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Solution to HJB equations with an elliptic integro-differential operator and gradient constraint

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Que para obtener el grado de

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con Orientación en

Probabilidad y Estadística

P r e s e n t a

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DOCTOR EN CIENCIAS
CON ORIENTACIÓN EN PROBABILIDAD Y ESTADÍSTICA

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en cumplimiento con lo establecido en los reglamentos y lineamientos de estudios de posgrado del Centro de Investigación en Matemáticas, A.C., mediante la presentación de la tesis

"SOLUTION TO HJB EQUATIONS WITH AN ELLIPTIC INTEGRO-DIFFERENTIAL OPERATOR AND GRADIENT CONSTRAINT".

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Solution to HJB equations with an elliptic integro-differential operator and gradient constraint

Harold A. Moreno-Franco

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Contents

Preliminaries: Notation and definitions

In this small chapter, we introduce some spaces of functions and basic definitions that will be used in this thesis. These functions are defined on an open set $\mathcal{O} \subseteq \mathbb{R}^d$, with $d \geq 2$, where the complement, closure, interior and boundary of $\mathcal{O}$ are denoted by $\mathcal{O}^{\mathrm{c}}$, $\overline{\mathcal{O}}$, $\operatorname{int} \mathcal{O}$ and $\partial\mathcal{O}$, respectively. We recall that $||\cdot||$ is the Euclidean norm and $\langle \cdot, \cdot \rangle$ is the inner product. When $\sigma = (\sigma_{ij})_{d \times d} \in \mathbb{R}^{d \times d}$, we denote $\operatorname{tr}(\cdot)$ as the trace of the matrix σ.

An open ball with radius $r > 0$ and center in an arbitrary $x \in \mathbb{R}^d$ is defined as B_r. In case that the center is fixed, we denote the open ball with radius $r > 0$ and center $x \in \mathbb{R}^d$ as $B_r(x)$. Let $\mathcal{O} \subseteq \mathbb{R}^d$ be an open set.

0.1 Spaces of continuous functions

Let $\mathrm{C}^0(\mathcal{O})$ denote the space of real valued continuous functions on $\mathcal{O}$. The set $\mathrm{C}^k(\mathcal{O})$ consists of real valued functions on $\mathcal{O}$ that are k-fold differentiable, i.e. $\partial^a f \in \mathrm{C}^0(\mathcal{O})$ for every $a \in \mathcal{D}_m$, $0 \leq m \leq k$, where $\mathcal{D}_m$ is the set of all multi-indices of order m. For instance, if $k = 2$, then $\partial^1 f \in \{\partial_i f : i = 1, \ldots, d\}$ and $\partial^2 f \in \{\partial^2_{ij} f : i, j = 1, \ldots, d\}$. Here $\mathrm{D}^1 u = (\partial_1 u, \ldots, \partial_d u)$, $\mathrm{D}^2 u = (\partial^2_{ij} u)_{d \times d}$. We define $\mathrm{C}^\infty(\mathcal{O}) = \bigcap_{k=0}^\infty \mathrm{C}^k(\mathcal{O})$. The sets $\mathrm{C}^k_{\mathrm{c}}(\mathcal{O})$ and $\mathrm{C}^\infty_{\mathrm{c}}(\mathcal{O})$ consist of functions in $\mathrm{C}^k(\mathcal{O})$ and $\mathrm{C}^\infty(\mathcal{O})$, whose support is compact and contained in $\mathcal{O}$, respectively. The following result gives an extension for uniformly continuous functions defined on open sets, which proof is in Section 0.3.

Proposition 0.1. *If $f : \mathcal{O} \longrightarrow \mathbb{R}$ is uniformly continuous function, then f has a unique extension to a continuous function $\overline{f} : \overline{\mathcal{O}} \longrightarrow \mathbb{R}$.*

If $f \in \mathrm{C}^0(\mathcal{O})$ is bounded and uniformly continuous on $\mathcal{O}$, by Proposition 0.1, it follows that it possesses a unique bounded continuous extension to $\overline{\mathcal{O}}$. The set $\mathrm{C}^k(\overline{\mathcal{O}})$ is defined as the set of real valued functions $f \in \mathrm{C}^k(\mathcal{O})$ for which $\partial^a f$ is bounded and uniformly continuous on $\mathcal{O}$ for every

$a \in \mathcal{D}_m$, with $0 \le m \le k$. This space is equipped with the following norm

$$||f||_{\mathrm{C}^k(\overline{\mathcal{O}})} = \sum_{m=0}^{k} \sum_{a \in \mathcal{D}_m} ||\partial^a f||_{\mathrm{C}^0(\overline{\mathcal{O}})} = \sum_{m=0}^{k} \sum_{a \in \mathcal{D}_m} \sup_{x \in \mathcal{O}} \{|\partial^a f(x)|\},$$

where $\sum_{a \in \mathcal{D}_m}$ denotes summation over all possible m-fold derivatives of f. For each $D \subseteq \mathbb{R}^d$, $f : D \longrightarrow \mathbb{R}$ and $0 < \alpha \le 1$, the operator $[\,\cdot\,]_{\mathrm{C}^{0,\alpha}(D)}$ is defined as

$$[f]_{\mathrm{C}^{0,\alpha}(D)} = \sup_{\substack{x,y \in D \\ x \neq y}} \left\{ \frac{|f(x) - f(y)|}{||x - y||^\alpha} \right\}. \tag{0.1}$$

Next we define different spaces of Hölder continuous functions that will be used in this work. Let $\mathcal{O} \subseteq \mathbb{R}^d$ be an open set, the set $\mathrm{C}^{0,\alpha}_{\mathrm{loc}}(\mathcal{O})$ is that of all functions in $\mathrm{C}^0(\mathcal{O})$ such that $[f]_{\mathrm{C}^{0,\alpha}(K)} < \infty$, for every compact set $K \subseteq \mathcal{O}$. The set $\mathrm{C}^{0,\alpha}(\overline{\mathcal{O}})$ is the set of all functions f in $\mathrm{C}^0(\overline{\mathcal{O}})$ that $||f||_{\mathrm{C}^{0,\alpha}(\overline{\mathcal{O}})} = ||f||_{\mathrm{C}^0(\overline{\mathcal{O}})} + [f]_{\mathrm{C}^{0,\alpha}(\mathcal{O})} < \infty$. Define $\mathrm{C}^{k,\alpha}_{\mathrm{loc}}(\mathcal{O})$ as the set of functions in $\mathrm{C}^k(\mathcal{O})$ that satisfy $[\partial^a f]_{\mathrm{C}^{0,\alpha}(K)} < \infty$, for every compact set $K \subseteq \mathcal{O}$ and every $a \in \mathcal{D}_m$, with $0 \le m \le k$. The set $\mathrm{C}^{k,\alpha}(\overline{\mathcal{O}})$ denotes the set of all functions in $\mathrm{C}^k(\overline{\mathcal{O}})$ such that $[\partial^a f]_{\mathrm{C}^{0,\alpha}(\mathcal{O})} < \infty$, for every $a \in \mathcal{D}_m$, with $0 \le m \le k$. This set is equipped with the following norm

$$||f||_{\mathrm{C}^{k,\alpha}(\overline{\mathcal{O}})} = ||f||_{\mathrm{C}^k(\overline{\mathcal{O}})} + \sum_{m=0}^{k} \sum_{a \in \mathcal{D}_m} [\partial^a f]_{\mathrm{C}^{0,\alpha}(\mathcal{O})}. \tag{0.2}$$

Taking $k = 2$ in (0.2), the norm for the space $\mathrm{C}^{2,\alpha}(\overline{\mathcal{O}})$ takes the following form

$$||f||_{\mathrm{C}^{k,\alpha}(\overline{\mathcal{O}})} = ||f||_{\mathrm{C}^{0,\alpha}(\overline{\mathcal{O}})} + \sum_{i=1}^{n} ||\partial_i f||_{\mathrm{C}^{0,\alpha}(\overline{\mathcal{O}})} + \sum_{i,j=1}^{n} ||\partial^2_{ij} f||_{\mathrm{C}^{0,\alpha}(\overline{\mathcal{O}})}.$$

The set $\mathrm{C}^{k,\alpha}_{\mathrm{c}}(\mathcal{O})$ consists of all functions in $\mathrm{C}^{k,\alpha}(\mathcal{O})$ whose support is compact and contained in $\mathcal{O}$. This space is equipped with the norm $||\cdot||_{\mathrm{C}^{k,\alpha}(\mathcal{O})}$. We understand $\mathrm{C}^{k,\alpha}(\mathbb{R}^d)$ as $\mathrm{C}^{k,\alpha}(\overline{\mathbb{R}^d})$, when $\mathcal{O} = \mathbb{R}^d$, in the sense that $[\partial^a f]_{\mathrm{C}^{0,\alpha}(\mathbb{R}^d)} < \infty$, for every $a \in \mathcal{D}_m$, with $0 \le m \le k$.

0.2 L^p and Sobolev spaces

As usual, $\mathrm{L}^p(\mathcal{O})$ with $1 \le p < \infty$, denotes the class of real valued functions on $\mathcal{O}$ with finite norm

$$||f||^p_{\mathrm{L}^p(\mathcal{O})} = \int_{\mathcal{O}} |f|^p \mathrm{d}x < \infty,$$

where $\mathrm{d}x$ denotes the Lebesgue measure. Also, let $\mathrm{L}^p_{\mathrm{loc}}(\mathcal{O})$ consist of functions whose L^p-norm is finite on any compact subset of $\mathcal{O}$. Define the Sobolev space $\mathrm{W}^{k,p}(\mathcal{O})$ as the class of functions $f \in \mathrm{L}^p(\mathcal{O})$ with weak or distributional partial derivatives $\partial^a f$, see [1, p. 22], and with finite norm

$$||f||^p_{\mathrm{W}^{k,p}(\mathcal{O})} = \sum_{m=0}^{k} \sum_{a \in \mathcal{D}_m} ||\partial^a f||^p_{\mathrm{L}^p(\mathcal{O})}, \text{ for all } f \in \mathrm{W}^{k,p}(\mathcal{O}). \tag{0.3}$$

The space $\mathrm{W}^{k,p}_{\mathrm{loc}}(\mathcal{O})$ consists of functions whose $\mathrm{W}^{k,p}$-norm is finite on any compact subset of $\mathcal{O}$. When $p = \infty$, the Sobolev and Lipschitz spaces are related. In particular, $\mathrm{W}^{k,\infty}_{\mathrm{loc}}(\mathcal{O}) = \mathrm{C}^{k-1,1}(\mathcal{O})$ for an arbitrary subset $\mathcal{O} \subseteq \mathbb{R}^d$, and $\mathrm{W}^{k,\infty}(\mathcal{O}) = \mathrm{C}^{k-1,1}(\overline{\mathcal{O}})$ for a sufficiently smooth domain $\mathcal{O}$, when it is Lipschitz; see Definition 2.1.

0.3 Proof of Proposition 0.1

In this section we shall give a proof of Proposition 0.1.

Proof. Let $f : \mathcal{O} \longrightarrow \mathbb{R}$ be a uniformly continuous function and $x \in \partial\mathcal{O}$. There exists a sequence $\{x_n\}_{n\geq 1} \subset \mathcal{O}$ such that $\lim\limits_{n\to\infty} x_n = x$. Thus, $\{x_n\}_{n\geq 1}$ is a Cauchy sequence, i.e. for all $\rho > 0$ there exists a positive integer N,

$$||x_n - x_m|| \leq \rho, \text{ for all } n, m \geq N.$$

Since f is uniformly continuous, we have that for each $\epsilon > 0$, there exists a positive integer N,

$$|f(x_n) - f(x_m)| \leq \epsilon, \text{ for all } n, m \geq N.$$

Then, $\{f(x_n)\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$. Therefore, by completeness in $\mathbb{R}$, $\overline{f}(x) := \lim\limits_{n\to\infty} f(x_n)$ exists. If $\{y_n\} \subset \mathcal{O}$ is another sequence that converges to x, and proceeding the same way as for $\overline{f}(x)$, we can see that $\overline{f}_1(x) := \lim\limits_{n\to\infty} f(y_n)$ exists. Now, we shall verify that $\overline{f}(x) = \overline{f}_1(x)$, showing that $\overline{f}(x)$ is well defined. Since $\lim\limits_{n\to\infty} x_n = x = \lim\limits_{n\to\infty} y_n$, we get that

$$||x_n - y_n|| \leq ||x_n + x|| + ||x - y_n|| \underset{n\to\infty}{\longrightarrow} 0$$

Then, for each $\rho > 0$, there exists a positive integer N_1,

$$||x_n - y_n|| \leq \rho, \text{ for all } n \geq N_1. \tag{0.4}$$

By uniform continuity of f, it follows

$$|f(x_n) - f(y_n)| \leq \frac{\epsilon}{3}, \text{ for all } n \geq N_1, \tag{0.5}$$

for each $\epsilon > 0$. We know that there exist positive integers N_2 and N_3 such that

$$\begin{cases} |\overline{f}(x) - f(x_n)| \leq \dfrac{\epsilon}{3} & \text{for all } n \geq N_2, \\ |\overline{f}_1(x) - f(y_n)| \leq \dfrac{\epsilon}{3} & \text{for all } n \geq N_3, \end{cases} \tag{0.6}$$

for each $\epsilon > 0$. Taking $N = \max\{N_1, N_2, N_3\}$ and from (0.5)–(0.6), it implies

$$|\overline{f}(x) - \overline{f}_1(x)| \leq |\overline{f}(x) - f(x_n)| + |f(y_n) - \overline{f}_1(x)| + |f(x_n) - f(y_n)| \leq \epsilon,$$

for all $n \geq N$. Thus, $\overline{f}(x) = \overline{f}_1(x)$. Defining $\overline{f} : \overline{\mathcal{O}} \longrightarrow \mathbb{R}$ as

$$\overline{f}(x) = \begin{cases} f(x), & \text{if } x \in \mathcal{O}, \\ \lim\limits_{n\to\infty} f(x_n), & \text{if } x \in \overline{\mathcal{O}} \text{ and } \{x_n\}_{n\geq 1} \subset \mathcal{O} \text{ such that } \lim\limits_{n\to\infty} x_n = x, \end{cases}$$

we now show that $\overline{f}$ is a continuous function. In case that $x \in \mathcal{O}$, this is strightforward. When $x \in \partial\mathcal{O}$, we have two cases,

$$\lim_{\substack{y_n \xrightarrow[n\to\infty]{} x \\ \{y_n\}_{n\geq 1} \subset \mathcal{O}}} f(y_n) = \overline{f}(x), \text{ and } \lim_{\substack{y_n \xrightarrow[n\to\infty]{} x \\ \{y_n\}_{n\geq 1} \subset \partial\mathcal{O}}} \overline{f}(y_n) = \overline{f}(x).$$

Let $\{y_n\}_{n\geq 1} \subset \mathcal{O}$ be a sequence such that $\lim\limits_{n\to\infty} y_n = x$. Since $\{x_n\}_{n\geq 1}$ is a sequence that converges to x, it yields (0.4). Then, by uniform continuity of f, it follows that for each $\epsilon > 0$, there exists a positive integer N_1 such that

$$|f(x_n) - f(y_n)| \leq \frac{\epsilon}{2}, \text{ for all } n \geq N_1.$$

By definition of $\overline{f}(x)$, there exists N_2 such that

$$|\overline{f}(x) - f(x_n)| \leq \frac{\epsilon}{2}, \text{ for all } n \geq N_2.$$

Then, taking $N = \max\{N_1, N_2\}$, we conclude

$$|\overline{f}(x) - f(y_n)| \leq |\overline{f}(x) - f(x_n)| + |f(x_n) - f(y_n)| \leq \epsilon, \text{ for all } n \geq N.$$

Thus,

$$\lim_{\substack{y_n \xrightarrow[n\to\infty]{} x \\ \{y_n\}_{n\geq 1} \subset \mathcal{O}}} f(y_n) = \overline{f}(x).$$

Let $\{y_n\}_{n\geq 1} \subset \partial\mathcal{O}$ be a sequence such that $\lim\limits_{n\to\infty} y_n = x$. This means that for any $\rho > 0$, there exists a positive integer N_1,

$$||x - y_n|| \leq \frac{\rho}{3}, \text{ for all } n \geq N_1. \tag{0.7}$$

Since

$$\begin{cases} \overline{f}(x) = \lim_{n\to\infty} f(x_n), \\ \overline{f}(y_n) = \lim_{m\to\infty} f(x_{n,m}), \text{ for all } n \geq 1, \end{cases}$$

with $\{x_n\}_{n\geq 1}, \{x_{n,m}\}_{m\geq 1} \subset \mathcal{O}$ sequences such that

$$\begin{cases} \lim_{n\to\infty} x_n = x, \\ \lim_{m\to\infty} x_{n,m} = y_n, \text{ for all } n \geq 1, \end{cases}$$

we have that for $\epsilon > 0$ and $\rho > 0$ given in (0.7), there exists a positive integer N_2 that satisfies

$$\begin{cases} |\overline{f}(x) - f(x_n)| \leq \dfrac{\epsilon}{3}, \\ ||x_n - x|| \leq \dfrac{\rho}{3}, \end{cases} \tag{0.8}$$

for all $n \geq N_2$, and for each $n \geq N_1$, there exists a positive integer N_n such that

$$\begin{cases} |\overline{f}(y_n) - f(x_{n,N_n})| \leq \dfrac{\epsilon}{3}, \\ ||x_{n,N_n} - y_n|| \leq \dfrac{\rho}{3}. \end{cases} \tag{0.9}$$

Then, taking $N = \max\{N_1, N_2\}$, we get

$$||x_n - x_{n,N_n}|| \leq ||x_n - x|| + ||x - y_n|| + ||y_n - x_{n,N_n}|| \leq \rho, \text{ for all } n \geq N.$$

By uniformly continuous of f, it follows for $\varepsilon > 0$ given in (0.8) and (0.9), that

$$|f(x_n) - f(x_{n,N_n})| \leq \frac{\epsilon}{3}, \text{ for all } n \geq N. \tag{0.10}$$

From (0.8)–(0.10), we conclude

$$|\overline{f}(x) - \overline{f}(y_n)| \leq |\overline{f}(x) - f(x_n)| + |f(x_n) - f(x_{n,N_n})| + |f(x_{n,N_n}) - \overline{f}(y_n)| \leq \epsilon,$$

for all $n \geq N$. Then

$$\lim_{\substack{y_n \xrightarrow[n\to\infty]{} x \\ \{y_n\}_{n\geq 1} \subset \partial\mathcal{O}}} f(y_n) = \overline{f}(x).$$

Therefore, it yields that $\overline{f}$ is a continuous function in $\overline{O}$. Finally, we shall prove uniqueness of the extension $\overline{f}$. Suppose $\overline{f}$ and $\overline{f}_1$ are two continuous extensions of f from $\mathcal{O}$ to $\overline{\mathcal{O}}$. If $x \in \mathcal{O}$, it is clear from the definition of extension that $\overline{f}(x) = f(x) = \overline{f}_1(x)$. If $x \in \partial\mathcal{O}$, there exists $\{x_n\}_{n\geq 1} \subset \mathcal{O}$ a sequence that converges to x. Then

$$\overline{f}(x) = \lim_{n\to\infty} f(x_n) = \overline{f}_1(x).$$

Thus, $\overline{f}$ is the unique continuous extension of f from $\mathcal{O}$ to $\overline{\mathcal{O}}$. ∎

Chapter 1

Introduction and main results

In this thesis, we study a Hamilton-Jacobi-Bellman (HJB) equation in the domain $B_R(0)$, with $R > 0$, whose operator associated is an elliptic integro-differential operator. The HJB equation analyzed in this work is closely related to singular stochastic control problems, where the controlled process is a d-dimensional Lévy process, whose components are a Brownian motion with drift and a compound Poisson process; see (1.6). We recall that a Lévy process is a càdlàg process with independent and stationary increments [30]. Our main goal is to establish the existence, uniqueness and regularity of the solution u to the HJB equation

$$\begin{cases} \max\{qu(x) - \Gamma u(x) - h(x), ||\,\mathrm{D}^1 u(x)||^2 - 1\} = 0, & \text{a.e. in } B_R(0), \\ u(x) = 0, & \text{on } \partial B_R(0), \end{cases} \tag{1.1}$$

where $B_R(0) \subset \mathbb{R}^d$, with $R > 0$ and $d \geq 2$ fixed. The components of this equation are:

(i) A constant $q > 0$ and a positive function $h : \overline{B_R(0)} \longrightarrow \mathbb{R}$.

(ii) An integro-differential operator Γ which has two parts, an elliptic partial differential operator and an integral operator, i.e.

$$\Gamma u(x) := \frac{1}{2}\operatorname{tr}(\sigma\,\mathrm{D}^2 u(x)) + \langle \mathrm{D}^1 u(x), \gamma\rangle + \int_{\mathbb{R}^*} (E(u)(x+z) - u(x) - \langle \mathrm{D}^1 u(x), z\rangle)\nu(\mathrm{d}z), \tag{1.2}$$

with $x \in B_R(0)$. Here $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{R}^d$, $\sigma = (\sigma_{ij})_{d\times d} \in \mathbb{R}^{d\times d}$ is a positive definite matrix, and ν is a finite non-trivial Lévy measure in $\mathbb{R}^* := \mathbb{R}^d \setminus \{0\}$ such that $\int_{\mathbb{R}^*} ||z||\nu(\mathrm{d}z) < \infty$. The operator $E : \mathrm{C}^{k,\alpha}(\overline{B_R(0)}) \longrightarrow \mathrm{C}^{k,\alpha}_{\mathrm{c}}(\mathbb{R}^d)$, with $k \geq 0$ and $\alpha \in [0, 1]$, is a continuous linear operator that has the following properties: there exist constants $C = C(k, R) > 0$ and $b > 0$ such that for every $w \in \mathrm{C}^{k,\alpha}(\overline{B_R(0)})$,

$$\begin{cases} E(w)\,\big|_{\overline{B_R(0)}} = w, \\ \mathrm{supp}[E(w)] \text{ is compact}, \\ \mathrm{supp}[E(w)] \subset B_{R+\frac{b}{2}}(0), \\ ||E(w)||_{\mathrm{C}^{k,\alpha}(\mathbb{R}^d)} \leq C||w||_{\mathrm{C}^{k,\alpha}(\overline{B_R(0)})}, \end{cases} \tag{1.3}$$

where $\mathrm{supp}[E(w)] := \{x \in \mathbb{R}^d : E(w)(x) \neq 0\}$. The norm $||\cdot||_{\mathrm{C}^{k,\alpha}(\cdot)}$ is as in (0.2).

Since $\int_{\mathbb{R}^*} ||z||\nu(\mathrm{d}z) < \infty$ and the continuous linear operator E satisfies (1.3), we see that Γ, given in (1.2), can be written as

$$\Gamma u(x) = \frac{1}{2}\,\mathrm{tr}(\sigma\,\mathrm{D}^2\,u(x)) + \langle \mathrm{D}^1\,u(x), \widetilde{\gamma}\rangle + \int_{\mathbb{R}^*} (E(u)(x+z) - u(x))\nu(\mathrm{d}z), \tag{1.4}$$

for all $x \in B_R(0)$, where

$$\widetilde{\gamma} := \gamma - \int_{\mathbb{R}^*} z\nu(\mathrm{d}z). \tag{1.5}$$

Note that the operator Γ as in (1.4) is the infinitesimal generator of the d-dimensional Lévy process $Y = \{Y_t : t \geq 0\}$ given by

$$Y_t = W_t + \widetilde{\gamma}t + \int_{[0,t]}\int_{\mathbb{R}^*} z\,\vartheta(\mathrm{d}s \times \mathrm{d}z), \text{ for all } t \geq 0, \tag{1.6}$$

where $W = \{W_t : t \geq 0\}$ is a d-dimensional Brownian motion with Gaussian covariance matrix σ, $\widetilde{\gamma} \in \mathbb{R}^d$ as in (1.5), and ϑ is a Poisson random measure in $[0, \infty) \times \mathbb{R}^*$ equipped of the σ-algebra $\mathcal{B}$ generated by $\mathcal{B}[0, \infty) \times \mathcal{B}(\mathbb{R}^*)$, with an intensity measure $\mathrm{d}t \times \nu(\mathrm{d}z)$. The last part on the right side in (1.6) is a compound Poisson process with rate $\nu(\mathbb{R}^*)$ and the distribution of its jumps is $\nu(\mathbb{R}^*)^{-1}\nu(\mathrm{d}z)$. Recall that the process Y has independent and stationary increments, whose paths are right continuous with left limits, and $Y_0 = 0$ almost surely. For background of Lévy processes we refer to [30], which will be our main reference.

The following hypotheses will be assumed throughout the thesis.

Hypotheses

(H1) The function $h \in \mathrm{C}^2(\overline{B_R(0)})$ *is positive. Then* $||h||_{\mathrm{C}^2(\overline{B_R(0)})} \leq C_0$, *for some constant* $C_0 > 0$.

(H2) The Lévy measure ν *satisfies* $\nu(\mathrm{d}z) = \kappa(z)\mathrm{d}z$, *with* $\kappa \in \mathrm{C}^{0,\alpha}(\mathbb{R}^*)$, *for some* $\alpha \in (0, 1)$ *fixed,* $\nu_0 := \nu(\mathbb{R}^*) < \infty$ *and* $\nu_1 := \int_{\mathbb{R}^*} ||z||\nu(\mathrm{d}z) < \infty$, *where* $\mathbb{R}^* = \mathbb{R}^d \setminus \{0\}$.

(H3) *There exist real numbers* $0 < \theta \leq \Theta$ *such that the coefficients of the differential part of* Γ *satisfy*

$$\theta||\zeta||^2 \leq \langle \sigma\zeta, \zeta \rangle \leq \Theta||\zeta||^2, \text{ for all } \zeta \in \mathbb{R}^d,$$

and define $\Lambda := ||\gamma||$.

(H4) *The discount parameter* q *is large enough such that*

$$2A_0 \int_{B_{R+\frac{b}{2}}(0)} \nu(\mathrm{d}z) < q + \nu_0 =: q',$$

where $A_0 \approx 1.03727$ *which is given in (2.25) and* b *is a constant small enough but fixed.* ■

The paper of Soner and Shreve [32] has been one of the main sources of inspiration of this work. In that paper the authors consider the following HJB equation

$$\max\{u - \Delta u - h, ||\mathrm{D}^1 u||^2 - 1\} = 0, \tag{1.7}$$

where $\Delta u := \partial^2_{11} u + \partial^2_{22} u$, $h \in \mathrm{C}^{2,1}_{\mathrm{loc}}(\mathbb{R}^d)$ is an strictly convex function and there exist positive constants C_0 and c_0 such that

$$\begin{cases} 0 = h(0) \leq h(x) \leq C_0(1 + ||x||^2), \\ ||\mathrm{D}^1 h(x)|| \leq C_0(1 + h(x)), \\ c_0||y||^2 \leq \langle \mathrm{D}^2 h(x)y, y \rangle \leq C_0||y||^2(1 + h(x)), \end{cases}$$

for all $x, y \in \mathbb{R}^2$. Soner and Shreve [32] proved that there exists a unique solution $u \in \mathrm{C}^{2,\alpha}(\mathbb{R}^2)$ to the problem (1.7), which is a non-negative convex function. Also, they showed that the value function of a stochastic control problem, where the controlled process is a two-dimensional standard Brownian motion, satisfies the HJB equation (1.7). When the controlled process is a d-dimensional standard Brownian motion, with $d > 2$, Kruk [21] showed that the value function of this stochastic control problem is related to the solution of the HJB equation (1.7), with $h : \mathbb{R}^d \longrightarrow \mathbb{R}$ and $\Delta u := \sum_{i=1}^d \partial^2_{ii} u$. In this case, the solution to the problem (1.7) is in $\mathrm{W}^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$; see [27]. The d-dimensional standard Brownian motion is a particular example of continuous Lévy process. In our setting the controlled process is allowed to be a more general d-dimensional Lévy process, it has a continuous component given by a Brownian motion with drift and a component with jumps given by a compound Poisson process, whose jumps occur at exponential times with parameters $\nu(\mathbb{R}^d \setminus \{0\})$ and jump sizes distributed as $\nu(\mathbb{R}^d \setminus \{0\})^{-1}\nu(\mathrm{d}z)$. This makes that our HJB equation, given in (1.1), differs from (1.7) by an integral term coming from the compound Poisson process in the controlled process, which is naturally related to the integral term in its infinitesimal generator, given by (1.4).

Recently, Davis et. al. [6], Bayraktar et. al. [2] and Menaldi and Robin [26] studied a class of HJB equations in $\mathbb{R}^d$, with an integro-differential operator. The first two works are interested in studying regularity properties of the value function for an infinite horizon discounted cost impulse control problem, where the controlled process is a non-degenerate multidimensional jump diffusion with infinite activity. By probabilistic, partial differential and viscosity methods, they proved that this value function belongs to $\mathrm{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, and it is associated to a HJB equation. Davis et. al. [6] study this problem when the jump process has finite variation, i.e. integro-differential operator of order $[0,1]$, and later Bayraktar et. al. [2] generalizes this work, when the jump process has infinite variation, i.e. integro-differential operator of order $(1,2]$.

In our case, we consider a HJB equation with constant coefficients, and ν as a finite non-trivial Lévy measure that satisfies (H2). The existence of the solution u to the HJB equation (1.1) is a strong sense, i.e., a strong solution of the equation

$$\max\{(qu(x)-\Gamma u(x)-h(x),||\,\mathrm{D}^1 u(x)||^2-1\}=0,\ \text{in } B_R(0), \tag{1.8}$$

is a twice weakly differentiable function on $B_R(0)$ that satisfies (1.8) almost everywhere in $B_R(0)$.

Under the assumptions (H1)–(H4), the main result obtained in this thesis is the following.

Theorem 1.1. *If $d<p<\infty$, there exists a unique nonnegative strong solution $u\in \mathrm{C}^{0,1}(\overline{B_R(0)})\cap \mathrm{W}^{2,p}_{\mathrm{loc}}(B_R(0))$ to the HJB equation*

$$\begin{cases}\max\{qu(x)-\Gamma u(x)-h(x),||\,\mathrm{D}^1 u(x)||^2-1\}=0, & \textit{a.e. in } B_R(0),\\ u(x)=0, & \textit{on } \partial B_R(0).\end{cases}$$

It is worth observing that the solution obtained in this thesis is in a strong sense which should be contrasted with recent results in the topic, where the solutions are established in the viscosity sense. This problem has only been studied in the case that Γ is an elliptic differential operator; see, e.g. [9, 18, 32, 27, 21, 17]. We establish our main result; Theorem 1.1, by probabilistic, integro-differential and PDE classical methods, which are inspired by Evans [9], Lenhart [23], Gimbert and Lions [14], Soner and Shreve [32], Garroni and Menaldi [12] and Hynd [17].

The closest to our work is the paper by Menaldi and Robin [26]. They study a singular control problem for a multidimensional Gaussian-Poisson process, and establish a relationship between the value function to this problem and the solution of the corresponding HJB equation. The multidimensional Gaussian-Poisson process is a Lévy process where it only has a d-dimensional standard Brownian motion and a jump process whose Lévy measure ν satisfies $\int_{\mathbb{R}^*}||z||^p\nu(\mathrm{d}z)<\infty$, for all $p\geq 2$. Although the proofs of their principal results are not provided in detail, and they left these to future works, they give enough arguments to show that the solution to the HJB equation associated with the value function to the singular control problem is in the classical sense.

To guarantee the existence and regularity of the HJB equation (1.1), first we have to analyze the existence, regularity and uniqueness of the solution u^ε to the non-linear integro-differential Dirichlet (NIDD) problem

$$\begin{cases} qu^\varepsilon(x)-\Gamma u^\varepsilon(x)+\psi_\varepsilon(||\,\mathrm{D}^1 u^\varepsilon(x)||^2)=h(x), & \text{in } B_R(0),\\ u^\varepsilon(x)=0, & \text{on } \partial B_R(0); \end{cases} \tag{1.9}$$

see Theorem 1.3. The *penalizing function* $\psi_\varepsilon:\mathbb{R}\longrightarrow\mathbb{R}$, with $\varepsilon\in(0,1)$, is defined by

$$\psi_\varepsilon(r):=\psi\left(\frac{r-1}{\varepsilon}\right),\ \text{for all } r\in\mathbb{R}, \tag{1.10}$$

with $\psi\in\mathrm{C}^\infty(\mathbb{R})$ such that

$$\begin{cases} \psi(r)=0, & \text{for all } r\leq 0,\\ \psi(r)>0, & \text{for all } r>0,\\ \psi(r)=r-1, & \text{for all } r\geq 2,\\ \psi'(r)\geq 0,\ \psi''(r)\geq 0, & \text{for all } r\in\mathbb{R}. \end{cases}$$

The penalty method used in the NIDD problem (1.9), was introduced by L. C. Evans to establish existence and regularity of solutions to second order elliptic equations with gradient constraints [9]. This method has also been used in other works, like [18, 32, 16, 17]. Deducing uniform estimates of the solutions to the NIDD problem (1.9) that allow us to pass to the limit as $\varepsilon\to 0$ in a weak sense in (1.9), it is obtained the existence and regularity of the solution to the HJB equation (1.1).

Although the NIDD problem (1.9) is a tool to guarantee the existence of the HJB equation (1.1), this turns out to be an independent problem of great interest because it can be related with optimal stochastic control problems where the state process is a controlled d-dimensional Lévy process as in (1.6). In this work the optimal stochastic control problem related to the HJB equation (1.1) is not developed, albeit we analyze it for the NIDD problem (1.9); see Section 1.2.

Previous to this work, Bony [4], Bensoussan and Lions [3], Lenhart [23] and [24], Gimbert and Lions [14] and Garroni and Menaldi [12], among others, studied the existence, uniqueness and regularity of the solutions to the linear Dirichlet problem with an integro-differential operator similar to (1.2), obtaining results in the spaces $\mathrm{W}^{2,p}$ and $\mathrm{W}^{1,\infty}\cap\mathrm{W}^{2,p}_{\mathrm{loc}}$, respectively. We note that the NIDD problem (1.9) is more general than the linear Dirichlet problem studied in the works mentioned above, in the sense that our problem has a non-linear part that is determined by $\psi_\varepsilon(||\mathrm{D}^1 u^\varepsilon(\cdot))$. Also, we can also highlight that for each $\varepsilon\in(0,1)$, the solution u^ε to the NIDD problem (1.9) is in $\mathrm{C}^{3,\alpha}(\overline{B_R(0)})$; see Lemma 3.9.

The integral part of the operator Γ given in (1.2) has an important component, this is the continuous linear operator E. The reason for introducing the operator E is that the integral part of Γ is an operator defined in $\mathbb{R}^d$, i.e. when $u:\mathbb{R}^d\longrightarrow\mathbb{R}$ the integral operator is given by $\int_{\mathbb{R}^*}(u(x+z)-u(x)-\langle\mathrm{D}^1 u(x),z\rangle\nu(\mathrm{d}z)$, and hence we can see that it is not well defined if the domain of a function u is restricted to a bounded set. For this reason, it is required that $E:\mathrm{C}^{k,\alpha}(\overline{B_R(0)})\longrightarrow\mathrm{C}^{k,\alpha}_{\mathrm{c}}(\mathbb{R}^d)$, be a continuous linear operator that satisfies the properties described in (1.3). Using this argument the integral operator $\int_{\mathbb{R}^*}(E(u)(x+z)-u(x)-\langle\mathrm{D}^1 u(x),z\rangle)\nu(\mathrm{d}z)$ is well defined and hence the HJB equation (1.1) is also well defined. From (1.3), we know that $\mathrm{supp}[E(u)]\subseteq B_{R+\frac{b}{2}}(0)$. Then, the solution to the HJB equation (1.1) depends also of the values of $E(u)(\cdot)$ on $B_{R+\frac{b}{2}}(0)\setminus B_R(0)$, where b is a fixed small constant, as it is explained in Section 2. Since $E:\mathrm{C}^{k,\alpha}(\overline{B_R(0)})\longrightarrow\mathrm{C}^{k,\alpha}_{\mathrm{c}}(\mathbb{R}^d)$, we can also verify that the solution u^ε to the NIDD problem belongs to $\mathrm{C}^{3,\alpha}(\overline{B_R(0)})$. The construction of the continuous linear operator will be given in Subsection 2.2.

We note that the HJB equation (1.1) and the NIDD problem (1.9) can be defined on any Lipschitz, bounded domain $\mathcal{O}$. Then, the results obtained in this thesis when the domains are open balls of the form $B_R(0)$, with $R>0$, are the same for Lipschitz, bounded domains $\mathcal{O}$. The reason that we restrict ourselves to the case where the domains are $B_R(0)$, with $R>0$, is because the NIDD problem (1.9) is related in the study of the existence, regularity and uniqueness of the HJB equation

$$\max\{qu(x)-\Gamma_1^{'}u(x)-h(x),||\mathrm{D}^1 u(x)||^2-1\}=0,\ \text{in }\mathbb{R}^d,\tag{1.11}$$

where

$$\Gamma_1^{'}u(x):=\frac{1}{2}\operatorname{tr}(\sigma\,\mathrm{D}^2 u(x))+\langle\mathrm{D}^1 u(x),\gamma\rangle+\int_{\mathbb{R}^*}(u(x+z)-u(x)-\langle\mathrm{D}^1 u(x),z\rangle)\nu(\mathrm{d}z).\tag{1.12}$$

The existence and regularity of the solution to the HJB equation (1.11) is obtained passing to limits in a weak sense in (1.9). For this is necessary to find bounds of $u^{\varepsilon,R}$, $\mathrm{D}^1 u^{\varepsilon,R}$ and $\mathrm{D}^2 u^{\varepsilon,R}$ that are independent of (ε, R), where $u^{\varepsilon,R}$ is the solution to the NIDD problem (1.9). The problem here is the constant that limits $|| \mathrm{D}^1 u^{\varepsilon,R}(\cdot)||$, since this constant grows exponentially fast with R; see Lemma 3.19. This is not suitable as it suggests that a bound function for $|| \mathrm{D}^1 u^{\varepsilon,R}(\cdot)||$ in $B_R(0)$ is of the exponential type, and hence it possesses technical issues when we estimate the first and second derivatives of this bound function; see more in the conclusions of this thesis, page 85.

The HJB equation (1.11) arises in the study of the minimization of an infinite horizon discounted running convex cost, where the state process is a controlled d-dimensional Lévy process which components are a d-dimensional Brownian motion with Gaussian covariance matrix σ, and a compound Poisson process with rate $\nu(\mathbb{R}^*)$ and the distribution of its jumps is $\nu(\mathbb{R}^*)^{-1}\nu(\mathrm{d}z)$; see Section 1.2.

Let us now some comments about the hypotheses (H1)–(H4). Hypotheses (H1) and (H4) ensures the existence and uniqueness to the positive solution u^ε of the NIDD problem (1.9) in $\mathrm{C}^{3,\alpha}(\overline{B_R(0)})$; see Theorem 3.8 and Propositions 3.9 and 3.13. The main reason of Hypothesis (H2) is because this is necessary to guarantee the existence of the solution $u^\varepsilon(\cdot; w)$ to the non-linear Dirichlet problem (3.5), when $w \in \mathrm{C}^0(\overline{B_R(0)})$; see Lemma 3.2. Defining the map T_ε as in (3.19) and using contraction fixed point Theorem; see Theorem 3.1, we can prove the existence and uniqueness to the solution u^ε of the NIDD problem (1.9), which is in $\mathrm{C}^{3,\alpha}(\overline{B_R(0)})$. Finally, Hypothesis (H3) is a classical assumption for differential operators called *ellipticity property*, see, e.g. [9, 18, 22, 14, 13, 12, 6, 16, 2].

In the following section, we shall state with an equivalent form of the main result; Theorem 1.1, and we shall give our main contribution concerning to the NIDD problem (1.9). After, we shall show a sketch of the proofs of Theorems 1.2 and 1.3. Finally, in Subsection 1.2, we shall explain the relationship that there exists between the equations (1.9), (1.11) and singular stochastic control problems; see Lemmas 1.4 and 1.5.

1.1 Main results and sketch of proof

Under the assumptions (H1)–(H4), we establish the existence of the unique strong solution to the HJB equation (1.1) in $\mathrm{C}^{0,1}(\overline{B_R(0)}) \cap \mathrm{W}^{2,p}_{\mathrm{loc}}(B_R(0))$, with $d < p < \infty$. First, since $\nu(\mathbb{R}^*) < \infty$, we have that the HJB equation (1.1) can be written as

$$\begin{cases} \max\{q' u(x) - \Gamma' u(x) - h(x), || \mathrm{D}^1 u(x)||^2 - 1\} = 0, & \text{in } B_R(0), \\ u(x) = 0, & \text{on } \partial B_R(0), \end{cases} \tag{1.13}$$

where

$$\begin{cases} q' = q + \nu(\mathbb{R}^*) = q + \nu_0, \\ \Gamma' u(x) := \frac{1}{2}\operatorname{tr}(\sigma \operatorname{D}^2 u(x)) + \langle \operatorname{D}^1 u(x), \widetilde{\gamma} \rangle + \int_{\mathbb{R}^*} E(u)(x+z)\nu(\mathrm{d}z) \\ \qquad\qquad =: \mathcal{L}' u(x) + \mathcal{I}\, E(u)(x). \end{cases} \tag{1.14}$$

The differential and integral part of Γ' are denoted by $\mathcal{L}'$ and $\mathcal{I}$, respectively. Then, our main result; Theorem 1.1, is equivalent to prove the following theorem.

Theorem 1.2. *If $d < p < \infty$, there exists a unique nonnegative strong solution $u \in \mathrm{C}^{0,1}(\overline{B_R(0)}) \cap \mathrm{W}^{2,p}_{\mathrm{loc}}(B_R(0))$ to the HJB equation*

$$\begin{cases} \max\{q'u(x) - \Gamma' u(x) - h(x), ||\operatorname{D}^1 u(x)||^2 - 1\} = 0, & \text{a.s. in } B_R(0), \\ u(x) = 0, & \text{on } \partial B_R(0). \end{cases}$$

Using (1.14) we see that the NIDD problem (1.9) is equivalent to the following NIDD problem

$$\begin{cases} q'u^\varepsilon(x) - \Gamma' u^\varepsilon(x) + \psi_\varepsilon(||\operatorname{D}^1 u^\varepsilon(x)||^2) = h(x), & \text{in } B_R(0), \\ u^\varepsilon(x) = 0, & \text{on } \partial B_R(0), \end{cases} \tag{1.15}$$

where ψ_ε is given in (1.10). We have the following result.

Theorem 1.3. *For each $\varepsilon \in (0,1)$, there exists a unique positive solution u^ε to the NIDD problem (1.15) in the space $\mathrm{C}^{3,\alpha}(\overline{B_R(0)})$.*

The key steps in the proofs of these theorems are the following. First, we guarantee the existence and uniqueness of a sequence of positive functions $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$, where for each $\varepsilon \in (0,1)$, u^ε is the solution to the NIDD problem (1.15). Define for each $\varepsilon \in (0,1)$, the operator $T_\varepsilon : \mathrm{C}^0(\overline{B_R(0)}) \longrightarrow \mathrm{C}^0(\overline{B_R(0)})$ as $T_\varepsilon(w) = V^\varepsilon(\cdot; w)$, whenever $w \in \mathrm{C}^0(\overline{B_R(0)})$, where the value function $V^\varepsilon(\cdot, w)$ is as in (3.16). From Lemma 3.5, it follows

$$T_\varepsilon(w) = V^\varepsilon(\cdot, w) \in \mathrm{C}^{2,\alpha}(\overline{B_R(0)}) \subset \mathrm{C}^0(\overline{B_R(0)}), \text{ for each } w \in \mathrm{C}^0(\overline{B_R(0)}).$$

Verifying that $V^\varepsilon(\cdot; w)$ satisfies

$$||V^\varepsilon(\cdot; w_1) - V^\varepsilon(\cdot; w_2)||_{\mathrm{C}^0(\overline{B_R(0)})} \leq \frac{2A_0}{q'} \int_{B_{R+\frac{b}{2}}(0)} \nu(\mathrm{d}z) ||w_1 - w_2||_{\mathrm{C}^0(\overline{B_R(0)})},$$

for each $w_1, w_2 \in \mathrm{C}^0(\overline{B_R(0)})$; see Lemma 3.6, by Hypothesis (H4), we obtain that T_ε is a contraction mapping in the Banach space $(\mathrm{C}^0(\overline{B_R(0)}), ||\cdot||_{\mathrm{C}^0(\overline{B_R(0)})})$. By contraction fixed point Theorem;

see Theorem 3.1, it yields that there exists a unique $w^* \in \mathrm{C}^0(\overline{B_R(0)})$ such that $T_\varepsilon(w^*) = w^*$; see Lemma 3.7. Using this and that $V^\varepsilon(;w)$ is related with the solution $u^\varepsilon(\cdot;w)$ to the non-linear Dirichlet problem (3.5); see Lemma 3.5, we obtain the existence, uniqueness and regularity of the solution $u^\varepsilon \in \mathrm{C}^{2,\alpha}(\overline{B_R(0)})$ to the NIDD problem (3.1); see Theorem 3.8. By Proposition 3.9, we obtain that $u^\varepsilon \in \mathrm{C}^{3,\alpha}(\overline{B_R(0)})$, and hence it is obtained the result of Theorem 1.3. Now, from Lemma 4.6, we know that there exist a decreasing subsequence $\{\varepsilon_{\kappa(\iota)}\}_{\iota\geq 1}$, with $\varepsilon_{\kappa(\iota)} \to 0$, and $u \in \mathrm{C}^{0,1}(\overline{B_R(0)}) \cap \mathrm{W}^{2,p}_{\mathrm{loc}}(B_R(0))$, such that

$$\begin{cases} u^{\varepsilon_{\kappa(\iota)}} \underset{\varepsilon_{\kappa(\iota)}\to 0}{\longrightarrow} u, & \text{in } \mathrm{C}^1_{\mathrm{loc}}(B_R(0)),\\ u^{\varepsilon_{\kappa(\iota)}} \underset{\varepsilon_{\kappa(\iota)}\to 0}{\longrightarrow} u, & \text{uniformly in } \overline{B_R(0)},\\ u^{\varepsilon_{\kappa(\iota)}} \underset{\varepsilon_{\kappa(\iota)}\to 0}{\longrightarrow} u, & \text{weakly in } \mathrm{W}^{2,p}_{\mathrm{loc}}(B_R(0)). \end{cases}$$

Moreover, the following limit holds

$$\mathcal{I}\,E(u^{\varepsilon_{\kappa(\iota)}})(x) \underset{\varepsilon_{\kappa(\iota)}\to 0}{\longrightarrow} \mathcal{I}\,E(u)(x), \text{ uniformly in } B_R(0).$$

Using Theorem 1.3 and Lemma 4.6, we conclude that u is the solution to the HJB equation (1.13) and hence it is also solution to (1.1). The proof of the uniqueness of u is given in Subsection 4.1.

1.2 Probabilistic interpretation

Through at this document, we will work on a filtered probabilistic space $(\Omega,\, \mathcal{F},\, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0},\, \mathbb{P})$, whose filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions of right-continuity and completeness with respect to $\mathbb{P}$. Let $Y = \{Y_t : t \geq 0\}$ be a d-dimensional Lévy process as in (1.6), which is adapted with respect to the filtration $\mathbb{F}$. By the Lévy-Khintchine formula [30, p. 37] it is well known that the Lévy process Y is determined by a triplet $(\widetilde{\gamma}, \sigma, \nu)$, where $\widetilde{\gamma} \in \mathbb{R}^d$ as in (1.5), σ is a positive definite matrix of size $d \times d$ that satisfies (H3) and ν is a measure on $\mathbb{R}^*$ that satisfies (H2). In the present case the characteristic exponent has the following form

$$\Psi(\lambda) = -\log(\mathbb{E}(\mathrm{e}^{\mathrm{i}\langle\lambda,Y\rangle})) = -\,\mathrm{i}\langle\widetilde{\gamma},\lambda\rangle + \frac{1}{2}\langle\lambda\sigma,\lambda\rangle - \nu(\mathbb{R}^*)\int_{\mathbb{R}^*}(\mathrm{e}^{\mathrm{i}\langle\lambda,z\rangle} - 1)\frac{\nu(\mathrm{d}z)}{\nu(\mathbb{R}^*)},$$

for all $\lambda \in \mathbb{R}^d$, and we recall that its infinitesimal generator is given by (1.4).

1.2.1 Probabilistic interpretation of the HJB equation on the whole space

In this part, it is established the relationship between the HJB equation (1.11) and the value function of an optimal stochastic control problem. The *state process* $X = \{X_t : t \geq 0\}$ is defined as

$$X_t = x + Y_t + \int_{[0,t]} N_s \, \mathrm{d}\xi_s, \text{ for all } t \geq 0, \tag{1.16}$$

where $x \in \mathbb{R}^d$ is the initial condition and Y is a d-dimensional Lévy process as in (1.6). Here the corresponding Lévy measure ν to the process Y satisfies

$$\int_{\mathbb{R}^*} \nu(\mathrm{d}z) < \infty \text{ and } \int_{\mathbb{R}^*} (||z|| \vee ||z||^2)\nu(\mathrm{d}z) < \infty. \tag{1.17}$$

The *control process* $(N, \xi) = \{(N_t, \xi_t) : t \geq 0\}$ is $\mathbb{F}$-adapted with

$$||N_t|| = 1, \text{ for all } t \geq 0 \text{ a.s.,}$$

and, with probability one, ξ is a nondecreasing, left-continuous process with $\xi_0 = 0$. The process N provides the direction and ξ the intensity of the push applied to the state process X. Observing that

$$\int_{[0,t]} N_s \mathrm{d}\xi_s = \int_0^t N_s \mathrm{d}\xi_s^{\mathrm{c}} + \sum_{0<s\leq t} N_s \Delta\xi_s, \text{ for all } t \geq 0,$$

where ξ^{c} is the continuous part of ξ, we can show that the state process X is a semimartingale [28, Ch. II] whose paths are right continuous and with left limits. Note that the jumps of the state process X are inherited from Y and ξ, and we assume that these processes do not jump at the same time t, i.e.

$$\Delta X_t = X_t - X_{t-} = \Delta Y_t \mathbb{1}_{\{\Delta Y_t \neq 0, \, \Delta\xi_t = 0\}} + N_t \Delta\xi_t \mathbb{1}_{\{\Delta\xi_t \neq 0, \, \Delta Y_t = 0\}}, \tag{1.18}$$

for all $t \geq 0$. For $q > 0$ and a control process (N, ξ), the corresponding *cost function* is defined as

$$V_{(N,\xi)}(x) = \mathbb{E}_x \bigg(\int_{[0,\infty)} \mathrm{e}^{-qt} (h(X_t) \, \mathrm{d}t + \mathrm{d}\xi_t) \bigg), \text{ for all } x \in \mathbb{R}^d,$$

where $h \in \mathrm{C}^{2,1}_{\mathrm{loc}}(\mathbb{R}^d)$ is an strictly convex function satisfying for some positive constants C_0 and c_0,

$$\begin{cases} 0 = h(0) \leq h(x) \leq C_0(1 + ||x||^2), \\ ||\,\mathrm{D}^1 h(x)|| \leq C_0(1 + h(x)), \\ c_0||y||^2 \leq \langle \mathrm{D}^2 h(x)y, y\rangle \leq C_0||y||^2(1 + h(x)), \end{cases} \tag{1.19}$$

for all $x, y \in \mathbb{R}^d$. From (1.17) and (1.19), we see $\mathbb{E}(h(Y_t)) < \infty$. The *value function* corresponding to the state process X is given by

$$V(x) = \inf_{(N,\xi)} V_{(N,\xi)}(x), \text{ for } x \in \mathbb{R}^d. \tag{1.20}$$

Note that the HJB equation (1.11) is equivalent to

$$\max\{qu(x) - \Gamma_1 u(x) - h(x), ||\,\mathrm{D}^1 u(x)||^2 - 1\} = 0, \text{ in } \mathbb{R}^d, \tag{1.21}$$

where

$$\Gamma_1 u(x) := \frac{1}{2}\operatorname{tr}(\sigma\,\mathrm{D}^2 u(x)) + \langle \mathrm{D}^1 u(x), \widetilde{\gamma}\rangle + \int_{\mathbb{R}^*} (u(x+z) - u(x))\nu(\mathrm{d}z). \tag{1.22}$$

The relationship between the value function (1.20) and the HJB equation (1.21) is described in the following result.

Lemma 1.4. *Suppose that (1.17) and (1.19) hold true. If u is a convex function in $\mathrm{C}^2(\mathbb{R}^d)$, which is a solution of the HJB equation (1.21), then*

(i) $u(x) \leq V(x)$*, for each* $x \in \mathbb{R}^d$*;*

(ii) given the initial condition $X^*_0 = x$, $x \in \mathbb{R}^d$, suppose that there exists a control process (N^*, ξ^*) such that $V_{(N^*,\xi^*)}(x) < \infty$ and the state process X^* satisfies

$$\begin{cases} (q - \Gamma_1)u(X^*_{t-}) - h(X^*_{t-}) = 0, \\ \displaystyle\int_{[0,t]} \mathbb{1}_{\{N^*_s = -\mathrm{D}^1 u(X^*_{s-})\}}\,\mathrm{d}\xi^*_s = \xi^*_t, \\ (u(X^*_{t-}) - u(X^*_{t+}))\mathbb{1}_{\{\Delta\xi^*_t \neq 0,\, \Delta Y_t = 0\}} = \xi^*_{t+} - \xi^*_t, \end{cases}$$

for all $t \in [0, \infty)$ a.s., with Γ_1 as in (1.4). Then,

$$u(x) = V(x) = V_{(N^*,\xi^*)}(x),$$

i.e. (N^*, ξ^*) is optimal at x.

Proof. Let us assume that u is a convex function in $\mathrm{C}^2(\mathbb{R}^d)$, such that it is a solution of the HJB equation (1.21).

(i) Let $x \in \mathbb{R}^d$ be an initial state and (N, ξ) a control process. Using integration by parts in $\mathrm{e}^{-qt} u(X_t)$ [28, Cor. 2, p. 68], it follows that

$$\mathrm{e}^{-qt} u(X_t) = \int_0^t \mathrm{e}^{-qs}\,\mathrm{d}u(X_s) - \int_0^t q\,\mathrm{e}^{-qs} u(X_s)\mathrm{d}s + [\mathrm{e}^{-qt}, u(X_t)],$$

where $[\mathrm{e}^{-qt}, u(X_t)]$ is the *quadratic covariation* of e^{-qt} and $u(X_t)$ [28, p. 66]. Since e^{-qt} is of bounded variation [28, Thm. 23, p. 68], it implies that

$$[\mathrm{e}^{-qt}, u(X_t)] = u(x).$$

Then

$$\mathrm{e}^{-qt} u(X_t) - u(x) = \int_0^t \mathrm{e}^{-qs} \mathrm{d}u(X_s) - \int_0^t q\,\mathrm{e}^{-qs} u(X_s)\mathrm{d}s. \tag{1.23}$$

Applying Itô's formula to u [28, Thm. 33, p. 81], we get that

$$\begin{aligned} u(X_t) - u(x) = \int_0^t \langle \mathrm{D}^1 u(X_{s-}), \mathrm{d}X_s\rangle + \frac{1}{2}\int_0^t \mathrm{tr}(\sigma\,\mathrm{D}^2 u(X_s))\mathrm{d}s \\ + \sum_{0<s\leq t} (u(X_s) - u(X_{s-}) - \langle \mathrm{D}^1 u(X_{s-}), \Delta X_s\rangle). \end{aligned} \tag{1.24}$$

From (1.6) and (1.16), we observe that

$$\begin{aligned} \mathrm{d}X_t &= \mathrm{d}Y_t + N_t \mathrm{d}\xi_t, \\ \mathrm{d}Y_t &= \mathrm{d}W_t + \widetilde{\gamma}\mathrm{d}t + \int_{\mathbb{R}^*} z\,\vartheta(\mathrm{d}t \times \mathrm{d}z), \end{aligned}$$

and hence, the first term on the right side of (1.24) has the following expression

$$\begin{aligned} \int_0^t \langle \mathrm{D}^1 u(X_{s-}), \mathrm{d}X_s\rangle = \int_0^t \langle \mathrm{D}^1 u(X_{s-}), \mathrm{d}W_s\rangle + \int_0^t \langle \mathrm{D}^1 u(X_s), \widetilde{\gamma}\rangle \mathrm{d}s \\ + \int_0^t \int_{\mathbb{R}^*} \langle \mathrm{D}^1 u(X_{s-}), z\rangle \vartheta(\mathrm{d}s \times \mathrm{d}z) \\ + \int_0^t \langle \mathrm{D}^1 u(X_s), N_s\rangle \mathrm{d}\xi_s^{\mathrm{c}} + \sum_{0<s\leq t} \langle \mathrm{D}^1 u(X_{s-}), N_s\rangle \Delta\xi_s, \end{aligned} \tag{1.25}$$

where ξ^{c} is the continuous part of ξ. Since the state process X jumps only at the times where the process Y or ξ and these processes do not jump at the same time (see (1.18)), hence

$$\begin{aligned} &\sum_{0<s\leq t} (u(X_s) - u(X_{s-}) - \langle \mathrm{D}^1 u(X_{s-}), \Delta X_s\rangle) \\ &= \sum_{0<s\leq t} (u(X_s) - u(X_{s-}) - \langle \mathrm{D}^1 u(X_{s-}), \Delta Y_s\rangle) \mathbb{1}_{\{||\Delta Y_s|| \neq 0,\, ||\Delta\xi_s|| = 0\}} \\ &\quad + \sum_{0<s\leq t} (u(X_s) - u(X_{s-}) - \langle \mathrm{D}^1 u(X_{s-}), N_s\rangle \Delta\xi_s) \mathbb{1}_{\{||\Delta\xi_s|| \neq 0,\, ||\Delta Y_s|| = 0\}}. \end{aligned} \tag{1.26}$$

In case that $||\Delta Y_t|| \neq 0$ and $||\Delta\xi_s|| = 0$, (1.16) implies that

$$X_t = X_{t-} + \Delta Y_t, \text{ for all } t \geq 0,$$

and, if $||\Delta\xi_t|| \neq 0$ and $||\Delta Y_s|| = 0$, defining the process $\{A_t : t \geq 0\}$ as

$$A_t = X_{t-} + \Delta Y_t, \text{ for all } t \geq 0,$$

it follows that

$$\begin{aligned} A_t &= x + Y_{t-} + \int_{[0,t)} N_s \mathrm{d}\xi_s + \Delta Y_t \\ &= x + Y_t + \int_{[0,t]} N_s \mathrm{d}\xi_s - N_t \Delta\xi_t \\ &= X_t - N_t \Delta\xi_t. \end{aligned} \tag{1.27}$$

Now, recalling that Γ_1 is as in (1.22) and combining the equalities (1.24)–(1.27), we get that

$$\begin{aligned} u(X_t) - u(x) = &\int_0^t \Gamma_1 u(X_s)\mathrm{d}s + \int_0^t \langle \mathrm{D}^1 u(X_{s-}), N_s\rangle \mathrm{d}\xi_s^{\mathrm{c}} + \int_0^t \langle \mathrm{D}^1 u(X_s), \mathrm{d}W_s\rangle \\ &+ \sum_{0<s\leq t} (u(A_s + N_s\Delta\xi_s) - u(A_s))\mathbb{1}_{\{||\Delta\xi_s||\neq 0,\, ||\Delta Y_s||=0\}} \\ &+ \int_0^t \int_{\mathbb{R}^*} (u(X_{s-} + z) - u(X_{s-}))(\vartheta(\mathrm{d}s \times \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}s). \end{aligned}$$

Then, the expression (1.23) has the following form

$$\begin{aligned} \mathrm{e}^{-qt} u(X_t) - u(x) = &\int_0^t \mathrm{e}^{-qs}((\Gamma_1 - q)u(X_s) + h(X_s))\mathrm{d}s \\ &- \int_0^t \mathrm{e}^{-qs} h(X_s)\mathrm{d}s + \int_0^t \mathrm{e}^{-qs}\langle \mathrm{D}^1 u(X_{s-}), N_s\rangle \mathrm{d}\xi_s^{\mathrm{c}} + M_t \\ &+ \sum_{0<s\leq t} \mathrm{e}^{-qs}(u(A_s + N_s\Delta\xi_s) - u(A_s))\mathbb{1}_{\{||\Delta\xi_s||\neq 0,\, ||\Delta Y_s||=0\}}, \end{aligned} \tag{1.28}$$

for all $t \geq 0$, where

$$\begin{aligned} M_t := &\int_0^t \mathrm{e}^{-qs}\langle \mathrm{D}^1 u(X_s), \mathrm{d}W_s\rangle \\ &+ \int_0^t \int_{\mathbb{R}^*} \mathrm{e}^{-qs}(u(X_{s-} + z) - u(X_{s-}))(\vartheta(\mathrm{d}s \times \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}s). \end{aligned}$$

Since the process $M = \{M_t : t \geq 0\}$ is a local martingale and defining the stopping time $\tau_{B_n(0)}$ as

$$\tau_{B_n(0)} = \inf\{t > 0 : X_t \notin B_n(0)\}, \text{ for all } n \geq 1,$$

the process $M^{\tau_{B_n}(0)} = \{M_{t\wedge\tau_{B_n(0)}} : t \geq 0\}$ is a $\mathbb{P}_x$-martingale with $M_0 = 0$. Then, taking expected value in (1.28), it follows that

$$\begin{aligned} u(x) =&\mathbb{E}_x\big(\mathrm{e}^{-q(t\wedge\tau_{B_n(0)})}\,u(X_{t\wedge\tau_{B_n(0)}})\big) + \mathbb{E}_x\left(\int_0^{t\wedge\tau_{B_n(0)}} \mathrm{e}^{-qs}((q-\Gamma_1)u(X_s) - h(X_s))\mathrm{d}s\right) \\ &+ \mathbb{E}_x\left(\int_0^{t\wedge\tau_{B_n(0)}} \mathrm{e}^{-qs}\,h(X_s)\mathrm{d}s\right) - \mathbb{E}_x\left(\int_0^{t\wedge\tau_{B_n(0)}} \mathrm{e}^{-qs}\langle \mathrm{D}^1\,u(X_{s-}), N_s\rangle \mathrm{d}\xi_s^{\mathrm{c}}\right) \\ &- \mathbb{E}_x\left(\sum_{0<s\leq t\wedge\tau_{B_n(0)}} \mathrm{e}^{-qs}(u(A_s + N_s\Delta\xi_s) - u(A_s))\mathbb{1}_{\{||\Delta\xi_s||\neq 0,\,||\Delta Y_s||=0\}}\right). \end{aligned} \tag{1.29}$$

Given that u is a convex solution to the HJB equation (1.21), we know that

$$\begin{cases} ||\,\mathrm{D}^1\,u(X_{t-})||^2 - 1 \leq 0, \\ (q-\Gamma_1)u(X_{t-}) - h(X_{t-}) \leq 0, \\ u(A_t + N_t\Delta\xi(t)) - u(A_t) \geq \langle \mathrm{D}^1\,u(A_t), N_t\rangle \Delta\xi(t). \end{cases}$$

Then,

$$u(x) \leq \mathbb{E}_x\big(\mathrm{e}^{-q(t\wedge\tau_{B_n(0)})}\,u(X_{t\wedge\tau_{B_n(0)}})\big) + \mathbb{E}_x\left(\int_0^{t\wedge\tau_{B_n(0)}} \mathrm{e}^{-qs}(h(X_s)\mathrm{d}s + \mathrm{d}\xi_s)\right).$$

Letting $n \to \infty$, it follows that $\tau_{B_n(0)} \to \infty$ a.s. and hence

$$u(x) \leq \mathbb{E}_x\big(\mathrm{e}^{-qt}\,u(X_t)\big) + \mathbb{E}_x\left(\int_0^t \mathrm{e}^{-qs}(h(X_s)\mathrm{d}s + \mathrm{d}\xi_s)\right). \tag{1.30}$$

Since

$$\lim_{t\to\infty} \mathbb{E}_x\left(\int_0^t \mathrm{e}^{-qs}(h(X_s)\mathrm{d}s + \mathrm{d}\xi_s)\right) = \mathbb{E}_x\left(\int_0^\infty \mathrm{e}^{-qs}(h(X_s)\mathrm{d}s + \mathrm{d}\xi_s)\right),$$

we only need to prove that

$$\varliminf_{t\to\infty} \mathbb{E}(\mathrm{e}^{-qt}\,u(X_t)) = 0. \tag{1.31}$$

Assume that $\mathbb{E}_x\big(\int_0^\infty \mathrm{e}^{-qt}\,h(X_t)\mathrm{d}t\big) < \infty$. Otherwise (1.30) is always true. This implies that

$$\varliminf_{t\to\infty} \mathbb{E}_x(\mathrm{e}^{-qt}\,h(X_t)) = 0.$$

By (1.19) and Taylor's Formula, we can observe that

$$\frac{c_0}{2}||y||^2 \leq \int_0^1 (1-\lambda)\langle \mathrm{D}^2\,h(\lambda y)y, y\rangle \mathrm{d}\lambda = h(y).$$

Then, using that u is a convex function and $||\,\mathrm{D}^1 u(y)||^2 < 1$, for all $y \in \mathbb{R}^d$, we can see

$$\begin{aligned} u(y) &\leq u(0) + \langle \mathrm{D}^1 u(y), y\rangle \\ &\leq u(0) + ||\,\mathrm{D}^1 u(y)||\,||y|| \\ &\leq u(0) + 1 + ||y||^2 \\ &\leq u(0) + 1 + \frac{2}{c_0} h(y), \end{aligned}$$

for all $y \in \mathbb{R}^d$. This implies that $\underline{\lim}_{t\to\infty} \mathbb{E}_x(\mathrm{e}^{-qt}\, u(X_t)) = 0$. It follows that $u(x) \leq V(x)$, for each $x \in \mathbb{R}^d$.

(ii) Let $x \in \mathbb{R}^d$ be an initial state and (N^*, ξ^*) a control process such that $V_{(N^*,\xi^*)}(x) < \infty$, and the state process X^* satisfies

$$(q - \Gamma_1)u(X^*_{t-}) - h(X^*_{t-}) = 0, \tag{1.32}$$

$$\int_{[0,t]} \mathbb{1}_{\{N^*_s = -\,\mathrm{D}^1 u(X^*_{s-})\}} \mathrm{d}\xi^*_s = \xi^*_t, \tag{1.33}$$

$$(u(X^*_{t-}) - u(X^*_{t+}))\mathbb{1}_{\{\Delta\xi^*_t \neq 0,\, \Delta Y_t = 0\}} = \xi^*_{t+} - \xi^*_t, \tag{1.34}$$

for all $t \in [0, \infty)$ a.s., with Γ_1 as in (1.22). Applying similar arguments as in the previous proof of $u \leq V$, (1.29) holds for X^*. From (1.33) and (1.34), it is easily verified for $\tau^*_{B_n(0)} = \inf\{t > 0 : X^*_t \notin \mathrm{B}_n(0)\}$, with $n \geq 1$, and $t \geq 0$, that

$$\int_0^{t\wedge\tau^*_{B_n(0)}} \langle \mathrm{D}^1 u(X^*_{s-}), N_s\rangle \mathrm{d}\xi^{*\,\mathrm{c}}_s = -\int_0^{t\wedge\tau^*_{B_n(0)}} \mathbb{1}_{\{N^*_s = -\,\mathrm{D}^1 u(X^*_{s-})\}} \mathrm{d}\xi^{*\,\mathrm{c}}_s, \tag{1.35}$$

and

$$\sum_{0<s\leq t\wedge\tau^*_{B_n(0)}} \Delta\xi^*_s = \sum_{0<s\leq t\wedge\tau^*_{B_n(0)}} (u(A_s + N_s\Delta\xi^*_s) - u(A_s))\mathbb{1}_{\{||\Delta\xi^*_s||\neq 0,\, ||\Delta Y_s|| = 0\}}. \tag{1.36}$$

Using (1.32), (1.35) and (1.36) in (1.30), it follows that

$$u(x) = \mathbb{E}_x\big(\mathrm{e}^{-q(t\wedge\tau^*_{B_n(0)})}\, u(X^*_{t\wedge\tau^*_{B_n(0)}})\big) + \mathbb{E}_x\left(\int_0^{t\wedge\tau^*_{B_n(0)}} \mathrm{e}^{-qs}(h(X^*_s)\mathrm{d}s + \mathrm{d}\xi^*_s)\right). \tag{1.37}$$

Letting $n \to \infty$ in (1.37) and by (1.31), we get $u(x) = V_{(N^*,\xi^*)}(x) = V(x)$. This means that (N^*, ξ^*) is the optimal control. ■

1.2.2 Probabilistic interpretation of the NIDD problem

Define the convex function $g_\varepsilon : \mathbb{R}^d \longrightarrow \mathbb{R}$ and its Legendre transform $l_\varepsilon : \mathbb{R}^d \longrightarrow \mathbb{R}$ by

$$\begin{cases} g_\varepsilon(\zeta) := \psi_\varepsilon(||\zeta||^2), \\ l_\varepsilon(\eta) := \sup_\zeta\{\langle \eta, \zeta\rangle - g_\varepsilon(\zeta)\}, \end{cases} \tag{1.38}$$

The Legendre transform l_ε satisfies

$$\begin{cases} l_\varepsilon(\eta) \geq \frac{\varepsilon}{2}||\eta||^2 - g_\varepsilon(\frac{\varepsilon}{2}\eta) \geq \frac{\varepsilon}{4}||\eta||^2, \\ l_\varepsilon(2\psi_\varepsilon'(||\zeta||^2)\zeta) = 2\psi_\varepsilon'(||\zeta||^2)||\zeta||^2 - \psi_\varepsilon(||\zeta||^2), \end{cases} \tag{1.39}$$

for all $\eta, \zeta \in \mathbb{R}^d$. Since g_ε is differentiable, it follows that $g_\varepsilon(\zeta) = \sup_\eta\{\langle \eta, \zeta\rangle - l_\varepsilon(\eta)\}$. Then, the NIDD problem (1.15) can be written as

$$\begin{cases} qu^\varepsilon(x) - \Gamma_1 E(u^\varepsilon)(x) + \sup_\eta\{\langle \mathrm{D}^1 u^\varepsilon(x), \eta\rangle - l_\varepsilon(\eta)\} = h(x), & \text{in } B_R(0), \\ u^\varepsilon(x) = 0, & \text{on } \partial B_R(0), \end{cases} \tag{1.40}$$

where

$$\begin{aligned} \Gamma_1 E(u^\varepsilon)(x) &= \frac{1}{2}\operatorname{tr}(\sigma \mathrm{D}^2 E(u^\varepsilon)(x)) + \langle \mathrm{D}^1 E(u^\varepsilon)(x), \widetilde{\gamma} + \int_{\mathbb{R}^*} (E(u^\varepsilon)(x+z) - E(u^\varepsilon)(x))\nu(\mathrm{d}z) \\ &= \frac{1}{2}\operatorname{tr}(\sigma \mathrm{D}^2 u^\varepsilon(x)) + \langle \mathrm{D}^1 u^\varepsilon(x), \widetilde{\gamma}\rangle + \int_{\mathbb{R}^*} (E(u^\varepsilon)(x+z) - u^\varepsilon(x))\nu(\mathrm{d}z), \end{aligned}$$

for all $x \in B_R(0)$. A *control process* is any d-dimensional, absolutely continuous process $\varrho = \{\varrho_t : t \geq 0\}$ $\mathbb{F}$-adapted and satisfying $\varrho_0 = 0$ almost surely. Given an initial state $x \in B_R(0)$, we define the state process $Z = \{Z_t : t \geq 0\}$ by

$$Z_t := x + Y_t - \varrho_t, \text{ for all } t \geq 0,$$

where $Y = \{Y_t : t \geq 0\}$, is a d-dimensional Lévy process as in(1.6). The *cost function* corresponding to ϱ is given by

$$V_\varrho^\varepsilon(x) := \mathbb{E}_x\bigg(\int_0^{\tau_{B_R(0)}} \mathrm{e}^{-qs}(h(Z_s) + l_\varepsilon(\dot{\varrho}_s))\,\mathrm{d}s\bigg),$$

for all $x \in B_R(0)$, with $\tau_{B_R(0)} := \inf\{t \geq 0 : Z_t \notin B_R(0)\}$ and $\dot{\varrho}_t := \frac{\mathrm{d}\varrho_t}{\mathrm{d}t}$. Finally, the *value function* is defined by

$$V^\varepsilon(x) := \inf_\varrho V_\varrho^\varepsilon(x).$$

Recalling that $u^\varepsilon \in \mathrm{C}^{2,\alpha}(\overline{B_R(0)})$ is the solution to the NIDD problem (3.15), the following result is obtained.

Lemma 1.5. *The solution u^{ε} to the NIDD problem (3.15) agrees with V^{ε} in $\overline{B_R(0)}$.*

Proof. Let ϱ be a control process and $x \in B_R(0)$ fix an initial state. Integration by parts and Itô's formula imply (see [28, Cor. 2 and Thm. 33, pp. 68 and 81, respectively]) that

$$\begin{aligned} &u^{\varepsilon}(x) - \mathrm{e}^{-q(t\wedge\tau_{B_R(0)})}\, u^{\varepsilon}(Z_{t\wedge\tau_{B_R(0)}}) \\ &= \int_0^{t\wedge\tau_{B_R(0)}} \mathrm{e}^{-qs}(qu^{\varepsilon}(Z_s)) - \Gamma_1 E(u^{\varepsilon})(Z_s) + \langle \mathrm{D}^1\, u^{\varepsilon}(Z_s), \dot{\varrho}_s\rangle\rangle \,\mathrm{d}s - M_{t\wedge\tau_{B_R(0)}}, \end{aligned} \tag{1.41}$$

for all $t \geq 0$, with

$$\begin{aligned} M_t := &\int_0^t \mathrm{e}^{-qs}\langle \mathrm{D}^1\, u^{\varepsilon}(Z_s), \mathrm{d}W_s\rangle \\ &+ \int_0^t \int_{\mathbb{R}^*} \mathrm{e}^{-qs}(E(u^{\varepsilon})(Z_{s-}+z) - u^{\varepsilon}(Z_{s-}))(\vartheta(\mathrm{d}s\times\mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}s), \end{aligned}$$

The process $M = \{M_t : t \geq 0\}$ is a local martingale with $M_0 = 0$. Then, the process $M^{\tau_{B_R(0)}} := \{M_{t\wedge\tau_{B_R(0)}} : t \geq 0\}$ is a $\mathbb{P}_x$-martingale with $M_0 = 0$. Then, taking the expected value in (1.41), it follows that

$$\begin{aligned} &u^{\varepsilon}(x) - \mathbb{E}_x(\mathrm{e}^{-q(t\wedge\tau_{B_R(0)})}\, u^{\varepsilon}(Z_{t\wedge\tau_{B_R(0)}})) \\ &= \mathbb{E}_x\bigg(\int_0^{t\wedge\tau_{B_R(0)}} \mathrm{e}^{-qs}(qu^{\varepsilon}(Z_s)) - \Gamma_1 E(u^{\varepsilon})(Z_s) + \langle \mathrm{D}^1\, u^{\varepsilon}(Z_s), \dot{\varrho}_s\rangle\rangle \,\mathrm{d}s\bigg). \end{aligned} \tag{1.42}$$

From (3.15), we get that

$$\mathbb{E}_x(\mathrm{e}^{-q(t\wedge\tau_{B_R(0)})}\, u^{\varepsilon}(Z_{t\wedge\tau_{B_R(0)}})) \geq u^{\varepsilon}(x) - \mathbb{E}_x\bigg(\int_0^{t\wedge\tau_{B_R(0)}} \mathrm{e}^{-qs}(h(Z_{s-}) + l_{\varepsilon}(\dot{\varrho}_s))\mathrm{d}s\bigg). \tag{1.43}$$

Note that $\tau_{B_R(0)} < \infty$ or $\tau_{B_R(0)} = \infty$. If we are over the event $\{\tau_{B_R(0)} < \infty\}$, then, letting $t \to \infty$ in (1.43), we have that

$$u^{\varepsilon}(x) \leq \mathbb{E}_x\bigg(\bigg(\int_0^{\tau_{B_R(0)}} \mathrm{e}^{-qs}(h(Z_s) + l_{\varepsilon}(\dot{\varrho}_s))\mathrm{d}s\bigg)\mathbb{1}_{\{\tau_{B_R(0)}<\infty\}}\bigg). \tag{1.44}$$

Now, if we are over $\{\tau_{B_R(0)} = \infty\}$, we observe that $\mathrm{e}^{-q(t\wedge\tau_{B_R(0)})} = 0$ and $Z_t \in B_R(0)$, for all $t > 0$. Since u^{ε} is a bounded continuous function, we have that

$$\mathbb{E}_x(\mathrm{e}^{-q(t\wedge\tau_{B_R(0)})}\, u^{\varepsilon}(Z_{t\wedge\tau_{B_R(0)}})\mathbb{1}_{\{\tau_{B_R(0)}=\infty\}}) = 0.$$

Then, by (1.43), it yields that

$$u^{\varepsilon}(x) \leq \mathbb{E}_x\bigg(\bigg(\int_0^{\infty} \mathrm{e}^{-qs}(h(Z_s) + l_{\varepsilon}(\dot{\varrho}_s))\mathrm{d}s\bigg)\mathbb{1}_{\{\tau_{B_R(0)}=\infty\}}\bigg). \tag{1.45}$$

From (1.44) and (1.45), we get $u^\varepsilon \leq V^\varepsilon$. Since $\psi'_\varepsilon(||\,\mathrm{D}^1 u^\varepsilon(x)||^2)\,\mathrm{D}^1 u^\varepsilon(x)$ is a Lipschitz continuous function [28, Thm. 6, p. 255], the process $\widetilde{Z} = \{\widetilde{Z}_t : 0 \leq t \leq \tau_{B_R(0)}\}$ is solution to

$$\widetilde{Z}_t = x + Y_t - \int_0^{t\wedge\tau_{B_R(0)}} 2\psi'_\varepsilon(||\,\mathrm{D}^1 u^\varepsilon(\widetilde{Z}_s)||^2)\,\mathrm{D}^1 u^\varepsilon(\widetilde{Z}_s)\mathrm{d}s, \tag{1.46}$$

for all $0 \leq t \leq \tau_{B_R(0)}$. Then, its corresponding control process is given by

$$\dot{\varrho}^R_t = 2\psi'_\varepsilon(||\,\mathrm{D}^1 u^\varepsilon(\widetilde{Z}_{s-})||^2)\,\mathrm{D}^1 u^\varepsilon(\widetilde{Z}_{s-}),\ \text{for all } 0 \leq t \leq \tau_{B_R(0)}. \tag{1.47}$$

The process $\widetilde{X}$ satisfies (1.42) and by (3.13), from a similar it follows that

$$\mathbb{E}_x(\mathrm{e}^{-q'(t\wedge\tau_{B_R(0)})}\,u^\varepsilon(\widetilde{Z}_{t\wedge\tau_{B_R(0)}})) = u^\varepsilon(x) - \mathbb{E}_x\bigg(\int_0^{t\wedge\tau_{B_R(0)}} \mathrm{e}^{-qs}(h(\widetilde{Z}_s) + l_\varepsilon(\dot{\varrho}^R_s))\mathrm{d}s\bigg),$$

Proceeding of a similar way that (1.44) and (1.45), we have that $u^\varepsilon(x) = V^{\varepsilon,R}(x)$. We finish the proof. ■

The rest of this thesis is organized as follows. Section 2 is devoted to the study of the extension operator E. We first recall an extension theorem for Hölder spaces (Theorem 2.10), whose proof can be found in [33, p. 353]. Then, Theorem 2.10 gives a continuous linear operator $E : \mathrm{C}^{k,\alpha}(\overline{B_R(0)}) \longrightarrow \mathrm{C}^{k,\alpha}_{\mathrm{c}}(\mathbb{R}^d)$, which is used to verify that $\mathcal{I}\,E(w)$ is well defined when $w \in \mathrm{C}^k(\overline{B_R(0)})$, where $k \geq 0$. Before that, we need to introduce the definition of Lipschitz domain, since the construction of the continuous linear operator E depends on the regularity of $\partial B_R(0)$. We also discuss properties of $\mathcal{I}\,E(w)$, when $w \in \mathrm{C}^k(\overline{B_R(0)})$. In Section 3 we present the existence, regularity and uniqueness to the non-linear Dirichlet problems (1.15) and (3.5); the former with an integro-differential operator, and the latter with a differential operator. We also discuss some properties of these solutions. In Section 4 we establish the existence and uniqueness of the HJB equation (1.13); see Theorem 1.2. We also present some properties of this solution.

Chapter 2

Extension theorem and properties of the integral operator

In the first part of this chapter, we give a brief introduction about Lipchitz domains and some of their properties. In particular, we focus in $B_R(0)$, with $R > 0$. The reason for doing this is because we need to have a complete description of the Lipschitz functions that are defined on the neighborhoods of the points in $\partial B_R(0)$. These functions are part of the sets where the continuous linear operator E is constructed. Recall that E was described in (1.3). After that, we present an extension theorem for Hölder spaces, where we focus in the construction and analysis of the continuous linear operator E, defined (2.36). Although Theorem 2.10 is valid for more general domains, see for instance [5], we are interested in the case that the domains are open balls in $\mathbb{R}^d$. At the end of the section, we show useful properties of the integral operator $\mathcal{I}$, which is defined as $\mathcal{I}\, w(x) = \int_{R^*} E(w)(x+z)\nu(\mathrm{d}z)$.

2.1 Lipschtz domains

It is well known that the regularity of the solutions to partial differential or integro-differential equations on a domain $\mathcal{O} \subseteq \mathbb{R}^d$ depends on the regularity of $\partial\mathcal{O}$, when $\partial\mathcal{O} \neq \varnothing$; see e.g. [9, 18, 22, 14, 23, 13, 12, 6, 5, 2, 17]. In our case, we are interested in having the property of Lipschitz domain, since the construction of the continuous linear operator E requires it. For more general domains, see e.g. [11, 15, 8, 7, 5].

Definition 2.1. *An open set $\mathcal{O} \subseteq \mathbb{R}^d$, with $\partial\mathcal{O} \neq \varnothing$, is said to be Lipschitz, or $\mathrm{C}^{0,1}$, if for every*

$x \in \partial\mathcal{O}$ there exist a neighborhood $\mathcal{U}_x$ of x and $\varphi_x \in \mathrm{C}^{0,1}(\mathbb{R}^{d-1})$ such that, up to rotation,

$$\mathcal{U}_x \cap \mathcal{O} = \mathcal{U}_x \cap \{y = (y', y_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : y_d > \varphi_x(y')\}.$$

The open balls $B_R(0)$, with $R > 0$, satisfies this definition as it is verified below. To show that $\partial B_R(0)$ satisfies the Definition 2.1, we use the following Lipschitz extension theorem; its proof is given in [10, p. 80].

Theorem 2.2. *Let $f : D \longrightarrow \mathbb{R}$ be a Lipschitz function, with $D \subseteq \mathbb{R}^d$. There exists a Lipschitz function $\overline{f} : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that*

$$\begin{cases} \overline{f} = f \text{ in } D, \\ [\overline{f}]_{\mathrm{C}^{0,1}(\mathbb{R}^d)} = [f]_{\mathrm{C}^{0,1}(D)}, \end{cases}$$

with $[\overline{f}]_{\mathrm{C}^{0,1}(\mathbb{R}^d)}$ and $[f]_{\mathrm{C}^{0,1}(D)}$ as in (0.1).

Let $e_d = (0, \ldots, 0, 1) \in \mathbb{R}^d$ and define $\mathcal{H} := \{e_d\}^{\perp}$, $x' := P_{\mathcal{H}}(x)$, and $x_d := \langle e_d, x\rangle$, for all $x \in \mathbb{R}^d$, where $\mathcal{H}$ and $P_{\mathcal{H}}$ are the orthogonal hyperplane to e_d and the orthogonal projection onto $\mathcal{H}$, respectively. We identify $\mathcal{H}$ with $R^{d-1} := \{(z', 0) \in \mathbb{R}^d : z' \in \mathbb{R}^{d-1}\}$, where this is a $d - 1$ dimension vectorial subspace, and write $x = x' + x_d e_d$, denoting this vector by (x', x_d). A *direction vector* ω in $\mathbb{R}^d$ is identified with a unit vector. Defining the *orthogonal subgroup* of $d \times d$ matrices as

$$\mathcal{O}_{d\times d} := \{A \in \mathbb{R}^{d\times d} : A^{\mathrm{T}} A = AA^{\mathrm{T}} = I\},$$

where A^{T} is the transposed matrix of A, the direction vector $\omega \in \mathbb{R}^d$ can be associated with some $A \in \mathcal{O}_{d\times d}$ such that $\omega = Ae_d$. An important property of the orthogonal subgroup is that it preserves the inner product $\langle \cdot, \cdot\rangle$, i.e. for each $A \in \mathcal{O}_{d\times d}$ and any $x_1, x_2 \in R^d$, we have that

$$\langle Ax_1, Ax_2\rangle = x_1^{\mathrm{T}} A^{\mathrm{T}} A x_2 = x_1^{\mathrm{T}} x_2 = \langle x_1, x_2\rangle.$$

This implies that $||Ax|| = ||x|| = ||A^{-1}x||$, for all $A \in \mathcal{O}_{d\times d}$ and $x \in \mathbb{R}^d$. Note that the hyperplane $A\mathcal{H}$ is orthogonal to the vector ω and hence for any element $x \in \mathbb{R}^d$ can be written in terms of the hyperplane $A\mathcal{H}$ and its orthogonal complement, i.e.

$$x = A(\zeta' + \zeta_d e_d), \text{ with } \zeta' = P_{\mathcal{H}}(A^{-1}x) \text{ and } \zeta_d = \langle A^{-1}x, e_d\rangle,$$

where $(\zeta', \zeta_d) \in \mathcal{H} \times \mathbb{R}$. Besides, for each $A \in \mathcal{O}_{d\times d}$, we have that $A^{-\mathrm{T}} = A$, then

$$\langle A^{-1}x, e_d\rangle = x^{\mathrm{T}} A^{-\mathrm{T}} e_d = x^{\mathrm{T}} A e_d = \langle x, Ae_d\rangle, \text{ with } x \in \mathbb{R}^d.$$

In case that $\omega = x_2 - x_1$, with $x_1, x_2 \in \mathbb{R}^d$ fixed, we have that there exists a matrix $A_{x_1} \in \mathcal{O}_{d\times d}$ such that $\omega = A_{x_1} e_d$ and for any $x \in \mathbb{R}^d$,

$$x = x_1 + A_{x_1}(\zeta' + \zeta_d e_d),$$

with $\zeta' = P_{\mathcal{H}}(A_{x_1}^{-1}(x - x_1))$ and $\zeta_d = \langle A_{x_1}^{-1}(x - x_1), e_d\rangle$, where $(\zeta', \zeta_d) \in \mathcal{H} \times \mathbb{R}$. We note that the element x_1 is the origin of the hyperplane $A_{x_1}\mathcal{H}$, whose is orthogonal to the vector $\omega = x_2 - x_1$.

Definition 2.3. *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^d$, with $\partial\mathcal{O} \neq \varnothing$. The set $\mathcal{O}$ is said to be a locally Lipschitz, or locally $\mathrm{C}^{0,1}$, if for each $x \in \partial\mathcal{O}$ there exist*

(i) an open neighborhood $\mathcal{U}_x$ of x;

(ii) a matrix $A_x \in \mathcal{O}_{d\times d}$;

(iii) a bounded open neighborhood $\mathcal{V}_x$ of 0 in $\mathcal{H}$ such that

$$\mathcal{U}_x \subseteq \{y \in \mathbb{R}^d : P_{\mathcal{H}}(A_x^{-1}(y - x)) \in \mathcal{V}_x\}; \tag{2.1}$$

(iv) a function $\varphi_x \in \mathrm{C}^{0,1}(\mathcal{H})$ such that $\varphi_x(0) = 0$ and

$$\mathcal{U}_x \cap \partial\mathcal{O} = \mathcal{U}_x \cap \{x + A_x(\zeta' + \zeta_d e_d) : \zeta' \in \mathcal{V}_x,\ \zeta_d = \varphi_x(\zeta')\}, \tag{2.2}$$

$$\mathcal{U}_x \cap \mathcal{O} = \mathcal{U}_x \cap \{x + A_x(\zeta' + \zeta_d e_d) : \zeta' \in \mathcal{V}_x,\ \zeta_d > \varphi_x(\zeta')\}. \tag{2.3}$$

Lemma 2.4. *The open ball $B_R(0)$, with $R > 0$, is Lipschitz, i.e. for any $x \in \partial B_R(0)$ there exists a Lipschitz function $\varphi_x : \mathcal{H} \longrightarrow \mathbb{R}$ such that $[\varphi_x]_{\mathrm{C}^{0,1}(\mathcal{H})} \leq 2$.*

The above result can be found in [7, Thm. 56, p. 93] for convex subsets of $\mathbb{R}^d$. Delfour and Zolésio [7, Thm. 56, p. 93] prove that any convex subset $\mathcal{O}$ of $\mathbb{R}^d$, with $\mathcal{O} \neq \mathbb{R}^d$ and $\partial\mathcal{O} \neq \varnothing$ is locally Lipschitz. We adapt their proof to show that $B_R(0)$ is Lipschitz, since for each point in $\partial B_R(0)$, it is defined explicit a locally Lipschitz function, given by (2.4), which is used to construct and analyze the continuous linear operator E.

Proof Lemma 2.4. Let $\mathcal{H}$ be the orthogonal hyperplane to $e_d = (0, \ldots, 0, 1) \in \mathbb{R}^d$. Taking $x \in \partial B_R(0)$ and $0 < r < 1$ fixed, we choose $x^+ = \frac{R-2r}{R} x \in B_R(0)$, and the direction vector $\omega_x = \frac{1}{2r}(x^+ - x)$. We associate a matrix $A_x \in \mathcal{O}_{d\times d}$ such that $\omega_x = A_x e_d$. Recall that the hyperplane $A_x\mathcal{H}$ is orthogonal to the direction vector ω_x. Choosing $x^- = \frac{R+2r}{R} x$, it follows that $x^- \notin B_R(0)$

and the minimum distance from x^- to $B_R(0)$ is $||x^- - x|| = 2r$. Then, for each $\zeta' \in \mathcal{H}$, with $||\zeta'|| < 2r$, the line

$$L_{\zeta'} := \{x + A_x(\zeta' + \zeta'_d e_d) : |\zeta'_d| \leq 2r\},$$

from $x^+ + A_x\zeta'$ to $x^- + A_x\zeta'$ in the direction to the vector ω_x has a point in $\overline{B_R(0)}$ and a point in its complement. Note that $x^+ + A_x\zeta' \in B_R(0)$ and $x^- + A_x\zeta' \notin B_R(0)$. Therefore, there exists $\widehat{\zeta}_d$, with $|\widehat{\zeta}_d| \leq 2r$, such that $\widehat{y} = x + A_x(\zeta' + \widehat{\zeta}_d e_d) \in \partial B_R(0) \cap L_{\zeta'}$ minimizes $\zeta_d = \langle A_x e_d, (y - x)\rangle$ over all $y \in \overline{B_R(0)} \cap L_{\zeta'}$. If $\widehat{y}_1$ and $\widehat{y}_2$ are two minimizing points such that $\widehat{\zeta}_{1d} = \widehat{\zeta}_{2d}$, then

$$\widehat{y}_1 = x + A_x(\zeta' + \widehat{\zeta}_{1d} e_d) = x + A_x(\zeta' + \widehat{\zeta}_{2d} e_d) = \widehat{y}_2.$$

Hence, the function $f_x : \{\zeta' \in \mathcal{H} : ||\zeta'|| < 2r\} \longrightarrow \mathbb{R}$ defined as

$$f_x(\zeta') := \inf_{y \in \overline{B_R(0)} \cap L_{\zeta'}} \langle A_x e_d, (y - x)\rangle = \inf_{y \in \overline{B_R(0)} \cap L_{\zeta'}} \langle e_d, A_x^{-1}(y - x)\rangle, \tag{2.4}$$

is finite, well defined and there exists a unique $\widehat{\zeta}_d$, with $|\widehat{\zeta}_d| < 2r$, such that $f_x(\zeta') = \widehat{\zeta}_d$ and $\widehat{y} = x + A_x(\zeta' + \widehat{\zeta}_d e_d)$ is the unique minimizer. When $\zeta' = 0$, we see that $x \in \partial B_R(0) \cap L_0$ minimizes over all $y \in \overline{B_R(0)} \cap L_0$, and hence

$$f_x(0) = 0.$$

Taking the neighborhoods

$$\mathcal{V}_x := \{\zeta' \in \mathcal{H} : ||\zeta'|| < 2r\},$$
$$\mathcal{U}_x := \{y \in \mathbb{R}^d : ||P_{\mathcal{H}}(A_x^{-1}(y - x))|| < 2r,\ |\langle A_x^{-1}(y - x), e_d\rangle| < 2r\},$$

we have that, by construction, these are convex open sets and satisfy the conditions (2.1)–(2.3) from Definition 2.3. Now, we shall verify that the function f defined in (2.4) is a Lipschitz function in $\widehat{\mathcal{V}}_x := \{\zeta' \in \mathcal{H} : ||\zeta'|| < r\}$. We shall show that f is a convex function in $\mathcal{V}_x$. Since $B_R(0)$ and $\mathcal{U}_x$ are convex sets, $\mathcal{U}_x \cap \overline{B_R(0)}$ is also a convex set. Then, for all $y_1, y_2 \in \mathcal{U}_x \cap \overline{B_R(0)}$, it follows that

$$ty_1 + (1 - t)y_2 \in \mathcal{U}_x \cap \overline{B_R(0)}, \text{ for all } t \in (0, 1). \tag{2.5}$$

From the conditions (2.1)–(2.3) that

$$\mathcal{U}_x \cap \overline{B_R(0)} = \mathcal{U}_x \cap \{x + A_x(\zeta' + \zeta'_d e_d) : \zeta' \in \mathcal{V}_x,\ \zeta'_d \geq f_x(\zeta')\}, \tag{2.6}$$

and hence, we get that for any $y_1, y_2 \in \mathcal{U}_x \cap \partial B_R(0)$,

$$y_i = x + A_x(\zeta'_i + \zeta'_{id} e_d), \text{ with } \zeta'_{id} = f_x(\zeta'_i) \text{ and } i = 1, 2. \tag{2.7}$$

Then, (2.5) and (2.6) imply that

$$ty_1 + (1-t)y_2 = x + A_x(t\zeta_1' + (1-t)\zeta_2' + (t\zeta_{1d}' + (1-t)\zeta_{2d}')e_d) \in \mathcal{U}_x \cap \overline{B_R(0)},$$

and (2.6) it follows that

$$t\zeta_{1d}' + (1-t)\zeta_{2d}' \geq f_x(t\zeta_1' + (1-t)\zeta_2'). \tag{2.8}$$

Hence, from (2.7) and (2.8), we get that

$$tf_x(\zeta_1') + (1-t)f_x(\zeta_2') \geq f_x(t\zeta_1' + (1-t)\zeta_2').$$

Therefore, f_x is a convex function in $\widehat{\mathcal{V}}_x$. Now, taking $\zeta_1', \zeta_2' \in \widehat{\mathcal{V}}_x$, such that $\zeta_1' \neq \zeta_2'$, and $\zeta_3' := \zeta_2' + \frac{r}{\rho}(\zeta_2' - \zeta_1')$, with $\rho = ||\zeta_2' - \zeta_1'||$, we note that $\zeta_3' \in \mathcal{V}_x$ and write

$$\zeta_2' = \frac{r}{r+\rho}\zeta_1' + \frac{\rho}{r+\rho}\zeta_3'. \tag{2.9}$$

Then, by convexity of f_x, we have that

$$\begin{aligned} f_x(\zeta_2') &\leq \frac{r}{r+\rho} f_x(\zeta_1') + \frac{\rho}{r+\rho} f_x(\zeta_3') \\ &= \frac{\rho}{r+\rho}(f_x(\zeta_3') - f_x(\zeta_1')) + f_x(\zeta_1') \\ &\leq \frac{\rho}{r}|f_x(\zeta_3') - f_x(\zeta_1')| + f_x(\zeta_1') \end{aligned}$$

Since $|f_x(\zeta')| < r$ for all $||\zeta'|| < r$, it follows that

$$f_x(\zeta_2') - f_x(\zeta_1') \leq \frac{\rho}{r}|f_x(\zeta_3') - f_x(\zeta_1')| \leq 2||\zeta_2' - \zeta_1'||.$$

Interchanging the roles of ζ_1' and ζ_2' in (2.9), we obtain

$$f_x(\zeta_1') - f_x(\zeta_2') \leq 2||\zeta_2' - \zeta_1'||.$$

Therefore f_x is a Lipschitz function in $\widehat{\mathcal{V}}_x$, with $[f_x]_{\mathrm{C}^{0,1}(\widehat{\mathcal{V}}_x)} \leq 2$. we conclude, by Theorem 2.2, that there exists a Lipschitz function $\varphi_x : \mathcal{H} \longrightarrow \mathbb{R}$ such that

$$\varphi_x = f_x \text{ in } \widehat{\mathcal{V}}_x, \text{ and } [\varphi_x]_{\mathrm{C}^{0,1}(\mathbb{R}^d)} = [f_x]_{\mathrm{C}^{0,1}(\widehat{\mathcal{V}}_x)} \leq 2. \qquad \blacksquare$$

Note that as a consequence of Lemma 2.4, the Lipschitz constant $[\varphi_x]_{\mathrm{C}^{0,1}(\mathcal{H})}$ is uniformly bounded. Since $\partial B_R(0)$ is a compact set, we can choose an integer $N \geq 1$ large enough, $x_\kappa \in \partial B_R(0)$ and $b_\kappa > 0$ small enough, with $\kappa \in \{1, \ldots, N\}$, such that

$$\partial B_R(0) \subseteq \bigcup_{\kappa=1}^{N} B_{b_\kappa}(x_\kappa). \tag{2.10}$$

Taking $0 < b < \min_{\kappa\in\{1,\ldots,N\}}\left\{\frac{1}{2^N}, b_\kappa\right\}$ such that $\partial B_R(0) \subseteq \bigcup_{\kappa=1}^N B_{b_\kappa-\frac{b}{2}}(x_\kappa)$, we assume that

$$\begin{cases} x_{\kappa'} \notin B_{b_\kappa}(x_\kappa), \text{ with } \kappa, \kappa' \in \{1,\ldots,N\} \text{ and } \kappa \neq \kappa', \\ B_{b_N-\frac{b}{2}}(x_N) \cap B_{b_1-\frac{b}{2}}(x_1) \neq \varnothing, \\ B_{b_\kappa-\frac{b}{2}}(x_\kappa) \cap B_{b_{\kappa+1}-\frac{b}{2}}(x_{\kappa+1}) \neq \varnothing, \text{ for any } \kappa \in \{1,\ldots,N-1\}. \end{cases} \tag{2.11}$$

We also know from the proof of Lemma 2.4 that for each $x_\kappa \in \partial B_R(0)$, with $\kappa \in \{1,\ldots,N\}$, there is a neighborhood $\mathcal{U}_{x_\kappa}$ defined as

$$\mathcal{U}_{x_\kappa} := \{y \in \mathbb{R}^d : ||P_{\mathcal{H}}(A_{x_\kappa}^{-1}(y-x_\kappa))|| < 2b_\kappa,\ |\langle A_{x_\kappa}^{-1}(y-x_\kappa), e_d\rangle| < 2b_\kappa\},$$

where the matrix $A_{x_\kappa} \in \mathcal{O}_{d\times d}$ is associated to the direction vector $\omega_{x_\kappa} = A_{x_\kappa}e_d$. Recall that $\omega_{x_\kappa} = \frac{1}{2b_\kappa}(x_\kappa^+ - x_\kappa)$, with $x_\kappa^+ = \frac{R-2b_\kappa}{R}x_\kappa \in B_R(0)$. It is easy to verify that $B_{b_\kappa}(x_\kappa) \subseteq \mathcal{U}_{x_\kappa}$, for all $\kappa \in \{1,\ldots,N\}$. Now, taking f_{x_κ} as in (2.4), we know that this function is Lipschitz on $\widehat{\mathcal{V}}_{x_\kappa} := \{\zeta' \in \mathcal{H} : ||\zeta'|| < b_\kappa\}$, and we can see that

$$B_{b_\kappa}(x_\kappa) \cap \overline{B_R(0)} = B_{b_\kappa}(x_\kappa) \cap \{x_\kappa + A_{x_\kappa}(\zeta' + \zeta_d e_d) : \zeta' \in \widehat{\mathcal{V}}_{x_\kappa},\ \zeta_d \geq f_{x_\kappa}(\zeta')\}, \tag{2.12}$$

for all $\kappa \in \{1,\ldots,N\}$. Since f_{x_κ} is a Lipschitz function on $\widehat{\mathcal{V}}_{x_\kappa}$, from Theorem 2.2, we have that the Lipschitz extension of f_{x_κ} is given by

$$\varphi_{x_\kappa}(\zeta') := \inf_{\zeta_1' \in \widehat{\mathcal{V}}_{x_\kappa}} (f_{x_\kappa}(\zeta_1') + [f_{x_\kappa}]_{\mathrm{C}^{0,1}(\widehat{\mathcal{V}}_{x_\kappa})}||\zeta' - \zeta_1'||), \tag{2.13}$$

for all $\zeta' \in \mathcal{H}$ and $\kappa \in \{1,\ldots,N\}$, where this extension satisfies

$$\varphi_{x_\kappa} = f_{x_\kappa} \text{ in } \widehat{\mathcal{V}}_{x_\kappa}, \text{ and } [\varphi_{x_\kappa}]_{\mathrm{C}^{0,1}(\mathbb{R}^d)} = [f_{x_\kappa}]_{\mathrm{C}^{0,1}(\widehat{\mathcal{V}}_{x_\kappa})} \leq 2, \text{ for all } \kappa \in \{1,\ldots,N\}.$$

Defining the set $\mathcal{O}_{\varphi_{x_\kappa}}$ as

$$\mathcal{O}_{\varphi_{x_\kappa}} := \{x_\kappa + A_{x_\kappa}(\zeta' + \zeta_d e_d) : \zeta_d > \varphi_\kappa(\zeta')\}, \tag{2.14}$$

from (2.12) we get

$$\overline{B_{b_\kappa}(x_\kappa)} \cap \overline{B_R(0)} = \overline{B_{b_\kappa}(x_\kappa)} \cap \overline{\mathcal{O}}_{\varphi_{x_\kappa}}, \text{ for any } \kappa \in \{1,\ldots,N\}. \tag{2.15}$$

In order to define the continuous linear operator $E : \mathrm{C}^{k,\alpha}(\overline{B_R(0)}) \longrightarrow \mathrm{C}_{\mathrm{c}}^{k,\alpha}(\mathbb{R}^d)$, given in (2.37), first, for each $\mathcal{O}_{\varphi_{x_\kappa}}$ defined in (2.14) with $\kappa \in \{1,\ldots,d\}$, we shall construct the continuous linear operator $E_\kappa' : \mathrm{C}^{k,\alpha}(\overline{\mathcal{O}}_{\varphi_\kappa}) \longrightarrow \mathrm{C}^{k,\alpha}(\mathbb{R}^d)$, given in (2.26). The contruction of the continuous linear operator E is given by the sequence the continuous linear operators $\{E'_\kappa\}_{\kappa=1}^N$. But before, we give some properties of the Hausdorff distance, which is denoted by $\varrho(x) := \varrho(x, \overline{\mathcal{O}}) = \inf\{||x-y|| : y \in \overline{\mathcal{O}}\}$, with $\mathcal{O}$ an arbitrary open set. The distance of $x = x_\kappa + A_{x_\kappa}(\zeta'_1 + \zeta_{d1} e_d)$ to $\overline{\mathcal{O}}_{\varphi_{x_\kappa}}$, for some $(\zeta'_1, \zeta_{d1}) \in \mathcal{H} \times \mathbb{R}$, is given by

$$\begin{aligned}\varrho_\kappa(x) &:= \inf\{||x-y|| : y = x_\kappa + A_{x_\kappa}(\zeta' + \zeta_d) \in \overline{\mathcal{O}}_{\varphi_{x_\kappa}}\}\\ &= \inf\{||\zeta'_1 - \zeta' - e_d(\zeta_{d1} - \zeta_d)|| : \zeta_d \geq \varphi_{x_\kappa}(\zeta')\},\end{aligned} \tag{2.16}$$

for each $\kappa \in \{1, \ldots, N\}$. The Hausdorff distance $\varrho_\kappa(\cdot) = \varrho_\kappa(\cdot, \overline{\mathcal{O}}_{\varphi_{x_\kappa}})$, with $\kappa \in \{1, \ldots, N\}$ and $\mathcal{O}_{\varphi_{x_\kappa}}$ as in (2.14), satisfies the following result, whose proof can be found in [5, Thm. 16.20, p. 346].

Lemma 2.5. *Let* $\varphi \in \mathrm{C}^{0,1}(\mathbb{R}^{d-1})$,

$$\mathcal{O}_\varphi := \{x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : x_d > \varphi(x')\}, \text{ and } \mathcal{O}_- := \overline{\mathcal{O}}_\varphi^{\mathrm{c}}. \tag{2.17}$$

Then

$$(1 + [\varphi]_{\mathrm{C}^{0,1}})\varrho_\varphi(x) \geq \varphi(x') - x_d, \text{ for any } x = (x', x_d) \in \mathcal{O}_-,$$

where $\varrho_\varphi(\cdot) = \varrho_\varphi(\cdot; \overline{\mathcal{O}}_\varphi)$*. Moreover, for every* $x, y \in \overline{\mathcal{O}}_-$ *with* $x \neq y$*, there exists* $z \in \mathcal{O}_-$ *such that*

$$[z, x) \cup [z, y) \subseteq \mathcal{O}_- \quad \text{and} \quad ||x - z|| + ||z - y|| \leq (2 + 4[\varphi]_{\mathrm{C}^{0,1}})||x - y||.$$

Where $[z, x) := \{\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d : z_i \leq \omega_i < x_i,\ i \in \{1, \ldots, d\}\}$*. This result is true for* $\mathcal{O}_-$ *replaced by* $\mathcal{O}_\varphi$*.*

Since the Hausdorff distance $\varrho(\cdot) = \varrho(\cdot; \overline{\mathcal{O}})$ is not differentiable in general on $\overline{\mathcal{O}}^{\mathrm{c}}$, with $\mathcal{O}$ an open ball, this is replaced by a regularized distance $\varrho^*(\cdot) := \varrho^*(\cdot; \overline{\mathcal{O}})$, which is smooth on $\overline{\mathcal{O}}^{\mathrm{c}}$ and essentially has the same profile as $\varrho(\cdot)$. The existence of this regularized distance is guaranteed by the following result. Its proof is in [33, 20, Thms. 2 and 17.21, pp.171 and 267, respectively].

Lemma 2.6. *Let* $\varphi \in \mathrm{C}^{0,1}(\mathbb{R}^{d-1})$, $\mathcal{O}_\varphi$, $\mathcal{O}_-$ *be defined as in (2.17) and* $\varrho(\cdot) = \varrho(\cdot; \overline{\mathcal{O}}_\varphi)$ *be the Hausdorf distance. Then, there exists* $\varrho^* := \varrho^*(\cdot; \overline{\mathcal{O}}_\varphi) \in \mathrm{C}^\infty(\mathcal{O}_-; [0, \infty))$ *such that for every* $x = (x', x_d), y = (y', y_d) \in \mathcal{O}_-$,

$$\begin{cases} \varrho^*(x) \geq 2(\varphi(x') - x_d), \\ \frac{1}{C}\varrho(x) \leq \varrho^*(x) \leq C\varrho(x), \\ ||\,\mathrm{D}^k\, \varrho^*(x)|| \leq C\varrho(x)^{1-k}, \\ ||\,\mathrm{D}^k\, \varrho^*(x) - \mathrm{D}^k\, \varrho^*(y)|| \leq C||x-y||^\alpha \max\{\varrho(x)^{1-k-\alpha}; \varrho(y)^{1-k-\alpha}\}, \end{cases} \tag{2.18}$$

for some constant $C = C(k, d, [\varphi]_{\mathrm{C}^{0,1}}) > 0$*, where* $k \geq 0$ *is an arbitrary integer and* $\alpha \in [0, 1]$ *is an arbitrary real number.*

The constant C that appears in (2.18) is of the form $C = 2(1 + [\varphi]_{\mathrm{C}^{0,1}})C_1 > 0$, where $C_1 = C_1(k, d) > 0$ is a constant independent of $\overline{\mathcal{O}}_\varphi$. For more details see [33, 5, pp. 183 and 347, respectively]. In our case, from Lemma 2.6, we have the following result.

Corollary 2.7. *Let* $\mathcal{O}_{\varphi_{x_\kappa}}$ *be as in (2.14),* $\mathcal{O}_-^{\varphi_\kappa} := \overline{\mathcal{O}}^{\mathrm{c}}_{\varphi_{x_\kappa}}$*, and* $\varrho_\kappa(\cdot) = \varrho_\kappa(\cdot; \overline{\mathcal{O}}_{\varphi_{x_\kappa}})$ *be the Hausdorf distance, with* $\kappa \in \{1, \ldots, N\}$ *and* N *as in (2.10). Then, there exists* $\varrho^*_\kappa(\cdot) = \varrho^*_\kappa(\cdot; \overline{\mathcal{O}}_{\varphi_{x_\kappa}}) \in \mathrm{C}^\infty(\mathcal{O}_-^{\varphi_\kappa}; [0, \infty))$*, such that for every* $x, y \in \mathcal{O}_-^{\varphi_\kappa}$*,*

$$
\begin{cases}
\varrho^*_\kappa(x) \geq 2(\varphi_\kappa(\zeta_1') - \zeta_{d1}), \\
\frac{1}{C}\varrho_\kappa(x) \leq \varrho^*_\kappa(x) \leq C\varrho_\kappa(x), \\
||\,\mathrm{D}^k\, \varrho^*_\kappa(x)|| \leq C\varrho_\kappa(x)^{1-k}, \\
||\,\mathrm{D}^k\, \varrho^*_\kappa(x) - \mathrm{D}^k\, \varrho^*_\kappa(y)|| \leq C||\zeta_1' - \zeta_2' + (\zeta_{d1} - \zeta_{d2})e_d||^\alpha \\
\qquad\qquad\qquad\qquad \times \max\{\varrho_\kappa(x)^{1-k-\alpha},\ \varrho_\kappa(y)^{1-k-\alpha}\},
\end{cases}
$$

for some constant $C_1 = C_1(k, d) > 0$ *independent of* $\overline{\mathcal{O}}_{\varphi_{x_\kappa}}$*, where* $k \geq 0$ *is an arbitrary integer,* $\alpha \in [0, 1]$ *is an arbitrary real number,* $C = 2C_1(1 + [\varphi_\kappa]_{\mathrm{C}^{0,1}(\mathcal{H})}) \leq 6C_1$ *and* $(\zeta_1', \zeta_{d1}), (\zeta_2', \zeta_{d2}) \in \mathcal{H} \times \mathbb{R}$ *are such that* $x = x_\kappa + A_{x_\kappa}(\zeta_1' + \zeta_{d1}e_d),\ y = x_\kappa + A_{x_\kappa}(\zeta_2' + \zeta_{d2}e_d)$*, with* $x_\kappa \in \partial B_R(0)$*.*

In case that $x \in B_{b_\kappa}(x_\kappa) \setminus \overline{B_R(0)}$, with $\kappa \in \{1, \ldots, N\}$ and N as in (2.10), we obtain that $\varrho^*_\kappa(x)$ is uniformly bounded with respect to $\overline{\mathcal{O}}_{\varphi_{x_\kappa}}$.

Corollary 2.8. *If* $\kappa \in \{1, \ldots, N\}$ *and* $x \in B_{b_\kappa}(x_\kappa) \setminus \overline{B_R(0)}$*, there exists a constant* $C_2 = C_2(k, d) > 0$ *independent of* $\overline{\mathcal{O}}_{\varphi_{x_\kappa}}$*, such that*

$$
\varrho^*_\kappa(x) \leq C_2 b_\kappa. \tag{2.19}
$$

Proof. Let $\kappa \in \{1, \ldots, N\}$ and $x \in B_{b_\kappa}(x_\kappa) \setminus \overline{B_R(0)}$ fixed. Then, there exists $(\zeta_1', \zeta_{d1}) \in \mathcal{H} \times \mathbb{R}$, with $||(\zeta_1', \zeta_{d1})|| < b_\kappa$ such that

$$
x = x_\kappa + A_{x_\kappa}(\zeta_1' + \zeta_{d1}e_d) \in L_{\zeta_1'},
$$

where $L_{\zeta_1'} = \{x_\kappa + A_x(\zeta_1' + \zeta_d' e_d) : |\zeta_d'| \leq 2b_\kappa\}$. Recalling the definition of f_{x_κ} in (2.4), i.e.

$$
f_{x_\kappa}(\zeta') = \inf_{y \in \overline{B_R(0)} \cap L_{\zeta'}} \langle A_{x_\kappa} e_d, (y - x_\kappa)\rangle,\ \text{with } ||\zeta'|| < 2b_\kappa,
$$

we know that there exists a unique $\widehat{\zeta}_d$, with $|\widehat{\zeta}_d| < b_\kappa$, such that

$$f_{x_\kappa}(\zeta_1') = \widehat{\zeta}_d \text{ and } \widehat{y} = x_\kappa + A_{x_\kappa}(\zeta_1' + \widehat{\zeta}_d e_d) \in \partial B_R(0) \cap L_{\zeta_1'}.$$

Besides, there exists a unique $\widehat{\zeta}_{d2}$, with $|\widehat{\zeta}_{d2}| < b_\kappa$, such that

$$\begin{cases} f_{x_\kappa}(\zeta_2') = \widehat{\zeta}_{d2}, \\ \widehat{z} = x_\kappa + A_{x_\kappa}(\zeta_2' + \widehat{\zeta}_{d2} e_d) \in \partial B_R(0) \cap L_{\zeta_2'}, \\ \varrho_\kappa(x) = ||(\zeta_1' + \zeta_{d1} e_d) - (\zeta_2' + \zeta_{d2} e_d)||. \end{cases}$$

Hence, using the triangle inequality, we get that

$$\begin{aligned} \varrho_\kappa(x) &= ||(\zeta_1' + \zeta_{d1} e_d) - (\zeta_2' + \zeta_{d2} e_d)|| \\ &\leq ||(\zeta_1' + \zeta_{d1} e_d) - (\zeta_1' + \widehat{\zeta}_d e_d)|| + ||(\zeta_1' + \widehat{\zeta}_d) e_d) - (\zeta_2' + \zeta_{d2} e_d)|| \\ &\leq ||(\zeta_{d1} - \widehat{\zeta}_d) e_d|| + ||\zeta_1' - \zeta_2'|| + ||(\widehat{\zeta}_d - \zeta_{d2}) e_d|| \\ &\leq 8 b_\kappa. \end{aligned}$$

Therefore, from Corollary 2.7, it follows that $\varrho_\kappa^*(x) \leq 8 C b_\kappa \leq 48 C_1 b_\kappa$. Thus, there exists a constant $C_2 = C_2(k, d) > 0$ such that it satisfies (2.19). ■

Remark 2.9.

(i) Let $x \notin \overline{\mathcal{O}}_{\varphi_{x_\kappa}}$, with $\kappa \in \{1, \ldots, N\}$ and N as in (2.10). Defining

$$x_\kappa(t) := x_\kappa + A_\kappa(\zeta' + (\zeta_d + t \varrho_\kappa^*(x)) e_d), \text{ for all } t \geq 1, \tag{2.20}$$

where $x = x_\kappa + A_{x_\kappa}(\zeta' + \zeta_d e_d)$, $A_{x_\kappa} \in \mathcal{O}_{d \times d}$, $(\zeta', \zeta_d) \in \mathcal{H} \times \mathbb{R}$ and $\varrho_\kappa^*(\cdot) = \varrho_\kappa^*(\cdot; \overline{\mathcal{O}}_{\varphi_{x_\kappa}})$ as in Corollary 2.7, we observe that $x_\kappa(t) \in \mathcal{O}_{\varphi_{x_\kappa}}$. Since $\varphi_{x_\kappa}(\zeta') - \zeta_d > 0$, by Corollary 2.7, we get

$$\zeta_d + t \varrho_\kappa^*(x) \geq \zeta_d + 2t(\varphi_{x_\kappa}(\zeta') - \zeta_d) \geq \varphi_{x_\kappa}(\zeta') + (\varphi_{x_\kappa}(\zeta') - \zeta_d) > \varphi_{x_\kappa}(\zeta'),$$

for all $t \geq 1$. Hence $x_\kappa(t) \in \mathcal{O}_{\varphi_{x_\kappa}}$.

(ii) Now, since $B_{b_\kappa}(x_\kappa) \cap B_R(0) = B_{b_\kappa}(x_\kappa) \cap \mathcal{O}_{\varphi_{x_\kappa}}$, we will need to know when a segment of $x_\kappa(t)$ is contained in $B_{b_\kappa}(x_\kappa)$, with $x_\kappa(t)$ given in (2.20). This occurs if and only if there exists $t' > 1$ such that

$$||x_\kappa - x_\kappa(t')|| = b_\kappa, \text{ when } x = x_\kappa + A_{x_\kappa}(\zeta' + \zeta_d e_d) \in B_{b_\kappa}(x_\kappa) \setminus \overline{\mathcal{O}}_{\varphi_{x_\kappa}}.$$

This is equivalent to

$$||\zeta' + (\zeta_d + t'\varrho^*_\kappa(x))e_d|| = b_\kappa, \text{ when } ||\zeta' + \zeta_d e_d|| < b_\kappa \text{ and } \varphi_{x_\kappa}(\zeta') > \zeta_d.$$

Then, it is easy to see that

$$t' = \frac{(b_\kappa^2 - ||\zeta'||^2)^{\frac{1}{2}} - |\zeta_d|}{\varrho^*_\kappa(x)}, \text{ when } x = x_\kappa + A_{x_\kappa}(\zeta' + \zeta_d e_d) \in B_{b_\kappa}(x_\kappa) \setminus \overline{\mathcal{O}}_{\varphi_{x_\kappa}}.$$

This implies that

$$t' \to \infty \text{ when } \varrho^*_\kappa(x) \to 0.$$ ■

The previous results that were studied in this section, shall be used extensively in the following section. Specially in the construction and properties of the continuous linear operator

$$E : \mathrm{C}^{k,\alpha}(\overline{B_R(0)}) \longrightarrow \mathrm{C}^{k,\alpha}_{\mathrm{c}}(\mathbb{R}^d), \text{ with } k \geq 0 \text{ and } \alpha \in [0,1],$$

which is defined in (2.36).

2.2 Extension theorem for Hölder spaces

The following result gives an extension between Hölder spaces $\mathrm{C}^{k,\alpha}(\overline{B_R(0)})$ and $\mathrm{C}^{k,\alpha}_{\mathrm{c}}(\mathbb{R}^d)$, when $k \geq 0$ is an integer and $\alpha \in [0,1]$, whose proof is given in [5, p. 353]. However, we will reproduce some parts of the proof of Csató et. al. [5], in order to describe in detail the construction of the continuous linear operator E that satisfies Theorem 2.10. This will be useful to study some fundamental properties of $\mathcal{I}\, E(w)$, when $w \in \mathrm{C}^{k,\alpha}(\overline{B_R(0)})$.

Theorem 2.10 (Extension theorem for Hölder spaces)**.** *For any integer $k \geq 0$ and any $0 \leq \alpha \leq 1$, there exists a continuous linear extension operator*

$$E : \mathrm{C}^{k,\alpha}(\overline{B_R(0)}) \longrightarrow \mathrm{C}^{k,\alpha}_{\mathrm{c}}(\mathbb{R}^d),$$

that satisfies

$$\begin{cases} E(w)\,|_{\overline{B_R(0)}} = w, \\ \mathrm{supp}[E(w)] \text{ is compact,} \\ \mathrm{supp}[E(w)] \subseteq B_{R+\frac{b}{2}}(0), \\ ||E(w)||_{\mathrm{C}^{k,\alpha}(\mathbb{R}^d)} \leq C||w||_{\mathrm{C}^{k,\alpha}(\overline{B_R(0)})}, \end{cases} \quad \text{for all } w \in \mathrm{C}^{k,\alpha}(\overline{B_R(0)}), \tag{2.21}$$

for some constants $C = C(k,R) > 0$ and $b > 0$. The norms $||\cdot||_{\mathrm{C}^{k,\alpha}(\mathbb{R}^d)}$ and $||\cdot||_{\mathrm{C}^{k,\alpha}(\overline{B_R(0)})}$ are as in (0.2).

2.2.1 Proof of Theorem 2.10

We shall show some steps of the proof of Theorem 2.10, where we shall give some previous results for the construction of the operator E; see (2.36).

Step 1. The proof of the following result can be found in [33, Lemma 1, p. 182].

Lemma 2.11. *There exists $\Psi \in \mathrm{C}^0([1,\infty))$ such that for every positive integer $k \geq 1$, there exists $A_k > 0$ such that*

$$|\Psi(t)| \leq \frac{A_k}{t^k}, \text{ for every } t \in [1,\infty), \tag{2.22}$$

and for every $k \geq 1$,

$$\int_1^\infty \Psi(t)\mathrm{d}t = 1 \quad \text{and} \quad \int_1^\infty t^k \Psi(t)\mathrm{d}t = 0.$$

Stein [33] proves that $\Psi : [1,\infty) \longrightarrow \mathbb{R}$ given by

$$\Psi(t) := \frac{\mathrm{e}}{\pi t} \operatorname{Im}(\exp(-\mathrm{e}^{\frac{-i\pi}{4}}(t-1)^{\frac{1}{4}})), \tag{2.23}$$

satisfies Lemma 2.11. Note that

$$A_0 := \int_1^\infty |\Psi(t)|\mathrm{d}t \approx 1.03727, \tag{2.24}$$

Defining A_k as

$$A_k := \sup_{t\in[1,\infty)} \{t^k|\Psi(t)|\}, \text{ for each } k \geq 1, \tag{2.25}$$

we can see that A_k satisfies (2.22).

Step 2. Taking $N \geq 1$ and Ψ as in (2.10) and (2.23), respectively, we define the continuous linear operator $E'_\kappa : \mathrm{C}^{k,\alpha}(\overline{\mathcal{O}}_{\varphi_{x_\kappa}}) \longrightarrow \mathrm{C}^{k,\alpha}(\mathbb{R}^d)$ by

$$E'_\kappa(w)(x) := \begin{cases} w(x), & \text{if } x = x_\kappa + A_{x_\kappa}(\zeta' + \zeta_d e_d) \in \overline{\mathcal{O}}_{\varphi_{x_\kappa}}, \\ \int_1^\infty \Psi(t) w(x_\kappa(t))\mathrm{d}t, & \text{if } x = x_\kappa + A_{x_\kappa}(\zeta' + \zeta_d e_d) \notin \overline{\mathcal{O}}_{\varphi_{x_\kappa}}, \end{cases} \tag{2.26}$$

for all $w \in \mathrm{C}^{k,\alpha}(\overline{\mathcal{O}}_{\varphi_{x_\kappa}})$ and $\kappa \in \{1,\ldots,N\}$. Here $\mathcal{O}_{\varphi_{x_\kappa}}$ and $x_\kappa(t)$ are given in (2.14) and (2.20), respectively. By Remark 2.9(i), we have E'_κ is well defined, for each $\kappa \in \{1,\ldots,N\}$. This operator is part of the construction of the continuous linear operator E, that was announced in Theorem 2.10, and satisfies the following result, whose proof is given in [5, p. 348].

Theorem 2.12. *Let $\varphi_{x_\kappa} \in \mathrm{C}^{0,1}(\mathbb{R}^{d-1})$ and $\mathcal{O}_{\varphi_{x_\kappa}}$ defined as in (2.14), with $\kappa \in \{1,\ldots,N\}$. For any integer $k \geq 0$ and any $0 \leq \alpha \leq 1$, the continuous linear operator E'_κ given in (2.26),* *satisfies* $E'_\kappa(w) \in \mathrm{C}^{k,\alpha}(\mathbb{R}^d)$*, for all* $w \in \mathrm{C}^{k,\alpha}(\overline{\mathcal{O}}_{\varphi_{x_\kappa}})$*. In particular, there exists a constant* $C = C(k, \mathcal{O}_{\varphi_{x_\kappa}}) > 0$ *such that for every* $w \in \mathrm{C}^{k,\alpha}(\overline{\mathcal{O}}_{\varphi_{x_\kappa}})$*,*

$$||E'_\kappa(w)||_{\mathrm{C}^{k,\alpha}(\mathbb{R}^d)} \leq C||w||_{\mathrm{C}^{k,\alpha}(\overline{\mathcal{O}}_{\varphi_\kappa})}.$$

We claim that the continuous linear operator E'_κ, with $\kappa \in \{1, \ldots, N\}$, satisfies the following inequality

$$|E'_\kappa(w)(x)| \leq A_0||w||_{\mathrm{C}^{k,\alpha}(\overline{\mathcal{O}}_{\varphi_{x_\kappa}})}, \tag{2.27}$$

for all $x \in \mathbb{R}^d$ and $w \in \mathrm{C}^{k,\alpha}(\overline{\mathcal{O}}_{\varphi_{x_\kappa}})$, with A_0 as in (2.24). When $x = x_\kappa + A_{x_\kappa}(\zeta' + \zeta_d e_d) \in \overline{\mathcal{O}}_{\varphi_\kappa}$, with $(\zeta', \zeta_d) \in \mathcal{H} \times \mathbb{R}$, it is easy to prove the inequality (2.27),

$$|E'_\kappa(w)(x)| = |w(x)| \leq ||w||_{\mathrm{C}^{k,\alpha}(\overline{\mathcal{O}}_{\varphi_{x_\kappa}})} < A_0||w||_{\mathrm{C}^{k,\alpha}(\overline{\mathcal{O}}_{\varphi_{x_\kappa}})}.$$

In the other case, when $x = x_\kappa + A_{x_\kappa}(\zeta', \zeta_d e_d) \notin \overline{\mathcal{O}}_{\varphi_\kappa}$, by definition of E'_κ, we get that

$$|E'_\kappa(w)(x)| \leq \int_1^\infty |\Psi(t)|\,|w(x_\kappa(t))|\mathrm{d}t \leq A_0||w||_{\mathrm{C}^{k,\alpha}(\overline{\mathcal{O}}_{\varphi_{x_\kappa}})}.$$

Therefore, it satisfies (2.27). Note that $E^{'}_\kappa$ is differentiable and it is given by (2.28).

Lemma 2.13. *Let* $\mathcal{O}_{\varphi_{x_\kappa}}$ *be as in (2.14), with* $\kappa \in \{1, \ldots, N\}$*. If* $x \notin \overline{\mathcal{O}}_{\varphi_{x_\kappa}}$ *and* $w \in \mathrm{C}^{k,\alpha}(\overline{O}_{\varphi_{x_\kappa}})$*, then*

$$\partial_i E'_\kappa(w)(x) = \int_1^\infty \Psi(t)(\partial_i w(x_\kappa(t)) + \lambda \partial_d w(x_\kappa(t))\partial_i \varrho^*_\kappa(x))\mathrm{d}t, \tag{2.28}$$

with $i \in \{1, \ldots, d\}$*.*

Proof. Let $\kappa \in \{1, \ldots, N\}$, $x = x_\kappa + A_{x_\kappa}(\zeta' + \zeta_d e_d) \notin \overline{\mathcal{O}}_{\varphi_{x_\kappa}}$, for some $(\zeta', \zeta_d) \in \mathcal{H} \times \mathbb{R}$, $A_{x_\kappa} \in \mathcal{O}_{d\times d}$ and $x_\kappa \in \partial B_R(0)$. Taking $w \in \mathrm{C}^{k,\alpha}(\overline{O}_{\varphi_{x_\kappa}})$, define $g(x,t) \colon = w(x_\kappa(t))$, where $x_\kappa(t) = x_\kappa + A_{x_\kappa}(\zeta' + (\zeta_d + t\varrho^*_\kappa(x))e_d),$ with $t \geq 1$, and $\varrho^*_\kappa(\cdot) = \varrho^*_\kappa(\cdot; \mathcal{O})$ as in Lemma 2.6. First we show that

$$\frac{|\Psi(t)|\,|g((x + \rho e_i), t) - g(x,t)|}{\rho},$$

is an integrable function with respect to the Lebesgue measure $\mathrm{d}t$. Computing first derivatives of $g(x,t)$, with respect to x, we have that

$$\partial_i g(x,t) = \partial_i w(x_\kappa(t)) + t\partial_d w(x_\kappa(t))\partial_i d^*_\kappa(x)), \text{ for all } i \in \{1, \ldots, d\}.$$

Then, for $\epsilon > 0$, there exists $\rho' \in (0,1)$ such that if $\rho \in (0,\rho')$, it follows that

$$\begin{aligned}\frac{|\Psi(t)|\,|g((x+\rho e_i),t)-g(x,t)|}{\rho} &\\ &\leq |\Psi(t)|(\epsilon + |\partial_i g(x,t)|)\\ &\leq |\Psi(t)|(\epsilon + |\partial_i w(x_\kappa(t))| + t|\partial_d w(x_\kappa(t))|\,|\partial_i \varrho^*_\kappa(x)|)\\ &\leq \frac{A_2(\epsilon + ||\partial_i w||_{\mathrm{C}^0(\overline{\mathcal{O}}_{\varphi_{x_\kappa}})}) + A_3|\partial_i \varrho^*_\kappa(x)|\,||\partial_d w||_{\mathrm{C}^0(\overline{\mathcal{O}}_{\varphi_{x_\kappa}})}}{t^2},\end{aligned} \tag{2.29}$$

for all $i \in \{1,\dots,d\}$. Here A_2 and A_3 are given in (2.25). Since $||\partial_i w||_{\mathrm{C}^0(\overline{\mathcal{O}}_\varphi)}$ and $||\partial_d w||_{\mathrm{C}^0(\overline{\mathcal{O}}_\varphi)}$ are finite, the last right part in inequality (2.29) is integrable with respect to the Lebesgue measure $\mathrm{d}t$. Thus, by the Dominate Convergence Theorem, we get

$$\partial_i E'_\kappa(w)(x) = \int_1^\infty \Psi(t)(\partial_i w(x_\kappa(t)) + t\partial_d w(x_\kappa(t))\partial_i \varrho^*_\kappa(x))\mathrm{d}t,$$

for all $i \in \{1,\dots,d\}$. ■

Step 3. Now, we proceed to construct the continuos linear extension operator E, which was announced in Theorem 2.10. First, recall that for any $R > 0$ fixed, we can choose an integer $N \geq 1$ large enough, $x_\kappa \in \partial B_R(0)$ and $b_\kappa > 0$ small enough, with $\kappa \in \{1,\dots,N\}$, such that $\partial B_R(0) \subseteq \bigcup_{\kappa=1}^N B_{b_\kappa}(x_\kappa)$. Taking $0 < b < \min_{\kappa\in\{1,\dots,N\}}\{\frac{1}{2^N}, b_\kappa\}$ such that $\partial B_R(0) \subseteq \bigcup_{\kappa=1}^N B_{b_\kappa - \frac{b}{2}}(x_\kappa)$, we assume that

$$\begin{cases} x_{\kappa'} \notin B_{b_\kappa}(x_\kappa), \text{ with } \kappa,\kappa' \in \{1,\dots,N\} \text{ and } \kappa \neq \kappa', \\ B_{b_N - \frac{b}{2}}(x_N) \cap B_{b_1 - \frac{b}{2}}(x_1) \neq \varnothing, \\ B_{b_\kappa - \frac{b}{2}}(x_\kappa) \cap B_{b_{\kappa+1} - \frac{b}{2}}(x_{\kappa+1}) \neq \varnothing, \text{ for any } \kappa \in \{1,\dots,N-1\}. \end{cases} \tag{2.30}$$

We define the following auxiliary functions. Let $\lambda_\kappa \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^d)$, with $\kappa \in \{1,\dots,N\}$, be such that

$$\lambda_\kappa \in [0,1], \quad \lambda_\kappa = 1, \text{ in } B_{b_\kappa - \frac{b}{2}}(x_\kappa), \quad \text{and} \quad \mathrm{supp}[\lambda_\kappa] \subseteq B_{b_\kappa - \frac{b}{4}}(x_\kappa), \tag{2.31}$$

and $\lambda_0, \lambda_+, \lambda_- \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^d)$ satisfying that

$$\begin{cases} \lambda_0, \lambda_+, \lambda_- \in [0,1], \\ \lambda_0 = 1, \text{ in } B_R(0), \text{ and } \mathrm{supp}[\lambda_0] \subseteq B_{R+\frac{b}{2}}(0), \\ \lambda_+ = 1, \text{ in } B_{R+\frac{b}{2}}(0) \setminus B_{R-\frac{b}{2}}(0) \text{ and } \mathrm{supp}[\lambda_+] \subseteq B_{R+b}(0) \setminus B_{R-b}(0), \\ \lambda_- = 1, \text{ in } B_{R-\frac{b}{2}}(0) \text{ and } \mathrm{supp}[\lambda_-] \subseteq B_R(0). \end{cases} \tag{2.32}$$

Defining the functions Λ_+, Λ_- as

$$\Lambda_+ := \lambda_0 \frac{\lambda_+}{\lambda_+ + \lambda_-}, \quad \text{and} \quad \Lambda_- := \lambda_0 \frac{\lambda_-}{\lambda_+ + \lambda_-},$$

we get that

$$\begin{aligned} &\Lambda_+ \leq 1, \text{ in } \operatorname{supp}[\Lambda_+] \subseteq B_{R+\frac{b}{2}}(0) \setminus B_{R-\frac{b}{2}}(0) \subseteq \bigcup_{\kappa=1}^{N} B_{b_\kappa - \frac{b}{2}}(x_\kappa), \qquad (2.33)\\ &\Lambda_- \leq 1, \text{ in } \operatorname{supp}[\Lambda_-] \subseteq B_R(0), \end{aligned}$$

since

$$\begin{aligned} &\lambda_+ + \lambda_- \geq 1, \text{ in } \operatorname{supp}[\lambda_+ + \lambda_-] \subseteq B_{R+b}(0),\\ &\operatorname{supp}[\lambda_0] \subseteq \{x \in \mathbb{R}^d : \lambda_+ + \lambda_- \geq 1\}. \end{aligned}$$

Then, $\Lambda_+, \Lambda_- \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ and $\Lambda_+ + \Lambda_- = \lambda_0$. Note that by (2.30),

$$1 \leq \sum_{\kappa=1}^{N} \lambda_\kappa^2 \leq 2, \text{ in } \bigcup_{\kappa=1}^{N} B_{b_\kappa - \frac{b}{2}}(x_\kappa), \qquad (2.34)$$

and hence

$$\frac{\Lambda_+}{\sum_{\kappa=1}^{N} \lambda_\kappa^2} \in \mathrm{C}_c^\infty(\mathbb{R}^d) \text{ and } \frac{\Lambda_+}{\sum_{\kappa=1}^{N} \lambda_\kappa^2} \leq 1 \text{ in } \operatorname{supp}[\Lambda_+] \subseteq \bigcup_{\kappa=1}^{N} B_{b_\kappa - \frac{b}{2}}(x_\kappa). \qquad (2.35)$$

Finally, we define the continuous linear extension operator

$$E : \mathrm{C}^{k,\alpha}(\overline{B_R(0)}) \longrightarrow \mathrm{C}_c^{k,\alpha}(\mathbb{R}^d),$$

as

$$E(w)(x) := \Lambda_+(x) \left(\frac{\sum_{\kappa=1}^{N} \lambda_\kappa(x) E'_\kappa(w_\kappa)(x)}{\sum_{\kappa=1}^{N} \lambda_\kappa^2(x)} \right) + \Lambda_-(x) w(x), \qquad (2.36)$$

for all $x \in \mathbb{R}^d$ and $w \in \mathrm{C}^{k,\alpha}(\overline{B_R(0)})$, with $k \geq 0$ an integer and $\alpha \in [0,1]$. For each $\kappa \in \{1, \ldots, N\}$, the operator E'_κ is defined in (2.26) and the function $w_\kappa : \overline{\mathcal{O}}_{\varphi_{x_\kappa}} \longrightarrow \mathbb{R}$ is given by

$$w_\kappa(x) := \begin{cases} \lambda_\kappa(x) w(x), & \text{if } x = x_\kappa + A_{x_\kappa}(\zeta' + \zeta_d e_d) \in \overline{\mathcal{O}}_{\varphi_{x_\kappa}} \cap \overline{B_{b_\kappa}(x_\kappa)},\\ 0, & \text{if } x = x_\kappa + A_{x_\kappa}(\zeta' + \zeta_d e_d) \in \overline{\mathcal{O}}_{\varphi_{x_\kappa}} \setminus \overline{B_{b_\kappa}(x_\kappa)}. \end{cases} \qquad (2.37)$$

To verify that $w_\kappa \in \mathrm{C}^{k,\alpha}(\overline{\mathcal{O}}_{\varphi_{x_\kappa}})$, see for instance [5, p. 355]. The continuous linear extension operator E is well defined by the properties previously reviewed and it is also possible to show that $\operatorname{supp}[E(w)] \subseteq B_{R+\frac{b}{2}}(0)$ and $E(w) = w$ in $\overline{B_R(0)}$.

Remark 2.14. From the construction of the linear continuous operator E, (2.36), we clearly observe that

$$E(w)(x) = \begin{cases} w(x), & \text{if } x \in \overline{B_R(0)}, \\ \frac{\lambda_0(x)\sum_{\kappa=1}^{N}\lambda_\kappa(x)E'_\kappa(w_\kappa)(x)}{\sum_{\kappa=1}^{N}\lambda_\kappa^2(x)}, & \text{if } x \in B_{R+\frac{b}{2}}(0)\setminus\overline{B_R(0)}, \\ 0, & \text{if } x \in B_{R+\frac{b}{2}}(0)^{\mathrm{c}}, \end{cases}$$

for each $w \in \mathrm{C}^{k,\alpha}(\overline{B_R(0)})$. ■

In the following two subsections, we shall give some properties of the continuous linear operator E and the integral operator $\mathcal{I}$.

2.2.2 Some properties of the continuous linear operator E

Define the following sets

$$\begin{aligned} &\mathcal{D}_1 := B_{b_N-\frac{b}{4}}(x_N)\cap B_{b_1-\frac{b}{4}}(x_1), \\ &\mathcal{D}_\kappa := B_{b_{\kappa-1}-\frac{b}{4}}(x_{\kappa-1})\cap B_{b_\kappa-\frac{b}{4}}(x_\kappa), \text{ with } \kappa = 2,\dots,N, \\ &\mathcal{G}_\kappa := B_{b_\kappa-\frac{b}{4}}(x_\kappa)\setminus\mathcal{D}_\kappa, \text{ with } \kappa=1,\dots,N, \\ &\mathcal{J}_\kappa := (B_{R+\frac{b}{2}}(0)\setminus\overline{B_R(0)})\cap\mathcal{G}_\kappa, \text{ with } \kappa=1,\dots,N, \qquad (2.38) \\ &\mathcal{J}'_1 := (B_{R+\frac{b}{2}}(0)\setminus\overline{B_R(0)})\cap\mathcal{D}_1, \qquad (2.39) \\ &\mathcal{J}'_\kappa := (B_{R+\frac{b}{2}}(0)\setminus\overline{B_R(0)})\cap\mathcal{D}_\kappa, \text{ with } \kappa=2,\dots,N. \qquad (2.40) \end{aligned}$$

Note that $\mathcal{B}' := B_{R+\frac{b}{2}}(0)\setminus\overline{B_R(0)} = \bigcup_{\kappa=1}^{N}\mathcal{J}_\kappa$, $\mathcal{D}_\kappa \neq \varnothing$, $\mathcal{J}_\kappa\cap\mathcal{J}_{\kappa'} = \varnothing$, for any $\kappa,\kappa' \in \{1,\dots,N\}$, with $\kappa \neq \kappa'$, and

$$\begin{cases} \mathcal{J}_\kappa = (\mathcal{J}_\kappa\setminus\mathcal{J}'_{\kappa+1})\cup\mathcal{J}'_{\kappa+1}, & \text{if } \kappa\in\{1,\dots,N-1\}, \\ \mathcal{J}_N = (\mathcal{J}_N\setminus\mathcal{J}'_1)\cup\mathcal{J}'_1, & \text{if } \kappa = N. \end{cases}$$

If $x \in \mathcal{B}'$, there exists $\kappa \in \{1,\dots,N\}$ such that $x \in \mathcal{J}_\kappa$. Then, when $\kappa \in \{1,\dots,N-1\}$, we get that

$$\begin{aligned} E(w)(x) &= \lambda_0(x)E'_\kappa(w_\kappa)(x)\mathbb{1}_{\mathcal{J}_\kappa\setminus\mathcal{J}'_{\kappa+1}}(x) \\ &\quad + \frac{\lambda_0(x)}{\lambda_\kappa^2(x)+\lambda_{\kappa+1}^2(x)}(\lambda_\kappa(x)E'_\kappa(w_\kappa)(x)+\lambda_{\kappa+1}(x)E'_{\kappa+1}(w_{\kappa+1})(x))\mathbb{1}_{\mathcal{J}'_{\kappa+1}}(x), \qquad (2.41) \end{aligned}$$

for all $x \in \mathcal{J}_\kappa$. When $\kappa = N$, we obtain that

$$E(w)(x) = \lambda_0(x)E'_N(w_N)(x)\mathbb{1}_{\mathcal{J}_N\setminus\mathcal{J}'_1}(x) \\ + \frac{\lambda_0(x)}{\lambda_N^2(x)+\lambda_1^2(x)}(\lambda_N(x)E'_N(w_N)(x)+\lambda_1(x)E'_1(w_1)(x))\mathbb{1}_{\mathcal{J}'_1}(x). \quad (2.42)$$

for all $x \in \mathcal{J}_N$. Recall that E'_κ and w_κ are given in (2.26) and (2.37), respectively.

Proposition 2.15. *If $w \in \mathrm{C}^0(\overline{B_R(0)})$, then*

$$|E(w)(x)| \leq 2A_0||w||_{\mathrm{C}^0(\overline{B_R(0)})}, \textit{ for all } x \in \mathbb{R}^d, \quad (2.43)$$

with A_0 as in (2.24).

Proof. Let $w \in \mathrm{C}^0(\overline{B_R(0)})$. Observe that when $x \in B_{R+\frac{b}{2}}(0)^{\mathrm{c}}$, (2.43) is trivially true. When $x \in \overline{B_R(0)}$ it follows that

$$|E(w)(x)| = |w(x)| \leq ||w||_{\mathrm{C}^0(\overline{B_R(0)})} \leq 2A_0||w||_{\mathrm{C}^0(\overline{B_R(0)})}.$$

If $x \in \mathcal{B}' = B_{R+\frac{b}{2}}(0) \setminus \overline{B_R(0)}$, we know there exits $\kappa \in \{1,\ldots,N-1\}$ such that $x \in \mathcal{J}_\kappa = (\mathcal{J}_\kappa \setminus \mathcal{J}'_{\kappa+1}) \cup \mathcal{J}'_{\kappa+1}$, or $x \in \mathcal{J}_N = (\mathcal{J}_N \setminus \mathcal{J}'_1) \cup \mathcal{J}'_1$, where $\mathcal{J}_\kappa$ and $\mathcal{J}'_\kappa$ are given in (2.38)–(2.40). Suppose that $x \in \mathcal{J}_\kappa = (\mathcal{J}_\kappa \setminus \mathcal{J}'_{\kappa+1}) \cup \mathcal{J}'_{\kappa+1}$, for some $\kappa \in \{1,\ldots,N-1\}$. From (2.26), (2.31), (2.32), (2.35), (2.37), (2.41), and taking $\mathcal{B}_\kappa := B_R(0) \cap B_{b_\kappa}(x_\kappa)$, with $\kappa \in \{1,\ldots,N\}$, it follows that

$$\begin{aligned} |E(w)(x)| &\leq |E'_\kappa(w_\kappa)(x)|\mathbb{1}_{\mathcal{J}_\kappa\setminus\mathcal{J}'_{\kappa+1}}(x) \\ &\quad + (|E'_\kappa(w_\kappa)(w_\kappa)(x)| + |E'_{\kappa+1}(w_{\kappa+1})(x)|)\mathbb{1}_{\mathcal{J}'_{\kappa+1}}(x) \\ &= |E'_\kappa(x)| + |E'_{\kappa+1}(w_{\kappa+1})(x)|\mathbb{1}_{\mathcal{J}'_{\kappa+1}}(x) \\ &\leq \int_1^\infty |\Psi(t)|\lambda_\kappa(x_\kappa(t))|w(x_\kappa(t))|\mathbb{1}_{\mathcal{B}_\kappa}(x_\kappa(t))\mathrm{d}t \\ &\quad + \int_1^\infty |\Psi(t)|\lambda_{\kappa+1}(x_{\kappa+1}(t))|w(x_{\kappa+1}(t))|\mathbb{1}_{\mathcal{B}_{\kappa+1}}(x_{\kappa+1}(t))\mathrm{d}t \qquad (2.44) \\ &\leq (||w||_{\mathrm{C}^0(\mathcal{B}_\kappa)} + ||w||_{\mathrm{C}^0(\mathcal{B}_{\kappa+1})})\int_1^\infty |\Psi(t)|\mathrm{d}t \qquad (2.45) \\ &\leq 2A_0||w||_{\mathrm{C}^0(\overline{B_R(0)})}, \end{aligned}$$

with A_0 as in (2.24). When $x \in \mathcal{J}_N = (\mathcal{J}_N \setminus \mathcal{J}'_1) \cup \mathcal{J}'_1$, we use (2.42) and proceeding in a similar way to obtain (2.43). ∎

Now, since $\Lambda_+,\ \lambda_\kappa \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^d)$, with

$$\mathrm{supp}[\lambda_\kappa] \subseteq B_{b_\kappa - \frac{b}{4}}(x_\kappa) \text{ and } \mathrm{supp}[\Lambda_+] \subseteq \bigcup_{\kappa=1}^{N} B_{b_\kappa - \frac{b}{2}}(x_\kappa),$$

we can assume that

$$|\partial_i \Lambda_+| \leq C_2 \text{ and } |\partial_i \lambda_\kappa| \leq C_2, \text{ for all } i \in \{1, \ldots, d\}, \tag{2.46}$$

for some constant $C_2 > 0$.

Lemma 2.16. *If $w \in \mathrm{C}^1(\overline{B_R(0)})$, there exists a constant $C_3 = C_3(k,d) > 0$ such that for each $x \in \mathcal{B}' = B_{R+\frac{b}{2}}(0) \setminus \overline{B_R(0)}$,*

$$|\partial_i E(w)(x)| < C_3 ||w||_{\mathrm{C}^1(\overline{B_R(0)})}, \textit{ for all } i \in \{1, \ldots, d\}. \tag{2.47}$$

Proof. Let $x \in \mathcal{B}'$ and $w \in \mathrm{C}^{k,\alpha}(\overline{B_R(0)})$ fixed. Then, there exists $\kappa \in \{1, \ldots, N\}$ such that $x \in \mathcal{J}_\kappa$. When $\kappa \in \{1, \ldots, N-1\}$, from (2.41), we know that

$$\begin{aligned} E(w)(x) &= \lambda_0(x) E'_\kappa(x) \mathbb{1}_{\mathcal{J}_\kappa \setminus \mathcal{J}'_{\kappa+1}}(x) \\ &\qquad + \frac{\lambda_0(x)}{\lambda_\kappa^2(x) + \lambda_{\kappa+1}^2(x)} (\lambda_\kappa(x) E'_\kappa(w_\kappa)(x) + \lambda_{\kappa+1}(x) E'_{\kappa+1}(w_{\kappa+1})(x)) \mathbb{1}_{\mathcal{J}'_{\kappa+1}}(x). \end{aligned}$$

Then, the first derivatives are given by

$$\begin{aligned} \partial_i E(w)(x) &= (E'_\kappa(w_\kappa)(x) \partial_i \lambda_0(x) + \lambda_0(x) \partial_i E'_\kappa(w_\kappa)(x)) \mathbb{1}_{\mathcal{J}_\kappa \setminus \mathcal{J}'_{\kappa+1}}(x) \\ &+ \Bigg(\frac{\lambda_\kappa(x) E'_\kappa(w_\kappa)(x) + \lambda_{\kappa+1}(x) E'_{\kappa+1}(w_{\kappa+1})(x)}{(\lambda_\kappa(x)^2 + \lambda_{\kappa+1}(x)^2)^2} \\ &\times ((\lambda_\kappa^2(x) + \lambda_{\kappa+1}^2(x)) \partial_i \lambda_0(x) - 2\lambda_0(x)(\lambda_\kappa(x) \partial_i \lambda_\kappa(x) + \lambda_{\kappa+1}(x) \partial_i \lambda_{\kappa+1}(x))) \\ &+ \frac{\lambda_0(x)}{\lambda_\kappa^2(x) + \lambda_{\kappa+1}^2(x)} (E'_\kappa(w_\kappa)(x) \partial_i \lambda_\kappa(x) + \lambda_\kappa(x) \partial_i E'_\kappa(w_\kappa)(x)) \\ &+ E'_{\kappa+1}(w_{\kappa+1})(x) \partial_i \lambda_{\kappa+1}(x) + \lambda_{\kappa+1}(x) \partial_i E'_{\kappa+1}(w_{\kappa+1})(x) \Bigg) \mathbb{1}_{\mathcal{J}'_{\kappa+1}}(x). \end{aligned}$$

Recall that the sets $\mathcal{J}_\kappa,\ \mathcal{J}'_\kappa$ are given in (2.38)–(2.40). By (2.37), for each $\iota \in \{\kappa, \kappa+1\}$, we know that $w_\iota(x_\iota(t)) = \lambda_\iota(x_\iota(t)) w(x_\iota(t))$, if $x_\iota(t) \in \mathcal{B}_\iota := B_R(0) \cap \overline{B_{b_\iota - \frac{b}{4}}(x_\iota)}$. Then, from Lemma 2.13 and computing the first derivatives of w_ι, it is follows that

$$\begin{aligned} \partial_i E'_\iota(w_\iota)(x) &= \int_1^\infty \Psi(t)(\partial_i w_\iota(x_\iota(t)) + t \partial_d w_\iota(x_\iota(t)) \partial_i \varrho_\iota^*(x)) \mathrm{d}t \\ &= \int_1^\infty \Psi(t)(\lambda_\iota(x_\iota(t)) \partial_i w(x_\iota(t)) + w(x_\iota(t)) \partial_i \lambda_\iota(x_\iota(t)) \\ &\qquad + t(\lambda_\iota(x_\iota(t)) \partial_d w(x_\iota(t)) + w(x_\iota(t)) \partial_d \lambda_\iota(x_\iota(t))) \partial_i \varrho_\iota^*(x)) \mathbb{1}_{\mathcal{B}_\iota}(x_\iota(t)) \mathrm{d}t, \end{aligned}$$

for all $i \in \{1, \ldots, d\}$. Where ϱ_ι^*, $x_\iota(t)$ and Ψ are given in Corollary 2.7, (2.20) and (2.23), respectively. Hence, using Corollary 2.7, Lemma 2.11, (2.24) and (2.46), we get

$$\begin{aligned}
|\partial_i E'_\iota(w_\iota)(x)| &\leq \int_1^\infty |\Psi(t)|(|\partial_i w(x_\iota(t))| + C_2|w(x_\iota(t))| \\
&\quad + tC(|\partial_d w(x_\iota(t))| + C_2|w(x_\iota(t))|))\mathbb{1}_{\mathcal{B}_\iota}(x_\iota(t))\mathrm{d}t \\
&\leq ||w||_{\mathrm{C}^1(\mathcal{B}'_1)}(1 + C_2)\int_1^\infty |\Psi(t)|(1 + Ct)\mathrm{d}t \\
&\leq ||w||_{\mathrm{C}^1(\mathcal{B}'_1)}(1 + C_2)(A_0 + CA_3),
\end{aligned} \tag{2.48}$$

with C, A_0, A_3, C_2 are constants given in Corollary 2.7, (2.24), (2.25), (2.46), respectively, and $\mathcal{B}'_1 := \overline{B_R(0)} \cap \bigcup_{\kappa=1}^{N} \overline{B_{b_\kappa - \frac{b}{4}}(x_\kappa)}$. Taking $C'_3 := (1 + C_2)(A_0 + CA_3)$, from (2.31)–(2.35) and (2.46)–(2.48), it follows that

$$\begin{aligned}
|\partial_i E(w)(x)| &\leq (C_2(|E'_\kappa(w_\kappa)(x)| + |E'_{\kappa+1}(w_{\kappa+1})(x)|) \\
&\quad + |\partial_i E'_\kappa(w_\kappa)(x)| + |\partial_i E'_{\kappa+1}(w_{\kappa+1})(x)|)\mathbb{1}_{\mathcal{J}_\kappa \setminus \mathcal{J}'_{\kappa+1}}(x) \\
&\quad + (7C_2(|E'_\kappa(w_\kappa)(x)| + |E'_{\kappa+1}(w_{\kappa+1})(x)|) \\
&\quad + |\partial_i E'_\kappa(w_\kappa)(x)| + |\partial_i E'_{\kappa+1}(w_{\kappa+1})(x)|)\mathbb{1}_{\mathcal{J}'_{\kappa+1}}(x) \\
&\leq 2(A_0C_2 + C'_3)||w||_{\mathrm{C}^1(\mathcal{B}'_1)}\mathbb{1}_{\mathcal{J}_\kappa \setminus \mathcal{J}'_{\kappa+1}}(x) \\
&\quad + 2(7A_0C_2 + C'_3)||w||_{\mathrm{C}^1(\mathcal{B}'_1)}\mathbb{1}_{\mathcal{J}'_{\kappa+1}}(x) \\
&\leq 2(7A_0C_2 + C'_3)||w||_{\mathrm{C}^1(\mathcal{B}'_1)},
\end{aligned} \tag{2.49}$$

Taking $C_3 = 2(7A_0C_2 + C'_3)$, it follows (2.47). When $x \in \mathcal{J}_N = (\mathcal{J}_N \setminus \mathcal{J}'_1) \cup \mathcal{J}'_1$, computing first derivatives in (2.42) and proceeding in a similar way, it yields (2.47). ■

2.3 Properties of the integral operator $\mathcal{I}$

Recall that the integral operator $\mathcal{I}$ is defined for each $w \in \mathrm{C}^0(\overline{B_R(0)})$ as

$$\mathcal{I}\, E(w)(x) = \int_{\mathbb{R}^*} E(w)(x + z)\nu(\mathrm{d}z),$$

for all $x \in \overline{B_R(0)}$. Some properties of $\mathcal{I}\, E(w)$ shall be analyzed below. These results will be helpful in order to show some properties of the solutions to the non-linear Dirichlet problems (3.5) and (3.21).

Lemma 2.17. *Let $B_R(0)$ be an open ball in $\mathbb{R}^d$.*

(i) If $w \in \mathrm{C}^0(\overline{B_R(0)})$, then

$$|\mathcal{I}\,E(w)(x)| \leq 2A_0\nu_0||w||_{\mathrm{C}^0(\overline{B_R(0)})}, \text{ for all } x \in \mathbb{R}^d,$$

where ν_0, A_0 are as in (H2) and (2.24), respectively.

(ii) If $w \in \mathrm{C}^0(\overline{B_R(0)})$, then $\mathcal{I}\,E(w) \in \mathrm{C}^{0,\alpha}(\mathbb{R}^d)$.

(iii) If $w \in \mathrm{C}^1(\overline{B_R(0)})$, then $\frac{\partial}{\partial x_i}\mathcal{I}\,E(w) \in \mathrm{C}^{0,\alpha}(\mathbb{R}^d)$ and $\frac{\partial}{\partial x_i}\mathcal{I}\,E(w) = \mathcal{I}\,\frac{\partial}{\partial x_i}E(w)$, for each $i \in \{1, \ldots, d\}$.

Proof. Let $w \in \mathrm{C}^0(\overline{B_R(0)})$ and $x, y \in \mathbb{R}^d$ such that $x \neq y$ fixed. The proof of (i) is as follows. By Proposition 2.15, we have

$$|\mathcal{I}\,E(u)(x)| \leq \int_{\mathbb{R}^*} |E(w)(x+z)|\nu(\mathrm{d}z) \leq 2A_0\nu_0||w||_{\mathrm{C}^0(\overline{B_R(0)})}. \tag{2.50}$$

To prove (ii) we use that $\nu(\mathrm{d}z) = \kappa(z)\mathrm{d}z$ with $\kappa \in \mathrm{C}^{0,\alpha}(\mathbb{R}^*)$, for some $\alpha \in (0,1)$ fixed. Then, from Proposition 2.15, doing variable chance and using that $\mathrm{supp}[E(w)] \subset B_{R+\frac{b}{2}}(0)$, we get

$$\begin{aligned}
|\mathcal{I}\,E(w)(x) - \mathcal{I}\,E(w)(y)| &= \left|\int_{\mathbb{R}^*} (E(w)(x+z) - E(w)(y+z))\nu(\mathrm{d}z)\right| \\
&= \left|\int_{\mathbb{R}^*} E(w)(z')(\kappa(z'-x) - \kappa(z'-y))\mathrm{d}z'\right| \\
&\leq \int_{\mathbb{R}^*} |E(w)(z')|\,|\kappa(z'-x) - \kappa(z'-y)|\mathrm{d}z' \\
&\leq ||\kappa||_{\mathrm{C}^{0,\alpha}(\mathbb{R}^*)}||x-y||^{\alpha} \int_{B_{R+\frac{b}{2}(0)}} |E(w)(z')|\mathrm{d}z' \\
&\leq K_1||x-y||^{\alpha}.
\end{aligned} \tag{2.51}$$

Here $K_1 = 2A_0||w||_{\mathrm{C}^0(\overline{B_R(0)})}||\kappa||_{\mathrm{C}^{0,\alpha}(\mathbb{R}^*)}\big(\int_{B_{R+\frac{b}{2}}(0)} \mathrm{d}z\big)$, where A_0 as in (2.24). This implies that $[\mathcal{I}\,E(w)]_{\mathrm{C}^{0,\alpha}(\overline{B_R(0)})} \leq K_1 < \infty$. Note that from (2.51), it follows that $\mathcal{I}\,E(w)$ is Hölder continuous. Using (2.50), we get

$$||\mathcal{I}\,E(w)||_{\mathrm{C}^0(\overline{B_R(0)})} \leq 2A_0\nu(B_{R+\frac{b}{2}}(0))||w||_{\mathrm{C}^0(\overline{B_R(0)})} < \infty. \tag{2.52}$$

By (2.51) and (2.52) we conclude that $\mathcal{I}\,E(w) \in \mathrm{C}^{0,\alpha}(\mathbb{R}^d)$. Let $w \in \mathrm{C}^1(\overline{B_R(0)})$. To prove (iii), we should first show that for each $i \in \{1, \ldots, d\}$,

$$\frac{E(w)(x+z+\rho e_i) - E(w)(x+z)}{\rho}, \tag{2.53}$$

is bounded by an integrable function with respect to the Lévy measure ν, for all $\rho \in (0,1)$. Since $E(w) \in \mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^d)$, we get that

$$|E(w)(x+z+\rho e_i) - E(w)(x+z)| \leq C||w||_{\mathrm{C}^1(\overline{B_R(0)})}\rho,$$

where $C = C(2,R) > 0$ is a constant as in (2.21). It follows that (2.53) is bounded by $C||w||_{\mathrm{C}^1(\overline{B_R(0)})}$, which is integrable with respect to the Lévy measure ν. Thus, by the dominated convergence Theorem, it follows that

$$\begin{aligned}\frac{\partial}{\partial x_i}\int_{\mathbb{R}^*} E(w)(x+z)\nu(\mathrm{d}z) &= \int_{\mathbb{R}^*} \lim_{\rho\to 0}\frac{E(w)(x+z+\rho e_i) - E(w)(x+z)}{\rho}\nu(\mathrm{d}z)\\ &= \int_{\mathbb{R}^*}\frac{\partial}{\partial x_i}E(w)(x+z)\nu(\mathrm{d}z).\end{aligned}$$

Finally, we proceed to show that $\frac{\partial}{\partial x_i}\mathcal{I}\,E(w) \in \mathrm{C}^{0,\alpha}(\mathbb{R}^d)$, for each $i \in \{1,\ldots,N\}$. Since $E(w) \in \mathrm{C}^1_{\mathrm{c}}(\overline{B_R(0)})$, then, by (i), we conclude $\frac{\partial}{\partial x_i}\mathcal{I}\,E(w) \in \mathrm{C}^{0,\alpha}(\mathbb{R}^d)$, for each $i \in \{1,\ldots,N\}$. ■

The following corollary is a consequence of the previous lemma. Recall that $\mathcal{D}_m$, with $0 \leq m \leq k$, is the set of all multi-indices of order m.

Corollary 2.18. *Let $B_R(0)$ be an open ball in $\mathbb{R}^d$ and $k \geq 0$ an integer. If $w \in \mathrm{C}^k(\overline{B_R(0)})$, then $\mathcal{I}\,E(w) \in \mathrm{C}^{k,\alpha}(\mathbb{R}^d)$.*

From Corollary 2.18, note that the integral operator $\mathcal{I}$ maps $\mathrm{C}^k(\overline{B_R(0)})$ into $\mathrm{C}^{k,\alpha}(\mathbb{R}^d)$, with $k \geq 0$. The following two lemmas describe the behavior of the integral of $E(w)(x+z)$ and $\partial_i E(w)(x+z)$ with respect to the Lévy measure $\nu(\mathrm{d}z)$ when $x+z \in \overline{B_R(0)}^{\mathrm{c}}$ and $x \in B_R(0)$.

Lemma 2.19. *If $w \in \mathrm{C}^0(\overline{B_R(0)})$, then*

$$\left|\int_{\{||x+z||>R\}} E(w)(x+z)\nu(\mathrm{d}z)\right| \leq 2A_0||w||_{\mathrm{C}^0(\mathcal{B}'_1)}\int_{\mathbb{R}^*}\mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z),$$

for all $x \in B_R(0)$, where A_0 is a constant given in (2.24), $\mathcal{B}' = B_{R+\frac{b}{2}}(0) \setminus \overline{B_R(0)}$ and $\mathcal{B}'_1 := \overline{B_R(0)} \cap \bigcup_{\kappa=1}^N \overline{B_{b_\kappa - \frac{b}{4}}(x_\kappa)}$.

Proof. Let $x \in B_R(0)$ and $w \in \mathrm{C}^0(\overline{B_R(0)})$ fixed. Then

$$\left|\int_{\{||x+z||>R\}} E(w)(x+z)\nu(\mathrm{d}z)\right| \leq \int_{\{||x+z||>R\}}|E(w)(x+z)|\nu(\mathrm{d}z). \tag{2.54}$$

By construction of the linear operator E (see (2.36)), we know that $E(w)(x+z) = 0$, for all $||x+z|| \geq R + \dfrac{b}{2}$. Then,

$$\int_{\{||x+z||>R\}} |E(w)(x+z)|\nu(\mathrm{d}z) = \int_{\mathbb{R}^*} |E(w)(x+z)|\mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z), \tag{2.55}$$

with $\mathcal{B}' = B_{R+\frac{b}{2}}(0) \setminus \overline{B_R(0)}$. Since $\mathcal{B}' = \bigcup_{\kappa=1}^{N} \mathcal{J}_\kappa$, where $\{\mathcal{J}_\kappa\}_{\kappa=1}^{N}$ is a sequence of disjoint sets given in (2.38), we have that

$$\int_{\mathbb{R}^*} |E(w)(x+z)|\mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z) = \sum_{\kappa=1}^{N} \int_{\mathbb{R}^*} |E(w)(x+z)|\mathbb{1}_{\mathcal{J}_\kappa}(x+z)\nu(\mathrm{d}z). \tag{2.56}$$

Using (2.45), we get

$$\int_{\mathbb{R}^*} |E(w)(x+z)|\mathbb{1}_{\mathcal{J}_\kappa}(x+z)\nu(\mathrm{d}z) \leq 2A_0||w||_{\mathrm{C}^0(\mathcal{B}_1')} \int_{\mathbb{R}^*} \mathbb{1}_{\mathcal{J}_\kappa}(x+z)\nu(\mathrm{d}z),$$

where $\mathcal{B}_1' = \overline{B_R(0)} \cap \bigcup_{\kappa=1}^{N} \overline{B_{b_\kappa-\frac{b}{4}}(x_\kappa)}$. Then,

$$\begin{aligned}\sum_{\kappa=1}^{N} \int_{\mathbb{R}^*} |E(w)(x+z)|\mathbb{1}_{\mathcal{J}_\kappa}(x+z)\nu(\mathrm{d}z) &\leq 2A_0||w||_{\mathrm{C}^0(\mathcal{B}_1')} \sum_{\kappa=1}^{N} \int_{\mathbb{R}^*} \mathbb{1}_{\mathcal{J}_\kappa}(x+z)\nu(\mathrm{d}z)\\ &= 2A_0||w||_{\mathrm{C}^0(\mathcal{B}_1')} \int_{\mathbb{R}^*} \mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z).\end{aligned} \tag{2.57}$$

From (2.54)–(2.57), it follows

$$\left|\int_{\{||x+z||>R\}} E(w)(x+z)\nu(\mathrm{d}z)\right| \leq 2A_0||w||_{\mathrm{C}^0(\mathcal{B}_1')} \int_{\mathbb{R}^*} \mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z). \qquad \blacksquare$$

Lemma 2.20. *If $w \in \mathrm{C}^1(\overline{B_R(0)})$, then*

$$\left|\int_{\{||x+z||>R\}} \partial_i E(w)(x+z)\nu(\mathrm{d}z)\right| \leq C_3||w||_{\mathrm{C}^1(\mathcal{B}_1')} \int_{\mathbb{R}^*} \mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z),$$

for all $x \in B_R(0)$, where C_3 is a constants given in (2.24) and Lemma 2.16, respectively, $\mathcal{B}' = B_{R+\frac{b}{2}}(0) \setminus \overline{B_R(0)}$ and $\mathcal{B}_1' := \overline{B_R(0)} \cap \bigcup_{\kappa=1}^{N} \overline{B_{b_\kappa-\frac{b}{4}}(x_\kappa)}$.

Proof. Let $x \in B_R(0)$, $i \in \{1, \ldots, d\}$, and $w \in \mathrm{C}^1(\overline{B_R(0)})$ fixed. Then

$$\left|\int_{\{||x+z||>R\}} \partial_i E(w)(x+z)\nu(\mathrm{d}z)\right| \leq \int_{\{||x+z||>R\}} |\partial_i E(w)(x+z)|\nu(\mathrm{d}z). \tag{2.58}$$

Proceeding in a similar way than (2.55) and (2.56), it yields that

$$\int_{\{||x+z||>R\}} |\partial_i E(w)(x+z)|\nu(\mathrm{d}z) = \sum_{\kappa=1}^{N} \int_{\mathbb{R}^*} |\partial_i E(w)(x+z)|\mathbb{1}_{\mathcal{J}_\kappa}(x+z)\nu(\mathrm{d}z), \tag{2.59}$$

where $\{\mathcal{J}_\kappa\}_{\kappa=1}^N$ is a sequence of disjoint sets given in (2.38). From (2.49), we get that

$$\int_{\mathbb{R}^*} |\partial_i E(w)(x+z)| \mathbb{1}_{\mathcal{J}_\kappa}(x+z)\nu(\mathrm{d}z) \leq C_3 ||w||_{\mathrm{C}^1(\mathcal{B}'_1)} \int_{\mathbb{R}^*} \mathbb{1}_{\mathcal{J}_\kappa}(x+z)\nu(\mathrm{d}z), \tag{2.60}$$

with C_3 given in Lemma 2.16 and $\mathcal{B}'_1 = \overline{B_R(0)} \cap \bigcup_{\kappa=1}^N \overline{B_{b_\kappa - \frac{b}{4}}(x_\kappa)}$. From (2.58)–(2.60), we conclude that

$$\begin{aligned} \left| \int_{\{||x+z||>R\}} \partial_i E(w)(x+z)\nu(\mathrm{d}z) \right| &\leq C_3 ||w||_{\mathrm{C}^1(\mathcal{B}'_1)} \sum_{\kappa=1}^N \int_{\mathbb{R}^*} \mathbb{1}_{\mathcal{J}_\kappa}(x+z)\nu(\mathrm{d}z) \\ &= C_3 ||w||_{\mathrm{C}^1(\mathcal{B}'_1)} \int_{\mathbb{R}^*} \mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z), \end{aligned}$$

with $\mathcal{B}' = B_{R+\frac{b}{2}}(0) \setminus \overline{B_R(0)}$. ■

Remark 2.21. Observe that when $N \to \infty$, $b \to 0$, since $0 < b < \frac{1}{2^N}$. It follows that $\mathcal{B}' = B_{R+\frac{b}{2}}(0) \setminus \overline{B_R(0)} \underset{b \to 0}{\longrightarrow} \varnothing$. Then,

$$\int_{\mathbb{R}^*} \mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z) \underset{b \to 0}{\longrightarrow} 0.$$

This implies that we can choose $N > 1$ arbitrary large in (2.10) but fixed, such that the arguments realized in this chapter are valid, with the difference that b is small enough, and the value of $\int_{\mathbb{R}^*} \mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z)$ can be taken arbitrarily small. From the condition (2.30), we can also note when $N > 1$ increases, the values $b_\kappa > 0$ decreases, for all $\kappa \in \{1, \dots, N\}$. This implies that $\mathcal{B}'_1 = \overline{B_R(0)} \cap \bigcup_{\kappa=1}^N \overline{B_{b_\kappa - \frac{b}{4}}(x_\kappa)} \underset{N \to \infty}{\longrightarrow} \partial B_R(0)$, and hence that for each $w \in \mathrm{C}^{0,1}(\overline{B_R(0)})$ fixed, we have that

$$||w||_{\mathrm{C}^0(\mathcal{B}'_1)} \underset{N \to \infty}{\longrightarrow} ||w||_{\mathrm{C}^0(\partial B_R(0))}.$$ ■

Chapter 3

Non-linear Dirichlet problems

In this chapter, we are interested in establishing the existence, uniqueness and regularity of the solution to the non-linear integro-differential Dirichlet (NIDD) problem given in (1.15), i.e.

$$\begin{cases} q' u^\varepsilon(x) - \Gamma' u^\varepsilon(x) + \psi_\varepsilon(\|\mathrm{D}^1 u^\varepsilon(x)\|^2) = h(x), & \text{in } B_R(0), \\ u^\varepsilon(x) = 0, & \text{on } \partial B_R(0). \end{cases} \tag{3.1}$$

We recall that

$$q' = q + \nu_0 > 0, \tag{3.2}$$

with $\nu_0 = \nu(\mathbb{R}^*)$ and

$$\begin{aligned} \Gamma' w(x) &= \frac{1}{2} \operatorname{tr}(\sigma \, \mathrm{D}^2 w(x)) + \langle \mathrm{D}^1 w(x), \widetilde{\gamma} \rangle + \int_{\mathbb{R}^*} E(w)(x+z)\nu(\mathrm{d}z) \\ &= \mathcal{L}' w(x) + \mathcal{I} \, E(w)(x), \end{aligned}$$

where $\widetilde{\gamma} = \gamma - \int_{\mathbb{R}^*} z\nu(\mathrm{d}z)$, the continuous linear extension $E : \mathrm{C}^{k,\alpha}(\overline{B_R(0)}) \longrightarrow \mathrm{C}^{k,\alpha}_{\mathrm{c}}(\mathbb{R}^d)$ is defined in (2.36), and the penalizing function $\psi_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$\psi_\varepsilon(r) = \psi\left(\frac{r-1}{\varepsilon}\right), \text{ for } r \in \mathbb{R}, \tag{3.3}$$

with $\psi \in \mathrm{C}^\infty(\mathbb{R})$ such that

$$\begin{cases} \psi(r) = 0, & \text{for } r \leq 0, \\ \psi(r) > 0, & \text{for } r > 0, \\ \psi(r) = r - 1, & \text{for } r \geq 2, \\ \psi'(r) \geq 0, \ \psi''(r) \geq 0, & \text{for } r \in \mathbb{R}. \end{cases} \tag{3.4}$$

Recall the hypotheses (H1)–(H4) in pages 8 and 9, are in force all thorough this chapter.

The arguments to guarantee existence, uniqueness and regularity of the solution to the NIDD problem (3.1) are based in the contraction fixed point Theorem which is recalled below; see [13, Thm. 5.1 p.74]. If $(\mathcal{B}, ||\cdot||_{\mathcal{B}})$ is a Banach space with norm $||\cdot||_{\mathcal{B}}$, a mapping $T:\mathcal{B}\longrightarrow\mathcal{B}$ is called a *contraction* in $(\mathcal{B}, ||\cdot||_{\mathcal{B}})$ if there exists a constant $0<K<1$ such that

$$||T(b_1)-T(b_2)||_{\mathcal{B}}\leq K||b_1-b_2||_{\mathcal{B}},\ \text{for all } b_1,b_2\in\mathcal{B}.$$

Theorem 3.1 (Contraction fixed point Theorem)**.** *A contraction mapping in a Banach space $(\mathcal{B}, ||\cdot||_{\mathcal{B}})$ has a unique fixed point, i.e., there exists a unique solution $b^*\in\mathcal{B}$ to the equation $T(b^*)=b^*$.*

To use this result, we define the operator $T_\varepsilon:\mathrm{C}^0(\overline{B_R(0)})\longrightarrow\mathrm{C}^0(\overline{B_R(0)})$ as

$$T_\varepsilon(w)=V^\varepsilon(\cdot;w),\ \text{for each } w\in\mathrm{C}^0(\overline{B_R(0)}),$$

where the value function $V^\varepsilon(\cdot,w)$ is as in (3.16). From Lemma 3.5, it follows

$$T_\varepsilon(w)=V^\varepsilon(\cdot,w)\in\mathrm{C}^{2,\alpha}(\overline{B_R(0)})\subset\mathrm{C}^0(\overline{B_R(0)}),\ \text{for each } w\in\mathrm{C}^0(\overline{B_R(0)}).$$

Verifying that $V^\varepsilon(\cdot;w)$ satisfies

$$||V^\varepsilon(\cdot;w_1)-V^\varepsilon(\cdot;w_2)||_{\mathrm{C}^0(\overline{B_R(0)})}\leq\frac{2A_0}{q'}\nu(B_{R+\frac{b}{2}}(0))||w_1-w_2||_{\mathrm{C}^0(\overline{B_R(0)})},$$

for each $w_1,w_2\in\mathrm{C}^0(\overline{B_R(0)})$; see Lemma 3.6. By Hypothesis (H4), we obtain that T_ε is a contraction mapping in the Banach space $(\mathrm{C}^0(\overline{B_R(0)}), ||\cdot||_{\mathrm{C}^0(\overline{B_R(0)})})$. By contraction fixed point Theorem; Theorem 3.1, it yields that there exists a unique $w^*\in\mathrm{C}^0(\overline{B_R(0)})$ such that $T_\varepsilon(w^*)=w^*$; see Lemma 3.7. Using this and that $V^\varepsilon(;w)$ is related with the solution $u^\varepsilon(\cdot;w)$ to the non-linear Dirichlet problem (3.5); see Lemma 3.5, we obtain the existence, uniqueness and regularity of the solution u^ε to the NIDD problem (3.1); see Theorem 3.8.

Finally, in Subsection 3.2.1, we shall show some properties of the solution u^ε to the NIDD problem (3.1), which shall be used in Chapter 4 to prove the existence and regularity of the solution u to the Hamilton-Jacobi-Bellman equation

$$\begin{cases}\max\{qu(x)-\Gamma u(x)-h(x), ||\mathrm{D}^1 u(x)||^2-1\}=0, & \text{in } B_R(0),\\ u(x)=0, & \text{on } \partial B_R(0).\end{cases}$$

3.1 Non-linear Dirichlet problem with an elliptic differential operator

For each $w \in \mathrm{C}^0(\overline{B_R(0)})$, define $\widetilde{h}(\cdot;w) : \overline{B_R(0)} \longrightarrow \mathbb{R}$ as

$$\widetilde{h}(x;w) = h(x) + \mathcal{I}\,E(w)(x), \text{ for all } x \in \overline{B_R(0)}.$$

Since $h \in \mathrm{C}^2(\overline{B_R(0)})$ and $\mathcal{I}\,E(w) \in \mathrm{C}^{0,\alpha}(\mathbb{R}^d)$, whenever $w \in \mathrm{C}^0(\overline{B_R(0)})$; see Hypothesis (H1) and Lemma 2.17(ii) respectively, we have that $\widetilde{h}(\cdot;w) \in \mathrm{C}^{0,\alpha}(\overline{B_R(0)})$. Then, from [13, Thm 15.10 p. 380], we have the following result.

Lemma 3.2. *For each $w \in \mathrm{C}^0(\overline{B_R(0)})$ and $\varepsilon \in (0,1)$ fixed, the non-linear Dirichlet problem*

$$\begin{cases} q'u^{\varepsilon}(x;w) - \mathcal{L}'u^{\varepsilon}(x;w) + \psi_{\varepsilon}(||\mathrm{D}^1 u^{\varepsilon}(x;w)||^2) = \widetilde{h}(x;w), & \text{in } B_R(0), \\ u^{\varepsilon}(x;w) = 0, & \text{on } \partial B_R(0), \end{cases} \tag{3.5}$$

has a solution $u^{\varepsilon}(\cdot;w) \in \mathrm{C}^{2,\alpha}(\overline{B_R(0)})$.

To guarantee the existence of the solution $u^{\varepsilon}(\cdot;w)$ to the non-linear Dirichlet problem (3.5), we only need to verify the conditions in [13, Thm 15.10 p.380], which is recalled below. A quasi-linear operator $\mathcal{Q}$ is an operator of the form

$$\mathcal{Q}v := \sum_{ij} a_{ij}(x,v,\mathrm{D}^1 v)\partial_{ij}v + b(x,v,\mathrm{D}^1 v), \text{ with } a_{ij} = a_{ji}. \tag{3.6}$$

We say that the operator $\mathcal{Q}$ is elliptic in $\mathcal{U}$, a subset of $\Omega \times \mathbb{R} \times \mathbb{R}^d$, if there exist functions $\lambda, \Lambda : \mathcal{U} \longrightarrow \mathbb{R}$, such that

$$0 < \lambda(x,\eta,\zeta)||\xi||^2 \leq \sum_{ij} a_{ij}(x,\eta,\zeta)\xi_i\xi_j \leq \Lambda(x,\eta,\zeta)||\xi||^2, \tag{3.7}$$

for all $\xi = (\xi_1,\ldots,\xi_d) \in \mathbb{R}^d \setminus \{0\}$ and for all $(x,\eta,\zeta) \in \mathcal{U}$. Define $\mathcal{E}(x,\eta,\zeta) := \sum_{ij} a_{ij}(x,\eta,\zeta)\zeta_i\zeta_j$. If $\mathcal{Q}$ is elliptic in $\mathcal{U}$, from (3.7), it follows that

$$\lambda(x,\eta,\zeta)||\zeta||^2 \leq \mathcal{E}(x,\eta,\zeta) \leq \Lambda(x,\eta,\zeta)||\zeta||^2, \text{ for all } (x,\eta,\zeta) \in \mathcal{U}. \tag{3.8}$$

Before stating Theorem 3.3, we introduce the conditions that should satisfy the coefficients a_{ij} and b of $\mathcal{Q}$. Namely,

$$\begin{cases} a_{ij} = \mathrm{O}(\lambda), \\ \langle \zeta, \mathrm{D}^1_\zeta\, a_{ij} \rangle = \mathrm{O}(\lambda), \\ \mathrm{D}^1_\eta\, a_{ij} + ||\zeta||^{-2} \langle \zeta, \mathrm{D}^1_\zeta\, a_{ij} \rangle = \mathrm{o}(\lambda), \\ b = \mathrm{O}(\lambda ||\zeta||^2), \\ \langle \zeta, \mathrm{D}^1_\zeta\, b \rangle \leq \mathrm{O}(\lambda ||\zeta||^2), \\ \mathrm{D}^1_\eta\, b + ||\zeta||^{-2} \langle \zeta, \mathrm{D}^1_\zeta\, b \rangle \leq \mathrm{o}(\lambda ||\zeta||^2). \end{cases} \tag{3.9}$$

as $||\zeta|| \to \infty$, uniformly for $x \in \Omega$ and bounded η. Recall that $f = \mathrm{O}(g)$ if only if there exists a constant M such that for some x_0, it satisfies $|f(x)| \leq M|g(x)|$, for all $x \geq x_0$, and $f = \mathrm{o}(g)$ if only if $\lim_{||x|| \to \infty} \frac{f(x)}{g(x)} = 0$. Furthermore, suppose that there exist non-negative constant μ_1 and μ_2 such that

$$\frac{b(x,\eta,\zeta)\,\mathrm{sgn}(\eta)}{\mathcal{E}(x,\eta,\zeta)} \leq \frac{\mu_1 ||\zeta|| + \mu_2}{||\zeta||^2}, \text{ for all } (x,\mu,\zeta) \in \Omega \times \mathbb{R} \times \mathbb{R}^d. \tag{3.10}$$

Theorem 3.3. *Let Ω be a bounded domain in $\mathbb{R}^d$ and suppose that the operator $\mathcal{Q}$ in (3.6) is elliptic, with coefficients $a_{ij}, b \in \mathrm{C}^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d)$, which satisfy (3.9) together with the condition (3.10). Then, if $\partial\Omega \in \mathrm{C}^{2,\alpha}$, and $\varphi \in \mathrm{C}^{2,\alpha}(\overline{\Omega})$, $0 < \alpha < 1$, there exists a solution $v \in \mathrm{C}^{2,\alpha}(\overline{\Omega})$ of the Dirichlet problem $\mathcal{Q}v = 0$, in Ω, and $v = \varphi$, on $\partial\Omega$.*

Although Theorem 3.3 is valid for more general quasi-linear elliptic operators, we are interested in the case when the coefficients of the quasi-linear elliptic operator are given by

$$\begin{cases} a_{ij}(x,\eta,\zeta) := \frac{1}{2}\sigma_{ij}, \\ b(x,\eta,\zeta) := \widetilde{h}(x) + \langle \zeta, \widetilde{\gamma} \rangle - q'\eta - \psi_\varepsilon(||\zeta||^2), \end{cases} \tag{3.11}$$

for all $(x,\eta,\zeta) \in \overline{B_R(0)} \times \mathbb{R} \times \mathbb{R}^d$.

Proof of Lemma 3.2. Existence. To guarantee the existence of the solution $u^\varepsilon(\cdot;w) \in \mathrm{C}^{2,\alpha}(\overline{B_R(0)})$ to the equation (3.5), with $w \in \mathrm{C}^0(\overline{B_R(0)})$ fixed, we only need to verify the conditions (3.9) and (3.10), when a_{ij} and b are given by (3.11). Recall from Hypothesis (H3) that there exist real numbers $0 < \theta \leq \Theta$ such that the coefficients of the differential part of Γ satisfy

$$\theta ||\zeta||^2 \leq \langle \sigma\zeta, \zeta \rangle \leq \Theta ||\zeta||^2, \text{ for all } \zeta \in \mathbb{R}^d.$$

In our case we have that λ, Λ are θ, Θ, respectively. Then, since a_{ij} is a constant, it follows that $a_{ij} = \mathrm{O}(\theta)$, $\langle \zeta, \mathrm{D}_\zeta\, a_{ij} \rangle = \mathrm{O}(\theta)$ and $\mathrm{D}^1_\eta\, a_{ij} + ||\zeta||^{-2} \langle \zeta, \mathrm{D}^1_\zeta\, a_{ij} \rangle = \mathrm{o}(\theta)$. Furthermore, by (3.4) and since $\widetilde{h}$ is in $\mathrm{C}^{0,\alpha}(\overline{B_R(0)})$, we have that there exists a positive constant M such that for some ζ_0 we have that $|b(x,\eta,\zeta)| \leq M\theta ||\zeta||^2$, for all $||\zeta|| \geq ||\zeta_0||$, i.e., $b = \mathrm{O}(\theta ||\zeta||^2)$. Note that

$$\begin{cases} \langle \zeta, \mathrm{D}^1_\zeta\, b(x,\eta,\zeta) \rangle = \langle \zeta, \widetilde{\gamma} \rangle - 2\psi'_\varepsilon(||\zeta||^2)||\zeta||^2, \\ \mathrm{D}^1_\eta\, b(x,\eta,\zeta) + ||\zeta||^{-2} \langle \zeta, \mathrm{D}^1_\zeta\, b(x,\eta,\zeta) \rangle = -q' - 2\psi'_\varepsilon(||\zeta||^2) + ||\zeta||^{-2} \langle \zeta, \widetilde{\gamma} \rangle. \end{cases}$$

Then, we can see that there exists a positive constant M, such that for some ζ_0, we have that $|\langle \zeta, \mathrm{D}^1_\zeta\, b(x,\eta,\zeta) \rangle| \leq M\theta ||\zeta||^2$, for all $||\zeta|| \geq ||\zeta_0||$, and

$$\lim_{||\zeta|| \to \infty} \frac{\mathrm{D}^1_\eta\, b(x,\eta,\zeta) + ||\zeta||^{-2} \langle \zeta, \mathrm{D}^1_\zeta\, b(x,\eta,\zeta) \rangle}{\theta ||\zeta||^2} = 0.$$

Finally, we prove that b, given by (3.11), satisfies the condition (3.10). By (H3), we have

$$\frac{1}{\Theta ||\zeta||^2} \leq \frac{1}{\mathcal{E}(x,\eta,\zeta)} \leq \frac{1}{\theta ||\zeta||^2}.$$

Then, since $q' > 0$ and $\psi_\varepsilon(||\zeta||^2) \geq 0$, we get

$$\begin{aligned} \frac{b(x,\eta,\zeta)\,\mathrm{sgn}(\eta)}{\mathcal{E}(x,\eta,\zeta)} &= \frac{-q'|\eta| - \psi_\varepsilon(||\zeta||^2)\,\mathrm{sgn}(\eta) + \langle \zeta, \widetilde{\gamma} \rangle\,\mathrm{sgn}(\eta) + \widetilde{h}(x;w)\,\mathrm{sgn}(\eta)}{\mathcal{E}(x,\eta,\zeta)} \\ &\leq \frac{\langle \zeta, \widetilde{\gamma} \rangle\,\mathrm{sgn}(\eta) + \widetilde{h}(x;w)\,\mathrm{sgn}(\eta)}{\mathcal{E}(x,\eta,\zeta)} \\ &\leq \frac{||\zeta||\,||\widetilde{\gamma}|| + ||\widetilde{h}(\cdot;w)||_{\mathrm{C}^0(\overline{B_R(0)})}}{\theta ||\zeta||^2}. \end{aligned}$$

Taking $\mu_1 = \frac{||\widetilde{\gamma}||}{\theta}$ and $\mu_2 = \frac{||\widetilde{h}(\cdot;w)||_{\mathrm{C}^0(\overline{B_R(0)})}}{\theta}$ we obtain the inequality given in (3.10). Therefore, from Theorem 3.3, we conclude the existence of the solution $u^\varepsilon(\cdot;w)$ to the Dirichlet problem (3.5). ■

The uniqueness of the solution $u^\varepsilon(\cdot;w)$ to the non-linear Dirichlet problem (3.5) is obtained in the following result.

Lemma 3.4. *Let $w \in \mathrm{C}^0(\overline{B_R(0)})$ and $\varepsilon \in (0,1)$ fixed. Then, the non-linear Dirichlet problem (3.5) has a unique solution.*

Proof. Let $w \in \mathrm{C}^0(\overline{B_R(0)})$ and $\varepsilon \in (0,1)$ be fixed. If $u^\varepsilon_1(\cdot;w)$ and $u^\varepsilon_2(\cdot;w)$ are two solutions to the non-linear Dirichlet problem (3.5), we define $f(\cdot) := u^\varepsilon_1(\cdot;w) - u^\varepsilon_2(\cdot;w)$ in $\overline{B_R(0)}$, which is in $\mathrm{C}^{2,\alpha}(\overline{B_R(0)})$ and

$$\begin{cases} q' f(x) - \mathcal{L}' f(x) + \psi_\varepsilon(||\mathrm{D}^1\, u^\varepsilon_1(x;w)||^2) - \psi_\varepsilon(||\mathrm{D}^1\, u^\varepsilon_2(x;w)||^2) = 0, & \text{in } B_R(0), \\ f(x) = 0, & \text{on } \partial B_R(0). \end{cases} \tag{3.12}$$

Let $x^* \in \overline{B_R(0)}$ be the point where f attains its maximum. If $x^* \in \partial B_R(0)$, from (3.12), it follows that $f(x) \leq f(x^*) = 0$. Suppose now that $x^* \in B_R(0)$. Then, we have

$$\begin{cases} \mathrm{D}^1 f(x^*) = \mathrm{D}^1 u_1^\varepsilon(x^*; w) - \mathrm{D}^1 u_2^\varepsilon(x^*; w) = 0, \\ \frac{1}{2}\operatorname{tr}(\sigma\, \mathrm{D}^2 f(x^*)) \leq 0, \end{cases}$$

which implies that $\psi_\varepsilon(||\,\mathrm{D}^1 u_1(x^*; w)||^2) - \psi_\varepsilon(||\,\mathrm{D}^1 u_2(x^*; w)||^2) = 0$. Evaluating x^* in (3.12), we get that $0 \geq \frac{1}{2}\operatorname{tr}(\sigma\, \mathrm{D}^2 f(x^*)) = q' f(x^*)$, and hence $u_1^\varepsilon(x; w) - u_2^\varepsilon(x; w) \leq f(x^*) \leq 0$ in $B_R(0)$. By symmetry we have also that $u_2^\varepsilon(\cdot; w) - u_1^\varepsilon(\cdot; w) \leq 0$ in $B_R(0)$. Therefore $u_1^\varepsilon(\cdot; w) = u_2^\varepsilon(\cdot; w)$, and then, the non-linear Dirichlet problem (3.5) has a unique solution. ■

The convex function $g_\varepsilon : \mathbb{R}^d \longrightarrow \mathbb{R}$ was defined in (1.38) as $g_\varepsilon(\zeta) = \psi_\varepsilon(||\zeta||^2)$, together with its Legendre transform $l_\varepsilon : \mathbb{R}^d \longrightarrow \mathbb{R}$, which is given by $l_\varepsilon(\eta) = \sup_\zeta\{\langle \eta, \zeta\rangle - g_\varepsilon(\zeta)\}$. The Legendre transform l_ε satisfies

$$\begin{cases} l_\varepsilon(\eta) \geq \frac{\varepsilon}{2}||\eta||^2 - g_\varepsilon\big(\frac{\varepsilon}{2}\eta\big) \geq \frac{\varepsilon}{4}||\eta||^2, \\ l_\varepsilon(2\psi_\varepsilon'(||\zeta||^2)\zeta) = 2\psi_\varepsilon'(||\zeta||^2)||\zeta||^2 - \psi_\varepsilon(||\zeta||^2), \end{cases} \tag{3.13}$$

for all $\eta \in \mathbb{R}^d$. Since g_ε is differentiable, it follows that

$$g_\varepsilon(\zeta) = \sup_\eta\{\langle \eta, \zeta\rangle - l_\varepsilon(\eta)\}. \tag{3.14}$$

Then, the non-linear Dirichlet problem (3.5) can be written as

$$\begin{cases} q' u^\varepsilon(x; w) - \mathcal{L}' E(u^\varepsilon)(x; w) \\ \qquad\qquad + \sup_\eta\{\langle \mathrm{D}^1 u^\varepsilon(x; w), \eta\rangle - l_\varepsilon(\eta)\} = \widetilde{h}(x; w), & \text{in } B_R(0), \\ u^\varepsilon(x; w) = 0, & \text{on } \partial B_R(0). \end{cases} \tag{3.15}$$

Next we describe the stochastic control problem associated with this equation. A *control process* is any d-dimensional, absolutely continuous process $\varrho = \{\varrho_t : t \geq 0\}$ $\mathbb{F}$-adapted and satisfying $\varrho_0 = 0$ almost surely. Given an initial state $x \in B_R(0)$, we define the state process $X = \{X_t : t \geq 0\}$ by

$$X_t := x + W_t + \widetilde{\gamma} t - \varrho_t, \ \text{for all } t \geq 0,$$

where $W = \{W_t : t \geq 0\}$ is a d-dimensional Brownian motion with Gaussian covariance matrix σ and drift $\widetilde{\gamma}$ is as in (3.2). The cost function corresponding of ϱ, depending on $w \in \mathrm{C}^0(\overline{B_R(0)})$, is given by

$$V_\varrho^\varepsilon(x; w) := \mathbb{E}_x\bigg(\int_0^{\tau_{B_R(0)}} \mathrm{e}^{-q's}(\widetilde{h}(X_s; w) + l_\varepsilon(\dot{\varrho}_s))\,\mathrm{d}s\bigg),$$

for all $x \in B_R(0)$, with $\tau_{B_R(0)} := \inf\{t \geq 0 : X_t \notin B_R(0)\}$ and $\dot{\varrho}_t = \frac{\mathrm{d}\varrho_t}{\mathrm{d}t}$. The constant $q' > 0$ is given in (3.2). Finally, the value function is defined by

$$V^\varepsilon(x; w) := \inf_{\varrho} V^\varepsilon_\varrho(x; w). \tag{3.16}$$

Recalling that $u^\varepsilon(\cdot; w) \in \mathrm{C}^{2,\alpha}(\overline{B_R(0)})$, with $w \in \mathrm{C}^0(\overline{B_R(0)})$, is the solution to the non-linear Dirichlet problem (3.15).

Lemma 3.5. *The solution $u^\varepsilon(\cdot; w)$ to the non-linear Dirichlet problem (3.15) agrees with $V^\varepsilon(\cdot; w)$ in $\overline{B_R(0)}$.*

Proof. Let ϱ and $x \in B_R(0)$ be a control process and an initial state, respectively. Since $u^\varepsilon(\cdot; w) \in \mathrm{C}^{2,\alpha}(\overline{B_R(0)})$, by integration by parts and Itô's formula (see [28, Cor. 2 and Thm. 33, pp. 68 and 81, respectively]), we have that

$$\begin{aligned}
&\mathrm{e}^{-q'(t\wedge\tau_{B_R(0)})}\, u^\varepsilon(X_{t\wedge\tau_{B_R(0)}}; w) - u^\varepsilon(x; w) \\
&= \int_0^{t\wedge\tau_{B_R(0)}} \mathrm{e}^{-q's}((\mathcal{L}' - q')u^\varepsilon(X_s; w)) - \langle \mathrm{D}^1 u^\varepsilon(X_s; w), \dot{\varrho}_s\rangle \mathrm{d}s + M_{t\wedge\tau_{B_R(0)}} \\
&= -\int_0^{t\wedge\tau_{B_R(0)}} \mathrm{e}^{-q's}(\widetilde{h}(X_s; w) + \langle \mathrm{D}^1 u^\varepsilon(X_s; w), \dot{\varrho}_s\rangle - \psi_\varepsilon(||\,\mathrm{D}^1 u^\varepsilon(X_s; w)||^2))\mathrm{d}s + M_{t\wedge\tau_{B_R(0)}},
\end{aligned} \tag{3.17}$$

with $M_t := \int_0^t \mathrm{e}^{-q's}\langle \mathrm{D}^1 u^\varepsilon(X_s; w), \mathrm{d}W_s\rangle$, for all $t \geq 0$. The process $M = \{M_t : t \geq 0\}$ is a local martingale. Then, the process $M^{\tau_{B_R(0)}} := \{M_{t\wedge\tau_{B_R(0)}} : t \geq 0\}$ is a $\mathbb{P}_x$-martingale with $M_0 = 0$. Taking the expected value in (3.17), it follows that

$$\begin{aligned}
\mathbb{E}_x(\mathrm{e}^{-q'(t\wedge\tau_{B_R(0)})}\, u^\varepsilon(X_{t\wedge\tau_{B_R(0)}}; w)) = u^\varepsilon(x; w) - \mathbb{E}_x\bigg(\int_0^{t\wedge\tau_{B_R(0)}} \mathrm{e}^{-q's}(\widetilde{h}(X_s; w) \\
+ \langle \mathrm{D}^1 u^\varepsilon(X_s; w), \dot{\varrho}_s\rangle - \psi_\varepsilon(||\,\mathrm{D}^1 u^\varepsilon(X_s; w))||^2))\mathrm{d}s\bigg).
\end{aligned} \tag{3.18}$$

By definition of l_ε, it implies

$$\mathbb{E}_x(\mathrm{e}^{-q'(t\wedge\tau_{B_R(0)})}\, u^\varepsilon(X_{t\wedge\tau_{B_R(0)}}; w)) \geq u^\varepsilon(x; w) - \mathbb{E}_x\bigg(\int_0^{t\wedge\tau_{B_R(0)}} \mathrm{e}^{-q's}(\widetilde{h}(X_s; w) + l_\varepsilon(\dot{\varrho}_s))\mathrm{d}s\bigg).$$

Proceeding in a similar way as in (1.44) and (1.45), we obtain that $u^\varepsilon(\cdot; w) \leq V^\varepsilon(\cdot; w)$. Since

$$\psi'_\varepsilon(||\,\mathrm{D}^1 u^\varepsilon(x; w)||^2)\,\mathrm{D}^1 u^\varepsilon(x; w),$$

is a Lipschitz continuous function [19, Thm. 2.5, p. 287], we define the process $\widetilde{X} = \{\widetilde{X} : 0 \leq t \leq \tau_{B_R(0)}\}$ as the unique strong solution to

$$\widetilde{X}_t = x + W_t + \widetilde{\gamma} t - \int_0^{t\wedge\tau_{B_R(0)}} 2\psi'_\varepsilon(||\,\mathrm{D}^1\, u^\varepsilon(\widetilde{X}_s;w)||^2)\,\mathrm{D}^1\, u^\varepsilon(\widetilde{X}_s;w)\mathrm{d}s,$$

for all $0 \leq t \leq \tau_{B_R(0)}$. Then, its corresponding control process is given by

$$\dot{\varrho}^R_t = 2\psi'_\varepsilon(||\,\mathrm{D}^1\, u^\varepsilon(\widetilde{X}_s;w)||^2)\,\mathrm{D}^1\, u^\varepsilon(\widetilde{X}_s;w),\ \text{for all } 0 \leq t \leq \tau_{B_R(0)}.$$

The process $\widetilde{X}$ satisfies (3.18) and by (3.14), it follows that

$$\mathbb{E}_x(\mathrm{e}^{-q'(t\wedge\tau_{B_R(0)})}\, u^\varepsilon(\widetilde{X}_{t\wedge\tau_{B_R(0)}};w)) = u^\varepsilon(x;w) - \mathbb{E}_x\bigg(\int_0^{t\wedge\tau_{B_R(0)}} \mathrm{e}^{-q's}(h(\widetilde{X}_s;w) + l_\varepsilon(\dot{\varrho}^R_s))\mathrm{d}s\bigg),$$

Proceeding of a similar way that (1.44) and (1.45), we have that $u^\varepsilon(x;w) = V^{\varepsilon,R}(x;w)$. This ends the proof. ∎

Defining $T_\varepsilon : \mathrm{C}^0(\overline{B_R(0)}) \longrightarrow \mathrm{C}^0(\overline{B_R(0)})$ as

$$T_\varepsilon(w) = V^\varepsilon(\cdot;w),\ \text{for each } w \in \mathrm{C}^0(\overline{B_R(0)}), \tag{3.19}$$

from Lemma 3.5, we see that T_ε is well defined. Now, by Hypothesis (H4) and using the following result; Lemma 3.6, we obtain that T_ε is a contraction mapping in $(\mathrm{C}^0(\overline{B_R(0)}), ||\cdot||_{\mathrm{C}^0(\overline{B_R(0)})})$, and hence, by contraction fixed point Theorem; seeTheorem 3.1, we have that T_ε has a unique point in $\mathrm{C}^0(\overline{B_R(0)})$; see Lemma 3.7.

Lemma 3.6. *If* $w_1, w_2 \in \mathrm{C}^0(\overline{B_R(0)})$*, then*

$$||V^\varepsilon(\cdot;w_1) - V^\varepsilon(\cdot;w_2)||_{\mathrm{C}^0(\overline{B_R(0)})} \leq \frac{2A_0\nu(B_{R+\frac{b}{2}}(0))}{q'}||w_1 - w_2||_{\mathrm{C}^0(\overline{B_R(0)})}.$$

Proof. Let $w_1, w_2 \in \mathrm{C}^0(\overline{B_R(0)})$. For each $x \in \overline{B_R(0)}$, we have

$$\begin{aligned} V^\varepsilon(x;w_1) &= \inf_\varrho\{V^\varepsilon_\varrho(x;w_1) - V^\varepsilon_\varrho(x;w_2) + V^\varepsilon_\varrho(x;w_2)\} \\ &\leq \inf_\varrho\bigg\{\sup_\varrho\{V^\varepsilon_\varrho(x;w_1) - V^\varepsilon_\varrho(x;w_2)\} + V^\varepsilon_\varrho(x;w_2)\bigg\} \\ &\leq \sup_\varrho\{V^\varepsilon_\varrho(x;w_1) - V^\varepsilon_\varrho(x;w_2)\} + V^\varepsilon(x;w_2). \end{aligned} \tag{3.20}$$

Therefore $V^{\varepsilon}(x;w_1) - V^{\varepsilon}(x;w_2) \leq \sup_{\varrho}\{V^{\varepsilon}_{\varrho}(x;w_1) - V^{\varepsilon}_{\varrho}(x;w_2)\}$. Proceeding of the same way than (3.20), it yields $V^{\varepsilon}(x;w_2) - V^{\varepsilon}(x;w_1) \leq \sup_{\varrho}(V^{\varepsilon}_{\varrho}(x;w_2) - V^{\varepsilon}_{\varrho}(x;w_1))$. Then, using Proposition 2.15 and that $\operatorname{supp}[E(w_2 - w_1)] \subset B_{R+\frac{b}{2}}(0)$, we conclude that

$$\begin{aligned}
|V^{\varepsilon}(x;w_2) - V^{\varepsilon}(x;w_1)| &\leq \sup_{\varrho} |V^{\varepsilon}_{\varrho}(x;w_2) - V^{\varepsilon}_{\varrho}(x;w_1)| \\
&\leq \sup_{\varrho} \mathbb{E}_x \int_0^{\tau_{B_R(0)}} \mathrm{e}^{-q's}\, |\widetilde{h}(X_s;w_2) - \widetilde{h}(X_s;w_1)|\,\mathrm{d}s \\
&= \sup_{\varrho} \mathbb{E}_x \int_0^{\tau_{B_R(0)}} \mathrm{e}^{-q's}\, |\mathcal{I}\, E(w_2 - w_1)(X_s)|\,\mathrm{d}s \\
&\leq \sup_{\varrho} \mathbb{E}_x \int_0^{\tau_{B_R(0)}} \mathrm{e}^{-q's} \int_{\mathbb{R}^*} |E(w_2 - w_1)(X_s + z)|\nu(\mathrm{d}z)\,\mathrm{d}s \\
&\leq \mathbb{E}_x \int_0^{\infty} \mathrm{e}^{-q's} \int_{B_{R+\frac{b}{2}}(0)} 2A_0 ||w_2 - w_1||_{\mathrm{C}^0(\overline{B_R(0)})} \nu(\mathrm{d}z)\,\mathrm{d}s \\
&\leq \frac{2A_0 \nu(B_{R+\frac{b}{2}}(0))}{q'} ||w_2 - w_1||_{\mathrm{C}^0(\overline{B_R(0)})},
\end{aligned}$$

■

Lemma 3.7. *Let $T_{\varepsilon} : \mathrm{C}^0(\overline{B_R(0)}) \longrightarrow \mathrm{C}^0(\overline{B_R(0)})$ be as in (3.19). Then, there exists a unique solution $w^* \in \mathrm{C}^0\,\overline{B_R(0)}$ to the equation $T_{\varepsilon}(w^*) = w^*$.*

Proof. Recall that $T_{\varepsilon} : \mathrm{C}^0(\overline{B_R(0)}) \longrightarrow \mathrm{C}^0(\overline{B_R(0)})$ is defined as $T_{\varepsilon}(w) = V^{\varepsilon}(\cdot;w)$, for each $w \in \mathrm{C}^0(\overline{B_R(0)})$, where $V^{\varepsilon}(\cdot;w)$ is given by (3.16). Then, by Hypothesis (H4) and Lemma 3.6, we obtain that T_{ε} is a contraction mapping in $(\mathrm{C}^0(\overline{B_R(0)}), ||\cdot||_{\mathrm{C}^0(\overline{B_R(0)})})$. Therefore, from contraction fixed point Theorem; see Theorem 3.1, there exists a unique solution $w^* \in \mathrm{C}^0(\overline{B_R(0)})$ to the equation $T_{\varepsilon}(w^*) = w^*$. ■

3.2 Non-linear Dirichlet problem with an elliptic integro-differential operator

We begin this section showing the existence, regularity and uniqueness of the solution u^{ε} to the non-linear integro-differential Dirichlet problem (NIDD) (3.21). To prove this, we use Lemmas 3.5–3.7, stated in the previous section.

Theorem 3.8. *For each $\varepsilon \in (0,1)$ fixed, there exists a unique solution $u^{\varepsilon} \in \mathrm{C}^{2,\alpha}(\overline{B_R(0)})$ to the*

NIDD problem

$$\begin{cases} q' u^\varepsilon(x) - \mathcal{L}' u^\varepsilon(x) - \mathcal{I}\, E(u^\varepsilon)(x) + \psi_\varepsilon(|| \mathrm{D}^1 u^\varepsilon(x)||^2) = h(x), & \textit{in } B_R(0), \\ u^\varepsilon(x) = 0, & \textit{on } \partial B_R(0). \end{cases} \tag{3.21}$$

Proof. From Lemma 3.7, we know that there exists a unique solution $w^* \in \mathrm{C}^0\, \overline{B_R(0)}$ to the equation $T_\varepsilon(w^*) = w^*$, where T_ε is given by (3.19). Furthermore, Lemmas 3.2 and 3.4 imply that there exists a unique solution $u^\varepsilon(\cdot; w^*) \in \mathrm{C}^{2,\alpha}(\overline{B_R(0)})$ to the Dirichlet problem

$$\begin{cases} q' u^\varepsilon(x; w^*) - \mathcal{L}' u^\varepsilon(x; w^*) + \psi_\varepsilon(|| \mathrm{D}^1 u^\varepsilon(x; w^*)||^2) = \widetilde{h}(x; w^*), & \text{in } B_R(0), \\ u^\varepsilon(x; w^*) = 0, & \text{on } \partial B_R(0), \end{cases}$$

and by Lemma 3.5, we obtain that

$$u^\varepsilon(\cdot; w^*) = V^*(\cdot; w^*) = T_\varepsilon(w^*) = w^*, \text{in } \overline{B_R(0)}.$$

Therefore, taking u^ε as w^*, we conclude that u^ε is in $\mathrm{C}^{2,\alpha}(\overline{B_R(0)})$, and it is the unique solution to the NIDD problem (3.21). ■

3.2.1 Some properties of the solution to the NIDD problem

In this subsection, we shall show some properties of the solution u^ε to the NIDD problem (3.21), such properties will in turn be used in Chapter 4 to establish the existence and regularity of the solution to the HJB equation (1.1). Since $h \in \mathrm{C}^2(\overline{B_R(0)})$, the proposition below, establishes that $u^\varepsilon \in \mathrm{C}^{3,\alpha}(\overline{B_R(0)})$, and it satisfies (3.22).

Proposition 3.9. *The solution u^ε to the NIDD problem (3.21) is in $\mathrm{C}^{3,\alpha}(\overline{B_R(0)})$ and it satisfies,*

$$\begin{aligned} \frac{1}{2}\operatorname{tr}(\sigma\, \mathrm{D}^2\, \partial_i u^\varepsilon(x)) &= q' \partial_i u^\varepsilon(x) - \langle \mathrm{D}^1\, \partial_i u^\varepsilon(x), \widetilde{\gamma}\rangle \\ &\quad - \partial_i h(x) - \mathcal{I}\, \partial_i E(u^\varepsilon)(x) + \psi'_\varepsilon(g(x))\partial_i g(x), \end{aligned} \tag{3.22}$$

with $i, j \in \{1, \ldots, d\}$. Where

$$g(x) := || \mathrm{D}^1 u^\varepsilon(x)||^2, \textit{ for all } x \in B_R(0),$$

and its first and second derivatives are, respectively,

$$\begin{cases} \partial_i g(x) = 2\sum_k \partial_k u^\varepsilon(x)\partial^2_{ik} u^\varepsilon(x), \\ \partial^2_{ji} g(x) = 2\sum_k \big(\partial^2_{kj} u^\varepsilon(x)\partial^2_{ki} u^\varepsilon(x) + \partial_k u^\varepsilon(x)\partial^3_{jik} u^\varepsilon(x)\big), \end{cases} \tag{3.23}$$

with $i, j \in \{1, \ldots, d\}$.

To verify this statement, we state three results on partial differential equations whose proofs can be found in [13, Thm. 3.3, Corollary 6.9 and Thm. 6.17 pp. 33, 101 and 109, respectively]. Although Theorems 3.10–3.12 are valid for more general differential operators, we are interested in the case that the differential operator $\mathcal{L}'$ is given by $\mathcal{L}' v = \frac{1}{2}\operatorname{tr}(\sigma\, \mathrm{D}^2 v) + \langle \mathrm{D}^1 v, \widetilde{\gamma}\rangle$.

Theorem 3.10 ([13], Thm. 3.3, p. 33)**.** *Suppose that* $u, v \in \mathrm{C}^2(B_R(0)) \cap \mathrm{C}^0(\overline{B_R(0)})$ *satisfying*

$$\begin{cases} q'u - \mathcal{L}' u = q'v - \mathcal{L}' v, & \textit{in } B_R(0), \\ u = v, & \textit{on } \partial B_R(0). \end{cases}$$

Then $u = v$ *in* $B_R(0)$.

Theorem 3.11 ([13], Corollary 6.9, p. 101)**.** *If* $f \in \mathrm{C}^{0,\alpha}(\overline{B_R(0)})$*, then the Dirichlet problem*

$$\begin{cases} q'v - \mathcal{L}' v = f, & \textit{in } B_R(0), \\ v = 0, & \textit{on } \partial B_R(0), \end{cases}$$

has a unique solution $v \in \mathrm{C}^{2,\alpha}(\overline{B_R(0)})$.

Theorem 3.12 ([13], Thm. 6.17, p. 109)**.** *Let* $v \in \mathrm{C}^2(B_R(0))$ *be a solution of the equation* $q'v - \mathcal{L}' v = f$ *in* $B_R(0)$*, where* $f \in \mathrm{C}^{k,\alpha}(B_R(0))$*. Then* $v \in \mathrm{C}^{k+2,\alpha}(B_R(0))$.

Proof of Proposition 3.9. Defining $f := h(x) + \mathcal{I}\, E(u^\varepsilon)(x) - \psi_\varepsilon(||\, \mathrm{D}^1 u^\varepsilon(x)||^2)$, we see that $f \in \mathrm{C}^{1,\alpha}(\overline{B_R(0)})$, since $h \in \mathrm{C}^2(\overline{B_R(0)})$ and $\psi_\varepsilon(||\, \mathrm{D}^1 u^\varepsilon||^2), \mathcal{I}\, E(w) \in \mathrm{C}^{1,\alpha}(\overline{B_R(0)})$. Then, by Theorem 3.11, we have

$$\begin{cases} q'v - \mathcal{L}' v = f, & \text{in } B_R(0), \\ v = 0, & \text{on } \partial B_R(0), \end{cases}$$

has a unique solution $v \in \mathrm{C}^{2,\alpha}(\overline{B_R(0)})$. Also, from Theorem 3.12, it follows that $v \in \mathrm{C}^{3,\alpha}(\overline{B_R(0)})$. Furthermore, we know that $u^\varepsilon \in \mathrm{C}^{2,\alpha}(\overline{B_R(0)})$ is the unique solution to

$$\begin{cases} q'u^\varepsilon(x) - \mathcal{L}' u^\varepsilon(x) = h(x) + \mathcal{I}\, E(u^\varepsilon)(x) - \psi_\varepsilon(||\, \mathrm{D}^1 u^\varepsilon(x)||^2), & \text{in } B_R(0), \\ u^\varepsilon = 0, & \text{on } \partial B_R(0). \end{cases}$$

Then, we get

$$\begin{cases} q'u^\varepsilon(x) - \mathcal{L}' u^\varepsilon(x) = q'v(x) - \mathcal{L}' v(x), & \text{in } B_R(0), \\ u^\varepsilon = v & \text{on } \partial B_R(0), \end{cases}$$

since $f = h(x) + \mathcal{I}\, E(u^\varepsilon)(x) - \psi_\varepsilon(||\, \mathrm{D}^1 u^\varepsilon(x)||^2)$. Using Theorem 3.10, we conclude that $u^\varepsilon = v$ and hence $u^\varepsilon \in \mathrm{C}^{3,\alpha}(\overline{B_R(0)})$. ∎

From Lemma 1.5, it is easy to verify that u^{ε} is a positive function. This fact is proved below.

Proposition 3.13. *The solution u^{ε} to the NIDD problem (3.21) is a positive function.*

Proof. From the proof of Lemma 1.5, it is known $u^{\varepsilon}(x) = \mathbb{E}_x\big(\int_0^{t\wedge\tau_{B_R(0)}} \mathrm{e}^{-qs}(h(\widetilde{Z}_s) + l_{\varepsilon}(\dot{\varrho}_s^R))\mathrm{d}s\big)$, where $\widetilde{Z}$ and $\dot{\varrho}^R$ are given by (1.46) and (1.47), respectively. Since h is a positive function, it follows $u^{\varepsilon}(x) \geq \mathbb{E}_x\big(\int_0^{t\wedge\tau_{B_R(0)}} \mathrm{e}^{-qs}\, h(\widetilde{Z}_s)\mathrm{d}s\big) > 0$. Therefore, $u^{\varepsilon} > 0$. ∎

Now, we shall establish estimates for u^{ε}, $||\,\mathrm{D}^1 u^{\varepsilon}||$, $\psi_{\varepsilon}(||\,\mathrm{D}^1 u^{\varepsilon}||^2)$ and $||\,\mathrm{D}^2 u^{\varepsilon}||_{\mathrm{L}^p(B_r)}$, with $B_r \subset B_R(0)$ an open ball, such that these estimates are independent of ε; see Lemmas 3.16, 3.21, 3.23 and 3.24. The reason for doing this is because in Chapter 4 we will need to extract a convergent subsequence $\{u^{\varepsilon_{\kappa}}\}_{\kappa\geq 1}$ of $\{u^{\varepsilon}\}_{\varepsilon\in(0,1)}$ such that $u{:=} \lim_{\varepsilon_{\kappa}\to 0} u^{\varepsilon_{\kappa}}$ is the solution of the HJB equation (1.1).

The following result is based in the weak maximum principle for integral-differential equations. Although Theorem 3.14 is valid for more general domains and integro-differential operators, see for instance [12, Thm. 3.1.3], we are interested in the case that the domain and integro-differential operator are $B_R(0) \subseteq \mathbb{R}^d$ and $q - \mathcal{L} - \mathcal{I}'\, E(\cdot)$, respectively, where $q > 0$ and

$$\begin{cases} \mathcal{L}u(x){:=} \frac{1}{2}\,\mathrm{tr}(\sigma\,\mathrm{D}^2 u(x)) + \langle \mathrm{D}^1 u(x), \gamma\rangle, \\ \mathcal{I}'\, u(x){:=} \int_{\mathbb{R}^*} (E(u)(x+z) - u(x) - \langle \mathrm{D}^1 u(x), z\rangle)\nu(\mathrm{d}z). \end{cases} \tag{3.24}$$

Theorem 3.14 (Weak maximum principle)**.** *If $w \in \mathrm{C}^2(B_R(0)) \cap \mathrm{C}^0_{\mathrm{c}}(B_{R+\frac{b}{2}}(0))$ satisfies $qw - \mathcal{L}w - \mathcal{I}'\, E(w) \leq 0,$ in $B_R(0)$, then $\sup_{\mathbb{R}^d} E(w) = \sup_{B_{R+\frac{b}{2}}(0)\setminus B_R(0)} [E(w)]^+$, where $[E(w)]^+{:=} \max\{E(w), 0\}$.*

Note that the NIDD problem (3.21) is equivalent to

$$\begin{cases} qu^{\varepsilon}(x) - \mathcal{L}u^{\varepsilon}(x) - \mathcal{I}'\, E(u^{\varepsilon})(x) + \psi_{\varepsilon}(||\,\mathrm{D}^1 u^{\varepsilon}(x)||^2) = h(x), & \text{in } B_R(0), \\ u^{\varepsilon}(x) = 0, & \text{on } \partial B_R(0). \end{cases} \tag{3.25}$$

Remark 3.15. Observe that the linear integro-differential Dirichlet problem

$$\begin{cases} q\eta(x) - \mathcal{L}\eta(x) - \mathcal{I}'\, E(\eta)(x) = h(x), & \text{in } B_R(0), \\ \eta(x) = 0, & \text{on } \partial B_R(0), \end{cases} \tag{3.26}$$

has a unique solution $\eta \in \mathrm{C}^{2,\alpha}(\overline{B_R(0)})$ [12, Thm. 3.1.12]. We can see that the linear integro-differential Dirichlet problem (3.26) is equivalent to

$$\begin{cases} q'\eta(x) - \mathcal{L}'\eta(x) - \mathcal{I}\, E(\eta)(x) = h(x), & \text{in } B_R(0), \\ \eta(x) = 0, & \text{on } \partial B_R(0). \end{cases}$$

Since $h+\mathcal{I}\,E(\eta)\in \mathrm{C}^{1,\alpha}(\overline{B_R(0)})$ and using similar arguments that the proof of Proposition 3.9, it is easy to verify that $\eta\in \mathrm{C}^{3,\alpha}(\overline{B_R(0)})$. ■

Lemma 3.16. *There exists a finite constant $K_5>0$, independent of (ε,R), such that*

$$u^{\varepsilon}(x)\leq K_5,\ \textit{in}\ B_R(0).$$

Proof. Let $u^{\varepsilon},\eta\in \mathrm{C}^{3,\alpha}(\overline{B_R(0)})$ be solutions to (3.21) and (3.26), respectively. Note that

$$\begin{aligned}
&qu^{\varepsilon}(x)-\mathcal{L}u^{\varepsilon}(x)-\mathcal{I}'\,E(u^{\varepsilon})(x)\\
&\qquad\qquad\leq qu^{\varepsilon}(x)-\mathcal{L}u^{\varepsilon}(x)-\mathcal{I}'\,E(u^{\varepsilon})(x)+\psi_{\varepsilon}(||\,\mathrm{D}^1\,u^{\varepsilon}(x)||^2)=h(x),
\end{aligned}$$

in $B_R(0)$. Then

$$\begin{cases} q(u^{\varepsilon}-\eta)(x)-\mathcal{L}(u^{\varepsilon}-\eta)(x)-\mathcal{I}'\,E(u^{\varepsilon}-\eta)(x)\leq 0, & \text{in } B_R(0),\\ (u^{\varepsilon}-\eta)(x)=0, & \text{on } \partial B_R(0). \end{cases} \tag{3.27}$$

From Theorem 3.14, it follows that $(u^{\varepsilon}-\eta)\leq \sup_{B_{R+\frac{b}{2}}(0)\setminus B_R(0)}[E(u^{\varepsilon}-\eta)]^{+},$ in $B_R(0)$. We prove below that $u^{\varepsilon}-\eta\leq 0$ in $B_R(0)$. Let $x^{*}\in\overline{B_R(0)}$ be the point where $u^{\varepsilon}-\eta$ in $B_R(0)$ attains its maximum. Observe that $(u^{\varepsilon}-\eta)(x^{*})\leq \sup_{B_{R+\frac{b}{2}}(0)\setminus B_R(0)}[E(u^{\varepsilon}-\eta)]^{+}$. If $x^{*}\in\partial B_R(0)$, we have trivially

$$(u^{\varepsilon}-\eta)\leq 0,\ \text{in } B_R(0). \tag{3.28}$$

Now, if $x^{*}\in B_R(0)$, we shall prove the statement (3.28) by contradiction. Suppose

$$(u^{\varepsilon}-\eta)(x^{*})>0. \tag{3.29}$$

Since $u^{\varepsilon}-\eta$ attains its maximum at $x^{*}\in B_R(0)$ and $u-\eta=0$ on $\partial B_R(0)$, we get that

$$\begin{cases} \mathrm{D}^1(u^{\varepsilon}-\eta)(x^{*})=0,\\ \frac{1}{2}\,\mathrm{tr}(\sigma\,\mathrm{D}^2(u^{\varepsilon}-\eta)(x^{*}))\leq 0,\\ (u^{\varepsilon}-\eta)(x^{*}+z)-(u^{\varepsilon}-\eta)(x^{*})\leq 0,\ \text{for all } x^{*}+z\in\overline{B_R(0)}. \end{cases} \tag{3.30}$$

Since $(u^{\varepsilon}-\eta)(x^{*}+z)-(u^{\varepsilon}-\eta)(x^{*})\leq 0$, for all $x^{*}+z\in\overline{B_R(0)}$, and b is small enough, it follows

$$\begin{aligned}
0\geq \mathcal{I}'\,E(u^{\varepsilon}-\eta)(x^{*})&=\int_{\{||x^{*}+z||\leq R\}}((u^{\varepsilon}-\eta)(x^{*}+z)-(u^{\varepsilon}-\eta)(x^{*}))\nu(\mathrm{d}z)\\
&\quad+\int_{\{R<||x^{*}+z||\leq R+\frac{b}{2}\}}(E(u^{\varepsilon}-\eta)(x^{*}+z)-(u^{\varepsilon}-\eta)(x^{*}))\nu(\mathrm{d}z).
\end{aligned}\tag{3.31}$$

From (3.27) and (3.30), we have $0 \geq \frac{1}{2}\operatorname{tr}(\sigma \mathrm{D}^2(u^\varepsilon - \eta)(x^*)) \geq q(u^\varepsilon - \eta)(x^*) - \mathcal{I}' E(u^\varepsilon - \eta)(x^*)$. Then, by (3.31), we get $q(u^\varepsilon - \eta)(x^*) \leq \mathcal{I}' E(u^\varepsilon - \eta)(x^*) \leq 0$, which is a contradiction of (3.29) and hence $u^\varepsilon - \eta \leq 0$ in $B_R(0)$. Since η is a function independent of ε, we conclude that $u^\varepsilon \leq K_5$ in $B_R(0)$, where $K_5 := ||\eta||_{\mathrm{C}^0(\overline{B_R(0)})}$. ∎

From Remark 2.21, we know that we can choose $N > 1$ large enough in (2.10) but fixed, such that $0 < b < \frac{1}{2^N}$ is small enough and the value of $\int_{\mathbb{R}^*} \mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z)$, is also arbitrarily small, where $\mathcal{B}' = B_{R+\frac{b}{2}}(0) \setminus \overline{B_R(0)}$. Using this and Proposition 3.13, we have the following result.

Lemma 3.17. *The solution u^ε to the NIDD problem (3.21) satisfies that $\mathcal{I}\, E(u^\varepsilon)(x) \geq 0$, for any $x \in B_R(0)$.*

Proof. Let us write $\mathcal{I}\, E(u^\varepsilon)(x)$ in the following way

$$
\begin{aligned}
\mathcal{I}\, E(u^\varepsilon)(x) &= \int_{\{||x+z|| \leq R\}} u^\varepsilon(x+z)\nu(\mathrm{d}z) + \int_{\{||x+z|| > R\}} E(u^\varepsilon)(x+z)\nu(\mathrm{d}z) \\
&= \int_{\mathbb{R}^*} u^\varepsilon(x+z)\mathbb{1}_{\mathcal{B}_2'}(x+z)\nu(\mathrm{d}z) + \int_{\mathbb{R}^*} u^\varepsilon(x+z)\mathbb{1}_{\mathcal{B}_1'}(x+z)\nu(\mathrm{d}z) \\
&\quad + \int_{\mathbb{R}^*} E(u^\varepsilon)(x+z)\mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z) \\
&= \int_{\mathbb{R}^*} u^\varepsilon(x+z)\mathbb{1}_{\mathcal{B}_2'}(x+z)\nu(\mathrm{d}z) + \int_{\mathbb{R}^*} u^\varepsilon(x+z)\mathbb{1}_{\mathcal{B}_1'}(x+z)\nu(\mathrm{d}z) \\
&\quad + \sum_{\kappa=1}^{N} \int_{\mathbb{R}^*} E(u^\varepsilon)(x+z)\mathbb{1}_{\mathcal{J}_\kappa}(x+z)\nu(\mathrm{d}z), \qquad (3.32)
\end{aligned}
$$

where $\mathcal{B}' = B_{R+\frac{b}{2}}(0) \setminus B_R(0) = \bigcup_{\kappa=1}^N \mathcal{J}_\kappa$, with $\{\mathcal{J}_\kappa\}_{\kappa=1}^N$ a sequence of disjoint sets given in (2.38), $\mathcal{B}_1' = \overline{B_R(0)} \cap \bigcup_{\kappa=1}^N \overline{B_{b_\kappa - \frac{b}{4}}(x_\kappa)}$ and $\mathcal{B}_2' := \overline{B_R(0)} \setminus \bigcup_{\kappa=1}^N \overline{B_{b_\kappa - \frac{b}{4}}(x_\kappa)}$. Recall that $0 < b < \min_{\kappa \in \{1,\ldots,N\}}\{\frac{1}{2^N}, b_\kappa\}$, such that $\partial B_R(0) \subseteq \bigcup_{\kappa=1}^N B_{b_\kappa - \frac{b}{2}}(x_\kappa)$, with $x_\kappa \in \partial B_R(0)$. Estimating the last term on the right hand side of (3.32), we have

$$
\begin{aligned}
\sum_{\kappa=1}^{N} \int_{\mathbb{R}^*} E(u^\varepsilon)(x+z)\mathbb{1}_{\mathcal{J}_\kappa}(x+z)\nu(\mathrm{d}z) &= \sum_{\kappa=1}^{N} \int_{\{E(u^\varepsilon)(x+z) \geq 0\}} E(u^\varepsilon)(x+z)\mathbb{1}_{\mathcal{J}_\kappa}(x+z)\nu(\mathrm{d}z) \\
&\quad - \sum_{\kappa=1}^{N} \int_{\{E(u^\varepsilon)(x+z) < 0\}} |E(u^\varepsilon)(x+z)|\mathbb{1}_{\mathcal{J}_\kappa}(x+z)\nu(\mathrm{d}z).
\end{aligned}
$$

Using (2.45), it follows

$$
\begin{aligned}
&\int_{\{E(u^\varepsilon)(x+z) < 0\}} |E(u^\varepsilon)(x+z)|\mathbb{1}_{\mathcal{J}_\kappa}(x+z)\nu(\mathrm{d}z) \\
&\qquad \leq 2A_0 ||u^\varepsilon||_{\mathrm{C}^0(\mathcal{B}_1')} \int_{\{E(u^\varepsilon)(x+z) < 0\}} \mathbb{1}_{\mathcal{J}_\kappa}(x+z)\nu(\mathrm{d}z), \quad (3.33)
\end{aligned}
$$

for each $\kappa \in \{1, \dots, N\}$. From (3.32) and (3.33), we get

$$\begin{aligned}
&\int_{\mathbb{R}^*} E(u^\varepsilon)(x+z)\nu(\mathrm{d}z) \\
&\geq \int_{\mathbb{R}^*} u^\varepsilon(x+z)\mathbb{1}_{\mathcal{B}_2'}(x+z)\nu(\mathrm{d}z) + \int_{\mathbb{R}^*} u^\varepsilon(x+z)\mathbb{1}_{\mathcal{B}_1'}(x+z)\nu(\mathrm{d}z) \\
&\quad + \sum_{\kappa=1}^{N} \int_{\{E(u^\varepsilon)(x+z)\geq 0\}} E(u^\varepsilon)(x+z)\mathbb{1}_{\mathcal{J}_\kappa}(x+z)\nu(\mathrm{d}z) \\
&\quad - 2A_0||u^\varepsilon||_{\mathrm{C}^0(\mathcal{B}_1')} \sum_{\kappa=1}^{N} \int_{\{E(u^\varepsilon)(x+z)<0\}} \mathbb{1}_{\mathcal{J}_N}(x+z)\nu(\mathrm{d}z) \\
&= \int_{\mathbb{R}^*} u^\varepsilon(x+z)\mathbb{1}_{\mathcal{B}_2'}(x+z)\nu(\mathrm{d}z) + \int_{\{E(u^\varepsilon)(x+z)\geq 0\}} E(u^\varepsilon)(x+z)\mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z) \\
&\quad + \int_{\mathbb{R}^*} u^\varepsilon(x+z)\mathbb{1}_{\mathcal{B}_1'}(x+z)\nu(\mathrm{d}z) - 2A_0||u^\varepsilon||_{\mathrm{C}^0(\mathcal{B}_1')} \int_{\{E(u^\varepsilon)(x+z)<0\}} \mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z). \qquad (3.34)
\end{aligned}$$

By Proposition 3.13, we know that $u^\varepsilon > 0$ in $B_R(0)$. This implies

$$\int_{\mathbb{R}^*} u^\varepsilon(x+z)\mathbb{1}_{\mathcal{B}_2'}(x+z)\nu(\mathrm{d}z) > 0 \text{ and } \int_{\mathbb{R}^*} u^\varepsilon(x+z)\mathbb{1}_{\mathcal{B}_1'}(x+z)\nu(\mathrm{d}z) > 0.$$

Observe that

$$\begin{aligned}
\int_{\mathbb{R}^*} u^\varepsilon(x+z)\mathbb{1}_{\mathcal{B}_2'}(x+z)\nu(\mathrm{d}z) \geq \Bigg| &\int_{\mathbb{R}^*} u^\varepsilon(x+z)\mathbb{1}_{\mathcal{B}_1'}(x+z)\nu(\mathrm{d}z) \\
&- 2A_0||u^\varepsilon||_{\mathrm{C}^0(\mathcal{B}_1')} \int_{\{E(u^\varepsilon)(x+z)<0\}} \mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z)\Bigg|, \qquad (3.35)
\end{aligned}$$

because $\int_{\mathbb{R}^*} \mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z)$ can be chosen arbitrarily small, and $||u^\varepsilon||_{\mathrm{C}^0(\mathcal{B}_1')} \leq ||u^\varepsilon||_{\mathrm{C}^0(\overline{B_R(0)})}$, where $||u^\varepsilon||_{\mathrm{C}^0(\overline{B_R(0)})}$ is bounded by a constant independent of ε; see Lemma 3.16. From (3.34) and (3.35), we conclude that $\int_{\mathbb{R}^*} E(u^\varepsilon)(x+z)\nu(\mathrm{d}z) \geq 0$. ■

Defining η_1 as

$$\eta_1(x) = \begin{cases} \mathrm{e}^{K_6 R^2} - \mathrm{e}^{K_6||x||^2}, & \text{if } x \in B_R(0), \\ E(u^\varepsilon)(x), & \text{if } x \in B_R(0)^{\mathrm{c}}, \end{cases} \qquad (3.36)$$

with $K_6 > 0$ a constant, we can see that $\eta_1 \in \mathrm{C}^2(B_R(0)) \cap \mathrm{C}^0(B_R(0)^{\mathrm{c}})$ is a positive concave function in $B_R(0)$. We have the following result.

Lemma 3.18. *Let η_1 be defined as in (3.36). Then, choosing $K_6 > 0$ large enough,*

$$q\eta_1(x) - \mathcal{L}\eta_1(x) - \mathcal{I}'\eta_1(x) \geq C_0(1+||x||^2) \geq h(x), \textit{ in } B_R(0). \qquad (3.37)$$

This statement will be helpful in finding a constant, independent of ε, which bounds by above $|\partial_\vartheta u^\varepsilon|$ in $\partial B_R(0)$. Recall that $\partial_\vartheta f$ denotes the directional derivative of the function f with respect to the unit vector $\vartheta \in \mathbb{R}^d$, i.e. $\partial_\vartheta f(x) := \lim_{\delta\to 0} \frac{f(x)-f(x-\delta\vartheta)}{h}$, with $x \in \mathbb{R}^d$.

Proof of Lemma 3.18. Let η_1 be as in (3.36). Calculating their first and second derivatives in $B_R(0)$,

$$\begin{cases} \partial_i \eta_1(x) &= -2K_6\, \mathrm{e}^{K_6||x||^2}\, x_i, \\ \partial^2_{ii}\eta_1(x) &= -2K_6\, \mathrm{e}^{K_6||x||^2}(1+2K_6x_i^2), \\ \partial^2_{ji}\eta_1(x) &= -4K_6^2\, \mathrm{e}^{K_6||x||^2}\, x_i x_j, \end{cases} \tag{3.38}$$

with $i, j \in \{1, \dots, d\}$ and $i \neq j$, by (H3) and (3.38), we see that

$$\begin{aligned} -\mathcal{L}\eta_1(x) &= 2K_6\, \mathrm{e}^{K_6||x||^2}\left(\frac{1}{2}\sum_i \sigma_{ii} + K_6\langle \sigma x, x\rangle + \langle x, \gamma\rangle\right) \\ &\geq 2K_6\, \mathrm{e}^{K_6||x||^2}\left(K_6\theta||x||^2 - \Lambda||x|| + \frac{\theta d}{2}\right). \end{aligned} \tag{3.39}$$

Since η_1 is a positive concave function in $B_R(0)$, we have that

$$\eta_1(x+z) - \eta_1(x) \leq \langle \mathrm{D}^1\, \eta_1(x), z\rangle, \text{ for all } ||x+z|| < R.$$

Then, using Lemma 2.19, we obtain the following inequalities

$$\begin{aligned} -\mathcal{I}'\, \eta_1(x) &= -\int_{\{||x+z||<R\}} (\eta_1(x+z) - \eta_1(x) - \langle \mathrm{D}^1\, \eta_1(x), z\rangle)\nu(\mathrm{d}z) \\ &\quad - \int_{\{||x+z||\geq R\}} (E(u^\varepsilon)(x+z) - \eta_1(x) - \langle \mathrm{D}^1\, \eta_1(x), z\rangle)\nu(\mathrm{d}z) \\ &\geq -\int_{\{||x+z||\geq R\}} (E(u^\varepsilon)(x+z) - \eta_1(x) - \langle \mathrm{D}^1\, \eta_1(x), z\rangle)\nu(\mathrm{d}z) \\ &\geq -\int_{\{||x+z||\geq R\}} |E(u^\varepsilon)(x+z)|\nu(\mathrm{d}z) + \eta_1(x)\int_{\{||x+z||\geq R\}} \nu(\mathrm{d}z) \\ &\quad + 2K_6\, \mathrm{e}^{K_6||x||^2}\int_{\{||x+z||\geq R\}} \langle x, z\rangle \nu(\mathrm{d}z) \\ &\geq -2K_6\left(A_0\int_{\mathbb{R}^*} \mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z) + \nu_0\, \mathrm{e}^{K_6||x||^2}\, ||x||\right). \end{aligned} \tag{3.40}$$

Recall that $\mathcal{B}' = B_{R+\frac{b}{2}}(0) \setminus \overline{B_R(0)}$ and ν_0, A_0 are constants given by (H2) and (2.24), respectively. Using (3.39)–(3.40), we get that

$$\begin{aligned} q\eta_1(x) - \mathcal{L}\eta_1(x) - \mathcal{I}'\, \eta_1(x) &\geq 2K_6\, \mathrm{e}^{K_6||x||^2}\left(\theta K_6||x||^2 - \left(\Lambda + \nu_0\right)||x|| + \frac{\theta d}{2}\right) \\ &\quad - 2K_6 A_0 \int_{\mathbb{R}^*} \mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z), \end{aligned}$$

for all $x \in B_R(0)$. From (H1) and choosing K_6 large enough, it implies (3.37). ∎

We obtain the following result as a consequence of the previous lemma.

Lemma 3.19. *Let $K_6 > 0$ be the constant given in Lemma 3.18. Then*

$$|\partial_\vartheta u^\varepsilon(x)| \leq 2K_6 R\, \mathrm{e}^{K_6 R^2}, \text{ in } \partial B_R(0).$$

Proof. Let $x \in \partial B_R(0)$, ϑ a unit vector and η_1 as in (3.36). Since

$$\begin{cases} q(u^\varepsilon - \eta_1) - \mathcal{L}(u^\varepsilon - \eta_1) - \mathcal{I}'(E(u^\varepsilon) - \eta_1) \leq 0, \text{ in } B_R(0), \\ \sup_{B_{R+\frac{b}{2}} \setminus B_R(0)} [E(u^\varepsilon) - \eta_1]^+ = 0, \end{cases}$$

by the weak maximum principle, Theorem 3.14, it follows that $u^\varepsilon \leq \eta_1$. Since these functions agree in $B_R(0)^{\mathrm{c}}$ and $u^\varepsilon > 0$, we get that

$$\partial_\vartheta u^{\varepsilon,R}(x) = \lim_{h\to 0} \frac{u^\varepsilon(x) - u^\varepsilon(x - h\vartheta)}{h} = \lim_{h\to 0} \frac{-u^\varepsilon(x - h\vartheta)}{h} \leq 0,$$
$$\partial_\vartheta \eta_1(x) = \lim_{h\to 0} \frac{-\eta_1(x - h\vartheta)}{h} \leq \lim_{h\to 0} \frac{-u^\varepsilon(x - h\vartheta)}{h} = \partial_\vartheta u^\varepsilon(x).$$

Then $\partial_\vartheta \eta_1(x) \leq \partial_\vartheta u^\varepsilon(x) \leq 0$. It implies that $|\partial_\vartheta u^\varepsilon(x)| \leq ||\,\mathrm{D}^1\, \eta_1(x)||$ in $\partial B_R(0)$. Recalling the definition of η_1 and its first derivatives, see (3.38), it follows that $|\partial_\vartheta u^\varepsilon(x)| \leq 2K_6 R\, \mathrm{e}^{K_6 R^2}$ in $\partial B_R(0)$. ■

Before showing that $||\,\mathrm{D}^1\, u^\varepsilon||$ is bounded by a positive constant in $\overline{B_R(0)}$, which is independent of ε; see Lemma 3.21, we establish an auxiliary result.

Lemma 3.20. *Define the auxiliary function $\varphi : \overline{B_R(0)} \longrightarrow \mathbb{R}$ as*

$$\varphi(x) := ||\,\mathrm{D}^1\, u^\varepsilon||^2 - M u^\varepsilon(x), \text{ for all } x \in \overline{B_R(0)}, \tag{3.41}$$

where $M := \max_{x\in\overline{B_R(0)}} ||\,\mathrm{D}^1\, u^\varepsilon(x)||$. Then

$$\frac{1}{2}\sum_{ij} \sigma_{ij}\partial^2_{ij}\varphi(x) \geq \psi'_\varepsilon(g(x))(2\langle \mathrm{D}^1\, \varphi(x), \mathrm{D}^1\, u^\varepsilon(x)\rangle + M||\,\mathrm{D}^1\, u^\varepsilon(x)||^2)$$
$$- (K_8 + MK_9)||\,\mathrm{D}^1\, u^\varepsilon(x)|| - M(K_7 + K_{10}) - \langle \mathrm{D}^1\, \varphi(x), \widetilde{\gamma}\rangle, \tag{3.42}$$

for all $x \in B_R(0)$, where the constants $K_7, \ldots, K_{10}$ are independent of ε.

Proof. Note that $\varphi \in \mathrm{C}^{2,\alpha}(\overline{B_R(0)})$, since $u^\varepsilon \in \mathrm{C}^{3,\alpha}(\overline{B_R(0)})$. Then, calculating first and second derivatives of φ in $B_R(0)$,

$$\begin{cases} \partial_i\varphi(x) = 2\sum_k \partial_k u^\varepsilon(x)\partial^2_{ki}u^\varepsilon(x) - M\partial_i u^\varepsilon(x), \\ \partial^2_{ij}\varphi(x) = 2\sum_k(\partial^2_{kj}u^\varepsilon(x)\partial^2_{ki}u^\varepsilon(x) + \partial_k u^\varepsilon(x)\partial^3_{kij}u^\varepsilon(x)) - M\partial^2_{ij}u^\varepsilon(x), \end{cases}$$

we have that

$$\frac{1}{2}\sum_{ij}\sigma_{ij}\partial^2_{ij}\varphi(x)=\sum_{kij}\sigma_{ij}\partial^2_{kj}u^\varepsilon(x)\partial^2_{ki}u^\varepsilon(x)+\sum_{kij}\sigma_{ij}\partial_k u^\varepsilon(x)\partial^3_{kij}u^\varepsilon(x))-\frac{M}{2}\sum_{ij}\sigma_{ij}\partial^2_{ij}u^\varepsilon(x).$$

Using (3.21) and (3.22), we get for each $x\in B_R(0)$,

$$\begin{cases}-\frac{M}{2}\sum_{ij}\sigma_{ij}\partial^2_{ij}u^\varepsilon(x)=M(h(x)-q'u^\varepsilon(x)-\psi_\varepsilon(g(x))+\langle \mathrm{D}^1 u^\varepsilon(x),\widetilde{\gamma}\rangle+\mathcal{I}\,E(u^\varepsilon)(x)),\\ \frac{1}{2}\sum_{kij}\sigma_{ij}\partial_k u^\varepsilon(x)\partial^3_{kij}u^\varepsilon(x))=q'||\,\mathrm{D}^1 u^\varepsilon(x)||^2-\langle \mathrm{D}^1 u^\varepsilon(x),\mathrm{D}^1 h(x)\rangle\\ \qquad\qquad +\psi'_\varepsilon(g(x))\langle \mathrm{D}^1 u^\varepsilon(x),\mathrm{D}^1 g(x)\rangle-\langle \mathrm{D}^2 u^\varepsilon(x)\,\mathrm{D}^1 u^\varepsilon(x),\widetilde{\gamma}\rangle\\ \qquad\qquad -\sum_i\partial_i u^\varepsilon(x)\int_{\mathbb{R}^*}E(\partial_i u^\varepsilon)(x+z)\nu(\mathrm{d}z),\end{cases}$$

where the first and second derivatives of $g(x)$ are given in (3.23). Then,

$$\begin{aligned}&\frac{1}{2}\sum_{ij}\sigma_{ij}\partial^2_{ij}\varphi(x)\\ &=\sum_{kij}\sigma_{ij}\partial^2_{kj}u^\varepsilon(x)\partial^2_{ki}u^\varepsilon(x)-2\langle \mathrm{D}^1 u^\varepsilon(x),\mathrm{D}^1 h(x)\rangle+Mh(x)\\ &\quad+q'(2||\,\mathrm{D}^1 u^\varepsilon(x)||^2-Mu^\varepsilon(x))+2\psi'_\varepsilon(g(x))\langle \mathrm{D}^1 u^\varepsilon(x),\mathrm{D}^1 g(x)\rangle-M\psi_\varepsilon(g(x))\\ &\quad+M\int_{\mathbb{R}^*}E(u^\varepsilon)(x+z)\nu(\mathrm{d}z)-2\sum_i\partial_i u^\varepsilon(x)\int_{\mathbb{R}^*}E(\partial_i u^\varepsilon)(x+z)\nu(\mathrm{d}z)\\ &\quad-2\langle \mathrm{D}^2 u^\varepsilon(x)\,\mathrm{D}^1 u^\varepsilon(x),\widetilde{\gamma}\rangle+M\langle \mathrm{D}^1 u^\varepsilon(x),\widetilde{\gamma}\rangle.\end{aligned}\tag{3.43}$$

Lemma 3.16 implies

$$q'(2||\,\mathrm{D}^1 u^\varepsilon(x)||^2-Mu^\varepsilon(x))\geq -MK_7,\tag{3.44}$$

where $K_7{:=}\,q'K_5$. The constant K_5 is as in Lemma 3.16. By (H1) and (H3), it follows

$$\begin{aligned}-K_8||\,\mathrm{D}^1 u^\varepsilon(x)||&\leq\theta||\,\mathrm{D}^2 u^\varepsilon(x)||^2-K_8||\,\mathrm{D}^1 u^\varepsilon(x)||\\ &\leq\sum_{kij}\sigma_{ij}\partial^2_{kj}u^\varepsilon(x)\partial^2_{ki}u^\varepsilon(x)-2\langle \mathrm{D}^1 u^\varepsilon(x),\mathrm{D}^1 h(x)\rangle+Mh(x),\end{aligned}\tag{3.45}$$

where $K_8{:=}\,2C_0$. Recall that the constants θ and C_0 are given in (H1) and (H3), respectively. Since $\psi_\varepsilon(g(x))\leq\psi'_\varepsilon(g(x))g(x)$ and $\partial_i\varphi(x)=2\sum_k\partial_k u^\varepsilon(x)\partial^2_{ki}u^\varepsilon(x)-M\partial_i u^\varepsilon(x)$ for all $i\in\{1,\dots,d\}$, we have that

$$\begin{cases}\langle \mathrm{D}^1\varphi(x),\widetilde{\gamma}\rangle=2\langle \mathrm{D}^2 u^\varepsilon(x)\,\mathrm{D}^1 u^\varepsilon(x),\widetilde{\gamma}\rangle-M\langle \mathrm{D}^1 u^\varepsilon(x),\widetilde{\gamma}\rangle,\\ 2\psi'_\varepsilon(g(x))\langle \mathrm{D}^1 u^\varepsilon(x),\mathrm{D}^1 g(x)\rangle-M\psi_\varepsilon(g(x))\\ \qquad\qquad\geq\psi'_\varepsilon(g(x))(2\langle \mathrm{D}^1\varphi(x),\mathrm{D}^1 u^\varepsilon(x)\rangle+M||\,\mathrm{D}^1 u^\varepsilon(x)||^2).\end{cases}\tag{3.46}$$

Since $\int_{\mathbb{R}^*} E(u^\varepsilon)(x+z)\nu(\mathrm{d}z) \geq 0$ and from Lemma 2.20, it follows that

$$-\,||\,\mathrm{D}^1 u^\varepsilon(x)||MK_9 - MK_{10} \\ \leq M\int_{\mathbb{R}^*} E(u^\varepsilon)(x+z)\nu(\mathrm{d}z) - 2\sum_i \partial_i u^\varepsilon(x)\int_{\mathbb{R}^*} E(\partial_i u^\varepsilon)(x+z)\nu(\mathrm{d}z), \quad (3.47)$$

where $K_9 := d\big(\nu_0 + dC_3\int_{\mathbb{R}^*} \mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z)\big)$, and $K_{10} := dC_3K_5\int_{\mathbb{R}^*} \mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z)$. Recall that the constants ν_0 and C_3 are as in (H2) and Lemma 2.20, respectively, and $\mathcal{B}' = B_{R+\frac{b}{2}}(0)\setminus\overline{B_R(0)}$. Applying (3.44), (3.45), (3.46) and (3.47) in (3.43), it yields (3.42). ■

Lemma 3.21. *There exists a constant $K_{11} > 0$ independent of ε such that*

$$||\,\mathrm{D}^1 u^\varepsilon(x)|| \leq K_{11},\ \textit{in } B_R(0).$$

Proof. Consider the auxiliary function φ as in (3.41). Observe that if

$$M = \sup_{x\in\overline{B_R(0)}} ||\,\mathrm{D}^1 u^\varepsilon(x)|| \leq 1,$$

we obtain a bound for M that is independent of ε. We assume henceforth that $M \geq 1$. Taking $x^* \in \overline{B_R(0)}$ as a point where φ attains its maximum on $B_R(0)$, it suffices to bound $||\,\mathrm{D}^1 u^\varepsilon(x^*)||^2$ for a constant independent of ε, since

$$||\,\mathrm{D}^1 u^\varepsilon(x)||^2 \leq ||\,\mathrm{D}^1 u^\varepsilon(x^*)||^2 + M(u^\varepsilon(x^*) + u^\varepsilon(x)) \leq ||\,\mathrm{D}^1 u^\varepsilon(x^*)||^2 + 2MK_5, \quad (3.48)$$

for all $x \in \overline{B_R(0)}$. The last inequality in (3.48) is obtained from Lemma 3.16. If $x^* \in \partial B_R(0)$, by Lemma 3.19, it is easy to deduce $\varphi(x^*) = ||\,\mathrm{D}^1 u^\varepsilon(x^*)||^2 \leq 2K_6R\,\mathrm{e}^{K_6R^2}$, where K_6 is as in Lemma 3.19. Then, from (3.48),

$$||\,\mathrm{D}^1 u^\varepsilon(x)||^2 \leq 2K_6R\,\mathrm{e}^{K_6R^2} + 2MK_5,\ \text{for all } x \in \overline{B_R(0)}.$$

Note that for all ϵ, there exists $x_0 \in \overline{B_R(0)}$ such that $(M-\epsilon)^2 \leq ||\,\mathrm{D}\,u^\varepsilon(x_0)||^2$. Then

$$(M-\epsilon)^2 \leq 2K_6R\,\mathrm{e}^{K_6R^2} + 2MK_5,\ \text{for all } x \in \overline{B_R(0)}. \quad (3.49)$$

Letting $\epsilon \to 0$ in (3.49), it follows $M \leq 2K_6R\,\mathrm{e}^{K_6R^2} + 2K_5$, where $2K_6R\,\mathrm{e}^{K_6R^2} + 2K_5$ is a constant independent of ε. When $x^* \in B_R(0)$, we have that $\mathrm{D}^1\varphi(x^*) = 0$ and $\frac{1}{2}\sum_{ij}\sigma_{ij}\partial_{ij}\varphi(x^*) \leq 0$. Then, from (3.42), we get

$$0 \geq M\psi_\varepsilon'(g(x^*))||\,\mathrm{D}^1 u^\varepsilon(x^*)||^2 - (K_8 + MK_9)||\,\mathrm{D}^1 u^\varepsilon(x^*)|| - M(K_7 + K_{10}). \quad (3.50)$$

If $\psi'_\varepsilon(g(x^*)) < 1 < \dfrac{1}{\varepsilon}$, by definition of ψ_ε, given in (3.3), we obtain that $\psi_\varepsilon(g(x^*)) \leq 1$. It follows that $||\,\mathrm{D}^1\, u^\varepsilon(x^*)|| \leq 2\varepsilon + 1 \leq 2$. Then, by (3.48) and arguing as in (3.49), we obtain $M \leq 4 + 2K_5$, where $4 + 2K_5$ is a constant independent of ε. If $\psi'_\varepsilon(g(x^*)) \geq 1$, from (3.50), we get

$$0 \geq M||\,\mathrm{D}^1\, u^\varepsilon(x^*)||^2 - (K_8 + MK_9)||\,\mathrm{D}^1\, u^\varepsilon(x^*)|| - M(K_7 + K_{10}),$$

and hence

$$\begin{aligned} 0 \geq &\left(||\,\mathrm{D}^1\, u^\varepsilon(x^*)|| - \frac{K_8 + MK_9 + ((K_8 + MK_9)^2 + 4M^2(K_7 + K_{10}))^{\frac{1}{2}}}{2M}\right) \\ &\times\left(||\,\mathrm{D}^1\, u^\varepsilon(x^*)|| - \frac{K_8 + MK_9 - ((K_8 + MK_9)^2 + 4M^2(K_7 + K_{10}))^{\frac{1}{2}}}{2M}\right). \end{aligned} \tag{3.51}$$

Since $K_8 + MK_9 - ((K_8 + MK_9)^2 + 4M^2(K_7 + K_{10}))^{\frac{1}{2}} \leq 0$, it implies

$$||\,\mathrm{D}^1\, u^\varepsilon(x^*)|| - \frac{K_8 + MK_9 - ((K_8 + MK_9)^2 + 4M^2(K_7 + K_{10}))^{\frac{1}{2}}}{2M} \geq 0.$$

From (3.51), it yields

$$\begin{aligned} ||\,\mathrm{D}^1\, u^\varepsilon(x^*)|| &\leq \frac{K_8 + MK_9 + ((K_8 + MK_9)^2 + 4M^2(K_7 + K_{10}))^{\frac{1}{2}}}{2M} \\ &\leq \frac{2(K_8 + MK_9) + 2M(K_7 + K_{10})^{\frac{1}{2}}}{2M} \\ &\leq K_8 + K_9 + (K_7 + K_{10})^{\frac{1}{2}}. \end{aligned}$$

Using (3.48) and a similar argument that (3.49), we conclude

$$M \leq (K_8 + K_9 + (K_7 + K_{10})^{\frac{1}{2}})^2 + 2K_5,$$

where $(K_8 + K_9 + (K_7 + K_{10})^{\frac{1}{2}})^2 + 2K_5$ is a constant independent of ε. Therefore, in this case we also have that there exists a constant $K_{11} > 0$, independent of ε, such that $||\,\mathrm{D}^1\, u^\varepsilon(x)|| \leq K_{11}$ in $B_R(0)$. ■

In Lemma 3.23, we shall establish that $\psi_\varepsilon(||\,\mathrm{D}^1\, u^\varepsilon||^2)$ is locally bounded by a constant independent of ε. Previous, we give an auxiliary result.

Lemma 3.22. *For each cutoff function ξ in $\mathrm{C}^\infty_c(B_r)$ satisfying $0 \leq \xi \leq 1$, with $B_r \subset B_R(0)$, define the function $\phi : \overline{B_r} \longrightarrow \mathbb{R}$ as*

$$\phi(x) = \xi(x)\psi_\varepsilon(g(x)). \tag{3.52}$$

Then,

$$\frac{1}{2}\sum_{ij}\sigma_{ij}\partial^2_{ij}\phi(x) \geq -K_{16}(K_{15}+K_{14}||\,\mathrm{D}^2\,u^{\varepsilon}(x)||) \\ + \psi'_{\varepsilon}(g(x))\big(\theta\xi(x)||\,\mathrm{D}^2\,u^{\varepsilon}(x)||^2-(2K_{11}||\widetilde{\gamma}||+K_{17})||\,\mathrm{D}^2\,u^{\varepsilon}(x)|| \\ -(2K_{18}+dK_{11}K_{13}+K_{11}K_{15}K_{16})+2\langle \mathrm{D}^1\,\phi(x),\mathrm{D}^1\,u^{\varepsilon}(x)\rangle\big), \qquad (3.53)$$

for all $x\in B_r$, where $K_{11},\dots,K_{18}$ are positive constants independent of ε.

Proof. Since $u^{\varepsilon}(\cdot)$ and $||\,\mathrm{D}^1\,u^{\varepsilon}(\cdot)||$ are uniformly bounded with respect ε in $B_R(0)$ (Lemmas 3.16 and 3.21), we have

$$\begin{cases} u^{\varepsilon}(x)\leq K_5, \\ ||\,\mathrm{D}^1(u^{\varepsilon})(x)||\leq K_{11}, \end{cases} \quad \text{in } x\in B_r. \qquad (3.54)$$

Furthermore, from Lemmas 2.19 and 2.20, we see that

$$\left|\int_{\mathbb{R}^*}E(u^{\varepsilon})(x+z)\nu(\mathrm{d}z)\right|\leq K_{12}, \qquad (3.55)$$

$$\left|\int_{\mathbb{R}^*}E(\partial_i u^{\varepsilon})(x+z)\nu(\mathrm{d}z)\right|\leq K_{13}, \qquad (3.56)$$

for all $x\in B_r$ and for each $i\in\{1,\dots,d\}$, where

$$K_{12}{:=}\ K_5\bigg(\nu_0+2A_0\int_{\mathbb{R}^*}\mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z)\bigg), \\ K_{13}{:=}\ K_{11}\nu_0+C_3(K_5+dK_{11})\int_{\mathbb{R}^*}\mathbb{1}_{\mathcal{B}'}(x+z)\nu(\mathrm{d}z),$$

are constants independent of ε. Recall ν_0, A_0, C_3, K_5 and K_{11} are constants given in (H2), (2.24) and Lemmas 2.16, 3.16 and 3.21, respectively. Then, using the Hypothesis (H1), (3.54) and (3.55) in (3.21), we have that

$$\psi_{\varepsilon}(g(x))=h(x)-q'u^{\varepsilon}(x)+\mathcal{L}'u^{\varepsilon}(x)+\int_{\mathbb{R}^*}E(u^{\varepsilon})(x+z)\nu(\mathrm{d}z) \\ \leq K_{14}||\,\mathrm{D}^2\,u^{\varepsilon}(x)||+K_{15}, \qquad (3.57)$$

for all $x\in B_r$, where $g(x)=||\,\mathrm{D}^1\,u^{\varepsilon}(x)||^2$ and

$$K_{14}{:=}\sum_{ij}|\sigma_{ij}|, \\ K_{15}{:=}\ (C_0+q'K_5)+K_{11}||\widetilde{\gamma}||+K_{12},$$

Then, calculating the first and second derivatives of ϕ in B_r,

$$\begin{cases} \partial_i\phi(x) = \psi_\varepsilon(g(x))\partial_i\xi(x) + \xi(x)\psi'_\varepsilon(g(x))\partial_i g(x), \\ \partial^2_{ji}\phi(x) = \psi_\varepsilon(g(x))\partial^2_{ji}\xi(x) + \psi'_\varepsilon(g(x))\partial_i\xi(x)\partial_j g(x) + \xi(x)\psi''_\varepsilon(g(x))\partial_j g(x)\partial_i g(x) \\ \qquad\qquad +\psi'_\varepsilon(g(x))\partial_j\xi(x)\partial_i g(x) + \xi(x)\psi'_\varepsilon(g(x))\partial^2_{ji} g(x), \end{cases}$$

where $i, j \in \{1, \dots, d\}$ and the derivatives of $g(x)$ are given in (3.23), by (3.43), we get

$$\begin{aligned} \frac{1}{2}\sum_{ji}\sigma_{ji}\partial^2_{ji}\phi(x) = {} & \frac{\psi_\varepsilon(g(x))}{2}\sum_{ji}\sigma_{ji}\partial^2_{ji}\xi(x) + \frac{\xi(x)\psi''_\varepsilon(g(x))}{2}\sum_{ji}\sigma_{ji}\partial_j g(x)\partial_i g(x) \\ & + \psi'_\varepsilon(g(x))\Bigg(\sum_{ji}\sigma_{ji}\partial_i\xi(x)\partial_j g(x) \\ & + \xi(x)\sum_{jik}\sigma_{ji}\big(\partial^2_{kj}u^\varepsilon(x)\partial^2_{kj}u^\varepsilon(x) + \partial_k u^\varepsilon(x)\partial^3_{jik}u^\varepsilon(x)\big)\Bigg) \\ = {} & \frac{\psi_\varepsilon(g(x))}{2}\sum_{ji}\sigma_{ji}\partial^2_{ji}\xi(x) + \frac{\xi(x)\psi''_\varepsilon(g(x))}{2}\sum_{ji}\sigma_{ji}\partial_j g(x)\partial_i g(x) \\ & + \psi'_\varepsilon(g(x))\Bigg(\sum_{ji}\sigma_{ji}\partial_i\xi(x)\partial_j g(x) + \xi(x)\sum_{jik}\sigma_{ji}\partial^2_{kj}u^\varepsilon(x)\partial^2_{kj}u^\varepsilon(x) \\ & + 2\xi(x)\Bigg(q'||\,\mathrm{D}^1 u^\varepsilon(x)||^2 - \langle \mathrm{D}^1 u^\varepsilon(x), \mathrm{D}^1 h(x)\rangle \\ & + \psi'_\varepsilon(g(x))\langle \mathrm{D}^1 u^\varepsilon(x), \mathrm{D}^1 g(x)\rangle - \langle \mathrm{D}^2 u^\varepsilon(x)\,\mathrm{D}^1 u^\varepsilon(x), \widetilde{\gamma}\rangle \\ & - \sum_i \partial_i u^\varepsilon(x)\int_{\mathbb{R}^*} E(\partial_i u^\varepsilon)(x+z)\nu(\mathrm{d}z)\Bigg)\Bigg). \end{aligned} \tag{3.58}$$

From Hypothesis (H3) and (3.57), it implies

$$\begin{aligned} & \frac{\psi_\varepsilon(g(x))}{2}\operatorname{tr}(\sigma\,\mathrm{D}^2\,\xi(x)) + \frac{\xi(x)\psi''_\varepsilon(g(x))}{2}\langle\sigma\,\mathrm{D}^1\,g(x), \mathrm{D}^1\,g(x)\rangle \\ & \qquad\qquad \geq -K_{16}(K_{15} + K_{14}||\,\mathrm{D}^2\,u^\varepsilon(x)||) + \frac{\theta\xi(x)\psi''_\varepsilon(g(x))}{2}||\,\mathrm{D}^1\,g(x)||^2, \end{aligned}$$

where $K_{16} > 0$ is a constant that only depends on ξ. Since $\xi(x) \geq 0$ and $\psi''_\varepsilon(x) \geq 0$, it follows

$$\begin{aligned} & -K_{16}(K_{15} + K_{14}||\,\mathrm{D}^2\,u^\varepsilon(x)||) \\ & \qquad \leq \frac{\psi_\varepsilon(g(x))}{2}\operatorname{tr}(\sigma\,\mathrm{D}^2\,\xi(x)) + \frac{\xi(x)\psi''_\varepsilon(g(x))}{2}\langle\sigma\,\mathrm{D}^1\,g(x), \mathrm{D}^1\,g(x)\rangle, \end{aligned} \tag{3.59}$$

Using Hypothesis (H3) and (3.54), it implies

$$\begin{aligned} & -K_{17}||\,\mathrm{D}^2\,u^\varepsilon(x)|| + \theta\xi(x)||\,\mathrm{D}^2\,u^\varepsilon(x)||^2 \\ & \qquad \leq \sum_{ji}\sigma_{ji}\partial_i\xi(x)\partial_j g(x) + \xi(x)\sum_{jik}\sigma_{ji}\partial^2_{kj}u^\varepsilon(x)\partial^2_{kj}u^\varepsilon(x), \end{aligned} \tag{3.60}$$

where $K_{17}:= 2dK_{11}K_{14}K_{16}$ is a constant that only depends on ξ. From Hypothesis (H1), it follows

$$-K_{18} \leq q' || \mathrm{D}^1 u^\varepsilon(x)||^2 - \langle \mathrm{D}^1 u^\varepsilon(x), \mathrm{D}^1 h(x)\rangle. \tag{3.61}$$

where $K_{18}:= K_{11}C_0$ is a constant independent of ε. Since

$$\partial_i\phi(x) = \psi_\varepsilon(g(x))\partial_i\xi(x) + \xi(x)\psi'_\varepsilon(g(x))\partial_i g(x),$$

we see

$$\xi(x)\psi'_\varepsilon(g(x))\langle \mathrm{D}^1 g(x), \mathrm{D}^1 u^\varepsilon(x)\rangle = \langle \mathrm{D}^1 \phi(x), \mathrm{D}^1 u^\varepsilon(x)\rangle - \psi_\varepsilon(g(x))\langle \mathrm{D}^1 \xi(x), \mathrm{D}^1 u^\varepsilon(x)\rangle.$$

From (3.57), it yields

$$\begin{aligned}\xi(x)\psi'_\varepsilon(g(x))&\langle \mathrm{D}^1 g(x), \mathrm{D}^1 u^\varepsilon(x)\rangle \\ &\geq \langle \mathrm{D}^1 \phi(x), \mathrm{D}^1 u^\varepsilon(x)\rangle - K_{16}K_{11}(K_{15} + K_{14}|| \mathrm{D}^2 u^\varepsilon(x)||).\end{aligned} \tag{3.62}$$

Finally, (3.54) and (3.56) implies

$$\begin{aligned}- K_{11}||\widetilde{\gamma}||\, &|| \mathrm{D}^2 u^\varepsilon(x)|| - dK_{11}K_{13} \\ &\leq -\langle \mathrm{D}^2 u^\varepsilon(x)\, \mathrm{D}^1 u^\varepsilon(x), \widetilde{\gamma}\rangle - \sum_i \partial_i u^\varepsilon(x) \int_{\mathbb{R}^*} E(\partial_i u^\varepsilon)(x+z)\nu(\mathrm{d}z).\end{aligned} \tag{3.63}$$

Then, applying (3.59), (3.60), (3.61), (3.62) and (3.63) in (3.58), we conclude

$$\begin{aligned}\frac{1}{2}\operatorname{tr}(\sigma\, \mathrm{D}^2 \phi(x)) &\geq -K_{16}(K_{15} + K_{14}|| \mathrm{D}^2 u^\varepsilon(x)||) \\ &\quad + \psi'_\varepsilon(g(x))(\theta\xi(x)|| \mathrm{D}^2 u^\varepsilon(x)||^2 - (2K_{11}||\widetilde{\gamma}||\xi(x) + K_{17})|| \mathrm{D}^2 u^\varepsilon(x)|| \\ &\quad - (2K_{18}\xi(x) + dK_{11}K_{13} + K_{11}K_{15}K_{16}) + 2\langle \mathrm{d}^1\phi(x), \mathrm{D}^2 u^\varepsilon(x)\rangle) \\ &\geq -K_{16}(K_{15} + K_{14}|| \mathrm{D}^1 u^\varepsilon(x)||) \\ &\quad + \psi'_\varepsilon(g(x))(\theta\xi(x)|| \mathrm{D}^2 u^\varepsilon(x)||^2 - (2K_{11}||\widetilde{\gamma}|| + K_{17})|| \mathrm{D}^2 u^\varepsilon(x)|| \\ &\quad - (2K_{18} + dK_{11}K_{13} + K_{11}K_{15}K_{16}) + 2\langle \mathrm{D}^1 \phi(x), \mathrm{D}^1 u^\varepsilon(x)\rangle).\end{aligned}$$

■

Lemma 3.23. *Let $B_r \subset B_R(0)$ be an open ball. For each $\xi \in \mathrm{C}^\infty_{\mathrm{c}}(B_r)$ satisfying $0 \leq \xi \leq 1$, there exist non-negative constants $K_{14}, K_{15}, K_{19}, K_{20}$ independent of ε, such that*

$$\xi(x)\psi_\varepsilon(|| \mathrm{D}^1 u^\varepsilon(x)||^2) \leq \frac{K_{14}(K_{19} + (\theta K_{20})^{\frac{1}{2}})}{\theta} + K_{15},$$

for all $x \in B_r$. The constant $\theta > 0$ is as in the Hypothesis (H3).

Proof. Let $B_r \subset B_R(0)$ and for each cutoff function ξ in $\mathrm{C}_\mathrm{c}^\infty(B_r)$ satisfying $0 \le \xi \le 1$, define ϕ as in (3.52). Taking $x^* \in \overline{B_r}$ as a point where ϕ attains its maximum on B_r, it suffices to bound $\phi(x^*)$ by a constant independent of ε. If $x^* \in \partial B_r$ then $\phi(x) \le \phi(x^*) = 0$. When $x^* \in B_R(0)$, we have

$$\mathrm{D}^1 \phi(x^*) = 0 \text{ and } \frac{1}{2}\sum_{ij} \sigma_{ij}\partial_{ij}^2\phi(x^*) \le 0.$$

Then, from (3.53), we get that

$$\begin{aligned} 0 \ge & -K_{16}(K_{15} + K_{14}|| \mathrm{D}^2 u^\varepsilon(x^*)||) + \psi_\varepsilon'(g(x^*))\big(\theta\xi(x^*)|| \mathrm{D}^2 u^\varepsilon(x^*)||^2 \\ & - (2K_{11}||\widetilde{\gamma}|| + K_{17})|| \mathrm{D}^2 u^\varepsilon(x^*)|| - (2K_{18} + dK_{11}K_{13} + K_{11}K_{15}K_{16})\big), \end{aligned} \tag{3.64}$$

where $K_{11}, \dots, K_{18}$ are constants independent of ε. If $\psi_\varepsilon'(g(x^*)) \le 1 < \frac{1}{\varepsilon}$, by the definition of ψ_ε, given in (3.3), we obtain that $\psi_\varepsilon(g(x^*)) \le 1$. Then,

$$\phi(x) \le \phi(x^*) = \xi(x^*)\psi_\varepsilon(g(x^*)) \le 1.$$

In the case where $\psi_\varepsilon'(g(x^*)) \ge 1$, from (3.64), we get that

$$0 \ge \psi_\varepsilon'(g(x^*))(\theta\xi(x^*)|| \mathrm{D}^2 u^\varepsilon(x^*)||^2 - K_{19}|| \mathrm{D}^2 u^\varepsilon(x^*)|| - K_{20}),$$

where

$$\begin{aligned} K_{19} &:= 2K_{11}||\widetilde{\gamma}|| + K_{17} + K_{14}K_{16}, \\ K_{20} &:= 2K_{18} + dK_{11}K_{13} + K_{11}K_{15}K_{16} + K_{15}K_{16}, \end{aligned}$$

are constants that only depend on ξ. Since $\psi_\varepsilon'(x^*) \ge 0$, this implies that

$$0 \ge \theta\xi(x^*)|| \mathrm{D}^2 u^\varepsilon(x^*)||^2 - K_{19}|| \mathrm{D}^2 u^\varepsilon(x^*)|| - K_{20},$$

and hence

$$\begin{aligned} 0 \ge & \left(|| \mathrm{D}^2 u^\varepsilon(x^*)|| - \frac{K_{19} + (K_{19}^2 + 4\theta\xi(x^*)K_{20})^{\frac{1}{2}}}{2\theta\xi(x^*)}\right) \\ & \times \left(|| \mathrm{D}^2 u^\varepsilon(x^*)|| - \frac{K_{19} - (K_{19}^2 + 4\theta\xi(x^*)K_{20})^{\frac{1}{2}}}{2\theta\xi(x^*)}\right). \end{aligned}$$

Since $K_{19} - (K_{19}^2 + 4\theta\xi(x^*)K_{20})^{\frac{1}{2}} \le 0$, it follows

$$|| \mathrm{D}^2 u^\varepsilon(x^*)|| \le \frac{K_{19} + (K_{19}^2 + 4\theta\xi(x^*)K_{20})^{\frac{1}{2}}}{2\theta\xi(x^*)}.$$

Therefore, from (3.57), we conclude that

$$\begin{aligned}\phi(x) &\leq \phi(x^*)\\ &= \xi(x^*)\psi_\varepsilon(g(x^*))\\ &\leq \xi(x^*)(K_{14}||\,\mathrm{D}^2\,u^\varepsilon(x^*)|| + K_{15})\\ &\leq \xi(x^*)\bigg(K_{14}\frac{K_{19}+(K_{19}^2+4\theta\xi(x^*)K_{20})^{\frac{1}{2}}}{2\theta\xi(x^*)}+K_{15}\bigg)\\ &\leq \frac{K_{14}(K_{19}+(\theta K_{20})^{\frac{1}{2}})}{\theta}+K_{15}.\end{aligned}$$

We finish the proof. ■

Lemma 3.24. *Let* $1 \leq p < \infty$ *and* $\beta \in (0,1)$ *such that* $B_{\beta' r} \subset B_R(0)$*, with* $\beta' = \frac{\beta+1}{2}$*. There exists a constant* $K_{23} = K_{23}(\beta r, p) > 0$ *independent of* ε *such that*

$$\begin{aligned}||\,\mathrm{D}^2\,u^\varepsilon||_{\mathrm{L}^p(B_{\beta r})} \leq K_{23}\big(&||h||_{\mathrm{L}^p(B_{\beta' r})} + ||\,\mathcal{I}\,E(u^\varepsilon)||_{\mathrm{L}^p(B_{\beta' r})}\\ &+ ||\xi\psi_\varepsilon(||\,\mathrm{D}^1\,u^\varepsilon||)||_{\mathrm{L}^p(B_{\beta' r})} + ||\,\mathrm{D}^1\,u^\varepsilon||_{\mathrm{L}^p(B_{\beta' r})} + ||u^\varepsilon||_{\mathrm{L}^p(B_{\beta' r})}\big).\end{aligned} \tag{3.65}$$

Proof. Let $r > 0$, $\beta \in (0,1)$ and $\xi \in \mathrm{C}_{\mathrm{c}}^\infty(B_r)$ a cutoff function such that $0 \leq \xi \leq 1$, $\xi = 1$ on $B_{\beta r}$ and $\xi = 0$ on $B_{\beta' r}^{\mathrm{c}}$, with $\beta' = \frac{\beta+1}{2}$. Suppose that $||\,\mathrm{D}^1\,\xi|| \leq K_{21}$ and $||\,\mathrm{D}^2\,\xi|| \leq K_{21}$, for some constant $K_{21} > 0$. Defining $w = \xi u^\varepsilon$, we obtain

$$||\,\mathrm{D}^2\,u^\varepsilon||_{\mathrm{L}^p(B_{\beta r})} \leq ||\,\mathrm{D}^2\,u^\varepsilon||_{\mathrm{L}^p(B_{\beta r})} + ||\,\mathrm{D}^2\,\xi u^\varepsilon||_{\mathrm{L}^p(B_{\beta' r}\setminus B_{\beta r})} = ||\,\mathrm{D}^2\,w||_{\mathrm{L}^p(B_{\beta' r})}. \tag{3.66}$$

Calculating first and second derivatives of w in $B_{\beta' r}$,

$$\begin{aligned}\partial_i w(x) &= u^\varepsilon(x)\partial_i\xi(x) + \partial_i u^\varepsilon(x)\xi(x),\\ \partial_{ji}^2 w(x) &= \partial_j u^\varepsilon(x)\partial_i\xi(x) + u^\varepsilon(x)\partial_{ji}^2\xi(x) + \partial_i u^\varepsilon(x)\partial_j\xi(x) + \xi(x)\partial_{ji}^2 u^\varepsilon(x),\end{aligned}$$

with $j, i \in \{1,\ldots,d\}$, by (3.21), we get

$$\begin{cases} q' w(x) - \mathcal{L}' w(x) = f(x), & \text{in } B_{\beta' r},\\ w(x) = 0, & \text{on } \partial B_{\beta' r},\end{cases} \tag{3.67}$$

where

$$\begin{aligned}f(x) := \xi(x)(h(x) + \int_{\mathbb{R}^*} E(u^\varepsilon)(x+z)\nu(\mathrm{d}z) - \psi_\varepsilon(||\,\mathrm{D}^1\,u^\varepsilon(x)||^2))\\ - u^\varepsilon(x)\bigg(\frac{1}{2}\sum_{ji}\sigma_{ji}\partial_{ji}^2\xi(x) + \langle \mathrm{D}^1\,\xi(x), \widetilde{\gamma}\rangle\bigg) - \langle \sigma\,\mathrm{D}^1\,\xi(x), \mathrm{D}^1\,u^\varepsilon(x)\rangle.\end{aligned} \tag{3.68}$$

We know that for the linear Dirichlet problem (3.67) (see [23, Lemma 3.1]), there exists a constant $K_{22} = K_{22}(\beta r, p) > 0$ independent of w, such that

$$|| \mathrm{D}^2 w||_{\mathrm{L}^p(B_{\beta' r})} \leq K_{22}||f||_{\mathrm{L}^p(B_{\beta' r})}.$$

Estimating the terms on the right hand side of (3.68) with the norm $|| \cdot ||_{\mathrm{L}^p(B_{\beta' r})}$ and by the choice of ξ, it follows

$$\begin{aligned} || \mathrm{D}^2 w||_{\mathrm{L}^p(B_{\beta' r})} \leq K_{23}\big(||h||_{\mathrm{L}^p(B_{\beta' r})} + || \mathcal{I}\, E(u^\varepsilon)||_{\mathrm{L}^p(B_{\beta' r})} \\ + ||\xi\psi_\varepsilon(|| \mathrm{D}^1 u^\varepsilon||^2)||_{\mathrm{L}^p(B_{\beta' r})} + || \mathrm{D}^1 u^\varepsilon||_{\mathrm{L}^p(B_{\beta' r})} + ||u^\varepsilon||_{\mathrm{L}^p(B_{\beta' r})}\big), \end{aligned} \tag{3.69}$$

for some constant $K_{23} = K_{23}(\beta r, p) > 0$ independent of ε. Hence, from (3.66) and (3.69), we have the inequality (3.65). ■

By (0.3) and Lemmas 3.24 it is easy to obtain the following result.

Lemma 3.25. *Let $1 \leq p < \infty$ and $\beta \in (0, 1)$ such that $B_{\beta' r} \subset B_R(0)$, with $\beta' = \frac{\beta+1}{2}$. There exists a constant $K_{24} > 0$ independent of ε such that*

$$\begin{aligned} ||u^\varepsilon||_{W^{2,p}(B_{\beta r})} \leq K_{24}\big(||h||_{\mathrm{L}^p(B_{\beta' r})} + || \mathcal{I}\, E(u^\varepsilon)||_{\mathrm{L}^p(B_{\beta' r})} \\ + ||\xi\psi_\varepsilon(|| \mathrm{D}^1 u^\varepsilon||)||_{\mathrm{L}^p(B_{\beta' r})} + || \mathrm{D}^1 u^\varepsilon||_{\mathrm{L}^p(B_{\beta' r})} + ||u^\varepsilon||_{\mathrm{L}^p(B_{\beta' r})}\big), \end{aligned}$$

with $\beta' = \dfrac{\beta + 1}{2}$.

Chapter 4

Existence, uniqueness and regularity to the HJB equation

As in Chapter 3, the hypotheses (H1)–(H4) are in force all through this chapter. Our main purpose here is to establish Theorem 1.2, that we recall below for case of reference. A strong solution of the equation

$$\max\{(q'u(x) - \Gamma' u(x) - h(x), ||\,\mathrm{D}^1 u(x)||^2 - 1\} = 0, \text{ in } B_R(0), \tag{4.1}$$

with

$$\begin{cases} q' = q + \nu(\mathbb{R}^*) = q + \nu_0, \\ \Gamma' u(x) = \frac{1}{2}\operatorname{tr}(\sigma\,\mathrm{D}^2 u(x)) + \langle \mathrm{D}^1 u(x), \widetilde{\gamma}\rangle + \int_{\mathbb{R}^*} E(u)(x+z)\nu(\mathrm{d}z) \\ \qquad\qquad = \mathcal{L}' u(x) + \mathcal{I}\, E(u)(x). \end{cases} \tag{4.2}$$

is a twice weakly differentiable function on $B_R(0)$ satisfying (4.1) almost everywhere in $B_R(0)$. With this at hand we can now recall the statement of Theorem 1.2.

Therem 1.2. *If $d < p < \infty$, there exists a unique nonnegative strong solution $u \in \mathrm{C}^{0,1}(\overline{B_R(0)}) \cap \mathrm{W}^{2,p}_{\mathrm{loc}}(B_R(0))$ to the HJB equation*

$$\begin{cases} \max\{(q'u(x) - \Gamma' u(x) - h(x), ||\,\mathrm{D}^1 u(x)||^2 - 1\} = 0, & \textit{a.e. in } B_R(0), \\ u(x) = 0, & \textit{on } \partial B_R(0), \end{cases} \tag{4.3}$$

Note that by the definition of q' and Γ', the HJB equation (4.3) can be written as

$$\begin{cases} \max\{(qu(x) - \mathcal{L}u(x) - \mathcal{I}'\, E(u)(x) - h(x), ||\,\mathrm{D}^1 u(x)||^2 - 1\} = 0, & \text{a.e. in } B_R(0), \\ u(x) = 0, & \text{on } \partial B_R(0), \end{cases} \tag{4.4}$$

where $q > 0$ is as in Hypothesis (H4), and

$$\begin{cases} \mathcal{L}u(x) = \dfrac{1}{2}\operatorname{tr}(\sigma \operatorname{D}^2 u(x)) + \langle \operatorname{D}^1 u(x), \gamma \rangle, \\ \mathcal{I}' u(x) = \int_{\mathbb{R}^*} (E(u)(x+z) - u(x) - \operatorname{D}^1 u^{\varepsilon}(x+z))\nu(\mathrm{d}z). \end{cases}$$

In order to show Theorem 1.2, first we shall prove the existence of the solution to HJB equation (4.3). Finally, we shall prove the uniqueness of the solution to the HJB equation (4.3). To verify this last part, we use Bony's maximum principle [25], which quoted next.

Theorem 4.1 (Bony's maximum principle, [25]). *Let Ω be an open set in $\mathbb{R}^d$ and let $u \in \mathrm{W}^{2,p}_{\mathrm{loc}}(\Omega)$. Then, if x_0 is a point of local maximum of u and $p > d$, we have*

$$\liminf_{y \to x_0} \operatorname{ess} \sum_{ij} a_{ij}(y) \partial_{ij} u(y) \leq 0,$$

where $(a_{ij})_{\mathbb{R}^d \times \mathbb{R}^d}$ is a positive definite matrix a.e. and $a_{ij} \in \mathrm{L}^{\infty}_{\mathrm{loc}}(\Omega)$.

Before proceeding to prove Theorem 1.2, we state without proof some auxiliary results. They shall help us to obtain the existence of a convergent subsequence of u^{ε}, whose limit is the solution to the HJB equation (4.3); see Lemma 4.6. The first can be seen in [29], and the last two in [1].

Recall that $\{f_n\}_{n \geq 1}$ is *uniformly bounded*, if there exists a positive constant M such that $\sup_{n \geq 1} \sup_{x \in K} |f(x)| \leq M$, and $\{f_n\}_{n \geq 1}$ is *equicontinuous*, if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$, for all $n \geq 1$ whenever $||x - y|| < \delta$.

Theorem 4.2 (Arzelà-Ascoli Theorem, [29], Thm. 7.25, p. 158). *Let $K \subset \mathbb{R}^d$ be compact and let $\{f_n\}_{n \geq 1} \subset \mathrm{C}^0(K)$. If $\{f_n\}_{n \geq 1}$ is uniformly bounded and equicontinuous, then there exists a subsequence $\{f_{n_\kappa}\}_{\kappa \geq 1}$ of $\{f_n\}_{n \geq 1}$ which converges uniformly.*

Theorem 4.3 (Reflexivity of $\mathrm{L}^p(B_r)$, [1], Thm. 2.46, p. 49). *The Banach space $(\mathrm{L}^p(B_r), ||\cdot||_{\mathrm{L}^p(B_r)})$ is reflexive if and only if $1 < p < \infty$. Then, for any bounded sequence in $(\mathrm{L}^p(B_r), ||\cdot||_{\mathrm{L}^p(B_r)})$ has a weakly convergent subsequence, i.e., let $\{f_n\}_{n \geq 1}$ be a bounded sequence $(\mathrm{L}^p(B_r), ||\cdot||_{\mathrm{L}^p(B_r)})$. Then, there exist a subsequence $\{f_{n_\kappa}\}_{\kappa \geq 1}$ of $\{f_n\}_{n \geq 1}$ and $f \in \mathrm{L}^p(B_r)$ such that*

$$\int_{B_r} f_{n_\kappa} \phi \mathrm{d}x \longrightarrow \int_{B_r} f \phi \mathrm{d}x, \text{ for any } \phi \in \mathrm{L}^{p'}(B_r),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 4.4 (Sobolev embedding theorem, [1], Thm. 4.12, p. 85)**.** *If $p > d$, then*

$$\mathrm{W}^{2,p}(B_r) \hookrightarrow \mathrm{C}^{1,\alpha'}(\overline{B_r}), \text{ with } \alpha' = 1 - \frac{d}{p}.$$

Moreover, there exists a positive constant $C = C(d,p,r)$ such that

$$||v||_{\mathrm{C}^{1,\beta}(\overline{B_r})} \leq C||v||_{\mathrm{W}^{2,p}(B_r)}, \text{ for all } v \in \mathrm{W}^{2,p}(B_r).$$

Having stated the auxiliary results, we can now proceed to prove Theorem 1.2, this proof will be carried on in several lemmas. For that end we start by noticing that by Lemmas 3.16 and 3.21, we obtain that there exists a constant $K_{25} > 0$ independent of ε, such that

$$||u^{\varepsilon}||_{\mathrm{C}^{0,1}(\overline{B_R(0)})} < K_{25}, \text{ for all } \varepsilon \in (0,1), \tag{4.5}$$

Moreover, Proposition 2.15 and Lemmas 3.16, 3.21, 3.23–3.25, guarantee that for each $B_r \subset B_R(0)$ there exist positive constants K_{26}, K_{27} independent of ε such that

$$\begin{cases} ||\,\mathrm{D}^2\, u^{\varepsilon}||_{\mathrm{L}^p(B_{\beta r})} \leq K_{26}, \\ ||u^{\varepsilon}||_{\mathrm{W}^{2,p}(B_{\beta r})} < K_{27}, \end{cases} \tag{4.6}$$

for all $\varepsilon \in (0,1)$, where $\beta \in (0,1)$ and $1 \leq p < \infty$ fixed. Finally, if we take $d < p < \infty$ in (4.6), then, from Theorem 4.4, we have that for each $B_r \subset B_R(0)$, there exists a positive constant K_{28} independent of ε such that

$$||u^{\varepsilon}||_{\mathrm{C}^{1,\alpha'}(\overline{B_{\beta r}})} \leq K_{28}, \text{ for all } \varepsilon \in (0,1), \tag{4.7}$$

with $\beta \in (0,1)$ fixed and $\alpha' = 1 - \frac{d}{p}$.

Recall that for each $\varepsilon \in (0,1)$, u^{ε} is the unique solution to the non-linear integro-differential Dirichlet problem (3.21). As a consequence of Theorems 4.2–4.3 and (4.5)–(4.7), we obtain the following key results.

Lemma 4.5. *Let $d < p < \infty$, $B_r \subset B_R(0)$ an open ball and $\beta \in (0,1)$ fixed. There exist a decreasing subsequence $\{\varepsilon_{\kappa(\iota)}\}_{\iota \geq 1}$, with $\varepsilon_{\kappa(\iota)} \underset{\iota \to \infty}{\longrightarrow} 0$, and $u_r \in \mathrm{C}^{0,1}(\overline{B_R(0)}) \cap \mathrm{W}^{2,p}(B_{\beta r})$ such that*

$$\begin{cases} u^{\varepsilon_{\kappa(\iota)}} \underset{\varepsilon_{\kappa(\iota)} \to 0}{\longrightarrow} u_r, & \text{in } \mathrm{C}^1(B_{\beta r}), \\ u^{\varepsilon_{\kappa(\iota)}} \underset{\varepsilon_{\kappa(\iota)} \to 0}{\longrightarrow} u_r, & \text{uniformly in } \overline{B_R(0)}, \\ u^{\varepsilon_{\kappa(\iota)}} \underset{\varepsilon_{\kappa(\iota)} \to 0}{\longrightarrow} u_r, & \text{weakly in } \mathrm{W}^{2,p}(B_{\beta r}). \end{cases} \tag{4.8}$$

Moreover, the following convergence also holds

$$\int_{\mathbb{R}^*} E(u^{\varepsilon_{\kappa(\iota)}})(\cdot + z)\nu(\mathrm{d}z) \underset{\varepsilon_{\kappa(\iota)}\to 0}{\longrightarrow} \int_{\mathbb{R}^*} E(u_r)(\cdot + z)\nu(\mathrm{d}z), \textit{ uniformly in } B_R(0). \tag{4.9}$$

Proof. Let $d < p < \infty$, $B_r \subseteq B_R(0)$ an open ball and $\beta \in (0,1)$ fixed. Since the sequence $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ satisfies (4.6), i.e. $||u^\varepsilon||_{\mathrm{W}^{2,p}(B_{\beta r})} \le K_{27}$, by Theorem 4.3, there exist a decreasing subsequence $\{\varepsilon_\kappa\}_{\kappa\ge 1}$, with $\varepsilon_\kappa \underset{\kappa\to\infty}{\longrightarrow} 0$, and $f_a \in \mathrm{L}^p(B_{\beta r})$, with $a \in \mathcal{D}_m$ and $m \in \{0,1,2\}$, such that

$$\int_{B_{\beta r}} \partial^a u^{\varepsilon_\kappa} \phi \mathrm{d}x \underset{\varepsilon_\kappa\to 0}{\longrightarrow} \int_{B_{\beta r}} f_a \phi \mathrm{d}x, \text{ for any } \phi \in \mathrm{L}^{p'}(B_{\beta r}), \tag{4.10}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Recall that $\mathcal{D}_m$ is the set of all multi-indices of order m. Observing that the sequence $\{u^{\varepsilon_\kappa}\}_{\kappa\ge 0}$ satisfies (4.7), i.e., $||u^{\varepsilon_\kappa}||_{\mathrm{C}^{1,\alpha'}(\overline{B_{\beta r}})} \le K_{28}$, for all $\kappa \ge 0$, we obtain

$$|\partial^a u^{\varepsilon_\kappa}(x)| \le ||u^{\varepsilon_\kappa}||_{\mathrm{C}^{1,\alpha'}(\overline{B_{\beta r}})} \le K_{28}, \text{ for all } \kappa \ge 1,\ a \in \mathcal{D}_m,\ m \in \{0,1\}, \text{ and } x \in \overline{B_R(0)}. \tag{4.11}$$

Taking $\epsilon > 0$ and $\rho \le (\frac{\epsilon}{K_{28}})^{1/\alpha'}$, it follows that if $||x-y|| \le \rho$, with $x, y \in \overline{B_{\beta r}}$, then

$$|\partial^a u^{\varepsilon_\kappa}(x) - \partial^a u^{\varepsilon_\kappa}(y)| \le ||u^{\varepsilon_\kappa}||_{\mathrm{C}^{1,\alpha'}(\overline{B_{\beta r}})} ||x-y||^{\alpha'} \le \epsilon, \tag{4.12}$$

for all $\kappa \ge 1$, $a \in \mathcal{D}_m$, $m \in \{0,1\}$. From (4.11) and (4.12), $\{\partial^a u^{\varepsilon_\kappa}\}_{n\ge 0}$ is uniformly bounded and equicontinuous, for all $a \in \mathcal{D}_m$, with $m \in \{0,1\}$, and hence Arzelà-Ascoli Theorem, Theorem 4.2, implies that for each $a \in \mathcal{D}_m$, with $m \in \{0,1\}$, there exists a subsequence $\{\partial^a u^{\varepsilon_{\kappa(\iota)}}\}_{\kappa(\iota)\ge 1} \subseteq \{\partial^a u^{\varepsilon_\kappa}\}_{\kappa\ge 1}$ such that

$$\partial^a u^{\varepsilon_{\kappa(\iota)}} \underset{\varepsilon^{\kappa(\iota)}\to 0}{\longrightarrow} f_a^{(1)}, \text{ uniformly in } B_{\beta r},$$

where $f_a^{(1)}$ is a continuous function in $\overline{B_{\beta r}}$; see, for instance, [29, Thorem 7.12]. We need to prove that $f_0^{(1)}$ is differentiable and $\partial^1 f_0^{(1)} = f_1^{(1)}$ in $\overline{B_{\beta r}}$. Define $\phi^i_{\kappa(\iota)}, \phi^i : \overline{B_{\beta r}} \times (0,1) \longrightarrow \mathbb{R}$ as

$$\phi^i_{\kappa(\iota)}(x,h) = \frac{u^{\varepsilon_{\kappa(\iota)}}(x+he_i) - u^{\varepsilon_{\kappa(\iota)}}(x)}{h} \text{ and } \phi^i(x,h) = \frac{f_0^{(1)}(x+he_i) - f_0^{(1)}(x)}{h},$$

with $i \in \{1, \ldots, d\}$. Hence,

$$\partial_i u^{\varepsilon_{\kappa(\iota)}}(x) = \lim_{h\to 0} \phi^i_{\kappa(\iota)}(x,h). \tag{4.13}$$

Since $u^{\varepsilon_{\kappa(\iota)}} \underset{\kappa(\iota)\to\infty}{\longrightarrow} f_0^{(1)}$ uniformly in $\overline{B_{\beta r}}$, then

$$\begin{aligned}
|\phi^i_{\kappa(\iota)}(x,h) - \phi^i(x,h)| &= \left| \frac{u^{\varepsilon_{\kappa(\iota)}}(x+he_i) - u^{\varepsilon_{\kappa(\iota)}}(x) - (f_0^{(1)}(x+he_i) - f_0^{(1)}(x))}{h} \right| \\
&\le \left| \frac{u^{\varepsilon_{\kappa(\iota)}}(x+he_i) - f_0^{(1)}(x+he_i)}{h} \right| + \left| \frac{u^{\varepsilon_{\kappa(\iota)}}(x) - f_0^{(1)}(x)}{h} \right| \\
&\underset{\varepsilon^{\kappa(\iota)}\to 0}{\longrightarrow} 0,
\end{aligned}$$

for each $x \in \overline{B_{\beta r}}$ and $h \in (0,1)$ such that $x + he_i \in B_R(0)$. This implies that

$$\phi^i_{\kappa(\iota)} \underset{\kappa(\iota)\to\infty}{\longrightarrow} \phi^i, \text{ uniformly, with } i \in \{1, \ldots, d\}. \tag{4.14}$$

Then, (4.13), (4.14) and Theeorem 7.11 in [29], imply that

$$\partial_i f_0^{(1)}(x) = \lim_{h\to 0} \phi^i(x,h) = \lim_{\varepsilon^{\kappa(\iota)}\to 0} \partial_i u^{\varepsilon_{\kappa(\iota)}}(x) = f_i^{(1)}, \text{ for all } x \in \overline{B_{\beta r}}.$$

Therefore $\partial^1 f_0^{(1)} = f_1^{(1)}$ in $\overline{B_{\beta r}}$. Now, the sequence $\{u^{\varepsilon_{\kappa(\iota)}}\}$ satisfies (4.5), i.e., $||u^{\varepsilon_{\kappa(\iota)}}||_{\mathrm{C}^{0,1}(\overline{B_R(0)})} \leq K_{25}$, for all $\kappa(\iota) \geq 1$. Then, of a similar way than (4.11) and (4.12), we can verify that $\{u^{\varepsilon_{\kappa(\iota)}}\}$ is uniformly bounded and equicontinuous in $\overline{B_R(0)}$. By Arzelà-Ascoli Theorem, we obtain that there exist a subsequence $\{u^{\varepsilon_{\kappa_1(\iota)}}\}_{\kappa_1(\iota)\geq 1}$ of $\{u^{\varepsilon_{\kappa(\iota)}}\}$ and $u_r \in \mathrm{C}^{0,1}(\overline{B_R(0)})$ such that

$$u^{\varepsilon_{\kappa_1(\iota)}} \underset{\varepsilon_{\kappa_1(\iota)}\to 0}{\longrightarrow} u_r, \text{ uniformly in } \overline{B_R(0)}.$$

Since $\{u^{\varepsilon_{\kappa_1(\iota)}}\}_{\kappa_1(\iota)\geq 1} \subset \{u^{\varepsilon_{\kappa(\iota)}}\}_{\kappa(\iota)\geq 1} \subset \{u^{\varepsilon_\kappa}\}_{\kappa\geq 1}$, we have that

$$f_0 = f_0^{(1)} = u_r \text{ and } f_1 = f_1^{(1)} = \partial^1 u_r, \text{ a.e. in } \overline{B_{\beta r}}. \tag{4.15}$$

Finally, from (4.10), we have

$$\int_{B_{\beta r}} \partial^2 u^{\varepsilon_{\kappa_1(\iota)}} \phi \mathrm{d}x \underset{\varepsilon_{\kappa_2(\iota)}\to 0}{\longrightarrow} \int_{B_{\beta r}} f_2 \phi \mathrm{d}x, \text{ for any } \phi \in \mathrm{L}^{p'}(B_{\beta r}), \tag{4.16}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Furthermore, by integration by parts, we get

$$\int_{B_{\beta r}} \partial^2 u^{\varepsilon_{\kappa_1(\iota)}} \phi \mathrm{d}x = \int_{B_{\beta r}} u^{\varepsilon_{\kappa_1(\iota)}} \partial^2 \phi \mathrm{d}x, \text{ for any } \phi \in \mathrm{C}^2_{\mathrm{c}}(B_{\beta r}). \tag{4.17}$$

Then, using (4.15), (4.16) and letting $\varepsilon_{\kappa_1(\iota)} \to 0$ in (4.17), we have

$$\int_{B_{\beta r}} f_2 \phi \mathrm{d}x = \int_{B_{\beta r}} u_r \partial^2 \phi \mathrm{d}x, \text{ for any } \phi \in \mathrm{C}^2_{\mathrm{c}}(B_{\beta r}),$$

which is the definition of weakly derivative of second order to u_r, and hence $\partial_{ij} u^{\varepsilon_{\kappa_1(\iota)}} \underset{\varepsilon_{\kappa_1(\iota)}\to 0}{\longrightarrow} \partial_{ij} u_r$ weakly in $\mathrm{L}^p(B_{\beta r})$, where $\partial_{ij} u_r$ represents the second weakly derivative of u, with $i, j \in \{1, \ldots, d\}$. Therefore

$$\begin{cases} u^{\varepsilon_{\kappa(\iota)}} \underset{\varepsilon_{\kappa(\iota)}\to 0}{\longrightarrow} u_r, & \text{in } \mathrm{C}^1(B_{\beta r}), \\ u^{\varepsilon_{\kappa(\iota)}} \underset{\varepsilon_{\kappa(\iota)}\to 0}{\longrightarrow} u_r, & \text{uniformly in } \overline{B_R(0)}, \\ u^{\varepsilon_{\kappa(\iota)}} \underset{\varepsilon_{\kappa(\iota)}\to 0}{\longrightarrow} u_r, & \text{weakly in } \mathrm{W}^{2,p}(B_{\beta r}). \end{cases}$$

Note that for each $x \in B_R(0)$, by Proposition 2.15, we have

$$\begin{aligned} |\mathcal{I}\,E(u^{\varepsilon_{\kappa_1(\iota)}})(x) - \mathcal{I}\,E(u_r)(x)| &\leq \int_{\mathbb{R}^*} |E(u^{\varepsilon_{\kappa_1(\iota)}} - u_r)(x+z)|\nu(\mathrm{d}z) \\ &\leq 2A_0\nu_0 ||u^{\varepsilon_{\kappa_1(\iota)}} - u_r||_{\mathrm{C}^0(\overline{B_R(0)})} \\ &\underset{\varepsilon_{\kappa(\iota)}\to 0}{\longrightarrow} 0, \end{aligned} \tag{4.18}$$

and hence $\mathcal{I}\,E(u^{\varepsilon_{\kappa(\iota)}}) \underset{\varepsilon_{\kappa_1(\iota)}\to 0}{\longrightarrow} \mathcal{I}\,E(u_r)$, uniformly in $B_R(0)$. We conclude that there exist a decreasing subsequence $\{\varepsilon_{\kappa(\iota)}\}_{\iota\geq 1}$, with $\varepsilon_{\kappa(\iota)} \underset{\iota\to\infty}{\longrightarrow} 0$, and $u_r \in \mathrm{C}^{0,1}(\overline{B_R(0)}) \cap \mathrm{W}^{2,p}(B_{\beta r})$ satisfying (4.8) and (4.9) ∎

Lemma 4.6. *Let $d < p < \infty$. There exists a decreasing subsequence $\{\varepsilon_{\kappa(\iota)}\}_{\iota\geq 1}$, with $\varepsilon_{\kappa(\iota)} \to 0$, and $u \in \mathrm{C}^{0,1}(\overline{B_R(0)}) \cap \mathrm{W}^{2,p}_{\mathrm{loc}}(B_R(0))$, such that*

$$\begin{cases} u^{\varepsilon_{\kappa(\iota)}} \underset{\varepsilon_{\kappa(\iota)}\to 0}{\longrightarrow} u, & \textit{in } \mathrm{C}^1_{\mathrm{loc}}(B_R(0)), \\ u^{\varepsilon_{\kappa(\iota)}} \underset{\varepsilon_{\kappa(\iota)}\to 0}{\longrightarrow} u, & \textit{uniformly in } \overline{B_R(0)}, \\ u^{\varepsilon_{\kappa(\iota)}} \underset{\varepsilon_{\kappa(\iota)}\to 0}{\longrightarrow} u, & \textit{weakly in } \mathrm{W}^{2,p}_{\mathrm{loc}}(B_R(0)). \end{cases} \tag{4.19}$$

Moreover, the following convergence also holds

$$\int_{\mathbb{R}^*} E(u^{\varepsilon_{\kappa(\iota)}})(\cdot + z)\nu(\mathrm{d}z) \underset{\varepsilon_{\kappa(\iota)}\to 0}{\longrightarrow} \int_{\mathbb{R}^*} E(u)(\cdot + z)\nu(\mathrm{d}z), \textit{ uniformly in } B_R(0). \tag{4.20}$$

Proof. Let $d < p < \infty$, $\beta \in (0,1)$ and $\{r_n\}_{n\geq 1}$ an increasing sequence of $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$ such that $\beta r_n \uparrow R$, when $n \to \infty$. This implies that $B_{\beta r_n}(0) \uparrow B_R(0)$, when $n \to \infty$. Taking $n = 1$, Lemma 4.5 implies that there exist a decreasing subsequence $\{\varepsilon_{\kappa_1(\iota)}\}$, with $\varepsilon_{\kappa_1(\iota)} \underset{\iota\to\infty}{\longrightarrow} 0$, and $u_{r_1} \in \mathrm{C}^{0,1}(\overline{B_R(0)}) \cap \mathrm{W}^{2,p}(B_{\beta r_1}(0))$ such that

$$\begin{cases} u^{\varepsilon_{\kappa_1(\iota)}} \underset{\varepsilon_{\kappa_1(\iota)}\to 0}{\longrightarrow} u_{r_1}, & \text{in } \mathrm{C}^1(B_{\beta r_1}(0)), \\ u^{\varepsilon_{\kappa_1(\iota)}} \underset{\varepsilon_{\kappa_1(\iota)}\to 0}{\longrightarrow} u_{r_1}, & \text{uniformly in } \overline{B_R(0)}, \\ u^{\varepsilon_{\kappa_1(\iota)}} \underset{\varepsilon_{\kappa_1(\iota)}\to 0}{\longrightarrow} u_{r_1}, & \text{weakly in } \mathrm{W}^{2,p}(B_{\beta r_1}(0)). \end{cases}$$

and

$$\mathcal{I}\,E(u^{\varepsilon_{\kappa_1(\iota)}})(\cdot + z)\nu(\mathrm{d}z) \underset{\varepsilon_{\kappa_1(\iota)}\to 0}{\longrightarrow} \mathcal{I}\,E(u_{r_1})(\cdot + z)\nu(\mathrm{d}z), \text{ uniformly in } B_R(0).$$

Now, taking $n=2$ and using Lemma 4.5 over $\{u^{\varepsilon_{\kappa_1(\iota)}}\}_{\iota>1}$, we extract a subsequence $\{\varepsilon_{\kappa_2(\iota)}\}_{\iota\geq 1}$ of $\{\varepsilon_{\kappa_1(\iota)}\}_{\iota\geq 1}$ such that

$$\begin{cases} u^{\varepsilon_{\kappa_2(\iota)}} \underset{\varepsilon_{\kappa_2(\iota)}\to 0}{\longrightarrow} u_{r_2}, & \text{in } \mathrm{C}^1(B_{\beta r_2}(0)), \\ u^{\varepsilon_{\kappa_2(\iota)}} \underset{\varepsilon_{\kappa_2(\iota)}\to 0}{\longrightarrow} u_{r_2}, & \text{uniformly in } \overline{B_R(0)}, \\ u^{\varepsilon_{\kappa_2(\iota)}} \underset{\varepsilon_{\kappa_2(\iota)}\to 0}{\longrightarrow} u_{r_2}, & \text{weakly in } \mathrm{W}^{2,p}(B_{\beta r_2}(0)). \end{cases}$$

and

$$\mathcal{I}\, E(u^{\varepsilon_{\kappa_2(\iota)}})(\cdot+z)\nu(\mathrm{d}z) \underset{\varepsilon_{\kappa_2(\iota)}\to 0}{\longrightarrow} \mathcal{I}\, E(u_{r_2})(\cdot+z)\nu(\mathrm{d}z), \text{ uniformly in } B_R(0).$$

where $u_{r_2} \in \mathrm{C}^{0,1}(\overline{B_R(0)}) \cap \mathrm{W}^{2,p}(B_{\beta r_2}(0))$. Continuing this process, it gives that there exists a subsequence $\{\varepsilon_{\kappa_n(\iota)}\}$ of $\{\varepsilon_{\kappa_{n-1}(\iota)}\}$ such that

$$\begin{cases} u^{\varepsilon_{\kappa_n(\iota)}} \underset{\varepsilon_{\kappa_n(\iota)}\to 0}{\longrightarrow} u_{r_n}, & \text{in } \mathrm{C}^1(B_{\beta r_n}(0)), \\ u^{\varepsilon_{\kappa_n(\iota)}} \underset{\varepsilon_{\kappa_n(\iota)}\to 0}{\longrightarrow} u_{r_n}, & \text{uniformly in } \overline{B_R(0)}, \\ u^{\varepsilon_{\kappa_n(\iota)}} \underset{\varepsilon_{\kappa_n(\iota)}\to 0}{\longrightarrow} u_{r_n}, & \text{weakly in } \mathrm{W}^{2,p}(B_{\beta r_n}(0)). \end{cases}$$

and

$$\mathcal{I}\, E(u^{\varepsilon_{\kappa_n(\iota)}}) \underset{\varepsilon_{\kappa_n(\iota)}\to 0}{\longrightarrow} \mathcal{I}\, E(u_{r_n}), \text{ uniformly in } B_R(0).$$

where $u_{r_n} \in \mathrm{C}^{0,1}(\overline{B_R(0)}) \cap \mathrm{W}^{2,p}(B_{\beta r_n}(0))$. Since $\{u^{\varepsilon_{\kappa_n(\iota)}}\}_{\iota>0}$ is a subsequence of $\{u^{\varepsilon_{\kappa_{n-1}(\iota)},R}\}_{\iota>0}$, it follows that

$$u_{r_n} = u_{r_{n-1}}, \text{ in } \overline{B_{r_{n-1}}(0)} \subseteq \overline{B_{r_n}(0)}.$$

Now, taking $u^{\varepsilon_{\kappa_n(n)}}$ of $\{u^{\varepsilon_{\kappa_n(\iota)}}\}_{\iota>0}$, for each $n\geq 1$, the sequence $\{u^{\varepsilon_{\kappa_n(n)}}\}_{n\geq 1}$ satisfies

$$\lim_{\varepsilon_{\kappa_n(n)}\to 0} u^{\varepsilon_{\kappa_n(n)}} = u_{r_{n'}}, \text{ in } B_{\beta r_{n'}(0)}, \text{ for each } n'\geq 1.$$

Defining

$$u := \lim_{\varepsilon_{\kappa_n(n)}\to 0} u^{\varepsilon_{\kappa_n(n)}},$$

we observe that $u \in \mathrm{C}^{0,1}(\overline{B_R(0)}) \cap \mathrm{W}^{2,p}_{\mathrm{loc}}(B_R(0))$ and for any compact set $K \subset B_R(0)$, there exists $r_{n'} \in \{r_n\}_{n\geq 1}$ such that $K \subset B_{r_{n'}}(0)$ and

$$u = u_{r_{n'}}, \text{ in } K.$$

Hence, the sequence $\{u^{\varepsilon_{\kappa_n(n)}}\}$ satisfies

$$\begin{cases} u^{\varepsilon_{\kappa_n(n)}} \underset{\varepsilon_{\kappa_n(n)}\to 0}{\longrightarrow} u, & \text{in } \mathrm{C}^1_{\mathrm{loc}}(B_R(0)), \\ u^{\varepsilon_{\kappa_n(n)}} \underset{\varepsilon_{\kappa_n(n)}\to 0}{\longrightarrow} u, & \text{uniformly in } \overline{B_R(0)}, \\ u^{\varepsilon_{\kappa_n(n)}} \underset{\varepsilon_{\kappa_n(n)}\to 0}{\longrightarrow} u, & \text{weakly in } \mathrm{W}^{2,p}_{\mathrm{loc}}(B_R(0)). \end{cases}$$

Since $u^{\varepsilon_{\kappa_n(n)}} \underset{\varepsilon_{\kappa_n(n)}\to 0}{\longrightarrow} u$ uniformly in $B_R(0)$ and proceeding of a similar way than in (4.18) and (4.20), we conclude (4.20). ■

4.1 Proof of Theorem 1.2

We proceed to show the existence and uniqueness to the solution of the HJB equation (4.3).

Remark 4.7. Note if $\int_{B_r} f\phi \mathrm{d}x = \int_{B_r} g\phi \mathrm{d}x$, for any non-negative function ϕ in $\mathrm{C}^\infty_{\mathrm{c}}(B_r)$ then $f = g$ almost everywhere in B_r. The same way we have that if $\int_{B_r} f\phi \mathrm{d}x \leq \int_{B_r} g\phi \mathrm{d}x$, for any non-negative function ϕ in $\mathrm{C}^\infty_{\mathrm{c}}(B_r)$ then $f \leq g$ almost everywhere in B_r. ■

4.1.1 Existence

Proof of Theorem 1.2. Existence. Let $d < p < \infty$. From Lemma 4.6, we know that there exist a decreasing subsequence $\{\varepsilon_{\kappa(\iota)}\}_{\iota\geq 1}$, with $\varepsilon_{\kappa(\iota)} \underset{\iota\to\infty}{\longrightarrow} 0$, and $u \in \mathrm{C}^{0,1}(\overline{B_R(0)}) \cap \mathrm{W}^{2,p}_{\mathrm{loc}}(B_R(0))$ satisfying (4.19) and (4.20). Since for each $\varepsilon_{\kappa(\iota)} \in (0,1)$, the function $u^{\varepsilon_{\kappa(\iota)}}$ is the unique solution of the NIDD problem (3.21), we get

$$q'u^{\varepsilon_{\kappa(\iota)}}(x) - \mathcal{L}' u^{\varepsilon_{\kappa(\iota)}}(x) - \mathcal{I}\, E(u^{\varepsilon_{\kappa(\iota)}})(x) \leq h(x), \text{ in } B_R(0). \tag{4.21}$$

Then, for each $B_r \subset B_R(0)$ and $\beta \in (0,1)$ fixed, we get

$$\int_{B_{\beta r}} (q'u^{\varepsilon_{\kappa(\iota)}} - \mathcal{L}' u^{\varepsilon_{\kappa(\iota)}} - \mathcal{I}\, E(u^{\varepsilon_{\kappa(\iota)}}))\phi \mathrm{d}x \leq \int_{\beta r} h\phi \mathrm{d}x, \tag{4.22}$$

for each non-negative function ϕ in $\mathrm{C}^\infty_{\mathrm{c}}(B_{\beta r})$. Letting $\varepsilon^{\kappa(\iota)} \to 0$ in (4.22), from Lemma 4.6, we obtain

$$\int_{B_{\beta r}} (q'u - \mathcal{L}' u - \mathcal{I}\, E(u))\phi \mathrm{d}x \leq \int_{B_{\beta r}} h\phi \mathrm{d}x,$$

for each non-negative function ϕ in $\mathrm{C}^\infty_{\mathrm{c}}(B_{\beta r})$. By Remark 4.7, it follows that

$$q'u(x) - \mathcal{L}' u(x) - \mathcal{I}\, E(u)(x) \leq h(x), \text{ a.e. in } B_R(0). \tag{4.23}$$

Now, since $\psi_{\varepsilon}(||\operatorname{D}^1 u^{\varepsilon_{\kappa(\iota)}}(x)||^2)$ is locally uniform bounded (Lemma 3.23), independent of $\varepsilon_{\kappa(\iota)}$, we have

$$||\operatorname{D}^1 u(x)||^2 \leq 1, \text{ in } B_R(0). \tag{4.24}$$

Suppose that $||\operatorname{D}^1 u(x^*)||^2 < 1$, for some $x^* \in B_R(0)$. Then, by the continuity of $\operatorname{D}^1 u$, there exists an small neighborhood $\mathcal{V}_{x^*}$ of x^* such that

$$||\operatorname{D}^1 u(x)||^2 < 1, \text{ for all } x \in \mathcal{V}_{x^*}.$$

Since $\operatorname{D}^1 u^{\varepsilon_{\kappa(\iota)}} \underset{\varepsilon_{\kappa(\iota)}\to 0}{\longrightarrow} \operatorname{D}^1 u$ uniformly in $\mathcal{V}_{x^*}$, we obtain that there exists $\varepsilon^{\kappa(\iota_0)} \in (0,1)$ such that

$$||\operatorname{D}^1 u^{\varepsilon_{\kappa(\iota)}}(x)||^2 < 1, \text{ for all } x \in \mathcal{V}_{x^*} \text{ and } \varepsilon_{\kappa(\iota)} \leq \varepsilon_{\kappa(\iota_0)}.$$

Then, from (3.21) and the definition of ψ_{ε}, it follows that

$$q' u^{\varepsilon_{\kappa(\iota)}}(x) - \mathcal{L}' u^{\varepsilon_{\kappa(\iota)}}(x) - \mathcal{I}\, E(u^{\varepsilon_{\kappa(\iota)}})(x+z)\nu(\mathrm{d}z) = h(x),$$

for all $x \in \mathcal{V}_{x^*}$ and $\varepsilon_{\kappa(\iota)} \leq \varepsilon_{\kappa(\iota_0)}$. Then,

$$\int_{\mathcal{V}_{x^*}} (q' u^{\varepsilon_{\kappa(\iota)}} - \mathcal{L}' u^{\varepsilon_{\kappa(\iota)}} - \mathcal{I}\, E(u^{\varepsilon_{\kappa(\iota)}}))\phi \mathrm{d}x = \int_{\mathcal{V}_{x^*}} h\phi \mathrm{d}x, \tag{4.25}$$

for each non-negative function ϕ in $\mathrm{C}^{\infty}_{\mathrm{c}}(B_{\beta r})$. Letting $\varepsilon^{\kappa(\iota)} \to 0$ in (4.25), from Lemma 4.6, we obtain

$$\int_{\mathcal{V}_{x^*}} (q' u - \mathcal{L}' u - \mathcal{I}\, E(u))\phi \mathrm{d}x = \int_{\mathcal{V}_{x^*}} h\phi \mathrm{d}x,$$

for each non-negative function ϕ in $\mathrm{C}^{\infty}_{\mathrm{c}}(B_{\beta r})$. By Remark 4.7, it yields

$$q' u(x) - \mathcal{L}' u(x) - \mathcal{I}'\, E(u)(x) = h(x), \text{ a.e. in } \mathcal{V}_{x^*}. \tag{4.26}$$

Finally, since $u^{\varepsilon_{\kappa(\iota)}}(x) = 0$ on $\partial B_R(0)$ and $u^{\varepsilon_{\kappa(\iota)}} \underset{\varepsilon_{\kappa(\iota)}\to 0}{\longrightarrow} u$ uniformly in $\overline{B_R(0)}$, it yields

$$u(x) = 0, \text{ on } \partial B_R(0). \tag{4.27}$$

From (4.23)–(4.27), we conclude that u is a solution to the HJB equation (4.3) a.e. in $B_R(0)$. ■

4.1.2 Uniqueness

Proof of Theorem 1.2. Uniqueness. To show the uniqueness of the HJB equation (4.3), we shall use the HJB equation (4.4) which is equivalent to it. Let $d < p < \infty$. Suppose that there exist

$u_1, u_2 \in \mathrm{C}^{0,1}(\overline{B_R(0)}) \cap \mathrm{W}^{2,p}_{\mathrm{loc}}(B_R(0))$ two solutions to the HJB equation (4.4). Let $x^* \in \overline{B_R(0)}$ be the point where $u_1 - u_2$ attains its maximum. If $x^* \in \partial B_R(0)$, it is easy to see

$$(u_1 - u_2)(x) \leq (u_1 - u_2)(x^*) = 0 \text{ in } B_R(0). \tag{4.28}$$

If $x^* \in B_R(0)$, we shall prove (4.28) by contradiction. Suppose $(u_1 - u_2)(x^*) > 0$. For $\rho > 0$ small enough, the function $(1-\rho)u_1 - u_2$ is positive at some point of $B_R(0)$, with $((1-\rho)u_1 - u_2)(x) = 0$ on $\partial B_R(0)$, and hence that $((1-\rho)u_1 - u_2)(x_1^*) > 0$, where $x_1^* \in B_R(0)$ is the point where $(1-\rho)u_1 - u_2$ attains its maximum. Besides, we have

$$\begin{cases} \mathrm{D}^1((1-\rho)u_1 - u_2)(x_1^*) = 0, \\ ((1-\rho)u_1 - u_2)(x_1^* + z) \leq ((1-\rho)u_1 - u_2)(x_1^*), \text{ for all } x_1^* + z \in B_R(0). \end{cases}$$

Since $((1-\rho)u_1 - u_2)(x_1^* + z) \leq ((1-\rho)u_1 - u_2)(x_1^*)$ for all $x_1^* + z \in B_R(0)$, it follows that

$$\begin{aligned} 0 &\geq \mathcal{I}' E((1-\rho)u_1 - u_2)(x_1^*) \\ &= \int_{\mathbb{R}^*} (((1-\rho)u_1 - u_2)(x_1^* + z) - ((1-\rho)u_1 - u_2)(x_1^*))\mathbb{1}_{B_R(0)}(x_1^* + z)\nu(\mathrm{d}z) \\ &\quad + \int_{\mathbb{R}^*} (E((1-\rho)u_1 - u_2)(x_1^* + z) - ((1-\rho)u_1 - u_2)(x_1^*))\mathbb{1}_{\mathcal{B}'}(x_1^* + z)\nu(\mathrm{d}z), \end{aligned}$$

with $\mathcal{B}' = B_{R+\frac{b}{2}}(0) \setminus \overline{B_R(0)}$. Since $\mathrm{D}^1((1-\rho)u_1 - u_2)(x_1^*) = 0$, $||\,\mathrm{D}^1 u_1(x_1)|| \leq 1$ and $\rho > 0$, we get that

$$||\,\mathrm{D}^1 u_2(x_1^*)|| = (1-\rho)||\,\mathrm{D}^1 u_1(x_1^*)|| < 1.$$

This implies that there exists $\mathcal{V}_{x_1^*}$ a neighborhood of x_1^* such that

$$\begin{cases} qu_2(x) - \mathcal{L}u_2(x) - \mathcal{I}' E(u_2)(x) = h(x), \\ qu_1(x) - \mathcal{L}u_1(x) - \mathcal{I}' E(u_1)(x) \leq h(x), \end{cases} \quad \text{for all } x \in \mathcal{V}_{x_1^*}.$$

Then,

$$q((1-\rho)u_1 - u_2)(x) - \mathcal{L}((1-\rho)u_1 - u_2)(x) - \mathcal{I}' E((1-\rho)u_1 - u_2)(x) \leq -\rho h(x),$$

for all $x \in \mathcal{V}_{x_1^*}$, and hence,

$$\begin{aligned} \frac{1}{2}\sum_{ij} \sigma_{ij}\partial^2_{ij}((1-\rho)u_1 - u_2)(x) &\geq q((1-\rho)u_1 - u_2)(x) \\ &\quad - \mathcal{I}' E((1-\rho)u_1 - u_2)(x) - \langle(\mathrm{D}^1(1-\rho)u_1 - u_2)(x), \gamma\rangle + \rho h(x), \end{aligned}$$

for all $x \in \mathcal{V}_{x_1^*}$. Using Bony's maximum principle; see Theorem 4.1, it yields

$$\begin{aligned} 0 &\geq \liminf_{x \to x_1^*} \operatorname{ess} \frac{1}{2} \sum_{ij} \sigma_{ij} \partial_{ij}^2 ((1-\rho)u_1 - u_2)(x) \\ &\geq q((1-\rho)u_1 - u_2)(x_1^*) - \mathcal{I}' E((1-\rho)u_1 - u_2)(x_1^*) + \rho h(x_1^*), \end{aligned}$$

which is a contradiction, since $((1-\rho)u_1 - u_2)(x_1^*) > 0$ implies that

$$q((1-\rho)u_1 - u_2)(x_1^*) - \mathcal{I}' E((1-\rho)u_1 - u_2)(x_1^*) + \rho h(x_1^*) > 0.$$

Therefore, we have

$$(u_1 - u_2)(x) \leq (u_1 - u_2)(x_1^*) \leq 0, \text{ for all } x \in B_R(0).$$

Taking $u_2 - u_1$ and proceeding of the similar way as before, it follows that

$$(u_2 - u_1)(x) \leq (u_2 - u_1)(x_1^*) \leq 0, \text{ for all } x \in B_R(0),$$

and hence we conclude that the solution u to the HJB equation (4.3) is unique. ■

Conclusions and some open problems

Let us start by reviewing the main results in the thesis and the techniques used, so that later we point towards which directions our results can be extend and discuss related problems. Under the hypotheses (H1)–(H4) given in pages 8 and 9, it was shown that the solution u to the HJB equation (4.3) there exists in $\mathrm{C}^{0,1}(\overline{B_R(0)}) \cap \mathrm{W}^{2,p}_{\mathrm{loc}}(B_R(0))$ and this is unique, if $d < p < \infty$; see Chapter 4. Recall that $R > 0$ is fixed, and the linear operator Γ' that appears in the HJB equation (4.3) is an elliptic integro-differential operator given in (4.2), where the principal ingredients of this operator are the integral operator $\mathcal{I}$ and the continuous linear operator E, which is defined in (2.36).

In order to obtain the existence and regularity of the solution u to the HJB equation (4.3), first we had to verify that the value function $V^{\varepsilon}(\cdot; w)$ related with the solution $u^{\varepsilon}(\cdot; w)$ to the non-linear Dirichlet problem (3.5) is in $\mathrm{C}^{2,\alpha}(\overline{B_R(0)}) \subset \mathrm{C}^{0}(\overline{B_R(0)})$ see Lemma 3.5, and it satisfies that if $w_1, w_2 \in \mathrm{C}^{0}(\overline{B_R(0)})$, then

$$||V^{\varepsilon}(\cdot; w_1) - V^{\varepsilon}(\cdot; w_2)||_{\mathrm{C}^{0}(\overline{B_R(0)})} \leq \frac{2A_0}{q'} ||w_1 - w_2||_{\mathrm{C}^{0}(\overline{B_R(0)})};$$

see Lemma 3.6. Then, defining the mapping $T_{\varepsilon} : \mathrm{C}^{0}(\overline{B_R(0)}) \longrightarrow \mathrm{C}^{0}(\overline{B_R(0)})$ as $T_{\varepsilon}(w) = V^{\varepsilon}(\cdot; w)$, for each $w \in \mathrm{C}^{0}(\overline{B_R(0)})$, we verified that T_{ε} is a contraction mapping in the Banach space $(\mathrm{C}^{0}(\overline{B_R(0)}), || \cdot ||_{\mathrm{C}^{0}(\overline{B_R(0)})})$, and hence, using contraction point fixed Theorem; see Theorem 3.1, we solved the NIDD problem (3.21), guaranteeing that its solution $u^{\varepsilon} \in \mathrm{C}^{3,\alpha}(\overline{B_R(0)})$ is unique, for each $\varepsilon \in (0, 1)$; see Theorem 3.8.

Note that to accomplish this step, it has been fundamental to transform the NIDD problem (3.21) in a classic non-linear Dirichlet problem, i.e. we fixed the integral part of the operator Γ', to guarantee the existence of its unique solution in $\mathrm{C}^{2,\alpha}(\overline{B_R(0)})$. Also, it has been very important to have a very good understanding of the continuous linear operator E defined in (2.36), which plays a crucial role in the definition of the non-linear Dirichlet problems (3.5) and (3.21), and to establish that their solutions belong to $\mathrm{C}^{2,\alpha}(\overline{B_R(0)})$.

After completing the above described step, for each $\varepsilon \in (0, 1)$, we verified, by probabilistic, integro-differential and partial differential methods, that the solution u^ε to the NIDD problem (3.21) is positive and bounded by a positive constant independent of ε in $B_R(0)$; see Proposition 3.13 and Lemma 3.16. We succeeded in proving that the norm of its gradient is bounded by a constant independent of ε in $B_R(0)$, see Lemma 3.21, and evaluating the matrix of its second derivatives in the norm of the Sobolev space $\mathrm{W}^{2,p}_{\mathrm{loc}}(B_R(0))$, with $d<p<\infty$, we obtained that the this estimation is locally bounded by a constant independent of ε; see Lemma 3.24. Using these, we established the convergence of a subsequence of $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ in $\mathrm{C}^{0,1}(\overline{B_R(0)})\cap\mathrm{W}^{2,p}_{\mathrm{loc}}(B_R(0))$, with $d<p<\infty$; see Lemma 4.6. Taking $\{u^{\varepsilon_{\kappa(\iota)}}\}_{\kappa(\iota)\geq1}$ and u as in Lemma 4.6, and using that $\psi_\varepsilon(||\,\mathrm{D}^1 u^\varepsilon(\cdot)||^2)$ is locally bounded by a constant independent of ε in $B_R(0)$, we obtained that u is the solution to the HJB equation (4.3), and finally, by Bony's maximum principle, see Theorem 4.1, it is shown that this solution is unique.

The closer to our work is to Menaldi and Robin [26]. They are interested in to study a singular control problem for a multidimensional Gaussian-Poisson process, and to establish a relationship between the value function to this problem and the solution a HJB equation. The multidimensional Gaussian-Poisson process is a Lévy process where it only has a d-dimensional standard Brownian motion and a jump process which Lévy measure ν satisfies

$$\int_{\mathbb{R}^d\setminus\{0\}}||z||^p\nu(\mathrm{d}z)<\infty,\ \text{for all } p\geq2.$$

Although the proofs of their principal results are not provided in detail, and they left these to future works, they give enough arguments to show that the solution to the HJB equation associated with the value function to the singular control problem is in the classic sense. Besides that our problem is also related to a singular control problem, there are several differences between this problem and ours. Our problem is related to an optimal stochastic control problem where the controlled process is a d-dimensional Lévy process as in (1.6).

Now, there are many lines of research in the theory of the optimal control and integro-differential equations that can be explored to extend the results obtained in the present work, we will next describe some of them.

First, observe that the solution u to the HJB equation (4.3) depends on the constants p and b. An improvement to this result would establish that the solution u to the HJB equation (4.3) belongs to $\mathrm{C}^{0,1}(\overline{B_R(0)})\cap\mathrm{W}^{2,\infty}_{\mathrm{loc}}(B_R(0))$. For this, we need to show that for each open ball $B_r\subset B_R(0)$, it satisfies $||\,\mathrm{D}^2 u^\varepsilon||_{\mathrm{W}^{2,\infty}(B_r)}\leq C$, for some constant C independent of ε, where u^ε is the solution to the NIDD problem (3.21).

Second, observe that the HJB equation (4.3) is stated in terms of the extension operator E, and it would be suitable to remove this from the equation to have a standard formulation of the HJB equation. For that end one should make b go to zero, which according to Remark 2.21 is equivalent to make N tend to infinity. For that end one should prove that the solution u to the HJB equation (4.3), their first and weakly second derivatives are uniformly bounded with respect to b, we could pass the limit in (4.3) when $b \to 0$, and guarantee the existence and regularity of the solution to the following HJB equation

$$\begin{cases} \max\{q'u(x) - \Gamma' u(x) - h(x), || \mathrm{D}^1 u(x)||^2 - 1\} = 0, & \text{in } B_R(0), \\ u(x) = 0, & \text{outside } B_R(0), \end{cases}$$

where

$$\Gamma' u(x) = \frac{1}{2} \operatorname{tr}(\sigma \mathrm{D}^2 u(x)) + \langle \mathrm{D}^1 u(x), \widetilde{\gamma} \rangle + \int_{\mathbb{R}^*} (u(x+z))\nu(\mathrm{d}z). \tag{4.29}$$

Another very important related problem is the HJB equation (1.11) defined in $\mathbb{R}^d$, i.e.

$$\max\{q'u(x) - \Gamma' u(x) - h(x), || \mathrm{D}^1 u(x)||^2 - 1\} = 0, \text{ in } \mathbb{R}^d, \tag{4.30}$$

with Γ' as in (4.29). In order to obtain the existence and regularity of the solution to this HJB equation, we need to pass to limits in a strong sense in (3.21). For this is necessary to find bounds of $u^{\varepsilon,R}$, $\mathrm{D}^1 u^{\varepsilon,R}$ and $\mathrm{D}^2 u^{\varepsilon,R}$ that are independent of (ε, R), where $u^{\varepsilon,R}$ is the solution to the NIDD problem (3.21). Now, in Lemma 3.19, it was shown that $|| \mathrm{D}^1 u^{\varepsilon,R}(\cdot)||$ is bounded by a positive constant on $\partial B_R(0)$. This constant is independent of ε but grows exponentially fast with R. This is not suitable as it suggests that a bound function for $|| \mathrm{D}^1 u^{\varepsilon,R}(\cdot)||$ in $B_R(0)$ is of the exponential type, and hence it possesses technical issues when estimating first and second derivatives of this bound function. In similar studies in the literature a polynomial bound has been obtained, here a bound of polynomial type with degree two would be enough to take limits in a strong sense in (3.21), which would allow to establish the existence of the HJB equation (4.30).

Finally, other topic of interest is when the integral operator of Γ', given in (1.2), is taken as

$$\mathcal{I}Eu(x) = \int_{\mathbb{R}^*} (E(u)(x+z) - u(x) - \langle \mathrm{D}^1 u(x), z \rangle \mathbb{1}_{\{||z||<1\}})\nu(\mathrm{d}z),$$

and ν is a Lévy measure in $\mathbb{R}^*$ that satisfies

$$\int_{\mathbb{R}^*} (1 \wedge ||z||^2)\nu(\mathrm{d}z) < \infty.$$

Then, the HJB equation for this problem is

$$\begin{cases} \max\{qu(x) - \Gamma_2 u(x) - h(x), || \mathrm{D}^1 u(x)||^2 - 1\} = 0, & \text{in } B_R(0), \\ u(x) = 0, & \text{on } \partial B_R(0), \end{cases} \tag{4.31}$$

where

$$\begin{aligned}\Gamma_2 u(x) &:= \frac{1}{2}\operatorname{tr}(\sigma \, \mathrm{D}^2 u(x)) + \langle \mathrm{D}^1 u(x), \gamma \rangle \\ &\quad + \int_{\mathbb{R}^*} (E(u)(x+z) - u(x) - \langle \mathrm{D}^1 u(x), z \rangle \mathbb{1}_{||z||<1}) \nu(\mathrm{d}z).\end{aligned}$$

The HJB equation (4.31) is of great interest because it can be related with an optimal stochastic control problem where the state process is a controlled d-dimensional Lévy process, i.e. a càdlàg process in $\mathbb{R}^d$, with independent and stationary increments [30].

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