Steklov Methods for Nonlinear Stochastic Differential Equations.

Ph.D. Dissertation

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Abstract

We propose a new way to construct numerical methods for stochastic differential equations (SDEs) via Steklov means. Here we present two schemes that evidence the properties of this new approach. First, we construct a scheme for scalar SDEs and prove its convergence and stability under standard globally Lipschitz and linear growth conditions. Moreover, we give sufficient conditions for the nonlinear asymptotic stability in both, multiplicative and additive cases. Finally, we showed the behavior of the explicit Steklov method for problems with stringent stability requirements as the logistic stochastic equation and the Langevin equation in Brownian dynamics. In all these studies, we established that the Steklov method is an accurate scheme for large time scales simulation. The second scheme extends the previous method towards a multidimensional set up and coefficients with locally Lipschitz and monotone grow conditions. This method is constructed on the basis that the drift function can be rewritten in a linearized form. Moreover, strong order one-half convergence has been proved for our explicit linear method and we have presented several applications formulated with the LS scheme. Also, we present numerical evidence that confirms our theoretical results and suggests an extension to super-linear growth diffusions. Finally, high-performance of the Linear Steklov method have been analyzed in diverse problems for which other methods have failed.

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Thesis Details

Title: Steklov Methods for non Linear Stochastic Differential Equations.
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The main body of this thesis consist of the following papers.

[3] Saúl Díaz-Infante, Silvia Jerez "Convergence and asymptotic stability of the explicit Steklov method for stochastic differential equations," *Journal of Computational and Applied Mathematics* DOI: 10.1016/j.cam.2015.01.016 14-FEB-2015 vol. 291, pp. 36-47, 2015.

[4] Saúl Díaz-Infante, Silvia Jerez, The Linear Steklov Method for SDEs with non-globally Lipschitz Coefficients: Strong convergence and simulation. Paper submitted at *Journal of Computational and Applied Mathematics.*

This thesis has been submitted for partial fulfillment of the PhD degree. The thesis is based on the submitted or published scientific papers which are listed above. Parts of the papers are used directly or indirectly in the extended summary of the thesis. As part of the assessment, co-author statements have been made available to the Ph. D. committee and are also available at the Faculty.

Chapter 1

Introduction and main results

1.1 Introduction

In the last decades, stochastic differential modeling has become a rapidly-growing research area. Historically, it appeared as an extension of the deterministic differential modeling of over-idealized situations with fluctuating behavior of the analyzed physical phenomenon. Actually, it is an important research area by itself that describes important phenomena such as turbulent diffusion, spread of diseases, genetic regulation, motion of particles etc. [1, 23, 67]. We can obtain the explicit solution of only few stochastic differential equations (SDEs), therefore developing accurate stochastic numerical approximations represent an option to analyze and confirm (by simulation) the nature of a stochastic model. Stochastic numerics allow the analysis of some model properties that are difficult or impossible to measure experimentally in laboratories, for example its long-time behavior. In this cases, we require that a numerical solvers be able to reproduce asymptotic behavior like mean square stability [29, 30, 59], usually, a linear analysis can be considered as the first step for understanding a method, but it is not an indicator of qualitative behavior on the nonlinear case [37]. Thus, some theoretical work on asymptotic stability has appeared for nonlinear SDEs Bokor [9] Buckwar et al. [12].

The first methods for solving SDEs were stochastic extensions of deterministic algorithms, for example schemes as the Euler-Maruyama (EM), Taylor and Runge-Kutta [9, 13, 36]. Unfortunately, sometimes their asymptotic stability conditions are very restrictive, considers for example the Brownian Dynamics Simulations, here the Euler-Maruyama discretization is the standard method to solve the Langevin equation that describe the motion of particles [10, 14, 20]. However, the operation time step size of this scheme has to be pint-size, otherwise the scheme becomes unstable. Now, the construction of methods focus on structural or dynamic properties of a specific SDE. Some examples are the balanced methods for stiff SDE [53] the quasi-symplectic schemes for stochastic Hamiltonian systems [51] and SDEs with small noise [11].

Numerical convergence and stability are well understood for SDE with globally Lipschitz continuous coefficients, which discard many important models from applications. Moreover, **?** report in [**?**] that if a SDE has drift or diffusion, which grows faster that a linear function, then the EM diverges in strong and weak sense. This result opens a new chapter on the design of numerical methods — stochastic models in applications as Finance, Biology and Physics use SDEs with locally Lipschitz coefficients. In addition, Giles proposes in [24] a new variance reducing technique that relies in strong numerical convergence, which optimizes the traditional Monte Carlo simulation. Thus, developing explicit schemes, which converges in strong sense with super-linear coefficients attracts the right now attention.

Recently research has been focused on modifying the EM method to obtain strong convergence under the previous conditions keeping its simple structure and its low computational cost. Several methods have been developed in this direction: the family of Tamed schemes [32, 34, 68, 70], a special type of balanced method [66], the stopped scheme [42] . For SDE with super-linear diffusion, Mao and Szpruch provided results for the strong convergence of implicit methods as the Backward-Euler-Maruyama. However, the convergence of explicit schemes for SDEs with super-linear growth is still under development. Works on this subject are Mao [45] with the truncated Euler method and [57] with a new kind of tamed scheme. There, the strong convergence of the proposed method is proved using the theory developed by in Higham, Mao, and Stuart [31] or by means of the new approach given by Hutzenthaler and Jentzen [32]. Both techniques prove strong convergence by verifying boundedness moments of the numerical and analytical solution of the underlying SDE. In spite of the recent work in this subject, it is still necessary to get more accurate numerical methods for SDE under super-linear growth and non-globally Lipschitz coefficients.

1.2 Main Results

Our main contribution follows two lines of research. The first one is to design an explicit numerical scheme with good stability properties. We focus on explicit methods because we are interested on applications of Brownian Dynamics, so we seek a simple and fast numerical solver. Also we require a stable scheme in order to obtain simulations for long periods of time. For example in Brownian Dynamics, the self-diffusion coefficient is an asymptotic property. We propose the Steklov method, which is a stochastic extension of an exact deterministic numerical scheme.

The second line consists in generalize the above scheme to a multidimensional setup and with more general coefficients. Thus, we propose the Linear Steklov (LS) scheme. This explicit method is based on a linear version of the Steklov average with a split-step formulation. We prove for this scheme a one-half convergence order with a one sided Lipschitz condition and polynomial growth on the drift; and a globally Lipschitz condition on the diffusion. Also we provide numerical evidence that this method is suitable for problems with super-linear growth diffusion where other methods have failed.

1.3 Methodology

Chapter 2 — Preliminaries

After the above introduction, in this chapter we present an overview of basic results from probability theory that we will need in order in order to set our framework. Next we state useful concepts and theorems from stochastic process and stochastic calculus. Finally, we give some important qualitative properties of numerical methods for stochastic differential equations.

Chapter 3 — Steklov method for scalar SDEs with Globally Lipschitz coefficients

The chapter contains our first contribution —the Steklov method. First we develop a new numerical method with asymptotic stability properties for solving stochastic differential equations (SDEs). The foundations for the new solver are the Steklov mean and an exact discretization for the deterministic version of the SDEs. Second strong consistency and convergence properties are demonstrated for the proposed method. Moreover, a rigorous linear and nonlinear asymptotic stability analysis is carried out for the multiplicative case in a mean-square sense and for the additive case in a path-wise sense using the pullback limit. In order to emphasize the characteristics of the Steklov discretization we use as benchmarks the stochastic logistic equation and the Langevin equation with a nonlinear potential of the Brownian dynamics. We show that the Steklov method has mild stability requirements and allows long-time simulations in several applications.

Chapter 4 — Steklov Method for SDEs with Non-Globally Lipschitz Continuous Drift

In this chapter we present an explicit numerical method for solving stochastic differential equations with non-globally Lipschitz coeffcients. A linear version of the Steklov average under a split-step formulation supports our new solver. The Linear Steklov method converges strongly with a standard one-half order. Also, we present numerical evidence that the explicit Linear Steklov reproduces almost surely stability solutions with high-accuracy for diverse application models even for stochastic differential systems with super-linear diffusion coefficients.

Chapter 5 — Conclusions and future work

Finally, in this Chapter we restate our contribution followed by conclusions and discuss possible future directions for our research.

Chapter 2

Preliminaries

Here we present some results from stochastic analysis. This chapter focus on provide the basic information and tools to understand the nature of a SDE and its numerical approximation. For reference see [5, 36, 52, 55, 69].

2.1 Probability theory and Stochastic Processes

Probability theory is the field that studies the random phenomena. A random event is the set of outcomes from an experiment conducted under the same conditions with a variability in results . Probability theory aims to describe this variability. We denote by Ω the set of observable outcomes, ω, from a experiment or phenomenon. However, not every observable event is measurable, so for the purpose of probability theory, a family of subsets from Ω with particular properties a —σ-algebra is needed. In the following, we formalize these concepts, for reference see e.g. [55].

Definition 2.1.1 (σ-algebra)**.** Let Ω a set and $\mathcal{F}$ a family of subset of Ω, we call $\mathcal{F}$ a σ- algebra if the following properties hold:

(i) $\emptyset \in \mathcal{F}$,

(ii) if $F \in \mathcal{F}$ then $F^c \in \mathcal{F}$ where $F^c = \Omega \setminus F$,

(iii) if $\{F_i\}_{i=1}^{\infty} \in \mathcal{F}$ then $\bigcup\limits_{i\geq 1} F_i \in \mathcal{F}$.

Let $\mathcal{C}$ a collection of subsets of Ω. The σ-algebra generated by $\mathcal{C}$ denoted by $\sigma(\mathcal{C})$, is the smallest σ-algebra which contains the collection $\mathcal{C}$, that is $\sigma(\mathcal{C}) \supset \mathcal{C}$, and if $\mathcal{B}$ is an other σ-algebra containing $\mathcal{C}$, then $\mathcal{B} \supset \sigma(\mathcal{C})$.

Definition 2.1.2 (The Borel σ-algebra)**.** If $\Omega = \mathbb{R}^d$ and $\mathcal{C}$ is the family of all open sets in $\mathbb{R}^d$, the $\mathcal{B}^d = \sigma(\mathcal{C})$ is called the Borel σ-algebra and its elements are called Borel sets.

A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where

- Ω is the set of all possible outcomes of an experiment.

- $\mathcal{F}$ is a chosen σ-algebra of subsets of Ω.
- $\mathbb{P}$ is a probability measure; that is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ such that
  (i) $\mathbb{P}(A) \geq 0$ for all $A \in \mathcal{F}$.
  (ii) $\mathbb{P}$ is σ-additive, that is: If $\{A_n, n \geq 1\}$ is a collection of disjoint events, then
$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$
  (iii) $\mathbb{P}(\Omega) = 1$.

Definition 2.1.3 (Random Variable)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{B}(\mathbb{R}^d)$ the Borel's σ-algebra. A function $X : \Omega \to \mathbb{R}^d$ is said to be a random variable if X is $(\mathcal{F}, \mathcal{B}(\mathbb{R}^d))$-measurable, that is $X^{-1}(\mathcal{B}(\mathbb{R}^d)) \subset \mathcal{F}$.

Every random variable X induces a probability measure μ_X on $\mathbb{R}^d$ by
$$\mu_X(B) = \mathbb{P}(X^{-1}(B)), \qquad B \in \mathcal{B}(\mathbb{R}^d).$$

Having two different measures $\mathbb{Q}$, $\mathbb{P}$, on a measurable space we can transform one measure into the other via Radon-Nikodym theorem (see for example [69, Thm. 10.1.2]).

Theorem 2.1.1 (Radon-Nikodym)**.** *Let $\mathbb{P}$ and $\mathbb{Q}$ probability measures on the measurable space $(\Omega, \mathcal{F})$. Suppose that for all $B \in \mathcal{F}$ $\mathbb{Q}(B) = 0$ implies $\mathbb{P}(B) = 0$. Then there exist a integrable random variable X such that*
$$\mathbb{Q}(E) = \int_E X d\mathbb{P}, \qquad \forall E \in \mathcal{F}.$$
X is $\mathbb{P}$-a.s. unique and is written as $X = \dfrac{d\mathbb{Q}}{d\mathbb{P}}$.

This important result describes the density p of a random variable X as the $\mathbb{P}$-a.s. unique Radon-Nikodym derivative of the induced distribution μ_X w.r.t. Lebesgue measure, in other words
$$\mu_X(B) = \int_B p(x)dx.$$

Definition 2.1.4 (Expectation)**.** Let X be a integrable random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the expectation of X is defined by
$$\mathbb{E}[X] = \int_\Omega X(\omega)\mathbb{P}(d\omega).$$

A stochastic process X is a system which could stay at each moment on any state of a given set S

Definition 2.1.5 (Stochastic Process)**.** A stochastic process is a collection of random variables $X = \{X_t : t \in T\}$ on $(\Omega, \mathcal{F})$, which takes values in a measurable space $(S, \mathcal{S})$, and where the index $t \in [0, \infty)$, conveniently receive an interpretation as time. Thus for a fixed $\omega \in \Omega$, the function $X_t(\omega)$, $t \geq 0$ is a sample path of the process X associated with ω, and for any fixed t , $X_t(\omega)$, $\omega \in \Omega$ is a random variable.

The main purpose of this thesis deals with the numerical approximation of sample paths.

Definition 2.1.6 (Measurable Process)**.** A stochastic process X is measurable if the mapping

$$(t,\omega) \to X_t(\omega) : ([0,\infty) \times \Omega, \mathcal{B}([0,\infty)) \otimes \mathcal{F}) \to \left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$$

is measurable.

We equip the underlying sample space $(\Omega, \mathcal{F})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ in order to keep track information about the past, present and future of a stochastic process. Formally, a filtration is a nondecreasing family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $0 \leq s \leq t < \infty$ and is called right continuous if $\mathcal{F}_t = \bigcap_{r>t} \mathcal{F}_r$ for all $t \geq 0$. Thus if the underlying probability space is complete, right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets, then we say that the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions. In the following, we will work only on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which verifies the usual conditions.

Given a stochastic process, the simplest choice of a filtration is that generated by the process itself, i.e. $\mathcal{F}_t^X := \sigma(X_s; 0 \leq s \leq t)$ the smallest σ-algebra with respect to which X_s is measurable for every $s \in [0,t]$. The introduction of this concept gives sense to the following.

Definition 2.1.7 (Adapted Process)**.** We call a process *adapted to the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *if, for each* $t > 0$ *fixed* X_t *is a* $\mathcal{F}_t$*-measurable random variable.*

Clearly, every process X is adapted to $\{\mathcal{F}_t^X\}$.

Definition 2.1.8 (Progressively Measurable Process)**.** The stochastic process X is progressively measurable if the mapping

$$(s,\omega) \to X_s(\omega) : ([0,t] \times \Omega, \mathcal{B}([0,t]) \otimes \mathcal{F}_t) \to \left(\mathbb{R}, \mathcal{B}\left(\mathbb{R}^d\right)\right)$$

is measurable for each $t \geq 0$, that is, if, for each $t > 0$ and $A \in \mathcal{B}\left(\mathbb{R}^d\right)$, the set

$$\{(\omega, s) : 0 \leq s \leq t, \omega \in \Omega, X_s(\omega) \in A\}$$

belongs to the product σ-algebra $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$.

2.1.1 Conditional Expectation

Conditional expectation plays a very important role in the modern probability theory. It gives foundation for the definitions of martingales and Markov processes. In fact, other areas of probability as stochastic dynamics, conditioning permits to describe and to analyze dynamical systems with randomness. Roughly speaking, the conditional expectation is an average that considers only a portion of information.

Definition 2.1.9. Let $X \in L_1(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra. Then the conditional expectation of the random variable X given $\mathcal{G}$ is the new random variable $Y = \mathbb{E}[X|\mathcal{G}]$ such that

(i) $\mathbb{E}[X|G]$ is $\mathcal{G}$-measurable and integrable.

(ii) For all event $G \in \mathcal{G}$ we have $\int_G X d\mathbb{P} = \int_G \mathbb{E}[X|\mathcal{G}]\, d\mathbb{P}$.

This new random variable is unique in the sense that if there is an other $\tilde{Y}$ satisfying the same two above properties, then $\mathbb{P}[Y \neq \tilde{Y}] = 0$. In this case $\tilde{Y}$ is said to be a version of the conditional expectation $\mathbb{E}[X|\mathcal{G}]$. Now we list some standard properties of the conditional expectation [69]. Here X_1, X_2, Z are integrable random variables, a_1,$a_2 \in \mathbb{R}$, and $\mathcal{G}$, $\mathcal{H}$ are sub-σ-algebras of $\mathcal{F}$.

(E1) If Y is any version of $\mathbb{E}[X|\mathcal{G}]$, then $E[X] = E[Y]$.

(E2) If X is $\mathcal{G}$ measurable, then $\mathbb{E}[X|\mathcal{G}] = X$, $\mathbb{P}$-a.s.

(E3) (Linearity) $\mathbb{E}[a_1 X_1 + a_2 X_2|\mathcal{G}] = a_1\mathbb{E}[X_1|\mathcal{G}] + a_2\mathbb{E}[X_2|\mathcal{G}]$ $\mathbb{P}$-a.s.
Clarification: if Y_1 is a version of $\mathbb{E}[X_1|\mathcal{G}]$ and Y_2 is a version of $\mathbb{E}[X_2|\mathcal{G}]$, then $a_1Y_1 + a_2Y2$ is a version of $\mathbb{E}[a_1X_1 + a_2X_2|\mathcal{G}]$.

(E4) (Positivity) If $X \geq 0$, then $\mathbb{E}[X|\mathcal{G}] \geq 0$, $\mathbb{P}$-a.s.

(E5) (Conditional Monotone Convergence) If $0 \leq X_n \uparrow X$, then $\mathbb{E}[X_n|\mathcal{G}] \uparrow \mathbb{E}[X|\mathcal{G}]$, $\mathbb{P}$-a.s.

(E6) (Conditional Fatou) If $X_n \geq 0$, then $\mathbb{E}\left[\liminf_{n\to\infty} X_n|\mathcal{G}\right] \leq \liminf_{n\to\infty} \mathbb{E}[X_n|\mathcal{G}]$.

(E7) (Conditional Dominated Convergence) If $|X_n(\omega)| \leq V(\omega)$ for all n, $\mathbb{E}[V] < \infty$, and $X_n \to X$ $\mathbb{P}$-a.s., then $\mathbb{E}[X_n|\mathcal{G}] \to \mathbb{E}[X|\mathcal{G}]$, $\mathbb{P}$-a.s.

(E8) (Conditional Jensen) If f is a real-valued convex function, then

$$f(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[f(X)|\mathcal{G}].$$

(E9) (Tower Property) If $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$, then

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}], \qquad \mathbb{P}\text{-a.s.}.$$

(E10) If Z is $\mathcal{G}$-measurable and bounded, then $\mathbb{E}[ZX|\mathcal{G}] = Z\mathbb{E}[X|\mathcal{G}]$.

(E11) If X is independent from $\mathcal{H}$, then $\mathbb{E}[X|\mathcal{H}] = \mathbb{E}[X]$, $\mathbb{P}$-a.s.

Now consider a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and define

$$\mathcal{F}_\infty := \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right) \subset \mathcal{F}.$$

Definition 2.1.10 (Martingale)**.** A process $\{M_t\}_{t\geq 0}$ is called a martingale (relative to $(\{F_t\}_{t\geq 0}, \mathbb{P})$ if

(i) M is adapted,

(ii) $\mathbb{E}[|M_t|] < \infty$ for all $t \geq 0$,

(iii) $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$, $\mathbb{P}$-a.s., $0 \leq s \leq t$.

In this way, a *supermartingale* (relative to $(\{F_t\}_{t\geq 0}, \mathbb{P})$ is defined similarly, except that (iii) is replaced by

$$\mathbb{E}\left[M_t|\mathcal{F}_s\right] \leq M_s \qquad \mathbb{P}\text{-a.s.}, \qquad 0 \leq s \leq t,$$

and a *submartingale* is defined with (iii) replaced by

$$\mathbb{E}\left[M_t|\mathcal{F}_s\right] \geq M_s \qquad \mathbb{P}\text{-a.s.}, \qquad 0 \leq s \leq t.$$

Theorem 2.1.2. *Let $\{M_t\}_{t\geq 0}$ be and $\mathbb{R}^d$-valued martingale with respect to $\{\mathcal{F}_t\}$, and let θ, ρ two finite stopping times. Then*

$$\mathbb{E}\left[M_\theta|\mathcal{F}_\rho\right] = M_{\theta\wedge\rho}.$$

Definition 2.1.11 (Local Martingale)**.** An $\mathbb{R}^d$-valued $\{F_t\}$-adapted integrable process $\{M_t\}_{t\geq 0}$ is said to be a *local martingale* if there exists a nondecreasing sequence $\{\tau_k\}_{k\geq 1}$ of stopping times with $\tau_k \uparrow \infty$ $\mathbb{P}$-a.s. such that $\{M_{\tau\wedge t} - M_0\}$ is a martingale.

A fundamental process in this thesis is the Brownian Motion.

Definition 2.1.12 (Brownian Motion)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$. A standard unidimensional Brownian motion is a real-valued continuous adapted process $\{W_t\}_{t\geq 0}$ which satisfies:

(i) $W_0 = 0, \qquad \mathbb{P} - \text{a. s.};$

(ii) the increments $W_t - W_s$ are normally distributed with mean zero and variance $t - s$ for $0 \leq s \leq t < \infty$;

(iii) $W_t - W_s$ is independent of $\mathcal{F}_s$.

Consider a Brownian motion $\{W_t\}_{t\geq 0}$ and a sequence of times $0 \leq t_0 < t_1 < \ldots < t_k < \infty$. Then $\{W_t\}_{t\geq 0}$ has independent increments, that is, the random variables $W_{t_i} - W_{t_{i-1}}$ $1 \leq i \leq k$ are independent. Moreover, the distribution of $W_{t_i} - Wt_{i-1}$ depends only on the difference $t_i - t_{i-1}$, in this sense, we say that the Brownian motion has stationary distribution. With this in mind, also we can say that this process is a martingale. As we will see, the above is fundamental for the numerical approximations of SDEs.

2.2 Stochastic Calculus and SDEs

In this section, we recall some basic results of the Itô integral (see for example [38])

$$\int_0^t f(s)dW(s).$$

with respect to an m-dimensional Brownian Motion, $\{W(t)\}$, for a class of $d \times m$-matrix-valued processes $\{f(t)\}$.

Definition 2.2.1. Let $0 \le a < b < \infty$. We denote by $\mathcal{M}^2([a,b];\mathbb{R})$ the space of all real-valued measurable $\{\mathcal{F}\}$-adapted processes $f = \{f(t)\}_{a\le t\le b}$ such that

$$\|f\|^2_{a,b} = \mathbb{E}\left[\int_a^b |f(t)|^2 dt\right] < \infty.$$

Theorem 2.2.1. *Assume $f \in \mathcal{M}\left([a,b];\mathbb{R}^{d\times m}\right)$ and let ρ, τ be two stopping times such that $0 \le \rho \le \tau \le T$. Then*

$$\mathbb{E}\left[\int_\rho^\tau f(t)dW(t)|\mathcal{F}_\rho\right] = 0,$$
$$\mathbb{E}\left[\left|\int_\rho^\tau f(t)dW(t)\right|^2|\mathcal{F}_\rho\right] = \mathbb{E}\left[\int_\rho^\tau |f(t)|^2 dt|\mathcal{F}_\rho\right].$$

Definition 2.2.2 (Itô process)**.** A d-dimensional Itô process is a $\mathbb{R}^d$-valued continuous adapted process $X(t) = (X_1(t),\dots,X_d(t))^T$ on $t \ge 0$ of the form

$$X(t) = X(0) + \int_0^t f(s)ds + \int_0^t g(s)dW(s),$$

where $f = (f_1,\dots,f_d)^T \in \mathcal{L}_1(\mathbb{R}_+;\mathbb{R}^d)$ and $g = (g_{ij})^{d\times m} \in \mathcal{L}_2(\mathbb{R}_+;\mathbb{R}^{d\times m})$. We will say that $X(t)$ has stochastic differential $dX(t)$ on $t \ge 0$ given by

$$dX(t) = f(t)dt + g(t)dW(t).$$

Theorem 2.2.2 (The multi-dimensional Itô formula)**.** *Let $X(t)$ be a d-dimensional Itô process on $t \ge 0$ and differential*

$$dX(t) = f(t)dt + g(t)dW(t),$$

where $f = (f_1,\dots,f_d)^T \in \mathcal{L}_1(\mathbb{R}_+;\mathbb{R}^d)$ and $g = (g_{ij})^{d\times m} \in \mathcal{L}_2(\mathbb{R}_+;\mathbb{R}^{d\times m})$. Let $V \in \mathcal{C}^{2,1}(\mathbb{R}^d \times \mathbb{R}_+;\mathbb{R})$. Then $V(X(t),t)$ is again an Itô process with stochastic differential given by

$$dV(X(t),t) = \left[V_t(X(t),t) + V_x(X(t),t)f(t) + \frac{1}{2}\operatorname{Tr}\left(g^T(t)V_{xx}(X(t),t)g(t)\right)\right]dt + V_x(X(t),t)g(t)dW(t) \qquad \mathbb{P}-\text{a.s.}$$

For simplicity of notation and with the same meaning as above, we define a diffusion generator L as

$$LV(X(t),t) = V_t(X(t),t) + V_x(X(t),t)f(t) + \frac{1}{2}\operatorname{Tr}\left(g^T(X)V_{xx}(X(t),t)g(t)\right). \tag{2.1}$$

2.3 Numerical Methods of SDEs

The topic of this thesis is the development of new numerical solutions for stochastic differential equations (SDEs)

$$dy(t) = f(y(t))dt + g(y(t))dW(t), \qquad t \in [0,T], \qquad y(0) = y_0. \tag{2.2}$$

Generally, we know the analytical solution for a few SDEs. So, we need numerical schemes in order to approximate the solutions of eq. (2.2). In this section we present classical numerical methods for SDEs and give fundamental results in numerical analysis of stochastic differential equations. In the following we consider the next setup: Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ a filtered and complete probability space with the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by the m-dimensional Brownian process $W_t = (W_t^{(1)} \dots W_t^{(m)})^T$. We denote the norm of a vector $y \in \mathbb{R}^d$ and the Frobenious norm of a matrix $G \in \mathbb{R}^{d\times m}$ by $|y|$ and $|G|$ respectively. The usual scalar product of two vectors $x, y \in \mathbb{R}^d$ is denoted by $\langle x, y\rangle$. Now we establish the definition of strong solution and the main theorems of existence and uniqueness.

Definition 2.3.1 (Strong Solution)**.** The strong solution of SDE (2.2) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, respect to a fixed Brownian motion B and initial condition y_0, is a continuous stochastic process $y = \{y(t) : 0 \leq t < \infty\}$ with the following properties:

(SS-1) y is adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$,

(SS-2) $\mathbb{P}[y(0) = y_0] = 1$,

(SS-3) $\mathbb{P}\left[\int_0^t f(s, y(s)) + g(s, y(s))ds < \infty\right] = 1$,

(SS-4) $y(t) = y_0 + \int_0^t f(y(s))ds + \int_0^t g(y(s))dW_s \qquad \text{a. s.}$

Now, consider SDE (2.2) where y_0 is a constant, f is a measurable $d-$vector valued function and g is a measurable $d \times m$-matrix-valued measurable function. In order to assure a unique solution we suppose the following.

Hypothesis 2.3.1. The coefficients of SDE (2.2) f, g satisfy:

(EU1) *Global Lipschitz condition.* There is a positive constant L such that

$$|f(x) - f(z)| \vee |g(x) - g(z)| \leq L|x - z|, \qquad \forall x, z \in \mathbb{R}^d.$$

(EU2) *Linear Growth condition*. There is a positive constant L such that

$$|f(x)|^2 \vee |g(x)|^2 \leq L(1 + |x|^2), \qquad \forall x \in \mathbb{R}^d.$$

Theorem 2.3.1 (Existence and uniqueness of solutions)**.** *If Hypothesis 2.3.1 holds, then exists a path-wise unique strong solution of the SDE* (2.2) *with initial condition* y_0 *on the time-interval* $[0, T]$ *and*

$$\sup_{t\in[0,T]} \mathbb{E}\left[|y(t)|^2\right] < \infty. \tag{2.3}$$

Here, path-wise uniqueness means that if $x(t)$ and $y(t)$ are two solutions of SDE (2.2), then

$$\mathbb{P}\left[\sup_{t\in[0,T]} |x(t) - y(t)| = 0\right] = 1.$$

It is worth mentioning that there exists a unique solution even when the linear growth conditions are removed, in [31], the authors have derived a existence and uniqueness result that depend on a weaker continuity condition on f and g than the Lipschitz condition. Here we enunciate the required hypothesis and two results which we will need in Chapter 4.

Hypothesis 2.3.2. The coefficients of SDE (2.2) satisfy the following:

(H-1) The functions f, g are in the class $C^1(\mathbb{R}^d)$.

(H-2) **Local, global Lipschitz condition**. For each integer n, there is a positive constant $L_f = L_f(n)$ such that

$$|f(u) - f(v)|^2 \le L_f |u - v|^2 \qquad \forall u, v \in \mathbb{R}^d, \qquad |u| \vee |v| \le n,$$

and there is a positive constant L_g such that

$$|g(u) - g(v)|^2 \le L_g |u - v|^2, \qquad \forall u, v \in \mathbb{R}^d.$$

(H-3) **Monotone condition.** There exist two positive constants α and β such that

$$\langle u, f(u) \rangle + \frac{1}{2}|g(u)|^2 \le \alpha + \beta |u|^2, \qquad \forall u \in \mathbb{R}^d. \tag{2.4}$$

Theorem 2.3.2 (Mao and Szpruch [46, Thm. 2.2])**.** *Let Hypothesis 2.3.2 holds. Then for all $y(0) = y_0 \in \mathbb{R}^d$ given, there exist a unique global solution $\{y(t)\}_{t \ge 0}$ to SDE(4.1). Moreover, the solution has the following properties for any $T > 0$,*

$$\mathbb{E}\,|y(T)|^2 < \left(|y_0|^2 + 2\alpha T\right) \exp(2\beta T),$$

and

$$\mathbb{P}\left[\tau_n \le T\right] \le \frac{\left(|y_0|^2 + 2\alpha T\right) \exp(2\beta T)}{n},$$

where n is any positive integer and $\tau_n := \inf\{t \ge 0 : |y(t)| > n\}$.

Theorem 2.3.3 (Mao [44, Thm. 2.4.1])**.** *Let $p \ge 2$ and $x_0 \in L^p(\Omega, \mathbb{R}^d)$. Assume that there exits a constant $C > 0$ such that for all $(x, t) \in \mathbb{R}^d \times [t_0, T]$,*

$$\langle x, f(x,t) \rangle + \frac{p-1}{2}|g(x,t)|^2 \le C(1 + |x|^2).$$

Then

$$\mathbb{E}|y(t)|^p \le 2^{\frac{p-2}{2}} \left(1 + \mathbb{E}|y_0|^p\right) \exp(Cpt) \quad \text{for all } t \in [0, T].$$

Lemma 2.3.1 ([31, Lem 3.2])**.** *Under Hypothesis 2.3.2, for each $p \ge 2$, there is a $C = C(p, T)$ such that*

$$\mathbb{E}\left[\sup_{0 \le t \le T} |y(t)|^p\right] \le C\left(1 + \mathbb{E}|y_0|^p\right).$$

Notice that we need to assure existence and uniqueness of the solution of SDE (2.1) in order to justify the development of a numerical approximation. Assuming that, we now propose several numerical approximations for this SDE.

2.3.1 Explicit and implicit schemes

Consider SDE (2.2) on time interval $[0,T]$, we define a time partition of the time interval $\mathcal{P}^N$ as a finite equidistant sequence of N points $t_k := kh$, for $0 \leq k \leq N$, taking the step size as $h = T/N$.

Definition 2.3.2 (discrete approximation)**.** We call a process $Y = \{Y(t), t \geq 0\}$, a discrete approximation of the solution of SDE (2.2) with step-size h over a partition $\mathcal{P}^N_{[0,T]} = \{0, h, 2h, \ldots, Nh\}$ if $Y(t_k)$ is $\mathcal{F}_{t_k}$-measurable and $Y(t_{k+1})$ can be expressed as a function of

$$Y(t_0) \ldots Y(t_k), 0, t_1, \ldots, t_k, t_{k+1}$$

and a finite number l of measurable random variables $Z_{k+1,j}$, $j = 1 \ldots l$.

We present some of the most known numerical schemes which will be useful to show the efficiency of our method (see for instance [36, 52]). Here, and in the next Chapter we will suppose Hypothesis 2.3.1.

Euler-Maruyama

The most easy implementable, popular and studied method is the *Euler-Maruyama* (EM) scheme. Given the SDE (2.2) and a time step-size h it is defined by taking

$$Y_{k+1} = Y_k + hf(Y_k) + g(Y_k)\Delta W_k, \qquad Y_0 = y_0, \tag{2.5}$$

where $\Delta W_k = W_{t_{k+1}} - W_{t_k}$. If we consider a implicit approximation for the drift coefficient, we obtain the *Backward-Euler-Maruyama* (BEM) [46], under the same notation as above, it has the recurrence:

$$Y_{k+1} = Y_k + hf(Y_{k+1}) + g(Y_k)\Delta W_k. \tag{2.6}$$

The θ-Maruyama scheme

This scheme generalizes the Euler-Maruyama algorithm using the parameter θ to weight contributions of the explicit and implicit approximations to the drift coefficient. Its recurrence is

$$Y_{k+1} = Y_k + h(1-\theta)f(Y_k) + \theta f(Y_{k+1}) + g(Y_k)\Delta W_k \qquad \theta \in [0,1]. \tag{2.7}$$

Note that if $\theta = 0$ we recover the explicit EM and if $\theta = 1$ we obtain the BEM.

Split Step Backward Euler

Also we will apply the split-step backward Euler (SSBE) method proposed by the authors in [31]. This scheme is defined by

$$Y^\star_k = Y_k + hf(Y^\star_k), \qquad Y_0 = y_0, \tag{2.8}$$
$$Y_{k+1} = Y^\star_k + g(Y^\star_k)\Delta W_k. \tag{2.9}$$

2.4 Theoretical Properties of Numerical Methods

It is always important in the construction of new algorithms to study the global discretization error and to give an estimate of the speed of convergence. Also we will study the stability of our numerical schemes. While convergence give us information about behavior of a scheme on a fixed time interval letting the time-step small, the stability analysis allow us to understand behavior of the approximation for a fixed step size when the time interval expands to infinity. For simplicity, we study these properties for a one-dimensional autonomous SDE

$$dy(t) = fy(t))dt + g(y(t))dW(t). \tag{2.10}$$

As a fist step, we suppose that Hypothesis 2.3.1 is fulfilled. But in Chapter 5 we will work under a more general setting. Let us state the classic definitions of these concepts (see e.g. [36]).

2.4.1 Strong consistency and convergence

As we mention above we analyze the global discretization error and convergence. Here, they are carried out with the study of the properties of consistency and convergence, see [36]. Here we state these concepts.

Definition 2.4.1. A time discrete approximation Y_n is strongly consistent if there is a nonnegative function $c = c(h)$ such that the following conditions hold for all fixed values $Y_n = y$, and $n = 0, 1, \ldots, N$,

1. $\lim_{h \to 0} c(h) = 0$,
2. $\mathbb{E}\left(\left|\mathbb{E}\left(\frac{Y_{n+1} - Y_n}{h}\, |\mathcal{F}_{\tau_n}\right) - f\left(Y_n\right)\right|^2\right) \leq c(h)$,
3. $\mathbb{E}\left(\frac{1}{h}\, \left|Y_{n+1} - Y_n - \mathbb{E}\left(Y_{n+1} - Y_n\, |\mathcal{F}_{\tau_n}\right) - g\left(Y_n\right) \Delta B_n\right|^2\right) \leq c(h)$.

On sake of clearness we define $n_t := \max_{n=1\ldots N} \{n : t_n \leq t\}$.

Definition 2.4.2. A time discrete approximation Y_n is strongly convergent if for the end time T is verified

$$\lim_{h \to 0} \mathbb{E}\, |y(T) - Y_{n_T}| = 0.$$

Now, we give a theorem that connects both concepts.

Theorem 2.4.1 ([36, Thm. 9.6.2])**.** *If Y_n is a strongly consistent time discrete approximation maximum step h of the solution of the SDE* (2.10) *with $Y_0 = y_0$. Then Y_n converges strongly to the solution y.*

Definition 2.4.1 (order)**.** A discrete approximation Y_k *converges strongly with order δ* at time T if there exist a positive constant C independent of the step size h, such that

$$\mathbb{E}\left[|y(T) - Y_{n_T}|\right] \leq Ch^{\delta}. \tag{2.11}$$

In addition, we say that a discrete approximation *converges strongly* with order δ *uniformly* on time if

$$\mathbb{E}\left[\sup_{1 \leq k \leq N} |y(t_k) - Y_k|\right] \leq Ch^{\delta}. \tag{2.12}$$

2.4.2 Higham-Mao-Stuart proof convergence technique

Now we discuss a technique reported by Higham, Mao, and Stuart [31] to prove strong convergence of stochastic numerical methods under non-globally Lipschitz conditions. This kind of analysis is useful whenever moment bounds can be established for the EM scheme and other method that can be shown to be "close" to it. Recently, several works have used this procedure to establish strong convergence for some particular scheme [8, 25, 32, 33, 34, 39, 46, 66], among others. To review this technique, we recall the definition of stopping time. Essentially, a stopping time provides a way to verify the first occurrence of an random event. This will be useful to justify the results presented on Chapter 4. We enunciate the formal definition and two results to assure its random meaning.

Definition 2.4.2 (Stopping Time)**.** A random variable $\tau : \Omega \to [0, \infty]$ is called an $\{\mathcal{F}_t\}$-stopping time if $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ for any $t \geq 0$.

Theorem 2.4.2. *If $\{X_t\}_{t\geq 0}$ is a progressively measurable process and τ is a stopping time, then $X_t \mathbf{1}_{\{\tau<\infty\}}$ is $\{\mathcal{F}_t\}$-measurable.*

Theorem 2.4.3. *Let $\{X_t\}_{t\geq 0}$ be and $\mathbb{R}^d$-valued continuous $\{\mathcal{F}_t\}$-adapted process and $D \subset \mathbb{R}^d$ an open set. Then $\tau := \inf\{t \geq 0 : X_t \notin D\}$ is an $\{\mathcal{F}_t\}$-stopping time.*

Now consider two conveniently versions for the continuous extension of the EM scheme,

$$\begin{aligned} \overline{Y}(t) &:= Y_{\eta(t)} + (t - t_{\eta(t)}) f(Y_{\eta(t)}) + g(Y_{\eta(t)})(W(t) - W_{\eta(t)}), \\ \eta(t) &:= k, \text{ for } t \in [t_k, t_{k+1}), \end{aligned} \tag{2.13}$$

and

$$\overline{Y}(t) := Y_0 + \int_0^t f(Y_{\eta(s)}) ds + \int_0^t g(Y_{\eta(s)}) dW(s).$$

So, with this notation we have $\overline{Y}(t_k) = Y_k$, see Figure 2.1. Using the continuous extension (2.13) and the uniform mean square norm, the authors use a stronger version of the ms-error

$$\mathbb{E}\left[\sup_{0\leq t\leq t} |y(t) - \overline{Y}(t)|^2\right].$$

In order to prove strong convergence of the EM method, the following assumptions are required.

Assumption 2.4.1. For each $R > 0$ there is a positive constant C_R, depending only on R, such that

$$|f(x) - f(y)|^2 \vee |g(x) - g(y)|^2 \leq C_R |x - y|^2, \quad \forall x, y \in \mathbb{R}^d \text{ with } |x| \vee |y| \leq R. \tag{2.14}$$

And for some $p > 2$, there is a constant A such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t)|^p\right] \vee \mathbb{E}\left[\sup_{0\leq t\leq T} |y(t)|^p\right] \leq A. \tag{2.15}$$

In [31], the authors prove that the Assumption 2.4.1 is sufficient to ensure strong convergence for the EM scheme, namely:

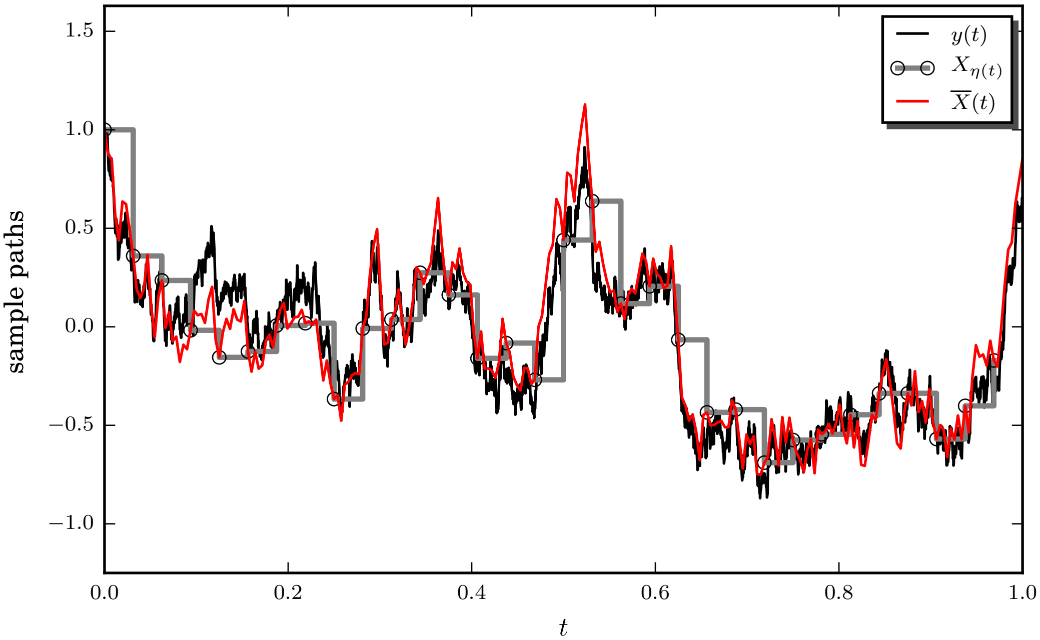

Figure 2.1: The red line represents the continuous extension of the EM scheme. The continuous gray line is the $Y_{\eta(t)}$ process defined in (2.5) and black line denotes the exact solution of SDE (2.2).

Theorem 2.4.4 ([31, Thm 2.2]). *Under Assumption 2.4.1, the EM scheme* (2.5) *with continuous extension* (2.13) *satisfies*

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t) - y(t)|^2\right] = 0. \tag{2.16}$$

Applying this result, the strong convergence of an implicit split-step variant of the EM, the SSEM method is proved . Their technique consist in proving each assertion of the following steps.

Step 1: The SSEM for SDE (4.1) is equivalent to the EM for the following conveniently SDE

$$dy_h(t) = f_h(y_h(t))dt + g_h(y_h(t))dW(t). \tag{2.17}$$

Step 2: The solution of the modified SDE (2.17) has bounded moments and it is "close" to y the sense of the uniform mean square norm $\mathbb{E}\left[\sup_{0\leq t\leq T}|\cdot|^2\right]$.

Step 3: Show that the SSEM method for the SDE (4.1) has bounded moments.

Step 4: There is a continuous extension of the SSEM, $\overline{Z}(t)$, with bounded moments.

Step 5: Use the above steps and Theorem 2.4.4 to conclude that

$$\lim_{h\to 0}\left\{\mathbb{E}\left[\sup_{0\leq t\leq T}|y_h(t) - y(t)|^2\right] + \mathbb{E}\left[\sup_{0\leq t\leq T}|\overline{Z}(t) - y_h(t)|^2\right]\right\} = 0. \tag{2.18}$$

In Chapter 4, we will use this technique. Moreover, if we are interested in simulating the solution of the SDE (2.2) for large periods of time, we need to use stable methods. We can interpret the stability of a numerical scheme, in some sense, as its capacity to preserve the dynamical structure of the solution in that sense. Here we recall the topics that we will work in the next chapter.

2.4.3 Numerical Stability

With a numerical stability one obtain the step sizes for which a method reproduces behavior of the solution for a SDE. Therefore, it is important to know some qualitative information about the solution, for example: if all solution paths tend to a fixed point, or if stay on a bounded set or reach an absorbent process. Usually the first step in this direction is a linear stability analysis. This study mimics the deterministic context, which is based in the following steps:

Step 1: Expand in Taylor series around a fixed point the right hand side of a nonlinear ordinary differential equation $x'(t) = f(t, x)$.

Step 2: Take a linear system with the Jacobian matrix of f evaluated at the equilibrium $x'(t) = Ax(t)$.

Step 3: Diagonalize to decouples the linear system and study equations of the form $x'(t) = \lambda x(t), \lambda \in \mathbb{C}$.

If all eigenvalues of A are different from zero, then the theorem of Hartman (see [27]) justifies the use of this last equation to study the behavior around a sufficient small neighborhood. So, one seek conditions to assure that the numerical methods preserves the dynamics of underlying test.

In stochastic numerics, the linearization procedure is analogous but here the linear SDE with multiplicative noise is the benchmark test. The advantage of this linear SDE is that has the same unique fixed point as its deterministic analogous, the origin [29] . Another benchmark equation is the linear SDE with additive noise. However, for these model the concepts of numerical stability were unclear. The first works with this test[6, 28, 52], differs about the meaning of fixed point and stability. Recently, the works of De la Cruz Cancino, Biscay, Jimenez, Carbonell, and Ozaki [18] and Buckwar, Riedler, and Kloeden [12] analyze the additive linear SDE using the theory of random dynamical systems, which in our opinion clarifies this issue.

Naturally the nonlinear case, is even more complex. Although Lyapunov theory is the usual approach in applications [35], a more general novel approach based on the theory of random dynamical systems [4] is a current topic of interest. In the following we provide this notions.

Linear Stability

Multiplicative noise

Consider the scalar linear SDE

$$dy(t) = \lambda y(t)dt + \xi y(t)dW(t), \quad X_0 = x_0, \quad \lambda, \xi \in \mathbb{C}. \tag{2.19}$$

The solutions of this SDE have the following property

$$\lim_{t\to\infty} \mathbb{E}\left[|y(t)|^2\right] = 0 \Leftrightarrow \mathrm{Re}(\lambda) + \frac{1}{2}|\xi|^2 < 0. \tag{2.20}$$

A solution that satisfies the previous limit is a *mean-square stable* solution. Note that for $\xi = 0$ we have, $\mathrm{Re}(\lambda) < 0$, which is the stability condition for the deterministic case.

Applying the EM method (2.5) to (2.19), we obtain

$$Y_{k+1} = \left(1 + h\lambda + \sqrt{h}\xi V_k\right) Y_k, \tag{2.21}$$

where each V_k is an independent $\mathcal{N}(0,1)$ random variable. In order to study the stability properties of the EM scheme, we must study the long time behavior of random variables of the form (2.21). Analogously, we will say that sequence (2.21) is mean-square stable if $\lim_{k\to\infty} \mathbb{E}\left[|Y_k|^2\right] = 0$. Note that the EM scheme depends upon the problem parameters λ and ξ, and the method parameter h. Then for a particular choice of parameters, we will say that the EM scheme is mean-square stable if it produces a mean-square stable sequence. Our interest lies in finding the parameter values for which the EM method is stable, and comparing results with the region $\mathrm{Re}(\lambda) + \frac{1}{2}|\xi|^2 < 0$ in (2.20) for the underlying SDE (see Figure 2.2). There is the following result.

Theorem 2.4.5. *Consider the EM method for the linear scalar SDE* (2.19)*. If the parameters* λ*,* ξ*, and the step size* h *satisfies*

$$\mathrm{Re}(\lambda) + \frac{1}{2}\left(|\xi|^2 + h|\lambda|^2\right) < 0.$$

Then the EM solution is mean square stable*.*

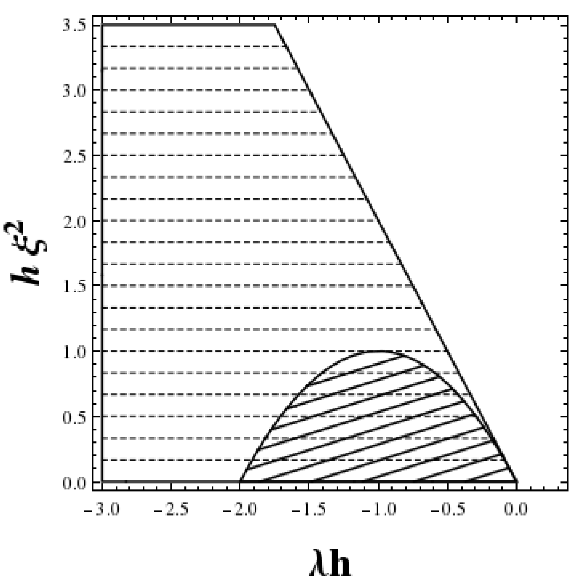

Figure 2.2: Mean square regions of stability. The horizontal lines represents the stability region of SDE (2.19) and diagonal lines for the EM solution.

Additive noise

Here we study the additive linear SDE:

$$dy(t) = \lambda y(t)dt + \xi dW(t), \qquad y_0 = y_(t_0), \qquad \lambda, \xi \in \mathbb{R}. \tag{2.22}$$

where λ, $\xi \in \mathbb{C}$ and X_{t_0} is the initial value of the process at time t_0. Equation (2.22) has the following exact solution:

$$y(t) = \exp(\lambda(t - t_0))y(t_0) + \xi \exp(\lambda t) \int_{t_0}^{t} \exp(-\lambda s)dW(s), \qquad t \geq t_0. \tag{2.23}$$

The stochastic process $y(t)$ defined in (2.23) is known as the *Ornstein-Uhlenbeck*'s (OU) process. According to [28], the OU process is *asymptotically mean stable* if $\lim_{t\to\infty} \mathbb{E}y(t) = 0$ and is *asymptotically mean square stable* if $\lim\limits_{t\to\infty} \mathbb{E}\left|y(t)\right|^2 = -\xi/2Re(\lambda)$. Both limits are verified if $\lambda < 0$. Analogous stability properties are given for stochastic difference equations with additive noise [58]. Now, if we consider $\lambda < 0$ then the OU solution (3.24) does not convergence as t tends to infinity but has the following pullback limit:

$$\lim_{t_0\to-\infty} y(t) = \widehat{O}_t := \exp(\lambda t) \int_{-\infty}^{t} \exp(-\lambda s)dW(s), \tag{2.24}$$

$W(t)$ is now defined for all $t \in \mathbb{R}$, see [4, 37]. Furthermore, the process (3.26) is a stationary solution of the additive linear SDE which attracts all other solutions in forward time and path-wise sense. Moreover, it is a finite process for all $t \geq T_{D(\omega)}$ ($\omega \in \Omega$) for appropriate families $D(\omega)$ of bounded sets of initial conditions, see [56]. Therefore, we can evaluate the numerical stability of a given stochastic method by examine if this scheme reproduce the pullback asymptotic behavior. For example, the explicit EM scheme for (2.22)

$$Y_{k+1} = (1 + \lambda h)Y_n + \xi\Delta W_n,$$

given a initial value Y_{k_0}, has the form

$$Y_{k+1} = (1 + \lambda h)^{k-k_0} Y_{k_0} + \xi \sum_{j=k_0}^{k-1} (1 + \lambda h)^{k-1-j} \Delta W_j.$$

So, the path-wise pullback limit (taking $k_0 \to \infty$ with k held fixed and $Y_{k_0} = Y_0$ for all Y_{k_0} and constant time step h) exists, provided that $0 < h < 2/(-\lambda)$, $\lambda < 0$, and is given by

$$\widehat{O}_k^{(h)} := \xi \sum_{j=-\infty}^{k} (1 + \lambda h)^{1-k-j} \Delta W_j,$$

for more details see the work of Buckwar, Riedler, and Kloeden [12].

Non-Linear Stability

Now we discuss the nonlinear case for multiplicative and additive noise.

Multiplicative Noise

We start with a notion of stability which emulates the continuity respect to initial conditions of deterministic ODEs.

Definition 2.4.3 (Baker and Buckwar [7]). Let Y_n and $\widehat{Y}_n$ two different numerical recurrences with corresponding initial process Y_0 and $\widehat{Y}_0$. We shall say that a discrete time, Y is *numerically zero-stable in quadratic mean-square sense* if given $\epsilon > 0$, there are positive constants h_0 and $\delta = \delta(\epsilon, h_0)$ such that for all $h \in (0, h_0)$ and positive integers $n \leq T/h$ whenever $\mathbb{E}\left|Y_0 - \widehat{Y}_0\right|^2 < \delta$ then

$$\rho_n := \mathbb{E}\left|Y_n - \widehat{Y}_n\right|^2 < \epsilon. \tag{2.25}$$

If the method is stable and $\rho_n \to 0$ when $n \to \infty$, then the method is *asymptotically zero-stable in the quadratic mean-square sense*.

Also, in [7] provides a result to characterizes this type of stability. Here we enunciated it for the EM.

Theorem 2.4.6 ([7, Thm. 4]). *Let C_1, C_2 and C_3 generic positive constants which not depends on h and V a $\mathcal{N}(0,1)$ random variable. If the coefficients of SDE (2.2) satisfies the estimates*

$$\left|\mathbb{E}\left[f(x)h + g(x)\sqrt{h}V - \left(f(x')h + g(x')\sqrt{h}V\right)\right]\right| \leq C_1 h\left(|x - x'|\right),$$
$$\mathbb{E}\left[\left|hf(x) + g(x)\sqrt{h}V - \left(f(x')h + g(x')\sqrt{h}V\right)\right|^2\right] \leq C_2 h\left(|x - x'|\right),$$

then the EM method (2.5) for (2.2) is zero-stable in the quadratic mean-square sense.

Additive noise

Nonlinear differential equations have more complex dynamics than the linear case and the same occurs for the finite difference equations. So, Caraballo and Kloeden in [15] extend the nonlinear stability theory of the deterministic numerical analysis given in [37] to the stochastic case. They propose and justify the use of the following SDE as a test equation with additive noise:

$$dy(t) = \left(Ay(ts) + f(y(t))\right)dt + \xi dW(t), \tag{2.26}$$

where A is a $d \times d$ stiff matrix and function $f : \mathbb{R}^d \to \mathbb{R}^d$ is a nonlinear and non-stiff function that satisfies, a *contractive one-sided Lipschitz* condition with constant $L_1 > 0$

$$\langle u - v, f(u) - f(v)\rangle \leq -L_1|u - v|^2 \qquad \forall u, v \in \mathbb{R}^d. \tag{2.27}$$

Also the authors give sufficient conditions to assure an asymptotically stable stochastic stationary solution of (2.26). In this context they establish the following result for the stability of θ-EM scheme.

Theorem 2.4.7 ([12, Thm. 3.1]). *Suppose that the drift coefficient satisfies a contractive one-sided Lipschitz condition, and that the vector field f satisfies a globally Lipschitz condition . Then the θ-EM scheme has a unique stochastic stationary solution which is pathwise asymptotically stable for all step sizes $h > 0$ if*

$$(1-\theta)(|A|+L) < -\theta(\mu[A] - L_1), \qquad \mu[A] = \lim_{\delta \to 0^+} \frac{(|Id + \delta A|)}{\delta},$$

where L refers to the Lipschitz condition and L_1 the to contractive one-sided Lipschitz condition.

The following chapter shows adaptations of these results for the construction of a new method, the Steklov method.

Chapter 3

Steklov method for scalar SDEs with Globally Lipschitz coefficients

In this chapter, we focus on the following scalar stochastic differential equation

$$dy(t) = f(t, y(t))dt + g(t, y(t))dW(t), \qquad y_0 = y(0), \tag{3.1}$$

considering the drift term as $f(t, y(t)) = f_1(t)f_2(y(t))$. Given this functional form of f, we propose an exact explicit algorithm for solving the deterministic equation linked to (3.1); details of this exact differentiation are given in [50]. So, the main characteristic of this new method is that it preserves qualitative features of the deterministic solution associated to the SDE. Next, we prove strong consistency, convergence and study the linear stability of the proposed method using properties of the *Steklov mean* [64]. Moreover, we analyze the nonlinear stability of the Steklov stochastic approximation specifically the asymptotic mean-square stability in the multiplicative case and the path-wise stability in the additive case. Finally, we show the efficiency of the new scheme in numerical problems with harsh requirements of stability like the logistic equation for the multiplicative case and the Langevin equation with a particular potential for the additive case.

In section 3.1, we construct the explicit Steklov method for the SDE (3.1) and show its development with some examples. In section 3.2, we prove strong consistency and convergence of the new explicit method. In section 3.3, sufficient conditions for the asymptotic mean and mean-square stability are given for both additive and multiplicative cases. A nonlinear stability analysis is carried out in section 3.4, where we prove that the explicit Steklov approximation is asymptoticly stable in square mean sense in the multiplicative case and it is path-wise stable under certain conditions in the additive case. In section 3.5, we test the Steklov method for the stochastic logistic equation in the multiplicative case and and for the Langevin equation in Brownian dynamics. Also, we show numerical evidence that the Steklov method is successful with step sizes significantly large reaching larger time scales of simulation. Finally, we give some conclusions.

3.1 Steklov Method

Under these considerations we construct the Steklov numerical scheme for the SDE (3.1) based on its integral formulation:

$$y(t) = X_0 + \int_0^t f(s, y(s))ds + \int_0^t g(s, y(s))dW(s), \quad t \in [0, T], \quad Y_0 = y_0, \tag{3.2}$$

where $y(t)$ denotes the value of the process at time t with initial value X_0. First we discretize the time domain with a uniform step size h such that $t_n = nh$ for $n = 0, 1, 2, \ldots, N$ and denote by Y_n the numerical solution at t_n. Now we approximate the stochastic integral of (3.2) with the usual form:

$$\int_{t_n}^{t_{n+1}} g(s, y(s))dW(s) \approx g(t_n, Y_n)\Delta W_n, \qquad \Delta W_n := (W(t_{n+1}) - W(t_n)) = \sqrt{h}V_n, \tag{3.3}$$

where $W(t_{n+1}) - W(t_n)$ is a discrete standard Brownian motion such that $V_n \sim \mathcal{N}(0, 1)$. We can obtain different schemes depending on the numerical integration used for the first integral of (3.2). For example, if we choose the Euler's approximation:

$$\int_{t_n}^{t_{n+1}} f(s, y(s))ds \approx f(t_n, Y_n)(t_{n+1} - t_n), \tag{3.4}$$

then we obtain the Euler-Maruyama scheme as follows:

$$Y_{n+1} = Y_n + f(t_n, Y_n)h + g(t_n, Y_n)\Delta W_n, \quad n = 1, \ldots, N-1, \qquad Y_0 = x_0. \tag{3.5}$$

Assuming that we can rewrite the function f as $f(t, y(t)) = f_1(t)f_2(y(t))$, we propose an alternative approach to (3.4) based on the construction of an exact discretization for the deterministic differential equation associated to (3.1):

$$\frac{dx}{dt} = f_1(t)f_2(x), \qquad x(0) = x_0. \tag{3.6}$$

Integrating this equation in the interval $[t_n, t_{n+1})$ and using the Steklov mean [50], we have

$$\int_{t_n}^{t_{n+1}} f_1(s)f_2(x)ds \approx \phi_1(t_n)\phi_2(y_n, y_{n+1})(t_{n+1} - t_n), \tag{3.7}$$

where

$$\phi_1(t_n) = \frac{1}{t_{n+1} - t_n}\int_{t_n}^{t_{n+1}} f_1(s)ds \qquad \text{and} \qquad \phi_2(y_n, y_{n+1}) = \left(\frac{1}{y_{n+1} - y_n}\int_{y_n}^{y_{n+1}} \frac{du}{f_2(u)}\right)^{-1}.$$

Thus, the exact scheme for (3.6) is given as:

$$y_{n+1} - y_n = \phi_1(t_n)\phi_2(y_n, y_{n+1})h, \qquad y_0 = x_0. \tag{3.8}$$

Notice that it is an implicit algorithm, so in order to get an explicit formulation we define the following function:

$$H(x) := \int_0^x \frac{du}{f_2(u)}, \tag{3.9}$$

and the exact scheme (3.8) is written as follows:

$$y_{n+1} - y_n = \phi_1(t_n)\,\frac{(y_{n+1} - y_n)}{H(y_{n+1}) - H(y_n)}\,h.$$

Now assuming the existence of the function H^{-1}, we can give the following compact formulation of the scheme (3.8):

$$y_{n+1} = \Psi_h(t_n, y_n), \qquad \Psi_h(t_n, y_n) := H^{-1}[H(y_n) + h\phi_1(t_n)]. \tag{3.10}$$

Finally, the numerical method for the SDE (3.1) is proposed as follows:

$$Y_{n+1} = \Psi_h(t_n, Y_n) + g(t_n, Y_n)\Delta W_n, \quad n = 1, \ldots, N-1, \quad Y_0 = x_0, \tag{3.11}$$

and we named it *Steklov* scheme due to the origin of its construction. An important feature of this new stochastic scheme (3.11) is that it preserves qualitative properties of the deterministic solution if the noise term does not become dominant. Notice that the main step to develop Steklov approximations is to obtain the function Ψ_h, so forthcoming examples show the procedure to construct this function. We choose as examples some SDEs which appear in important applications and for which harsh conditions of stability are required for their numerical approximations.

Example 3.1.1. We consider the linear Itô equation

$$dy(t) = \lambda y(t)dt + \xi y(t)dW(t), \qquad Y_0 = y_0, \tag{3.12}$$

where λ, $\xi \in \mathbb{C}$ and $x_0 \neq 0$ with probability one. We construct the function Ψ_h for (3.12) using its integral form and approximating the deterministic integral by (3.7) as:

$$\int_{y_n}^{y_{n+1}} \lambda u du \approx \left(\frac{1}{\lambda(y_{n+1} - y_n)} \ln\left(\frac{y_{n+1}}{y_n}\right) \right)^{-1} h, \quad n = 1, \ldots, N-1.$$

In order to obtain a explicit Steklov approximation, we consider the exact finite difference algorithm associated to $dx/dt = \lambda x$:

$$y_{n+1} - y_n = \lambda h \frac{(y_{n+1} - y_n)}{\ln\left(\frac{y_{n+1}}{y_n}\right)}.$$

By algebraic manipulations, the previous equation is equivalent to the equation

$$y_{n+1} = \exp(\lambda h) y_n$$

and the explicit function Ψ_h for the linear SDE is

$$\Psi_h(y) = \exp(\lambda h) y. \tag{3.13}$$

Notice that we obtain the same function Ψ_h that for an additive linear SDE.

Example 3.1.2. Now we consider the logistic growth SDE proposed by Schurz in [61]:

$$dy(t) = \lambda y(t)(K - y(t))dt + \xi y(t)^{\alpha} |K - y(t)|^{\beta} dW(t), \tag{3.14}$$

where λ, K, α, β and ξ are nonnegative real coefficients. So using (3.7) we approximate the deterministic integral of the integral form of (3.14) as:

$$\int_{y_n}^{y_{n+1}} \lambda u(K-u) du \approx \frac{y_{n+1} - y_n}{\frac{1}{\lambda K} \ln\left(\frac{y_{n+1}(K - y_n)}{y_n(K - y_{n+1})}\right)} h, \quad n = 1, \ldots, N-1.$$

Analogously to the previous example, we develop the Steklov function from the exact finite difference equation associated to the deterministic counterpart of (3.14), obtaining:

$$\Psi_h(y) = \frac{Ky}{K - y + \exp(\lambda K h)}. \tag{3.15}$$

Example 3.1.3. As a final example, we consider the following SDE with additive noise:

$$dy(t) = -y(t)^3 dt + \xi dW(t), \tag{3.16}$$

where ξ is a positive coefficient. Using (3.7), we get

$$\int_{y_n}^{y_{n+1}} -u^3 du \approx 2 \frac{(y_{n+1} y_n)^2}{y_{n+1} + y_n} h, \quad n = 1, ..N-1.$$

By algebraic manipulations on the associated deterministic exact algorithm, we obtain the following Steklov function

$$\Psi_h(y) = \frac{y}{\sqrt{1+2y^2h}}. \tag{3.17}$$

In the section of numerical results, we will show the behavior of the new scheme (3.11) in these three examples and compare it with standard methods. As a next step, we prove important qualitative properties of the explicit Steklov method.

3.2 Strong consistency and convergence

It is always important in the construction of new algorithms to study the global discretization error and give an estimation of the speed of convergence. Here, they are carried out with the analysis of the properties of consistency and convergence, see [36]. For simplicity, we study these properties for a one-dimensional autonomous SDE

$$dy(t) = f(y(t))dt + g(y(t))dW(t), \tag{3.18}$$

satisfying the necessary conditions of existence and uniqueness of solution.

So, considering Definition 2.4.1 and Theorem 2.4.1 we prove convergence of the explicit Steklov approximation via strong consistency.

Theorem 3.2.1. *A time discrete approximation of SDE* (3.18) *generated with the explicit Steklov method* (3.11) *is strongly convergent.*

Proof. We substitute the Steklov recurrence (3.11) in the left hand side of the inequality (2). Given that F, G and Ψ_h are continuous functions adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and using standard conditional expectation properties [69], it follows that:

$$\begin{aligned}\mathbb{E}\left(\left|\mathbb{E}\left(\frac{Y_{n+1}-Y_n}{h}\,|\mathcal{F}_{t_n}\right) - f\left(Y_n\right)\right|^2\right) &= \mathbb{E}\left(\left|\frac{\Psi_h(Y_n)-Y_n}{h} - f(Y_n)\right|^2\right) \\ &= \mathbb{E}\left(\left|\frac{H^{-1}(H(Y_n)+h) - H^{-1}(H(Y_n))}{h} - f(Y_n)\right|^2\right).\end{aligned}$$

Since the functions F and Ψ_h are Lipschitz we can apply the *Inverse Function* theorem and from (3.9), we have that

$$(H^{-1})'(H(Y_n)) = f(Y_n),$$

then given any $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that whenever $0 < h < \delta(\epsilon)$ then

$$\left|\frac{H^{-1}(H(Y_n)+h) - H^{-1}(H(Y_n))}{h} - f(Y_n)\right| < \epsilon.$$

So, taking $\epsilon = \sqrt{h}$ and $c(h) = h(\delta(\sqrt{h}))^2$, the condition (ii) is satisfied. With an analogous procedure, the condition (iii) is verified and the condition (i) follows straightforward from the definition of $c(h)$. □

Thus, we can ensure that the explicit Steklov scheme converges on bounded time intervals [56]. However, if we are interested in simulating the solution of the SDE (3.1) for large periods of time, we need to use stable methods. We can interpret the stability of a numerical method, in some sense, as its capacity to preserve the dynamical structure of the solution in that sense. In the next two sections, we study the stability of the explicit Steklov method (3.11) in mean and mean square sense and extend this study in a path-wise sense for the additive case.

3.3 Linear Stability

We start the stability analysis for the linear case since the stability conditions for the solution of the linear SDE in both additive and multiplicative cases are well known. So, we first recall these conditions for the continuous case and later obtain sufficient conditions to ensure stability and asymptotic stability in mean and mean square for the explicit Steklov method (3.11). Moreover in the additive case, we analyze the stability in a path-wise sense based on the work of Buckwar et al. [12].

3.3.1 Multiplicative noise

For the linear multiplicative SDE (3.12), its zero equilibrium solution is called *mean stable* if $\lim_{t\to\infty} \mathbb{E}y(t) = 0$, and it is said to be *mean square stable* if $\lim_{t\to\infty} \mathbb{E}\left|y(t)\right|^2 = 0$. Then the zero solution of (3.12) is mean stable if $\lambda < 0$ and it is mean square stable if $Re(\lambda) + \frac{1}{2}|\xi|^2 < 0$, see [29]. In order to obtain the explicit Steklov approximation (3.11) for equation (3.12), we use the function Ψ_h defined in (3.13) so the linear Steklov discretization is written as follows:

$$Y_{n+1} = \exp(\lambda h)Y_n + \xi Y_n \Delta W_n. \tag{3.19}$$

Similarly, we say that the method (3.19) is *mean stable* if $\lim_{n\to\infty} \mathbb{E}Y_n = 0$, and we called it *mean square stable* if $\lim_{n\to\infty} \mathbb{E}\left|Y_n\right|^2 = 0$. Moreover, a stochastic numerical method is $\mathcal{A}$*-stable* in some sense, if it is stable for all step size h when its associated continuous SDE is stable in the same sense.

Proposition 3.3.1. *Let $\lambda < 0$, then the explicit Steklov method (3.19) for the SDE (3.12) is $\mathcal{A}$-stable in mean. Moreover, it is mean square stable if*

$$\exp\left(2Re(\lambda h)\right) + |\xi|^2 h < 1. \tag{3.20}$$

Proof. Denoting by $p = \exp(\lambda h)$ and $q = \xi\sqrt{h}$, we can rewrite the Steklov method (3.19) as

$$Y_{n+1} = (p + qV_n)Y_n. \tag{3.21}$$

Taking expectation in (3.21) and iterating this recurrence until the initial step, we obtain

$$\mathbb{E}Y_n = (p)^{n+1}\mathbb{E}Y_0, \tag{3.22}$$

thus the limit of the sequence (3.22) as n approaches infinity is zero for $\lambda < 0$. Now, applying square modulus to (3.21) and carrying out an analogous procedure, it follows that:

$$\mathbb{E}\left|Y_n^h\right|^2 = (|p|^2+|q|^2)^{n+1}\mathbb{E}\left|Y_0^h\right|^2.$$

Therefore the sequence $\mathbb{E}\left|Y_n^h\right|^2$ approaches to zero as n tends to infinity if and only if $|p|^2+|q|^2 < 1$. □

In Figure 3.1, we show a comparison between the mean square stability region of the zero solution for the linear SDE and the associated explicit Steklov and Euler-Maruyama approximations [30].

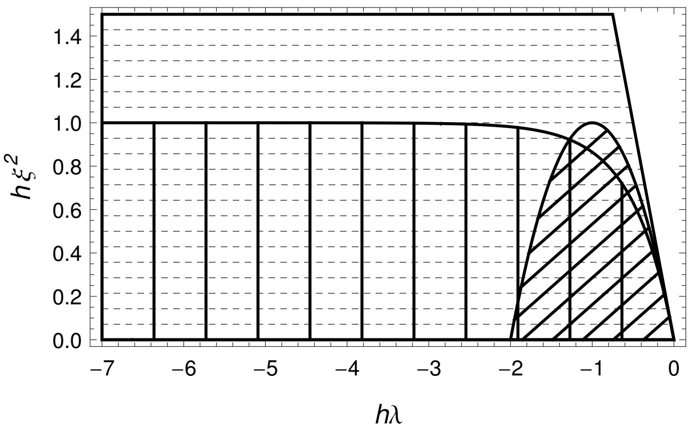

Figure 3.1: Mean square stability regions: horizontal lines represent the region for the linear SDE (3.12), the vertical lines form the explicit Steklov region and the diagonal lines draw the Euler-Maruyama region.

3.3.2 Additive noise

Here we study the additive linear SDE:

$$dy(t) = \lambda y(t)dt + \xi dW(t), \qquad X_{t_0} = x_{t_0}. \tag{3.23}$$

where λ, $\xi \in \mathbb{C}$ and X_{t_0} is the initial value of the process at time t_0. Equation (3.23) has the following exact solution:

$$y(t) = \exp(\lambda(t - t_0))y(t_0) + \xi \exp(\lambda t) \int_{t_0}^{t} \exp(-\lambda s)dW(s), \qquad t \geq t_0. \tag{3.24}$$

The stochastic process $y(t)$ defined in (3.24) is known as the *Ornstein-Uhlenbeck*'s (OU) process. According to [28], the OU process is *asymptotically mean stable* if $\lim_{t\to\infty} \mathbb{E}y(t) = 0$ and is *asymptotically mean square stable* if $\lim\limits_{t\to\infty} \mathbb{E}\,|y(t)|^2 = -\xi/2Re(\lambda)$. Both limits are verified if $\lambda < 0$. Now, the explicit Steklov recurrence to solve additive linear SDE is

$$Y_{n+1} = \exp(\lambda h)Y_n + \xi \Delta W_n. \tag{3.25}$$

Analogous stability properties are given for stochastic difference equations with additive noise [58]. Next, we prove *mean-square consistency* for the explicit Steklov, that is,

$$\lim_{h\to 0} \left(\lim_{n\to\infty} \mathbb{E}\,|Y_n|^2 \right) = -\xi/2Re(\lambda).$$

Proposition 3.3.2. *Let $\lambda < 0$, the explicit Steklov method (3.25) for the additive linear SDE (3.23) is $\mathcal{A}$-stable in mean and mean-square consistent.*

Proof. Taking the expected value of (3.25) and iterating backwards this recurrence we obtain the identity (3.22), so the $\mathcal{A}$-stability in mean is verified for $\lambda < 0$. Now, taking the mean square of the recurrence (3.25) and after some algebraic manipulations we get

$$\begin{aligned}\mathbb{E}\,|Y_{n+1}|^2 &= \exp(2Re(\lambda)h)\mathbb{E}\,|Y_n|^2 + |\xi|^2 h \\ &= \mathbb{E}\,|Y_0|^2 + |\xi|^2 h\{1 + \cdots + \exp(2nRe(\lambda)h)\}\mathbb{E}\,|Y_{n+1}|^2 \\ &= \exp(2nRe(\lambda)h)\mathbb{E}\,|Y_0|^2 + \xi^2 h \frac{[\exp(2Re(\lambda)h)]^{n+1} - 1}{\exp(2Re(\lambda)h) - 1}.\end{aligned}$$

Given that $\lambda < 0$

$$\lim_{\substack{n\to\infty \\ h\to 0}} \mathbb{E}\,|Y_{n+1}|^2 = \lim_{\substack{n\to\infty \\ h\to 0}} \frac{-|\xi|^2 h}{\exp(2Re(\lambda)h) - 1} = -\frac{|\xi|^2}{2Re(\lambda)}.$$

□

So far we have analyzed the asymptotic behavior of the forward motion for the explicit Steklov method (3.25). Now, if we consider $\lambda < 0$ then the OU solution (3.24) does not convergence as t tends to infinity but has the following pullback limit:

$$\lim_{t_0 \to -\infty} y(t) = \widehat{O}_t := \exp(\lambda t) \int_{-\infty}^{t} \exp(-\lambda s) dW(s), \tag{3.26}$$

B_t is now defined for all $t \in \mathbb{R}$, see [4, 37]. Furthermore, the process (3.26) is a stationary solution of the additive linear SDE which attracts all other solutions in forward time and path-wise sense. Moreover, it is a finite process for all $t \geq T_{D(\omega)}$ ($\omega \in \Omega$) for appropriate families $D(\omega)$ of bounded sets of initial conditions, see [56]. Therefore, a study of the pullback asymptotic behavior for the Steklov stochastic method (3.25) is important in the additive case and in the next subsection we carry it out based on Caraballo and Kloeden's work [15].

Path-wise linear stability

Here we obtain a stationary discrete process $\widehat{O}_n^{(h)}$ for the linear explicit Steklov and prove that converges to the continuous process (3.26).

Proposition 3.3.3. *Let $\lambda < 0$, the explicit Steklov method* (3.25) *for the additive linear SDE* (3.23) *has the following attractor:*

$$\widehat{O}_n^{(h)} := \xi \sum_{j=-\infty}^{n-1} \exp(\lambda h(n-1-j))\Delta W_j, \tag{3.27}$$

for any positive step size h. Moreover, it converges (path-wise) to the Ornstein-Uhlenbeck's process (3.26).

Proof. We consider a recurrence given by the Steklov method (3.25) and iterate it backwards, obtaining the explicit numerical solution

$$Y_n = \exp(\lambda h(n - n_0)) + \xi \sum_{j=n_0}^{n-1} \exp(\lambda h(n-1-j))\Delta W_j, \tag{3.28}$$

where n_0 is the initial point of this recurrence. Taking the path-wise pullback limit of Y_n given in (3.28), i.e. $n_0 \to -\infty$ for each n fixed, we get

$$\begin{aligned} \widehat{O}_n^{(h)} &:= \lim_{n_0 \to -\infty} Y_n \\ &= \xi \sum_{j=-\infty}^{n-1} \exp(\lambda h(n-1-j))\Delta W_j. \end{aligned}$$

Now, we take other explicit Steklov recurrence $\widehat{Y}_n$ and subtract it from the recurrence (3.28). It follows that

$$Y_n - \widehat{Y}_n = \exp(\lambda h(n-n_0))(Y_{n_0} - \widehat{Y}_{n_0}).$$

For any fixed n_0 letting $n \to \infty$ we deduce that $Y_n - \widehat{Y}_n \to Y_{n_0} - \widehat{Y}_{n_0}$. So replacing $\widehat{Y}_n$ by the discrete process (3.27), we have that this process attracts all explicit Steklov approximations forwards in time in the path-wise sense. Furthermore, notice that as $h \to 0$ then the series $\widehat{O}_0^h$ approaches $\widehat{O}_0$ and hence, for each n. □

3.4 Nonlinear Stability

To continue the stability analysis of the explicit Steklov method, we now discuss the nonlinear case since a linear stable numerical stochastic method does not imply that is stable under same conditions for any nonlinear problem. So, we study sufficient conditions for the nonlinear stability of the explicit stochastic method (3.11) applied on the autonomous SDE (3.18) in both multiplicative and additive cases.

3.4.1 Multiplicative Noise

Here we prove the nonlinear asymptotic stability in a quadratic mean-square sense for the Steklov approximation.

Definition 3.4.1 (Baker and Buckwar [7]). Let Y_n and $\widehat{Y}_n$ two different numerical recurrences with corresponding initial process Y_0 and $\widehat{Y}_0$. We shall say that a discrete time, Y is numerically zero-stable in quadratic mean-square sense if given $\epsilon > 0$, there are positive constants h_0 and $\delta = \delta(\epsilon, h_0)$ such that for all $h \in (0, h_0)$ and positive integers $n \leq T/h$ whenever $\mathbb{E}\left|Y_0 - \widehat{Y}_0\right|^2 < \delta$ then

$$\rho_n := \mathbb{E}\left|Y_n - \widehat{Y}_n\right|^2 < \epsilon. \tag{3.29}$$

If the method is stable and $\rho_n \to 0$ when $n \to \infty$, then the method is asymptotically zero-stable in the quadratic mean-square sense.

In order to prove that the Steklov method satisfies the definition 3.4.1, we will follow the idea of the proof given in [7, Thm. 4].

Theorem 3.4.1. *If the functions Ψ_h and G of the Steklov method (3.11) are Lipschitz with constant L, then the Steklov method for the multiplicative SDE (3.18) is zero-stable in quadratic mean square sense. In addition, if $L < 1$ then the Steklov method is asymptotically zero-stable stable in quadratic mean-square sense.*

Proof. Given two Steklov sequences Y_n and $\widehat{Y}_n$ we have

$$\begin{aligned}\left(Y_{n+1}-\widehat{Y}_{n+1}\right)^2 \leq & \left(\Psi_h(Y_n)-\Psi_h(\widehat{Y}_n)\right)^2 \\ & +2\left(\Psi_h(Y_n)-\Psi_h(\widehat{Y}_n)\right)\left(G(Y_n)-G(\widehat{Y}_n)\right)\Delta W_n \\ & +(G(Y_n)-G(\widehat{Y}_n))^2(\Delta W_n)^2,\end{aligned}$$

for $0<n<N$ with $T=Nh$. Now, taking expected values conditioned on the σ-algebra $\mathcal{F}_{t_0}$ of the above inequality and applying properties of the conditional expectation we get

$$\begin{aligned}\mathbb{E}\left|Y_{n+1}-\widehat{Y}_{n+1}\right|^2 \leq & \mathbb{E}\left[\left|\Psi_h(Y_n)-\Psi_h(\widehat{Y}_n)\right|^2|\mathcal{F}_{t_0}\right] \\ & +2\left|\mathbb{E}\left[\left(\Psi_h(Y_n)-\Psi_h(\widehat{Y}_n)\right)\left(G(Y_n)-G(\widehat{Y}_n)\right)\Delta W_n|\mathcal{F}_{t_0}\right]\right| \\ & +\mathbb{E}\left[|G(Y_n)-G(\widehat{Y}_n)|^2|\mathcal{F}_{t_0}\right]\mathbb{E}\left[|\Delta W_n|^2|\mathcal{F}_{t_0}\right].\end{aligned}$$

The second term in this expression is zero due to the independence properties of Brownian motion. Next, using the Lipschitz condition for Ψ_h and G, we obtain:

$$\mathbb{E}\left[|Y_{n+1}-\widehat{Y}_{n+1}|^2|\mathcal{F}_{t_0}\right] \leq L(1+h)\mathbb{E}\left[|Y_n-\widehat{Y}_n|^2|\mathcal{F}_{t_0}\right]. \tag{3.30}$$

The sequence $\{R_n\}_{n\geq 0}$ defined by

$$R_n = \max_{0\leq r\leq n} \mathbb{E}\left[|Y_r-\widehat{Y}_r|^2|\mathcal{F}_{t_0}\right],$$

is monotonically non-decreasing. Furthermore, by (3.30) we have

$$R_n \leq L(1+h)R_{n-1}. \tag{3.31}$$

First suppose $0<L<1$, since $1+h\leq \exp(h)$ it follows that

$$R_n \leq L\exp(T)R_0, \qquad n=0,\ldots,N. \tag{3.32}$$

Hence, given $\epsilon>0$ if we take $\delta=\epsilon L^{-1}\exp(-T)$ then for all $0<h<h_0\leq T$ and any integer n such that $0\leq n\leq N$

$$\mathbb{E}\left|Y_0-\widehat{Y}_0\right|^2 \leq \delta \Rightarrow \mathbb{E}\left|Y_n-\widehat{Y}_n\right|^2 \leq \epsilon.$$

On the other hand, if $1<L<+\infty$ and with $h_0 := \frac{L-1}{L}$ then for $0<h<h_0$ we get

$$L(1+h) < 1+2Lh_0.$$

Thus, it follows that

$$R_n \leq \exp(2LNh_0)R_0 = \exp(2LT)R_0.$$

Hence, given $\epsilon>0$ if we take $h\in(0,(L-1)/L)$, and $\delta=\epsilon\exp(-2LT)$ then for all integers n such that $0\leq n\leq N$ we obtain

$$\mathbb{E}\left|Y_0-\widehat{Y}_0\right|^2 \leq \delta \Rightarrow \mathbb{E}\left|Y_n-\widehat{Y}_n\right|^2 \leq \epsilon.$$

So far we have proved the quadratic mean square stability for the explicit Steklov method. Notice that the asymptotic mean-square stability for the method (3.11) is verified for any $h\in(0,T]$ if $0<L<1$. □ □

3.4.2 Additive noise

Nonlinear differential equations have more complex dynamics than the linear case and the same occurs for the finite difference equations. So, Caraballo and Kloeden in [15] extend the nonlinear stability theory of the deterministic numerical analysis given in [37] to the stochastic numerical case. Following their work, we consider the non-autonomous additive SDE:

$$dy(t) = f(y(t))dt + \xi dW_t, \tag{3.33}$$

where f satisfies a *contractive one-sided Lipschitz* condition with constant $L_1 > 0$ as follows

$$\langle x - z, f(x) - f(z) \rangle \leq -L_1 |x - z|^2 \qquad \forall x, z \in \mathbb{R}, \tag{3.34}$$

and study the path-wise stability for the Steklov method (3.11) for the SDE (3.33).

Theorem 3.4.2. *If the Steklov function Ψ_h satisfies*

*(A1) (**Contractive Lipschitz condition**) There exists a constant $K_1 \in (0, 1)$ such that*

$$|\Psi_h(x) - \Psi_h(z)| \leq K_1 |x - z| \qquad \forall x, z \in \mathbb{R},$$

*(A2) (**Contractive one sided Lipschitz condition**) There exists a constant K_2 such that*

$$\langle \Psi_h(x) - \Psi_h(z), x - z \rangle \leq -K_2 |x - z|^2 \qquad \forall x, z \in \mathbb{R},$$

*(A3) (**Linear growth bound**) There exists a constant K_3 such that*

$$|\Psi_h(x)| \leq K_3(1 + h + |x|) \qquad \forall x \in \mathbb{R},$$

and the condition

$$\frac{K_3}{1 + K_2 - K_3} < 1, \tag{3.35}$$

is verified. Then there exists $h^* > 0$ such that for all $0 < h < h^*$ the Steklov method (3.11) *has a unique stochastic stationary solution which is path-wise asymptotically stable for an additive SDE* (3.33)*.*

Proof. In order to obtain the path-wise asymptotic stability for the explicit Steklov method we will show: *(i)* the path-wise contractive Lipschitz property for the Steklov numerical solution and *(ii)* the existence of a random attractor for the Steklov approximations.

(i) Let Y_{n+1} and $\widehat{Y}_{n+1}$ two different solutions of the Steklov method (3.11) for the additive SDE (3.33) and using the Lipschitz condition (A1) we get the following upper bound:

$$\begin{aligned} |Y_{n+1} - \widehat{Y}_{n+1}|^2 &= \left\langle Y_n - \widehat{Y}_n, \Psi_h(Y_n) - \Psi_h(\widehat{Y}_n) \right\rangle \\ &\leq K_1 |Y_{n+1} - \widehat{Y}_{n+1}| |Y_n - \widehat{Y}_n|. \end{aligned}$$

From this, we deduce that

$$|Y_n - \widehat{Y}_n| \leq K_1^{n-n_0} |Y_{n_0} - \widehat{Y}_{n_0}|. \tag{3.36}$$

then for $0 < K_1 < 1$ the path-wise contractivity. Moreover taking the limit of (3.36) as $n_0 \to -\infty$ for fixed n we have that $|Y_n - \widehat{Y}_n| \to 0$.

(ii) Defining a new variable by $Z_n := Y_n - \widehat{O}_n^{(h)}$ where Y_n is the Steklov approximation and $\widehat{O}_n^{(h)}$ is the Steklov OU process (3.27) we obtain the numerical scheme

$$Z_{n+1} = \Psi_h(Z_n + \widehat{O}_n^{(h)}) - \exp(\lambda h)\widehat{O}_n^{(h)}. \tag{3.37}$$

Taking the inner product with Z_{n+1} in (3.37) and adding convenient terms we get

$$\begin{aligned}|Z_{n+1}|^2 &= \left\langle Z_n + \widehat{O}_n^{(h)} - (Z_n + \widehat{O}_n^{(h)} + Z_{n+1}), \Psi_h(Z_n + \widehat{O}_n^{(h)}) - \Psi_h(Z_n + \widehat{O}_n^{(h)} + Z_{n+1})\right\rangle \\ &\quad + \left\langle Z_{n+1}, \Psi_h(Z_n + \widehat{O}_n^{(h)} + Z_{n+1})\right\rangle + \left\langle Z_{n+1}, \exp\{(\lambda h)\}\widehat{O}_n^h\right\rangle \\ &\leq -K_2|Z_{n+1}|^2 + |Z_{n+1}|\left|\Psi_h(Z_n + \widehat{O}_n^{(h)} + Z_{n+1})\right| + \exp\{(\lambda h)\}\,|Z_{n+1}|\left|\widehat{O}_n^h\right|.\end{aligned}$$

From the linear growth condition (A3) we deduce that

$$\begin{aligned}|Z_{n+1}|^2 &\leq (K_3 - K_2)|Z_{n+1}|^2 + K_3|Z_n||Z_{n+1}| \\ &\quad + K_3(1+h)|Z_{n+1}| + (K_3 + \exp(\lambda h))|Z_{n+1}||\widehat{O}_n^{(h)}|.\end{aligned}$$

Thus, we obtain

$$|Z_{n+1}| \leq \frac{K_3}{1 + K_2 - K_3}|Z_n| + \frac{K_3(1+h)}{1 + K_2 - K_3} + \frac{(K_3 + \exp(\lambda h))}{1 + K_2 - K_3}|\widehat{O}_n^{(h)}|. \tag{3.38}$$

Taking

$$\alpha := \frac{K_3}{1 + K_2 - K_3} \qquad \text{and} \qquad \beta := \frac{(K_3 + \exp(\lambda h))}{1 + K_2 - K_3},$$

we can rewrite (3.38) as

$$|Z_n| \leq \alpha^{n-n_0}|Z_{n_0}| + (1+h)\alpha \sum_{j=n_0}^{n-1} \alpha^{n-1-j} + \beta \sum_{j=n_0}^{n-1} \alpha^{n-1-j}|\widehat{O}_n^{(h)}|. \tag{3.39}$$

Then taking the limit as $n_0 \to -\infty$ for n fixed and assuming the condition (3.35) the first series of (3.39) converges. From [56] we have that for h small enough and considering the set of the bounded initial conditions $D(\omega)$ for the continuous OU process (3.26), the iterates Z_n remain in a ball with center the origin and random radius:

$$R_h(\omega) = C + \beta \sum_{j=n_0}^{n-1} \alpha^{n-1-j}|\widehat{O}_n^{(h)}|,$$

where C is a bound for the first terms of the right hand of the inequality (3.39). Thus, from theory of random numerical dynamical systems [37] and since Z_n inherits the contractivity from Y_n we conclude the existence of a random attractor for the sequence (3.37) defined by a unique stationary stochastic process. So, transforming back to the original variables we can assure that the explicit Steklov method for the SDE (3.33) has a stationary stochastic process $\widehat{Y}_n = \widehat{Z}_n + \widehat{O}_n$, which is a pathwise-attractor for all Steklov approximations in both pullback and forward senses. □

□

3.5 Numerical Results

Here, we analyze the efficiency of the explicit Steklov method (3.11) for SDEs for which a step size of the usual stochastic algorithms has to be small enough to preserve numerical stability. In particular, we consider as benchmarks the examples given in section 3.1 to show the behavior of the Steklov scheme and compare it with the EM approximation, the CBD method [10] and a balanced implicit method [61]. Moreover, long-time simulations of the new method are carried out in order to evidence its good asymptotic dynamical properties. But before, we start by evaluating the accuracy of the Steklov method for the linear SDE where the analytical solution is known.

3.5.1 Linear SDE

We apply the explicit Steklov approximation to the multiplicative (3.12) and additive (3.23) linear SDEs and study its accuracy showing its strong error which is determined by

$$\varepsilon = \mathbb{E}\left(\left|y(T) - Y_{n_T}\right|\right), \tag{3.40}$$

where $y(t)$ is the exact solution and Y_n is a time discretization approximation for the linear SDEs. Moreover, we also present numerical results for the EM scheme for the same equations. Numerical results for the Steklov and Euler-Maruyama (EM) approximations for both additive and multiplicative cases are shown in tables 3.1 and 3.2 respectively. The confidence interval for the strong error is obtained for 20 samples of 100 trajectories each. We also estimate the mean square error at a discrete time $t_n = T$ as follows:

$$\varepsilon_{MS}(T) = \left(\frac{1}{N}\sum_{k=1}^{N}\left(y^{[k]}(T) - Y^{h}_{n_T,k}\right)^2\right)^{\frac{1}{2}}, \tag{3.41}$$

for $N = 100\,000$ paths. Table 3.3 shows the results for both Steklov and Euler-Maruyama schemes. Notice that the Steklov method maintains its accuracy even when the step size is close to one while the Euler-Maruyama approximation is no longer stable from $h = 0.5$.

h	EM	Steklov
0.250 00	$4.1463 \times 10^{-2} \pm 2.9553 \times 10^{-3}$	$4.1076 \times 10^{-2} \pm 2.5145 \times 10^{-3}$
0.500 00	$1.2815 \times 10^{2} \pm 1.3437 \times 10^{-1}$	$5.5109 \times 10^{-2} \pm 3.6455 \times 10^{-3}$
0.750 00	$7.8644 \times 10^{2} \pm 5.9516 \times 10^{-1}$	$6.8446 \times 10^{-2} \pm 3.7039 \times 10^{-3}$
1.000 00	$1.2800 \times 10^{3} \pm 5.7282 \times 10^{-1}$	$7.8523 \times 10^{-2} \pm 6.0528 \times 10^{-3}$

Table 3.1: Intervals at 95% of confidence of the strong error for the additive linear SDE with $\lambda = -5$, $\xi = 0.1$ and initial condition $x_0 = 5$.

3.5.2 Logistic equation

Here we reconsider the stochastic logistic equation (3.14)

$$dy(t) = \lambda y(t)(K - y(t))dt + \xi y(t)^{\alpha}|K - y(t)|^{\beta}dW(t),$$

h	EM	Steklov
0.125 00	$1.8376 \times 10^{-2} \pm 8.3217 \times 10^{-4}$	$1.8376 \times 10^{-2} \pm 8.3217 \times 10^{-4}$
0.250 00	$1.7452 \times 10^{-2} \pm 1.3495 \times 10^{-3}$	$1.7452 \times 10^{-2} \pm 1.3495 \times 10^{-3}$
0.500 00	$1.2824 \times 10^{2} \pm 1.4210$	$1.7774 \times 10^{-2} \pm 1.3205 \times 10^{-3}$

Table 3.2: Intervals at 95% of confidence of the strong error for the multiplicative linear SDE with $\lambda = -5.0$, $\xi = 0.1$ and initial condition $x_0 = 5$.

	Additive noise		Multiplicative noise	
h	EM	Steklov	EM	Steklov
0.2500	2.1300×10^{-1}	2.0367×10^{-1}	5.4261×10^{-3}	9.4396×10^{-7}
0.5000	3.5206×10^{2}	3.0370×10^{-1}	2.7560×10^{2}	1.0752×10^{-3}
0.7500	8.1368×10^{2}	3.9055×10^{-1}	8.5490×10^{2}	7.1843×10^{-2}
1.0000	1.2930×10^{3}	4.5875×10^{-1}	1.3337×10^{3}	2.8987×10^{-1}

Table 3.3: MS-Error at time $T = 4.0$ for a linear SDE with $\lambda = -5$, $\xi = 0.1$ and initial condition $x_0 = 5$.

where $y(t)$ represents the number of individuals of certain specie with growth rate λ into an environment with limited natural resources and K is the maximum capacity population; α, β and ξ are nonnegative coefficients linked with the random contribution that models the influence of the environmental fluctuations or measurement errors [54, 61, 63]. The analytical solution of this equation in general is unknown. Thus it is necessary to obtain numerical solutions. In order to get an accuracy approximation it is desirable that the stochastic numerical method preserves the dynamic properties of the solution of (3.14). We choose this example to emphasize the structural dynamical consistency between the explicit Steklov defined by the function Ψ_h (3.15) and the SDE (3.14). In figure 3.2, we show the numerical results of the Steklov and Euler-Maruyama schemes and a balanced implicit method (BIM) developed to solve the equation (3.14) in [61]. For step sizes greater than 0.01, we observe that the Euler scheme is outside of its stability region and the BIM method has a slow convergence. On the other hand, the Steklov preserves the deterministic solution profile which is consistent with its structural foundation.

3.5.3 Langevin equation in Brownian dynamics

Finally, we study the Langevin equation (LE)

$$dy(t) = -y(t)^3 dt + \xi dW(t),$$

where $y(t)$ is the position of a particle at time t which is exposed to deterministic and random forces. This equation is used in Brownian dynamics like a benchmark test, see [10]. As in the logistic SDE, the analytical solution for the Langevin equation is only obtained under special conditions. The most common Brownian dynamics algorithm is the CBD method of Ermak and McCammon [20] which is

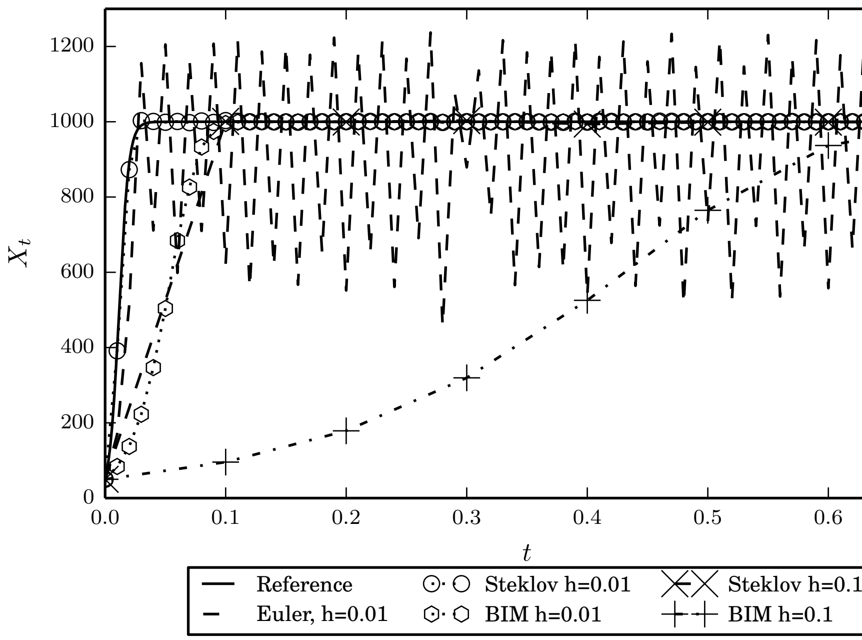

Figure 3.2: Paths obtained with the Euler, Steklov and BIM methods for the logistic SDE (3.14) with $X_0 = 50$ and taking $K = 1000$, $\alpha = 1$, β=0.5, λ=0.25, $\rho = 0$ and σ=0.05.

based on the Euler discretization of the LE. Although this method is easy to implement, a small time step size is required, therefore this algorithm runs in relatively small temporal windows. So, to study the asymptotic behavior of the solution of the LE it is convenient to apply methods with good asymptotic stability properties and simple structure. Therefore we show the behavior of the Steklov method defined by the function (3.17) for short-time and long-time dynamics by computing the *self-diffusion* coefficient D/D_0 associated to the LE, for details of the derivation of this coefficient see [10, 43]. In figure3.3, we compare the profiles of the Steklov and CBD approximations for several step sizes. According to the notation in Brownian dynamics, we take Dta=0.00001 as time step size and use 10 000 sample paths to calculate the self-diffusion coefficient. The Steklov and CBD methods have the same behavior at short time with small step sizes. However, for step sizes greater than 1 000 Dta the Euler method diverges and the Steklov method preserves its numerical stability. Thus, it can be used for long-time dynamics with big step sizes as it is shown in figure 3.3.

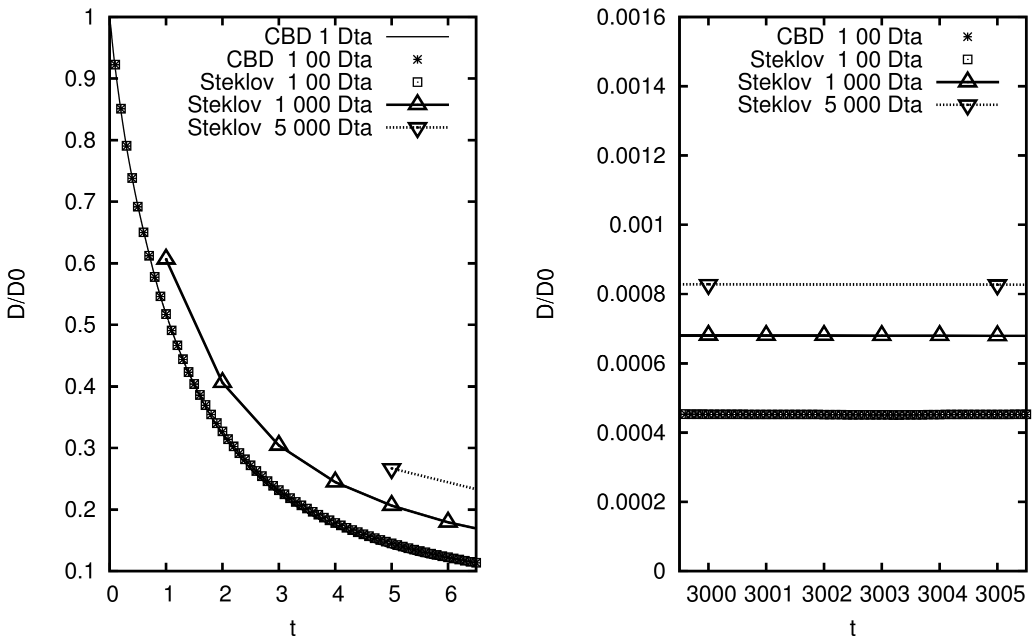

Figure 3.3: Numerical results of the Steklov and CBD methods for the self-diffusion coefficient of the LE with $\xi = 1$: the graph to the left shows short-time simulations and the graph to the right shows long-time simulations.

Chapter 4

Steklov Method for SDEs with Non-Globally Lipschitz Continuous Drift

4.1 Introduction

In this chapter, we develop an explicit method based on a linear version of the Steklov method proposed in Chapter 3. We consider the vector Itô stochastic differential equation

$$dy(t) = f(y(t))dt + g(y(t))dW(t), \quad 0 \leq t \leq T, \quad y(0) = y_0, \tag{4.1}$$

where $(f^{(1)}, \ldots, f^{(d)}) : \mathbb{R}^d \to \mathbb{R}^d$ is one sided Lipschitz and $g = (g^{(j,i)})_{j\in\{1,\ldots,d\},i\in\{1,\ldots,m\}} : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is global Lipschitz. Also we assume that each component function $f^{(j)}$ can be written of the form

$$f^{(j)}(x) = a_j(x)x^{(j)} + b_j(x^{(-j)}), \tag{4.2}$$

where a_j and b_j are two scalar functions in $\mathbb{R}^d$ and $x^{(-j)} = \left(x^{(1)}, \ldots, x^{(j-1)}, x^{(j+1)}, \ldots x^{(d)}\right)$. We work with a standard multidimensional setup, that is, $y(t) \in \mathbb{R}^d$ for each t and $W(t)$ is a m-dimensional standard Brownian motion on a filtered and complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, with the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by the Brownian process. Moreover, the complement of a set E is denoted by E^c and the indicator function of the set E is denoted by $\mathbf{1}_{\{E\}}$.

We recall that the following set of hypotheses assure existence and uniqueness of the solution of the stochastic differential system (4.1) and a explicit bound for its moments, see Theorem 2.3.3 and Lemma 2.3.1.

Hypothesis 4.1.1. The coefficients of SDE (4.1) satisfy the conditions:

(H-1) The functions f, g are in the class $C^1(\mathbb{R}^d)$.

(H-2) **Local, global Lipschitz condition**. For each integer n, there is a positive constant $L_f = L_f(n)$ such that

$$|f(u) - f(v)|^2 \leq L_f|u - v|^2 \qquad \forall u, v \in \mathbb{R}^d, \qquad |u|\vee|v| \leq n,$$

and there is a positive constant L_g such that

$$|g(u) - g(v)|^2 \leq L_g|u - v|^2, \qquad \forall u, v \in \mathbb{R}^d.$$

(H-3) **Monotone condition.** There exist two positive constants α and β such that

$$\langle u, f(u)\rangle + \frac{1}{2}|g(u)|^2 \leq \alpha + \beta|u|^2, \qquad \forall u \in \mathbb{R}^d. \tag{4.3}$$

This chapter is organized as follows: In section 4.2, we construct the new explicit method and prove the always existence of a succession of the Linear Steklov approximation as well as local Lipschitz conditions for its coefficients. In section 4.3, we prove the strong convergence of the LS method with one-half order using the Higham, Stuart and Mao (HSM) technique and in section 4.6, its convergence rate is obtained. In section 4.6, we analyze numerically the accuracy and efficiency of the proposed method applied to stochastic differential equations with super-linear growth and locally Lipschitz coefficients.

4.2 Construction of the Linear Steklov Method

For simplicity, we begin the construction of the Linear Steklov (LS) method considering the scalar case of SDE (4.1), that is, when $d = m = 1$, also, to shorten notation we use a, b instead a_j, b_j. Let $0 = t_0 < t_1 < \cdots < t_N = T$ a partition of the interval $[0, T]$ with constant step-size $h = T/N$ and such that $t_k = kh$ for $k = 0, \ldots, N$. The main idea of the LS approximation consists in estimating the drift coefficient of (4.1) by

$$f(y(t)) \approx \varphi_f(y(t_{\eta_+(t)})) = \left(\frac{1}{y(t_{\eta_+(t)}) - y(t_{\eta(t)})} \int_{y(t_{\eta(t)})}^{y(t_{\eta_+(t)})} \frac{du}{a(y(t_{\eta(t)}))u + b} \right)^{-1}, \qquad t \in [0, T], \tag{4.4}$$

where

$$\eta(t) := k \text{ for } t \in [t_k, t_{k+1}), \quad k \geq 0,$$
$$\eta_+(t) := k+1 \text{ for } t \in [t_k, t_{k+1}), \quad k \geq 0.$$

So we define the LS method for the scalar version of the SDE (4.1) using a split-step formulation as follows

$$Y_k^\star = Y_k + h\varphi_f(Y_k^\star), \tag{4.5}$$
$$Y_{k+1} = Y_k^\star + g(Y_k^\star)\Delta W_k, \tag{4.6}$$

with $Y_0 = y_0$ and $\varphi_f\left(Y_k^\star\right)$ defined by

$$\varphi_f\left(Y_k^\star\right) = \left(\frac{1}{Y_k^\star - Y_k} \int_{Y_k}^{Y_k^\star} \frac{du}{a(Y_k)u + b} \right)^{-1}, \tag{4.7}$$

where φ_f is the linearized Steklov average [19, 50]. For higher dimensions, we adapt the same split step scheme (4.5)–(4.6) as follows. For each component equation $j \in \{1, \ldots, d\}$, on the iteration $k \in \{1, \ldots, N\}$ take

$$a_{j,k} = a_j\left(Y_k^{(1)}, \ldots, Y_k^{(d)}\right), \qquad b_{j,k} = b_j\left(Y_k^{(-j)}\right). \tag{4.8}$$

So, define $\varphi_f(Y_k^\star) = \left(\varphi_{f^{(1)}}(Y_k^\star), \ldots, \varphi_{f^{(d)}}(Y_k^\star)\right)$ by

$$\varphi_{f^{(j)}}\left(Y_k^\star\right) = \left(\frac{1}{Y_k^{\star(j)} - Y_k^{(j)}} \int_{Y_k^{(j)}}^{Y_k^{\star(j)}} \frac{du}{a_{j,k}u + b_{j,k}} \right)^{-1}. \tag{4.9}$$

It is worth mentioning that even this formulation is semi implicit, we always can derive a explicit version. The next result deals with this issue. To simplify notation, we define $A^{(1)} = A^{(1)}(h, u)$, $A^{(2)} = A^{(2)}(h, u)$ and

$b = b(u)$ by

$$A^{(1)} := \begin{pmatrix} e^{ha_1(u)} & & 0 \\ & \ddots & \\ 0 & & e^{ha_d(u)} \end{pmatrix},$$

$$A^{(2)} := \begin{pmatrix} \left(\frac{e^{ha_1(u)}-1}{a_1(u)}\right)\mathbf{1}_{\{E_1^c\}} & & 0 \\ & \ddots & \\ 0 & & \left(\frac{e^{ha_d(u)}-1}{a_d(u)}\right)\mathbf{1}_{\{E_d^c\}} \end{pmatrix} + h \begin{pmatrix} \mathbf{1}_{\{E_1\}} & & 0 \\ & \ddots & \\ 0 & & \mathbf{1}_{\{E_d\}} \end{pmatrix}, \tag{4.10}$$

$$E_j := \{x \in \mathbb{R}^d : a_j(x) = 0\}, \qquad b(u) := \left(b_1(u^{(-1)}), \ldots, b_d(u^{(-d)})\right)^T.$$

Also we will need the following results from [40, Thm 2.1], [21, Thm. 1]. The first theorem will help us with the singularities of set E_j in the case where all elements of this set are limit points. Here we enunciate the results for $\mathbb{R}^2$ but the same theorem holds for real-valued functions of d variables.

Theorem 4.2.1 (Multivariate L'hôpital's Rule)**.** *Let $\mathcal{N}$ be a neighborhood in $\mathbb{R}^2$ containing a point $\mathbf{q}$ at which two differentiable functions $f : \mathcal{N} \to \mathbb{R}$ and $g : \mathcal{N} \to \mathbb{R}$ are zero. Set*

$$C = \{x \in \mathcal{N} : f(x) = g(x) = 0\},$$

and suppose that C is a smooth curve through $\mathbf{q}$. Suppose there exist a vector $\mathbf{v}$ not tangent to C at $\mathbf{q}$ such that the directional derivative $D_{\mathbf{v}}g$ of g in the direction of $\mathbf{v}$ is never zero within $\mathcal{N}$. Also we assume that $\mathbf{q}$ is a limit point of $\mathcal{N} \setminus C$. Then

$$\lim_{(x,y)\to\mathbf{q}} \frac{f(x,y)}{g(x,y)} = \lim_{\substack{(x,y)\to\mathbf{q} \\ (x,y)\in\mathcal{N}\setminus C}} \frac{D_{\mathbf{v}}f}{D_{\mathbf{v}}g},$$

if the latter limit exists.

For the second theorem we will need the following concepts.

Definition 4.2.1 (Directional derivative referred at a point)**.** Let $u, \mathbf{q} \in \mathbb{R}^2$ and α the positive angle respect to the x-axis and the segment $\overline{u\mathbf{q}}$. We denote by

$$f_\alpha(u) = \cos(\alpha)\frac{\partial f}{\partial u^{(1)}}(u) + \sin(\alpha)\frac{\partial f}{\partial u^{(2)}}(u) = \frac{\langle q - u, \nabla f(u)\rangle}{|u - q|}$$

the *directional derivative respect to the point* $\mathbf{q}$ *on* u.

Definition 4.2.2 (Star-like set)**.** A set $S \subset \mathbb{R}^2$ is *star-like* with respect a point $\mathbf{q}$, if for each point $s \in S$ the open segment $\overline{s\mathbf{q}}$ is in S.

Whit this in mine, second theorem give us a way to analyze isolated singularities.

Theorem 4.2.2. *Let $\mathbf{q} \in \mathbb{R}^2$ and let f,g be functions whose domains include a set $S \subset \mathbb{R}^2$ which is star-like with respect to the point $\mathbf{q}$. Suppose that on S the functions are differentiable and that the directional derivative of g with respect to $\mathbf{q}$ is never zero. With the understanding that all limits are taken from within on S at $\mathbf{q}$ and if* *(i)* $f(\mathbf{q}) = g(\mathbf{q}) = 0$,

(ii) $\lim_{x \to \mathbf{q}} \frac{f_\alpha(x)}{g_\alpha(x)} = L$,

then

$$\lim_{x \to \mathbf{q}} \frac{f(x)}{g(x)} = L.$$

With this on mind, we additionally require the following.

Hypothesis 4.2.1. For each component function $f^{(j)} : \mathbb{R}^d :\to \mathbb{R}$ with $j \in \{1, \dots, d\}$:

(A-1) There are two locally Lipschitz functions of class $C^1(\mathbb{R}^d)$ denoted by $a_j : \mathbb{R}^d \to \mathbb{R}$ and $b_j : \mathbb{R}^{d-1} \to \mathbb{R}$ such that the j-component of the drift function can be rewritten as in (4.2).

(A-2) There is a positive constant L_a such that

$$a_j(x) \leq L_a, \qquad \forall x \in \mathbb{R}^d.$$

(A-3) Each function $b_j(\cdot)$ satisfies the linear growth condition

$$|b_j(x^{(-j)})|^2 \leq L_b(1 + |x|^2), \qquad \forall x \in \mathbb{R}^d.$$

Hypothesis 4.2.2. The set $E_j := \{x \in \mathbb{R}^d : a_j(x) = 0\}$ satisfies either:

(i) All point $q \in E_j$ is a non isolated zero of a_j and:

- the set
$$D := \{u \in B_r(q) : e^{ha_j(u)} - 1 = a_j(u) = 0\},$$
is a smooth curve through q.
- The canonical vector e_j is not tangent to D.
- For each $q \in E_j$, there is an open ball with center on q and radio r $B_r(q)$, such that and
$$a_j \neq 0, \qquad \frac{\partial a_j(u)}{\partial u^{(j)}} \neq 0, \qquad \forall u \in B_r(q) \setminus D.$$

(ii) All point $q \in E_j$ is a isolated zero of a_j and:

- For each $q \in E_j$, q is not a limit point of the set $E_\alpha := \{x \in \mathbb{R}^d : (a_j)_\alpha(x) = 0\}$.
- For each $q \in E_j$ there is a star-like set respect to q E_q, such that the directional derivative respect to q satisfies
$$(a_j)_\alpha(x) \neq 0, \qquad \forall x \in E_q.$$

By Hypothesis 4.2.1 there is a unique linear Steklov approximation and by Hypothesis 4.2.2 we can apply Theorem 4.2.1 or Theorem 4.2.2 to deals with possible singularities of the matrix function $A^{(2)}$ defined on (4.10). Under the previous assumptions we will show that the explicit Linear Steklov approximation (4.5)–(4.6) exists, the function φ_f is bounded by the drift function f and also the coefficients φ_f and g satisfy a monotone condition. First, we will give the following lemma.

Lemma 4.2.1. *Assume Hypotheses 4.1.1, 4.2.1 and 4.2.2 hold. The function* $\Phi_j(x) = \Phi(h, a_j)(x)$ *defined by*

$$\Phi_j(x) := \frac{e^{ha_j(x)} - 1}{ha_j(x)}, \tag{4.11}$$

is bounded on $\mathbb{R}^d$ *for each* $j \in \{1, \ldots, d\}$ *by a positive constant* L_Φ*, which could depend on* h*.*

Proof. By Hypothesis 4.1.1, the operator Φ is continuous on E_j^c, thus

$$\lim_{h \to 0} \frac{e^{ha_j(x)} - 1}{ha_j} = 1, \tag{4.12}$$

for each fixed $x \in E_j^c$. If $x^* \in E_j$ and fixing any h, by Hypothesis 4.2.2, we obtain one of the following cases:

$$\lim_{\substack{x \to x^* \\ x \in E_j^c}} \Phi(h, a_j)(x) = \lim_{\substack{x \to x^* \\ x \in E_j^c}} \frac{\frac{\partial a_j(x)}{\partial x^{(j)}} h e^{ha_j(x)}}{h \frac{\partial a_j(x)}{\partial x^{(j)}}} = 1, \tag{4.13}$$

or

$$\lim_{\substack{x \to x^* \\ x \in E_j^c}} \Phi(h, a_j)(x) = \lim_{\substack{x \to x^* \\ x \in E_j^c}} \frac{\left(e^{ha_j(x)} - 1\right)_\alpha}{\left(ha_j(x)\right)_\alpha} = 1, \qquad \alpha = 0, \pi, 2\pi, \ldots \tag{4.14}$$

From (4.12), (4.13) and (4.14) we can deduce that

$$\left| \frac{e^{ha_j(x)} - 1}{ha_j(x)} \right| \leq \left| \frac{e^{hL_a} - 1}{ha_j^*} \right|, \qquad \forall x \in \mathbb{R}^d, \tag{4.15}$$

where $a_j^* := \inf_{x \in E_j^c} \{|a_j(x)|\}$. So, for each h fixed by inequality (4.15) we can deduce that there a positive constant $L_\Phi = L_\Phi(h)$ such that

$$|\Phi_j(x)| \leq L_\Phi, \qquad \forall j \in \{1, \ldots, d\}.$$

Finally, if $a_j^* = 0$, then we can use an argument similar to (4.13)–(4.14). □

Now we can state the following result.

Lemma 4.2.2. *Let Hypotheses 4.1.1, 4.2.1 and 4.2.2 holds, and* $A^{(1)}$*,* $A^{(2)}$*,* b *defined by (4.10). Then given* $u \in \mathbb{R}^d$*, the equation*

$$v = u + h\varphi_f(v), \tag{4.16}$$

has a unique solution

$$v = A^{(1)}(h, u)u + A^{(2)}(h, u)b(u). \tag{4.17}$$

If we define the functions $F_h(\cdot)$*,* $\varphi_{f_h}(\cdot)$ *and* $g_h(\cdot)$ *by*

$$F_h(u) = v, \qquad \varphi_{f_h}(u) = \varphi_f(F_h(u)), \qquad g_h(u) = g(F_h(u)), \tag{4.18}$$

then $F_h(\cdot)$*,* $\varphi_{f_h}(\cdot)$*,* $g_h(\cdot)$ *are local Lipschitz functions and for all* $u \in \mathbb{R}^d$ *and each* h *fixed, there is a positive constant* L_Φ *such that*

$$|\varphi_{f_h}(u)| \leq L_\Phi |f(u)|. \tag{4.19}$$

Moreover, for each h fixed, there are positive constants α^* and β^* such that

$$\left\langle \varphi_{f_h}(u), u \right\rangle \vee |g_h(u)|^2 \leq \alpha^* + \beta^* |u|^2, \qquad \forall u \in \mathbb{R}^d. \tag{4.20}$$

Proof. Let us first prove that (4.17) is solution of equation (4.16). Note that

$$v^{(j)} = u^{(j)} + h\, \varphi_{f^{(j)}}(v), \tag{4.21}$$

for each $j \in \{1, \dots, d\}$ and using the linear Steklov function (4.4), we can derive that

$$v^{(j)} = e^{h a_j(u)} u^{(j)} + \left[h\Phi_j(u)\mathbf{1}_{\{E_j^c\}} + h\mathbf{1}_{\{E_j\}} \right] b_j(u^{(-j)}), \tag{4.22}$$

which is the j-component of the vector $A^{(1)}u + A^{(2)}b(u)$. Now let us prove inequality (4.19). Given that $v = \varphi_f(F_h(u))$, we can also rewrite (4.21) as

$$\varphi_{f_h}^{(j)}(u) = \frac{F_h^{(j)}(u) - u^{(j)}}{\displaystyle\int_{u^{(j)}}^{F_h^{(j)}(u)} \frac{dz}{a_j(u)z + b_j(u^{(-j)})}}.$$

If $u \in E_j$ then $\varphi_{f_h}^{(j)}(u) = b_j(u^{(-j)}) = f^j(u)$, so $L_\Phi \geq 1$ fulfills (4.19). On the other hand, if $u \in E_j^c$ then

$$\varphi_{f_h}^{(j)}(u) = \frac{(F_h^{(j)}(u) - u^{(j)})a_j(u)}{\underbrace{\ln\left(a_j(u)F_h^{(j)}(u) + b_j(u^{(-j)}) \right)}_{:=R_1} - \ln\left(a_j(u)u^{(j)} + b_j(u^{(-j)}) \right)} = \Phi_j(u) f^j(u), \tag{4.23}$$

where

$$R_1 = \ln\left\{ a_j(u) \left[e^{h a_j(u)} u^{(j)} + h\Phi_j(u) b_j(u^{(-j)}) \right] + b_j\left(u^{(-j)} \right) \right\} = h a_j(u) + \ln\left(f^{(j)}(u) \right).$$

By lemma 4.2.1, inequality (4.19) is satisfied for all $u \in E_j \cup E_j^c$. As $g_h(x) = g\left(F_h(x)\right)$ by Hypothesis 4.1.1, then

$$|g_h(u) - g_h(v)|^2 \leq L_g |F_h(u) - F_h(v)|^2 \leq 2L_g \underbrace{|A^{(1)}u - A^{(1)}v|^2}_{:=R_2} + 2L_g \underbrace{|A^{(2)}b(u) - A^{(2)}b(v)|^2}_{:=R_3}. \tag{4.24}$$

Let us consider each term of the right hand of inequality (4.24). First, note that $A^{(1)}$ is a continuous differentiable function on all $\mathbb{R}^d$, so using the mean value theorem, we have

$$R_2 \leq L_{A^{(1)}} |u - v|^2, \quad u, v \in \mathbb{R}^d, \quad |u| \vee |v| \leq n, \tag{4.25}$$

for a positive constant $L_{A^{(1)}} \geq \sup_{0 \leq t \leq 1} |\partial A^{(1)}(h, u + t(v-u))|^2$. Meanwhile,

$$\begin{aligned}
R_3 \quad &= \quad \sum_{j=1}^{d} \Big[\mathbf{1}_{\{E_j^c\}}(u)\Phi_j(u) b_j(u^{(-j)}) + h\mathbf{1}_{\{E_j\}}(u) b_j(u^{(-j)}) - \mathbf{1}_{\{E_j^c\}}(v)\Phi_j(v) b_j(u^{(-j)}) \\
&- \quad h\mathbf{1}_{\{E_j\}}(v) b_j(v^{(-j)}) \Big]^2 \leq 4\sum_{j=1}^{d} \Big[\left(\mathbf{1}_{\{E_j^c\}}(u) L_\Phi b_j(u^{(-j)}) \right)^2 + \left(h\mathbf{1}_{\{E_j\}}(u) b_j(u^{(-j)}) \right)^2 \\
&+ \quad \left(\mathbf{1}_{\{E_j^c\}}(v) L_\Phi b_j(v^{(-j)}) \right)^2 + \left(h\mathbf{1}_{\{E_j\}}(v) b_j(v^{(-j)} \right)^2 \Big].
\end{aligned} \tag{4.26}$$

Since $b_j^2(\cdot)$ is a function of class $C^1(\mathbb{R}^d)$, there is a constant $L_b = L_b(n)$ such that

$$|b_j(u)|^2 \leq L_b, \quad \forall u \in \mathbb{R}^d, \quad |u| \vee |v| \leq n, \tag{4.27}$$

for each $j \in \{1, \cdots, d\}$. Using this bound in (4.26), we obtain

$$R_3 \leq 4 \sum_{j=1}^{d} \left[2L_\Phi L_b + 2h^2 L_b \right] \leq L_0, \qquad \forall u, v \in \mathbb{R}^d, \quad |u| \vee |v| \leq n, \tag{4.28}$$

where $L_0 = 8dL_b(n)(L_\Phi + h^2)$. By inequalities (4.25) and (4.28), we get

$$|g_h(u) - g_h(v)|^2 \leq L_{g_h}(n)|u - v|^2, \qquad \forall u, v \in \mathbb{R}^d, \quad |u| \vee |v| \leq n, \tag{4.29}$$

where $L_{g_h}(n) \geq n^2 + 1 + L_0 + L_{A^{(1)}}$. Then $g_h(\cdot)$ is a locally Lipschitz function. Furthermore, note that under some modifications this argument can be used to prove that $F_h(\cdot)$ is also a locally Lipschitz function, which implies that φ_{f_h} is a locally Lipschitz function. Finally, we will demonstrate inequality (4.20). By Hypotheses 4.1.1 and 4.2.1, we have

$$\langle f(u), u \rangle = \sum_{j=1}^{d} a_j(u) \left(u^{(j)} \right)^2 + \sum_{j=1}^{d} b_j(u) u^{(j)} \leq \alpha + \beta |u|^2,$$

and

$$\langle b(u), u \rangle \leq \alpha + (\beta + L_a)|u|^2.$$

Using these inequalities and (4.19), we deduce that

$$\left\langle \varphi_{f_h}(u), u \right\rangle = \sum_{j=1}^{d} \Phi_j(u) f^{(j)}(u) u^{(j)} \leq L_\Phi L_a |u| + L_\Phi(\alpha + (L_a + \beta)|u|^2) \leq L_{\varphi_{f_h}}(1 + |u|^2). \tag{4.30}$$

where $L_{\varphi_{f_h}} \geq 2L_\Phi \max\{L_a, \alpha, \beta\} + 1$. Meanwhile, g is globally Lipschitz then

$$|g_h(u)|^2 \leq 2|g(F_h(u)) - g(F_h(0))|^2 + 2|g(F_h(0))|^2 \leq 4L_g|F_h(u)|^2 + 8L_g|F_h(0)|^2 + 4|g(0)|^2. \tag{4.31}$$

Now, we bound each term on the right-hand side of (4.31). By the monotone condition (4.3), $|g(0)|^2 \leq 2\alpha$. Moreover,

$$|F_h^{(j)}(0)| = h\Phi_j(0)\,|b_j(0)|\mathbf{1}_{\{E_j^c\}}(0) + h\,|b_j(0)|\mathbf{1}_{\{E_j\}}(0) \leq \frac{b_0^*}{a_0^*} e^{hL_a}(1 + h), \quad \forall j \in \{1, \cdots, d\}.$$

where $a_0^* := \min_{\substack{j \in \{1, \cdots, d\} \\ a_j(0) \neq 0}} \left\{ |a_j(0)| \right\}$ and $b_0^* := \max_{j \in \{1, \cdots, d\}} \left\{ |b_j(0)| \right\}$. Then

$$|F_h(0)|^2 \leq d \left(\frac{b_0^*}{a_0^*} \right)^2 e^{2hL_a}(1 + h)^2. \tag{4.32}$$

Since Φ_j is bounded, from (4.22) we get

$$F_h^{(j)}(u) \leq e^{ha_j(u)}|u^{(j)}| + hL_\Phi|b_j(u)|\mathbf{1}_{\{E_j^c\}}(u) + h|b_j(u)|\mathbf{1}_{\{E_j\}}(u).$$

And by Hypothesis 4.2.1,

$$|F_h^{(j)}(u)|^2 \leq 3e^{2hL_a}|u|^2 + (3h^2 L_\Phi^2 L_b + 3h^2 L_b)(1+|u|^2) \leq L_F(1+|u|^2), \tag{4.33}$$

where $L_F \geq 3d \max\{e^{2hL_a}, h^2 L_b(L_\Phi^2+1)\}$. Using (4.32) and (4.33) in inequality (4.31) yields

$$|g_h(u)|^2 \leq 4L_g L_F(1+|u|^2) + 8L_g d \left(\frac{b_0^*}{a_0^*}\right)^2 e^{2hL_a}(1+h)^2 + 8\alpha.$$

Therefore, if $L_{g_h} \geq 4L_g L_F + 8L_g d \left(\frac{b_0^*}{a_0^*}\right)^2 e^{2hL_a}(1+h)^2 + 8\alpha$ then

$$|g_h(u)|^2 \leq L_{g_h}(1+|u|^2). \tag{4.34}$$

Hence, from inequalities (4.30) and (4.34) and taking for each fixed $h > 0$, $\alpha^* := L_{\varphi_{f_h}} \vee L_{g_h}$ and $\beta^* := 2\alpha^*$, we obtain inequality (4.20). □

Remark 4.2.1. Note that if $b_j = 0$, then Hypothesis 4.2.1 and Hypothesis 4.2.2 are unnecessary to prove Lemma 4.2.2 which is the case for stochastic Lotka-Volterra systems [48, 49], the Ginzburg-Landau SDE [36] or the damped Langevin equations where the potential lacks of a constant term [34]. On the other hand, there are several applications with $b_j \neq 0$ among others the stochastic SIR [65], the noisy Duffing-Van der Pol oscillator [60] and the stochastic Lorenz equation [22].

Remark 4.2.2. Note that by Lemma 4.2.2, we have that $\lim_{h\to 0}|f(x) - \varphi_{f_h}(x)| = 0$. Hence it is convenient to consider the following modified SDE

$$dy_h(t) = \varphi_{f_h}(y_h(t))dt + g_h(y_h(t))dW(t), \qquad y_h(0) = y_0, \qquad t \in [0,T],$$

as a perturbation of SDE (4.1). Moreover, the functions $\varphi_{f_h}(\cdot)$ and $g_h(\cdot)$ in (4.18) are respectively defined as the functions φ_f and g, but evaluated in the solution of $c = d + h\varphi(c)$, then we can rewrite the LS method (4.5)–(4.6) as

$$\begin{aligned} Y_k^\star &= Y_k + h\varphi_{f_h}(Y_k), \\ Y_{k+1} &= Y_k^\star + g_h(Y_k)\Delta W_k. \end{aligned}$$

We formalize these ideas in the following sections.

4.3 Strong Convergence of the Linear Steklov method

Here, we state and prove the main result of this chapter, the strong convergence of the LS method (4.5)–(4.6) for the solution of SDE (4.1). The main idea of the proof consists in applying the technique discussed in Section 2.4.2. We begin establishing the underlying convergence theorem.

Theorem 4.3.1. *Let Hypotheses 4.1.1 and 4.2.1 hold, consider the LS method (4.5)–(4.6) for the SDE* (4.1). *Then there is a continuous-time extension $\overline{Y}(t)$ of the LS solution $\{Y_k\}$ for which $\overline{Y}(t_k) = Y_k$ and*

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t) - y(t)|^2\right] = 0.$$

To proof this result, we initiate with the first step of the HMS technique, that is, we will show that the LS method for SDE (4.1) is equivalent to the EM scheme applied to the conveniently modified SDE

$$dy_h(t) = \varphi_{f_h}(y_h(t))dt + g_h(y_h(t))dW(t), \qquad y_h(0) = y_0, \qquad t \in [0, T]. \tag{4.35}$$

We formalize this as a Corollary of Lemma 4.2.2.

Corollary 4.3.1. *Let Hypotheses 4.1.1 and 4.2.1 hold, then the LS method for SDE (4.1) is equivalent to the EM scheme applied to the modified SDE (4.35).*

Proof. Using the functions $\varphi_{f_h}(\cdot)$ and $g_h(\cdot)$ defined in (4.18) of Lemma 4.2.2, we can rewrite the LS method (4.5)–(4.6) as

$$Y_{k+1} = Y_k + h\varphi_{f_h}(Y_k) + g_h(Y_k)\Delta W_k,$$

which is the EM approximation for the modified SDE (4.35). □

Now we proceed with the Step 2, that is, we will prove that the solution of the modified SDE (4.35) has bounded moments and is close in uniform mean square norm to the solution of the SDE (4.1). In what follows we denote by C a universal constant, that is, a positive constant independent on h which value could change in occurrences.

Lemma 4.3.1. *Let Hypotheses 4.1.1, 4.2.1 and 4.2.2 hold, then there is a universal constant $C = C(p, T) > 0$ and a sufficiently small step size h, such that for all $p > 2$*

$$\mathbb{E}\left[\sup_{0 \le t \le T} |y_h(t)|^p\right] \le C\left(1 + \mathbb{E}|y_0|^p\right). \tag{4.36}$$

Moreover

$$\lim_{h \to 0} \mathbb{E}\left[\sup_{0 \le t \le T} |y(t) - y_h(t)|^2\right] = 0. \tag{4.37}$$

Proof. By theorem 2.3.3 and inequality (4.20), we have bound (4.36). On the other hand, to prove (4.37) we will use the properties of φ_{f_h} and the Higham's stopping time technique employed in [31, Thm 2.2]. Note that by relation (4.23) of Lemma 4.2.2 we have

$$\varphi_{f_h}(x) = \Phi(h, a_j)(u) f^{(j)}(u) \mathbf{1}_{\{E_j^c\}}(u) + f^{(j)}(u)\mathbf{1}_{\{E_j\}}(u).$$

By Hypothesis 4.2.2 and since $f \in C^1(\mathbb{R}^d)$, $\Phi(h, a_j)(\cdot)$ is bounded, hence, there is a positive constant R_n which depends on n such that

$$\begin{aligned} |\varphi^{(j)}_{f_h}(u) - f^{(j)}(u)| &\le \mathbf{1}_{\{E_j^c\}}(u)|f^{(j)}(u)|\left|\Phi(h, a_j)(u) - 1\right| \\ &\le \mathbf{1}_{\{E_j^c\}}(u)\left(L_\Phi + 1\right)|f(u)| \\ &\le \mathbf{1}_{\{E_j^c\}}(u) R_n (L_\Phi + 1), \quad \forall u \in \mathbb{R}^d, \quad |u| \le n, \end{aligned}$$

for each $j \in \{1, \dots, d\}$. Moreover, we know by the proof of Lemma 4.2.2 that

$$\lim_{\substack{h \to 0 \\ u \in E_j^c}} \Phi(h, a_j)(u) = 1.$$

Also, we note that for each $j \in \{1, \dots, d\}$

$$\lim_{h\to 0} F_h^{(j)}(u) = \lim_{h\to 0} e^{ha_j(u)} u^{(j)} + \lim_{h\to 0} \left(\frac{e^{ha_j(u)} - 1}{a_j(u)} \mathbf{1}_{\{E_j^c\}}(u) + h\mathbf{1}_{\{E_j\}}(u) \right) b_j(u^{(j)}) = u^{(j)},$$

hence $\lim_{h\to 0} F_h(u) = u$. Consequently, given $n > 0$ there is a function $K_n(\cdot) : (0,\infty) \to (0,\infty)$, such that $K_n(h) \to 0$ when $h \to 0$ and

$$|\varphi_{f_h}(u) - f(u)|^2 \vee |g_h(u) - g(u)|^2 \leq K_n(h) \qquad \forall u \in \mathbb{R}^d, \quad |u| \leq n. \tag{4.38}$$

Now, using that both f, g are C^1, there is a constant $H_n > 0$ such that

$$|f(u) - f(v)|^2 \vee |g(u) - g(v)|^2 \leq H_n |u - v|^2 \qquad \forall u, v \in \mathbb{R}^d, |u| \vee |v| \leq n. \tag{4.39}$$

On the other hand, by Lemma 2.3.1 and inequality (4.36) we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |y(t)|^p\right] \vee \mathbb{E}\left[\sup_{0\leq t\leq T} |y_h(t)|^p\right] \leq K := C\left(1 + \mathbb{E}|y_0|^p\right).$$

Now, we define the stopping times

$$\tau_n := \inf\{t \geq 0 : |y(t)| \geq n\}, \qquad \rho_n := \inf\{t \geq 0 : |y_h(t)| \geq n\}, \qquad \theta_n := \tau_n \wedge \rho_n, \tag{4.40}$$

and the difference function

$$e_h(t) := y(t) - y_h(t).$$

From the Young's inequality (A.2), we deduce that for any $\delta > 0$

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t)|^2\right] &= \mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t)|^2 \mathbf{1}_{\{\tau_n > T, \rho_n > T\}}\right] + \mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t)|^2 \mathbf{1}_{\{\tau_n \leq T \text{ or } \rho_n \leq T\}}\right] \\
&\leq \mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t\wedge\theta_n)|^2 \mathbf{1}_{\{\theta_n \geq T\}}\right] + \frac{2\delta}{p}\mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t)|^p\right] \\
&\quad + \frac{1 - 2/p}{\delta^{2/(p-2)}} \mathbb{P}\left[\tau_n \leq T \text{ or } \rho_n \leq T\right].
\end{aligned} \tag{4.41}$$

We proceed to bound each term on the right-hand side of inequality (4.41). By Lemma 2.3.1, $y(t)$ has bounded moments, hence there is a positive constant A such that

$$\mathbb{P}\left[\tau_n \leq T\right] = \mathbb{E}\left[\mathbf{1}_{\{\tau_n < T\}} \frac{|y(\tau_n)|^p}{n^p}\right] \leq \frac{1}{n^p}\mathbb{E}\left[\sup_{0\leq t\leq T} |y(t)|^p\right] \leq \frac{A}{n^p}, \qquad \text{for } p \geq 2. \tag{4.42}$$

The same conclusion can be drawn for ρ_n, then

$$\mathbb{P}\left[\tau_n \leq T \text{ or } \rho_n \leq T\right] \leq \mathbb{P}\left[\tau_n \leq T\right] + \mathbb{P}\left[\rho_n \leq T\right] \leq \frac{2A}{n^p}. \tag{4.43}$$

Now, using inequality (A.4) and Lemma 2.3.1 we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t)|^p\right] \leq 2^{p-1}\mathbb{E}\left[\sup_{0\leq t\leq T}\left(|y(t)|^p+|y_h(t)|^p\right)\right] \leq 2^p A. \tag{4.44}$$

So, combining bound (4.43) with (4.44) in inequality (4.41) we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t)|^2\right] \leq \mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t\wedge\theta_n)|^2\mathbf{1}_{\{\theta_n\geq T\}}\right] + \frac{2^{p+1}\delta A}{p} + \frac{2(p-2)A}{p\delta^{2/(p-2)}n^p}. \tag{4.45}$$

Next, we show that the first term of (4.45) is bounded. Adding conveniently terms yields

$$\begin{aligned} e_h(t\wedge\theta_n) = &\int_0^{t\wedge\theta_n}\left[f(y(s)) - f(y_h(s)) + f(y_h(s)) - \varphi_{f_h}(y_h(s))\right] ds \\ &+ \int_0^{t\wedge\theta_n}\left[g(y(s)) - g(y_h(s)) + g(y_h(s)) - g_h(y_h(s))\right] dW(s). \end{aligned}$$

Using bounds (4.38) and (4.39), the Cauchy-Schwarz, and Doob martingale inequalities, we get

$$\mathbb{E}\left[\sup_{0\leq t\leq \tau}|e_h(t\wedge\theta_n)|^2\right] \leq 4H_n(T+4)\int_0^{\tau}\mathbb{E}\left[\sup_{0\leq t\leq \tau}|e_h(t\wedge\theta_n)|^2\right] ds + 4T(T+4)K_n(h).$$

The Gronwall inequality now yields

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t\wedge\theta_R)|^2\right] \leq 4T(T+4)K_n(h)\exp(4H_n(T+4)T).$$

Hence, given $\epsilon > 0$ for any $\delta > 0$ such that $2^{p+1}\delta A/p < \epsilon/3$, we can take $n > 0$ verifying $(p-2)2A/(p\delta^{2/(p-2)}n^p) < \epsilon/3$. Moreover, we can take h sufficiently small such that $4T(T+4)K_n(h)e^{4H_n(T+4)T} < \epsilon/3$. It follows immediately that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t)|^2\right] < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon,$$

which is the desired conclusion. □

Next, we proceed with Step 3, in which we establish that LS method has bounded moments.

Lemma 4.3.2. *Let Hypotheses 4.1.1, 4.2.1 and 4.2.2 hold. Then for each $p \geq 2$ there is a universal positive constant $C = C(p,T)$ such that the explicit LS method*

$$\mathbb{E}\left[\sup_{kh\in[0,T]}|Y_k|^{2p}\right] \leq C.$$

Proof. Denoting by $A_k^{(i)} := A^{(i)}(h, Y_k)$ for $i = 1, 2$ and $b_k := b(Y_k)$, we use a split formulation of the LS scheme (4.5)–(4.6) as follows:

$$\begin{aligned} Y_k^\star &= A_k^{(1)}Y_k + A_k^{(2)}b_k, \\ Y_{k+1} &= Y_k^\star + g(Y_k^\star)\,\Delta W_k, \end{aligned}$$

from the first step of this split scheme, using (A-3) and the Cauchy-Schwartz inequality, we get

$$\begin{aligned} |Y_k^\star|^2 &\leq |A_k^{(1)}|^2|Y_k|^2+2\left\langle A_k^{(1)}Y_k, A_k^{(2)}Y_k b_k\right\rangle + |A_k^{(2)}|^2|b_k|^2 \\ &\leq |A_k^{(1)}|^2|Y_k|^2+2\sqrt{L_b}d|A_k^{(1)}||A_k^{(2)}||Y_k|(1+|Y_k|) + L_b|A_k^{(2)}|^2(1+|(Y_k)|^2). \end{aligned} \tag{4.46}$$

From (A-2), we can deduce that

$$|A_k^{(1)}|^2 = \left|\operatorname{diag}\left(e^{ha_1(Y_k)},\ldots,e^{ha_d(Y_k)}\right)\right|^2 \leq L_{A^{(1)}}, \tag{4.47}$$

where $L_{A^{(1)}} = d\, e^{2TL_a}$ and also by (4.15), we can derive that

$$\begin{aligned} |A^{(2)}(h,Y_k)|^2 &= \left|h\operatorname{diag}\left(\mathbf{1}_{\{E_1\}}(Y_k)+\mathbf{1}_{\{E_1^c\}}(Y_k)\Phi_1(Y_k),\ldots,\mathbf{1}_{\{E_d\}}(Y_k)+\mathbf{1}_{\{E_d^c\}}(Y_k)\Phi_d(Y_k)\right)\right|^2 \\ &\leq \sum_{j=1}^{d}\left(\mathbf{1}_{\{E_j^c\}}|h\Phi_j(Y_k)|^2+h^2\right) \leq 2e^{2L_aT}\sum_{j=1}^{d}\frac{1}{a_j^*}+dT^2 \leq L_{A^{(2)}}. \end{aligned} \tag{4.48}$$

Substituting (4.47) and (4.48) on inequality (4.46) yields

$$|Y_k^\star|^2 \leq L_{A^{(1)}}|Y_k|^2+2d\sqrt{L_{A^{(1)}}L_{A^{(2)}}L_b}\,|Y_k|(1+|Y_k|)+L_{A^{(2)}}L_b(1+|Y_k|^2) \leq C(1+|Y_k|^2),$$

where $C \geq L_{A^{(1)}}+2d\sqrt{L_{A^{(1)}}L_{A^{(2)}}L_b}+L_{A^{(2)}}L_b$. Applying bound (4.49) in the second step of the split scheme, we get

$$|Y_{k+1}|^2 \leq C\left(|Y_k|^2+1\right)+2\left\langle Y_k^\star, g(Y_k^\star)\Delta W_k\right\rangle+|g(Y_k^\star)\Delta W_k|^2.$$

Now, we choose two integers N, M such that $Nh \leq Mh \leq T$. So, adding backwards we obtain

$$|Y_N|^2 \leq S_N\left(\sum_{j=0}^{N-1}(1+|Y_j|^2)+2\sum_{j=0}^{N-1}\left\langle Y_j^\star, g(Y_j^\star)\Delta W_j\right\rangle+\sum_{j=0}^{N-1}\left|g(Y_j^\star)\Delta W_j\right|^2\right),$$

where $S_N := \sum_{j=0}^{N-1}C^{N-j}$. Raising both sides to the power p, we get

$$|Y_N|^{2p} \leq 6^p S_N^p\left(N^{p-1}\sum_{j=0}^{N-1}(1+|Y_j|^{2p})+\left|\sum_{j=0}^{N-1}\left\langle Y_j^\star, g(Y_j^\star)\Delta W_j\right\rangle\right|^p+N^{p-1}\sum_{j=0}^{N-1}\left|g(Y_j^\star)\Delta W_j\right|^{2p}\right). \tag{4.49}$$

Now we will show that the second and third terms of inequality (4.49) are bounded. We denote by $C = C(p,T)$ a generic positive constant which does not depend on the step size h and whose value may change between occurrences. Next, applying the Bunkholder-Davis-Gundy inequality [44], we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq N\leq M}\left|\sum_{j=0}^{N-1}\left\langle Y_j^\star, g(Y_j^\star)\Delta W_j\right\rangle\right|^p\right] &\leq C\mathbb{E}\left[\sum_{j=0}^{N-1}|Y_j^\star|^2|g(Y_j^\star)|^2 h\right]^{p/2} \\
&\leq Ch^{p/2}M^{p/2-1}\mathbb{E}\sum_{j=0}^{M-1}|Y_j^\star|^p(\alpha+\beta|Y_j^\star|^2)^{p/2} \\
&\leq 2^{p/2-1}CT^{p/2-1}h\mathbb{E}\sum_{j=0}^{M-1}(\alpha^{p/2}|Y_j^\star|^p+\beta^{p/2}|Y_j^\star|^{2p}) \\
&\leq Ch\mathbb{E}\sum_{j=0}^{M-1}(1+2|Y_j^\star|^p+|Y_j^\star|^{2p}) \\
&\leq C+Ch\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}, \qquad (4.50)
\end{aligned}$$

Now, using the Cauchy-Schwartz inequality, the monotone condition (4.3) and bound (4.49), we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq N\leq M}\sum_{j=0}^{N-1}\left|g(Y_j^\star)\Delta W_j\right|^{2p}\right] &\leq \sum_{j=0}^{M-1}\mathbb{E}\left|g(Y_j^\star)\right|^{2p}\mathbb{E}\left|\Delta W_j\right|^{2p} \\
&\leq Ch^p\sum_{j=0}^{M-1}\mathbb{E}\left[\alpha+\beta|Y_j^\star|^2\right]^p \\
&\leq Ch^p\sum_{j=0}^{M-1}\mathbb{E}\left[\alpha^p+\beta^p|Y_j^\star|^{2p}\right] \\
&\leq Ch^{p-1}+Ch^p\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}. \qquad (4.51)
\end{aligned}$$

Thus, combining bounds (4.50) and (4.51) with inequality (4.49), we can assert that

$$\mathbb{E}\left[\sup_{0\leq N\leq M}|Y_N|^{2p}\right]\leq C+C(1+h)\sum_{j=0}^{M-1}\mathbb{E}\left[\sup_{0\leq N\leq j}|Y_N|^{2p}\right]. \qquad (4.52)$$

Finally, using the discrete-type Gronwall inequality [44], we conclude that

$$\mathbb{E}\left[\sup_{0\leq N\leq M}|Y_N|^{2p}\right]\leq Ce^{C(1+h)M}\leq Ce^{C(1+T)}<C,$$

since the constant C does not depend on h, the proof is complete. □

Since the LS scheme has bounded moments, we now proceed whit Step 4, that is, we will obtain a continuous extension of the LS method with bounded moments. Let $\{Y_k\}$ denote the LS solution of SDE (4.1). By Corollary 4.3.1, we conveniently made a continuous extension for the LS approximation, from the time continuous extension of the EM method (2.13). Also, we prove that the moments of the Linear Steklov extension remains bounded.

Corollary 4.3.2. *Let Hypotheses 4.1.1, 4.2.1 and 4.2.2 hold and suppose* $0 < h < 1$ *and* $p \geq 2$*. Then there is a continuous extension* $\overline{Y}(t)$ *of* $\{Y_k\}$ *and a universal constant* $C = C(T, p)$ *such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|\overline{Y}(t)|^{2p}\right] \leq C.$$

Proof. We take $t = s + t_k$ in $[0, T]$, $\Delta W_k(s) := W(t_k + s) - W(t_k)$ and $0 \leq s < h$. Then we define

$$\overline{Y}(t_k + s) := Y_k + s\varphi_{f_h}(Y_k) + g_h(Y_k)\Delta W_k(s), \tag{4.53}$$

as a continuous extension of the LS scheme. We proceed to show that $\overline{Y}(t)$ has bounded moments. By Lemma 4.2.2, we have $Y_k^\star = Y_k + h\varphi_{f_h}(Y_k)$. Then for $\gamma = s/h$, it follows that

$$\begin{aligned} Y_k + s\varphi_{f_h}(Y_k) &= \gamma(Y_k + h\varphi_{f_h}(Y_k)) + (1-\gamma)Y_k \\ &= \gamma Y_k^\star + (1-\gamma)Y_k. \end{aligned}$$

Hence, we can rewrite the continuous extension (4.53) as

$$\overline{Y}(t) = \gamma Y_k^\star + (1-\gamma)Y_k + g_h(Y_k)\Delta W_k(s).$$

Combining this relation with the inequalities (4.49) and (A.4), we get

$$\begin{aligned} |\overline{Y}(t_k + s)|^2 &\leq 3\left[\gamma C + (\gamma C + 1 - \gamma)\,|Y_k|^2 + |g_h(Y_k)\Delta W_k(s)|^2\right] \\ &\leq C + C\left(|Y_k|^2 + |g_h(Y_k)\Delta W_k(s)|^2\right). \end{aligned}$$

Thus,

$$\begin{aligned} \sup_{0\leq t\leq T}|\overline{Y}(t)|^{2p} &\leq \sup_{0\leq kh\leq T}\left[\sup_{0\leq s\leq h}|\overline{Y}(t_k+s)|^{2p}\right] \\ &\leq \sup_{0\leq kh\leq T}\left[\sup_{0\leq s\leq h} C\left(1 + |Y_k|^{2p} + |g_h(Y_k)\Delta W_k(s)|^{2p}\right)\right], \end{aligned} \tag{4.54}$$

for $t \in [0, T]$. Now taking a non negative integer $0 \leq k \leq N$ such that $0 \leq Nh \leq T$. From the bond (4.54), we get

$$\sup_{0\leq t\leq T}|\overline{Y}(t)|^{2p} \leq C\left(1 + \sup_{0\leq kh\leq T}|Y_k|^{2p} + \sup_{0\leq s\leq h}\sum_{j=0}^{N}|g_h(Y_j)\Delta W_j(s)|^{2p}\right). \tag{4.55}$$

So, using the Doob's Martingale inequality (A.5), Lemma 4.3.2 and that g_h is a locally Lipschitz function, we can bound each term of the inequality (4.55), as follows

$$\begin{aligned} \mathbb{E}\left[\sup_{0\leq s\leq h}|g(Y_j)\Delta W_j(s)|^{2p}\right] &\leq \left(\frac{2p}{2p-1}\right)^{2p}\mathbb{E}|g_h(Y_j)\Delta W_j(h)|^{2p} \\ &\leq C\mathbb{E}|g_h(Y_j)|^{2p}\mathbb{E}|\Delta W_j(h)|^{2p} \\ &\leq Ch^p\left(1 + \mathbb{E}|Y_j|^{2p}\right) \\ &\leq Ch, \end{aligned} \tag{4.56}$$

for each $j \in \{0, \dots, N\}$. Since $Nh \leq T$, combining the bounds (4.55) and (4.56) we get the desired conclusion. □

Once we have carried out all the previous steps, we can prove Theorem 4.3.1 by Step 5.

Proof of Theorem 4.3.1. First, note that by inequality (A.4), we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t) - y(t)|^2\right] \leq 2\mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t) - y_h(t)|^2\right] + 2\mathbb{E}\left[\sup_{0\leq t\leq T} |y_h(t) - y(t)|^2\right]. \tag{4.57}$$

Using Lemma 4.3.1, which was established in the Step 2, yields

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T} |y_h(t) - y(t)|^2\right] = 0. \tag{4.58}$$

It remains to prove that the first term of the right hand side in inequality (4.57) decreases to zero when h tends to zero. Recalling that:

i) By Lemma 4.3.1, the solution of the modified SDE (4.35), y_h, has p-bounded moments ($p \geq 2$).

ii) By Corollary 4.3.2, the LS continuous extension for the SDE (4.1), $\overline{Y}(t)$, has bounded moments and it is equivalent to the EM extension for the modified SDE (4.35).

Hence, we can apply Theorem 2.4.4 to conclude that

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t) - y_h(t)|^2\right] = 0. \tag{4.59}$$

Finally, combining the limits (4.58) and (4.59) with inequality (4.57) gives

$$\begin{aligned}\lim_{h\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t) - y(t)|^2\right] &\leq 2\lim_{h\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t) - y_h(t)|^2\right] \\ &\quad + 2\lim_{h\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T} |y_h(t) - y(t)|^2\right] = 0,\end{aligned}$$

which proves the theorem. □

4.4 Convergence Rate

In this section we show that the Linear Steklov method (4.5)–(4.6) converges with a standard one-half order. For that, we use a similar procedure as in [31]. In addition to Hypotheses 4.1.1, 4.2.1 and 4.2.2 we also require the following.

Hypothesis 4.4.1. There exist constants $L_f, D \in \mathbb{R}$ and $q \in \mathbb{Z}^+$ such that $\forall u, v \in \mathbb{R}^d$

$$\langle u - v, f(u) - f(v)\rangle \leq L_f |u - v|^2, \tag{4.60}$$

$$|f(u) - f(v)|^2 \leq D(1 + |u|^q + |v|^q)|u - v|^2. \tag{4.61}$$

Hypothesis 4.4.2. The SDE (4.1), the EM solution and its continuous extension satisfy

$$\mathbb{E}\left[\sup_{0\le t\le T}|y(t)|^p\right],\quad \mathbb{E}\left[\sup_{0\le t\le T}|Y(t)|^p\right],\quad \mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}(t)|^p\right]<\infty,\qquad \forall p\ge 1. \tag{4.62}$$

Theorem 4.4.1. *[Higham et al. [31, Thm 4.4]] Under Hypotheses 4.2.1–4.4.1 the EM solution with continuous extension* (2.13) *satisfies*

$$\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}(t)-y(t)|^2\right]=\mathcal{O}(h^2). \tag{4.63}$$

Lemma 4.4.1. *Under Hypotheses 4.4.1 and 4.4.2 and sufficiently small h, there exist constants $D'\in\mathbb{R}$ and $q'\in\mathbb{Z}$ such that for all $u,v\in\mathbb{R}^d$*

$$|\varphi_{f_h}(u)-\varphi_{f_h}(v)|^2\le D'\left(1+|u|^{q'}+|v|^{q'}\right)|u-v|^2, \tag{4.64}$$

$$|f(u)-\varphi_{f_h}(u)|^2\le D'\left(1+|u|^{q'}\right)h^2, \tag{4.65}$$

$$|g(u)-g_h(u)|^2\le D'\left(1+|u|^{q'}\right)h^2. \tag{4.66}$$

Proof. From inequality (4.19), we have

$$|\varphi_{f_h}(u)-\varphi_{f_h}(v)|^2\le(2+L_\Phi)|f(u)-f(v)|^2\le(2+L_\Phi)D(1+|u|^q+|v|^q).$$

Moreover, if $u\in E_j$ then $\varphi_{f_h}(u)=f^{(j)}(u)$. On the other hand, if $u\in E_j^c$ then

$$|f(u)-\varphi_{f_h}(u)|^2=\sum_{j=1}^{d}|1-\Phi(h,a_j)(u)|^2|f^{(j)}(u)|^2,$$

By the L'Hôpital theorem, we get

$$\lim_{h\to 0}|1-\Phi(h,a_j)(u)|=\left|1-\lim_{h\to 0}\frac{e^{ha_j(u)}-1}{ha_j(u)}\right|\le\left|1-\lim_{h\to 0}e^{hL_a}\right|=0.$$

Thus, there is a sufficiently small $h>0$ such that $|1-\Phi_j(u)|<Ch$ for all $u\in E_j^c$ and

$$|f(u)-\varphi_{f_h}(u)|^2\le Ch^2|f(u)|^2\le D'(1+|u|^q)h^2,$$

as we require. Given that $g_h(u)=g(F_h(u))$ from theorem 4.2.2 we get

$$|g(u)-g_h(u)|^2\le L_g|u-u+h\varphi f_h(u)|^2\le 2(1+L_\Phi)h^2|f(u)|^2\le 2(1+L_\Phi)D(1+|u|^q)h^2.$$

□

Lemma 4.4.2. *Assume Hypotheses 4.4.1 and 4.4.2 hold then the solution $y_h(t)$ of the modified SDE* (2.17) *satisfies*

$$\mathbb{E}\left[\sup_{0\le t\le T}|y_h(t)-y(t)|^2\right]=\mathcal{O}(h^2). \tag{4.67}$$

Proof. We define $e(t) := y(t) - y_h(t)$ where

$$\begin{aligned} y(t) &= y_0 + \int_0^t f(y(s))ds + \int_0^t g(y(s))dW(s), \\ y_h(t) &= y_0 + \int_0^t \varphi_{f_h}(y_h(s))ds + \int_0^t g_h(y_h(s))dW(s). \end{aligned}$$

Using Itô's formula over the function $V(t,x,y) = |x-y|^2$ for all $x, y \in \mathbb{R}^d$, we obtain

$$de(t) = \left(f(y(t)) - \varphi_{f_h}(y(t))dt\right) + \left(g(y(t)) - g_h(y_h(t))\right) dW(t),$$

Thus,

$$\begin{aligned} |e(t)|^2 = {} & \underbrace{2\int_0^t \left\langle e(s), f(y(s)) - \varphi_{f_h}(y_h(s))\right\rangle ds}_{:=I_1} + \underbrace{\int_0^t |g(y(s)) - g_h(y_h(s))|^2 ds}_{:=I_2} \\ & + \underbrace{2\int_0^t \left\langle e(s), \left[g(y(s)) - g_h(y_h(s))\right] dW(s)\right\rangle}_{:=I_3}. \end{aligned} \tag{4.68}$$

Now we proceed to bound each integral of inequality (4.68). By Hypothesis 4.4.1 and the Young inequality, we get

$$\begin{aligned} I_1(t) &\leq 2\int_0^t \langle y(s) - y_h(s), f(y(s)) - f(y_h(s))\rangle\, ds + \int_0^t \left\langle y(s) - y_h(s), f(y_h(s)) - \varphi_f(y_h(s))\right\rangle ds \\ &\leq 3\int_0^t |y(s) - y_h(s)|^2 ds + D'h^2 \int_0^t 1 + |y_h(s)|^{q'} ds. \end{aligned}$$

Since $y_h(t)$ has bonded moments, there exists a universal constant L which does not depends on h such that

$$\mathbb{E}\left[I_1(s)\right] \leq L\int_0^t \mathbb{E}|e(s)|^2 ds + Lh^2. \tag{4.69}$$

Using Hypotheses 4.1.1 and 4.4.1 it is followed

$$I_2(t) \leq 2L_g \int_0^t |y(s) - y_h(s)|^2 ds + 2D'h^2 \int_0^t 1 + |y_h(s)|^q ds,$$

thus

$$\mathbb{E}\left[I_2(s)\right] \leq L\int_0^t \mathbb{E}|e(s)|^2 ds + Lh^2. \tag{4.70}$$

Note that $\mathbb{E}\left[I_3(t)\right] \leq \mathbb{E}\left[\sup_{0\leq s\leq t}|I_3(s)|\right]$. From the Burkholder-Davis-Gaundy inequality, Hypotheses 4.1.1

and 4.4.1 and as $y_h(t)$ has bounded moments, we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le s\le t}|I_3(s)|\right] &\le 2^4\mathbb{E}\left[\sup_{0\le s\le t}|e(s)|^2\int_0^t|g(y(s))-g_h(y(s))|^2ds\right]^{1/2}\\ &\le 2^4\mathbb{E}\left[\frac{1}{2\cdot 2^9}\left(\sup_{0\le s\le t}|e(s)|^2\right)+\frac{2^9}{2}\left(\int_0^t|g(y(s))-g_h(y_h(s))|^2ds\right)^2\right]\\ &\le 2L_g\mathbb{E}\left[\int_0^t|y(s)-y_h(s)|^2ds\right]+D'Th^2+D'Th^2\int_0^t\mathbb{E}|y_h(s)|^{q'}ds\\ &\le L\int_0^t\mathbb{E}|e(s)|^2ds+Lh^2. \end{aligned} \tag{4.71}$$

Substituting inequalities (4.69), (4.70) and (4.71) on equation (4.68), we deduce that

$$\mathbb{E}\left[\sup_{0\le s\le t}|e(t)|^2\right]\le L\int_0^t\mathbb{E}|e(s)|^2ds+Lh^2\le L\int_0^t\mathbb{E}\left[\sup_{0\le r\le s}|e(s)|^2\right]ds+Lh^2.$$

By the Gronwall inequality, we conclude that

$$\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2\right]\le L\exp(LT)h^2\le Ch^2.$$

□

We can now obtain the convergence rate of the explicit Linear Steklov method.

Theorem 4.4.2. *Under Hypotheses 4.1.1–4.4.1 and consider the explicit LS method (4.10) for the SDE (4.1). Then there exists a continuous-time extension $\overline{Y}(t)$ of the LS numerical approximation for which*

$$\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}(t)-y(t)|^2\right]=\mathcal{O}(h). \tag{4.72}$$

Proof. Using bound (4.57) then by lemma 4.4.2 and since the LS continuous-time extension (4.53) is equivalent to the EM continuous-time extension (2.13), we can use Theorem 4.4.1 and conclude that the LS has order one-half. □

4.5 Almost Sure Stability

In this section we study the globally almost surely asymptotic stability (as-stability) of the Linear Steklov method (4.5)–(4.6), in the scalar case. For simplicity we assume that the drift coefficient satisfies

$$f(x)=a(x)x,$$

for some suitable nonlinear function $a:\mathbb{R}\to\mathbb{R}$. Here, we will follow the same technique reported by Mao and Szpruch in [46]. First, we need sufficient conditions to characterize when the solution of the SDE (4.1) is as-stable. The following result deals with it.

Theorem 4.5.1 (Mao and Szpruch [46, Thm. 2.2]). *Let hypothesis 4.1.1 hold and suppose that there exists a function $z \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}_+)$ such that*

$$\langle x, f(x)\rangle + \frac{1}{2}|g(x)|^2 \leq -z(x), \qquad \forall x \in \mathbb{R}^d,$$

then

(i) For any $y_0 \in \mathbb{R}^n$ the solution of the SDE (4.1), $y(t)$, satisfies

$$\limsup_{t\to\infty} |y(t)|^2 \leq \infty \qquad \text{a. s.} \qquad \textit{and} \qquad \lim_{t\to\infty} z(y(t)) = 0 \qquad \text{a. s.}$$

(ii) additionally, if $z(x) = 0$ only when it is evaluated at $x = 0$, then

$$\lim_{t\to\infty} y(t) = 0 \qquad \text{a. s.} \qquad \forall y_0 \in \mathbb{R}^d.$$

Next, we prove that LS method verifies the as-stability. The proof of this result depends on the Lemma 4.5.1 see for instance [41, Th. 7, pg. 139]. We will denote by $\{Z \to\}$ the set of all $\omega \in \Omega$ for which the scalar process Z has the property that $\lim_{k\to\infty} Z_k$ exists and is finite.

Lemma 4.5.1 ([41, Thm. 7, pg. 139]). *Let $Z = \{Z_k\}$ be a nonnegative semimartingale with $\mathbb{E}|Z| < \infty$ and Doob decomposition*

$$Z = Z_0 + A^{(1)} - A^{(2)} + M,$$

where $A^{(1)} := \{A_k^{(1)}\}_{k\in\mathbb{N}}$ and $A^{(2)} := \{A_k^{(2)}\}_{k\in\mathbb{N}}$ are a. s. *nondecreasing predictable processes with $A_0^{(1)} = A_0^{(2)} = 0$ and $M := \{M_k\}_{k\in\mathbb{N}}$ is a local $\{\mathcal{F}_k\}$-martingale with $M_0 = 0$. Then*

$$\left\{A^{(1)} \to\right\} \subseteq \left\{A^{(2)} \to\right\} \cap \{Z \to\} \qquad \text{a. s.}$$

Theorem 4.5.2. *Let Hypothesis 4.1.1 hold. Suppose that there is a function $z \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}_+)$ and a step size $h^* > 0$ such that for all $x \in \mathbb{R}$ and for all h in $(0, h^*)$,*

$$\langle x, f(x)\rangle + \frac{1}{2}|g(x)|^2 \leq -z(x), \tag{4.73}$$

$$|x|^2 \frac{(\exp(2ha(x)) - 1)}{h} + |g_h(x)|^2 \leq -z(x), \tag{4.74}$$

Then the LS method defined by (4.5)–(4.6) satisfies

$$\limsup_{k\to\infty} |Y_k|^2 < \infty \quad \textit{and} \quad \lim_{k\to\infty} w(Y_k) = 0.$$

In addition, if $z(x) = 0$ only when $x = 0$, then $\lim_{k\to\infty} Y_k = 0$.

Proof. Taking advantage of Lemma 4.5.1, we proceed to construct a conveniently semimartingale. To

$$\begin{aligned}|Y_{k+1}|^2 = |Y_k|^2 + h^2|\varphi_{f_h}(Y_k)|^2 + |g_h(Y_k)\Delta W_k|^2 + 2h\left\langle Y_k, \varphi_{f_h}(Y_k)\right\rangle \\ + 2\left\langle Y_k, g_h(Y_k)\Delta W_k\right\rangle + 2h\left\langle \varphi_{f_h}(Y_k), g_h(Y_k)\Delta W_k\right\rangle.\end{aligned} \tag{4.75}$$

Let

$$\Delta M_{k+1} := |g_h(Y_k)\Delta W_{k+1}|^2 - |g_h(Y_k)|^2 h$$
$$+ 2\langle Y_k, g_h(Y_k)\Delta W_{k+1}\rangle + 2h\left\langle \varphi_{f_h}(Y_k), g_h(Y_k)\Delta W_{k+1}\right\rangle,$$

which is a local martingale. Taking $B_j := -\left[2\left\langle Y_j, \varphi_{f_h}(Y_j)\right\rangle + |g_h(Y_j)|^2 + h|\varphi_{f_h}(Y_j)|^2\right]$, and fixing $N \in \mathbb{N}$, we can rewrite (4.75) as

$$|Y_{N+1}|^2 = |Y_0|^2 - \sum_{j=0}^{N} B_j h + \sum_{j=0}^{N} \Delta M_{j+1}. \tag{4.76}$$

To prove that (4.76) is the required decomposition to apply Lemma 4.5.1, we use that

$$\varphi_{f_h}(x) = x\frac{(\exp(ha(x)) - 1)}{h}. \tag{4.77}$$

By algebraic manipulations, we obtain

$$B_j = -\left[|Y_j|^2\frac{(\exp(2ha(Y_j)) - 1)}{h} + |g_h(Y_j)|^2\right], \qquad j = 0, \ldots, N.$$

Given that inequality (4.74) holds, we can deduce that

$$B_j \geq z(Y_j) \geq 0, \quad j = 0, \ldots N.$$

Consequently, $A_k^{(2)} := \sum_{j=0}^{k} B_j h$ is a non decreasing process. Finally, taking $A^{(1)} = 0$, $Z = |Y_k|^2$ and $M_k = \sum_{j=0}^{k} \Delta M_{j+1}$. We can deduce by Lemma 4.5.1 that $\{A^{(1)} \to\} = \Omega$, thus

$$\limsup_{k\to\infty} |Y_k|^2 < \infty \quad \text{a.s.,} \quad \text{and} \quad \sum_{j=0}^{\infty} z(Y_k) \leq \sum_{j=0}^{\infty} B_j h < \infty.$$

Consequently $\lim_{k\to\infty} z(Y_k) = 0$, and the theorem follows. □

4.6 Numerical Simulations

Here we analyze the behavior of the explicit LS method for scalar and vector SDEs. The tests confirm the convergence order 1/2 for stochastic differential systems with locally Lipschitz drift and suggest that the LS scheme reproduces almost surely stability (a.s.). We validate the efficiency of the new method by comparing with other actual methods like the Euler-Maruyama, Backward-Euler-Maruyama (BEM) [46] and Tamed-Euler-Maruyama (TEM) [34]. All simulations are implemented in Python 2.7 and we use the Mersenne random number generator with fixed seed 100.

Example 4.6.1. Here we illustrate the stability Theorem 4.5.2 through an numerical example presented in [2, sec 7, pg. 420]. Here, Appleby and Kelly, proved that the EM method does not satisfies the almost sure stability of the test SDE

$$dy(t) = -\beta y(t)|y(t)|^p dt + \sigma(t)|y(t)|^\rho dW(t). \tag{4.78}$$

So, the EM approximation explodes to infinity on finite time when $p+1 > 2\rho$. But, with the same parameters $\lim_{t\to\infty} y(t) = 0$ a. s., see [2, 3] for more details. In particular for the following SDE

$$dy(t) = -y^3 dt + \frac{1}{[\log(t+1)]^{1.1}} dW_t, \qquad t > 0, \tag{4.79}$$

and deduce conditions for the step-size h and initial condition $y(t_0) = y_0$ in order to claim with high probability when the EM scheme for SDE (4.79) is as-stable or diverge [2, Cor 7.1 pg. 421]. More specifically, given $h < 0.0384$ and the EM for SDE (4.79)

$$X_{k+1} = X_k - hX_k^3 + \frac{1}{[\log(n+1)]^{1.1}} \Delta W_k, \qquad X_0 = y(t_0). \tag{4.80}$$

(i) If $X_0 \in \left(-\sqrt{\frac{2}{h}} + 7\sqrt{h}, \sqrt{\frac{2}{h}} - 7\sqrt{h}\right)$, then $\mathbb{P}\left[\lim_{k\to\infty} X_k = 0\right] > 0.95$.

(ii) If $X_0 \in \left(-\infty, -\sqrt{\frac{2}{h}} - 7\sqrt{h}\right) \bigcup \left(\sqrt{\frac{2}{h}} + 7\sqrt{h}, \infty\right)$, then

$$\mathbb{P}\left[\limsup_{n\to\infty} X_k = \infty \text{ or } \liminf_{n\to\infty} X_k = -\infty\right] > 0.95 \quad .$$

Thus we perform a simulation with step size $h = 0.2$ using the EM, Tamed Euler-Maruyama (TEM) and the LS schemes with unstable EM initial conditions. Figure 4.1 shows how the EM scheme produces spurious solutions. Meanwhile, the TEM ans LS approximations reproduce the asymptotic behavior, also we observe a better initial precision of the LS approximation.

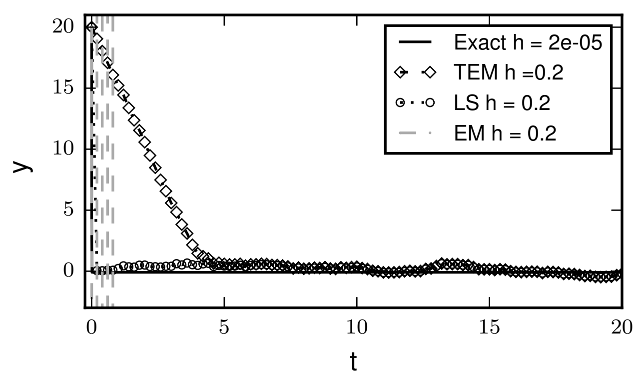

Figure 4.1: Likening between the EM, TEM and LS approximations with unstable EM conditions. Here "exact" means a BEM solution with step size $h = 2 \times 10^{-5}$.

Example 4.6.2. We examine the LS method using a SDE with super-linear grow diffusion. We consider the SDE reported by Tretyakov and Zhang in [66, Eq. (5.6)]

$$dy(t) = \left(1 - y^5(t) + y^3(t)\right) dt + y^2(t) dW(t), \qquad y_0 = 0. \tag{4.81}$$

Tretyakov and Zhang show via simulation of (4.81) that the increment-tamed scheme [32, Eq(1.5)]

$$X_{k+1} = X_k + \frac{f(X_k)h + g(X_k)\Delta W_k}{\max\left(1, h\left|hf(X_k) + g(X_k)\Delta W_k\right|\right)} \tag{4.82}$$

produces spurious oscillations. Hutzenthaler and Jentzen prove the convergence of this scheme under linear growth condition over diffusion. So, this suggests us that only certain kind of explicit schemes with convergence under globally Lipschitz and linear growth diffusion conditions can extended their convergence to a locally Lipschitz diffusion and other kind of growth bound. Using $a(x) := -x^4 + x^2$, $b := 1$ and $E = \{-1, 0, 1\}$, we construct the LS method

$$Y_{k+1} = \exp(ha(Y_k))Y_k + \frac{\exp(ha(Y_k)) - 1}{a(Y_k)} \mathbf{1}_{\{E^c\}} + h\mathbf{1}_{\{E\}} + Y_k^2 \Delta W_k. \tag{4.83}$$

Figure 4.2 shows the numerical solution of SDE (4.81) with the Increment-Tamed (I-TEM) (4.82), LS method (4.83), and the Tamed (TEM) scheme. We consider the implicit Midpoint scheme [66, Eq.(5.3)] with $h = 10^{-4}$ as reference.

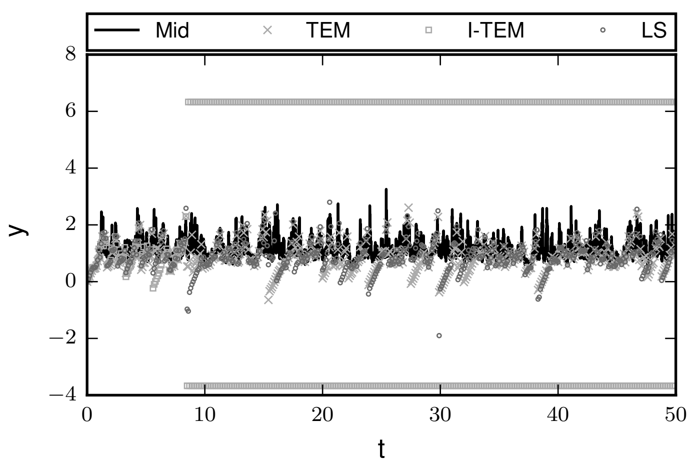

Figure 4.2: Numerical solution of SDE (4.81) using the I-TEM (4.82), LS method (4.83) and TEM with $h = 0.1$. The reference solution is a Midpoint rule approximation with $h = 10^{-4}$.

Example 4.6.3. Now we compare the order of convergence and the run time of the LS method with the TEM scheme as in [34]. That is, we consider a Langevin equation under the d-dimensional potential $U(x) = \frac{1}{4}|x|^4 - \frac{1}{2}|x|^2$, and d-dimensional Brownian additive noise. The corresponding SDE reads

$$dy(t) = \left(y(t) - |y(t)|^2 \cdot y(t)\right) dt + dW(t), \qquad y(0) = 0. \tag{4.84}$$

	TEM		LS		BEM	
h	ms-error	ECO	ms-error	ECO	ms-error	ECO
2^{-2}	1.703 88	—	1.553 94	—	1.381 57	—
2^{-3}	1.169 77	0.54	1.107 75	0.48	1.053 09	0.39
2^{-7}	0.278 95	0.48	0.277 95	0.48	0.276 895	0.48
2^{-11}	0.070 10	0.50	0.070 09	0.50	0.070 07	0.50
2^{-15}	0.017 39	0.51	0.017 39	0.51	0.017 39	0.51

Table 4.1: Mean square errors and the experimental convergence order (ECO) for the SDE (4.84) with a TEM with $h = 2^{-19}$ as reference solution.

This model describes the motion of a Brownian particle of unit mass immersed on the potential $U(x)$. Taking $a_j(x) := 1 - |x|$ and $b_j = 0$, $j \in \{1, \dots, d\}$ we obtain the LS method

$$Y_{k+1} = \operatorname{diag}\left[e^{ha_1(Y_k)}, \dots, e^{ha_d(Y_k)})\right] Y_k + \Delta W_k. \tag{4.85}$$

Table 4.1 shows the root means square errors at a final time $T = 1$, which is approximated by

$$\sqrt{\mathbb{E}\left[|Y_N - y(T)|^2\right]} \approx \frac{1}{M}\left(\sum_{i=1}^{M}|y_i(T) - Y_{N,i}|^2\right)^{1/2}, \tag{4.86}$$

over a sample of M =10 000 trajectories of the TEM, LS and BEM solutions to SDE (4.84) with dimension $d = 10$. We consider the TEM solution with step $h = 2^{-19}$ as reference solution. In this experiment we confirm that the LS method converges with standard order 1/2 and is almost equal accurate as the TEM approximation.

In some applicationa as in Brownian Dynamics Simulations [16], the dimension of a SDE increases considerable the complexity and computational cost — this excludes the use of implicit methods. In Figure 4.3, we observe that the runtime of the BEM method grows quadratically depending on the dimension, meanwhile the LS and TEM methods grow linearly.

Example 4.6.4. Let us recall the following stochastic model for internal HIV dynamics given by Dalal et al. in [17]:

$$\begin{aligned}
dy_1(t) &= (\lambda - \delta y_1(t) - (1-\gamma)\beta y_1(t)y_3(t))\, dt - \sigma_1 y_1(t) dW_t^{(1)},\\
dy_2(t) &= ((1-\gamma)\beta y_1(t)y_3(t) - \alpha y_2(t))\, dt - \sigma_1 y_2(t) dW_t^{(1)},\\
dy_3(t) &= ((1-\eta)N_0\alpha y_2(t) - \mu y_3(t) - (1-\gamma)\beta y_1(t)y_3(t))\, dt - \sigma_2 y_3(t) dW_t^{(2)}.
\end{aligned} \tag{4.87}$$

Under certain conditions Dalal et al. prove that system (4.87) has a unique almost surely exponential stability solution, that is, $y = (y_1, y_2, y_3)$ tends exponentially to an equilibrium $(\bar{y}_1, 0, 0)$ with probability 1. Now, we want to verify if the EM, TEM, TEM-Sabanis [57] and LS approximations can reproduce this

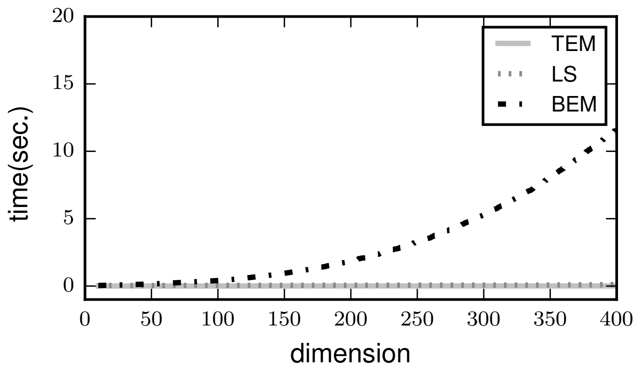

Figure 4.3: Runtime calculation of Y_N with $h = 2^{-17}$, using the BEM, LS and TEM methods for SDE (4.84).

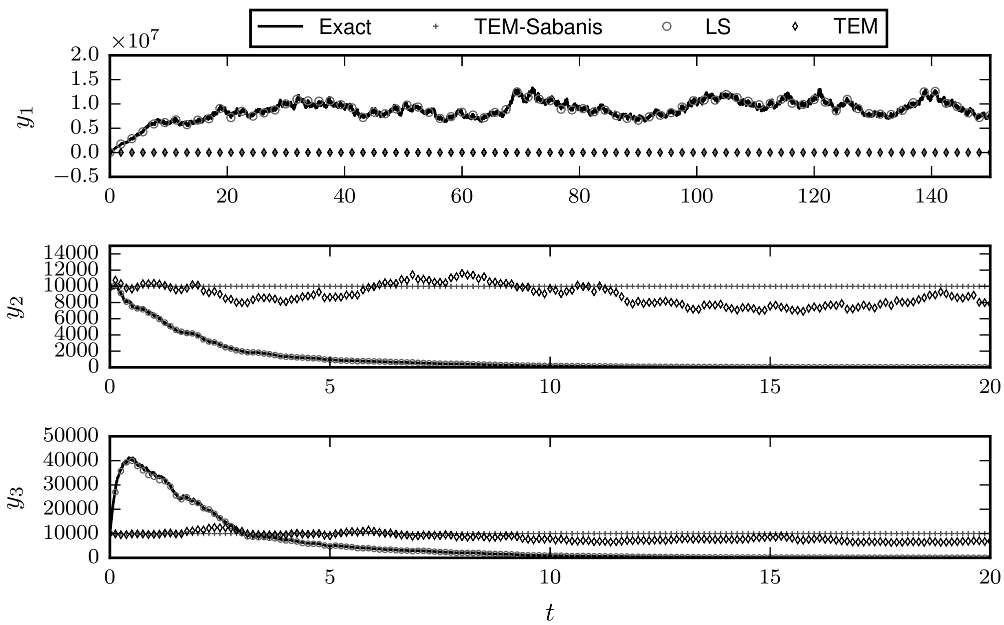

Figure 4.4: Likening between EM, LS , TEM approximations for SDE (4.87) with $\gamma = 0.5$, $\eta = 0.5$, $\lambda = 10^6$, $\delta = 0.1$, $\beta = 10^{-8}$, $\alpha = 0.5$, $N_0 = 100$, $\mu = 5$, $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $y_0 = (10\,000, 10\,000, 10\,000.)^T$, $h = 0.125$. Here the reference solution means a BEM simulation with the same parameters but with a step-size $h = 10^{-5}$.

property of the solution. Taking

$$E_1 := \left\{ (x, y, z)^T \in \mathbb{R}^3 : z = 0 \text{ or } z = 0\frac{-\delta}{\beta(1-\gamma)} \right\}, \quad E_2 := \emptyset,$$
$$E_3 := \left\{ (x, y, z)^T \in \mathbb{R}^3 : x = 0 \text{ or } x = \frac{-\mu}{\beta(1-\gamma)} \right\}.$$

The LS method for (4.87) is given by

$$
\begin{aligned}
a_1(Y_k)) &:= -\left(\delta + (1-\gamma)\beta Y_k^{(3)}\right), & b_1(Y_k^{(-1)}) &:= \lambda, \\
a_2(Y_k) &:= -\alpha, & b_2(Y_k^{(-2)}) &:= (1-\gamma)\beta Y_k^{(1)} Y_k^{(3)}, \\
a_3(Y_k) &= -\left(\mu + (1-\gamma)\beta Y_k^{(1)}\right), & b_3(Y_k^{(-3)}) &:= (1-\eta)\, N_0 \alpha Y_k^{(2)},
\end{aligned}
$$

and its explicit form reads,

$$
\begin{aligned}
Y_{k+1} &= A^{(1)}(h, Y_k)\, Y_k + A^{(2)}(h, Y_k) b(Y_k) + g(Y_k)\, \Delta W_k, \qquad \Delta W_k = \left(W_k^{(1)}, W_k^{(2)}\right)^T, \\
A^{(1)}(h, Y_k) &:= \begin{pmatrix} e^{h a_1(Y_k)} & 0 & 0 \\ 0 & e^{h a_2(Y_k))} & 0 \\ 0 & 0 & e^{h a_3(Y_k))} \end{pmatrix}, \\
A^{(2)} &:= \begin{pmatrix} h\Phi_1(Y_k)\mathbf{1}_{\{E_1^c\}} & 0 & 0 \\ 0 & \left(\dfrac{e^{-h\alpha} - 1}{\alpha}\right) & 0 \\ 0 & 0 & h\Phi_3(Y_k)\mathbf{1}_{\{E_3^c\}} \end{pmatrix} + h \begin{pmatrix} \mathbf{1}_{\{E_1\}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{\{E_3\}} \end{pmatrix}, \\
b(Y_k) &:= \begin{pmatrix} b_1(Y_k^{(-1)}) \\ b_2(Y_k^{(-2)}) \\ b_3(Y_k^{(-3)}) \end{pmatrix}, \qquad g(Y_k) := \begin{pmatrix} -\sigma_1 Y_k^{(1)} & 0 \\ -\sigma_1 Y_k^{(2)} & 0 \\ 0 & -\sigma_2 Y_k^{(3)} \end{pmatrix}.
\end{aligned}
\tag{4.88}
$$

Figure 4.4 shows the LS , TEM and TEM-Sabanis approximations with the parameters reported in [17]. The EM approximation blows up so it is not drawn. We observe how the TEM approximation (components y_2 and y_3) oscillates about the initial condition and the TEM-Sabanis approximation (components y_2 and y_3) is almost constant, while the LS method reproduces the asymptotic behavior of the solution. It is important to remark that the Tamed family methods improve convergence of the Euler method by taming the drift increment term with the factor $1/(1 + h|f(Y_k)|)$, bounding the norm of $hf(Y_k)/(1 + h|f(Y_k)|)$ by 1. This norm controls the drift contribution of the Tamed methods at each step. Such modification is recommended for SDEs with drift contributions and initial conditions with similar scales. We observe that for models where such terms have different scales the TEM over damps the drift contribution.

Chapter 5

Conclusions and future work

5.1 Conclusions

We have constructed a new way to design numerical methods for SDEs based on the Steklov average. First we presented a scalar scheme originated in an exact discretization for the deterministic version of the SDEs with desired stability properties — the Steklov method. We verified its convergence and stability over a standard globally Lipschitz setup and compared its performance with a competitive solvers. Also, we have extended the explicit Steklov scheme for vector SDE by developing a new version based on a linearized Steklov average. This method is constructed on the basis that the drift function can be rewritten in the linearized form. Moreover, strong order one-half convergence has been proved for our explicit linear method and we have presented several applications formulated with the LS scheme. Finally, high-performance of the Linear Steklov method have been analyzed in diverse problems, even for SDEs with super-linear diffusion. Future work will be focused on the following problems:

- Numerical evidence suggests that the Steklov methods are suitable for SDEs with super- linear growth diffusion. So, one should prove this claim.
- Since we have proved strong convergence, we would to apply the Multilevel Monte Carlo approach to the Brownian Dynamics Simulation using Steklov type schemes.
- The schemes presented here have a simple structure like the Euler-Maruyama family, so it is possible to formulate versions of the Tamed, Milstein, Balanced, Theta, Runge-Kutta methods by approximating the drift term by its Steklov average.
- Also, it is viable to study stability of the LS method usibg the theory of random dynamical systems.
- Furthermore, a natural extension of this work would be to design Steklov type schemes for more general SDEs, that is, SDEs with delay, Poisson jumps or partial derivatives.

Appendix A

Useful Inequalities

In this appendix we enunciate basic results that are extensively used through our analysis. Here the main reference are [62] and [44].

Hölder ([62, pg. 193])**.**

$$\mathbb{E}[X^T Y] \leq \left(\mathbb{E}|X|^p\right)^{\frac{1}{p}} \left(\mathbb{E}|X|^q\right)^{\frac{1}{q}}. \tag{A.1}$$

Young ([26, pg. 111])**.**

$$|a||b| \leq \frac{\delta}{p}|a|^p + \frac{\delta}{q\delta^{q/p}}|b|^q. \tag{A.2}$$

Minkowski ([62, pg. 194])**.**

$$\left(\mathbb{E}|X+Y|^p\right)^{\frac{1}{p}} \leq \left(\mathbb{E}|X|^p\right)^{\frac{1}{p}} + \left(\mathbb{E}|Y|^p\right)^{\frac{1}{p}}. \tag{A.3}$$

A standard inequality. Fix $1 < p < \infty$ and consider a sequence of real numbers $\{a_i\}_{i=1}^N$ with $N \in \mathbb{N}$. Then one can formulate this usefully inequality

$$\left(\sum_{j=1}^N a_j\right)^p \leq N^{p-1} \sum_{j=1}^N a_j^p. \tag{A.4}$$

Doob's Martingale Inequality ([44, Thm. 3.5])**.** Let $\{M_t\}_{t\geq 0}$ be a $\mathbb{R}^d$-valued martingale. Let $[a,b]$ be a bounded interval in $\mathbb{R}_+$. If $p > 1$ and $M_t \in L^p(\Omega;\mathbb{R}^d)$ then

$$\mathbb{E}\left(\sup_{a\leq t\leq b} |M_t|^p\right) \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|M_b|^p. \tag{A.5}$$

Burkholder–Davis–Gundy inequality ([44, Thm. 7.3])**.** Let $g \in \mathcal{L}(\mathbb{R}_+;\mathbb{R}^{d\times m})$. Define for $t \geq 0$

$$x(t) = \int_0^t g(s)dW(s) \quad \text{and} \qquad A(t) = \int_0^t |g(s)|^2 ds. \tag{A.6}$$

Then for all $p > 0$, there exist universal positive constants c_p, C_p such that

$$c_p \mathbb{E}|A(t)|^{\frac{p}{2}} \leq \mathbb{E}\left[\sup_{0\leq s\leq t} |x(s)|^p\right] \leq C_p \mathbb{E}|A(t)|^{\frac{p}{2}}, \tag{A.7}$$

for all $t \geq 0$. In particular, one may take

$$\begin{aligned}
c_p &= (p/2)^p, & C_p &= (32/p)^{\frac{p}{2}} & &\text{if } 0 < p < 2;\\
c_p &= 1, & C_p &= (32/p)^{\frac{p}{2}} & &\text{if } p = 2;\\
c_p &= (2p)^{-\frac{p}{2}}, & C_p &= \frac{p+1}{2(p-1)^{\frac{p}{2}}} & &\text{if } p > 2.
\end{aligned}$$

Gronwall inequality ([44, Thm. 8.1]). Let $T > 0$ and $c \geq 0$. Let $u(\cdot)$ be a Borel measurable bounded nonnegative function on $[0, T]$, and let v be a nonnegative integrable function on $[0, T]$ If

$$u(t) \leq c + \int_0^t v(s)u(s)ds \qquad \forall t \in [0, T],$$

then

$$u(t) \leq c \exp\left(\int_0^t v(s)ds\right) \qquad \forall t \in [0, T]. \tag{A.8}$$

Discrete Gronwall Inequality ([46, Lm. 3.4]). Let M be a positive integer. Let u_k and v_k be non-negative numbers for $k = 0, 1, \ldots, M$. If

$$u_k \leq u_0 + \sum_{j=0}^{k-1} u_j v_j$$

then

$$u_k \leq u_0 \exp\left(\sum_{j=0}^{k-1} v_j\right). \tag{A.9}$$

References

References

[1] E. Allen. *Modeling With Itô Stochastic Differential Equations*. Springer, Mar 2007. ISBN 978-14-0205-953-7.

[2] J. Appleby and C. Kelly. On the local dynamics of polynomial difference equations with fading stochastic perturbations. *Dynamics of Continuous, Discrete and Impulsive Systems*, pages 401–430, 2010. URL http://strathprints.strath.ac.uk/id/eprint/29102.

[3] J. Appleby, X. Mao, and A. Rodkina. Stabilization and destabilization of nonlinear differential equations by noise. *IEEE Transactions on Automatic Control*, 53(3):683–691, 2008. ISSN 00189286. doi: 10.1109/TAC.2008.919255.

[4] L. Arnold. *Random Dynamical Systems*. Springer, Jan 1998. ISBN 978-35-4063-758-5.

[5] L. Arnold, W. Horsthemke, and J. W. Stucki. The Influence of External Real and White Noise on the LOTKA-VOLTERRA Model. *Biometrical Journal*, 21(5):451–471, 1979. ISSN 03233847. doi: 10.1002/bimj.4710210507. URL http://doi.wiley.com/10.1002/bimj.4710210507.

[6] S. S. Artemiev and T. A. Averina. *Numerical Analysis of Systems of Ordinary and Stochastic Differential Equations*. VSP, 1997. ISBN 9789067642507.

[7] C. T. H. Baker and E. Buckwar. Numerical Analysis of Explicit One-Step Methods for Stochastic Delay Differential Equations. *LMS Journal of Computation and Mathematics*, 3(January):315–335, Feb. 2010. ISSN 1461-1570. doi: 10.1112/S1461157000000322. URL http://journals.cambridge.org/abstract_S1461157000000322http://www.journals.cambridge.org/abstract_S1461157000000322.

[8] W.-j. Beyn, E. Isaak, and R. Kruse. Stochastic c-stability and b-consistency of explicit and implicit euler-type schemes. pages 1–29, 2010.

[9] R. H. Bokor. Stochastically stable one-step approximations of solutions of stochastic ordinary differential equations. *Applied Numerical Mathematics*, 44(3):299–312, Feb. 2003. ISSN 01689274. doi: 10.1016/S0168-9274(02)00141-1. URL http://linkinghub.elsevier.com/retrieve/pii/S0168927402001411.

[10] A. C. Brańka and D. M. Heyes. Algorithms for Brownian dynamics simulation. *Physical Review E*, 58 (2):2611–2615, Aug. 1998. ISSN 1063-651X. doi: 10.1103/PhysRevE.58.2611. URL http://pre.aps.org/abstract/PRE/v58/i2/p2611_1.

[11] E. Buckwar and R. Winkler. Multistep methods for SDEs and their application to problems with small noise. *SIAM Journal on Numerical Analysis*, 44(2):779–803, 2006. doi: 10.1137/040602857. URL http://dx.doi.org/10.1137/040602857.

[12] E. Buckwar, M. G. Riedler, and P. E. Kloeden. The Numerical Stability of Stochastic Ordinary Differential Equations With Additive Noise. *Stochastics and Dynamics*, 11(02n03):265–281, Sept. 2011. ISSN 0219-4937. doi: 10.1142/S0219493711003279. URL http://www.worldscientific.com/doi/abs/10.1142/S0219493711003279.

[13] K. Burrage, P. M. Burrage, and T. Tian. Numerical methods for strong solutions of stochastic differential equations: an overview. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 460(2041):373–402, Jan. 2004. ISSN 1364-5021. doi: 10.1098/rspa.2003.1247. URL http://rspa.royalsocietypublishing.org/cgi/doi/10.1098/rspa.2003.1247.

[14] G. Bussi and M. Parrinello. Accurate sampling using Langevin dynamics. *Physical Review E*, 75(5): 056707, May 2007. ISSN 1539-3755. doi: 10.1103/PhysRevE.75.056707. URL http://link.aps.org/doi/10.1103/PhysRevE.75.056707.

[15] T. Caraballo and P. E. Kloeden. The Pathwise Numerical Approximation of Stationary Solutions of Semilinear Stochastic Evolution Equations. *Applied Mathematics and Optimization*, 54(3):401–415, Oct. 2006. ISSN 0095-4616. doi: 10.1007/s00245-006-0876-z. URL http://link.springer.com/10.1007/s00245-006-0876-z.

[16] C. Cruz, F. Chinesta, and G. Régnier. Review on the Brownian Dynamics Simulation of Bead-Rod-Spring Models Encountered in Computational Rheology. *Archives of Computational Methods in Engineering*, 19(2):227–259, June 2012. ISSN 1134-3060. doi: 10.1007/s11831-012-9072-2. URL http://link.springer.com/10.1007/s11831-012-9072-2.

[17] N. Dalal, D. Greenhalgh, and X. Mao. A stochastic model for internal HIV dynamics. *Journal of Mathematical Analysis and Applications*, 341(2):1084–1101, May 2008. ISSN 0022247X. doi: 10.1016/j.jmaa.2007.11.005. URL http://linkinghub.elsevier.com/retrieve/pii/S0022247X07013170.

[18] H. De la Cruz Cancino, R. J. Biscay, J. C. Jimenez, F. Carbonell, and T. Ozaki. High order local linearization methods: An approach for constructing A-stable explicit schemes for stochastic differential equations with additive noise. *BIT Numerical Mathematics*, 50(3):509–539, 2010. ISSN 0006-3835. doi: 10.1007/s10543-010-0272-6. URL http://dx.doi.org/10.1007/s10543-010-0272-6.

[19] S. Díaz-Infante and S. Jerez. Convergence and asymptotic stability of the explicit Steklov method for stochastic differential equations. *Journal of Computational and Applied Mathematics*, 291:36–47, 2015. ISSN 03770427. doi: 10.1016/j.cam.2015.01.016. URL http://linkinghub.elsevier.com/retrieve/pii/S037704271500028X.

[20] D. L. Ermak and J. A. McCammon. Brownian dynamics with hydrodynamic interactions. *The Journal of Chemical Physics*, 69(4):1352, 1978. ISSN 00219606. doi: 10.1063/1.436761. URL http://link.aip.org/link/JCPSA6/v69/i4/p1352/s1&Agg=doihttp://scitation.aip.org/content/aip/journal/jcp/69/4/10.1063/1.436761.

[21] A. Fine and S. Kass. Indeterminate forms for multi-place functions. *Annales Polonici Mathematici*, 18 (1):59–64, 0 1966.

[22] J. B. Gao, W.-w. Tung, and N. Rao. Noise-Induced Hopf-Bifurcation-Type Sequence and Transition to Chaos in the Lorenz Equations. *Phys. Rev. Lett.*, 89:254101, Nov 2002. doi: 10.1103/PhysRevLett.89.254101. URL http://link.aps.org/doi/10.1103/PhysRevLett.89.254101.

[23] C. Gardiner. *Stochastic Methods: A Handbook for the Natural and Social Sciences*. Springer Series in Synergetics. Springer, 2009. ISBN 9783540707127.

[24] M. B. Giles. Multilevel Monte Carlo Path Simulation. *Operations Research*, 56(3):607–617, 2008. doi: 10.1287/opre.1070.0496.

[25] Q. Guo, H. Li, and Y. Zhu. The improved split-step θ methods for stochastic differential equation. *Mathematical Methods in the Applied Sciences*, 37(August 2013):2245–2256, 2014. ISSN 01704214. doi: 10.1002/mma.2972. URL http://doi.wiley.com/10.1002/mma.2972.

[26] H. Hardy, E. J. Littlewood, and G. Pólya. *Inequalities*. Cambridge University Press, 2 edition, 1934. ISBN 0521358809,9780521358804. URL http://gen.lib.rus.ec/book/index.php?md5=9ACC936E3629BB6CCC8333861C484A1E.

[27] P. Hartman. A lemma in the theory of structural stability of differential equations. *Proceedings of the American Mathematical Society*, 11(4):610–610, apr 1960. ISSN 0002-9939. doi: 10.1090/S0002-9939-1960-0121542-7. URL http://www.jstor.org/stable/2034720http://www.ams.org/jourcgi/jour-getitem?pii=S0002-9939-1960-0121542-7.

[28] D. B. Hernandez and R. Spigler. A-stability of Runge-Kutta methods for systems with additive noise. *BIT*, 32(4):620–633, Dec. 1992. ISSN 0006-3835. doi: 10.1007/BF01994846. URL http://link.springer.com/10.1007/BF01994846.

[29] D. J. Higham. A-Stability and Stochastic Mean-Square Stability. *BIT*, 40(2):404–409, 2000. doi: 10.1023/A:1022355410570.

[30] D. J. Higham. Mean-Square and Asymptotic Stability of the Stochastic Theta Method. *SIAM Journal on Numerical Analysis*, 38(3):753–769, Jan. 2000. ISSN 0036-1429. doi: 10.1137/S003614299834736X. URL http://epubs.siam.org/doi/pdf/10.1137/S003614299834736Xhttp://epubs.siam.org/doi/abs/10.1137/S003614299834736X.

[31] D. J. Higham, X. Mao, and A. M. Stuart. Strong Convergence of Euler-Type Methods for Nonlinear Stochastic Differential Equations. *SIAM Journal on Numerical Analysis*, 40(3):1041–1063, Jan. 2002. ISSN 0036-1429. doi: 10.1137/S0036142901389530. URL http://epubs.siam.org/doi/abs/10.1137/S0036142901389530.

[32] M. Hutzenthaler and A. Jentzen. Numerical approximations of stochastic differential equations with non-globally Lipschitz continuous coefficients. *Memoirs of the American Mathematical Society*, 236(1112), July 2015. ISSN 1947-6221. doi: 10.1090/memo/1112. URL http://www.ams.org.

[33] M. Hutzenthaler, A. Jentzen, and P. E. Kloeden. Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 467(2130):1563–1576, Dec. 2010. ISSN 1364-5021. doi: 10.1098/rspa.2010.0348. URL http://rspa.royalsocietypublishing.org/cgi/doi/10.1098/rspa.2010.0348.

[34] M. Hutzenthaler, A. Jentzen, and P. E. Kloeden. Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients. *The Annals of Applied Probability*, 22(4):1611–1641, Aug. 2012. ISSN 1050-5164. doi: 10.1214/11-AAP803. URL http://projecteuclid.org/euclid.aoap/1344614205.

[35] R. Khasminskii and G. N. Milstein. *Stochastic Stability of Differential Equations*. Stochastic Modelling and Applied Probability. Springer Berlin Heidelberg, 2011. ISBN 9783642232800.

[36] P. E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations*. Springer-Verlag, Jan 1992. ISBN 978-03-8754-062-7.

[37] P. E. Kloeden, H. Keller, and B. Schmalfuß. Towards a Theory of Random Numerical Dynamics. In H. Crauel and M. Gundlach, editors, *Stochastic Dynamics*, pages 259–282. Springer New York, 1999. ISBN 978-0-387-98512-1.

[38] H.-H. Kuo. *Introduction to stochastic integration*. Springer, 2006 edition, 2005. ISBN 0387287205. URL http://www.worldcat.org/isbn/0387287205.

[39] H. Lamba, J. C. Mattingly, and A. M. Stuart. An adaptive Euler-Maruyama scheme for SDEs: Convergence and stability. *IMA Journal of Numerical Analysis*, 27:479–506, 2007. ISSN 02724979. doi: 10.1093/imanum/drl032.

[40] G. R. Lawlor. A L'hospital's rule for multivariable functions. *arXiv preprint arXiv:1209.0363*, 2012.

[41] R. Liptser and N. A. Shiryaev. *Theory of Martingales*. Springer, Oct 1989. ISBN 978-94-010-7600-5.

[42] W. Liu and X. Mao. Strong convergence of the stopped Euler-Maruyama method for nonlinear stochastic differential equations. *Applied Mathematics and Computation*, 223:389–400, 2013. ISSN 00963003. doi: 10.1016/j.amc.2013.08.023. URL http://dx.doi.org/10.1016/j.amc.2013.08.023.

[43] H. Lowen and G. Szamel. Long-time self-diffusion coefficient in colloidal suspensions: theory versus simulation. *Journal of Physics: Condensed Matter*, 5(15):2295–2306, Apr. 1993. ISSN 0953-8984. doi: 10.1088/0953-8984/5/15/003. URL http://iopscience.iop.org/0953-8984/5/15/003http://stacks.iop.org/0953-8984/5/i=15/a=003?key=crossref.b20e8001a77af6b95b10e4510a9eab3b.

[44] X. Mao. *Stochastic Differential Equations and Application*. Horwood Pub., Dec 2007. ISBN 978-19-0427-534-3.

[45] X. Mao. The truncated Euler-Maruyama method for stochastic differential equations. *Journal of Computational and Applied Mathematics*, 290:370–384, 2015. ISSN 0377-0427. doi: 10.1016/j.cam.2015.06.002. URL http://www.sciencedirect.com/science/article/pii/S0377042715003210.

[46] X. Mao and L. Szpruch. Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally Lipschitz continuous coefficients. *Journal of Computational and Applied Mathematics*, 238:14–28, Jan. 2013. ISSN 03770427. doi: 10.1016/j.cam.2012.08.015. URL http://linkinghub.elsevier.com/retrieve/pii/S0377042712003378http://strathprints.strath.ac.uk/13892/.

[47] X. Mao and L. Szpruch. Strong convergence rates for backward Euler-Maruyama method for non-linear dissipative-type stochastic differential equations with super-linear diffusion coefficients. *Stochastics An International Journal of Probability and Stochastic Processes*, 85(1):144–171, Feb. 2013. ISSN 1744-2508. doi: 10.1080/17442508.2011.651213. URL http://www.tandfonline.com/doi/abs/10.1080/17442508.2011.651213.

[48] X. Mao, G. Marion, and E. Renshaw. Environmental Brownian noise suppresses explosions in population dynamics. *Stochastic Processes and their Applications*, 97(1):95–110, Jan. 2002. ISSN 03044149. doi: 10.1016/S0304-4149(01)00126-0. URL http://linkinghub.elsevier.com/retrieve/pii/S0304414901001260.

[49] X. Mao, S. Sabanis, and E. Renshaw. Asymptotic behaviour of the stochastic Lotka-Volterra model. *Journal of Mathematical Analysis and Applications*, 287(1):141–156, 2003. ISSN 0022247X. doi: 10.1016/S0022-247X(03)00539-0.

[50] P. Matus, U. Irkhin, and M. Lapinska-Chrzczonowicz. Exact Difference Schemes for Time-dependent Problems. *Computational Methods in Applied Mathematics*, 5(4):422–448, 2005. ISSN 1609-4840. doi: 10.2478/cmam-2005-0020. URL http://old.cmam.info/issues/?Vol=5&Num=4&ItID=133&Type=pdf&Aid=133http://www.degruyter.com/view/j/cmam.2005.5.issue-4/cmam-2005-0020/cmam-2005-0020.xml.

[51] G. N. Milstein. Quasi-symplectic methods for Langevin-type equations. *IMA Journal of Numerical Analysis*, 23(4):593–626, Oct. 2003. ISSN 0272-4979. doi: 10.1093/imanum/23.4.593. URL http://imanum.oupjournals.org/cgi/doi/10.1093/imanum/23.4.593.

[52] G. N. Milstein and M. Tretyakov. *Stochastic Numerics for Mathematical Physics*. Scientific Computation. Springer Berlin Heidelberg, 2004. ISBN 9783540211105.

[53] G. N. Milstein, E. Platen, and H. Schurz. Balanced implicit methods for stiff stochastic systems. *SIAM Journal on Numerical Analysis*, 35(3):1010–1019, 1998. URL http://www.jstor.org/stable/2587119.

[54] S. Pasquali. The stochastic logistic equation : stationary solutions and their stability. *Rendiconti del Seminario Matematico della Università di Padova*, 106:165–183, 2001. URL http://eudml.org/doc/108561https://eudml.org/doc/108561.

[55] S. Resnick. *A Probability Path*. A Probability Path. Birkhäuser Boston, 2003. ISBN 9780817640552.

[56] J. C. Robinson. Stability of random attractors under perturbation and approximation. *Journal of Differential Equations*, 186(2):652–669, Dec. 2002. ISSN 00220396. doi: 10.1016/S0022-0396(02)00038-4. URL http://linkinghub.elsevier.com/retrieve/pii/S0022039602000384.

[57] S. Sabanis. Euler approximations with varying coefficients : the case of superlinearly growing diffusion coefficients. To appear in Annals of Applied Probability, 2015.

[58] Y. Saito. Stability analysis of numerical methods for stochastic systems with additive noise. http://www.shotoku.ac.jp/data/facilities/library/publication/education-keizai08_2_06.pdf.

[59] Y. Saito and T. Mitsui. Stability Analysis of Numerical Schemes for Stochastic Differential Equations. *SIAM Journal on Numerical Analysis*, 33(6):2254–2267, Dec. 1996. ISSN 0036-1429. doi: 10.1137/S0036142992228409. URL http://epubs.siam.org/doi/abs/10.1137/S0036142992228409.

[60] K. R. Schenk-Hoppé. Stochastic Hopf bifurcation: An example. *International Journal of Non-Linear Mechanics*, 31(5 SPEC. ISS.):685–692, 1996. ISSN 00207462. doi: 10.1016/0020-7462(96)00030-3.

[61] H. Schurz. Modeling, analysis and discretization of stochastic logistic equations. *International Journal of Numerical Analysis and Modeling*, 4(2):178–197, 2007. URL http://www.math.ualberta.ca/ijnam/Volume-4-2007/No-2-07/2007-02-02.pdf.

[62] N. A. Shiryaev and R. P. Boas. *Probability*. Graduate texts in mathematics Springer series in Soviet mathematics 95. Springer, 2nd ed edition, 1996. ISBN 9780387945491,0387945490.

[63] X. Sun and Y. Wang. Stability analysis of a stochastic logistic model with nonlinear diffusion term. *Applied Mathematical Modelling*, 32(10):2067–2075, Oct. 2008. ISSN 0307904X. doi: 10.1016/j.apm.2007.07.012. URL http://linkinghub.elsevier.com/retrieve/pii/S0307904X07001746.

[64] A. F. Timan. *Theory of Approximation of Functions of a Real Variable*. Dover Publications, Feb 1963. ISBN 978-04-8667-830-6.

[65] E. Tornatore, S. M. Buccellato, and P. Vetro. Stability of a stochastic {SIR} system. *Physica A: Statistical Mechanics and its Applications*, 354:111–126, 2005. ISSN 0378-4371. doi: 10.1016/j.physa.2005.02.057. URL http://www.sciencedirect.com/science/article/pii/S0378437105001093.

[66] M. Tretyakov and Z. Zhang. A Fundamental Mean-Square Convergence Theorem for SDEs with Locally Lipschitz Coefficients and Its Applications. *SIAM Journal on Numerical Analysis*, 51(6):3135–3162, Jan. 2013. ISSN 0036-1429. doi: 10.1137/120902318. URL http://epubs.siam.org/doi/abs/10.1137/120902318.

[67] N. Van Kampen. *Stochastic Processes in Physics and Chemistry*. North-Holland Personal Library. Elsevier Science, 1992. ISBN 9780080571386.

[68] X. Wang and S. Gan. The tamed Milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients. *Journal of Difference Equations and Applications*, 19(3):466–490, Mar. 2013. ISSN 1023-6198. doi: 10.1080/10236198.2012.656617. URL http://arxiv.org/abs/1102.0662http://www.tandfonline.com/doi/abs/10.1080/10236198.2012.656617.

[69] D. Williams. *Probability with Martingales*. Cambridge University Press, 1991. ISBN 9781139642989.

[70] X. Zong, F. Wu, and C. Huang. Convergence and stability of the semi-tamed Euler scheme for stochastic differential equations with non-Lipschitz continuous coefficients. *Applied Mathematics and Computation*, 228(2013):240–250, 2014. ISSN 00963003. doi: 10.1016/j.amc.2013.11.100. URL http://dx.doi.org/10.1016/j.amc.2013.11.100.