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Homological Ideals of Finite Dimensional Algebras

T H E S I S

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Homological Ideals of Finite Dimensional Algebras

Contents

Introduction

We will be working with a finite dimensional connected and basic k-algebra A over an algebraically closed field k, all modules will be finite dimensional modules and all ideals will be two sided ideals unless stated otherwise.

In this dissertation we study some special ideals of an algebra, called "homological ideals", for which the corresponding quotient map induces a full and faithfull functor between derived categories. This ideals seem to have really interesting properties and they appear in a very big branch of algebras. They were introduced by Auslander, Platzeck and Todorov; they called them *strong idempotent ideals*, they have been studied also by Gatica, Lanzillota and Platzeck, and independently by Xu and Xi with some relations to the so called *finitistic dimension conjecture*.

The first interesting property stated in this thesis is Lemma 1, which has been a handy tool to identify homological epimorphisms (and hence homological ideals). We use this lemma to prove some other interesting properties of homological ideals such as obtaining new homological ideals in one-point (co)extension algebras from given homological ideals, and identifying some inclusion maps which are homological epimorphisms on quotient algebras.

Other result we give is a briefly view through the lattices of homological ideals of an algebra, the proof is very easy given Lemma 1, and it says that the correspondence of ideals stated in The Correspondence Theorem restricts to the sets of homological ideals when the ideal that we are quotienting is homological.

There is an interesting result due to Happel. It says that if M is an exceptional A-module, then the Hochschild cohomologies of A and $A[M]$ are isomorphic:

$$H^n(A) \cong H^n(A[M]) \qquad \forall \ n \geq 2$$

We have generalized this theorem, and weakend the hypothesis in just asking for a homological ideal I such that the category $mod(A/I)$ has an exceptional module. And even more, there are isomorphisms for all homological ideals $J \subset I$ of A:

$$H^n(A/J) \cong H^n\big((A/J)[M]\big)$$

The following results are original of this thesis: Proposition 2, Lemma 1, Theorem 2, Theorem 9 and Theorem 10.

We assume the reader is familiar with the standard language in representation theory of finite dimensional associative algebras and the tools of homological algebra.

1 Homological Ideals

1.1 First Properties

Definition 1. *A morphism of k-algebras $\varphi : A \to B$ is called an epimorphism if for all k-algebra morphisms $f,g : B \to C$ the fact that $f \circ \varphi = g \circ \varphi$ implies that $f = g$.*

Of course, surjective algebra morphisms are epimorphisms, such as quotient maps by a two sided ideal $\pi_I : A \to A/I$ and the map $A[M] \to A$ given by $\begin{bmatrix} a & 0 \\ m & \lambda \end{bmatrix} \mapsto a$.

But they do not have to be surjective maps, for example the inclusion map $\begin{bmatrix} k & 0 \\ k & k \end{bmatrix} \hookrightarrow \begin{bmatrix} k & k \\ k & k \end{bmatrix}$ is an epimorphism, and even for rings, the inclusion map $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism.

Given a morphism of algebras $\varphi : A \to B$, we have an induced functor $\varphi^* : mod(B) \to mod(A)$ that gives to $M \in mod(B)$ an action from A given by $a \cdot m := \varphi(a)m$.

The next result is probably well known:

Proposition 1. *Let $\varphi : A \to B$ be a morphism of algebras. The following statements are equivalent:*

1. *φ is an epimorphism.*
2. *For all $M,N \in mod(B)$, every A-linear map $f : M \to N$ is also B-linear.*
3. *The induced functor $\varphi^* : mod(B) \to mod(A)$ is full and faithfull.*

Proof: 1. $\Rightarrow$ 2.): Let $M,N \in mod(B)$, and $f : M \to N$ an A-linear map (in terms of the action given by φ). Let $C = \{ \begin{bmatrix} b & 0 \\ g & b \end{bmatrix} : b \in B, g \in Hom_k(M,N)\}$ and define two morphisms $\chi, \psi : B \to C$ given by $\chi(b) = \begin{bmatrix} b & 0 \\ bf - fb & b \end{bmatrix}$ and $\psi(b) = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$. Then $\chi \circ \varphi = \psi \circ \varphi$, hence $\chi = \psi$ and $bf - fb = 0$.

2. $\Rightarrow$ 1.): Let $\chi, \psi : B \to C$ be morphisms of algebras such that $\chi \circ \varphi = \psi \circ \varphi$. Then C is a B-module via ψ, and χ is B-linear, that is, for all $b, b' \in B$ we have $\chi(bb') = b \cdot \chi(b') = \psi(b)\chi(b')$, taking $b' = 1$ the unity of B gives: $\chi(b) = \psi(b)$ for all $b \in B$.

2 $\Leftrightarrow$ 3): Is immediate.$\blacksquare$

This last result motivates the following definition.

Definition 2. *An epimorphism of algebras $\varphi : A \to B$ is called a* ***homological epimorphism*** *if it induces a full and faithfull functor:*

$$D^b(\varphi^*) : D^b(B) \to D^b(A)$$

We will wait until we have proven some properties of this morphisms to give examples, since proving that a morphism satisfies this definition its not easy.

Let I be a two sided ideal of A. Since the quotient map $\pi : A \to A/I$ is an epimorphism, the induced functor $\pi^* : mod(A/I) \to mod(A)$ is full and faithfull.

Definition 3. *A two sided ideal I of A is **homological** if the quotient map $\pi : A \to A/I$ is a homological epimorphism.*

Observe that the kernel of a homological epimorphism is a homological ideal.

We have found a very useful property of these maps, stated by the following lemma.

Lemma 1. *Let $A, A_1, \dots, A_n, A'_1, \dots A'_m$ and B be finite dimensional algebras such that the following is a commutative diagram of algebra maps:*

$$\begin{tikzcd}
 & A_1 \arrow[r, "f_1"] & \cdots \arrow[r, "f_{n-1}"] & A_n \arrow[dr, "f_n"] & \\
A \arrow[ur, "f_0"] \arrow[dr, "g_0"'] & & & & B \\
 & A'_1 \arrow[r, "g_1"'] & \cdots \arrow[r, "g_{m-1}"'] & A'_m \arrow[ur, "g_m"'] &
\end{tikzcd}$$

If the maps $f_0, \dots, f_n, g_0, \dots, g_{m-1}$ are homological epimorphisms, then g_m is a homological epimorphisms.

Proof: Let X, Y be objects of $D^b(B)$, then we have a commutative diagram of vector spaces:

$$\begin{tikzcd}
Hom_{D^b(A_1)}(X,Y) \arrow[d, "D^b(f_0^*)"'] & \cdots \arrow[l, "D^b(f_1^*)"'] & Hom_{D^b(A_n)}(X,Y) \arrow[l, "D^b(f_{n-1}^*)"'] \\
Hom_{D^b(A)}(X,Y) & & Hom_{D^b(B)}(X,Y) \arrow[u, "D^b(f_n^*)"'] \arrow[d, "D^b(g_m^*)"] \\
Hom_{D^b(A'_1)}(X,Y) \arrow[u, "D^b(g_0^*)"] & \cdots \arrow[l, "D^b(g_1^*)"] & Hom_{D^b(A'_m)}(X,Y) \arrow[l, "D^b(g_{m-1}^*)"]
\end{tikzcd}$$

where all maps but $D^b(g_m^*)$ are isomorphisms, hence all of them are isomorphisms. This proves that g_m is a homological epimorphisms.■

We can apply this lemma, for example in the following proposition.

Proposition 2. *Let $I \subset J \subset A$ be homological ideals in A, then the inclusion map $A/I \to A/J$ is a homological epimorphism.*

Proof: In deed, we have a commutative diagram of algebras:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A/I \arrow[dl] \\ A/J & \end{tikzcd}$$

where all maps but one are homological epimorphisms, hence the last lemma applies.■

There is a well known and important theorem in algebra called **The Correspondence Theorem**, namely:

Theorem 1. *Let I be a two sided ideal of A. There exist a one to one correspondence induced by the quotient map $\pi : A \to A/I$*

$$\{Ideals\ of\ A\ containing\ I\} \longleftrightarrow \{Ideals\ of\ A/I\}$$

Which is actually an isomorphism of partially ordered sets.

We have proven a homological version of this theorem, using Lemma 1.

Theorem 2. *If I is a homological ideal of A, then the correspondence on the correspondence theorem restricts to the sets of homological ideals.*

That is, the quotient map $\pi : A \to A/I$ induces a one to one correspondence between the homological ideals of A that contain I and the homological ideals of A/I.

Proof: Let J be a homological ideal of A that contain I. The correspondence theorem sends J to the ideal J/I. We have to show that the quotient map $A/I \to (A/I)/(J/I)$ is homological. Because of the isomorphism theorem we have a commutative diagram of algebras:

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & A/I \arrow[r] & (A/I)/(J/I) \\ & A/J \arrow[ur, "\cong"'] & \end{tikzcd}$$

where all maps but $A/I \to (A/I)/(J/I)$ are homological epimorphisms (isomorphisms are homological epimorphisms), since $(A/I)/(J/I)$ is the sink of the diagram, Lemma 1 applies.

Now, if we have a homological ideal of A/I, say J/I, then we have a commutative diagram of algebras:

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & A/I \arrow[r] & (A/I)/(J/I) \arrow[dl, "\cong"] \\ & A/J & \end{tikzcd}$$

where all maps but $A \to A/J$ are homological epimorphisms and A/J is the sink of the diagram, hence all of them are homological epimorphisms, by Lemma 1.■

This theorem gives us a view to the structure of the lattice of homological ideals of an algebra. Observe that the lattice of homological ideals behaves in some way like the lattice of ideals, but lets do this carefully.

We do know that the correspondence respects contentions, but also that the intersection of ideals is an ideal, while intersection of homological ideals does not have to be a homological ideal. Observation which can be appreciated if we take two homological ideals I and J of A, and look at the following diagram of algebra maps:

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr] & A/(I\cap J) \arrow[d] \arrow[dl] \\ A/J & A/I \end{tikzcd}$$

In which there is no easy way (and maybe there is not) to embbed the map $A \to A/(I\cap J)$ into some commutative diagram that has all of the other maps as homological epimorphisms and use Lemma 1, unless one of the two inclusions $I \to I\cap J$ or $J \to I\cap J$ induces a full and faithfull functor between the respective derive categories. That is, either

$$D^b\big(A/(I\cap J)\big) \to D^b(A/I) \quad or \quad D^b\big(A/(I\cap J)\big) \to D^b(A/J)$$

is a full and faithfull functor. Observe that we can not apply proposition 2 in this situation.

An example for which the intersection of homological ideals is not a homological ideal is given in page 18.

1.2 Functorial and Derived Properties

In order to determine wheter an ideal or an epimorphism are homological, we need some other tools we give in this section.

Definition 4. *A subcategory $\mathscr{C}$ of an abelian category $\mathscr{A}$ is called **thick** if given a short exact sequence:*

$$0 \to X \to Y \to Z \to 0$$

in which two of the terms are objects of $\mathscr{C}$, then the third is an object of $\mathscr{C}$

Definition 5. *A subcategory $\mathscr{C}$ of an abelian category $\mathscr{A}$ **covers** (resp. **finitely covers**) $\mathscr{A}$ if the smallest thick subcategory of $\mathscr{A}$ containing $\mathscr{C}$ which is closed under arbitrary (resp. finite) direct sums is $\mathscr{A}$*

Definition 6. *A subcategory $\mathscr{C}$ of an abelian category $\mathscr{A}$ **weakle covers** (resp **finitely weakly covers**) $\mathscr{A}$ if the smallest thick subcategory of $\mathscr{A}$ containing all objects admitting a resolution by arbitrary (resp. finite) direct sums of objects from $\mathscr{C}$ is $\mathscr{A}$.*

Lemma 2. *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories and $\eta : (G_n)_{n\in\mathbf{Z}} \to (F_n)_{n\in\mathbf{Z}}$ be a morphism of connected sequences of additive functors $(G_n)_{n\in\mathbf{Z}}, (F_n)_{n\in\mathbf{Z}} : \mathscr{A} \to \mathscr{B}$*

1. *Suppose that $(G_n)_{n\in\mathbf{Z}}, (F_n)_{n\in\mathbf{Z}}$ are exact and that $\mathscr{A}'$ finitely covers $\mathscr{A}$ and $\eta_n(A) : G_n(A) \to F_n(A)$ is an isomorphism for all $A \in \mathscr{A}'$ and all n, then η is an isomorphism.*

2. *Suppose that $(G_n)_{n\in\mathbf{Z}}, (F_n)_{n\in\mathbf{Z}}$ are right exact and that $\mathscr{A}'$ finitely weakly covers $\mathscr{A}$, $G_n = F_n = 0$ for all $n < 0$, and that for any $A \in \mathscr{A}'$ the morphism $\eta_0(A)$ is an isomorphism and $G_n(A) = F_n(A) = 0$ for all $n \neq 0$, then η is an isomorphism.*

Proof:

1. Let $\mathscr{C}$ be the subcategory of $\mathscr{A}$ consisting of all objects $A \in \mathscr{A}$ such that $\eta_n(A)$ is an isomorphism for all n. By hypothesis, every object of $\mathscr{A}'$ is an object of $\mathscr{C}$. Given a short exact sequence in $\mathscr{A}$,

$$0 \to X \to Y \to Z \to 0$$

since G_n and F_n are exact functors, we have a commutative diagram with exact rows for every n:

$$\begin{tikzcd}
0 \arrow[r] & G_n(X) \arrow[r] \arrow[d, "\eta_n(X)"'] & G_n(Y) \arrow[r] \arrow[d, "\eta_n(Y)"'] & G_n(Z) \arrow[r] \arrow[d, "\eta_n(Z)"'] & 0 \\
0 \arrow[r] & F_n(X) \arrow[r] & F_n(Y) \arrow[r] & F_n(Z) \arrow[r] & 0
\end{tikzcd}$$

Then by the Five Lemma, if two of the $\eta_n's$ maps are isomorphisms, then the third is an isomorphism. Hence $\mathscr{C}$ is a thick subcategory, and since $\mathscr{A}'$ finitely covers $\mathscr{A}$, we have that $\mathscr{A} = \mathscr{C}$.

2. We will proceed by induction on n. If $n = 0$, let

$$X_1 \to X_0 \to A \to 0$$

be an exact sequence of an object $A \in \mathscr{A}$ with $X_0, X_1 \in \mathscr{A}'$. Since G_0 and F_0 are right exact, we have a commutative diagram with exact rows:

$$\begin{tikzcd}
G_0(X_1) \arrow[r] \arrow[d, "\eta_0(X_1)"'] & G_0(X_0) \arrow[r] \arrow[d, "\eta_0(X_0)"'] & G_0(A) \arrow[r] \arrow[d, "\eta_0(A)"'] & 0 \\
F_0(X_1) \arrow[r] & F_0(X_0) \arrow[r] & F_0(A) \arrow[r] & 0
\end{tikzcd}$$

in which $\eta_0(X_1)$ y $\eta_0(X_0)$ are isomorphisms, and by the Five Lemma, $\eta_0(A)$ is an isomorphism.

Now suppose that $n > 0$ and η_{n-1} is an isomorphism. Let $A \in \mathscr{A}$ and let

$$0 \to K \to X \to A \to 0$$

be a short exact sequence, where X is a finite direct sum of objects in $\mathscr{A}'$. Since $G_i(X) = F_i(X) = 0$ for all $i > 0$ we get a commutative diagram with exact rows.

$$\begin{tikzcd}
0 = G_n(X) \arrow[r] & G_n(A) \arrow[r, "\partial_{G_n}"] \arrow[d, "\eta_n(A)"] & G_{n-1}(K) \arrow[r] \arrow[d, "\eta_{n-1}(K)"] & G_{n-1}(X) \arrow[d, "\eta_{n-1}(X)"] \\
0 = F_n(X) \arrow[r] & F_n(A) \arrow[r, "\partial_{F_n}"'] & F_{n-1}(K) \arrow[r] & F_{n-1}(X)
\end{tikzcd}$$

where ∂_{G_n} and ∂_{F_n} are the connecting morphisms of the connected sequences of functors. By the Five Lemma and the induction hypothesis, we get that $\eta_n(A)$ is an isomorphism.■

What we want now is to give these notions at the level of derived categories, substituting the exact sequences by the exact triangles. We consider $\mathscr{A}$ as a full subcategory of $D^b(\mathscr{A})$, viewing every object $A \in \mathscr{A}$ as a complex concentrated in degree 0. We denote the translation functor in $D^b(\mathscr{A})$ as T.

Definition 7. *Let $\mathscr{C}$ be a triangulated category, a subcategory $\mathscr{D}$ is called* ***thick*** *if for each triangle*

$$X \to Y \to Z \to T(X)$$

in $\mathscr{C}$ the fact that two terms belong to $\mathscr{D}$ implies that the third term also belongs to $\mathscr{D}$.

Definition 8. *Let $\mathscr{C}$ be a triangulated category, a subcategory $\mathscr{D}$* ***covers*** *(resp.* ***finitely covers****) $\mathscr{C}$ if the smallest thick subcategory of $\mathscr{C}$ containing $\mathscr{D}$ which is closed under arbitrary (resp. finite) direct sums is $\mathscr{C}$.*

Definition 9. *Let $\mathscr{C}$ be a triangulated category, $\mathscr{D}$ a subcategory of $\mathscr{C}$. An object $C \in \mathscr{C}$* ***admits a resolution by objects from*** *$\mathscr{D}$ if there is a sequence of triangles:*

$$K_{i+1} \to D_i \to K_i \to T(K_{i+1}) \;\; i \geq -1$$

with $K_{-1} = C$ and $D_i \in \mathscr{D}$.

Definition 10. *Let $\mathscr{C}$ be a triangulated category, a subcategory $\mathscr{D}$* ***weakly covers*** *(resp.* ***finitely weakly covers****) $\mathscr{C}$ if the smallest thick subcategory of $\mathscr{C}$ containing all objects admitting a resolution by arbitrary (resp. finite) direct sums of objects from $\mathscr{D}$ is $\mathscr{C}$.*

Lemma 3. *Let $\mathscr{C}$ be a triangulated category, $\mathscr{B}$ an abelian category, $G, F : \mathscr{C} \to \mathscr{B}$ exact functors and $\eta : G \to F$ a natural transformation. Denote $G_n = G \circ T^{-n}$ and $F_n = F \circ T^{-n}$.*

1. *Suppose $\mathscr{C}'$ finitely covers $\mathscr{C}$ and that $\eta_n(C)$ is an isomorphism for all $C \in \mathscr{C}'$ and all n, then η is an isomorphism.*
2. *Suppose that $\mathscr{C}'$ finitely weakly covers $\mathscr{C}$, $G_n = F_n = 0$ for all $n < 0$, and that for any $C \in \mathscr{C}'$ we have that η_0 is an isomorphism and $G_n(C) = F_n(C) = 0$ for all $n \neq 0$, then η is an isomorphism.*

Proof: The proof is analogous to the one of the last lemma.■

Proposition 3. *Let $j : \mathscr{A} \to \mathscr{B}$ be an exact embedding of abelian categories. The following statements are equivalent:*

1. *The morphism induced by j*

$$\tilde{j} : Ext^n_{\mathscr{A}}(X,Y) \to Ext^n_{\mathscr{B}}(jX, jY)$$

   is an isomorphism for all $X, Y \in \mathscr{A}$ and all $n \geq 0$

2. *The induced functor of derived categories*

$$D^b(j) : D^b(\mathscr{A}) \to D^b(\mathscr{B})$$

   is a full embedding.

Proof: $1. \Rightarrow 2.$): For each $A \in \mathscr{A}$ we have a morphism of exact functors:

$$\eta_A : Hom_{D^b(\mathscr{A})}(A, -) \to Hom_{D^b(\mathscr{B})}(jA, -) \circ D^b(j)$$

and $\eta_A(X)$ is an isomorphism for all $X \in \mathscr{A}$. Since $\mathscr{A}$ finitely covers $D^b(\mathscr{A})$, by the last lemma we have that η_A is an isomorphism.

Now, for $X \in D^b(\mathscr{A})$ we have a morphism of exact functors:

$$\eta_X : Hom_{D^b(\mathscr{A})}(-, X) \to Hom_{D^b(\mathscr{B})}(-, jX) \circ D^b(j)$$

which is an isomorphism for objects of $\mathscr{A}$, and since $\mathscr{A}$ finitely weakly covers $D^b(\mathscr{A})$, by lemma 3, η is an isomorphism.

$2. \Rightarrow 1.$): Since $Ext^n_{\mathscr{A}}(X,Y) \cong_\varphi Hom_{D^b(\mathscr{A})}(X, T^nY)$, $Ext^n_{\mathscr{B}}(X,Y) \cong_\psi Hom_{D^b(\mathscr{B})}(X, T^nY)$, and by hypothesis we have that $Hom_{D^b(\mathscr{A})}(X, T^nY)$ and $Hom_{D^b(\mathscr{B})}(X, T^nY)$ are isomorphic under $D^b(j)$. Given that $\tilde{j} = \psi^{-1} \circ D^b(j) \circ \varphi$ we have that $\tilde{j}$ is an isomorphism.■

Theorem 3. *For a morphism of artinian rings $\varphi : A \to B$ the following statements for finitely generated modules are equivalent:*

1. *The multiplication map $B \otimes_A B \to B$ is an isomorphism and $Tor_i^A(B,B) = 0$ for all $i \geq 1$.*
2. *For all M_B the multiplication map $M \otimes_A B \to M$ is an isomorphism and $Tor_i^A(M,B) = 0$ for all $i \geq 1$.*
3. *For all ${}_BN$ the multiplication map $B \otimes_A N \to N$ is an isomorphism and $Tor_i^A(B,N) = 0$ for all $i \geq 1$.*
4. *For all M_B and all ${}_BN$, the induced map $Tor_i^B(M,N) \to Tor_i^A(M,N)$ is an isomorphism for all $i \geq 0$.*
5. *For all M_B the map $Hom_A(B,M) \to M$ is an isomorphism and $Ext_A^i(B,M) = 0$ for all $i \geq 1$.*
6. *For all ${}_BN$ the map $Hom_A(B,N) \to N$ is an isomorphism and $Ext_A^i(B,N) = 0$ for all $i \geq 1$.*
7. *For all M_B and M'_B the map $Ext_B^i(M,M') \to Ext_A^i(M,M')$ is an isomorphism for all $i \geq 0$.*
8. *For all ${}_BN$ and ${}_BN'$ the map $Ext_B^i(N,N') \to Ext_A^i(N,N')$ is an isomorphism for all $i \geq 0$.*
9. *The functor*
$$D^b(\varphi^*) : D^b(mod\ B) \to D^b(mod\ A)$$
is a full embedding.
10. *The functor*
$$D^b\big((\varphi^{op})^*\big) : D^b(mod\ B^{op}) \to D^b(mod\ A^{op})$$
is a full embedding.

Proof:
1. $\Rightarrow$ 2.): For each M_B we have a sequence of isomorphisms:

$$M \otimes_A B \cong M \otimes_B B \otimes_A B \cong M \otimes_B B \cong M$$

whose composition is the multiplication map induced by the action.
Now, for each M_B we have a short exact sequence:

$$0 \to X \to B^m \to M \to 0$$

And a commutative diagramm:

$$
\begin{array}{ccccccccc}
 & & X \otimes_A B & \longrightarrow & B^m \otimes_A B & \longrightarrow & M \otimes_A B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & B^m & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

where the vertical arrows are the isomorphisms we have just proven. Hence we have a short exact sequence:

$$0 \to X \otimes_A B \to B^m \otimes_A B \to M \otimes_A B \to 0$$

Therefore $Tor_1^A(M,B) = 0$, and also $Tor_{n+1}^A(M,B) \cong Tor_n^A(X,B)$ for all $n \geq 2$, and finally:

$$Tor_n^A(M,B) = 0 \quad \forall n \geq 1$$

2. $\Rightarrow$ 4.): For each M_B we have functors $M \otimes_A -$ and $M \otimes_B -$ from $mod(B)$ to **Ab**. The natural transformation:

$$\tau : M \otimes_A - \to M \otimes_B -$$

is such that τ_B is an isomorphism and since the functors are right exact we may use lemma 2. We already have that $Tor_n^A(M,-) = Tor_n^B(M,-) = 0$ for all $n < 0$; Also the derived functors are anihilated by the projectives in every $n \neq 0$, and since $proj(B)$ finitely weakly covers $mod(B)$, we have that τ is an isomorphism, in particular the induced maps:

$$Tor_n^A(M,N) \to Tor_n^B(M,N)$$

are isomorphisms for all M_B, all $_BN$ and all $n \geq 0$.

4. $\Rightarrow$ 1.): Since $Tor_n^A(M,N) \cong Tor_n^B(M,N)$ for all M_B, $_BN$ and all n, we have that:

$$Tor_n^A(B,B) \cong Tor_n^B(B,B) = 0$$

for all $n > 0$, since B is B-projective. Then we have a sequence of isomorphisms:

$$
\begin{array}{lcrcc}
B \otimes_A B = Tor_0^A(B,B) & \to & Tor_0^B(B,B) = B \otimes_B B & \to & B \\
b \otimes b' & \mapsto & b \otimes b' & \mapsto & bb'
\end{array}
$$

1. $\Rightarrow$ 3.): Analogous to 1. $\Rightarrow$ 2.
3. $\Rightarrow$ 4.): Analogous to 2. $\Rightarrow$ 4.
5. $\Rightarrow$ 7.): Analogous to 2. $\Rightarrow$ 4.
6. $\Rightarrow$ 8.): Analogous to 2. $\Rightarrow$ 4.
7. $\Rightarrow$ 5.): Analogous to 4. $\Rightarrow$ 1.
8. $\Rightarrow$ 6.): Analogous to 4. $\Rightarrow$ 1.
3. $\Rightarrow$ 5.): For M_B we have a sequence of isomorphisms:

$$\begin{array}{rcll} Hom_A(B,M) & \cong & Hom_A(B,Hom_B(B,M)) & \\ & \cong & Hom_B(B\otimes_A B,M) & \textit{by the adjoint isomorphism} \\ & \cong & Hom_B(B,M) & \textit{by hypothesis} \\ & \cong & M & \end{array}$$

whose composition is the map $Hom_A(B,M) \to M$.

In particular the functor $Hom_A(B,-)$ is exact in $mod(B)$. We denote DM for the left B-module $Hom_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$, and by [16]:

$$Ext^i_A(B,DDM) \cong DTor^B_i(B,DM) = D0 = 0$$

for all $i \geq 1$. Since M is a submodule of DDM we have that $Ext^1_A(B,M) = 0$ because $Ext^1_A(B,-)$ is left exact, and we can conclude by induction on i.

5. $\Rightarrow$ 3.): For ${}_BN$ we have

$$DTor^A_i(B,N) \cong Ext^i_A(B,DN)$$

for all $i \geq 0$, again by [16]. Hence $Tor^A_i(B,N) = 0$ for all $i \geq 1$ and we get:

$$Hom_{\mathbb{Z}}(N,\mathbb{Q}/\mathbb{Z}) \cong Hom_A(B,DN) \cong Hom_{\mathbb{Z}}(B\otimes_A N,\mathbb{Q}/\mathbb{Z})$$

This composition of isomorphisms is the multipication map.

2. $\Leftrightarrow$ 6.): Analogous to $3 \Leftrightarrow 5$.

7. $\Leftrightarrow$ 9.): Follows from proposition 2.

8. $\Leftrightarrow$ 10.): Follows from proposition 2.■

Corollary 1. *Let $\varphi : A \to B$ be a homological epimorphism of algebras. Then*

1. *φ^{op} is a homologial epimorphism.*
2. *gl.dim(B) $\leq$ gl.dim(A).*

Proof: It is immediate from last theorem.■

Corollary 2. *We have the following:*

1. *Let $\varphi : A \to B$ be an epimorphism and suppose that B is a flat B-module. Then φ is a homological epimorphism.*
2. *If A is commutative and S is a subset of A, the map $A \to S^{-1}A$ is a homological epimorphism.*

Proof: It is immediate from last theorem.■

Proposition 4. *Let I be an ideal of A, then*

1. *I is a homological ideal of A if and only if $Tor^A_n(I,A/I) = 0$ for all $n \geq 0$. In this case, I is idempotent.*
2. *If I is idempotent and A-projective, then I is homological.*
3. *If I is idempotent then I is homological if and only if $Ext^n_A(I,A/I) = 0$ for all $n \geq 0$.*

Proof:

1. Apply the functor $-\otimes_A A/I$ to the exact sequence of A-modules: $0 \to I \to A \to A/I \to 0$ to get another exact sequence:

   $$Tor_1^A(A, A/I) = 0 \to Tor_1^A(A/I, A/I) \to I \otimes_A A/I \to A \otimes_A A/I \to A/I \otimes_A A/I \to 0$$

   Now, since I is the syzygy of $A \to A/I$ we get that $Tor_n^A(A/I, A/I) \cong Tor_{n-1}^A(I, A/I)$ for $n \geq 2$. We have that $I \otimes_A A/I \cong I/I^2$ (because I/I^2 satisfies the universal property of $I \otimes_A A/I$) and $A/I \cong A/I \otimes_A A/I$, substituting this in the last exact sequence we get:
   $$Tor_1^A(A/I, A/I) \cong I/I^2$$

2. If I is A-projective, then $Tor_n^A(I, A/I) = 0$ for all $n \geq 1$, and also

   $$I \otimes_A A/I \cong I/I^2 \cong 0$$

   from 1, we get that I is homological.

3. It is obvious.■

Proposition 5. *Let $\varphi : A \to B$ be a (homological) epimorphism of k-algebras. Then:*

1. *$\varphi^e : A^e \to B^e$ is a (homological) epimorphism.*
2. *$\varphi_n^e : Ext_{B^e}^n(B, B) \to Ext_{A^e}^n(B, B)$ is an isomorphism for all $n \geq 0$.*

Proof: We are going to use the following results given in [16]. For k-algebras Λ, Γ, Σ and modules $X_{\Lambda-\Gamma}, {}_\Lambda Y_\Sigma, {}_{\Gamma-\Sigma} Z$:

$$\begin{array}{ll} (a) & (X \otimes_\Lambda Y) \otimes_{(\Gamma \otimes_k \Sigma)} Z \cong X \otimes_{(\Lambda \otimes_k \Gamma)} (Y \otimes_\Sigma Z) \\ (b) & If\ Tor_n^\Lambda(X, Y) = 0 = Tor_n^\Sigma(Y, Z)\ for\ all\ n > 0 \\ & Tor_n^{(\Lambda \otimes_k \Sigma)}(X \otimes_\Lambda Y, Z) \cong Tor_n^{(\Lambda \otimes_k \Gamma)}(X, Y \otimes_\Sigma Z) \end{array}$$

1. $B^e \otimes_{A^e} B^e = (B \otimes_k B^{op}) \otimes_{A \otimes_k A^{op}} B^e \cong B \otimes_A (B^{op} \otimes_{A^{op}} B^e)$ for (a).

   Now, since $B^{op} \otimes_{A^{op}} (B \otimes_k B^{op}) \cong (B^{op} \otimes_{A^{op}} B) \otimes_k B^{op} \cong B \otimes_k B^{op}$, we have:

   $$\begin{array}{rcll} B^e \otimes_{A^e} B^e & \cong & (B \otimes_A B) \otimes_k B^{op} & \\ & \cong & B \otimes_k B^{op} & \varphi\ is\ an\ epimorphism \\ & \cong & B^e & \end{array}$$

   Hence φ^e is an epimorphism. Finally:

   $$Tor_n^{A^e}(B^e, B^e) = Tor_n^{A \otimes_k A^{op}}(B \otimes_k B, B^e) \cong Tor_n^A(B, B^{op} \otimes_{A^{op}} B^e) = 0$$

   Where the last Tor is cero because φ is homological. Hence φ^e is homological.

2. φ^e is a homological epimorphism and B is a B^e-module.■

1.3 Examples

Now with all this theory it is easier to give some examples.

1. Let A be the path algebra of the following quiver:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

That is, A is the matrix algebra:

$$\begin{bmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{bmatrix}$$

whose Auslander-Reiten quiver is:

$$\begin{tikzcd}[column sep=small]
 & & P(1)=I(3) \arrow[dr] & & \\
 & P(2) \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & I(2) \arrow[dr] & \\
P(3) \arrow[ur] \arrow[rr, dashed, no head] & & S(2) \arrow[ur] \arrow[rr, dashed, no head] & & I(1)
\end{tikzcd}$$

Since A is hereditary, the cluster category is:

$$\begin{tikzcd}[column sep=small]
 & & P(1)=I(3) \arrow[dr] & & P(3)[1] \arrow[dr] & & \\
 & P(2) \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & I(2) \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & P(2)[1] \arrow[dr] & \\
P(3) \arrow[ur] \arrow[rr, dashed, no head] & & S(2) \arrow[ur] \arrow[rr, dashed, no head] & & I(1) \arrow[ur] \arrow[rr, dashed, no head] & & P(1)[1]
\end{tikzcd}$$

And also every ideal is projective, hence every idempotent ideal is homological. Define:

$$I_1 = \begin{bmatrix} 0 & 0 & 0 \\ k & k & 0 \\ k & k & k \end{bmatrix}, \quad I_2 = \begin{bmatrix} k & 0 & 0 \\ k & 0 & 0 \\ k & k & k \end{bmatrix}, \quad I_3 = \begin{bmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & 0 \end{bmatrix}.$$

It is easy to see that these three ideals are two sided idempotent ideals, and also that $A/I_n \cong k$ for $n = 1,2,3$. Hence the subcategories given by these homological ideals in $mod(A)$ are:

$$A \to A/I_1 \;\; induces: \quad mod(k) \cong add(I(1)) \hookrightarrow mod(A)$$

$$A \to A/I_2 \;\; induces: \quad mod(k) \cong add(S(2)) \hookrightarrow mod(A)$$

$$A \to A/I_3 \;\; induces: \quad mod(k) \cong add(P(3)) \hookrightarrow mod(A)$$

Observe that the subcategories on the cluster category are obvious.

Now, from [9], all idempotent ideals are traces of projective modules, and the traces of the indecomposable projectives are:

$$J_1 := tr_{P(1)}(A) = \begin{bmatrix} k & 0 & 0 \\ k & 0 & 0 \\ k & 0 & 0 \end{bmatrix}, \quad J_2 := tr_{P(2)}(A) = \begin{bmatrix} 0 & 0 & 0 \\ k & k & 0 \\ k & k & 0 \end{bmatrix},$$
$$J_3 := tr_{P(3)}(A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k & k & k \end{bmatrix}.$$

We have that $A/J_i \cong \begin{bmatrix} k & 0 \\ k & k \end{bmatrix}$ for $i = 1,3$, and $A/J_2 \cong k \times k$.

Hence the subcategory of $mod(A)$ given by $A \to A/J_1$ is $add(P(3) \oplus P(2) \oplus S(2))$, i.e. is the one given by the subquiver of the Auslander-Reiten quiver of A:

$$\begin{tikzcd}
 & P(2) \arrow[dr] & \\
P(3) \arrow[ur] \arrow[rr, dashed, no head] & & S(2)
\end{tikzcd}$$

The subcategory of $mod(A)$ given by $A \to A/J_3$ is $add(S(2) \oplus I(2) \oplus I(1))$, i.e. is the one given by the subquiver of the Auslander-Reiten quiver:

$$\begin{tikzcd}
 & I(2) \arrow[dr] & \\
S(2) \arrow[ur] \arrow[rr, dashed, no head] & & I(1)
\end{tikzcd}$$

And obviously, the subcategory of $mod(A)$ given by $A \to A/J_2$ is $add(P(3) \oplus I(1))$.

It appears that we are breaking down into pieces from below to above the category $mod(A)$ if we go from the maximal homological (the $I's$) ideals to the minimal homological (the $J's$) ideals, at least in the hereditary case. Observe that the zero ideal gives the whole category and it is homological.

Now, the cluster category of A/J_1 is:

$$\begin{tikzcd}
& P(2) \arrow[dr] \arrow[rr, dash, dashed] & & P(3)[1] \arrow[dr] & \\
P(3) \arrow[ur] \arrow[rr, dash, dashed] & & S(2) \arrow[ur] \arrow[rr, dash, dashed] & & P(2)[1]
\end{tikzcd}$$

Inside the cluster category of A the shift should be the one from $D^b(A)$, hence inside the cluster category of A the subcategory given by J_1 is:

$$add\Big(P(3) \oplus P(2) \oplus S(2) \oplus P(3)[1] \oplus P(2)[1]\Big)$$

Since in the cluster category $S(2)[1] = P(2)$ and $I(2)[1] = P(2)$, the subcategory given by J_3 inside the cluster category of A is given by the following subquiver:

$$\begin{tikzcd}
& P(2) \arrow[dr] \arrow[rr, dash, dashed] & & I(2) \arrow[dr] & \\
P(3) \arrow[ur] \arrow[rr, dash, dashed] & & S(2) \arrow[ur] \arrow[rr, dash, dashed] & & I(1)
\end{tikzcd}$$

For the ideal J_3 it is analogous, and for J_2 and the $I_i's$ it is trivial. We finish with this example by observing that we have analyzed all the homological ideals of A.

2. The Kronecker algebra A:

$$\begin{tikzcd}
2 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 1
\end{tikzcd}$$

We can write:

$$A = \begin{bmatrix} k & 0 \\ k^2 & k \end{bmatrix}$$

Again is hereditary, hence all homological ideals are the idempotent ideals, by [9], they are all the traces of the indecomposable projectives on A. It is easy to see that:

$$
\begin{aligned}
tr_{P(1)}(P(2)) &= P(1) \oplus P(1) \\
tr_{P(2)}(P(1)) &= 0
\end{aligned}
$$

Hence the homological ideals are:

$$
\begin{aligned}
I_1 := tr_{P(1)}(A) &= \begin{bmatrix} k & 0 \\ k^2 & 0 \end{bmatrix} \\
I_2 := tr_{P(2)}(A) &= \begin{bmatrix} 0 & 0 \\ k^2 & k \end{bmatrix}
\end{aligned}
$$

Observe that $A/I_i \cong k$ for all i, and then as epimorphisms, I_1 gives the category $add(P(1))$, and I_2 gives $add(P(2))$. Also in the derived categories these induced subcategories are again trivial.

Finally, $I_1 \cap I_2 = rad(A)$, which is not idempotent, and hence not homological. Then the intersection of homological ideals does not have to be a homological ideal.

2 Hochschild Cohomology and One Point Extensions

2.1 Hochschild Cohomology Spaces

Definition 11. *Let M be a finite dimensional left A^e-module (where $A^e = A \otimes_k A^{op}$), we define the* ***Hochschild Complex*** *of A with coefficients in M, $C^\bullet = (C^n, d^n)_{n\in\mathbb{Z}}$, as follows: $C^n = 0$ and $d^n = 0$ for all $n < 0$; $C^0 = M$, $C^n = Hom_k(A^{\otimes n}, M)$ for $n > 0$, $d^0(m)(a) = am - ma$ for $m \in M$ and $a \in A$, and $d^n : C^n \to C^{n+1}$ by:*

$$\begin{aligned}(d^n f)(a_1 \otimes \cdots \otimes a_{n+1}) \quad = \quad & a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) \\ & + \sum_{j=1}^{n} (-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{n+1}) \\ & + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1}\end{aligned}$$

for $f \in C^n$ and $a_1, \ldots, a_{n+1} \in A$.

$$\cdots \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \cdots$$

Definition 12. *The* ***n-th Hochschild Cohomology*** *of A with coefficients in M is:*

$$H^n(A, M) = H^n(C^\bullet)$$

In the case where $M = A$, we denote $H^n(A) = H^n(A, A)$ and call it the **n-th Hochschild Cohomology** of A.

It is well known and easy to prove that $H^0(A) = Z(A)$ and that $H^1(A)$ is the quotient of the derivations group over the subgroup of inner derivations.

An interpretation for the second Hochschild Cohomology space of an algebra is the following. For $f \in Hom_k(A \otimes_k A, M)$ we define the algebra $A \times_f M$ which is $A \oplus M$ as k-vector spaces with product:

$$(a, m)\dot{(}a', m') = (aa', am' + ma' + f(a \otimes a'))$$

If $f \in ker(d^2)$, then $A \times_f M$ is an associative algebra with unity.

Theorem 4. *Let $f, g \in ker(d^2)$. Then $A \times_f M \cong A \times_g M$ if $\hat{f} = \hat{g}$ in $H^2(A, M)$.*

Another equivalent way of constructing the Hochschild Cohomology spaces is to construct first the following A^e-projective resolution of A. We denote $S_n(A) = A^{\otimes n+2}$ for $n \geq -1$, which is a left A^e-module, and define $\delta_n : S_n(A) \to S_{n-1}(A)$ by:

$$\delta_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{j=0}^{n} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{n+1}$$

In the computation of the Hochschild Cohomology spaces we use:

$$\begin{aligned}Hom_{A^e}(S_n(A), M) \quad & = \quad Hom_{A^e}(A \otimes_k A^{\otimes n} \otimes_k A, M) \\ & = \quad Hom_{A^e}(A^e \otimes_k A^{\otimes n}, M) \\ & = \quad Hom_k(A^{\otimes n}, M) \\ & = \quad C^n\end{aligned}$$

Hence $H^n(A,M) \cong Ext_{A^e}(A,M)$.

And actually it is stronger than this:

Theorem 5. *There is an isomorphism*

$$H^n(A,-) \cong Ext_{A^e}(A,-)$$

of functors from $mod(A^e)$ to $mod(k)$.

One of the most important properties of Hochschild Cohomology is its derived invariance:

Theorem 6. *Let A and B be finite dimensional k-algebras. If there is an equivalence of triangulated categories $F : D^b(A) \to D^b(B)$ such that $F(A) = B$, then*

$$H^n(A) \cong H^n(B)$$

Proof: Indeed, since we can define a triangulated equivalence $\hat{F} : D^b(A^e) \to D^b(B^e)$ ([11]) sending A^e to B^e:

$$\begin{array}{rcl} H^n(A) & \cong & Ext_{A^e}(A,A) \\ & \cong & Hom_{D^b(A^e)}(A,T^n(A)) \\ & \cong & Hom_{D^b(B^e)}(B,T^n(B)) \\ & \cong & Ext_{B^e}(B,B) \\ & \cong & H^n(B)\blacksquare \end{array}$$

Now we pass to One-Point Extensions and their relation to Hochschild Cohomology.

2.2 One Point Extensions

Definition 13. *Let $M \in mod(A)$, the **One-Point Extension** algebra of A by M is the matrix algebra:*

$$A[M] = \left(\begin{array}{cc} A & 0 \\ M & k \end{array} \right)$$

For example the algebra of upper triangular matrices over k, $T_n(k)$, acts by left multiplication on an n-dimensional vector space. Then $T_n(k)[M] \cong T_{n+1}(k)$. For an algebra B to be of the form $A[M]$ it is necessary that there exist a simple injective B-module, say S. If $P(S)$ is the projective cover of S and $b \in B$ is an idempotent such that $P(S) = bB$, then $B = A[M]$ for $A = B/I$, $I =< b >$ and $M = radP(S)$.

Proposition 6. *The map $p : A[M] \to A$ given by $\left[\begin{array}{cc} a & 0 \\ m & \lambda \end{array} \right] \to a$ is a homological epimorphism.*

Proof: The morphism is surjective and has kernel $\left[\begin{array}{cc} 0 & 0 \\ M & k \end{array} \right]$, which is idempotent and projective, hence a homological ideal.$\blacksquare$

Now we use the following lemma stated in [11] to prove Happel's long exact sequence.

Lemma 4. *If $P(b,b')$ denotes the indecomposable projective $A[M]^e$-module corresponding to the idempotent $b \otimes b'$, where b' denotes the corresponding element in the opposite algebra $A[M]^{op}$ of b. Then*

1. $I \cong P(b,b') \cong Hom_k(S(b),P(b))$ *as left $A[M]^e$-modules.*
2. $Ext^j_{A^e}(A,A) \cong Ext^j_{A[M]^e}(A,A)$ *for all $j \geq 0$.*
3. $Ext^j_{A[M]}(S(b),P(b)) \cong Ext^{j-1}_A(M,M)$ *for all $j \geq 2$.*
4. $Ext^1_{A[M]}(S(b),P(b)) \cong Hom_A(M,M)/<1_M>$, *and since $<1_M> \cong k$, we write $Hom_A(M,M)/k$ insted of $Hom_A(M,M)/<1_M>$.*
5. $Hom_{A[M]}(S(b),P(b)) = 0$.

Theorem 7. *(**Happel**) There exist a long exact sequence:*

$$0 \to H^0(A[M]) \to H^0(A) \to Hom_A(M,M)/k \to H^1(A[M]) \to H^1(A) \to Ext^1_A(M,M) \to \dots$$

Proof: Let I be the kernel of the $A[M]^e$-map, $A[M] \to A$ given by $\begin{pmatrix} a & 0 \\ m & \lambda \end{pmatrix} \mapsto a$. We have an exact sequence of $A[M]^e$-modules:

$$0 \to I \to A[M] \to A \to 0$$

By the last lemma, $I \cong P(b,b')$ is $A[M]^e$-projective and hence $Ext^i_{A[M]^e}(I,A) = 0$ for $i \geq 1$. Applying the functor $Hom_{A[M]^e}(-,A)$ to the last short exact sequence we get that $Ext^i_{A[M]^e}(A,A) \cong Ext^i_{A[M]^e}(A[M],A)$ for $i \geq 1$. Now:

$$\begin{aligned} Ext^i_{A[M]^e}(A[M],I) &\cong H^i(A[M],I) \\ &\cong H^i\big(A[M],Hom_k(S(b),P(b))\big) \\ &\cong Ext^i_{A[M]}(S(b),P(b)) \\ &\cong \begin{cases} Hom_A(M,M)/k & for\ i=1 \\ Ext^{i-1}_A(M,M) & for\ i \geq 2 \end{cases} \end{aligned}$$

Applying the functor $Hom_{A[M]^e}(A[M],-)$ to the first short exact sequence, we get a long exact sequence:

$$Hom_{A[M]^e}(A[M],I) = 0 \to Hom_{A[M]^e}(A[M],A[M]) \to Hom_{A[M]^e}(A[M],A)$$

$$\to Ext^1_{A[M]^e}(A[M],I) \to Ext^1_{A[M]^e}(A[M],A[M]) \to Ext^1_{A[M]^e}(A[M],A) \to \dots$$

Now, by Theorem 6, $Hom_{A[M]^e}(A[M],A[M]) = H^0(A[M])$; by the first assertion in this proof we get $Hom_{A[M]^e}(A[M],A) \cong Hom_{A[M]^e}(A,A)$ and by the last lemma part 2

$$Hom_{A[M]^e}(A,A) \cong Hom_{A^e}(A,A) \cong H^0(A)$$

We already proved that $Ext^1_{A[M]^e}(A[M],I) \cong Ext^1_{A[M]}(S(b),P(b))$ and by the last lemma part 4, $Ext^1_{A[M]}(S(b),P(b)) \cong Hom_A(M,M)/k$; All the other isomorphisms to obtain the desired long exact sequence are obtained this way.■

There is an important result that follows from this last theorem.

Definition 14. *An A-module M is exceptional if* $Ext^n_A(M,M) = 0$ *for all* $n > 0$ *and* $Hom_A(M,M)$ *is a one dimensional vector space.*

Theorem 8. *Let M be an exceptional A-module, then* $H^n(A) \cong H^n(A[M])$ *for all* $n \geq 0$.

Proof: It is clear from Happel's long exact sequence.■

2.3 Connections with Homological Ideals

We have the following result, we give a direct proof using lemma 1.

Theorem 9. *Let M be an A-module and I a homological ideal of A such that* $IM = MI = 0$. *Then* $I[M]$ *is a homological ideal of* $A[M]$.

Proof: Let $\varphi : A[M] \to A/I$ be given by $\begin{pmatrix} a & m \\ 0 & \lambda \end{pmatrix} \mapsto a + I$, then by the isomorphism theorem we get:

$$A[M]/I[M] \cong A/I$$

And we get a commutative diagram of algebras:

$$\begin{tikzcd} A[M]/I[M] & A/I \arrow[l, "\hat{\varphi}^{-1}"'] \\ A[M] \arrow[u, "\pi_{I[M]}"] \arrow[r, "p"'] & A \arrow[u, "\pi_I"'] \end{tikzcd}$$

where $\hat{\varphi}^{-1}$ is an isomorphism, π_I is a homological epimorphism by hypothesis, and by Proposition 6, p is a homological epimorphism, hence all maps but $\pi_{I[M]}$ are homological epimorphisms, and by Lemma 1, $\pi_{I[M]}$ is a homological epimorphism.■

We have proved a generalization of Theorem 8.

Theorem 10. *If there exist a homological ideal I of A such that the category* $mod(A/I)$ *has an exceptional module M, then:*

$$H^n(A/J) \cong H^n\Big((A/J)[M]\Big)$$

for all $n \geq 1$ *and all homological ideals J of A with* $J \subset I$.

In particular, for the trivial homological ideal we get $H^n(A) \cong H^n\big(A[M]\big)$ *for all* $n \geq 1$.

Proof: Let I be a homological ideal of A and M an exceptional A/I-module. If $J \subset I$ is a homological ideal of A, then the inclusion map induces morphisms of algebras $A/J \to A/I$, and then M is an A/J-module, now $(A/J)[M]$ makes sense. By Theorem 3, $Ext^n_{A/J}(M,M) \cong Ext^n_A(M,M) \cong Ext^n_{A/I}(M,M) = 0$, applying Theorem 8 for the algebra A/J and the now exceptional A/J-module M, gives the desired result.■

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