beta(i)}{[\Delta_{-b}+K+\Lambda^{[n]}I]_{i,j}}\int_0^\infty u(\mathrm{e}^{-u\langle V_0^{[n],t}f,\pi_i\rangle}-1)\mathrm{n}(i,\mathrm{d}u)+1$$

and

$$\mathbb{P}^x_{x\mu}(\cdot) = \sum_{i\in[n]}\frac{x(i)\mu(i)}{\langle x,\mu\rangle}\mathbb{P}^x_i(\cdot),$$

with an obviously associated expectation operator $\mathbb{E}^x_{x\mu}(\cdot)$.

Proof. We start by noting that

$$\widetilde{\mathbf{E}}^{[n]}_{\mu}\left[\mathrm{e}^{-\langle f, X^{[n],t}\rangle}\frac{\langle \bar{x}\circ\bar{g}, X^{[n],t}\rangle}{\langle \bar{x}, X^{[n],t}\rangle}\right] = \frac{\mathrm{e}^{\Lambda^{[n]}t}}{\langle x,\mu\rangle}\mathbf{E}_{\mu}\left[\langle \bar{x}\circ\bar{g}, X^{[n],t}\rangle \mathrm{e}^{-\langle f, X^{[n],t}\rangle}\right].$$

Replacing f by $f+\lambda\bar{x}\circ\bar{g}$ in (5.18) and (5.19) and differentiating with respect to λ and then setting $\lambda = 0$, we obtain

$$\begin{aligned}\widetilde{\mathbf{E}}^{[n]}_{\mu}\left[\mathrm{e}^{-\langle f, X^{[n],t}\rangle}\frac{\langle \bar{x}\circ\bar{g}, X^{[n],t}\rangle}{\langle \bar{x}, X^{[n],t}\rangle}\right] &= \mathbf{E}_{\mu}\left[\mathrm{e}^{-\langle f, X^{[n],t}\rangle}\right]\frac{\langle \theta^t_0, x\circ\mu\rangle}{\langle x,\mu\rangle}\\ &= \mathbf{E}_{\mu}\left[\mathrm{e}^{-\langle f, X^{[n],t}\rangle}\right]\sum_{i\le n}\frac{x(i)\mu_i}{\langle x,\mu\rangle}\theta^t_0(i),\end{aligned} \tag{5.22}$$

where $\circ$ denotes element wise multiplication of vectors and, for $t\ge r\ge 0$, θ^t_r is the vector with entries

$$\theta^t_r(i) := \frac{1}{x(i)}\mathrm{e}^{\Lambda^{[n]}(t-r)}\left.\frac{\partial}{\partial\lambda}V^{[n],t}_r[f+\lambda\bar{x}\circ\bar{g}](i)\right|_{\lambda=0}, \qquad i\in[n].$$

So that, in particular, $\theta^t_t(i) = g(i)$, $i\in[n]$, and additionally, $\theta^t_r(i) = 0$ for $i>n$ and $r\le t$. Note that the temporal homogeneity property (5.20) implies that $\theta^t_r(i) = \theta^{t-r}_0(i)$, $i\in[n]$, $t\ge r\ge 0$. Moreover, $\theta^t_r(i)$, $i\in[n]$, is also the unique solution to

$$\begin{aligned}\theta^t_r(i) = g(i) &- \int_r^t \theta^t_s(i)\left[2c(i)V^{[n],t-s}_0 f(i) + \int_0^\infty u(1-\mathrm{e}^{V^{[n],t-s}_0 f(i)})\ell(i,\mathrm{d}u)\right]\mathrm{d}s\\ &+ x(i)^{-1}\int_r^t \left[(\Delta_{-b}+K+\Lambda^{[n]}I)x\circ\theta^t_s\right](i)\mathrm{d}s\\ &+ \int_r^t \langle\theta^t_s, \pi^x_i\rangle\beta(i)\int_0^\infty u(\mathrm{e}^{-u\langle V^{[n],t-s}_0 f, \pi_i\rangle}-1)\mathrm{n}(i,\mathrm{d}u)\mathrm{d}s,\end{aligned}$$

where

$$\pi^x_i(j) := \frac{x(j)}{x(i)}\pi_i(j), \qquad ,i,j\in[n].$$

A straightforward integration by parts now ensures that

$$\begin{aligned}[\mathrm{e}^{\widetilde{L}^{[n]}r}\theta^t_r](i) =& [\mathrm{e}^{\widetilde{L}^{[n]}t}g](i)\\ &- \int_r^t \mathrm{e}^{\widetilde{L}^{[n]}s}\left[\theta^t_s\circ\left[2c(\cdot)V^{[n],t-s}_0 f(\cdot) + \int_0^\infty u(1-\mathrm{e}^{V^{[n],t-s}_0 f(\cdot)})\ell(\cdot,\mathrm{d}u)\right]\right](i)\mathrm{d}s\\ &+ \int_r^t \mathrm{e}^{\widetilde{L}^{[n]}s}\left[\langle\theta^t_s, \pi^x_\cdot\rangle\beta(\cdot)\int_0^\infty u(\mathrm{e}^{-u\langle V^{[n],t-s}_0 f, \pi_\cdot\rangle}-1)\mathrm{n}(\cdot,\mathrm{d}u)\mathrm{d}s\right](i)\mathrm{d}s.\end{aligned}$$

Then appealing to temporal homogeneity, and the fact that $\{\mathrm{e}^{\widetilde{L}^{[n]}t} : t\ge 0\}$ is the semigroup of $(\eta, \mathbb{P}^x_\cdot)$,

$$\begin{aligned}\theta_0^t(i) =& \ \mathbb{E}_i^x[g(\eta_t)] - \mathbb{E}_i^x\left[\int_0^t \theta_0^{t-s}(\eta_s)\left[2c(\eta_s)V_0^{[n],t-s}f(\eta_s) + \int_0^\infty u(1-\mathrm{e}^{V_0^{[n],t-s}f(\eta_s)})\ell(\eta_s,\mathrm{d}u)\right]\mathrm{d}s\right] \\ &+\mathbb{E}_i^x\left[\int_0^t \langle\theta_0^{t-s},\, \pi^x_{\eta_s}\rangle\beta(\eta_s)\int_0^\infty u(\mathrm{e}^{-u\langle V_0^{[n],t-s}f,\,\pi_{\eta_s}\rangle}-1)\mathrm{n}(\eta_s,\mathrm{d}u)\mathrm{d}s\right] \\ =& \ \mathbb{E}_i^x[g(\eta_t)] - \mathbb{E}_i^x\left[\int_0^t \theta_0^{t-s}(\eta_s)\left[2c(\eta_s)V_0^{[n],t-s}f(\eta_s) + \int_0^\infty u(1-\mathrm{e}^{V_0^{[n],t-s}f(\eta_s)})\ell(\eta_s,\mathrm{d}u)\right]\mathrm{d}s\right] \\ &+\mathbb{E}_i^x\left[\int_0^t \sum_j \mathbf{1}_{(\widetilde{L}^{[n]}_{\eta_s,j}\neq 0)}\theta_0^{t-s}(j)\left(\frac{\pi^x_{\eta_s}(j)\beta(\eta_s)}{\widetilde{L}^{[n]}_{\eta_s,j}}\int_0^\infty u(\mathrm{e}^{-u\langle V_0^{[n],t-s}f,\,\pi_{\eta_s}\rangle}-1)\mathrm{n}(\eta_s,\mathrm{d}u)\right)\widetilde{L}^{[n]}_{\eta_s,j}\mathrm{d}s\right]\end{aligned}$$

(Note, in the last equality, we have used that $\widetilde{L}^{[n]}_{\eta_s,j} = 0$ if and only if $\pi_{\eta_s}(j) = 0$). We now see from Lemma 19 in the appendix that

$$\theta_0^t(i) = \mathbb{E}_i^x\left[\exp\left\{-\int_0^t\left(2c(\eta_s)V_0^{[n],t-s}f(\eta_s) + \int_0^\infty u(1-\mathrm{e}^{-uV_0^{[n],t-s}f(\eta_s)})\ell(\eta_s,\mathrm{d}u)\right)\mathrm{d}s\right\}\prod_{s\le t}\Theta^{[n],t-s}_{\eta_{s-},\eta_s}\right],$$

as required. □

Fix μ as a finite measure with support in $[n]$. Proposition 9 suggests that the process $(X^{[n],\cdot}, \widetilde{\mathbf{P}}_\mu)$ is equal in law to a process $\{\Gamma_t : t\ge 0\}$, whose law is henceforth denoted by P_μ, where

$$\Gamma_t = X'_t + \sum_{s\le t:c} X^{\mathrm{c},s}_{t-s} + \sum_{s\le t:\mathrm{d}} X^{\mathrm{d},s}_{t-s} + \sum_{s\le t:\mathrm{j}} X^{\mathrm{j},s}_{t-s}, \qquad t\ge 0, \tag{5.23}$$

such that X' is an independent copy of $(X^{[n],\cdot}, \mathbf{P}_\mu)$ and the processes $X^{\mathrm{c},s}_\cdot$, $X^{\mathrm{d},s}_\cdot$ and $X^{\mathrm{j},s}_\cdot$ are defined through a process of immigration as follows: Given the path of the Markov chain $(\eta, \mathbb{P}^x_{x\mu})$,

[**continuous immigration**] in a Poissonian way an $(\psi^{[n]}, \phi^{[n]})$-MCBP $X^{\mathrm{c},s}_\cdot$ is immigrated at (s,η_s) with rate $\mathrm{d}s\times 2c(\eta_s)d\mathbf{N}_{\eta_s}$,

[**discontinuous immigration**] in a Poissonian way an $(\psi^{[n]}, \phi^{[n]})$-MCBP $X^{\mathrm{d},s}_\cdot$ is immigrated at (s,η_s) with rate $\mathrm{d}s\times\int_0^\infty u\ell(\eta_s,\mathrm{d}u)\mathbf{P}_{u\delta_{\eta_s}}$

[**jump immigration**] at each jump time s of η, an $(\psi^{[n]}, \phi^{[n]})$-MCBP $X^{\mathrm{j},s}_\cdot$ is immigrated at (s,η_s) with law $\int_0^\infty \nu_{\eta_{s-},\eta_s}(\mathrm{d}u)\mathbf{P}_{u\pi_{\eta_{s-}}}$, where, for i,j in the range of η,

$$\nu_{i,j}(\mathrm{d}u) = \frac{[\Delta_{-b}+I\Lambda^{[n]}]_{i,j}+\beta(i)d(i)\pi_i(j)}{[\Delta_{-b}+K+I\Lambda^{[n]}]_{i,j}}\delta_0(\mathrm{d}u) + \frac{\pi_i(j)\beta(i)}{[\Delta_{-b}+K+I\Lambda^{[n]}]_{i,j}}u\mathrm{n}(i,\mathrm{d}u).$$

Given η, all the processes are independent.

We remark that we suppressed the dependence on n of the processes X', $X^{\mathrm{c},s}_\cdot$, $X^{\mathrm{d},s}_\cdot$, $X^{\mathrm{j},s}_\cdot$ and Γ in order to have a nicer notation. Moreover, in the above description, the quantity $\mathbf{N}_i$ is the excursion measure of the $(\psi^{[n]}, \phi^{[n]})$-MCBP corresponding to $\mathbf{P}_{\delta_i}$. To be more precise, Dynkin and Kuznetsov ([38]) showed that associated to the laws $\{\mathbf{P}_{\delta_i} : i\in\mathbb{N}\}$ are the measures $\{\mathbf{N}_i : i\in\mathbb{N}\}$, defined on the same measurable space, which satisfy

$$\mathbf{N}_i(1-\mathrm{e}^{-\langle f, X^{[n],t}\rangle}) = -\log\mathbf{E}_{\delta_i}(\mathrm{e}^{-\langle f, X^{[n],t}\rangle}),$$

for all non-negative bounded function f on $\mathbb{N}$ and $t \geq 0$. A particular feature of $\mathbf{N}_i$ that we shall use later is that

$$\mathbf{N}_i(\langle f, X^{[n],t}\rangle) = \mathbf{E}_{\delta_i}[\langle f, X^{[n],t}\rangle]. \tag{5.24}$$

Observe that the processes $X^{\mathbf{c}}$, $X^{\mathbf{d}}$ and $X^{\mathbf{j}}$ are initially zero valued, therefore, if $\Gamma_0 = \mu$ then $X_0' = \mu$. Moreover (η, P_μ) is equal in distribution to $(\eta, \mathbb{P}^x_{x\mu})$. The following result corresponds to a classical spine decomposition, albeit now for the setting of an $(\psi^{[n]}, \phi^{[n]})$-MCBP. Note, we henceforth refer to the process η as the *spine*.

Remark 2. The inclusion of the immigration process indexed by j appears to be a new feature not seen before in previous spine decompositions and is a consequence of non-local branching. Simultaneously to our work, we learnt that a similar phenomenon has been observed by Chen, Ren and Song [25].

Theorem 10 (Spine decomposition)**.** *Suppose that μ as a finite measure with support in $[n]$. Then (Γ, P_μ) is equal in law to $(X^{[n],\cdot}, \widetilde{\mathbf{P}}_\mu)$.*

Proof. The proof is designed in two steps. First we show that Γ is a Markov process. Secondly we show that Λ has the same semigroup as $X^{[n],\cdot}$. In fact the latter follows immediately from Proposition 9 and hence we focus our attention on the first part of the proof. Observe that $((\Gamma_t, \eta_t), \mathrm{P}_\mu)$ is a Markov process. By the same argument that appeared on Theorem 5.2 in [65], if we prove

$$\mathrm{E}_\mu[\eta_t = i \mid \Gamma_t] = \frac{x(i)\Gamma_t(i)}{\langle \bar{x}, \Gamma_t\rangle}, \qquad i \leq n, \tag{5.25}$$

then, $(\Gamma_t, \mathrm{P}_\mu)$ is a Markov process. By conditioning over η, using the definition of Γ, the equation 5.21 and the fact that $(\Gamma_t, \mathrm{P}_\mu)$ is equal in law to $(X_t, \widetilde{\mathbf{P}}_\mu)$, for each t, we obtain

$$\mathrm{E}_\mu\left[\mathrm{e}^{-\langle f, \Gamma_t\rangle} g(\eta_t)\right] = \mathrm{E}_\mu\left[\mathrm{e}^{-\langle f, \Gamma_t\rangle}\frac{\langle \bar{x}\circ \bar{g}, \Gamma_t\rangle}{\langle x, \Gamma_t\rangle}\right], \qquad \text{for all } f,\ g \text{ measubles.}$$

The definition of conditional expectation implies (5.25). □

5.5 Martingale convergence

An important consequence of the spine decomposition in Theorem 10 is that we can establish an absolute continuity between the measures $\mathbf{P}_\mu$ and $\widetilde{\mathbf{P}}^{[n]}_\mu$.

Theorem 11. *Fix $n \in \mathbb{N}$ and $\mu \in \mathcal{M}(\mathbb{N})$ such that $\sup\{k : \mu(k) > 0\} \leq n$. The martingale $Y^{[n]}$ converges almost surely and in $L^1(\mathbf{P}_\mu)$ if and only if $\Lambda^{[n]} < 0$ and that*

$$\sum_{i\in[n]}\int_1^\infty (x\log x)\ell(i, \mathrm{d}x) + \sum_{i\in[n]}\int_1^\infty (x\log x)\mathrm{n}(i, \mathrm{d}x) < \infty, \tag{5.26}$$

Moreover, when these conditions fail, $\mathbf{P}_\mu(\lim_{t\to\infty} Y^{[n]}_t = 0) = 1$.

Proof. We follow a well established line of reasoning. Firstly we establish sufficient conditions. We know that $1/Y_t^{[n]}$ is a positive $\widetilde{\mathbf{P}}_\mu^{[n]}$-supermartingale and hence $\lim_{t\to\infty} Y_t^{[n]}$ exists $\widetilde{\mathbf{P}}_\mu^{[n]}$-almost surely. The statement of the theorem follows as soon as we prove that $\widetilde{\mathbf{P}}_\mu^{[n]}(\lim_{t\to\infty} Y_t^{[n]} < \infty) = 1$.

To this end, consider the spine decomposition in Theorem 10. Suppose, given the trajectory of the spine η, that we write $(s, \Delta_s^{\rm d}, \Delta_s^{\rm j})$, $s \geq 0$, for the process of immigrated mass along the spine, so that $(s, \Delta_s^{\rm d})$ has intensity ${\rm d}s \times u\ell(\eta_s, {\rm d}u)$ and, at s such that $\eta_{s-} \neq \eta_s$, $\Delta_s^{\rm j}$ is distributed according to ν_{η_{s-},η_s}.

Let $\mathcal{S} = \sigma(\eta, (s, \Delta_s^{\rm d}, \Delta_s^{\rm j}), s \geq 0)$ be the sigma algebra which informs the location of the spine and the volume of mass issued at each immigration time along the time and write

$$Z_t^{[n]} = {\rm e}^{\Lambda^{[n]} t} \frac{\langle \bar{x}, \Gamma_t \rangle}{\langle x, \mu \rangle}.$$

Our objective now is to use Fatou's Lemma and show that

$$\mathrm{E}_\mu[\lim_{t\to\infty} Z_t^{[n]} | \mathcal{S}] \leq \liminf_{t\to\infty} \mathrm{E}_\mu[Z_t^{[n]} | \mathcal{S}] < \infty.$$

Given that (Γ, P_μ) is equal in law to $(X^{[n],\cdot}, \widetilde{\mathbf{P}}_\mu)$, this ensures that $\widetilde{\mathbf{P}}_\mu^{[n]}(\lim_{t\to\infty} Y_t^{[n]} < \infty) = 1$, thereby completing the proof.

It therefore remains to show that $\liminf_{t\to\infty} \mathrm{E}_\mu[Z_t^{[n]} | \mathcal{S}] < \infty$. Taking advantage of the spine decomposition, we have, with the help of (5.24) and the fact that $\mathbf{E}_\mu[Y_t^{[n]}] = 1$, for $t \geq 0$ and μ such $\mu \in \mathcal{M}(\mathbb{N})$ such that $\sup\{k : \mu(k) > 0\} \leq n$,

$$\begin{aligned}\liminf_{t\to\infty} \mathrm{E}_\mu[Z_t^{[n]} | \mathcal{S}] =& \langle x, \mu \rangle + \int_0^\infty 2c(\eta_s) {\rm e}^{\Lambda^{[n]} s} \frac{x_{\eta_s}}{\langle x, \mu \rangle} {\rm d}s \\ &+ \sum_{s>0} {\rm e}^{\Lambda^{[n]} s} \Delta_s^{\rm d} \frac{x_{\eta_s}}{\langle x, \mu \rangle} + \sum_{s>0} {\rm e}^{\Lambda^{[n]} s} \Delta_s^{\rm j} \frac{\langle x, \pi_{\eta_s -} \rangle}{\langle x, \mu \rangle}.\end{aligned}$$

Recalling that $\Lambda^{[n]} < 0$ and that η lives on $[n]$, the first integral on the right-hand side above can be bounded above by a constant. The two sums on the right-hand side above can be dealt with almost identically.

It suffices to check that

$$\begin{aligned}\sum_{s>0} {\rm e}^{\Lambda^{[n]} s} \mathbf{1}_{(\Delta_s^{\rm d} < 1)} \Delta_s^{\rm d} + \sum_{s>0} {\rm e}^{\Lambda^{[n]} s} \mathbf{1}_{(\Delta_s^{\rm j} < 1)} \Delta_s^{\rm j} \\ + \sum_{s>0} {\rm e}^{\Lambda^{[n]} s} \mathbf{1}_{(\Delta_s^{\rm d} \geq 1)} \Delta_s^{\rm d} + \sum_{s>0} {\rm e}^{\Lambda^{[n]} s} \mathbf{1}_{(\Delta_s^{\rm j} \geq 1)} \Delta_s^{\rm j} < \infty.\end{aligned} \tag{5.27}$$

We first note that

$$\begin{aligned}&\mathrm{E}_\mu \left[\sum_{s>0} {\rm e}^{\Lambda^{[n]} s} \mathbf{1}_{(\Delta_s^{\rm d} < 1)} \Delta_s^{\rm d} + \sum_{s>0} {\rm e}^{\Lambda^{[n]} s} \mathbf{1}_{(\Delta_s^{\rm j} < 1)} \Delta_s^{\rm j} \right] \\ &= \mathrm{E}_\mu \left[\int_0^\infty {\rm e}^{\Lambda^{[n]} s} \int_{(0,1)} u^2 \ell(\eta_s, {\rm d}u) {\rm d}s \right] + \mathrm{E}_\mu \left[\int_0^\infty {\rm e}^{\Lambda^{[n]} s} \int_{(0,1)} u^2 L_{\eta_{s-},\eta_s}^{[n]} \nu_{\eta_{s-},\eta_s}(du) {\rm d}s \right] \\ &\leq \int_0^\infty {\rm e}^{\Lambda^{[n]} s} {\rm d}s \left\{ \sup_{i \in [n]} \int_{(0,1)} u^2 \ell(i, {\rm d}u) {\rm d}s + \sup_{i,j \in [n]} \pi_i^x(j) \int_{(0,1)} u^2 {\rm n}(i, {\rm d}u) {\rm d}s \right\} < \infty.\end{aligned}$$

Next, note that the condition (5.26) ensures that, P_μ almost surely,

$$\limsup_{s\to\infty} s^{-1}\mathbf{1}_{(\Delta^{\rm d}_s\geq 1)}\log\Delta^{\rm d}_s + \limsup_{s\to\infty} s^{-1}\mathbf{1}_{(\Delta^{\rm d}_s\geq 1)}\log\Delta^{\rm j}_s = 0,$$

so that both sequences $\Delta^{\rm d}_s$ and $\Delta^{\rm j}_s$ in the last two sums of (5.27) grow subexponentially. (Note that both of the aforesaid sequences are indexed by a discrete set of times when we insist $\{\Delta^{\rm d}_s\geq 1\}$.) Hence the second term in (5.27) converges.

To establish necessary conditions, let us suppose that τ is the set of times at which the mass $(s,\Delta^{\rm d}_s,\Delta^{\rm j}_s)$, $s\geq 0$, immigrates along the spine. We note that for $t\in\tau$,

$$Z^{[n]}_t \geq {\rm e}^{\Lambda^{[n]}t}\Delta^{\rm d}_t\frac{x_{\eta_t}}{\langle x,\mu\rangle} + {\rm e}^{\Lambda^{[n]}t}\Delta^{\rm j}_t\frac{\langle x,\pi_{\eta_{t-}}\rangle}{\langle x,\mu\rangle}. \tag{5.28}$$

If $\Lambda^{[n]}>0$ and (5.26) holds then

$$\widetilde{\mathbf{P}}_\mu(\limsup_{t\to\infty} Y^{[n]}_t=\infty) = \mathrm{P}_\mu(\limsup_{t\to\infty} Z^{[n]}_t=\infty)=1 \tag{5.29}$$

on account of the term ${\rm e}^{\Lambda^{[n]}t}$, the remaining terms on the righ-hand side of (5.28) grow subexponentially. If $\Lambda^{[n]}=0$ and (5.26) holds then, although there is subexponential growth of $(\Delta^{\rm j}_t,\Delta^{\rm d}_t)$, $t\geq 0$,

$$\limsup_{t\to\infty}\mathbf{1}_{(\Delta^{\rm d}_s\geq 1)}\Delta^{\rm d}_s + \limsup_{s\to\infty}\mathbf{1}_{(\Delta^{\rm d}_s\geq 1)}\Delta^{\rm j}_s=\infty$$

nonetheless. This again informs us that (5.29) holds. Finally if $\Lambda^{[n]}<0$ but (5.26) fails, then there exists an $i\in[n]$ such that $\int_1^\infty(x\log x)\ell(i,{\rm d}x)=\infty$ or $\int_1^\infty(x\log x){\rm n}(i,{\rm d}x)=\infty$. Suppose it is the latter. Recalling that η is ergodic, another straightforward Borel-Cantelli Lemma tells us that

$$\limsup_{s\to\infty} s^{-1}\mathbf{1}_{(\eta_{s-}=i,\,\Delta^{\rm d}_s\geq 1)}\log\Delta^{\rm j}_s>c,$$

for all $c>0$, which implies superexponential growth. In turn, (5.29) holds. The proof of the theorem is now complete as soon as we recall that (5.29) implies that $\mathbf{P}_\mu$ and $\widetilde{\mathbf{P}}_\mu$ are singular and hence $\widetilde{\mathbf{P}}_\mu(\lim_{t\to\infty}Y^{[n]}_t=0)=1$. □

5.6 Local and global extinction

Lemma 8. *For any finite $A\subset\mathbb{N}$ and any μ,*

$$\mathbf{P}_\mu\left(\limsup_{t\to\infty}\langle\mathbf{1}_A,X_t\rangle\in\{0,\infty\}\right)=1.$$

Proof. It is enough to prove the lemma for $A=\{i\}$. The branching property implies that $X_1(i)$ is an infinitely divisible random variable and consequently, its distribution has unbounded support on $\mathbb{R}_+$, (see Chapter 2 in Sato [84]). Therefore, for all $\epsilon>0$,

$$\mathbb{P}_{\epsilon\delta_i}(X_1(i)>K)>0. \tag{5.30}$$

Let us denote by Ω_0 the event $\limsup_{t\geq\infty}X_t(i)>0$ and, for each $\epsilon>0$, denote by Ω_ϵ the event $\limsup_{t\geq\infty}X_t(i)>\epsilon$. Define the sequence of stopping times as follows. On Ω_ϵ, let $T_0=\inf\{t>$

$0 : X_t(i) \geq \epsilon\}$ and $T_{n+1} = \inf\{t > T_n + 1 : X_t(i) \geq \epsilon\}$ and for Ω_ϵ^c extend T_n in a way such that the T_n's are bounded stopping times. Fix $K > 0$ and let $A_n = \Omega_\epsilon \cap \{X_{T_n+1}(i) \leq K\}$ and $\Omega^1 = \{\omega : \omega \in A_n \text{ i.o.}\}$. Thus by (5.30) and the strong Markov property,

$$\sum_{n=1}^{\infty} \mathbb{P}_\mu(A_n \mid X_{T_1}, \cdots X_{T_n}) = \infty \qquad \mathbb{P}_\mu\text{-a.s. on } \Omega_\epsilon.$$

By the extended Borel-Cantelli lemma [see Corollary 5.29 in [21]], $\mathbb{P}_\mu$-a.s. $\Omega_\epsilon \subset \Omega^1$. Observe that $\Omega_\epsilon \uparrow \Omega_0$ as $\epsilon \downarrow 0$. Therefore, for K arbitrary large, $\limsup_{t\to\infty} X_t(i) \geq K$, P_μ-a.s. on Ω_0, and the claim is true. □

Recall that we say that X under $\mathbf{P}_\mu$ *exhibits local extinction* for the finite set $A \subset \mathbb{N}$ if

$$\mathbf{P}_\mu\left(\lim_{t\uparrow\infty}\langle \mathbf{1}_A, X_t\rangle = 0\right) = 1.$$

Now, we have all the preliminary results needed for the Proof of Theorem 7

Proof of Theorem 7. (i) Let $0 \leq \Lambda$. By Propositions 15 and 16, there exists x a positive right subinvariant Λ-vector. Proposition 17 yields that $W_t = \mathrm{e}^{\Lambda t}\langle x, X\rangle$ is a non-negative supermartingale. By Doob's convergence theorem, there is a non-negative finite random variable W such that a.s.

$$W_t \longrightarrow W \qquad \text{as } \ t \to \infty.$$

When $\Lambda > 0$, since $\mathrm{e}^{\Lambda t} \to \infty$ as $t \to \infty$, and $x(i) > 0$ for any $i \in \mathbb{N}$, we have that $\mathbf{P}_\mu$-a.s. $\lim_{t\to\infty} X_t(i) = 0$, and hence, $\mathbf{P}_\mu$-a.s., $\lim_{t\to\infty}\langle \mathbf{1}_A, X_t\rangle = 0$. When $\Lambda = 0$, Lemma 8 yields the claim.

(ii) Now suppose that $\Lambda < 0$, using Lemma 7 there exits $n \geq i$ such that $\Lambda^{[n]} < 0$. Next, consider the conclusion of Theorem 10. Let $\mathbf{1}$ be n-dimensional vector whose entries are all 1 and let $\mathbf{0}$ be similarly defined. Note that $\widetilde{L}^{[n]}\mathbf{1} = \mathbf{0}$ and hence, together with irreducibility of $\pi|_{[n]}$, it follows that $(\eta, \mathbb{P}^x_\cdot)$ is ergodic. As a consequence, of the spine decomposition (5.23), we now see that, $\widetilde{\mathbf{P}}^{[n]}_\mu$-almost surely, mass is deposited by η infinitely often in state i. Thanks to the assumption (5.6) and Theorem 11, we have that $\widetilde{\mathbf{P}}^{[n]}_\mu \ll \mathbf{P}_\mu$ and hence there is no local extinction.

Next, recall that for a finite set of types $A \subset \mathbb{N}$

$$v_A(i) = -\log \mathbf{P}_{\delta_i}(\mathcal{L}_A).$$

It is a trivial consequence of the fact that $\mathcal{E} \subseteq \mathcal{L}_A$ that $v_A(i) \leq w(i)$, $i \in \mathbb{N}$. By independence, it follows that, for all finite $\mu \in \mathcal{M}(\mathbb{N})$,

$$\mathbf{P}_\mu(\mathcal{L}_A) = \exp\{-\langle v_A, \mu\rangle\}, \qquad t \geq 0.$$

By conditioning the event $\mathcal{L}_A$ on $\mathcal{F}_t$, we obtain that for all $t \geq 0$,

$$\mathbf{E}_\mu(\mathrm{e}^{-\langle v_A, X_t\rangle}) = \exp\{-\langle v_A, \mu\rangle\}. \tag{5.31}$$

Now recalling (5.4), v_A must satisfy the semigroup evolution, see

$$\psi(i, v_A(i)) + \phi(i, v_A) = 0.$$

Formally speaking, to pursue the reasoning, we need v_A to be a bounded vector, but this is not necessarily the case. To get round this problem, we can define $v_A^K(i) = K \wedge v_A(i)$, $i \in \mathbb{N}$, and observe by monotonicity and continuity that $V_t v_A^K(i) \uparrow v_A(i)$, $i \in \mathbb{N}, t \geq 0$, as $K \uparrow \infty$. When seen in the context of (5.4) (also using continuity and monotonicity), the desired reasoning can be applied. □

Proof of Lemma 4. The proof that w solves (5.5) is the same as the proof of (5.31). □

5.7 Examples

This section is devoted to some examples, where we find explicitly the global and local extinction probabilities. First we start with a remark of Kingman (see [61]).

Proposition 18. *Let $P_{ij}(t)$ be the transition probabilities of an irreducible continuous-time Markov chain on the countable state space E. Then there exists $\kappa \geq 0$ such that for each $i, j \in E$,*

$$t^{-1}\log(P_{ij}(t)) \to -\kappa.$$

Moreover, for each $i \in E$ and $t > 0$

$$P_{ii}(t) \leq \mathrm{e}^{-\kappa t}$$

and there exist finite constants K_{ij} such that

$$P_{ij}(t) \leq K_{ij}\mathrm{e}^{-\kappa t}, \qquad \textit{for all } i, j \in E, \quad t > 0.$$

If $Q = (q_{ij})$ is the associated Q-matrix, then

$$\kappa \leq -\sup\{q_{ii} : i \in E\}.$$

Observe that if the Markov chain is recurrent then $\kappa = 0$. When it is transient, κ could be greater than 0. In this case, we will say that the chain is geometrically transient with κ its decay parameter. Kingman provided a random walk example where $\kappa > 0$. The example is the following. Let ξ a random walk with Q-matrix given by

$$q_{i,i=1} = p, \qquad q_{ii} = -1, \qquad q_{i,i-1} = q = 1 - p,$$

where $p \in (0, 1)$. Then, ξ is an irreducible process with decay parameter $\kappa = 1 - 2\sqrt{pq}$. In particular, the process is geometrically transient except when $p = 1/2$.

Now, we can provide some examples.

Example 11. When ψ and ϕ don't depend on the underlying type, it is easy to show that $(\langle 1, X_t\rangle, t \geq 0)$ is a CBP with branching mechanism given by

$$\widetilde{\psi}(z) = \left(b - \beta d - \beta\int_0^\infty u\mathrm{n}(\mathrm{d}u)\right)z + cz^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu)(\ell + \beta\mathrm{n})(\mathrm{d}u), \qquad z \geq 0.$$

In this case, the global extinction probability is given by

$$\mathbf{P}_{\delta_i}(\mathcal{E}) = \mathrm{e}^{-\widetilde{\Phi}(0)},$$

where $\widetilde{\Phi}(0) = \sup\{z \leq 0 : \widetilde{\psi}(z) = 0\}$.

Define $a = \beta d + \beta \int_0^\infty u \mathrm{n}(\mathrm{d}u)$, then, our process X has global extinction a.s. if and only if $b - a \geq 0$. On the other hand, let $(\xi, \mathbb{P}_i)$ be an irreducible chain with Q-matrix given by

$$Q_{ij} = a(\pi(j) - \delta_{i=j}).$$

Then, by equation (5.16), the linear semigroup of X is

$$M_t f(i) = \mathbb{E}_i \left[f(\xi_t) \exp\left\{ \int_0^t (a-b)(\xi_s) \mathrm{d}s \right\} \right].$$

In particular,

$$H_{ij}(\lambda) = \int_0^\infty \mathrm{e}^{(\lambda + a - b)t} P_{ij}(t) \mathrm{d}t.$$

If $(\xi, \mathbb{P}_i)$ is geometrically transient, then $\kappa \in (0, a)$ and $\lambda < b - a + \kappa$ implies $H_{ij}(\lambda) < \infty$. In particular if $a - \kappa < b$, the spectral radius of M satisfies $\Lambda > 0$ and, by Theorem 7, X presents local extinction a.s.
In summary, if $a - \kappa < b < a$ then the process presents local extinction a.s. but global extintion with probability less than one.

Example 12. Define $a(i) = \beta(i) d(i) + \beta(i) \int_0^\infty u \mathrm{n}(i, \mathrm{d}u)$. Suppose that there exists a constant $c > 0$ such that $b(i) - a(i) \geq c > 0$. Let $(\xi, \mathbb{P}_i)$ the associated irreducible chain in Lemma 6. Let $0 \leq \lambda < c$. By equation (5.16) we have

$$H_{ij}(\lambda) = \int_0^\infty \mathrm{e}^{\lambda t} \mathbb{E}_i \left[\delta_j(\xi_t) \exp\left\{ \int_0^t (a-b)(\xi_s) \mathrm{d}s \right\} \right] \mathrm{d}t \leq \int_0^\infty \mathrm{e}^{(\lambda - c)t} \mathrm{d}t < \infty.$$

Then, $\Lambda > 0$ and the process presents local extinction a.s.

Example 13. Suppose now that there exists a constant $c > 0$ such that $b(i) - a(i) \leq -c < 0$ and $(\xi, \mathbb{P}_i)$ is a recurrent Markov chain. Then, for $-c < \lambda < 0$,

$$H_{ij}(\lambda) = \int_0^\infty \mathrm{e}^{\lambda t} \mathbb{E}_i \left[\delta_j(\xi_t) \exp\left\{ \int_0^t (a-b)(\xi_s) \mathrm{d}s \right\} \right] \mathrm{d}t \geq \int_0^\infty P_{ij}(t) \mathrm{d}t = \infty.$$

It follows that $\Lambda < 0$. If

$$\sup_{i \in \mathbb{N}} \int_1^\infty (x \log x) \ell(i, \mathrm{d}x) + \sup_{i \in \mathbb{N}} \int_1^\infty (x \log x) \mathrm{n}(i, \mathrm{d}x) < \infty,$$

then the process presents local extinction in each bounded subset of $\mathbb{N}$ with probability less than one.

Appendix A

A.1 Lemmas of Chapter 1

The following theory and lemmas will be useful in the proof of Theorem 1.
Given a differentiable function f, we write

$$\Delta_x f(a) = f(x+a) - f(a) \qquad \text{and} \qquad D_x f(a) = \Delta_x f(a) - f'(a)x.$$

Let $(a_m, n \geq 1)$ a sequence of positive real numbers such that $a_0 = 1$, $a_m \downarrow 0$ and

$$\int_{a_m}^{a_{m-1}} z\mathrm{d}z = m$$

for each $m \in \mathbb{N}$. Let $x \mapsto \kappa_m(x)$ be a non-negative continuous function supported on (a_m, a_{m-1}) such that $\kappa_m(x) \leq 2(mx)^{-1}$ for every $x > 0$, and $\int_{a_m}^{a_{m-1}} \kappa_m(x)\mathrm{d}x = 1$. For $m \geq 0$, let us define

$$f_m(z) = \int_0^{|z|} \mathrm{d}y \int_0^y \kappa_m(x)\mathrm{d}x, \qquad z \in \mathbb{R}.$$

Observe that f_m is a non-decreasing sequence of functions that converges to the function $x \mapsto |x|$ as $m \uparrow \infty$. For all $a, x \in \mathbb{R}$, we have $|f'_m(a)| \leq 1$ and $|f_m(a+x) - f_m(a)| \leq |x|$. Moreover, by Taylor's expansion, we get

$$\left| D_x f_m(a) \right| \leq x^2 \int_0^1 \kappa_m(|a+xu|)(1-u)\mathrm{d}u \leq \frac{2}{m} x^2 \int_0^1 \frac{(1-u)}{|a+xu|}\mathrm{d}u.$$

The proof of the following lemma can be found in [71] (Lemma 3.1)

Lemma 9. *Suppose that $x \mapsto x + h(x,v)$ is non-decreasing for $v \in \mathcal{V}$. Then, for any $x \neq y \in \mathbb{R}$,*

$$D_{l(x,y,v)} f_m(x-y) \leq \frac{2}{m} \int_0^1 \frac{l(x,y,u)^2(1-u)}{|x-y=tl(x,y,v)|} d\mathrm{d}u \leq \frac{2l(x,y,v)^2}{m|x-y|},$$

where $l(x,y,v) = h(x,v) - h(y,v)$.

Pathwise uniqueness.

Suppose that the parameters $(b, (\sigma_k)_{k\in K}, (h_i)_{i\in I}, (g_j)_{j\in J})$ are admissible and satisfies conditions a), b) and c).

Lemma 10. *The pathwise uniqueness holds for the positive solutions of*

$$Z_t^{(n)} = Z_0 + \int_0^t b(Z_s^{(n)} \wedge n)\mathrm{d}s + \sum_{k\in K}\int_0^t \sigma_k(Z_s^{(n)} \wedge n)\mathrm{d}B_s^{(k)} + \sum_{i\in I}\int_0^t\int_{W_i}\big(g_i(Z_{s-}^{(n)}\wedge n, u_i)\wedge n\big)M_i(\mathrm{d}s,\mathrm{d}u_i) + \sum_{j\in J}\int_0^t\int_{V_j}\big(h_j(Z_{s-}^{(n)}\wedge n, v_j)\wedge n\big)\widetilde{N}_j(\mathrm{d}s,\mathrm{d}v_j), \tag{A.1}$$

for every $n \in \mathbb{N}$.

Proof. We consider Z_t and Z'_t two solutions of (A.1) and let $Y_t = Z_t - Z'_t$. Therefore, Y_t satisfies the SDE

$$Y_t = Y_0 + \int_0^t \Big(b(Z_s\wedge n) - b(Z'_s\wedge n)\Big)\mathrm{d}s + \sum_{k\in K}\int_0^t\Big(\sigma_k(Z_s\wedge n) - \sigma_k(Z'_s\wedge n)\Big)\mathrm{d}B_s^{(k)} + \sum_{i\in I}\int_0^t\int_{W_i}\widetilde{g}_i^{(n)}(Z_{s-}, Z'_{s-}, u_i)M_i(\mathrm{d}s,\mathrm{d}u_i) + \sum_{j\in J}\int_0^t\int_{V_j}\widetilde{h}_j^{(n)}(Z_{s-}, Z'_{s-}, v_j)\wedge n\Big)\widetilde{N}_j(\mathrm{d}s,\mathrm{d}v_j),$$

with

$$\widetilde{g}_i^{(n)}(x,y,u_i) = g_i(x,u_i)\wedge n - g_i(y\wedge n, u_i)\wedge n,$$
$$\widetilde{h}_j^{(n)}(x,y,v_j) = h_j(x\wedge n, v_j)\wedge n - h_j(y\wedge n, v_j)\wedge n.$$

By applying Itô's formula to the functions defined at the beginning of this section, we deduce

$$f_m(Y_t) = f_m(Y_0) + M_t + \int_0^t f'_m(Y_s)\Big(b(Z_s\wedge n) - b(Z'_s\wedge n)\Big)\mathrm{d}s + \sum_{k\in K}\frac{1}{2}\int_0^t f''_m(Y_s)\Big(\sigma_k(Z_s\wedge n) - \sigma_k(Z'_s\wedge n)\Big)^2\mathrm{d}s + \sum_{i\in I}\int_0^t\int_{W_i}\Big(f_m(Y_{s-} + \widetilde{g}_i^{(n)}(Z_{s-}, Z'_{s-}, u_i)) - f_m(Y_{s-})\Big)\mu_i(\mathrm{d}u_i)\mathrm{d}s + \sum_{j\in J}\int_0^t\int_{V_j}\Big(f_m(Y_{s-} + \widetilde{h}_i^{(n)}(Z_{s-}, Z'_{s-}, u_i)) - f_m(Y_{s-}) - f'_m(Y_{s-})\widetilde{h}_i^{(n)}(Z_{s-}, Z'_{s-}, u_i)\Big)\nu_j(\mathrm{d}v_j)\mathrm{d}s, \tag{A.2}$$

where M_t is a martingale term. Using that $b = b_1 - b_2$ with b_2 a non-decreasing function and condition (b), we have for

$$|f'_m(x-y)||b(x\wedge n) - b(y\wedge n)| \le |b_1(x\wedge n) - b_1(y\wedge n)| \le r_n(|x-y|\wedge n), \tag{A.3}$$

and

$$\sum_{i\in I}\int_0^t\int_{W_i}\Delta_{\widetilde{g}_i^{(n)}(x,y,u_i)}f_m(x-y)\mu_i(\mathrm{d}u_i) \le \sum_{i\in I}\int_{W_i}|\widetilde{g}_i^{(n)}(x,y,u_i)|\mu_i(\mathrm{d}u_i) \le r_n(|x-y|\wedge n). \tag{A.4}$$

Since $x \mapsto x + h_j(x \wedge n, v) \wedge n$ is non decreasing for all $j \in J$, by Lemma 9 and condition (c), we have

$$\sum_{j\in J}\int_0^t \int_{V_j} D_{\widetilde{h}_j^{(n)}(x,y,v_j)} f_m(x-y)\nu_j(\mathrm{d}v_j) \leq \sum_{j\in J}\int_0^t \int_{V_j} \frac{2l_j(x,y,v)^2}{m|x-y|}\nu_j(\mathrm{d}v_j) \leq \frac{2B_n}{m}, \tag{A.5}$$

and

$$\sum_{k\in K} f''_m(x-y)(\sigma_k(x)-\sigma_k(y))^2 \leq \sum_{k\in K} \kappa_m(x-y)|\sigma_k(x)^2-\sigma_k(y)^2| \leq \frac{2B_n}{m}, \tag{A.6}$$

Next, we take expectation in equation (A.2), by (A.3, A.4, A.5 and A.6) we obtain

$$\mathbb{E}\left[f_m(Y_t)\right] \leq \mathbb{E}\left[f_m(Y_0)\right] + 2\int_0^t \mathbb{E}\left[r_n(|Y_s| \wedge n)\right]\mathrm{d}s + 4m^{-1}B_n.$$

Since $f_m(z) \to |z|$ increasingly as $m \to \infty$, we have

$$\mathbb{E}\left[|Y_t|\right] \leq \mathbb{E}\left[|Y_0|\right] + 2\int_0^t \mathbb{E}\left[r_n(|Y_s| \wedge n)\right]\mathrm{d}s.$$

Finally from Gronwall's inequality, we can deduce that pathwise uniqueness holds for the positive solutions of (A.1) for every $n \in \mathbb{N}$. □

Growth spaces

Suppose that the parameters $(b, (\sigma_k)_{k\in K}, (h_i)_{i\in I}, (g_j)_{j\in J})$ are admissible and satisfies conditions a), b) and c). Additionally suppose that $\mu_i(U_i \setminus W_i) < \infty$, for all $i \in I$. Our next result shows that if there is a unique strong solution to (A.1), we can replace the spaces $(W_i)_{i\in I}$ by $(U_i)_{i\in I}$ on the SDE and the unique strong solution still exists for the extended SDE. Its proof follows from similar arguments as those used in Proposition 2.2 in [45] but we provide its proof for sake of completeness.

Lemma 11. *If there is a unique strong solution to (A.1) and $\mu_i(U_i \setminus W_i) < \infty$ for all $i \in I$, then there is also a strong solution to*

$$\begin{aligned} Z_t^{(n)} = Z_0 &+ \int_0^t b(Z_s^{(n)} \wedge n)\mathrm{d}s + \sum_{k\in K}\int_0^t \sigma_k(Z_s^{(n)} \wedge n)\mathrm{d}B_s^{(k)} \\ &+ \sum_{i\in I}\int_0^t \int_{U_i} \big(g_i(Z_{s-}^{(n)} \wedge n, u_i) \wedge n\big) M_i(\mathrm{d}s, \mathrm{d}u_i) \\ &+ \sum_{j\in J}\int_0^t \int_{V_j} \big(h_j(Z_{s-}^{(n)} \wedge n, v_j) \wedge n\big) \widetilde{N}_j(\mathrm{d}s, \mathrm{d}v_j). \end{aligned} \tag{A.7}$$

Proof. It is enough to prove the result when $0 < \sum_{i\in I} \mu_i(U_i \setminus W_i) < \infty$. Suppose that (A.1) has a strong solution $(X_0(t), t \geq 0)$. Let $(S_r)_{r\geq 1}$ be the set of jump times of the Poisson process

$$t \mapsto \sum_{i\in I}\int_0^t \int_{U_i\setminus W_i} M_i(\mathrm{d}s, \mathrm{d}u_i).$$

Note that $S_r \to \infty$ as $r \to \infty$. By induction, we define the following process: for $0 \le t < S_1$, let $Y_t = X_0(t)$. Suppose that Y_t has been defined for $0 \le t < S_r$ and let

$$A = Y_{S_r-} + \sum_{i\in I} \int_{\{S_r\}} \int_{U_i \setminus W_i} \big(g_i(Y_{S_r-} \wedge n, u_i) \wedge n\big) M_i(\mathrm{d}s, \mathrm{d}u_i). \tag{A.8}$$

By our assumptions, there is also a strong solution $(X_r(t), t \ge 0)$ to

$$\begin{aligned} X_r(t) = A &+ \int_0^t b(X_r(s) \wedge n)\mathrm{d}s + \sum_{k\in K} \int_0^t \sigma_k(X_r(s) \wedge n)\mathrm{d}B^{(k)}_{S_r+s} \\ &+ \sum_{i\in I} \int_0^t \int_{W_i} \big(g_i(X_r(s-) \wedge n, u_i) \wedge n\big) M_i(S_r + \mathrm{d}s, \mathrm{d}u_i) \\ &+ \sum_{j\in J} \int_0^t \int_{V_j} \big(h_j(X_r(s-) \wedge n, v_j) \wedge n\big) \widetilde{N}_j(S_r + \mathrm{d}s, \mathrm{d}v_j). \end{aligned} \tag{A.9}$$

For $S_r \le t < S_{r+1}$ we set $Y_t = X_r(t - S_r)$. Then, Y_t is a strong solution to (A.7). On the other hand, if $(Y_t, t \ge 0)$ is a solution of (A.7), then it satisfies (A.1) for $0 \le t < S_1$ and the process $(Y_{S_k+t}, t \ge 0)$ satisfies (A.9) for $0 \le t < S_{r+1} - S_r$ with A given by (A.8). Then, the uniqueness for (A.7) follows from the uniqueness for (A.1) and (A.9). □

Tight sequence

Recall that for each $n, m \in \mathbb{N}$, the process $Z_t^{(n,m)}$ was defined as the unique non-negative strong solution to

$$\begin{aligned} Z_t^{(n,m)} = Z_0 &+ \int_0^t b(Z_s^{(n,m)} \wedge n)\mathrm{d}s + \sum_{k\in K} \int_0^t \sigma_k(Z_s^{(n,m)} \wedge n)\mathrm{d}B^{(k)}_s \\ &+ \sum_{i\in I} \int_0^t \int_{W_i^m} \big(g_i(Z_{s-}^{(n,m)} \wedge n, u_i) \wedge n\big) M_i(\mathrm{d}s, \mathrm{d}u_i) \\ &+ \sum_{j\in J} \int_0^t \int_{V_j^m} \big(h_j(Z_{s-}^{(n,m)} \wedge n, v_j) \wedge n\big) \widetilde{N}_j(\mathrm{d}s, \mathrm{d}v_j). \end{aligned} \tag{A.10}$$

Following the proof of Lemma 4.3 in [45] we will prove that for each $n \in \mathbb{N}$, the sequence $\{Z_t^{n,m} : t \ge 0\}$ is tight in $D([0,\infty), \mathbb{R}_+)$.

Lemma 12. *For each $n \in \mathbb{N}$ the sequence $\left\{Z_t^{(n,m)} : t \ge 0\right\}$ given by (A.10) is tight in the Skorokhod space $D([0,\infty), \mathbb{R}_+)$.*

Proof. First observe that since b is a continuous function in $[0, n]$ and by hypothesis b) and c), there exist a constant $K_n > 0$ such that for each $x \le n$

$$\begin{aligned} b(x) &+ \sum_{i\in I} \int_{W_i} |g_i(x, u_i) \wedge n| \mu_i(\mathrm{d}u_i) + \sum_{i\in I} \int_{W_i} |g_i(x, u_i) \wedge n|^2 \mu_i(\mathrm{d}u_i) \\ &+ \sum_{k\in K} \sigma_k^2(x) + \sum_{j\in J} \int_{V_j} |h_j(x, v_j) \wedge n|^2 \nu_j(\mathrm{d}v_j) \le K_n. \end{aligned} \tag{A.11}$$

Note that if C_n is the maximum of b in $[0,n]$, then $K_n = C_n + nB_n + (n+1)r_n(n)$. By applying Doob's inequality to the martingale terms in (A.10), we have

$$
\begin{aligned}
\mathbb{E}\left[\sup_{s\le t}(Z_s^{(n,m)})^2\right] \le &(2+2|I|+|J|+|K|)^2\left((Z_0)^2 + \mathbb{E}\left[\left(\int_0^t b(Z_s^{(n,m)}\wedge n)\mathrm{d}s\right)^2\right]\right.\\
&+\sum_{i\in I}\mathbb{E}\left[\left(\int_0^t\int_{W_i}|g(Z_s^{(n,m)}\wedge n,u_i)\wedge n|\mu_i(\mathrm{d}u_i)\right)^2\right]\\
&+4\left(\sum_{k\in K}\int_0^t \sigma_k^2(Z_s^{(n,m)}\wedge n)\mathrm{d}s + \sum_{i\in I}\int_0^t\int_{W_i}|g_i(Z_s^{(n,m)}\wedge n,u_i)\wedge n|^2\mu_i(\mathrm{d}u_i)\right.\\
&\left.\left.+\sum_{j\in J}\int_0^t\int_{V_j}|h_j(Z_s^{(n,m)}\wedge n,v_j)\wedge n|^2\nu_j(\mathrm{d}u_j)\right)\right).
\end{aligned}
$$

By (A.11), we obtain that

$$
t\mapsto \sup_{m\ge1}\mathbb{E}\left[\sup_{s\le t}(Z_s^{(n,m)})^2\right] \le (2+2|I|+|J|+|K|)^2\left((Z_0)^2 + (1+|I|)K_n^2t^2 + 4K_nt\right)
$$

is a function locally bounded. Then for every fixed $t\ge0$ the sequence of random variables $Z_t^{(n,m)}$ is tight. In a similar way, if $\{\tau_m : m\ge1\}$ is a sequence of stopping times bounded above by $T\ge0$, we have

$$
\mathbb{E}\left[|Z_{\tau_m+t}^{(n,m)} - Z_{\tau_m}^{(n,m)}|^2\right] \le (2+2|I|+|J|+|K|)^2\left((1+|I|)K_n^2t^2+4K_nt\right)
$$

Consequently, as $t\to0$

$$
\sup_{m\ge1}\mathbb{E}\left[|Z_{\tau_m+t}^{(n,m)} - Z_{\tau_m}^{(n,m)}|^2\right]\to 0.
$$

By Aldous' criterion [4], for all $n\in\mathbb{N}$, $\left\{Z_t^{(n,m)} : t\ge0\right\}$ is tight in the Skorokhod space $D([0,\infty),\mathbb{R}_+)$.

□

Martingale problem

For each $n,m\in\mathbb{N}$, $x\ge0$ and $f\in C^2(\mathbb{R})$ we define

$$
\begin{aligned}
\mathcal{L}^{(n)}f(x) =&b(x\wedge n)f'(x) + \frac{1}{2}f''(x)\sum_{k\in K}\sigma_k^2(x\wedge n) + \sum_{i\in I}\int_{W_i}\Big(f(x+g_i(x\wedge n,u_i)\wedge n) - f(x)\Big)\mu_i(\mathrm{d}u_i)\\
&+\sum_{j\in J}\int_{V_j}\Big(f(x+h_j(x\wedge n,v_j)\wedge n) - f(x) - f'(x)(h_j(x\wedge n,v_j)\wedge n)\Big)\nu_j(\mathrm{d}v_j).
\end{aligned}
\tag{A.12}
$$

and

$$
\begin{aligned}
\mathcal{L}^{(n,m)}f(x) =&f'(x)b(x\wedge n) + \frac{1}{2}f''(x)\sum_{k\in K}\sigma_k^2(x\wedge n) + \sum_{i\in I}\int_{W_i^m}\Big(f(x+g_i(x\wedge n,u_i)\wedge n) - f(x)\Big)\mu_i(\mathrm{d}u_i)\\
&+\sum_{j\in J}\int_{V_j^m}\Big(f(x+h_j(x\wedge n,v_j)\wedge n) - f(x) - f'(x)(h_j(x\wedge n,v_j)\wedge n)\Big)\nu_j(\mathrm{d}v_j).
\end{aligned}
\tag{A.13}
$$

In this section we prove the existence of the weak solution of a SDE by considering the corresponding martingale problem.

Lemma 13. *A cádlág process* $\{Z_t^{(n)} : t \geq 0\}$ *is a weak solution of (A.1) if and only if for every* $f \in C^2(\mathbb{R})$,

$$f(Z_t^{(n)}) - f(Z_0^{(n)}) - \int_0^t \mathcal{L}^{(n)} f(Z_s^{(n)}) \mathrm{d}s \tag{A.14}$$

is a locally bounded martingale. And a cádlág process $\{Z_t^{(n,m)} : t \geq 0\}$ *is a weak solution of (A.10) if and only if for every* $f \in C^2(\mathbb{R})$,

$$f(Z_t^{(n,m)}) - f(Z_0^{(n,m)}) - \int_0^t \mathcal{L}^{(n,m)} f(Z_s^{(n,m)}) \mathrm{d}s \tag{A.15}$$

is a locally bounded martingale.

Proof. We will just prove the first statement. The second one is analogous. If $\{Z_t : t \geq 0\}$ is a solution of (A.1), by Itô's formula we can see that (A.14) is a locally bounded martingale. Conversely, suppose that (A.14) is is a locally bounded martingale for every $f \in C^2(\mathbb{R})$. By a stopping time argument, we have

$$Z_t = Z_0 + \int_0^t b(Z_t \wedge n) \mathrm{d}s + \sum_{i \in I} \int_0^t \int_{W_i} (g_i(Z_{s-} \wedge n, u_i) \wedge n) \mu_i(\mathrm{d}u_i) \mathrm{d}s + M_t \tag{A.16}$$

for a square-integrable martingale $\{M_t : t \geq 0\}$. Let $N(\mathrm{d}s, \mathrm{d}z)$ be the optional random measure on $[0, \infty) \times \mathbb{R}$ defines by

$$N(\mathrm{d}s, \mathrm{d}z) = \sum_{s>0} \mathbf{1}_{\{\Delta Z_s \neq 0\}} \delta_{(s, \Delta Z_s)}(\mathrm{d}s, \mathrm{d}z),$$

with $\Delta Z_s = Z_s - Z_{s-}$. Denote by $\hat{N}$ its predictable compensator and $\tilde{N}$ the compensated random variable. Then

$$Z_t = Z_0 + \int_0^t b(Z_t \wedge n) \mathrm{d}s + \sum_{i \in I} \int_0^t \int_{W_i} (g_i(Z_{s-} \wedge n, u_i) \wedge n) \mu_i(\mathrm{d}u_i) \mathrm{d}s + M_t^c + M_t^d \tag{A.17}$$

where $\{M_t^c : t \geq 0\}$ is a continuous martingale and

$$M_t^d = \int_0^t \int_R z \tilde{N}(ds, dz)$$

is a purely discontinuous martingale. (See [29] p. 276). Let denote by C_t the quadratic variation process of M_t^c. By Itô's formula in the previous equation, we have

$$\begin{aligned} f(Z_t) =& f(Z_0) + \int_0^t f'(Z_s) b(Z_t \wedge n) \mathrm{d}s + \sum_{i \in I} \int_0^t \int_{W_i} f'(Z_s)(g_i(Z_{s-} \wedge n, u_i) \wedge n) \mu_i(\mathrm{d}u_i) \mathrm{d}s \\ &+ \frac{1}{2} \int_0^t f''(x) \mathrm{d}C_s + \int_0^t \int_{\mathbb{R}} D_z f(Z_{s-}) \hat{N}(\mathrm{d}s, \mathrm{d}z) + \text{martingale} \end{aligned} \tag{A.18}$$

The uniqueness of the canonical decomposition of semi-martingales, equations (A.14) and (A.18) implies $\mathrm{d}C_s = \underset{k\in K_k}{\sigma}{}^2(Z_s \wedge n)\mathrm{d}s$ and

$$\int_0^t \int_R F(s,z)\hat{N}(\mathrm{d}s,\mathrm{d}z) = \sum_{i\in I}\int_0^t \int_{W_i} F(s, g_i(Z_{s-}\wedge n, u_i)\wedge n), u_i)\mu_i(\mathrm{d}u_i)\mathrm{d}s + \sum_{j\in J}\int_0^t \int_{V_j} F(s, h_j(Z_{s-}\wedge n, v_j)\wedge n), v_j)\nu_j(\mathrm{d}v_j)\mathrm{d}s$$

for any non-negative Borel function F on $\mathbb{R}_+ \times \mathbb{R}$. Then we obtain A.1 by applying martingale representation theorems to (A.17). (See [57], Section II.7) □

Weak solutions

By lemma12, $\left\{Z_t^{(n,m)} : t \geq 0\right\}$ is tight in the Skorokhod space $D([0,\infty), \mathbb{R}_+)$. Then, there exists a subsequence $\left\{Z_t^{(n,m_k)} : t \geq 0\right\}$ that converges to some process $\{Z_t^{(n)} : t \geq 0\}$ in the Skorokhod sense. By the Skorokhod representation theorem, we may assume those processes defined in the same probability space and $\left\{Z_t^{(n,m_k)} : t \geq 0\right\}$ converges to $\{Z_t^{(n)} : t \geq 0\}$ almost surely in $D([0,\infty), \mathbb{R})$. Let $D(Z^{(n)}) = \left\{t > 0 : \mathbb{P}\left(Z_{t-}^{(n)} = Z_t^{(n)}\right) = 1\right\}$. Then, $[0,\infty) \setminus D(Z^{(n)})$ is at most numerable and $\lim_{k\to\infty} Z_t^{(n,m_k)} = Z_t^{(n)}$ almost surely for every $t \in D(Z^n)$. (see [43] p. 131). Therefore, in order to prove that the weak limit point $\{Z_t^{(n)}; t \geq 0\}$ is a weak solution of (A.1) it is enough to prove

Lemma 14. *If $Z_m \to Z$ as $m \to \infty$, then $\mathcal{L}^{(n,m)} f(Z_m) \to \mathcal{L}^{(n)} f(Z)$ as $m \to \infty$, for every $f \in C_b^2(\mathbb{R}_+)$. In particular, if $\{Z_t^{(n)} : t \geq 0\}$ is the weak limit point of $\left\{Z_t^{(n,m_k)} : t \geq 0\right\}$, then $\{Z_t^{(n)} : t \geq 0\}$ is a weak solution of (A.1). (or (1.5)).*

Proof. Let $M > 0$ a constant such that $|Z|, |Z_m| \leq M$, for all $m \in \mathbb{N}$. By condition b) and c). We have that for each k, the function

$$x \mapsto \sum_{i\in i}\int_{W_i\setminus W_i^k} (g_i(x\wedge n, u_i)\wedge n)\mu_i(\mathrm{d}u_i) + \sum_{j\in J}\int_{V_j\setminus V_j^k} (h_j(x\wedge n, v_j)\wedge n)^2\nu_j(\mathrm{d}v_j)$$

is continuous. By Dini's theorem, we know that, as $k \to \infty$,

$$\epsilon_k := \sup_{|x|\leq M}\int_{W_i\setminus W_i^k} (g_i(x\wedge n, u_i)\wedge n)^2\mu_i(\mathrm{d}u_i) + \sum_{j\in J}\int_{V_j\setminus V_j^k} (h_j(x\wedge n, v_j)\wedge n)\nu_j(\mathrm{d}v_j) \to 0.$$

Just in this proof and in order to simplify the notation, for a $f \in C(\mathbb{R})$, we will denote

$$\|f\| := \max\{|f(x)| : |x| \leq n + M\}.$$

By continuity, $\|f\| < \infty$. Observe that this norm depends on n and M but it doesn't affect the result of the proof.

For $m \geq k$ and $j \in J$, by applying the Mean Value Theorem several times we have

$$\begin{aligned}
&\left| \int_{V_j} D_{h(Z\wedge n, v_j)\wedge n} f(Z) \nu_j(\mathrm{d}v_j) - \int_{V_j^m} D_{h(Z_m \wedge n, v_j)\wedge n} f(Z_m) \nu_j(\mathrm{d}v_j) \right| \\
&\leq \|f''\| \epsilon_k + \int_{V_j^k} |D_{h(Z\wedge n, v_j)\wedge n} f(Z) - D_{h(Z_m\wedge n, v_j)\wedge n} f(Z_m)| \nu_j(\mathrm{d}v_j) \\
&\leq \|f''\| \epsilon_k + \int_{V_j^k} |f(Z) - f(Z_m)| \nu_j(\mathrm{d}v_j) \\
&\quad + \int_{V_j^k} |f(Z + h(Z\wedge n, v_j)\wedge n) - f(Z_m + h(Z_m \wedge n, v_j)\wedge n)| \nu_j(\mathrm{d}v_j) \\
&\quad + \int_{V_j^k} |f'(Z)(h(Z\wedge n, v_j)\wedge n)) - f'(Z_m)(h(Z_m\wedge n, v_j)\wedge n)| \nu_j(\mathrm{d}v_j) \\
&\leq \|f''\| \epsilon_k + \int_{V_j^k} |f(Z) - f(Z_m)| \nu_j(\mathrm{d}v_j) \\
&\quad + \|f'\| \int_{V_j^k} |Z + (h(Z\wedge n, v_j)\wedge n) - Z_m - (h(Z_m\wedge n, v_j)\wedge n)| \nu_j(\mathrm{d}v_j) \\
&\quad + \|f'\| \int_{V_j^k} |(h(Z\wedge n, v_j)\wedge n)) - (h(Z_m\wedge n, v_j)\wedge n)| \nu_j(\mathrm{d}v_j) \\
&\quad + \int_{V_j^k} |f'(Z) - f'(Z_m)| h(Z\wedge n, v_j)\wedge n| \nu_j(\mathrm{d}v_j).
\end{aligned}$$

Then, by Hölder inequality we have

$$\begin{aligned}
&\left| \int_{V_j} D_{h(Z\wedge n, v_j)\wedge n} f(Z) \nu_j(\mathrm{d}v_j) - \int_{V_j^m} D_{h(Z_m \wedge n, v_j)\wedge n} f(Z_m) \nu_j(\mathrm{d}v_j) \right| \\
&\leq \|f''\| \epsilon_k + |f(Z) - f(Z_m)| \nu_j(V_j^k) + \|f'\| |Z - Z_m| \nu_j(V_j^k) \\
&\quad + 2\|f'\| \int_{V_j^k} |(h(Z\wedge n, v_j)\wedge n) - (h(Z_m\wedge n, v_j)\wedge n)| \nu_j(\mathrm{d}v_j) \\
&\quad + |f'(Z) - f'(Z_m)| \left(\int_{V_j} |h(Z\wedge n, v_j)\wedge n|^2 \nu_j(\mathrm{d}v_j) \nu_j(V_j^k) \right)^{1/2}.
\end{aligned}$$

By letting $m \to \infty$ and $m \to \infty$, and using hypothesis c), we can prove

$$\lim_{m\to\infty} \int_{V_j^m} D_{h_j(Z_m\wedge n, v_j)\wedge n} f(Z_m) \nu_j(\mathrm{d}v_j) = \int_{V_j} D_{h_j(Z\wedge n, v_j)\wedge n} f(Z) \nu_j(\mathrm{d}v_j), \qquad \text{for } j \in J. \quad \text{(A.19)}$$

In a similar way, for each $m \geq k$ and $i \in J$

$$\begin{aligned}
&\left| \int_{W_i} \Delta_{g_i(Z\wedge n, u_i)\wedge n} f(Z) \mu_i(\mathrm{d}u_i) - \int_{W_i^m} \Delta_{g_i(Z_m\wedge n, u_i)\wedge n} f(Z_m) \mu_i(\mathrm{d}u_i) \right| \\
&\leq 2\|f'\| \epsilon_k + \|f'\| \int_{W_i} |(g_i(Z\wedge n, u_i)\wedge n) - (g_i(Z_m\wedge n, u_i)\wedge n)| \mu_i(\mathrm{d}u_i) \\
&\quad + \mu_i(W_i^k)(\|f'\| |Z_m - Z| + |f(Z_m) - f(z)|).
\end{aligned}$$

Now, by hypothesis b), when $m \to \infty$ and $k \to \infty$, we have

$$\lim_{m\to\infty} \int_{W_i^m} \Delta_{g_i(Z_m \wedge n, u_i)\wedge n} f(Z_m)\mu_i(\mathrm{d}u_i) = \int_{W_i} \Delta_{g_i(Z\wedge n, u_i)\wedge n} f(Z)\mu_i(\mathrm{d}u_i), \qquad \text{for } i \in I. \quad \text{(A.20)}$$

Therefore, by (A.19) and (A.20) it follows that $\mathcal{L}^{(n,m)} f(Z_m) \to \mathcal{L}^{(n)} f(Z)$ as $m \to \infty$.
Finally, if $\{Z_t^{(n,m)} : t \geq 0\}$ is a weak solution of (A.10). We know, by Lemma 13, that (A.15) is a locally bounded martingale. Since $\mathcal{L}^{(n,m)} f(Z_m) \to \mathcal{L}^{(n)} f(Z)$ as $m \to \infty$, by Dominate Convergence Theorem, (A.14) is a locally bounded martingale. By applying again Lemma 13, we obtain that $\{Z_t^{(n)} : t \geq 0\}$ is a weak solution of (A.1). □

Backward differential equation

The following result shows the a.s. existence and uniqueness of a solution of (1.12) and it is needed for the proof of Proposition 1.

Lemma 15. *Suppose that* $\int_{[1,\infty)} x\mu(\mathrm{d}x) < \infty$ *and let* $K = (K_t \geq 0)$ *be a Lévy process. Then for every* $\lambda \geq 0$, $v_t : s \in [0,t] \mapsto v_t(s,\lambda,K)$ *is the a.s. unique solution of the backward differential equation,*

$$\frac{\partial}{\partial s} v_t(s,\lambda,K) = e^{K_s}\psi_0(v_t(s,\lambda,K)e^{-\delta_s}), \qquad v_t(t,\lambda,K) = \lambda, \qquad \text{(A.21)}$$

where

$$\psi_0(\theta) = \psi(\theta) - \theta\psi'(0) = \gamma^2\theta^2 + \int_{(0,\infty)} \big(e^{-\theta x} - 1 + \theta x\big)\mu(\mathrm{d}x), \qquad \theta \geq 0.$$

Proof. Our proof will use a convergence argument for Lévy processes. Let K be a Lévy process with characteristic (α, σ, π) where $\alpha \in \mathbb{R}$ is the drift term, $\sigma \geq 0$ is the Gaussian part and π is the so-called Lévy measure satisfying

$$\int_{\mathbb{R}\setminus\{0\}} (1 \wedge z^2)\pi(\mathrm{d}z) < \infty.$$

From the Lévy-Itô decomposition (see for instance [64]), the process K can be decomposed as the sum of three independent Lévy processes $X^{(1)}$ a Brownian motion with drift, $X^{(2)}$ a compound Poisson process and $X^{(3)}$ a square-integrable martingale with an a.s. countable number of jumps on each finite time interval with magnitude less than unity.

Let $B_\epsilon = (-1,-\epsilon) \cup (-\epsilon, 1)$ and M be a Poison random measure with characteristic measure $\mathrm{d}t\pi(\mathrm{d}x)$. Observe that the process

$$X_t^{(3,\epsilon)} = \int_{[0,t]} \int_{B_\epsilon} xM(\mathrm{d}s,\mathrm{d}x) - t\int_{B_\epsilon} x\pi(\mathrm{d}x), \qquad t \geq 0$$

is a martingale. According to Theorem 2.10 in [64], for any fixed $t \geq 0$, there exists a deterministic subsequence $(\epsilon_n)_{n\in\mathbb{N}}$ such that $(X_s^{3,\epsilon_n}, 0 \leq s \leq t)$ converges uniformly to $(X_s^3, 0 \leq s \leq t)$ with probability one. We now define

$$K_s^{(n)} = X_s^{(1)} + X_s^{(2)} + X_s^{(3,\epsilon_n)}, \qquad s \leq t.$$

In the sequel, we work on the space $\widetilde{\Omega}$ such that $K^{(n)}$ converges uniformly to K on $[0,t]$. Note that ψ_0 is locally Lipschitz and $K^{(n)}$ is a piecewise continuous function with a finite number of discontinuities. Hence from the Cauchy-Lipschitz Theorem, we can define a unique solution $v_t^n(\cdot,\lambda,K^{(n)})$ of the backward differential equation:

$$\frac{\partial}{\partial s}v_t^n(s,\lambda,K^{(n)}) = e^{K_s^{(n)}}\psi_0(v_t^n(s,\lambda,K^{(n)})e^{-K_s^{(n)}}), \qquad v_t^n(t,\lambda,K^{(n)}) = \lambda.$$

In order to prove our result, we show that the sequence $(v^n(s) := v_t^n(s,\lambda,K^{(n)}),\ s \leq t)_{n\in\mathbb{N}}$ converges to a unique solution of (A.21) on $\widetilde{\Omega}$. With this purpose in mind, we define

$$S = \sup_{s\in[0,t],\ n\in\mathbb{N}} \left\{ e^{K_s^{(n)}}, e^{-K_s^{(n)}}, e^{K_s}, e^{-K_s} \right\}, \tag{A.22}$$

which turns out to be finite from the uniform convergence of $K^{(n)}$ to K. Since $\psi_0 \geq 0$, we necessarily have that v^n is increasing and moreover for every $n \in \mathbb{N}$,

$$v^n(s) \leq \lambda \qquad \text{for} \quad s \leq t. \tag{A.23}$$

On the other hand, since ψ_0 is a convex and increasing, we deduce that for any $0 \leq \zeta \leq \eta \leq \lambda S$, the following inequality holds

$$0 \leq \frac{\psi_0(\eta) - \psi_0(\zeta)}{\eta - \zeta} \leq \psi_0'(\eta) \leq \psi_0'(\lambda S) =: C. \tag{A.24}$$

For simplicity, we denote for all $v \geq 0$,

$$\psi^n(s,v) = e^{K_s^{(n)}}\psi_0(ve^{-K_s^{(n)}}) \quad \text{and} \quad \psi^\infty(s,v) = e^{K_s}\psi_0(ve^{-K_s}).$$

We then observe that for any $0 \leq s \leq t$ and $n,m \in \mathbb{N}$, we get

$$\begin{aligned} |v^n(s) - v^m(s)| &= \left| \int_s^t \psi^n(u,v^n(u))\mathrm{d}u - \int_s^t \psi^m(u,v^m(u))\mathrm{d}u \right| \\ &\leq \int_s^t (R^n(u) + R^m(u))\mathrm{d}u + \int_s^t |\psi^\infty(u,v^n(u)) - \psi^\infty(u,v^m(u))|\mathrm{d}u, \end{aligned}$$

where for any $u \in [0,t]$,

$$\begin{aligned} R^n(u) :=& |\psi^n(u,v^n(u)) - \psi^\infty(u,v^n(u))| \\ &\leq e^{K_u^{(n)}}|\psi_0(v^n(u)e^{-K_u^{(n)}}) - \psi_0(v^n(u)e^{-K_u})| + \psi_0(v^n(u)e^{-K_u})|e^{K_u^{(n)}} - e^{K_u}|. \end{aligned}$$

Next, using (A.22), (A.23) and (A.24), we deduce

$$\begin{aligned} R^n(u) &\leq SC\lambda|e^{-K_u^{(n)}} - e^{-K_u}| + \psi_0(S\lambda)|e^{K_u^{(n)}} - e^{K_u}| \\ &\leq (SC\lambda + S\psi_0(S\lambda)) \sup_{u\in[0,t]} \left\{ |e^{K_u^{(n)}} - e^{K_u}|, |e^{-K_u^{(n)}} - e^{-K_u}| \right\} =: s_n. \end{aligned}$$

From similar arguments, we obtain

$$|\psi^\infty(u,v^n(u)) - \psi^\infty(u,v^m(u))| \leq C|v^n(u) - v^m(u)|.$$

Therefore,

$$|v^n(s) - v^m(s)| \leq R_{n,m}(s) + C\int_s^t |v^n(u) - v^m(u)|\mathrm{d}u,$$

where

$$R_{n,m}(s) = \int_s^t (R^n(u) + R^m(u))\mathrm{d}u.$$

Gronwall's lemma yields that for all $0 \leq s \leq t$,

$$|v^n(s) - v^m(s)| \leq R_{n,m}(s) + C\int_s^t R_{n,m}(u)e^{C(u-s)}\mathrm{d}u.$$

Now, recalling that $R^n(u) \leq s_n$ and $R_{n,m}(u) \leq (s_n + s_m)t$, we get that for every $N \in \mathbb{N}$,

$$\sup_{n,m\geq N, s\in[0,t]} |v^n(s) - v^m(s)| \leq te^t \sup_{n,m\geq N}(s_n + s_m).$$

Moreover since $s_n \to 0$, we deduce that $(v^n(s), s \leq t)_{n\in\mathbb{N}}$ is a Cauchy sequence under the uniform norm on $\widetilde{\Omega}$. In other words, for any $\omega \in \widetilde{\Omega}$ there exists a continuous function v^* on $[0,t]$ such that $v^n \to v^*$ as n goes to ∞. We define the function $v : \Omega \times [0,t] \to [0,\infty]$ as follows

$$v(s) = \begin{cases} v^*(s) & \text{if } \omega \in \widetilde{\Omega}, \\ 0 & \text{elsewhere.} \end{cases}$$

The following argument proves that v in $\widetilde{\Omega}$ is solution to (A.21). More precisely, let $s \in [0,t]$ and $n \in \mathbb{N}$, then

$$\begin{aligned} \left| v(s) - \int_s^t \psi^\infty(s, v(s))\mathrm{d}s - \lambda \right| \leq & |v(s) - v^n(s)| + \int_s^t |\psi^n(s, v(s)) - \psi^n(s, v^n(s))|\mathrm{d}s \\ & + \int_s^t |\psi^\infty(s, v(s)) - \psi^n(s, v(s))|\mathrm{d}s \\ \leq & (1 + Ct) \sup_{s\in[0,t]} \{|v(s) - v^n(s)|\} + ts_n. \end{aligned}$$

By letting $n \to \infty$, we obtain our claim. The uniqueness of the solution of (A.21) follows from Gronwall's lemma. The proof is now complete. □

A.2 Lemmas of Chapter 3

Now, we recall two technical Lemmas whose proof can be found in [9]:

Lemma 16. *Assume that F satisfies one of the Assumptions (**A1**) or (**A2**). Then there exist two positive finite constants η and M such that for all (x,y) in $\mathbb{R}_+^2$ and ε in $[0,\eta]$,*

$$\begin{aligned} \left| F(x) - Ax^{-p} \right| &\leq Mx^{-(1+\varepsilon)p}, \\ \left| F(x) - F(y) \right| &\leq M\left| x^{-p} - y^{-p} \right|. \end{aligned}$$

Lemma 17. *Assume that the non-negative sequences* $(a_{n,q})_{(n,q)\in\mathbb{N}^2}$, $(a'_{n,q})_{(n,q)\in\mathbb{N}^2}$ *and* $(b_n)_{n\in\mathbb{N}}$ *satisfy for every* $(n,q)\in\mathbb{N}^2$*:*

$$a_{n,q} \le b_n \le a'_{n,q},$$

and that there exist three sequences $(a(q))_{q\in\mathbb{N}}$, $(c^-(q))_{q\in\mathbb{N}}$ *and* $(c^+(q)_{q\in\mathbb{N}}$ *such that*

$$\lim_{n\to\infty} a_{n,q} = c^-(q)a(q), \quad \lim_{n\to\infty} a'_{n,q} = c^+(q)a(q), \quad \textit{and} \quad \lim_{q\to\infty} c^-(q) = \lim_{q\to\infty} c^+(q) = 1.$$

Then there exists a non-negative constant a *such that*

$$\lim_{q\to\infty} a(q) = \lim_{n\to\infty} b_n = a.$$

A.3 Lemmas of Chapter 4

Lemma 18. *Let* $\eta \in \mathbb{R}$ *and* $p \ge 0$*. Then for every* $t > 0$*, we have*

$$\begin{aligned} i) \qquad & \mathbb{E}\left[\left(I_t^{(\eta)}\right)^{-p}\right] = e^{(2p^2-2p\eta)t}\mathbb{E}\left[\left(I_t^{(-(\eta-2p))}\right)^{-p}\right], \\ ii) \qquad & \mathbb{E}\left[\left(I_t^{(\eta)}\right)^{-2p}\right] \le e^{(2p^2-2p\eta)t}\mathbb{E}\left[\left(I_{t/2}^{(-(\eta-2p))}\right)^{-p}\right]\mathbb{E}\left[\left(I_{t/2}^{((\eta-2p))}\right)^{-p}\right]. \end{aligned}$$

Proof. Using the time reversal property for Brownian motion, we observe that the process $(\eta t + B_t - \eta(t-s) - B_{t-s}, 0 \le s \le t)$ has the same law as $(\eta s + B_s, 0 \le s \le t)$. Then, we deduce that

$$\int_0^t e^{2(\eta s + B_s)}\mathrm{d}s \quad \text{has the same law as} \quad e^{2(\eta t + B_t)}\int_0^t e^{-2(\eta s + B_s)}\mathrm{d}s.$$

Recall that the Esscher transform 3.3 for a Brownian motion is given by

$$\left.\frac{\mathrm{d}\mathbb{P}^{(\lambda)}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = e^{\lambda B_t - \frac{\lambda^2}{2}t}, \qquad \text{for} \quad \lambda \in \mathbb{R},$$

And, that under $\mathbb{P}^{(\lambda)}$, the process B is a Brownian motion with drift λ. Hence, taking $\lambda = -2p$, we deduce

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^t e^{2(\eta s + B_s)}\mathrm{d}s\right)^{-p}\right] &= \mathbb{E}\left[e^{-2p(\eta t + B_t)}\left(\int_0^t e^{-2(\eta s + B_s)}\mathrm{d}s\right)^{-p}\right] \\ &= e^{-2p\eta t}e^{2p^2 t}\mathbb{E}^{(-2p)}\left[\left(\int_0^t e^{-2(\eta s + B_s)}\mathrm{d}s\right)^{-p}\right] \\ &= e^{-2p\eta t}e^{2p^2 t}\mathbb{E}\left[\left(\int_0^t e^{-2((\eta-2p)s + B_s)}\mathrm{d}s\right)^{-p}\right], \end{aligned}$$

which implies the first identity, thanks to the symmetry property of Brownian motion.

In order to get the second identity, we observe

$$\int_0^t e^{2(\eta s + B_s)}\mathrm{d}s = \int_0^{t/2} e^{2(\eta s + B_s)}\mathrm{d}s + e^{\eta t + 2B_{t/2}}\int_0^{t/2} e^{2(\eta s + \tilde{B}_s)}\mathrm{d}s,$$

where $\tilde{B}_s = B_{s+t/2} - B_{t/2}$, $s \geq 0$, is a Brownian motion which is independent of $(B_u, 0 \leq u \leq t/2)$. Therefore, using part (i), we deduce

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t e^{2(\eta s+B_s)}\mathrm{ds}\right)^{-2p}\right] &\leq \mathbb{E}\left[\left(e^{\eta t+2B_{t/2}}\int_0^{t/2} e^{2(\eta s+B_s)}\mathrm{ds}\right)^{-p}\right]\mathbb{E}\left[\left(\int_0^{t/2} e^{2(\eta s+B_s)}\mathrm{ds}\right)^{-p}\right]\\ &\leq e^{(p^2-\eta p)t}\mathbb{E}\left[\left(e^{\eta t+2B_{t/2}}\int_0^{t/2} e^{2(\eta s+B_s)}\mathrm{ds}\right)^{-p}\right]\mathbb{E}\left[\left(I_{t/2}^{(-(\eta-2p))}\right)^{-p}\right].\end{aligned}$$

On the other hand from the Esscher transform with $\lambda = -2p$, we get

$$\begin{aligned}\mathbb{E}\left[\left(e^{\eta t+2B_{t/2}}\int_0^{t/2} e^{2(\eta s+B_s)}\mathrm{ds}\right)^{-p}\right] &= e^{-p\eta t}e^{p^2 t}\mathbb{E}^{(-2p)}\left[\left(\int_0^{t/2} e^{2(\eta s+B_s)}\mathrm{ds}\right)^{-p}\right]\\ &= e^{-p\eta t}e^{p^2 t}\mathbb{E}\left[\left(\int_0^{t/2} e^{2((\eta-2p)s+B_s)}\mathrm{ds}\right)^{-p}\right].\end{aligned}$$

Putting all the pieces together, we deduce

$$\mathbb{E}\left[\left(\int_0^t e^{2(\eta s+B_s)}\mathrm{ds}\right)^{-2p}\right] \leq e^{(2p^2-\eta 2p)t}\mathbb{E}\left[\left(I_{t/2}^{((\eta-2p))}\right)^{-p}\right]\mathbb{E}\left[\left(I_{t/2}^{(-(\eta-2p))}\right)^{-p}\right].$$

This completes the proof. □

A.4 Lemmas of Chapter 5

Lemma 5 is the analogue of a result proved for linear semigroups of MGW processes. Now, we provide a proof.

Lemma 5. *If, for some λ, $H_{ij}(\lambda) < \infty$ for a pair i, j, then $H_{ij}(\lambda) < \infty$ for all $i, j \in \mathbb{N}$. In particular, the parameter*

$$\Lambda_{ij} = \sup\{\lambda \geq \infty : H_{ij}(\lambda) < \infty\},$$

does not depend on i and j. The common value, $\Lambda = \Lambda_{ij}$, is called the spectral radius of M.

Proof. Since $\mathbf{M}(t)$ is irreducible, for each $i, j \in \mathbb{N}$ there exists t_0 and t_1 such that $M(t_0)_{ij}$ and $M(t_1)_{ji}$ are positive. By applying the semigroup property, we get that

$$\begin{aligned}M(t+t_0)_{ij} &\geq M(t_0)_{ij}M(t)_{jj},\\ M(t+t_1)_{jj} &\geq M(t_0)_{ij}M(t)_{jj}.\end{aligned}$$

The first inequality implies that

$$\Lambda_{ij} \leq \Lambda_{jj},$$

while the second implies

$$\Lambda_{jj} \leq \Lambda_{ij}.$$

Thus we have that $\Lambda_{ij} = \Lambda_{jj}$ for all $i, j \in \mathbb{N}$. In a similar way we can prove that $\Lambda_{ij} = \Lambda_{ii}$ for all $i, j \in \mathbb{N}$. □

We provide here a technical lemma pertaining to an extended version of the Feynman-Kac formula that is used in the main body of Chapter 5. Note that similar formulae have previously appeared in the literature e.g. in the work of Chen and Song [26].

Lemma 19. *Let $(\xi_t, \mathbb{P})$ be a Markov chain on a finite state space E with Q matrix $Q = (q_{ij})_{i,j\in E}$. Let $v : E \times \mathbb{R}_+ \to \mathbb{R}$ be a measurable function and $F : E \times E \times \mathbb{R}_+ \to \mathbb{R}$ be a Borel function vanishing on the diagonal of E. For $i \in E$ and $t \geq 0$ and $f : E \to \mathbb{R}$, define*

$$h(i,t) := \mathcal{T}_t[f](i) = \mathbb{E}_i\left[f(\xi_t)\mathbb{E}\left[\int_0^t v(\xi_s, t-s)\mathrm{d}s\right]\mathbb{E}\left[\sum_{s\leq t} F(\xi_{s-}, \xi_s, t-s)\right]\right].$$

Then $\mathcal{T}_t$ is a semigroup and for each $(i,t) \in E \times \mathbb{R}_+$, h satisfies

$$\begin{aligned} h(i,t) =& \mathbb{E}_i\left[f(\xi_t)\right] + \mathbb{E}_i\left[\int_0^t h(\xi_s, t-s)v(\xi_s, t-s)\mathrm{d}s\right] \\ &+ \mathbb{E}_i\left[\int_0^t \sum_{j\in E} h(j, t-s)(\mathrm{e}^{F(\xi_s, j, t-s)} - 1)q_{\xi_s, j}\mathrm{d}s\right]. \end{aligned} \tag{A.25}$$

Moreover, if v and F do not depend on t, the semigroup has infinitesimal matrix $\mathbf{P}$ given by,

$$p_{ij} = q_{ij}\mathrm{e}^{F(i,j)} + v(i)\mathbf{1}_{\{i=j\}}. \tag{A.26}$$

Proof. The Markov property implies the semigroup property. For each $0 \leq s \leq t$ define

$$A_{s,t} := \int_s^t v(\xi_r, t-r)\mathrm{d}r \sum_{s<r\leq t} F(\xi_{r-}, \xi_r, t-r).$$

Then,

$$\mathrm{e}^{A_{0,t}} - \mathrm{e}^{A_{t,t}} = \int_0^t v(\xi_s, t-s)\mathrm{e}^{A_{s,t}}\mathrm{d}s + \sum_{s\leq t} \mathrm{e}^{A_{s-,t}}(\mathrm{e}^{F(\xi_{s-}, \xi_s, t-s)} - 1).$$

This implies,

$$h(i,t) = \mathbb{E}_i\left[f(\xi_t)\right] + \mathbb{E}_i\left[\int_0^t f(\xi_t)v(\xi_s, t-s)\mathrm{e}^{A_{s,t}}\mathrm{d}s\right] + \mathbb{E}_i\left[\sum_{s\leq t} f(\xi_t)\mathrm{e}^{A_{s-,t}}(\mathrm{e}^{F(\xi_{s-}, \xi_s, t-s)} - 1)\right].$$

By the Markov property

$$h(i,t) = \mathbb{E}_i\left[f(\xi_t)\right] + \mathbb{E}_i\left[\int_0^t v(\xi_s, t-s)h(\xi_s, t-s)\mathrm{d}s\right] + \mathbb{E}_i\left[\sum_{s\leq t} h(\xi_s, t-s)(\mathrm{e}^{F(\xi_{s-}, \xi_s, t-s)} - 1)\right].$$

The Lévy formula says that for any non-negative Borel function G on $E \times E \times \mathbb{R}_+$ vanishing on the diagonal and any $i \in E$,

$$\mathbb{E}_i\left[\sum_{s\leq t} G(\xi_{s-}, \xi_s, s)\right] = \mathbb{E}_i\left[\int_0^t \sum_{y\in E} G(\xi_s, y, s)q_{\xi_s, y}\mathrm{d}s\right].$$

Therefore, h satisfies (A.25). Using this expression, we can obtain the infinitesimal matrix. □

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