Centro de Investigación en Matemáticas, A.C.

CIMAT

On the Signature of algebraically defined degenerate bilinear forms on Complete Intersection Algebras of Finite Dimension and its Application to the Relative Index of Vector Fields tangent to a Hypersurface with an isolated Singularity in R^n

T H E S I S

To obtain the degree of

Doctor in Science

With orientation in

Basic Math

P r e s e n t s

María de la Paz Suárez Fernández

Co-Directors of Thesis:

Dr. Xavier Gómez-Mont Ávalos

Dr. Pedro Luis del Ángel Rodríguez

December 12th, 2015

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Report submitted by

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To my husband and son

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Contents

Abstract

Let $\mathcal{A}_{\mathbb{R}^n,0}$ be the ring of germs of real analytic functions on $\mathbb{R}^n$ at 0 and consider $n+1$ germs of real analytic functions $f, f_1, ..., f_n : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ such that $(f, f_2, ..., f_n)$ and $(f_1, f_2, ..., f_n)$ are regular sequences (see [10]). We denote by f^{m+1} the $(m+1)^{th}$ power of f. For $m \geq 0$ let us introduce the $\mathbb{R}$-algebras

$$B_m := \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f^{m+1}, f_2, ..., f_n)}. \tag{1}$$

They are finite dimensional vector spaces over $\mathbb{R}$. Their dimensions are $(m+1)dim_{\mathbb{R}} B_0$.

Introduce the symmetric bilinear forms

$$\langle\ ,\rangle_m : B_m \times B_m \stackrel{\cdot}{\longrightarrow} B_m \stackrel{L_m}{\longrightarrow} \mathbb{R} \tag{2}$$

$$([a]_m, [b]_m) \mapsto [ab]_m \mapsto L_m([ab]_m),$$

where $\cdot$ denotes multiplication in the algebra B_m and L_m is an $\mathbb{R}$-linear map with $L_m([J]_m) > 0$, where $[J]_m$ in B_m is the class of the Jacobian determinant of $f, f_2, \ldots, f_n$. It is a theorem of Eisenbud and Levine (see [11]) that this bilinear form is nondegenerate and its signature, denoted by σ_m, is independent of the chosen L_m.

Theorem 0.1. *Let $\{f, f_2, ..., f_n\} : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be germs of real analytic functions forming a regular sequence, and σ_m the signature of the nondegenerate bilinear form in* (2)*, then we have*

$$\sigma_m = \begin{cases} \sigma_0 & if \quad m \quad is \quad even \\ 0 & if \quad m \quad is \quad odd. \end{cases} \tag{3}$$

Now for $m \geq 0$ define the m^{th} relative (symmetric degenerate) bilinear form

$$\langle\ ,\rangle_m^{rel} : B_m \times B_m \stackrel{\cdot}{\longrightarrow} B_m \stackrel{f_1}{\longrightarrow} B_m \stackrel{L_m}{\longrightarrow} \mathbb{R} \tag{4}$$

$$([a]_m, [b]_m) \mapsto [ab]_m \mapsto [f_1 ab]_m \mapsto L_m([f_1 ab]_m),$$

where $L_m : B_m \to \mathbb{R}$ is any linear map such that $L_m([f^m J]_m) > 0$ and we are using the multiplication in B_m in the expression $f_1 ab$. Its degeneracy locus is the annihilator of f_1 on B_m:

$$Ann_{B_m}(f_1) := \{b \in \mathcal{C} : [bf_1]_m = 0 \text{ on } B_m\}$$

and we denote by $\tilde{\sigma}_m^{rel}$ its signature.

Let $\langle\ ,\ \rangle_{m,Ann}^{rel}$ be the degenerate symmetric bilinear form obtained by restricting $\langle\ ,\rangle_m^{rel}$ to $Ann_{B_{m-1}}(f_1) \times f^m B_0$:

$$\langle\ ,\ \rangle_{m,Ann}^{rel} : (Ann_{B_{m-1}}(f_1) \oplus f^m B_0) \times (Ann_{B_{m-1}}(f_1) \oplus f^m B_0)$$

$$\stackrel{\cdot}{\longrightarrow} B_m \stackrel{f_1}{\longrightarrow} B_m \stackrel{L_m}{\longrightarrow} \mathbb{R}. \tag{5}$$

and denote by $\tilde{\sigma}_{m,Ann}^{rel}$ its signature.

Our main result is:

Theorem 0.2. *Let $\{f, f_2, ..., f_n\}$ and $\{f_1, f_2, ..., f_n\}$ be germs of real analytic functions on $\mathbb{R}^n$ forming regular sequences, then*

1) We have for even $m \geq 0$:

$$\tilde{\sigma}_m^{rel} = \tilde{\sigma}_0^{rel} + \tilde{\sigma}_{2,Ann}^{rel} + \tilde{\sigma}_{4,Ann}^{rel} + \cdots + \tilde{\sigma}_{m,Ann}^{rel},$$

and for odd $m \geq 1$:

$$\tilde{\sigma}_m^{rel} = \tilde{\sigma}_1^{rel} + \tilde{\sigma}_{3,Ann}^{rel} + \tilde{\sigma}_{5,Ann}^{rel} + \cdots + \tilde{\sigma}_{m,Ann}^{rel}.$$

2) For m large enough, $\tilde{\sigma}_{m,Ann}^{rel} = 0$.

3) For $m \geq 0$ we have the recursive formulas:

$$\sigma_{m+1}^{rel} = \sigma_{m-1}^{rel} + \sigma_{m+1,Ann}^{rel}$$

with $\sigma_{-1}^{rel} := 0$.

Chapter 1

Introduction

The objective of this thesis is to shed light on the following fact:

For a germ of a real analytic function $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with an algebraically isolated singularity the most basic topological invariant of the Milnor fiber $V_t := \{f = t\}$, its Euler characteristic, can change depending on the sign of t, if n is odd.

For example, for $f = x^2 + y^2 - z^2$ we pass from 2 2-dimensional discs to a 2-dimensional annulus, changing the Euler characteristic from 2 to 0.

Our tool to analyse the mathematics around this jump in the Euler characteristic is through a family of vector fields X_t on $\mathbb{R}^n$ with isolated singularities, each of multiplicity one for $t \neq 0$, and it is tangent to the Milnor fiber V_t. The tangency condition may be written as

$$\sum_{j=0}^{n} \frac{\partial(f-t)}{\partial x_j} X_t^j = (f-t)h_t$$

for a real analytic function $h_t(x)$ called the cofactor.

By the Poincaré-Hopf index theorem, we have for t fixed the sum

$$\sum_{X_t(p)=0} Ind_{\mathbb{R}^n}(X_t, p)$$

is independent of t. At a singular point of the vector field X_t at $p \in V_t$, with $t \neq 0$, besides the Poincaré-Hopf index, we have the relative Poincaré-Hopf index which is the Poincaré-Hopf index of the restriction $X_t|_{V_t}$ to the $n-1$ dimensional manifold V_t. For $t \neq 0$ fixed, the sum

$$\sum_{X_t(p)=0=f(p)-t} Ind_{V_t}(X_t|_{V_t}, p)$$

is locally constant so we have a value for $t > 0$ and another one for $t < 0$. The relation between the 2 indices is, for $t \neq 0$

$$Ind_{\mathbb{R}^n}(X_t, p) = \pm Ind_{V_t}(X_t|_{V_t}, p)$$

where they coincide if $h_t(p) > 0$ and they differ by the sign if $h_t(p) < 0$.

For example, the family of contact vector fields

$$X_t := (f - t)\frac{\partial}{\partial x_1} + \sum_{i=1}^{N}(\frac{\partial f}{\partial x_{2i+1}}\frac{\partial}{\partial x_{2i}} - \frac{\partial f}{\partial x_{2i}}\frac{\partial}{\partial x_{2i+1}}) \tag{1.1}$$

satisfies the tangency relation

$$d(f - t)(X_t) = \frac{\partial f}{\partial x_1}(f - t)$$

in an odd dimensional ambient space. All the singular points of X_t are contained in this case in V_t, so the subindex in the above sums is over the same set, so what is involved are the sign properties of $\frac{\partial f}{\partial x_1}$ when restricted to the connected components of the smooth curve $\{\frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_n} = 0\} - \{0\} \subset \mathbb{R}^n$.

In this thesis, we describe an algebraic method to compute from 'infinitesimal information' of the family of vector fields X_t at the point $t = 0$ the index of the family to the right and to the left. The vector field $X_0|_{V_0}$ determines only part of this index, and one must look at higher order terms

$$\frac{\partial^j X_t}{\partial t^j}|_{V_0}$$

to see the other contributions.

More generally, let $\mathcal{A}_{\mathbb{R}^n,0}$ be the ring of germs of real analytic functions on $\mathbb{R}^n$ at 0 and consider $n+1$ germs of real analytic functions $f, f_1, ..., f_n : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ such that $(f, f_2, ..., f_n)$ and $(f_1, f_2, ..., f_n)$ are regular sequences (see [10]). We denote by f^{m+1} the $(m+1)^{th}$ power of f. For $m \geq 0$ let us introduce the $\mathbb{R}$-algebras

$$B_m := \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f^{m+1}, f_2, ..., f_n)}. \tag{1.2}$$

They are finite dimensional vector spaces over $\mathbb{R}$. Their dimensions are $(m+1)dim_{\mathbb{R}}B_0$.

Let

$$J := Det\begin{pmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \\ & \cdots & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \qquad J_m := [J]_m \in B_m$$

denote the Jacobian J and its class $[J]_m$ in B_m.

The absolute index

Introduce the symmetric bilinear forms

$$\langle\ ,\rangle_m : B_m \times B_m \stackrel{\cdot}{\longrightarrow} B_m \xrightarrow{L_m} \mathbb{R} \tag{1.3}$$

$$([a]_m, [b]_m) \mapsto [ab]_m \mapsto L_m([ab]_m),$$

where $\cdot$ denotes multiplication in the algebra B_m and L_m is an $\mathbb{R}$-linear map with $L_m([J]_m) > 0$. (See [11]) that this bilinear form is nondegenerate and its signature, denoted by $\tilde{\sigma}_m$, is independent of the chosen L_m.

Our first result is:

Theorem 1.1. *Let* $\{f, f_2, ..., f_n\} : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *be germs of real analytic functions forming a regular sequence, and* $\tilde{\sigma}_m$ *the signature of the nondegenerate bilinear form in* (1.3)*, then we have*

$$\tilde{\sigma}_m = \begin{cases} \tilde{\sigma}_0 & if \quad m \quad is \quad even \\ 0 & if \quad m \quad is \quad odd. \end{cases} \tag{1.4}$$

The proof relies on choosing $v_1, \ldots, v_s \in \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f_2,\ldots,f_n)}$ such that $[v_1]_0, \ldots, [v_s]_0 \in B_0$ are an $\mathbb{R}$-basis with $[v_1]_0 = 1$ and $[v_s] = [J]_0$, and considering the basis of B_m:

$$[v_1]_m, \ldots, [v_s]_m, [fv_1]_m, \ldots, [fv_s]_m, \ldots, [f^m v_1]_m, \ldots, [f^m v_s]_m \tag{1.5}$$

that provide $\mathbb{R}$-vector space isomorphisms

$$B_m = B_0 \bigoplus fB_0 \bigoplus \cdots \bigoplus f^m B_0 \tag{1.6}$$

and $\mathbb{R}$-vector space inclusions

$$B_0 \hookrightarrow B_1 \hookrightarrow \ldots \hookrightarrow B_{m-1} \hookrightarrow B_m.$$

We also choose for $L_m : B_m \longrightarrow \mathbb{R}$ the map sending all the base elements to 0, except the last where $L_m([f^m J]_m) = 1$. Using this block decomposition of B_m, the multiplication table

$$\mu_m : B_m \times B_m \longrightarrow B_m$$

takes the form

$$Q_m = \left[\begin{array}{c|c|c|c} Q_0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \vdots & \ddots & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] + f \left[\begin{array}{c|c|c|c} H_1 & Q_0 & \cdots & 0 \\ \hline Q_0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \vdots & \ddots & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] + \cdots$$

$$\cdots + f^m \left[\begin{array}{l|c|c|c|c} H_m & H_{m-1} & \cdot & H_1 & Q_0 \\ \hline H_{m-1} & \cdots & H_1 & Q_0 & 0 \\ \hline \vdots & \ddots & \ddots & 0 & 0 \\ \hline H_1 & Q_0 & 0 & 0 & 0 \\ \hline Q_0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{1.7}$$

where $Q_0, H_1, \ldots, H_m$ are symmetric $(s\times s)$-matrices with entries in B_0. The expression of these matrices can be obtained from the restriction of μ_m to B_0 and using the isomorphism $B_m \simeq \bigoplus_{j=0}^m f^j B_0$ we obtain a bilinear form

$$\mu_m : B_0 \times B_0 \longrightarrow B_m$$

with matrix expression

$$Q_0 + fH_1 + \cdots + f^m H_m.$$

Q_0 is the matrix expression of the multiplication μ_0 on B_0 and the H_j are the higher order terms in the multiplication μ_m restricted to $B_0 \hookrightarrow B_m$. These terms contain all the information needed for describing μ_m, as can be seen from the expression (1.7). Applying to (1.7) the chosen L_m, the matrix for the bilinear form (1.3) is

$$L_{m*}Q_m = \left[\begin{array}{l|c|c|c|r} L_{0*}H_m & L_{0*}H_{m-1} & \cdot & L_{0*}H_1 & L_{0*}Q_0 \\ \hline L_{0*}H_{m-1} & \cdots & L_{0*}H_1 & L_{0*}Q_0 & 0 \\ \hline \vdots & \ddots & \ddots & 0 & 0 \\ \hline L_{0*}H_1 & L_{0*}Q_0 & 0 & 0 & 0 \\ \hline L_{0*}Q_0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{1.8}$$

Here $L_{m*} : Sym(B_m) \to Sym(\mathbb{R})$ is the operation on matrices with entries in B_m to matrices with real entries obtained by applying L_m to the entries.

Observing the anti-triangular form of (1.8) and the fact that the anti-diagonal terms are non-singular matrices, we may do then a change basis for the $\mathbb{R}$-vector space $\langle v_1, \ldots, f^m v_s\rangle \hookrightarrow \mathcal{A}_{\mathbb{R}^n,0}$ to obtain a matrix representation of $< \, , \, >_m$ as an anti-diagonal matrix by blocks, with all the anti-diagonal terms being the matrix $L_{0*}Q_0$:

$$L_{m*}Q_m = \left[\begin{array}{l|c|c|c|r} 0 & 0 & \cdot & 0 & L_{0*}Q_0 \\ \hline 0 & \cdots & 0 & L_{0*}Q_0 & 0 \\ \hline \vdots & \ddots & \ddots & 0 & 0 \\ \hline 0 & L_{0*}Q_0 & 0 & 0 & 0 \\ \hline L_{0*}Q_0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{1.9}$$

The matrix (1.9) suggests that we consider the decomposition of B_m as:

$$B_m = [B_0 \oplus f^m B_0] \bigoplus [fB_0 \oplus f^{m-1}B_0] \bigoplus \cdots , \tag{1.10}$$

It is an $< , >_m$-orthogonal direct sum. The contribution to the signature of each vector space within a bracket is 0, since they have the form

$$\begin{pmatrix} 0 & L_{0*}Q_0 \\ L_{0*}Q_0 & 0 \end{pmatrix}. \tag{1.11}$$

Therefore, if m is odd, the brackets in (1.10) are paired off, giving a signature $\sigma_m = 0$. If m is even, in the above pairing (1.10), we are left with 1 block $L_{0*}Q_0$ which does not have a term to pair with. This term then gives the only non-zero contribution σ_0 to the signature σ_m of B_m.

The relative index

We introduce now the main object of analysis in this thesis; the relative bilinear form. Let $\{f, f_1, ..., f_n\}$, B_m as before (we did not use f_1 for Theorem 1.1). For $m \geq 0$ define the m^{th} relative (symmetric degenerate) bilinear form

$$\langle\ ,\rangle_m^{rel} : B_m \times B_m \xrightarrow{\cdot} B_m \xrightarrow{f_1} B_m \xrightarrow{L_m} \mathbb{R} \tag{1.12}$$

$$([a]_m, [b]_m) \mapsto [ab]_m \mapsto [f_1 ab]_m \mapsto L_m([f_1 ab]_m),$$

where $L_m : B_m \to \mathbb{R}$ is any linear map such that $L_m[f^m J_m]_m > 0$ and we are using the multiplication in B_m in the expression $f_1 ab$. Its degeneracy locus is the annihilator of f_1 on B_m:

$$Ann_{B_m}(f_1) := \{b \in \mathcal{C} : [bf_1]_m = 0 \text{ on } B_m\}$$

and we denote by $\tilde{\sigma}_m^{rel}$ its signature.

Let $\langle\ ,\ \rangle_{m,Ann}^{rel}$ be the degenerate symmetric bilinear form obtained by restricting $\langle\ ,\rangle_m^{rel}$ to $Ann_{B_{m-1}}(f_1) \oplus f^m B_0$:

$$\langle\ ,\ \rangle_{m,Ann}^{rel} : (Ann_{B_{m-1}}(f_1) \oplus f^m B_0) \times (Ann_{B_{m-1}}(f_1) \oplus f^m B_0)$$

$$\xrightarrow{\cdot} B_m \xrightarrow{f_1} B_m \xrightarrow{L_m} \mathbb{R}. \tag{1.13}$$

and denote by $\tilde{\sigma}_{m,Ann}^{rel}$ its signature.

Our main result is:

Theorem 1.2. *Let $\{f, f_2, ..., f_n\}$ and $\{f_1, f_2, ..., f_n\}$ be germs of real analytic functions on $\mathbb{R}^n$ forming regular sequences, then*

1) We have for even $m \geq 0$:

$$\tilde{\sigma}_m^{rel} = \tilde{\sigma}_0^{rel} + \tilde{\sigma}_{2,Ann}^{rel} + \tilde{\sigma}_{4,Ann}^{rel} + \ldots + \tilde{\sigma}_{m,Ann}^{rel},$$

and for odd $m \geq 1$:

$$\tilde{\sigma}_m^{rel} = \tilde{\sigma}_1^{rel} + \tilde{\sigma}_{3,Ann}^{rel} + \tilde{\sigma}_{5,Ann}^{rel} + \ldots + \tilde{\sigma}_{m,Ann}^{rel}.$$

2) For m large enough, $\tilde{\sigma}_{m,Ann}^{rel} = 0$.

3) For $m \geq 0$ we have the recursive formulas:

$$\sigma_{m+1}^{rel} = \sigma_{m-1}^{rel} + \sigma_{m+1,Ann}^{rel}$$

with $\sigma_{-1}^{rel} := 0$.

Thus we introduce the invariants σ_{even}^{rel} and σ_{odd}^{rel} as the corresponding sums above, for m sufficiently large.

For the proof, we begin as in the absolute case, by giving a matrix expression in blocks to the map

$$\mu_m^{rel} : B_m \times B_m \longrightarrow B_m \qquad\qquad \mu_m^{rel}([a]_m, [b]_m) = [f_1 ab]_m$$

$$Q_m^{rel} = \left[\begin{array}{c|c|c|c} Q_0^{rel} & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \vdots & \ddots & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] + f \left[\begin{array}{c|c|c|c} H_1^{rel} & Q_0^{rel} & \cdots & 0 \\ \hline Q_0^{rel} & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \vdots & \ddots & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] + \cdots$$

$$\cdots + f^m \left[\begin{array}{c|c|c|c|c} H_m^{rel} & H_{m-1}^{rel} & \cdot & H_1^{rel} & Q_0^{rel} \\ \hline H_{m-1}^{rel} & \cdots & H_1^{rel} & Q_0^{rel} & 0 \\ \hline \vdots & \ddots & \ddots & 0 & 0 \\ \hline H_1^{rel} & Q_0^{rel} & 0 & 0 & 0 \\ \hline Q_0^{rel} & 0 & 0 & 0 & 0 \end{array}\right], \tag{1.14}$$

where $Q_0^{rel}, H_1^{rel}, \ldots H_m^{rel}$ are symmetric $(s \times s)$-matrices with entries in B_0. These matrices are obtained from the restriction of μ_m^{rel} to B_0 and using the isomorphism (1.10):

$$\mu_m^{rel}|_{B_0 \times B_0} : B_0 \times B_0 \longrightarrow B_m = \bigoplus_{j=0}^{m} f^j B_0$$

with matrix expression

$$Q_0^{rel} + fH_1^{rel} + \cdots + f^m H_m^{rel}.$$

Now $L_{0*}Q_0^{rel}$ is the basis expression of $\langle\ ,\rangle_0^{rel}$ on B_0, so we make a change of $\mathbb{R}$-basis in $\langle v_1, \ldots, v_s\rangle$ so that

$$L_{0*}Q_0^{rel} = \begin{pmatrix} I_{p_0} & 0 & 0 \\ 0 & -I_{q_0} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where we have chosen maximal orthogonal subspaces where $\langle\ ,\rangle_0^{rel}$ is positive and negative definite, but the 3^{rd} summand, which is the Annihilator of $\langle\ ,\ \rangle_0^{rel}$ is canonically determined.

In the basis (1.5), the expression of the bilinear form $\langle\ ,\ \rangle_1^{rel}$ on B_1 takes the block form:

$$Q_1^{rel} = \left(\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{13} & I_{p_0} & 0 & 0 \\ A_{12}^t & A_{22} & A_{23} & 0 & -I_{q_0} & 0 \\ A_{13}^t & A_{23}^t & E_1 & 0 & 0 & 0 \\ \hline I_{p_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_{q_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{1.15}$$

We take a new basis of B_1 by applying the product of elementary matrices, as in the Jordan-Gauss elimination method to the bilinear form Q_1^{rel}, applied simultaneously to the rows and columns so as to preserve the symmetry of the matrix expression. In this basis one has the expression

$$Q_1^{rel} = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & I_{p_0} & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_{q_0} & 0 \\ 0 & 0 & E_1 & 0 & 0 & 0 \\ \hline I_{p_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_{q_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{1.16}$$

From this expression we see that the contribution given by the antidiagonal blocks to the signature is again 0 (even though it is degenerate) and the only contribution to the signature comes from the matrix E_1, which is defined where the bilinear form $<\ ,\ >_0^{rel}$ is degenerate.

Making a new change of basis of the third summand $Ker(Q_0^{rel})$, by choosing maximal subspaces where E_1 is positive and negative definite, but the third summand is canonically determined by $KerE_1 \cap KerQ_0^{rel}$. The block representation of $<\ ,\ >_2^{rel}$ takes the form

$$
Q_2^{rel} = \left(\begin{array}{cccc|cccc|cccc}
A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & 0 & I_{p_0} & 0 & 0 & 0 \\
A_{12}^t & A_{22} & A_{23} & A_{24} & A_{16}^t & A_{26} & A_{27} & 0 & 0 & -I_{q_0} & 0 & 0 \\
A_{13}^t & A_{23}^t & A_{33} & A_{34} & A_{17}^t & A_{27}^t & \begin{matrix} I_{p_1} & 0 \\ 0 & -I_{q_1} \end{matrix} & 0 & 0 & 0 & 0 & 0 \\
A_{14}^t & A_{24}^t & A_{34}^t & E_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
A_{15} & A_{16} & A_{17} & 0 & I_{p_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_{16}^t & A_{26} & A_{27} & 0 & 0 & -I_{q_0} & 0 & 0 & 0 & 0 & 0 & 0 \\
A_{17}^t & A_{27}^t & \begin{matrix} I_{p_1} & 0 \\ 0 & -I_{q_1} \end{matrix} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
I_{p_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -I_{q_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \tag{1.17}
$$

After doing a change of basis using the Jordan-Gauss method as before, we obtain the matrix expression

$$
Q_2^{rel} = \left(\begin{array}{cccc|cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{p_0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I_{q_0} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \begin{matrix} I_{p_1} & 0 \\ 0 & -I_{q_1} \end{matrix} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & E_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & I_{p_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -I_{q_0} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \begin{matrix} I_{p_1} & 0 \\ 0 & -I_{q_1} \end{matrix} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
I_{p_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -I_{q_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \tag{1.18}
$$

The antidiagonal terms consist of 3 blocks, the 2 extreme ones give a 0 contribution to the index, and the middle one gives then only 1 contribution to the signature by $p_0 - q_0$. The 2 blocks 1 step above the antidiagonal give no contribution to the signature since they have the form (1.11), and the upper right block gives a contribution to the signature of Q_2^{rel} by the signature of E_2.

So we see the same pattern as in the absolute index, having a distinction between even and odd, but with the difference that the new contribution is not 0 as in the nondegenerate absolute index where we had $\tilde{\sigma}_m = \tilde{\sigma}_{m-2} + 0$, but a new term appears in the Annihilator of the previous form, which the theorem asserts that it eventually becomes 0.

What remains to be explained is how these new contributions $\tilde{\sigma}^{rel}_{j,Ann}$ are organized. To do this, we use the method describe in [13] to transport the signatures $\tilde{\sigma}^{rel}_{j,Ann}$ to the signatures of the primitive components of the algebra $A := \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f_1,\ldots,f_n)}$ with its canonical bilinear form, with respect to multiplication by f. This gives then that $\tilde{\sigma}^{rel}_{j,Ann} = 0$ for $j > dim(B_0)$.

In chapter 2 we describe the topological properties of vector fields that we will use. In chapter 3 we describe the algebraic properties of vector fields that we will use. In chapter 4 we analyse the absolute index and prove Theorem (1.1). In chapter 5 we describe the relative index and prove Theorem (1.2). In Chapter 6 we transport the signatures $\sigma^{rel}_{j,Ann}$ to the algebra A. In chapter 7 we give applications of our theorems for 1-parameter families of vector fields tangent to the Milnor fibres and to contact vector fields.

Theorem (1.1) and Theorem (1.2) are the original contributions of this thesis, and the material although motivated by [13], is not contained in it. The paper [13] centers in analysing the Taylor series expansion of the family of bilinear forms $< \, , \, >_t$ and how far in this expansion one has to go to know the index on the right and the left, and this Thesis centers in the analysis of the family of bilinear forms in the truncated algebras B_m.

Chapter 2

Topological properties of vector fields

Our purpose in this chapter is to give the basic theory of the Poincaré-Hopf index and the relative index (abbreviated as GSV). We explain a procedure to compute the GSV index. We provide a natural example, it has been selected to illustrate an interesting and important phenomena.

2.1 Real analytic germs of vector fields

$\mathcal{C}^\omega$ denotes the class of real analytic functions.

Definition 2.1. *Let $U \subset \mathbb{R}^n$ be an open subset. A real analytic vector field in U is a map $X : U \to \mathbb{R}^n$ of class $\mathcal{C}^\omega$. We define a singularity of a vector field X to be a point p such that $X(p) = 0$. Furthermore, the linear part of the vector field X in p is $DX(p)$, where $DX(p) = \frac{\partial(X^1,...,X^n)}{\partial(x_1,...,x_n)}$ i.e. It is the Jacobian matrix valued at p.*

Definition 2.2. *A germ of a real analytic function at p is an equivalence class of pairs (U, f_1), where U is an open neighborhood of p and f_1 is a real analytic function of U.*

We recall that two pairs (U, f_1), (V, f_2) are equivalent if there exists an open neighborhood $W \subset U \cap V$ of p such that $f_1 |_W = f_2 |_W$. Where f_2 is a real analytic function.

Moreover, let U be a neighborhood of p in $\mathbb{R}^n$ such that p is an isolated singularity of the real analytic vector field X. If $f : U \subset (\mathbb{R}^n, p) \to (\mathbb{R}, p)$ is a germ of a real analytic function, then $df : TU \to T\mathbb{R}$, and

$$df \cdot X := df_p(X(p)).$$

Definition 2.3. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a germ of a real analytic function and $V(f) := \{a \in \mathbb{R}^n : f(a) = 0\}$ be a hypersurface. We say that, p is an isolated singularity of f, or of the hypersurface $V(f)$ if p is an isolated point of $V(f, f_1, ..., f_n)$, where $V(f, f_1, f_2, ..., f_n) := \{a \in \mathbb{R}^n : f(a) = f_1(a), ..., f_n(a) = 0\}$, with $f_i := \frac{\partial f}{\partial_{x_i}}$ $i = 1, ..., n$.*

Definition 2.4. *If f and $V(f)$ are as in the previous definition, we define a singularity algebraically isolated p of f or $V(f)$ to be an isolated point p of $V(f, f_1, ..., f_n)$ after we complexify the function f.*

Proposition 2.1.1. *Let f be a real analytic function and X be a germ of a real analytic vector field with an isolated singularity at 0, such that it is tangent to a smooth hypersurface*

$$V(f) := \{(a_1, ..., a_n) \in \mathbb{R}^n \mid f(a_1, ..., a_n) = 0\}.$$

Then $df \cdot X \mid_{V(f)} = 0$ if and only if $df \cdot X = hf$, for some analytic function h. The function h will be called the cofactor.

Proof. Since 0 is an isolated singularity of the vector field X, there exists a change of coordinates $(w_1, ..., w_n)$, and functions $\varphi, \psi \in \mathcal{C}^\omega$ with $\psi = \varphi^{-1}$ such that

$$f(\varphi(w_1, ..., w_n)) = w_1. \text{ (It is an immediate consequence of the implicit theorem.)}$$

On the other hand, for the vector field X, let us define the vector field $\tilde{X}$ by $\tilde{X} = D\psi X$. Therefore, $df \cdot \tilde{X} = \sum_{j=1}^{n} \tilde{X}_j \frac{\partial f}{\partial w_j} = \tilde{X}_1$. Since $\tilde{X}$ is a vector field of class $\mathcal{C}^\omega$, then the Taylor series expansion of $\tilde{X}_1$ is

$$\tilde{X}_1 = w_1 h_1(w_2, \ldots, w_n) + w_1^2 h_2(w_2, \ldots, w_n) + \ldots + w_1^n h_n(w_2, \ldots, w_n) + R_n.$$

h_1 is the cofactor, and R_n is the residue of the Taylor series.

Conversely, suppose that $df \cdot X = hf$ for some analytic function h then $df \cdot X \mid_{V(f)} = h(a_1, ..., a_n) f(a_1, ..., a_n) = 0$ and $df \cdot X \mid_{V(f)} = 0$ for all $(a_1, ..., a_n) \in V(f)$. □

Proposition 2.1.2. *Let f, $V(f)$ and X be as in Proposition (2.1.1), then the cofactor is invariant under a change of coordinates, i.e.*

$$\tilde{h}(p) = h(\varphi(p)) \text{ with } p \in V(f) - \{0\}. \tag{2.1}$$

Proof. Since p is a regular point of f, there exist local functions φ, $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with $\varphi(p) = q$ such that

$$\tilde{f} = f \circ \varphi(y_1, ..., y_n) = y_n \text{ and } \psi = \varphi^{-1}. \tag{2.2}$$

It is a consequence of the implicit theorem (see [26], p. 401). So, by the chain rule, we get

$$(\nabla \tilde{f})_p = (\nabla f)_{\varphi(p)} \cdot (D\varphi)_p. \tag{2.3}$$

Moreover, if we push forward the real analytic vector field X with $D\psi$, then we get $\tilde{X}$. i.e.

$$\tilde{X}(q) = (D\psi)_{\varphi(p)} X_{\varphi(p)}. \tag{2.4}$$

If we see the Proposition (2.1.1). It follows that

$$\nabla f(p)X(p) = h(p)f(p), \tag{2.5}$$

and

$$\nabla \tilde{f}(p) \cdot \tilde{X}(p) = \tilde{h}(p)\tilde{f}(p). \tag{2.6}$$

From (2.3) and (2.4), we obtain

$$\nabla \tilde{f}(p) \cdot \tilde{X}(p) = \nabla(f)_{\varphi(p)} D\varphi_p D\psi_{\varphi(p)} X_{\varphi(p)}. \tag{2.7}$$

Since, $D\varphi D\psi = D(\varphi \circ \psi) = DI = I$ then

$$\nabla \tilde{f}(p) \cdot \tilde{X}(p) = \nabla(f)_{\varphi(p)} X_{\varphi(p)}. \tag{2.8}$$

If we consider (2.5), then the previous equation is

$$\nabla \tilde{f}(p) \cdot \tilde{X}(p) = h(\varphi(p))f(\varphi(p)). \tag{2.9}$$

By (2.2), we get

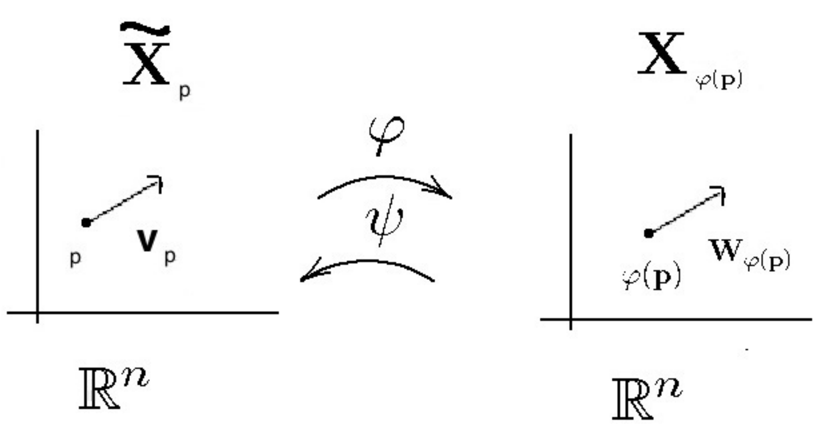
$$\nabla \tilde{f}(p) \cdot \tilde{X}(p) = h(\varphi(p))\tilde{f}(p). \tag{2.10}$$

Thus, if we see the equations (2.6) and (2.10), then

$$\tilde{h}(p) = h(\varphi(p)).$$

□

Graphically



In this case, $\tilde{X}_p = D\psi X_q$**,** $\varphi(p) = q$.

2.2 The absolute index; Poincaré-Hopf index

We define the sphere of radius ϵ to be

$$S_\epsilon^{n-1} = \{x \in \mathbb{R}^n | \parallel x \parallel = \epsilon\}.$$

Indeed, S_1^{n-1} denotes the sphere of radius equal to 1. If X be a real analytic vector field with an isolated singularity at 0, and $\eta : S_\epsilon^{n-1} \to S_1^{n-1}$ is a continuous map, then η induces a homomorphism

$$\eta_* : H_{n-1}(S_\epsilon^{n-1}) \to H_{n-1}(S_1^{n-1}).$$

Since, $H_{n-1}(S_\epsilon^{n-1}) \simeq \mathbb{Z}$ then $\eta_* : x \mapsto dx$ for some fixed $d \in \mathbb{Z}$. The integer d will be called the *degree* of η.

Remark 2.1. *If η is a real analytic function with an isolated singularity at 0, then we define the degree of η, denoted $deg(\eta)$, to be the sum of the signs of the Jacobian of η, at all regular values $q \in S_1^{n-1}$. (See [28] Lemma 3, p. 36, Lemma 4, p. 37).*

Definition 2.5. *Let X be a real analytic vector field with isolated singularity at 0 then d is the Poincaré-Hopf index, denoted ($Ind_{\mathbb{R}^n}(X,0)$).*

Remark 2.2. *Since X is a real analytic vector field, then*

$$DX = \begin{pmatrix} \frac{\partial X^1}{\partial x_1} & \cdots & \frac{\partial X^1}{\partial x_n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial X^n}{\partial x_1} & \cdots & \frac{\partial X^n}{\partial x_n} \end{pmatrix}. \tag{2.11}$$

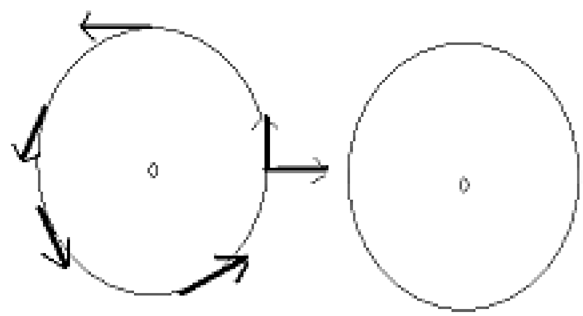
Note that, if p is an isolated singularity of $X := (X^1, ..., X^n)$ such that $|DX(p)| \neq 0$, then the Poincaré-Hopf index of X at p can be computed as the sign$|DX(p)|$. (See [3], [27]).

Remark 2.3. *We consider X as a real analytic vector field and 0 an isolated singularity of X. If X_t is a small perturbation of X of class C^ω, then*

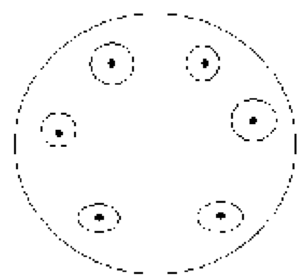
$$Ind_{\mathbb{R}^n}(X, 0) = \sum_{X_t(p_i)=0,\ p_i \in B} Ind_{\mathbb{R}^n}(X_t, p_i). \tag{2.12}$$

We will interpret B as a neighborhood of 0, and the points p_i are isolated singularities of X_t in B, with $i = 1, \ldots, \ell$, $\ell \in \mathbb{Z}^{>0}$.

Graphically, we have the following situation,



Zero is an isolated singularity.



In this case, $Ind_{\mathbb{R}^n}(X, 0) = \sum_{i=1}^{6} Ind_{\mathbb{R}^n}(X_t, p_i), p_i \in B.$

2.3 The relative index; GSV-index

Definition 2.6. *Let X be germs of a real analytic vector field, with an algebraic isolated singularity at 0, such that it is tangent to*

$$H := \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_n = 0\}.$$

Let $Y := X \mid_H$. We define the relative index of the vector field X, to be the Poincaré-Hopf index of the real analytic vector field Y.

We will interpret the vector field X in coordinates by defining $X = (X^1, ..., X^{n-1}, x_n h)$, where h is the real analytic function and $x_n \in \mathbb{R}$.

Furthermore, since X is tangent to H and $Y = X\mid_H$, then, DX and DY can be expressed as

$$DX = \begin{pmatrix} \frac{\partial X^1}{\partial x_1} & \cdots & \frac{\partial X^1}{\partial x_n} \\ . & & . \\ . & & . \\ . & & . \\ \frac{\partial X^{n-1}}{\partial x_1} & \cdots & \frac{\partial X^{n-1}}{\partial x_n} \\ x_n \frac{\partial h}{\partial x_1} & \cdots & h + x_n \frac{\partial h}{\partial x_n} \end{pmatrix} ; DY = \begin{pmatrix} \frac{\partial X^1}{\partial x_1} & \cdots & \frac{\partial X^1}{\partial x_{n-1}} \\ . & & . \\ . & & . \\ . & & . \\ \frac{\partial X^{n-1}}{\partial x_1} & & \frac{\partial X^{n-1}}{\partial x_{n-1}}. \end{pmatrix}$$

It follows that

$$DX\mid_H = \begin{pmatrix} DY & \star \\ 0 & h \end{pmatrix}.$$

Finally, $|DX(0)|$ is equal to

$$\mid DX(0) \mid = h(0) \mid DY(0) \mid . \tag{2.13}$$

Remark 2.4. *According to the previous paragraph, we define the* **relative index** *of X, to be sign $\mid DY(0) \mid$.*

Lemma 2.3.1. *Let X be germs of a real analytic vector field with an isolated singularity at* 0*, and tangent to H, such that $\mid DX(0) \mid \neq 0$. If Y is the restriction of X to H, then*

$$Ind_{\mathbb{R}^n}(X, 0) = sign(h(0)) Ind_H(Y, 0).$$

Proof. Since, $|DX(0)| \neq 0$ and $\mid DX(0) \mid = h(0) \mid DY(0) \mid$, then from the equation (2.13) it follows that $h(0) \neq 0$ and $|DY(0)| \neq 0$. Namely, if $Y := X|_H$ and X has an isolated singularity at 0, clearly Y has an isolated singularity at 0. Now, let us consider the sign on (2.13). Then, we get

$$Ind_{\mathbb{R}^n}(X, 0) = sign(h(0)) Ind_H(Y, 0).$$

□

Corollary 2.1. *Assuming the hypothesis of Lemma,* (2.3.1)*, and if p is an isolated singularity of the vector field X, then*

a) $Ind_H(Y, p) = Ind_{\mathbb{R}^n}(X, p) \iff h(p) > 0.$

b) $Ind_H(Y, p) = -Ind_{\mathbb{R}^n}(X, p) \iff h(p) < 0.$

Proof. The proof of *a*) and *b*) follows immediately if we consider

$$Ind_{\mathbb{R}^n}(X, p) = sign(h(p)) Ind_H(Y, p).$$

Indeed, if $h(p) > 0$ then $Ind_H(Y, p) = Ind_{\mathbb{R}^n}(X, p)$ and $Ind_H(Y, p) = -Ind_{\mathbb{R}^n}(X, p)$, when $h(p) < 0$. □

As an illustration of a change in the topology of the Milnor fibers, we will use the calculus of the GSV-index. To show this, we will exhibit in an example the structure of a singular hypersurface, using a uniparametric non-singular hypersurface when it passes through 0.

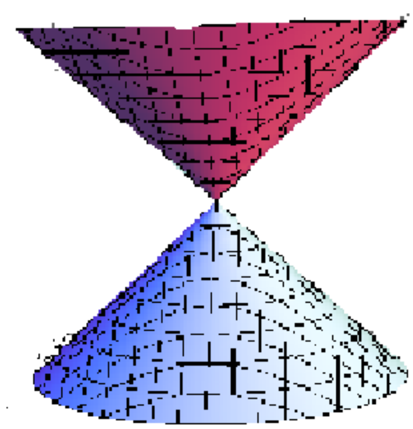
Example 2.3.1.

Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ be a germ of a real analytic function, and

$$X_t = (f - t, -f_z, f_y)$$

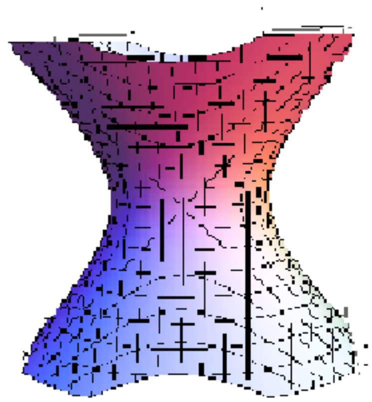
be a uniparametric family of the vector fields of class C^ω, (with $f_y = \frac{\partial f}{\partial y}$, $f_z = \frac{\partial f}{\partial z}$). If X_t is tangent to the hypersurface $V_t(f) = \{(a, b, c) \in \mathbb{R}^3 : f(a, b, c) = t\}$, then $df \cdot X_t = \nabla f \cdot X_t = (f_x, f_y, f_z) \cdot ((f - t), -f_z, f_y) = f_x(f - t)$. So, $f_x = \frac{\partial f}{\partial x}$ is the cofactor. In particular, if $f = x^2 + y^2 - z^2$ then the cofactor is $f_x = 2x$.

Now, we will study the topology of our smooth family of hypersurfaces. To do so, we consider different values of t in $V_t = \{(a, b, c) \in \mathbb{R}^3 : f(a, b, c) = t\}$. Indeed, if $t = 0$ then hypersurface $V_0(f) = \{(a, b, c) \in \mathbb{R}^3 : f(a, b, c) = 0\}$ is



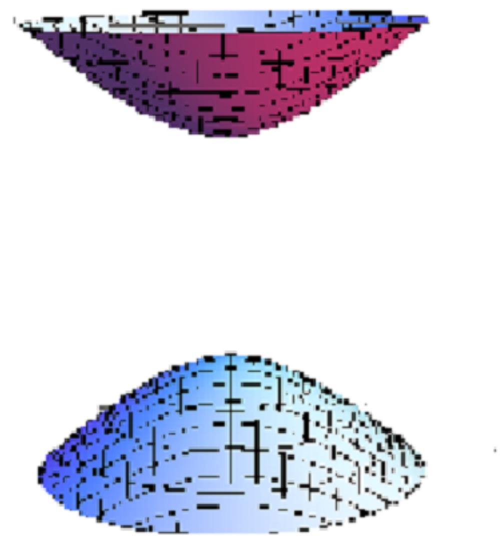
The hypersurface V_0.

Thus, if $t = 1$, then we will obtain



The hypersurface $V_1 = \{f = 1\}$.

Finally, when $t = -1$, then we will get



The hypersurface 3 $V_{-1} = \{f = -1\}$.

So, let us consider the GSV-index of the vector field X_t, for different values of t, and we will observe an interesting phenomenon. Indeed, if $t = 0$, then the origin $(0, 0, 0)$ is an isolated singularity of X_0.

We note that, if $t = 1$, then the points $p_1 = (1, 0, 0)$, $p_2 = (-1, 0, 0)$ are the isolated singularities of X_1. On the other hand, if DX_1 is defined to be

$$|DX_1| = \left| \begin{pmatrix} 2x & 2y & -2z \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \right| = -8x$$

then, the sign$| DX(p_1) |= 1$, and the sign$| DX(p_2) |= -1$. Hence, $Ind_{\mathbb{R}^3}(X, 0) = 0$. It is a direct consequence of the additivity property of the Poincare-Hopf index.

Since, $2x$ is the cofactor, then $h(p_1) = 2(1) > 0$ and $h(p_2) = 2(-1) < 0$. Therefore, from Corollary (2.1) we have $Ind_{\mathbb{R}^2}(Y, p_1) = -1$ and $Ind_{\mathbb{R}^2}(Y, p_2) = -1$. Thus $Ind_{\mathbb{R}^2}(Y, 0) = -2$.

If $t = -1$, then the set of singularities of vector field $X_{-1} = (x^2 + y^2 - z^2 + 1, 2z, 2y)$ are not considered to be real. So, $Ind_{\mathbb{R}^3}(X, 0) = Ind_{\mathbb{R}^2}(Y, 0) = 0$. Hence we conclude that:

1. If $t > 0$, then $Ind_{\mathbb{R}^3}(X, 0) = 0$ and $Ind_{V_t}(Y, 0) = -2$.

2. If $t < 0$, then $Ind_{\mathbb{R}^3}(X, 0) = 0$ and $Ind_{V_t}(Y, 0) = 0$.

Consequently, by the previous paragraph we have that the relative index changed, but the Poincare-Hopf index of X_t for different values of t did not change.

Chapter 3

Algebraic properties of vector fields and symmetric forms

Eisenbud-Levine used a symmetric bilinear form to compute the degree of the function f. Denoted $deg(f)$, with $f \in \mathcal{C}^{\omega}$. In this chapter, we define the general notion of algebra and study in detail an example of degenerate bilinear form. We will work with Zarisky topology.

Definition 3.1. *A sequence of elements* $\{x_1, \ldots, x_d\}$ *on a ring* $\mathcal{R}$ *is a regular sequence if the ideal* $(x_1, \ldots, x_d)$ *is proper, and the image of* x_{i+1} *is a nonzero divisor in* $\frac{\mathcal{R}}{(x_1,\ldots,x_i)}$.

Definition 3.2. *A local ring* $\mathcal{N}$ *is a complete intersection ring if there exists a regular local Noetherian ring* $\mathcal{R}$*, and a regular sequence* $\{x_1, x_2, \ldots, x_n\}$ *in* $\mathcal{R}$ *such that*

$$\mathcal{N} \simeq \frac{\mathcal{R}}{(x_1, \ldots, x_n)}.$$

Remark 3.1. *Let* $\mathbb{K}$ *be a field of characteristic 0 and* $\mathbb{K}[x_1, \ldots, x_n]$ *be the ring of polynomials with* n *variables. Namely, if* I *is an ideal in* $\mathbb{K}[x_1, \ldots, x_n]$*, then*

$$\mathcal{A} = \frac{\mathbb{K}[x_1, \ldots, x_n]}{I}$$

is an algebra and the following are equivalent:

1. $\mathcal{A}$ *is a finite-dimensional over* $\mathbb{K}$.
2. *The variety* $V(I) \in \mathbb{K}^n$ *is a finite set*

   $$V(I) := \{(a_1, \ldots, a_n) \in \mathbb{K}^n : f_\lambda(a_1, \ldots, a_n) = 0, \text{ and } f_\lambda \in I, \lambda \in \Lambda\}$$

   where, Λ *is a finite set of indices.*
3. *The ideal* I *is zero-dimensional.*

Definition 3.3. *Let V be an affine variety. If $I(V)$ is the ideal of polynomials vanishing on V, then, we define the coordinate ring of V, to be*

$$\mathbb{K}[V] = \frac{\mathbb{K}[x_1, \ldots, x_n]}{I(V)}.$$

An element $f \in \mathbb{K}[V]$ will be called a regular function.

Definition 3.4. *Let $\mathcal{A}_{\mathbb{R}^n,0}$ be the ring of germs of regular funcions of $\mathbb{R}^n$ at 0, and $\mathbb{R}\{\{x_1, \ldots, x_n\}\}$ be the set of convergent power series. If $f_1, \ldots, f_n \in \mathbb{R}\{\{x_1, \ldots, x_n\}\}$, then 0 is called an algebraically isolated singularity of $\{f_1, \ldots, f_n\}$, when $\mathcal{O} = \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f_1,\ldots,f_n)}$ it is finite dimensional.*

Moreover, let $\mathcal{O}_{V,p}$ be the ring of germs of regular functions of V at p. If V is an affine variety with a coordinate ring $\mathbb{K}[V]$, then:

1. $\mathcal{O}_{V,p}$ is a local ring, with maximal ideal

$$m_p = \{f \in \mathcal{O}_{V,p} \mid f(p) = 0\}. \tag{3.1}$$

2. $\mathcal{O}_{V,p} \simeq \mathbb{K}[V]_{m_p}$, where

$$\mathbb{K}[V]_{m_p} = \{\frac{f}{g} \mid f, g \in \mathbb{K}[V],\ \mathrm{g} \notin m_p\}. \tag{3.2}$$

.

3. $dim(\mathcal{O}_{V,p}) = max\{dimV_i \mid p \in V_i\}$, where V_i are the irreducible components of V, $i \in \Lambda$ and Λ denotes a finite set of indices.

(See [19] page 469.)

3.1 Symmetric bilinear forms and Sylvester's theorem.

In this section, we will give some basic definitions of the symmetric bilinear forms. Next, we will remember the inertia theorem.

Definition 3.5. *Let V be a vector space over field $\mathbb{K}$. The bilinear form on V $\phi : V \times V \to \mathbb{K}$ is said to be symmetric if $\phi(u, v) = \phi(v, u)$ for all $u, v \in V$.*

Remark 3.2. *If $\phi : V \times V \to \mathbb{K}$ is a symmetric bilinear form, then the associated matrix is symmetric.*

Theorem 3.1. *Sylvester's Law of inertia.*
If V is a real vector space of dimension $n \geq 1$, and $\phi : V \times V \to \mathbb{R}$ is a symmetric bilinear form on V of rank $r \leq n$, then, there is an integer p with $0 \leq p \leq r$ depending only on ϕ, and a basis $\mathcal{B} = \{e_1, .., e_n\}$ of V. Therefore, the associated matrix to ϕ has the following form

$$\begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0_{n-(p+q)} \end{pmatrix}.$$

The symbol $0_{n-(p+q)}$, denotes the zero matrix of suitable size.

3.2 Examples of bilinear forms in commutative algebra

In this section, we motivate our next results with an example of a degenerate symmetric bilinear form on $\mathbb{R}^3$. We will compute the signature in different subspaces of $\mathbb{R}^3$, and the one parameter family. Finally, we will give a specific example that arises from commutative algebra.

Let us consider the degenerate symmetric bilinear form $\langle\ ,\rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, with associated matrix Q given by

$$Q = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If $v_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, v_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \in \mathbb{R}^3$. They are arbitrary fixed vectors. Then, $v_1 Q v_2$ or $\langle v_1, v_2\rangle$ is

$$\langle \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \rangle = -a_1 b_2 - b_1 a_2. \tag{3.3}$$

Since, v_2 is an arbitrary vector, we get $\langle v_1, v_2\rangle = 0$ if and only if -$a_1 b_2 - b_1 a_2 = 0$, and $(a_1, b_1) = (0, 0)$. On the other hand, we define the annihilator of Q, denoted $Ann\langle\ ,\rangle$, to be

$$Ann\langle\ ,\rangle := \{\bar{v} \in \mathbb{R}^3 : \bar{v} Q \bar{v_1} = 0\}. \tag{3.4}$$

With $\bar{v_1} \in \mathbb{R}^3$, it is a vector fixed in $\mathbb{R}^3$. Therefore, the annihilator of the matrix Q is $Ann\langle\ ,\rangle = (0, 0) \times \mathbb{R}$.

Analysis of the bilinear form restricted to 1-dimensional subspaces

We begin by investigating the symmetric bilinear form when we have a one-dimensional subspace. So, let $\mathcal{L}$ be a line in $\mathbb{R}^3$, namely

$\mathcal{L} = \{(a_1, b_1, c_1) \in \mathbb{R}^3 : (a_1, b_1, c_1) = \lambda(a, b, c)$, where $\lambda \in \mathbb{R} - \{0\}$ and (a, b, c) is fixed vector $\}$.

In fact, if $\lambda, \mu \in \mathbb{R}$ then $\lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\mu \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathcal{L}$. Thus, we can write

$$\langle \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \mu \begin{pmatrix} a \\ b \\ c \end{pmatrix} \rangle = -2ab\lambda\mu = 0 \text{ if and only if } ab = 0. \tag{3.5}$$

In particular, if $a = 0$ or $b = 0$, we can define two planes in $\mathbb{R}^3$ to be

$$\mathcal{P}_1 = gen\{ \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \}, \text{and } \mathcal{P}_2 = gen\{ \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \}.$$

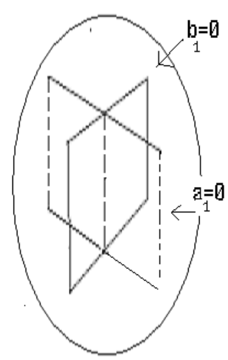
Hence, if $\mathcal{L} \subset \mathcal{P}_1 \cup \mathcal{P}_2$, then $\langle \ , \rangle \mid_{\mathcal{L}} = 0$, and if $\mathcal{L} \not\subset \mathcal{P}_1 \cup \mathcal{P}_2$, then the bilinear form restricted to $\mathcal{L}$ is $\langle (\lambda a, \lambda b, \lambda c), (\mu a, \mu b, \mu c) \rangle = -2\lambda a \mu b = -2a_1 b_2$.

So, we conclude that, $Q \mid_{\mathcal{L}}$ is nondegenerate and its signature σ is :

1) $\sigma = -1$ If $a_1, b_2 > 0$ or $a_1, b_2 < 0$.

2) $\sigma = 1$ If $a_1 > 0, b_2 < 0$ or $a_1 < 0, b_2 > 0$.

The intersection of the previous planes divides $\mathbb{R}^3$ in four connected components. See the following figure:



We have four connected components: $\mathbb{R}^3 - \{a_1 = b_1 = 0\}$.

Analysis of the bilinear form restricted to 2-dimensional subspaces in $\mathbb{R}^3$

Our next objective is to analyse the degenerate symmetric bilinear form, restricted to planes in $\mathbb{R}^3$.
Let $\mathcal{P}$ be a plane on $\mathbb{R}^3$, such that it is generated by the vectors $\{v_1, v_1\}$, where

$$v_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}.$$

Namely,

$$\mathcal{P} = gen \left\{ \lambda \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \mu \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \right\}.$$

If we recall the equation (3.4), then $Ann\langle \ , \rangle := gen \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Thus, if $Ann\langle \ , \rangle \subset \mathcal{P}$, and let us consider the vectors $v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then the bilinear form is

$$\left\langle \lambda_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \mu_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle = -2\lambda_2\mu_2 ab.$$

If a or b is zero, then

$$\langle \ , \rangle \mid_{\mathcal{P}} = 0. \tag{3.6}$$

Moreover, if $a, b \neq 0$, then $\langle \ , \rangle = -2\lambda_2\mu_2 ab$. It has rank 1, and its signature is equal to ± 1 as in the case of one-dimensional subspace.
Indeed, the associated matrix is

$$\begin{pmatrix} \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle & \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle \\ \\ \left\langle \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle & \left\langle \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -2ab \end{pmatrix}.$$

If $\mathcal{P} \cap Ann\langle\ ,\rangle = \{0\}$, and let

$$v_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, v_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

be independent linear vectors in $\mathbb{R}^3$, such that $\mathcal{P} := gen\{v_1, v_2\}$, then

$$det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \neq 0.$$

In particular, we may choose generators of $\mathcal{P}$ as

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ c_1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ c_2 \end{pmatrix}.$$

So, we get

$$\langle \lambda_1 v_1 + \lambda_2 v_2, \mu_1 v_1 + \mu_2 v_2 \rangle = -\lambda_2\mu_1 - \lambda_1\mu_2 = (\lambda_1, \lambda_2) \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}.$$

Therefore, $\mathcal{P} = \mathbb{R}v_1 \times \mathbb{R}v_2$, and $\langle\ ,\rangle$ is nondegenerate with rank 2 and its signature is 0. Consequently, from the previous discussion we get the following lemma,

Lemma 3.2.1. *If $\mathcal{P}$ is any plane such that $\langle\ ,\rangle\,|_{\mathcal{P}}$ has rank 2 then the signature is* 0.

Analysis of the bilinear form on a one-parameter family

In this case, we will illustrate that the contribution of the signature corresponds to the dimension of the annihilator of the matrix Q.
We start with a one-parameter family of bilinear forms in $\mathbb{R}^3$, with associated matrix defined by

$$Q_t = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + t \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_4 & \alpha_5 \\ \alpha_3 & \alpha_5 & \alpha_6 \end{pmatrix}.$$

With $0 < t << 1$, it has the property

$$\{t \in \mathbb{R} - \{0\} \mid \langle\ ,\rangle_t, \text{ it is nondegenerate } \}. \tag{3.7}$$

Thus, we can write

$$Q_t = \begin{bmatrix} t\alpha_1 & -1 + t\alpha_2 & t\alpha_3 \\ -1 + t\alpha_2 & t\alpha_4 & t\alpha_5 \\ t\alpha_3 & t\alpha_5 & t\alpha_6 \end{bmatrix}.$$

Hence,

$det(Q_t) = t^3\alpha_1\alpha_4\alpha_6 - t^3\alpha_1\alpha_5^2 - t^3\alpha_2^2\alpha_6 + 2t^3\alpha_2\alpha_3\alpha_5 - t^3\alpha_3^2\alpha_4 + 2t^2\alpha_2\alpha_6 - 2t^2\alpha_3\alpha_5 - t\alpha_6.$

Indeed, if $det(Q_t) \neq 0$, then we can factorize t. Consequently, we obtain

$$det(Q_t) = t[t^2(\alpha_1\alpha_4\alpha_6 - \alpha_1\alpha_5^2 - \alpha_2^2\alpha_6 + \alpha_2\alpha_3\alpha_5 - \alpha_3^2\alpha_4) + 2t(\alpha_2\alpha_6 - \alpha_3\alpha_5) - \alpha_6]. \quad (3.8)$$

Note that the term t inside of the square brackets in the previous formula is independent from t. So, if $t << 1$, the sign of $det(Q_t)$ depends only on the sign of α_6. Therefore, we have the following result.

Lemma 3.2.2. *Let us consider α_6 in* (3.8)*, then*

- *If $\alpha_6 > 0$, then for small values of t we have that $det(Q_t) < 0$, and the signature is $+1$.*
- *If $\alpha_6 < 0$, then for small values of t we have that $det(Q_t) > 0$, and the signature is -1.*
- *If $t = 0$, then the rank is 2 and the signature is 0.*

In the previous example, we motivated our main objects of study. In fact, we will explore the relationship between the new contribution to the signature in the relative case. We will get the answer in our main theorem.

Our next example arises from commutative algebra. We will obtain a degenerate symmetric bilinear form with associated matrix equal to the matrix Q defined in the beginning.

Example 3.2.1. Let us consider the real analytic function $f : \mathbb{R}^2 \to \mathbb{R}$ defined to be

$$f(x, y) = (x - y^2) \cdot (x^2 - y).$$

Thus, $\{x, y\}$ is a reduced Groebner basis of ∇f and $(0, 0)$ is an algebraically isolated singularity of f.

Since $\mathbb{R}[x, y]$ is the ring of polynomials on the variables x, y. Then B_0 is defined by

$$B_0 = \frac{\mathbb{R}[x, y]}{(f, f_y)}, \quad (3.9)$$

and $\{1, y, y^2\}$ is a basis of B_0. Note that the Jacobian class of $\{f, f_y\}$ in B_0 is

$$[J(f, f_y)]_{B_0} := \left[\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{pmatrix}\right]_{B_0} = y^2.$$

Furthermore, if we define the relative symmetric bilinear form to be

$$\langle\ ,\ \rangle_0^{rel} : B_0 \times B_0 \xrightarrow{\ \cdot\ } B_0 \xrightarrow{f_x} B_0 \xrightarrow{L_0} \mathbb{R},$$

and the linear map $L : B_0 \to \mathbb{R}$ by

$$L_0(y^2) = 1,\ L_0(y) = L_0(1) = 0,$$

then we get

Multiplication after $[f_x]_{B_0=-y}$			
base	1	y	y^2
1	$L_0(-y)$	$L_0(-y^2)$	0
y	$L_0(-y^2)$	0	0
y^2	0	0	0

.

Hence, the associated matrix is

$$Q = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We conclude that $Ann\langle\ ,\rangle_0^{rel} = Ann_{B_0}(f_x)$.

Chapter 4

The absolute index

In this chapter, we will describe an algebraic method to compute the signature of non-degenerate symmetric bilinear forms.

Let V be a finite dimensional vector space, and $\langle\ ,\rangle$ be a symmetric bilinear form, defined by $\langle\ ,\rangle : V \times V \to \mathbb{R}$. If we consider the inertia theorem, then the associated matrix to the symmetric bilinear form, it is equivalent to a diagonal matrix. So, we define the signature to be

$$\tilde{\sigma} = (p, q, r) \tag{4.1}$$

or simply

$$\sigma = p - q \in \mathbb{Z} \tag{4.2}$$

where, p, q, r represent the positive numbers, the negative numbers and the number of zeros.

4.1 B_m as an $\mathbb{R}$-vector space

Let $\mathcal{A}_{\mathbb{R}^n,0}$ be the algebra of germs of real analytic functions on $\mathbb{R}^n$, with coordinates $(x_1, \ldots, x_n)$. Thus, if $(f, f_2, \ldots, f_n), (f_1, f_2, \ldots, f_n)$ are regular sequences, where $f, f_2, \ldots, f_n \in \mathcal{A}_{\mathbb{R}^n,0}$, and $f_i = \frac{\partial f}{\partial x_i}$, then we define the local algebra $\mathcal{C}$, to be

$$\mathcal{C} = \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f_2, ..., f_n)}, \tag{4.3}$$

and the finite vector space B_m, by

$$B_m = \frac{\mathcal{C}}{(f^{m+1})} \simeq \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f^{m+1}, f_2, ..., f_n)}. \tag{4.4}$$

$m \in \Lambda$. Λ denotes a finite set of indices. The function f^{m+1} is f to the power $m + 1$.

Lemma 4.1.1. *Let $v_1, ..., v_s$ be real analytic functions, such that $v_1, ..., v_s \in \mathcal{C}$ and $\{[v_1]_0, ..., [v_s]_0\}$ is a basis of B_0 as an $\mathbb{R}$-vector space. $[v_j]_i$ denotes the class of v_j in B_i, with $j = 1, ...s$, and $i = 1, ..., m$. So, if $m \geq 0$ then*

$$\{[v_1]_m, ..., [v_s]_m, ..., [f^m v_1]_m, ..., [f^m v_s]_m\} \text{ is a basis of } B_m.$$

Proof. We will proceed by induction. Hence, from the hypothesis, the lemma is true for $m = 0$. Furthermore, we suppose

$$\{[v_1]_{m-1}, ..., [v_s]_{m-1}, ..., [f^{m-1}v_1]_{m-1}, ..., [f^{m-1}v_s]_{m-1}\} \tag{4.5}$$

is a basis of B_{m-1}. So, we will prove the lemma for case m. If

$$\sum_{i=1}^{s} \alpha_i^1 [v_i]_m + \sum_{i=1}^{s} \alpha_i^2 [fv_i]_m + ... + \sum_{i}^{s} \alpha_i^m [f^m v_i]_m = 0 \in B_m, \tag{4.6}$$

then,

$$\sum_{i=1}^{s} \alpha_i^1 v_i + \sum_{i=1}^{s} \alpha_i^2 f v_i + ... + \sum_{i=1}^{s} \alpha_i^m f^m v_i = g_m f^{m+1} \in \mathcal{C}. \tag{4.7}$$

If we reduce $mod(f^m)$ in (4.7), we have,

$$\sum_{i=1}^{s} \alpha_i^1 [v_i]_{m-1} + \sum_{i=1}^{s} \alpha_i^2 [fv_i]_{m-1} + \ldots + \sum_{i=0}^{s} \alpha_i^{m-1} [f^{m-1} v_i]_{m-1} = 0 \in B_{m-1}, \tag{4.8}$$

but $\{[v_1]_{m-1}, \ldots, [v_s]_{m-1}, \ldots, [f^{m-1}v_1]_{m-1}, \ldots, [f^{m-1}v_s]_{m-1}\}$ is a basis of B_{m-1}, then

$$\alpha_1^1 = \alpha_2^1 = \ldots = \alpha_s^1 = \ldots = \alpha_1^{m-1} = \ldots = \alpha_s^{m-1} = 0. \tag{4.9}$$

Thus, from (4.7) and (4.9) we get,

$$\sum_{i=1}^{s} \alpha_i^m f^m v_i = g_m f^{m+1} \in \mathcal{C} \tag{4.10}$$

then,

$$f^m (\sum_{i=1}^{s} \alpha_i^m v_i - g_m f) = 0 \in \mathcal{C}. \tag{4.11}$$

Since, f^m is not a zero divisor in $\mathcal{C}$, because $(f, f_1, \ldots, f_m)$ is a regular sequence, then,

$$\sum_{i=1}^{s} \alpha_i^m v_i = g_m f. \tag{4.12}$$

If we reduce $mod(f)$, we obtain

$$0 = \sum_{i=1}^{s} \alpha_i^m [v_i]_0 \in B_0 \tag{4.13}$$

Since, $\{[v_1]_0, \ldots, [v_s]_0\}$ is a basis of B_0, then

$$\alpha_1^m = \alpha_2^m = \ldots = \alpha_s^m = 0. \tag{4.14}$$

Indeed, from (4.6) and (4.14), the set $\{[v_1]_m, \ldots, [v_s]_m, \ldots, [f^m v_1]_m, \ldots, [f^m v_s]_m\}$ is a set of linearly independent vectors.
Now, we will prove that $dim(B_m) \leq sm$, with $dim(B_0) = s$. So, we will proceed in several steps. First, we will define an element $[a]_m \in B_m$, to be $a = \alpha_0 + \alpha_1 f + \alpha_2 f^2 + \ldots + \alpha_m f^m$ then $\varphi([a]_m) = \alpha_0$. Hence, we have an exact sequence

$$0 \to ker\varphi \to B_m \overset{\varphi}{\to} B_0 \to 0 \tag{4.15}$$

and, $B_m = B_0 \times ker(\varphi)$. Furthermore, the previous exact sequence is equivalent to

$$0 \to \frac{(f)}{(f^{m+1})} \to \frac{\mathcal{C}}{(f^{m+1})} \overset{\varphi}{\to} \frac{(\mathcal{C})}{(f)} \to 0. \tag{4.16}$$

Moreover, let us define a map $\psi : B_{m-1} \to \frac{(f)}{(f^{m+1})}$ to be

$$\psi([a]_{m-1}) = [af]_m. \tag{4.17}$$

It is clearly subjective because, if $[hf] \in \frac{(f)}{(f^{m+1})}$, then there exists $[h]_{m-1} \in B_{m-1}$ such that $\psi([h]_{m-1}) = [hf]_m$. It is well defined, since $[a]_{m-1} = [b]_{m-1}$, then $a - b \in (f^m)$. Indeed $(a-b)f = af - bf \in f^m f = f^{m+1}$ and $[af]_m = [bf]_m$.

Since, $B_m = B_0 \oplus ker(\varphi)$, then $dim(B_m) = dim(B_0) + dim\ ker(\varphi)$.
If we consider (4.15) and (4.16), then $ker(\varphi) = \frac{(f)}{(f^{m+1})}$, and by (4.17) we get

$$dim\ ker(\varphi) \leq dim B_{m-1}. \tag{4.18}$$

Therefore,

$dimB_m$	$\leq$	$dim\ B_0$	$+$	$dim\ B_{m-1}$	and from (4.18)
$dimB_m$	$=$	s	$+$	s(m-1)	by the induction hypothesis, we get
$dimB_m$	$= sm.$				

We note that, B_m has sm linearly independent vectors with $dim(B_m) \leq sm$. Hence, the set $\{[v_1]_m, \ldots, [v_s]_m, \ldots, [f^m v_1]_m, \ldots, [f^m v_s]_m\}$ is a basis of B_m.

□

4.2 The socle and the bilinear forms in B_m

In any local algebra, the annihilator of the maximal ideal is called the socle.
Let $f_1, f_2, \ldots, f_n \in \mathcal{A}_{\mathbb{R}^n,0}$, such that $(f_1, \ldots, f_n)$ is a regular sequence. If $\mathbf{A} := \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f_1,\ldots,f_n)}$ is the algebra, then the class of the Jacobian of $\{f_1, \ldots, f_n\}$ denoted $Jac(f_1, \ldots, f_n)$ in the algebra $\mathbf{A}$ is the socle. (See [11]).
In this section, we will compute the socle in the local algebra B_m, where $B_m := \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f^{m+1},\ldots,f_n)}$, $m = 1, \ldots, \ell$,$\ell \in \mathbb{Z}^{\geq 0}$.

Remark 4.1.

- *Let I be an ideal in B_0, then $I^{\perp}$ is an ideal in B_0, and $Ann_{B_0}(I) = I^{\perp}$.*
- *If B_0 is a local algebra, then B_0 has a unique minimal ideal called the socle.*
- *The socle is generated by the residue class of Jacobian of $\{f, f_2, \ldots, f_n\}$ in B_0, denoted J_0 ([11] Proposition 3.2 P.25, Corollary 3.3 P.25, Corollary 4.5 P.35).*

Lemma 4.2.1. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a germ of a real analytic function, and $(f, f_2, \ldots, f_n)$ be a regular sequence. If $J_0 = [J]_0$, is the class of the Jacobian in B_0, then $J_m = [J]_m$ denotes the class of the Jacobian in the local algebra B_m. If J_0 generates the socle in B_0 then $[f^m J]_m$ generates the socle of B_m.*

Proof. Since J_0 is the Jacobian class of $\{f_1, \ldots, f_n\}$, we have

$$J_0 = \left[det \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_{21} & f_{22} & \cdots & f_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix} \right]_0 ,$$

where $f_i = \frac{\partial f}{\partial x_i}$ and $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. Moreover, we consider the Jacobian class of $\{f^{m+1}, f_2, \ldots, f_n\}$ defined by

$$Jac(f^{m+1}, f_2, \ldots, f_n) = det \begin{pmatrix} (m+1)f^m f_1 & (m+1)f^m f_2 & \cdots & (m+1)f^m f_n \\ f_{21} & f_{22} & \cdots & f_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix} .$$

Since $Jac(f^{m+1}, f_2, \ldots, f_n)$ is a determinant we have

$$Jac(f^{m+1}, f_2, \ldots, f_n) = (m+1) f^m Jac(f, f_2, \ldots, f_n). \tag{4.19}$$

If we consider the class of B_m in (4.19), then we obtain

$[(m+1)f^m Jac(f, f_2, \ldots, f_n)]_m = [(m+1)f^m J]_m = (m+1)[f^m J]_m$. Hence, $[f^m J]_{B_m}$ is a positive generator of the socle.
Since, B_m is a local algebra, then it has a unique minimal ideal. Therefore, the socle in B_m is generated by $[f^m J]_m$. □

4.3 Multiplicative structure of B_m

In this section, we will get a Taylor series decomposition of the spaces B_m, in terms of the space B_0. We will also introduce symmetric bilinear forms, and we will obtain an algebraic method to compute the signature in the nondegenerate case.

Let $f, f_2, \ldots, f_n : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be germs of real analytic functions, such that $(f, f_2, \ldots, f_n)$, is a regular sequence. If $\mathcal{A}_{\mathbb{R}^n,0}$ is the set of germs of real analytic functions, then we define the local algebra $\mathcal{C}$ by

$$\mathcal{C} := \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f_2, \ldots, f_n)}. \tag{4.20}$$

We also, define the finite vector B_m to be

$$B_m := \frac{\mathcal{C}}{(f^{m+1})} = \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f^{m+1}, f_2, \ldots, f_n)}. \tag{4.21}$$

Lemma 4.3.1. *If f^m and f^{m+1} denote f to the power m and $m+1$ respectively, then* $\frac{(f^m)}{(f^{m+1})} \simeq f^m B_0$, *for* $m = 1, 2, \ldots, \ell, \ell \in \mathbb{Z}^{\geq 0}$.

Proof. First, we define a map $\varphi : f^m B_0 \to \frac{(f^m)}{(f^{m+1})}$, to be $\varphi(b) := \frac{b}{f^{m+1}}$. (We recall that, $(f^{m+1}) \subset (f^m) \subset, \ldots, \subset (f))$. It is injective. Thus, if $b \in f^m B_0$ with $\varphi(b) = 0$ then $\frac{b}{f^{m+1}} = 0$ and $b \in (f^{m+1})$. Since $b \in f^m B_0$, then we can take $b_0 \in B_0$ to define $b = f^m b_0$. Thus, using the fact $b \in (f^{m+1})$ we get $f^m b_0 \in (f^{m+1})$ and $b_0 \in (f)$. Since, $b_0 \in B_0$ and $b_0 \in (f)$, then $b = 0$ and the map φ is injective. To see that the map φ is surjective, if α is an element of $\frac{(f^m)}{(f^{m+1})}$, then $\varphi(f^{m+1}\alpha) = \alpha$. Therefore, φ is a surjective map, and we get the result.

□

Lemma 4.3.2. *If B_m is defined as in* (4.21)*, then $B_m \simeq B_0 \oplus f B_0 \oplus \ldots \oplus f^m B_0$, as $\mathbb{R}$-vector spaces.*

Proof. First, we will prove the lemma when $m = 1$, since $\frac{(f)}{(f^2)} \simeq f B_0$ and we define an exact sequence of $\mathbb{R}$-vector spaces, to be

$$0 \longrightarrow \frac{(f)}{(f^2)} \xrightarrow{i} B_1 \xrightarrow{\pi} B_0 \longrightarrow 0,$$

then, $B_1 \simeq B_0 \oplus \frac{(f)}{(f)^2}$,and by Lemma (4.3.1) we have $B_1 \simeq B_0 \oplus f B_0$.

So proceeding by induction on $m = k$, with $k \in \mathbb{Z}^{\geq 0}$, we obtain

$$B_k \simeq B_0 \oplus fB_0 \oplus f^2 B_0 \oplus \ldots \oplus f^k B_0. \tag{4.22}$$

then, we will prove the lemma for $m = k + 1$. Thus, let us consider

$$0 \longrightarrow \frac{(f^{k+1})}{(f^{k+2})} \xrightarrow{i} B_{k+1} \xrightarrow{\pi} B_k \longrightarrow 0$$

be an exact sequence. Therefore, from Lemma (4.3.1) and the previous equation we get

$$B_{k+1} = B_k \oplus f^{k+1} B_0. \tag{4.23}$$

Using (4.22) and (4.23), it follows that

$$B_{k+1} \simeq B_0 \oplus fB_0 \oplus f^2 B_0 \oplus \ldots \oplus f^{k+1} B_0.$$

□

Lemma 4.3.3. *Let B_m be defined as in* (4.21)*. If $\mu_m : B_m \times B_m \longrightarrow B_m$ are symmetric bilinear forms, such that for all $v_i, v_j \in B_m$ we have $\mu_m([v_i]_m [v_j]_m) = [v_i v_j]_m$, (where $m = 1, 2, \ldots, \ell, \quad i, j = 1, \ldots, s, \quad \ell, s \in \mathbb{Z}^{\geq 0}$). Then, μ_m has the associated matrix given by*

$$Q_m = \left[\begin{array}{l|c|c|r} Q_0 + fH_1 + \cdots + f^m H_m & fQ_0 + f^2 H_1 + \cdots + f^m H_{m-1} & \cdots & f^m Q_0 \\ \hline fQ_0 + f^2 H_1 + \cdots + f^m H_{m-1} & \cdots & f^m Q_0 & 0 \\ \hline \vdots & \ddots & 0 & 0 \\ \hline f^m Q_0 & 0 & 0 & 0 \end{array}\right] =$$

$$\left[\begin{array}{c|c|c|c} Q_0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \vdots & \ddots & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] + f \left[\begin{array}{c|c|c|c} H_1 & Q_0 & \cdots & 0 \\ \hline Q_0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \vdots & \ddots & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] + \cdots + f^m \left[\begin{array}{c|c|c|c|c} H_m & H_{m-1} & \cdot & H_1 & Q_0 \\ \hline H_{m-1} & \cdots & H_1 & Q_0 & 0 \\ \hline \vdots & \ddots & \ddots & 0 & 0 \\ \hline H_1 & Q_0 & 0 & 0 & 0 \\ \hline Q_0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Proof. We will discuss in detail the cases $m = 0, 1$. And, we will consider the same steps to get the general case.
Let $\{[v_1]_0, \ldots, [v_s]_0\}$ be an $\mathbb{R}$-basis of B_0, such that $v_1 = 1$ and $v_s \in J_0$. Since, $B_m \simeq B_0 \oplus fB_0 \oplus \cdots \oplus f^m B_0$, then $\{[v_1]_m, \ldots, [v_s]_m, [fv_1]_m, \ldots, [fv_s]_m, \ldots, [f^m v_1]_m, \ldots, [f^m v_s]_m\}$ is an $\mathbb{R}$-basis of B_m.

Case $m = 0$

Let $\mu_0 : B_0 \times B_0 \longrightarrow B_0$, be the nondegenerate symmetric bilinear form.
Since, $\{[v_1]_0, \ldots, [v_s]_0\}$ is an $\mathbb{R}$-basis of B_0, then we define the associated matrix to the bilinear form μ_0, denoted Q_0, to be

$$\mu_0([v_i]_0, [v_j]_0) = [v_i \cdot v_j]_0 = (q_{ij}) = Q_0, \tag{4.24}$$

where $i, j = 1, \ldots, s$, $s \in \mathbb{Z}^{\geq 0}$.

Case $m = 1$

Let $\mu_1 : B_1 \times B_1 \longrightarrow B_1$ be the nondegenerate symmetric bilinear form in $B_1 := \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f^2, f_2, \ldots, f_n)}$. Since, $B_1 \simeq B_0 \oplus fB_0$ and $\{[v_1]_0, \ldots, [v_s]_0\}$ is an $\mathbb{R}-$basis of B_0, then $\{[v_1]_1, \ldots, [v_s]_1, [fv_1]_1, \ldots, [fv_s]_1\}$ is an $\mathbb{R}$-basis of B_1. Thus, we can define the symmetric bilinear form in B_1, to be

1) $\mu_1([v_i]_1, [v_j]_1) = [v_i \cdot v_j]_1 = q_{ij} + fh^1_{ij} = (Q_0 + fH_1)_{ij}$

2) $\mu_1([fv_i]_1, [v_j]_1) = \mu_1([v_i]_1, fv_j]_1) = [f \cdot v_i \cdot v_j]_1 = f(q_{ij} + fh^1_{ij}) = (fQ_0)_{ij}$

3) $\mu_1([fv_i]_1, [fv_j]_1) = 0$.

Hence, $\mu_1(\ ,\ )$ can be represented by the following matrix:

$$Q_1 = \left[\begin{array}{c|r} Q_0 + fH_1 & fQ_0 \\ \hline fQ_0 & 0 \end{array}\right] = \left[\begin{array}{c|c} Q_0 & 0 \\ \hline 0 & 0 \end{array}\right] + f\left[\begin{array}{c|r} H_1 & Q_0 \\ \hline Q_0 & 0 \end{array}\right]. \tag{4.25}$$

In general, from Lemma (4.3.2) we have, $B_m \simeq B_0 \oplus fB_0 \oplus ... \oplus f^m B_0$. Now, if $\{[v_1]_0, \ldots, [v_s]_0\}$ is an $\mathbb{R}-$basis of B_0, then $\{[v_1]_m, \ldots, [v_s]_m, [fv_1]_m, \ldots, [fv_s]_m, \ldots, [f^m v_1]_m, \ldots, [f^m v_s]_m\}$ is an $\mathbb{R}$-basis of B_m. If $\mu_m : B_m \times B_m \longrightarrow B_m$ are the nondegenerate symmetric bilinear forms, where $m = 1, \ldots, \ell$, $\ell \in \mathbb{Z}^{\geq 0}$. Then we can define the symmetric bilinear forms μ_m, to be

1) $\mu_m([v_i]_m, [v_j]_m) = [v_i \cdot v_j]_m = q_{ij} + fh^1_{ij} + f^2 h^2_{ij} +, \ldots, +f^m h^m_{ij} = (Q_0 + fH_1 + f^2 H_2 +, \ldots, f^m H_m)_{i,j}$

2) $\mu_m(f[v_i]_m, [v_j]_m) = \mu_m([v_i]_m, [fv_j]_m) = [f \cdot v_i \cdot v_j]_m = (fq_{ij} + f^2 h^1_{ij} + f^3 h^2_{ij} +, \ldots, +f^m h^{m-1}_{ij}) = (fQ_0 + f^2 H_1 + f^3 H_2 +, \ldots, f^m H_{m-1})_{i,j}$

3) $\mu_m([fv_i]_m, [fv_j]_m) = \mu_m([f^2 v_i]_m, [v_j]_m) = \mu_m([v_i]_m, [f^2 v_j]_m) = [f^2 \cdot v_i \cdot v_j] = (f^2 Q_0 + f^3 H_1 +$
$, \ldots, + f^m H_{m-2})_{i,j}$.

$\vdots$

m) $\mu_m([f^i v_i]_m, [f^j v_j]_m) = [f^m \cdot v_i \cdot v_j]_m = (f^m Q_0)_{ij}$, with $i+j=m$. $\mu_m(\ ,\ )$ is zero in other cases.

Hence, the associated matrix to the symmetric bilinear form $\mu_m(\ )$ has the following representation,

$$Q_m = \left[\begin{array}{l|c|c|r} Q_0 + fH_1 + \cdots + f^m H_m & fQ_0 + f^2 H_1 + \cdots + f^m H_{m-1} & \cdots & f^m Q_0 \\ \hline fQ_0 + f^2 H_1 + \cdots + f^m H_{m-1} & \cdots & f^m Q_0 & 0 \\ \hline \vdots & \ddots & 0 & 0 \\ \hline f^m Q_0 & 0 & 0 & 0 \end{array}\right] =$$

$$\left[\begin{array}{c|c|c|c} Q_0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \vdots & \ddots & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] + f \left[\begin{array}{c|c|c|c} H_1 & Q_0 & \cdots & 0 \\ \hline Q_0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \vdots & \ddots & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] + \cdots + f^m \left[\begin{array}{c|c|c|c|c} H_m & H_{m-1} & \cdot & H_1 & Q_0 \\ \hline H_{m-1} & \cdots & H_1 & Q_0 & 0 \\ \hline \vdots & \ddots & \ddots & 0 & 0 \\ \hline H_1 & Q_0 & 0 & 0 & 0 \\ \hline Q_0 & 0 & 0 & 0 & 0 \end{array}\right].$$

□

Remark 4.2. *The matrices $Q_0, H_1, \ldots, H_m$ with coefficients in B_0 are symmetrics.*

If $w, w' \in B_m$, then $w = a_0 + fa_1 + \cdots + f^m a_m$ and $w' = b_0 + fb_1 + \cdots + f^m b_m$. Therefore,

$\mu_m(w, w') = \mu_m((a_0 + fa_1 + \cdots + f^m a_m), (b_0 + fb_1 + \cdots + f^m b_m))$=$\mu_m(a_0, b_0) + \mu_m(a_0, fb_1) + \cdots + \mu_m(a_0, f^m b_m) + \mu_m(fa_1, b_0) + \mu_m(fa_1, fb_1) + \cdots + \mu_m(f^m a_m, b_0)$.

On the other hand,

$\mu_m(w', w) = \mu_m((b_0 + ba_1 + \cdots + f^m b_m), (a_0 + fa_1 + \cdots + f^m a_m))$=$\mu_m(b_0, a_0) + \mu_m(b_0, fa_1) + \cdots + \mu_m(b_0, f^m a_m) + \mu_m(fb_1, a_0) + \mu_m(fb_1, fa_1) + \cdots + \mu_m(f^m b_m, a_0)$.

Hence,
$a_0(Q_0 + fH_1 + \cdots + f^m H_m)b_0 + a_0(fQ_0 + \cdots + f^m H_{m-1})b_1 + a_1(fQ_0 + \cdots + f^m H_{m-1})b_0 + \cdots + a_m(f^m Q_0)b_0 = b_0(Q_0 + fH_1 + \cdots + f^m H_m)a_0 + b_0(fQ_0 + \cdots + f^m H_{m-1})a_1 + b_1(fQ_0 + \cdots + f^m H_{m-1}a_0 + \cdots + b_m(f^m Q_0)a_0$.
So,

$$a_0 Q_0 b_0 = b_0 Q_0 a_0,$$

then $Q_0 = Q_0^t$. Similarly, $H_i = H_i^t$, for $i = 1, \cdots, m$.

4.4 Calculus of the bilinear forms in B_m

Lemma 4.4.1. *Let V be a finite vector space of dimension $(m+1)s$, and $\varphi : V \times V \to \mathbb{R}$ be a symmetric bilinear form with an associated matrix*

$$Q_V = \begin{pmatrix} 0 & . & . & . & C \\ . & . & . & & \\ . & . & & & \\ . & & & 0 & \\ C & & & & \end{pmatrix}. \tag{4.26}$$

Then,

$$\sigma(Q_V) = \begin{cases} \sigma(C) & if \quad m \quad is \quad even, \\ 0 & if \quad m \quad is \quad odd. \end{cases} \tag{4.27}$$

Proof. Let us consider a matrix S such that C is equivalent to the diagonal matrix D, i.e.

$$D = S^tCS = \begin{pmatrix} 1 & & & & & & & & \\ & . & & & & & & & \\ & & . & & & & & & \\ & & & . & & & & & \\ & & & & 1 & & & & \\ & & & & & -1 & & & \\ & & & & & & . & & \\ & & & & & & & . & \\ & & & & & & & & -1 \end{pmatrix}.$$

Therefore, the signature of the matrix D is $\tilde{\sigma} = (p, q, r)$ or $\sigma = p - q$, where p, q, r denote, the positive, the negative, and zero numbers in the diagonal.
In general, we get

$$\begin{pmatrix} S^t & & & \\ & . & & \\ & & . & \\ & & & . \\ & & & & S^t \end{pmatrix} \begin{pmatrix} & & & C \\ & & . & \\ & . & & \\ . & & & \\ C & & & \end{pmatrix} \begin{pmatrix} S & & & \\ & . & & \\ & & . & \\ & & & . \\ & & & & S \end{pmatrix} \tag{4.28}$$

$$= \begin{pmatrix} & & & S^tCS \\ & & . & \\ & . & & \\ . & & & \\ S^tCS & & & \end{pmatrix} = \begin{pmatrix} & & & \pm 1 \\ & & . & \\ & . & & \\ . & & & \\ \pm 1 & & & \end{pmatrix}. \tag{4.29}$$

Note that the matrix (4.29) has ± 1 in the anti diagonal. To see this, we can consider the orthogonal basis on

$$\mathbb{R}^{(m+1)s} = \underbrace{\mathbb{R}^s \oplus \mathbb{R}^s \oplus \ldots \oplus \mathbb{R}^s}_{(m+1)-times},$$

defined by

$$v_{jr} = (0, \dots, \frac{v_r}{\sqrt{2}}, \dots, 0, \frac{v_r}{\sqrt{2}}, \dots, 0) \tag{4.30}$$

and

$$u_{jr} = (0, \dots, \frac{v_r}{\sqrt{2}}, \dots, 0, \frac{-v_r}{\sqrt{2}}, \dots, 0) \tag{4.31}$$

$$1 \leq j \leq [\frac{m+1}{2}],$$

where, $[\frac{m+1}{2}]$ denotes the minor integer of $\frac{m+1}{2}$. The position of $\frac{v_r}{\sqrt{2}}$ is in the $j-th$ and $(m+1-\{j-1\})-th$ respectively. Moreover, if m is odd

$$w_r = (0, \dots, 0, v_r, 0, \dots, 0) \tag{4.32}$$

$$1 \leq r \leq s.$$

In this case the position of v_r is in the middle. So, if m is even we get

$$v_{jr}^t \begin{pmatrix} & & & S^tCS \\ & & \cdot & \\ & \cdot & & \\ \cdot & & & \\ S^tCS & & & \end{pmatrix} v_{jr} \tag{4.33}$$

$$= \frac{1}{2}(e_r^t S^t CSe_r + e_r^t S^t CSe_r) = e_r^t S^t CSe_r = \pm 1 \tag{4.34}$$

and

$$u_{jr}^t \begin{pmatrix} & & & S^tCS \\ & & \cdot & \\ & \cdot & & \\ \cdot & & & \\ S^tCS & & & \end{pmatrix} u_{jr} \tag{4.35}$$

$$= \frac{1}{2}(e_r^t S^t CS(-e_r) + (-e_r^t) S^t CSe_r) = -e_r^t S^t CSe_r = \mp 1. \tag{4.36}$$

Finally,

$$w_r^t \begin{pmatrix} & & & S^tCS \\ & & \cdot & \\ & \cdot & & \\ \cdot & & & \\ S^tCS & & & \end{pmatrix} w_r = w_r^t S^t CSw_r \tag{4.37}$$

$$= e_r^t DS^t CSe_r = \pm 1. \tag{4.38}$$

Hence, from (4.34),(4.36) and (4.38) we have $\sigma(Q_V) = \sigma(C)$ for m even and $\sigma(Q_V) = 0$ for m odd. □

4.5 The absolute index of B_m

Now, we give a proof of Theorem (1.1).

Proof. Since $\{f, f_2, ..., f_m\}$ is a regular sequence, then B_m is a vector space of finite dimension. If $v_1, \dots, v_s \in \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f_2,\dots,f_n)}$ such that $\{[v_1]_0, \dots, [v_s]_0\} \in B_0$, it is an $\mathbb{R}$-basis with $v_1 = 1$ and $v_s = J_0$, then

$$\{[v_1]_m, \dots, [v_s]_m, [fv_1]_m, \dots, [fv_s]_m, \dots, [f^m v_1]_m, \dots, [f^m v_s]_m\}$$

is an $\mathbb{R}$-basis of B_m. From Lemma (4.3.2), we get

$$B_m = B_0 \bigoplus fB_0 \bigoplus \cdots \bigoplus f^m B_0 \tag{4.39}$$

We choose for $L_m : B_m \longrightarrow \mathbb{R}$ the map sending all the base elements to 0, except the last where $L_m(f^m J_m) = 1$. Using this block decomposition of B_m, the multiplication table

$$\mu_m : B_m \times B_m \longrightarrow B_m$$

takes the form, (see Lemma 4.3.3)

$$Q_m = \left[\begin{array}{c|c|c|c} Q_0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \vdots & \ddots & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] + f \left[\begin{array}{c|c|c|c} H_1 & Q_0 & \cdots & 0 \\ \hline Q_0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \vdots & \ddots & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] + \cdots$$

$$\cdots + f^m \left[\begin{array}{c|c|c|c|c} H_m & H_{m-1} & \cdots & H_1 & Q_0 \\ \hline H_{m-1} & \cdots & H_1 & Q_0 & 0 \\ \hline \vdots & \ddots & \ddots & 0 & 0 \\ \hline H_1 & Q_0 & 0 & 0 & 0 \\ \hline Q_0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{4.40}$$

where $Q_0, H_1, \dots, H_m$ are symmetric $(s \times s)$-matrices with entries in B_0. The expression of these matrices can be obtained from the restriction of μ_m to B_0 and using the isomorphism (4.39), we obtain a bilinear form

$$\mu_m : B_0 \times B_0 \longrightarrow B_m = \bigoplus_{j=0}^{m} f^j B_0$$

with a matrix expression

$$Q_0 + fH_1 + \cdots + f^m H_m.$$

Q_0 is the matrix expression of the multiplication μ_0 on B_0 and the H_j are the higher order terms in the multiplication μ_m restricted to $B_0 \hookrightarrow B_m$. These terms contain all the information needed for describing μ_m, as can be seen from the expression (4.40). Applying L_m to (4.40), L_mQ_m is

$$\left[\begin{array}{l|r|r|r|r} L_mQ_0+,\cdots,+L_m(f^mH_m) & & \cdots & L_m(f^mH_1) & L_m(f^mQ_0) \\ \hline L_m(fQ_0)+,\cdots,+L_m(f^mH_{m-1}) & \cdots & & L_mf^mQ_0 & 0 \\ \hline \vdots & \ddots & \ddots & 0 & 0 \\ \hline L_m(f^mH_1) & L_mf^mQ_0 & 0 & 0 & 0 \\ \hline L_mf^mQ_0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{4.41}$$

Moreover, L_m is a linear map such that $L_m\langle f^jv_k, f^\ell v_s\rangle = L_m(f^{j+\ell}v_kv_s)$, and

1) If $j+\ell > m$ then $L_m(0) = 0$.

2) If $j+\ell < m$ then $L_m(f^{j+\ell}v_jv_s) = 0$.

3) If $j+\ell = m$ $L_m(f^{j+\ell}v_jv_s) = 1$.

Thus, the anti-diagonal terms of the matrix (4.41) are non-singular. Indeed, we can a change a basis of the $\mathbb{R}$-vector space to obtain matrix representation of $<\ ,\ >_m$ as an anti-diagonal matrix by blocks, with all the anti-diagonal terms. Hence, we get

$$L_mQ_m = \left[\begin{array}{l|c|c|c|r} 0 & 0 & \cdot & 0 & L_mf^mQ_0 \\ \hline 0 & \cdots & 0 & L_mf^mQ_0 & 0 \\ \hline \vdots & \ddots & \ddots & 0 & 0 \\ \hline 0 & L_mf^mQ_0 & 0 & 0 & 0 \\ \hline L_mf^mQ_0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{4.42}$$

The matrix (4.42) suggests that we consider the decomposition of B_m as:

$$B_m = [B_0 \oplus f^mB_0]\bigoplus[fB_0 \oplus f^{m-1}B_0]\bigoplus,\cdots,\bigoplus[f^mB_0 \oplus B_0]. \tag{4.43}$$

It is an $<\ ,\ >_m$-orthogonal direct sum. The contribution to the signature of each vector space within a bracket is 0, since they have the form

$$\begin{pmatrix} 0 & 0 & \cdots & L_mf^mQ_0 \\ 0 & \cdots & L_mf^mQ_0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ L_mf^mQ_0 & \cdots & 0 & 0 \end{pmatrix}. \tag{4.44}$$

Therefore, if we consider V as B_m in the lemma (4.4.1), then the bilinear form $\langle\ ,\rangle_m : B_m \times B_m \xrightarrow{\cdot} B_m \xrightarrow{L_m} \mathbb{R}$ has the associated matrix (4.44). Thus, if we consider the lemma (4.4.1), we get the proof of the theorem. $\square$

4.6 An example

Example 4.6.1. Let $f:(\mathbb{R}^2,0)\longrightarrow(\mathbb{R},0)$ be a germ of real analytic function defined to be $f(x,y)=x^2+y^2$. Moreover, if $B_0:=\frac{\mathbb{R}[x,y]}{(f,f_y)}$, then $\{1,x\}$ is a basis of B_0.
Thus, we consider the nondegenerate bilinear form

$$\mu_0 : B_0\times B_0 \stackrel{\cdot}{\longrightarrow} B_0.$$

Hence,the bilinear product is given by

base	1	x
1	1	x
x	x	0

.

If $L_0: B_0\to\mathbb{R}$ is any linear map, such that $L_0(1)=0$ and $L_0(x)=0$ then the bilinear form μ_0 is

$$\mu_0 : B_0\times B_0 \longrightarrow B_0 \stackrel{L_0}{\longrightarrow}\mathbb{R}.$$

It has the associated matrix defined by

$$Q_0=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

Now, we will construct an orthogonal basis of the finite vector space B_0. Thus, if $\{v_1,v_2\}\in\mathbb{R}^2$, it is a basis with $v_1=\frac{1}{\sqrt{2}}(\{1,1\})$ and $v_2=\frac{1}{\sqrt{2}}(\{1,-1\})$, then $v_1C_0v_1=1$, $v_1C_0v_2=0$, $v_2C_0v_1=0$, $v_2C_0v_2=-1$. So, the matrix Q_0 is equivalent to the following matrix

$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Its signature is $\sigma_0=0$. If $B_1:=\frac{\mathbb{R}[x,y]}{(f^2,f_2)}$ the symmetric bilinear form

$$\mu_1 : B_1\times B_1 \stackrel{\cdot}{\longrightarrow} B_1 \stackrel{L_1}{\longrightarrow}\mathbb{R}$$

has the following representation

$$Q_1=\left(\begin{array}{cc|cc}0&0&0&1\\0&0&1&0\\\hline 0&1&0&0\\1&0&0&0\end{array}\right). \tag{4.45}$$

The next step is to construct the orthogonal basis of $\mathbb{R}^{(1+1)2}=\mathbb{R}^4$.
Indeed, $\mathbb{R}^4\simeq\mathbb{R}^2\oplus\mathbb{R}^2$ and we can define the orthogonal basis $\{v_{11},v_{12},u_{11},u_{12}\}$, to be

$$\{v_{11}=(\frac{1}{\sqrt{2}}(v_1),\frac{1}{\sqrt{2}}(v_1)),v_{12}=(\frac{1}{\sqrt{2}}(v_2),\frac{1}{\sqrt{2}}(v_2)),$$

$$u_{11} = (\frac{1}{\sqrt{2}}(v_1), -\frac{1}{\sqrt{2}}(v_1)), u_{21} = (\frac{1}{\sqrt{2}}(v_2), -\frac{1}{\sqrt{2}}(v_2))\},$$

and it is equal to:

$$\{v_{11} = \frac{1}{2}(1,1,1,1), v_{12} = \frac{1}{2}(1,-1,1,-1), u_{11} = \frac{1}{2}(1,1,-1,-1), v_{11} = \frac{1}{2}(1,-1,-1,1)\}.$$

Then, this basis changes the matrix (4.45) to the diagonal matrix given by

$$Q_1 = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

Therefore, $\sigma_1 = 0$.
If we consider $m = 2$ and $B_2 := \frac{\mathbb{R}[x,y]}{(f^3, f_2)}$, then it has a basis defined as $\{1, x, x^2, x^3, x^4, x^5\}$, and $\mathbb{R}^6 \simeq \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2$. Thus, we define the the bilinear product μ_2, to be

$$\mu_2 : B_2 \times B_2 \xrightarrow{\cdot} B_2 \xrightarrow{L_2} R$$

with $L_2(x^5) = 1, L_2(x^4) = L_2(x^3) = L_2(x^2) = L_2(x) = L_2(1) = 0$. Therefore, it has the following matricial representation

$$Q_2 = \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{4.46}$$

Let $\{v_{11} = \frac{1}{\sqrt{2}}(v_1, 0, v_1), v_{12} = \frac{1}{\sqrt{2}}(v_2, 0, v_2), u_{11} = \frac{1}{\sqrt{2}}(v_1, 0, -v_1), u_{21} = \frac{1}{\sqrt{2}}(v_2, 0, -v_2),\ w_1 = (0, v_1, 0), w_2 = (0, v_2, 0)\}$ be an orthogonal basis, and it is equal to
$\{\frac{1}{2}(1,1,0,0,1,1), \frac{1}{2}(1,-1,0,0,1,-1), \frac{1}{2}(1,1,0,0,-1,-1), \frac{1}{2}(1-1,0,0,-1,1), (0,0,1,1,0),$
$(0,0,1,-1,0)\}$. So, it changes the matrix (4.46) by,

$$Q_2 = \left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{array}\right).$$

Hence $\sigma_2 = 0$, and so on.

Chapter 5

The relative index

Our purpose in this chapter is to provide an algebraic formula to compute the signature of degenerate symmetric bilinear forms on the finite vector spaces $B_m \simeq \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f^{m+1},f_2,\cdots,f_n)}$, $m = 1, 2, \cdots, \ell$, $\ell \in \mathbb{Z}^{\geq 0}$.
Before starting the next result, we will introduce some examples and lemmas, that help us understand our main result.

5.1 An example; computation of the relative index in the local algebra B_m

Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be a germ of real analytic function defined to be $f = x^3 + y^2$. Since

$$B_0 = \frac{\mathbb{R}[x, y]}{(f, f_2)} = \frac{\mathbb{R}[x, y]}{(x^3 + y^2, 2y)}$$

is a local algebra and (f, f_2) is a regular sequence in B_0, where $f_2 = \frac{\partial f}{\partial y}$, then B_0 is a finite vector space. Hence, $\{[1]_0, [x]_0, [x^2]_0\}$ is a basis of B_0. Moreover, if we define $[f_1]_0 = [3x^2]_0$ and the degenerate symmetric bilinear form to be

$$\mu_0^{rel} : B_0 \times B_0 \xrightarrow{\cdot} B_0 \xrightarrow{f_1} B_0, \tag{5.1}$$

$$([v_i]_0, [v_j]_0) \mapsto [v_i \cdot v_j]_0 \mapsto [f_1 \cdot v_i \cdot v_j]_0$$

then, it is represented by

The bilinear form μ_0 in B_0			
$[3x^2]_0$	$[1]_0$	$[x]_0$	$[x^2]_0$
$[1]_0$	$[3x^2]_0$	$[0]_0$	$[0]_0$
$[x]_0$	$[0]_0$	$[0]_0$	$[0]_0$
$[x^2]_0$	$[0]_0$	$[0]_0$	$[0]_0$

.

If we define a linear map $L_0 : B_0 \to \mathbb{R}$ to be, $L_0([x^2]_0) = \frac{1}{3},\ L_0([x]_0) = 0,\ L_0([1]_0) = 0,$ then the associated matrix to the degenerate symmetric bilinear form is

$$\langle\ ,\rangle_0^{rel} : B_0 \times B_0 \stackrel{\cdot}{\longrightarrow} B_0 \stackrel{f_1}{\longrightarrow} B_0 \stackrel{L_0}{\longrightarrow} \mathbb{R}.$$

It has the following matricial representation

$$Q_0^{rel} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

therefore, its signature is $\tilde{\sigma}_0^{rel} = (1, 0, 2)$.
Now, let us consider the finite vector space B_1. Namely,

$$B_1 = \frac{\mathbb{R}[x,y]}{(f^2, f_2)} = \frac{\mathbb{R}[x,y]}{((x^3+y^2)^2, 2y)} \simeq \frac{\mathbb{R}[x]}{(x^6)},$$

where $f^2 = x^6 + 2x^3y^2 + y^4$. Thus, we define a degenerate symmetric bilinear form to be

$$\mu_1^{rel} : B_1 \times B_1 \stackrel{\cdot}{\longrightarrow} B_1 \stackrel{f_1}{\longrightarrow} B_1.$$

So, we get

The bilinear form μ_1 in B_1						
$[3x^2]_1$	$[1]_1$	$[x]_1$	$[x^2]_1$	$[f]_1$	$[fx]_1$	$[fx^2]_1$
$[1]_1$	$[3x^2]_1$	$[3x^3]_1$	$[3x^4]_1$	$[3x^5]_1$	$[0]_1$	$[0]_1$
$[x]_1$	$[3x^3]_1$	$[3x^4]_1$	$[3x^5]_1$	$[0]_1$	$[0]_1$	$[0]_1$
$[x^2]_1$	$[3x^4]_1$	$[3x^5]_1$	$[0]_1$	$[0]_1$	$[0]_1$	$[0]_1$
$[f]_1$	$[3x^5]_1$	$[0]_1$	$[0]_1$	$[0]_1$	$[0]_1$	$[0]_1$
$[fx]_1$	$[0]_1$	$[0]_1$	$[0]_1$	$[0]_1$	$[0]_1$	$[0]_1$
$[fx^2]_1$	$[0]_1$	$[0]_1$	$[0]_1$	$[0]_1$	$[0]_1$	$[0]_1$

if we define a degenerate symmetric bilinear form to be

$$\langle,\rangle_1^{rel} : B_1 \times B_1 \stackrel{\cdot}{\longrightarrow} B_1 \stackrel{f_1}{\longrightarrow} B_1 \stackrel{L_1}{\longrightarrow} \mathbb{R}, \tag{5.2}$$

where the linear map $L_1 : B_1 \to \mathbb{R}$, is $L_1([x^j]_1) = 0,\ j = 0,1,2,3,4,\ L_1([x^5]) = 1$ and $[f]_1 = ([x^3]_1)$, then, its associated matrix is given by

$$Q_1^{rel} = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{5.3}$$

Furthermore, we can write the reduced matrix (5.3) as

$$Q_1^{rel}\mid_{red}=\left(\begin{array}{c|cc|c} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array}\right)=\left(\begin{array}{ccc} & & \tilde{D}_o \\ & E_1 & \\ \tilde{D}_o & & \end{array}\right).$$

Hence, $\tilde{D}_0=(1)$ and

$$E_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Moreover, we define an orthogonal basis of $\mathbb{R}^6$, to be

$$\{v_1=\frac{1}{\sqrt{2}}(1,0,0,1,0,0), v_2=\frac{1}{\sqrt{2}}(1,0,0,-1,0,0), v_3=\frac{1}{\sqrt{2}}(0,0,1,1,0,0),$$

$$v_4=\frac{1}{\sqrt{2}}(0,0,1,-1,0,0), v_5=(0,0,0,0,1,0), v_6=(0,0,0,0,0,1)\}.$$

Consequently, the matrix (5.3) is equivalent to

$$Q_1^{rel}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, it has signature $\tilde{\sigma}_1^{rel}=(2,2,2)$.

Since the signature in $\tilde{\sigma}_0^{rel}$ is equal to $(1,0,2)$, and the signature $\tilde{\sigma}_1^{rel}$ is equal to $(2,2,2)$, then, the number of zeros in $\tilde{\sigma}_0^{rel}=\tilde{\sigma}_1^{rel}=2$. Thus, if we consider

$$\tilde{\sigma}_1^{rel}-\tilde{\sigma}_0^{rel}=(1,2,0),$$

then, we have

$$\tilde{\sigma}_2^{rel}=(4,3,2), \tag{5.4}$$

and

$$(4,3,2)=(2,2,2)+(2,1,0).$$

i.e.

$$\tilde{\sigma}_2^{rel}=\tilde{\sigma}_1^{rel}+(2,1,0).$$

It is easy to see that in this example, the nexts signatures are constructed via the following algorithm. Therefore,

$$\tilde{\sigma}_3^{rel}=(5,5,2)=(4,3,2)+(1,2,0). \tag{5.5}$$

And

$$\tilde{\sigma}_4^{rel} = (5,5,2) + (2,1,0) = (7,6,2), \tag{5.6}$$

similarly,

$$\tilde{\sigma}_5^{rel} = (7,6,2) + (1,2,0) = (8,8,2), \tag{5.7}$$

and so on. Indeed, we get a pattern.

The following two lemmas describe a method to simplify the matrix associated to degenerate symmetric bilinear forms.

5.2 Calculus of the relative index in the local algebra B_1

Lemma 5.2.1. *Let $\tilde{\varphi}_1 : V \times V \to \mathbb{R}$ be a symmetric bilinear form, with V a finite vector space, defined by*

$$Q_1^{rel} = \left(\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{13} & I_p & 0 & 0 \\ A_{12}^t & A_{22} & A_{23} & 0 & -I_q & 0 \\ A_{13}^t & A_{23}^t & E_1 & 0 & 0 & 0 \\ \hline I_p & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{5.8}$$

Then it is equivalent to the matrix

$$\begin{pmatrix} 0 & 0 & \tilde{D}_0 \\ 0 & E_1 & 0 \\ \tilde{D}_0 & 0 & 0 \end{pmatrix}. \tag{5.9}$$

Where $\tilde{D}_0$ is given by

$$\tilde{D}_0 = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}. \tag{5.10}$$

Here, the matrices $A_{i,j}$ are any symmetric matrices, i.e. $A_{i,j} = A_{ij}^t$, $i,j = 1,2,3$.

Proof. If $Q_1^{rel}|_{red}$ denotes the reduced matrix corresponding to the matrix (5.8), namely,

$$Q_1^{rel}\,|_{red} = \left(\begin{array}{ccc|cc} A_{11} & A_{12} & A_{13} & I_p & 0 \\ A_{12}^t & A_{22} & A_{23} & 0 & -I_q \\ A_{13}^t & A_{23}^t & E_1 & 0 & 0 \\ \hline I_p & 0 & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 & 0 \end{array}\right), \tag{5.11}$$

then, using Gaussian elimination, it suggests that we can define the matrices, D_{11}, D_{12}, D_{21}, D_{22}, D_{31}, D_{32}, to be

$$D_{11} = -\frac{A_{11}}{2}, \quad D_{12} = \frac{A_{12}}{2}, \quad D_{21} = -\frac{A_{12}^t}{2}, \quad D_{22} = \frac{A_{22}}{2}, \quad D_{31} = -A_{13}^t, \quad D_{32} = A_{23}^t.$$

Since, the bilinear form is symmetric then $A_{11} = A_{11}^t$, $A_{22} = A_{22}^t$. Therefore,

$$\begin{pmatrix} I & 0 & 0 & D_{11} & D_{12} \\ 0 & I & 0 & D_{21} & D_{22} \\ 0 & 0 & I & D_{31} & D_{32} \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} & I_p & 0 \\ A_{12}^t & A_{22} & A_{23} & 0 & -I_q \\ A_{13}^t & A_{23}^t & E_1 & 0 & 0 \\ I_p & 0 & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ D_{11}^t & D_{21}^t & D_{31}^t & I & 0 \\ D_{12}^t & D_{22}^t & D_{32}^t & 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} \frac{A_{11}}{2} & \frac{A_{12}}{2} & A_{13} & I_p & 0 \\ \frac{A_{12}^t}{2} & \frac{A_{22}}{2} & A_{23} & 0 & -I_q \\ 0 & 0 & E_1 & 0 & 0 \\ I_p & 0 & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ D_{11}^t & D_{21}^t & D_{31}^t & I & 0 \\ D_{12}^t & D_{22}^t & D_{32}^t & 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \tilde{D}_0 \\ 0 & E_1 & 0 \\ \tilde{D}_0 & 0 & 0 \end{pmatrix}. \tag{5.12}$$

□

Remark 5.1. *Remember that*

$$Q_1^{rel} = \left(\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & \\ A_{12}^t & A_{22} & A_{23} & * \\ A_{13}^t & A_{23}^t & E_1 & \\ \hline & * & & 0 \end{array}\right)$$

so $rank(E_1) \leq rank(Q_1^{rel})$.

Remark 5.2. *The previous lemma is true when* $V := B_1$.

5.3 Calculus of the relative index in the local algebra B_2

Lemma 5.3.1. *If $\tilde{\varphi}_2 : V \times V \to \mathbb{R}$ is a degenerate symmetric bilinear form, where V is a finite vector space, then the associated matrix given by*

$$Q_2^{rel} = \left(\begin{array}{cccc|cccc|cccc} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & 0 & I_p & 0 & 0 & 0 \\ A_{12}^t & A_{22} & A_{23} & A_{24} & A_{16}^t & A_{26} & A_{27} & 0 & 0 & -I_q & 0 & 0 \\ A_{13}^t & A_{23}^t & A_{33} & A_{34} & A_{17}^t & A_{27}^t & \tilde{E}_1 & 0 & 0 & 0 & 0 & 0 \\ A_{14}^t & A_{24}^t & A_{34}^t & E_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline A_{15} & A_{16} & A_{17} & 0 & I_p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{16}^t & A_{26} & A_{27} & 0 & 0 & -I_q & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{17}^t & A_{27}^t & \tilde{E}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline I_p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \quad (5.13)$$

A_{ij}, $i, j = 1.2, ..., 7$ *are symmetric matrices. So, The matrix (5.13) is equivalent to the following matrix*

$$\left(\begin{array}{ccc|ccc|c} 0 & 0 & 0 & 0 & 0 & 0 & \tilde{D}_0 \\ 0 & 0 & 0 & 0 & \tilde{E}_1 & 0 & 0 \\ 0 & 0 & E_2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \tilde{D}_0 & 0 & 0 & 0 \\ 0 & \tilde{E}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \tilde{D}_0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \quad (5.14)$$

Proof. Since Q_2^{rel} is the associated matrix to the degenerate symmetric bilinear form $\varphi_2 : V \times V \to \mathbb{R}$, then the reduced matrix denoted $Q_2^{rel}|_{red}$ is

$$Q_2^{rel}|_{red} = \left(\begin{array}{cccc|cccc|cc} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & 0 & I_p & 0 \\ A_{12}^t & A_{22} & A_{23} & A_{24} & A_{16}^t & A_{26} & A_{27} & 0 & 0 & -I_q \\ A_{13}^t & A_{23}^t & A_{33} & A_{34} & A_{17}^t & A_{27}^t & \tilde{E}_1 & 0 & 0 & 0 \\ A_{14}^t & A_{24}^t & A_{34}^t & E_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline A_{15} & A_{16} & A_{17} & 0 & I_p & 0 & 0 & 0 & 0 & 0 \\ A_{16}^t & A_{26} & A_{27} & 0 & 0 & -I_q & 0 & 0 & 0 & 0 \\ A_{17}^t & A_{27}^t & \tilde{E}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline I_p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \quad (5.15)$$

Moreover, we consider the submatrix of the matrix (5.15) equivalent to

$$\left(\begin{array}{cccc|cccc} A_{15} & A_{16} & A_{17} & 0 & I_p & 0 & 0 & 0 \\ A_{16}^t & A_{26} & A_{27} & 0 & 0 & -I_q & 0 & 0 \\ A_{17}^t & A_{27}^t & \tilde{E}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline I_p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{5.16}$$

Indeed, it is similar to

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_q & 0 & 0 \\ 0 & 0 & \tilde{E}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline I_p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{5.17}$$

Finally, if we consider the matrices $\tilde{E}_1$, I_p and $-I_q$ as pivots from gaussian elimination, then the matrix (5.15) is equivalent to

$$\left(\begin{array}{ccc|ccc|c} 0 & 0 & 0 & 0 & 0 & 0 & \tilde{D}_0 \\ 0 & 0 & 0 & 0 & \tilde{E}_1 & 0 & 0 \\ 0 & 0 & E_2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \tilde{D}_0 & 0 & 0 & 0 \\ 0 & \tilde{E}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \tilde{D}_0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{5.18}$$

□

The previous lemmas provide a technique to simplify the matrices associated to the degenerate symmetric bilinear forms, thus, we can consider the general lemma.

Lemma 5.3.2. *Let V be a real vector space of dimension n, and $\varphi_m : V \times V \to \mathbb{R}$ be a symmetric bilinear form with associated matrix defined as*

$$Q_m^{rel} = \begin{pmatrix} E_m & \tilde{E}_{m-1} & \tilde{E}_{m-2} & \tilde{E}_{m-3} & \tilde{E}_{m-4} & & \cdots & \tilde{D}_0 \\ \tilde{E}_{m-1} & \tilde{E}_{m-2} & \tilde{E}_{m-3} & \tilde{E}_{m-4} & \cdots & \tilde{E}_1 & \tilde{D}_0 & 0 \\ \tilde{E}_{m-2} & \tilde{E}_{m-3} & \tilde{E}_{m-4} & \cdots & \tilde{E}_1 & \tilde{D}_0 & 0 & 0 \\ \tilde{E}_{m-4} & & \cdots & \tilde{E}_1 & \tilde{D}_0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{E}_1 & \tilde{D}_0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \tilde{D}_0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.19}$$

Hence, the matrix $\tilde{D}_0$ is as in the Lemma (5.2.1) and $\tilde{E}_i, (i = 0, 1, 2, ..., m,\ m \in \mathbb{Z}^{\geq 0})$ is a diagonal matrix given by

$$E_i = \begin{pmatrix} I_{s_i{}^+} & 0 & 0 \\ 0 & -I_{s_i{}^-} & 0 \\ 0 & 0 & 0_{s_i^0} \end{pmatrix} \tag{5.20}$$

with signature $\tilde{\sigma}(E_i) = (s_i^+, s_i^-, s_i^0)$ and s_i^+, s_i^-, s_i^0 are the positive, the negative and the zero numbers. In particular $E_0 = \tilde{D}_0$. Furthermore,

$$s_i^+ + s_i^- + s_i^0 = s_{i-1}^0. \tag{5.21}$$

If $\tilde{\sigma}(Q_m^{rel}) = (p_m, q_m, r_m)$, it is the signature associated to the matrix Q_m^{rel}, then

$$r_m = r_{m-1} + s_m^0,$$

and

$$\tilde{\sigma}(Q_m^{rel}) = \begin{cases} \tilde{\sigma}(Q_{m-1}^{rel}) + (p_0, q_0, 0) + (s_1^-, s_1^+, 0) + \ \ldots \ + (s_{m-1}^-, s_{m-1}^+, 0) + (s_m^+, s_m^-, s_m^0) \text{ if } m \text{ is even.} \\ \tilde{\sigma}(Q_{m-1}^{rel}) + (q_0, p_0, 0) + (s_1^+, s_1^-, 0) + \ \ldots \ + (s_{m-1}^-, s_{m-1}^+, 0) + (s_m^+, s_m^-, s_m^0) \text{ if } m \text{ is odd.} \end{cases}$$

$m = 0, 1, 2, \cdots, \ell,\ \ell \in \mathbb{Z}^{\geq 0}$.

Proof. We will use induction on m to show the lemma.

Case m=0. Let Q_0^{rel} be the matrix defined in (5.19), then it is given by

$$Q_0^{rel} = \tilde{D}_0 = \begin{pmatrix} I_{p_0} & 0 & 0 \\ 0 & -I_{q_0} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.22}$$

It has signature

$$\tilde{\sigma}(Q_0^{rel}) = (p_0, q_0, r_0). \tag{5.23}$$

With

$$p_0 + q_0 + r_0 = n.$$

Then,

$$r_0 = n - p_0 - q_0. \tag{5.24}$$

Case m=1. The matrix Q_1^{rel} has the following form

$$Q_1^{rel} = \left(\begin{array}{ccccc|ccc} 0 & 0 & 0 & 0 & 0 & I_{p_0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I_{q_0} & 0 \\ 0 & 0 & Is_1^+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -Is_1^- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0s_1^0 & 0 & 0 & 0 \\ \hline I_{p_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_{q_0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{5.25}$$

Hence, $\tilde{\sigma}(Q_1^{rel}) = (p_1, q_1, r_1)$, and

$$p_1 + q_1 + r_1 = 2n. \tag{5.26}$$

Since, $s_1^+ + s_1^- + s_1^0 = s_0 = r_0$ then,

$$r_1 = 2n - p_1 - q_1 \tag{5.27}$$

$$s_1^+ + s_1^- = r_0 - s_1^0. \tag{5.28}$$

If we make an orthogonal change in the basis in the matrix (5.25), we have

$$p_1 = p_0 + q_0 + s_1^+, \quad q_1 = q_0 + p_0 + s_1^-. \tag{5.29}$$

From (5.29) and (5.27), we obtain

$$r_1 = 2n - p_0 - q_0 - s_1^+ - q_0 - p_0 - s_1^-.$$

So,

$$r_1 = n - p_0 - q_0 + n - p_0 - q_0 - s_1^+ - s_1^-, \tag{5.30}$$

substituing (5.24) and (5.28) in (5.30) we have

$$r_1 = r_0 + s_1^0. \tag{5.31}$$

Then, from (5.29) and (5.31), it follows that

$$\tilde{\sigma}(Q_1^{rel}) = (p_0 + q_0 + s_1^+, q_0 + p_0 + s_1^-, r_0 + s_1^0)$$

and

$$\tilde{\sigma}(Q_1^{rel}) = (p_0, q_0, r_0) + (q_0, p_0, 0) + (s_1^+, s_1^-, s_1^0),$$

hence,

$$\tilde{\sigma}(Q_1^{rel}) = \tilde{\sigma}(Q_0^{rel}) + (q_0, p_0, 0) + (s_1^+, s_1^-, s_1^0). \tag{5.32}$$

General case. If these formulas are valid for $m-1$ and m is even, then

$$\tilde{\sigma}(Q_m^{rel}) = (p_m, q_m, r_m).$$

Where

$$p_m = p_{m-1} + p_0 + s_1^- + ... + s_{m-1}^- + s_m^+ \tag{5.33}$$

$$q_m = q_{m-1} + q_0 + s_1^+ + ... + s_{m-1}^+ + s_m^- \tag{5.34}$$

and

$$s_m^+ + s_m^- + s_m^0 = s_{m-1}^0$$

$$s_m^+ + s_m^- = s_{m-1}^0 - s_m^0 \tag{5.35}$$

$$p_m + q_m + r_m = (m+1)n \tag{5.36}$$

then

$$r_m = (m+1)n - p_m - q_m. \tag{5.37}$$

So,

$$r_m = mn - p_{m-1} - q_{m-1} + n - p_0 - q_0 - \sum_{i=1}^{m}(s_i^+ + s_i^-)$$

if we consider the previous equations and (5.28), (5.33) and (5.34), we get

$$r_m = r_{m-1} + s_m^0. \tag{5.38}$$

Then, it follows that

$$\tilde{\sigma}(Q_m^{rel}) = (p_m, q_m, r_m) = (p_{m-1}+p_0+s_1^-+...+s_{m-1}^-+s_m^+, q_{m-1}+q_0+s_1^++...+s_{m-1}^++s_m^-, r_{m-1}+s_m^0)$$

so

$$\tilde{\sigma}(Q_m^{rel}) = (p_{m-1}, q_{m-1}, r_{m-1}) + (p_0, q_0, 0) + (s_1^-, s_1^+, 0) + ...$$

$$+(s_{m-1}^-, s_{m-1}^+, 0) + (s_m^+, s_m^-, s_m^0). \tag{5.39}$$

Therefore,

$$\tilde{\sigma}(Q_m^{rel}) = \tilde{\sigma}(Q_{m-1}^{rel}) + (p_0, q_0, 0) + (s_1^-, s_1^+, 0) + ...$$

$$+(s_{m-1}^-, s_{m-1}^+, 0) + (s_m^+, s_m^-, s_m^0). \tag{5.40}$$

Analogously, if m is odd

$$\tilde{\sigma}(Q_m^{rel}) = (p_{m-1}, q_{m-1}, r_{m-1}) + (q_0, p_0, 0) + (s_1^+, s_1^-, 0)+$$

$$... + (s_{m-1}^-, s_{m-1}^+, 0) + (s_m^+, s_m^-, s_m^0) \tag{5.41}$$

$$\tilde{\sigma}(Q_m^{rel}) = \tilde{\sigma}(Q_{m-1}^{rel}) + (q_0, p_0, 0) + (s_1^+, s_1^-, 0)+$$

$$... + (s_{m-1}^-, s_{m-1}^+, 0) + (s_m^+, s_m^-, s_m^0). \tag{5.42}$$

□

Corollary 5.1. *Under the hypothesis of the previous lemma, if we define* $\tilde{\sigma}(Q_m^{rel}) = \tilde{\sigma}_m^{rel}$ *and* $\tilde{\sigma}_m^{rel} = (p_m, q_m, r_m)$*, so* $\sigma_m^{rel} = p_m - q_m$ *then, we have that*

$$\sigma_m^{rel} = \sigma_{m-2}^{rel} + \sigma^{rel}(E_m).$$

Proof. If m is even, we have

$$\tilde{\sigma}_m^{rel} = \tilde{\sigma}_{m-1}^{rel} + (p_0, q_0, 0) + (s_1^-, s_1^+, 0) + ... + (s_{m-1}^-, s_{m-1}^+, 0) + (s_m^+, s_m^-, s_m^0). \tag{5.43}$$

Hence, m is even, so $m-1$ is odd then

$$\tilde{\sigma}_{m-1}^{rel} = \tilde{\sigma}_{m-2}^{rel} + (q_0, p_0, 0) + (s_1^+, s_1^-, 0) + ... + (s_{m-2}^-, s_{m-2}^+, 0) + (s_{m-1}^+, s_{m+1}^-, s_{m+1}^0). \tag{5.44}$$

Substituing (5.44) in (5.43) we get

$$\tilde{\sigma}_m^{rel} = \tilde{\sigma}_{m-2}^{rel} + (q_0, p_0, 0) + (s_1^+, s_1^-, 0) + ... + (s_{m-2}^-, s_{m-2}^+, 0) + (s_{m-1}^+, s_{m+1}^-, s_{m+1}^0)$$
$$+ (p_0, q_0, 0) + (s_1^-, s_1^+, 0) + ... + (s_{m-1}^-, s_{m-1}^+, 0) + (s_m^+, s_m^-, s_m^0).$$

So, if

$$\sigma_m = p_{m-2} - q_{m-2} + s_m^+ - s_m^-$$

then

$$\sigma_m = \sigma_{m-2} + \sigma(E_m).$$

The proof is similar when m is odd. □

5.4 Another example of computation of the relative index

In this example, we will compute the signature using the routine siggen.lib with the singular package. We will obtain a flag, and the signature of the degenerate symmetric bilinear form in the relative case.
Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ be a germ of real analytic function defined by

$$f = (x^3 + y^2)(x^2 + y^3) + z^2.$$

If (f, f_y, f_z) is a regular sequence of $B_m := \frac{\mathbb{R}[x,y,z]}{(f^{m+1}, f_y, f_z)}$, where $f_y = \frac{\partial f}{\partial y}$, $f_z = \frac{\partial f}{\partial z}$, then B_m is a finite vector space.

Moreover, if we consider the degenerate symmetric bilinear form

$$\langle\ ,\rangle_m^{rel} : B_m \times B_m \overset{\cdot}{\longrightarrow} B_m \xrightarrow{f_x} B_m \xrightarrow{L_m} \mathbb{R}$$

and we use the routine bilirelamod, we get

Results of program relative bilinearform.lib			
	size	rank	signature
B_0	15	5	$(3, 2, 10)$
B_1	30	19	$(9, 10, 11)$
B_2	45	34	$(17, 17, 11)$

.

In the previous table, $(3, 2, 10)$ is the signature of the symmetric bilinear form
$\langle\ ,\rangle_0^{rel} : B_0 \times B_0 \overset{\cdot}{\longrightarrow} B_0 \xrightarrow{f_1} B_0 \xrightarrow{L_0} \mathbb{R}$, such that $L_0([Jac(f, f_2, f_3)]_0) > 0$.
Hence, with an appropriate basis we get

$$\left(\begin{array}{c|c} \begin{pmatrix} I_3 & \\ & -I_2 \end{pmatrix} & 0_{5\times 10} \\ \hline 0_{10\times 5} & \begin{pmatrix} 0_{5\times 5} & \\ & 0_{5\times 5} \end{pmatrix} \end{array}\right). \tag{5.45}$$

If

$$\tilde{D}_0 = \begin{pmatrix} I_3 & \\ & -I_2 \end{pmatrix}$$

represents the non singular part of the matrix (5.45), then $\sigma_0^{rel} = (3, 2, 10)$. Indeed, $(9, 10, 11)$ is the signature of the symmetric bilinear form
$\langle\ ,\ \rangle_1^{rel} : B_1 \times B_1 \overset{\cdot}{\longrightarrow} B_1 \xrightarrow{f_1} B_1 \xrightarrow{L_1} \mathbb{R}$ with $Jac((f^2, f_2, f_3)) > 0$. And, if we consider an orthogonal basis, then we get the following matrix

$$
\left[\begin{array}{c|c|c|c}
0_{5\times 5} & 0_{5\times 10} & \begin{smallmatrix} I_3 & \\ & -I_2 \end{smallmatrix} & 0_{5\times 10} \\ \hline
0_{10\times 5} & \begin{smallmatrix} I_4 & & \\ & -I_5 & \\ & & 0_{1\times 1} \end{smallmatrix} & 0_{10\times 5} & \begin{smallmatrix} 0_{5\times 5} & \\ & 0_{5\times 5} \end{smallmatrix} \\ \hline
\begin{smallmatrix} I_3 & \\ & -I_2 \end{smallmatrix} & 0_{5\times 10} & 0_{5\times 5} & 0_{5\times 10} \\ \hline
0_{10\times 5} & \begin{smallmatrix} 0_{5\times 5} & \\ & 0_{5\times 5} \end{smallmatrix} & 0_{10\times 15} & \begin{smallmatrix} 0_{5\times 5} & \\ & 0_{5\times 5} \end{smallmatrix}
\end{array}\right],
$$

or equivalently,

$$
\left[\begin{array}{c|c|c}
0_{5\times 5} & 0_{5\times 10} & \tilde{D}_0 \\ \hline
0_{10\times 5} & \begin{smallmatrix} I_4 & & \\ & -I_5 & \\ & & 0_{1\times 1} \end{smallmatrix} & 0_{10\times 5} \\ \hline
\tilde{D}_0 & 0_{5\times 5} & 0_{5\times 5}
\end{array}\right]. \tag{5.46}
$$

If we define the matrix $\tilde{E}_1$, to be

$$
\tilde{E}_1 = \begin{pmatrix} I_4 & \\ & -I_5 \end{pmatrix},
$$

then the matrix(5.46) is equal to

$$
\left[\begin{array}{c|c|c}
0 & 0 & \tilde{D}_0 \\ \hline
0 & E_1 & 0 \\ \hline
\tilde{D}_0 & 0 & 0
\end{array}\right].
$$

Hence, $\tilde{\sigma}_1^{rel} = (3+2+4, 2+3+5, 10+1) = (9, 10, 11)$.
Similarly, $(17, 17, 11)$ is the signature associated to the nondegenerate symmetric bilinear form $\langle\ ,\ \rangle_1^{rel} : B_1 \times B_1 \overset{\cdot}{\longrightarrow} B_1 \xrightarrow{f_1} B_1 \xrightarrow{L_1} \mathbb{R}$ with $Jac((f^3, f_2, f_3)) > 0$. So, in an appropriate basis we have

$$
\left[\begin{array}{c|c|c|c|c|c}
0_5 & 0 & 0_5 & 0 & \begin{smallmatrix} I_3 & \\ & -I_2 \end{smallmatrix} & 0 \\ \hline
0 & \begin{smallmatrix} 0_4 & & \\ & 0_5 & \\ & & -1 \end{smallmatrix} & 0 & \begin{smallmatrix} I_4 & & \\ & -I_5 & \\ & & 0_{1\times 1} \end{smallmatrix} & 0 & \begin{smallmatrix} 0_5 & \\ & 0_5 \end{smallmatrix} \\ \hline\hline
0_5 & 0 & \begin{smallmatrix} I_3 & \\ & -I_2 \end{smallmatrix} & 0 & 0_5 & 0 \\ \hline
0 & \begin{smallmatrix} I_4 & & \\ & -I_5 & \\ & & 0_{1\times 1} \end{smallmatrix} & 0 & \begin{smallmatrix} 0_5 & \\ & 0_5 \end{smallmatrix} & 0 & 0 \\ \hline\hline
\begin{smallmatrix} I_3 & \\ & -I_2 \end{smallmatrix} & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & \begin{smallmatrix} 0_5 & \\ & 0_5 \end{smallmatrix} & 0 & 0 & 0 & 0
\end{array}\right).
$$

If we define $\tilde{E}_1$ as the nonsingular part of the matrix E_1, namely

$$\tilde{E}_1 = \begin{pmatrix} I_4 & \\ & -I_5 \end{pmatrix},$$

then the reduced matrix is

$$\left(\begin{array}{c|c|c|c|c} 0_{5\times 5} & 0_{5\times 10} & 0 & 0 & \tilde{D}_0 \\ \hline 0_{10\times 5} & \begin{smallmatrix} 0_4 & & \\ & 0_5 & \\ & & -1 \end{smallmatrix} & \tilde{E}_1 & 0 & 0 \\ \hline\hline 0 & \tilde{E}_1 & 0 & \tilde{D}_0 & 0 \\ \hline \tilde{D}_0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Thus, the $\tilde{\sigma}_2^{rel}$ is computed as $\tilde{\sigma}_2^{rel} = (3+2+3+4+5+0, 2+3+2+5+4+1, 11) = (17, 17, 11)$, and so on. In this case, we will see that the matrices E_i represent the flag in the algebra B_0.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a germ of real analytic function such that $(f, f_2, \cdots, f_n)$ is a regular sequence on $B_m := \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f^{m+1}, f_2, \cdots, f_n)}$, where $\mathcal{A}_{\mathbb{R}^n,0}$ denotes the germs of real analytic functions with isolated singularity at 0. $f_i = \frac{\partial f}{\partial x_i}$ $i = 1, 2, \cdots, n$, $n \in \mathbb{Z}^{\geq 0}$, and f^{m+1} denotes f to the power $m+1$, $m \in \mathbb{Z}^{\geq 0}$. Furthermore, we define the annihilator of f_1 in B_m, to be

$$Ann_{B_m}(f_1) := \{b \in \mathcal{C} : [f_1 b]_m = 0 \text{ in} B_m\}.$$

Now, we consider the following lemma

Lemma 5.4.1. *Let $\langle\ ,\rangle_{m,Ann}^{rel}$ be the relative bilinear form restricted to the annihilator, namely,*

$$\langle\ ,\rangle_{m,Ann}^{rel} : (Ann_{B_{m-1}}(f_1) \oplus f^m B_0) \times (Ann_{B_{m-1}}(f_1) \oplus f^m B_0) \xrightarrow{\cdot} B_m \xrightarrow{f_1} B_m. \quad (5.47)$$

It is nondegenerate in $a_{m-1}Q_{m-1}b_{m-1}Q_{m-1}c_m$, where Q_m is the matrix defined by Lemma (4.3.3). $a_{m-1}, b_{m-1} \in Ann_{B_{m-1}}(f_1)$ and c_m is defined in the following context. Let $[f_1]_m \in B_m$ be the class of f_1 in the algebra B_m. Thus, if $\{[v_1]_0, ...[v_s]_0\}$ is a basis of algebra B_0, then by Lemma (4.1.1) $[f_1]_{B_m} := [f_1]_m$. It is given by

$$[f_1]_m = \sum_i^s c_i^0 [v_i]_m + \sum_i^s c_i^1 [f v_i]_m + ... + \sum_i^s c_i^m [f^m v_i]_m.$$

So, if $c_i = (c_1^i, ..., c_s^i)$, then $[f_1]_m \simeq c_0 + f c_1 + ... + f^m c_m$.

Proof. First, we show the lemma in little cases to help us understand the general proof.

Case m=0.

Let $\langle\ ,\rangle^{rel}_{0,Ann}$ be a symmetric bilinear form, defined to be

$$\langle\ ,\rangle^{rel}_{0,Ann} : B_0 \times B_0 \overset{\cdot}{\longrightarrow} B_0 \xrightarrow{f_1} B_0.$$

If $v_1, ..., v_s \in \mathcal{C}$, such that

1) $\{[v_{\ell+1}]_0, ..., [v_s]_0\}$ is a basis of $Ann_{B_0}(f_1)$,

2) $\{[v_1]_0..., [v_s]_0\}$ is a basis of B_0,

then $[v_1]_0, ..., [v_\ell]_0$ generates a transversal to $Ann_{B_0}(f_1)$. If $q_{ij} = \mu_0(v_i, v_j)$ is the matrix defined as before, then the matrix $(q_{ij})_{i,j=1,\cdots,\ell}$ is a nondegenerate symmetric matrix. And if $i > \ell$ or $j > \ell$ then $q_{ij} = 0$. Since $[f_1]_{B_0} = [c_0]_{B_0}$, then

$$\langle w\ , w'\rangle^{rel}_{\mu_0} = [a_0]_0 Q_0 [b_0]_0 Q_0 [c_0]_0.$$

By simplicity, we can write

$$\langle w\ , w'\rangle^{rel}_{\mu_0} = a_0 Q_0 b_0 Q_0 c_0,$$

which is a degenerate symmetric bilinear form. Hence, we observe the associated matrix Q_0 to $\langle\ ,\rangle^{rel}_{\mu_0}$ degenerates in $Ann_{B_0}(f_1)$.

Case m=1.

Since, $B_1 \simeq B_0 \oplus fB_0$ then $Ann_{B_0}(f_1) \oplus fB_0 \subset B_1$. So, we define $\langle\ ,\rangle^{rel}_{1,Ann}$, to be

$$\langle\ ,\rangle^{rel}_{1,Ann} : (Ann_{B_0}(f_1) \oplus fB_0) \times (Ann_{B_0}(f_1) \oplus fB_0) \overset{\cdot}{\longrightarrow} B_1 \xrightarrow{f_1} B_1. \tag{5.48}$$

Indeed, if $w, w' \in (Ann_{B_0}(f_1) \oplus fB_0)$ then $w = \sum_{i=1}^n \alpha_i^0 [v_i]_0 + \sum_{i=1}^n \alpha_i^1 f[v_i]_0$ and $w' = \sum_{i=1}^n \beta_i^0 [v_i]_0 + \sum_{i=1}^n \beta_i^1 f[v_i]_0$. Therefore,

$$w = (\alpha_1^0, ..., \alpha_n^0, \alpha_1^1, ..., \alpha_n^1) = a_0 + fa_1, \quad w' = (\beta_1^0, ..., \beta_n^0, \beta_1^1, ..., \beta_n^1) = b_0 + fb_1.$$

On the other hand, let $[f_1]_{B_1} = \sum_{i=1}^n c_i^0 [v_i] + \sum_{i=1}^n c_i^1 [fv_i]$, thus $[f_1]_{B_1} = (c_1^0, ..., c_n^0, c_1^1, ..., c_n^1) \simeq c_0 + fc_1$ in B_1. Hence,

$$\langle w, w'\rangle^{rel}_{1,Ann} = (a_0 + fa_1) Q_1 (b_0 + fb_1) Q_1 (c_0 + fc_1),$$

where the matrix Q_1 is defined as in Lemma (4.3.3). Indeed, if

$$Q_1 = \left[\begin{array}{c|c} Q_0 + fH_1 & fQ_0 \\ \hline fQ_0 & 0 \end{array}\right] = \left[\begin{array}{c|c} Q_0 & 0 \\ \hline 0 & 0 \end{array}\right] + f \left[\begin{array}{c|c} H_1 & Q_0 \\ \hline Q_0 & 0 \end{array}\right],$$

with

$$Q_0^{'} = \left[\begin{array}{c|c} Q_0 & 0 \\ \hline 0 & 0 \end{array}\right] \tag{5.49}$$

and

$$H_1^{'} = \left[\begin{array}{c|c} H_1 & Q_0 \\ \hline Q_0 & 0 \end{array}\right]$$

then $Q_1 = Q_0^{'} + fH_1^{'}$. Therefore,

$$(a_0 + fa_1)(Q_0^{'} + fH_1^{'})(b_0 + fb_1)(Q_0^{'} + fH_1^{'})(c_0 + fc_1)$$

is equal to

$$a_0Q_0^{'}b_0Q_0^{'}c_0 + f(a_0Q_0^{'}b_0H_1^{'}c_0 + [a_0H_1^{'}b_0 + a_1Q_0^{'}b_0 + a_0Q_0^{'}b_1]Q_0^{'}c_0) + a_0Q_0^{'}b_0Q_0^{'}c_1.$$

Since, $a_0, b_0 \in Ann_{B_0}(f_1)$ and B_0 is a commutative algebra, then from (5.49) we get

$$c_0Q_0a_0 = 0, \quad c_0Q_0b_0 = 0. \tag{5.50}$$

Hence, we have that

$$\langle w, w^{'}\rangle_{1,Ann}^{rel} = (0, a_0Q_0b_0Q_0c_1). \tag{5.51}$$

We observe that the expression in (5.51), the first term is 0 and the next term in $\langle\ ,\ \rangle_{1,Ann}^{rel}$ in B_1 is

$$a_0Q_0b_0Q_0c_1.$$

Indeed, we have only the matrix Q_0, therefore so we can consider the algebra B_0.

In general.

Let $\langle\ ,\rangle_{m,Ann}^{rel}$ be the symmetric bilinear form restricted to the annihilator, defined to be

$$\langle\ ,\rangle_{m,Ann}^{rel} : (Ann_{B_{m-1}}(f_1) \oplus f^m B_0) \times (Ann_{B_{m-1}}(f_1) \oplus f^m B_0) \stackrel{\cdot}{\longrightarrow} B_m \stackrel{f_1}{\longrightarrow} B_m. \tag{5.52}$$

If $w = a_{m-1} + f^m a_m$, $w^{'} = b_{m-1} + f^m b_m$, with $a_{m-1}, b_{m-1} \in Ann_{B_{m-1}}(f_1)$, and $[f_1]_m = c_{m-1} + f^m c_m$ where $c_{m-1} \in B_{m-1}$ then

$$\langle w, w^{'}\rangle_{m,Ann}^{rel} = ((a_{m-1} + f^m a_m)Q_m(b_{m-1} + f^m b_m)Q_m(c_{m-1} + f^m c_m)) \tag{5.53}$$

where Q_m is the matrix defined in (4.3.3). If we define the matrix $Q_{m-1}^{'}$, to be

$$Q_{m-1}^{'} = \left[\begin{array}{l|c|c|r} Q_0 + fH_1 + \cdots + f^{m-1}H_{m-1} & fQ_0 + f^2H_1 + \cdots + f^{m-1}H_{m-2} & \cdots & f^{m-1}Q_0 \\ \hline fQ_0 + f^2H_1 + \cdots + f^{m-1}H_{m-2} & \cdots & f^{m-1}Q_0 & 0 \\ \hline \vdots & \ddots & 0 & 0 \\ \hline f^{m-1}Q_0 & 0 & 0 & 0 \end{array}\right],$$

and

$$H_m^{'} = \left[\begin{array}{l|c|c} H_m & \cdots & Q_0 \\ \hline H_{m-1} & \cdots & Q_0 \\ \hline \vdots & \ddots & 0 \\ \hline Q_0 & 0 & 0 \end{array}\right],$$

then $\langle w, w^{'}\rangle_{m,Ann}^{rel} = (a_{m-1} + f^m a_m)Q_m(b_{m-1} + f^m b_m)Q_m(c_{m-1} + f^m c_m))$

$= (a_{m-1} + f^m a_m)(Q_{m-1}^{'} + f^m H_m^{'})((b_{m-1} + f^m b_m)(Q_{m-1}^{'} + f^m H_m^{'})(c_{m-1} + f^m c_m))$

$= (a_{m-1}Q_{m-1}^{'}b_{m-1}Q_{m-1}^{'}c_{m-1} + f^m(a_{m-1}Q_{m-1}^{'}b_{m-1}H_m^{'}c_{m-1} + a_{m-1}H_m^{'}b_m Q_{m-1}^{'}c_{m-1} + a_m Q_{m-1}^{'}b_{m-1}Q_{m-1}^{'}c_{m-1} + a_{m-1}Q_{m-1}^{'}b_m Q_{m-1}^{'}c_{m-1} + a_{m-1}Q_{m-1}^{'}b_{m-1}Q_{m-1}^{'}c_m)$.
So, by definition of $Q_{m-1}^{'}$ and the commutativity of algebra B_m, we get $a_{m-1}Q_{m-1}c_{m-1} = 0$, $b_{m-1}Q_{m-1}c_{m-1} = 0$, and

$$\langle w, w^{'}\rangle_{m,Ann}^{rel} = (0, a_{m-1}Q_{m-1}b_{m-1}Q_{m-1}c_m). \tag{5.54}$$

We want to make some observations about the bilinear form $\langle\ ,\rangle_{m,Ann}^{rel}$ in (5.47) and its expression (5.54). The first term in (5.54) is 0 in the decomposition $B_m = B_{m-1} \oplus f^m B_0$. It is clear, since we are restricting to the degeneracy locus of $\langle\ ,\rangle_{\mu_{m-1}}^{rel}$. The next term in $\langle\ ,\rangle_{\mu_m}^{rel}$ in B_m is

$$a_{m-1}Q_{m-1}b_{m-1}Q_{m-1}c_m.$$

The remarkable thing is, the matrix Q_{m-1} which means that we are considering the algebra B_{m-1}. The new term comes from multiplication in B_0 by c_m, which is the second in the expansion of $c = c_{m-1} + f^m c_m$. We also note that the term H_m which determines the extension B_m of B_{m-1}, does not enter into the formula (5.54).

□

Flag in the finite vector space B_0.

In the following paragraph, we get a flag in the finite vector space B_0.

Remark 5.3. *Let*

$$B_j \xrightarrow{\pi_j} B_{j-1} \xrightarrow{\pi_{j-1}} B_{j-2} \xrightarrow{\pi_{j-2}} \ldots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 \tag{5.55}$$

be surjective morphisms, such that π_j are maps given by $\pi_j(v) = v \ mod(f^{j-1})$, with $(j = 1, \cdots, m,\ m \in \mathbb{Z}^{\geq 0})$. If $B_j = \frac{\mathcal{C}}{(f^{j+1})}$ are finite dimensional vector spaces, and $Ann_{B_j}(f_1) = \{b \in \mathcal{C} : [bf_1]_j = 0 \ \ in \ B_j\}$ is the annihilator of f_1 in B_j, then

$$\rho_j = \pi_j \circ \pi_{j-1} \circ \ldots \circ \pi_1 \text{ are surjective maps from } B_j \text{ to } B_0. \tag{5.56}$$

Proposition 5.4.1. *If we consider the surjective map defined in* (5.55)*, then*

$$\pi_j(Ann_{B_j}(f_1)) \subset Ann_{B_{j-1}}(f_1).$$

Proof. If $l \in Ann_{B_j}(f_1)$, then $[lf_1]_j = 0 \in B_j$. Indeed, $\tilde{l} \cdot f_1 \in (f^{j+1}) \subset (f^j)$ and $[\tilde{l}f_1]_{j-1} = 0 \in B_{j-1}$, then $\pi_j l \in Ann_{B_{j-1}}(f_1)$.

□

We observe that the map π_j carries $Ann_{B_j}(f_1)$ to $Ann_{B_{j-1}}(f_1)$ and the map ρ_i carries the $Ann_{B_i}(f_1)$ to algebra B_0. So, we get the following flag

$$B_0 \supset \rho_1(Ann_{B_1}(f_1)) \supset \rho_2(Ann_{B_2}(f_1)) \supset \rho_3(Ann_{B_3}(f_1))... \supset \rho_\ell(Ann_{B_\ell}(f_1)), \quad (5.57)$$

for some $\ell \in \mathbb{Z}^{\geq 0}$.

Proof to the theorem (1.2).

Proof. Let us consider the matrix defined in lemma (4.3.3) and $\langle\rangle_m^{rel} : B_m \times B_m \overset{\cdot}{\longrightarrow} B_m \overset{f_1}{\to} B_m \overset{L_m}{\longrightarrow} \mathbb{R}$. So, it has the associated matrix given by

$$\left[\begin{array}{l|c|c|c} L_m f_1 Q_0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \vdots & \ddots & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{l|c|c|c} L_m f_1 f H_1 & L_m f_1 f Q_0 & \cdots & 0 \\ \hline L_m f_1 f Q_0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \vdots & \ddots & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] + \cdots$$

$$+ \left[\begin{array}{l|c|c|c|c|c} L_m f_1 f^{m-1} H_{m-1} & L_m f_1 f^{m-1} H_{m-2} & \cdot & L_m f_1 f^{m-1} H_1 & L_m f_1 f^{m-1} Q_0 & 0 \\ \hline L_m f_1 f^{m-1} H_{m-2} & \cdots & L_m f_1 f^{m-1} H_1 & L_m f_1 f^{m-1} Q_0 & 0 & 0 \\ \hline \vdots & \ddots & \ddots & 0 & 0 & 0 \\ \hline L_m f_1 f^{m-1} H_1 & L_m f_1 f^{m-1} Q_0 & 0 & 0 & 0 & 0 \\ \hline L_m f_1 f^{m-1} Q_0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$+ \left[\begin{array}{l|c|c|c|r} L_m f_1 f^m H_m & L_m f_1 f^m H_{m-1} & \cdot & L_m f_1 f^m H_1 & L_m f_1 f^m Q_0 \\ \hline L_m f_1 f^m H_{m-1} & \cdots & L_m f_1 f^m H_1 & L_m f_1 f^m Q_0 & 0 \\ \hline \vdots & \ddots & \ddots & 0 & 0 \\ \hline L_m f_1 f^m H_1 & L_m f_1 f^m Q_0 & 0 & 0 & 0 \\ \hline L_m f_1 f^m Q_0 & 0 & 0 & 0 & 0 \end{array}\right]. \quad (5.58)$$

Hence, from Lemma (5.4.1) the bilinear form is nondegenerate in $a_{m-1}Q_{m-1}b_{m-1}Q_{m-1}c_m$ if $a_{j-1}Q_{j-1}b_{j-1}Q_{j-1}c_j \in (f^m J_0)$, $(j = 0, ..., (m-1))$, then $L_m(a_{j-1}Q_{j-1}b_{j-1}Q_{j-1}c_j) > 0$. So, we get the flag given in (5.57) and the matrix has the form of Lemma (5.19).

Hence, the matrix (5.58) is equivalent to:

$$Q_m^{rel} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & \tilde{D_0} \\ 0 & 0 & 0 & 0 & \dots & 0 & \tilde{D_0} & 0 \\ 0 & 0 & 0 & \dots & 0 & \tilde{D_0} & 0 & 0 \\ 0 & 0 & \dots & 0 & \tilde{D_0} & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \tilde{D_0} & \cdots & 0 & 0 & 0 & 0 & 0 \\ \tilde{D_0} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \tilde{E_1} & 0 \\ 0 & 0 & 0 & 0 & \dots & \tilde{E_1} & 0 & 0 \\ 0 & 0 & 0 & \dots & \tilde{E_1} & 0 & 0 & 0 \\ 0 & 0 & \dots & \tilde{E_1} & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{E_1} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \dots + \begin{pmatrix} E_m & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.59}$$

Using the theorem (1.1) in (5.59), we get
1) If $m \geq 0$ is even then

$$\tilde{\sigma}_m^{rel} = \tilde{\sigma}_0^{rel} + \tilde{\sigma}_{2,Ann}^{rel} + \tilde{\sigma}_{4,Ann}^{rel} + \cdots + \tilde{\sigma}_{m,Ann}^{rel},$$

and if $m \geq 1$ is odd then

$$\tilde{\sigma}_m^{rel} = \tilde{\sigma}_1^{rel} + \tilde{\sigma}_{3,Ann}^{rel} + \tilde{\sigma}_{5,Ann}^{rel} + \cdots + \tilde{\sigma}_{m,Ann}^{rel}.$$

2) For m large enough, $\tilde{\sigma}_{m,Ann}^{rel} = 0$.
From the Corollary (5.1) we get 3) of Theorem (1.2).

□

Corollary 5.2. *Under the hypothesis of theorem (1.2), we consider* $\tilde{\sigma}_m^{rel} = (p_m, q_m, r_m)$ *and* $\tilde{\sigma}_{m,Ann}^{rel} = (s_m^+, s_m^-, s_m^0)$ *such that* $r_m = r_{m-1}$*. If we define* $(p^{'}, q^{'}, 0)$ *to be*

$$(p^{'}, q^{'}, 0) = \tilde{\sigma}_m^{rel} - \tilde{\sigma}_{m-1}^{rel}. \tag{5.60}$$

Then, we have:

$$\tilde{\sigma}_{m+1}^{rel} = \begin{cases} \tilde{\sigma}_m^{rel} + (p^{'}, q^{'}, 0) & if \quad m \quad is \quad even \\ \tilde{\sigma}_m^{rel} + (q^{'}, p^{'}, 0) & if \quad m \quad is \quad odd. \end{cases} \tag{5.61}$$

Proof. From (5.38), we have that

$$r_m = r_{m-1} + s_m^0.$$

Since, $r_m = r_{m-1}$ then $s_{m+k}^0 = s_m^0 = 0$, $k = 1, \cdots \ell$, $\ell \in \mathbb{Z}^{>0}$.
Thus, by (5.60) and if we consider $m = 2n$, then

$$(p^{'}, q^{'}, 0) = (\tilde{\sigma}_{2n}^{rel} - \tilde{\sigma}_{2n-1}^{rel}) = (p_{2n}, q_{2n}, r_{2n-1}) - (p_{2n-1}, q_{2n-1}, r_{2n-1}). \tag{5.62}$$

From (5.33) we obtain

$$p_{2n} = p_{2n-1} + p_0 + s_1^- + ... + s_{2n}^+, \tag{5.63}$$

and by (5.34), we get

$$p^{'} = p_{2n} - p_{2n-1} = p_0 + s_1^- + ... + s_{2n}^+. \tag{5.64}$$

Similarly,

$$q^{'} = q_{2n} - q_{2n-1} = q_0 + s_1^+ + ... + s_{2n}^-. \tag{5.65}$$

So,

$$(p^{'}, q^{'}, 0) = (p_0 + s_1^- + ... + s_{2n}^+, q_0 + s_1^+ + ... + s_{2n}^-, 0). \tag{5.66}$$

Moreover, from Lemma (5.3.2) we have

$$\tilde{\sigma}_{2n}^{rel} = \tilde{\sigma}_{2n-1}^{rel} + (p_0, q_0, 0) + (s_1^-, s_1^+, 0) + ... + (s_{2n}^{,} s_{2n}^-, 0).$$

Therefore, from the previous equation and (5.66) we get

$$\tilde{\sigma}_{2n}^{rel} = \tilde{\sigma}_{2n-1}^{rel} + (p^{'}, q^{'}, 0). \tag{5.67}$$

Similarly for $m = 2n + 1$

$$\tilde{\sigma}_{2n+1}^{rel} = \tilde{\sigma}_{2n}^{rel} + (q^{'}, p^{'}, 0). \tag{5.68}$$

□

Chapter 6

The finiteness of the algorithm

In this section, we get an argument for the stabilization of the formula of the theorem (1.2).

6.1 Transporting the primitive invariants of the relative index, and flags from B_0 to the algebra $\mathbf{A}$.

Let $\{f_1, f_2, ..., f_n\}$ be a regular sequence of germs at 0 of class $\mathcal{C}^\omega$, $f_i = \frac{\partial f}{\partial x_i}$.
If $\mathcal{C} = \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f_2,...,f_n)}$ is a local algebra, and $\mathcal{A}_{\mathbb{R}^n,0}$ is the ring of germs of real analytic functions at 0, then $\mathbf{A} = \frac{\mathcal{C}}{(f_1)}$ is a finite dimensional vector space.

On the other hand, let $f \in \mathcal{A}_{\mathbb{R}^n,0}$ and $M_f(a)$ be the action to multiply by f. It is defined to be $M_f(a) = f \cdot a$, for all $a \in \mathbf{A}$.

L.Giraldo, X.Gómez-Mont and P.Mardeŝic (see [14]), proved the following result:
For $j = 1, \ldots, \ell+1$, there are linear subspaces P_j of A, called primitive subspaces, such that

$$A = \bigoplus_{j=1}^{\ell+1}[\bigoplus_{k=0}^{j-1} M_f^k P_j]. \tag{6.1}$$

with $M_f^{j-1} : P_j \to A$ injective map and $M_f^j(P_j) = 0$. The mapping $M_f : A \to A$ is a Jordan cononical form in any basis obtained by choosing bases of each of the spaces P_j and extending them to a basis of A by the action of M_f as in (6.1).

Hence, it is convenient to present the direct sum decomposition (6.1) by the matrix:

$$A = \begin{pmatrix} P_1 & P_2 & P_3 & P_4 & ... & P_\ell & P_{\ell+1} \\ 0 & M_f P_2 & M_f P_3 & M_f P_4 & ... & M_f P_\ell & M_f P_{\ell+1} \\ 0 & 0 & M_f^2 P_3 & M_f^2 P_4 & ... & M_f^2 P_\ell & M_f^2 P_{\ell+1} \\ \vdots & \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & 0 & 0 & ... & M_f^{\ell-1} P_\ell & M_f^{\ell-1} P_{\ell+1} \\ 0 & 0 & 0 & 0 & ... & 0 & M_f^{\ell} P_{\ell+1} \end{pmatrix}. \tag{6.2}$$

If $Ann_{\mathbf{A}}(f) := \{a \in \mathcal{C} \mid af = 0, \text{ in } \mathbf{A}\}$ is the annihilator of f in the algebra $\mathbf{A}$, then

$$M_f^\ell(P_{\ell+1}) \oplus M_f^{\ell-1}(P_\ell) \oplus \cdots \oplus M_f^2(P_3) \oplus M_f(P_2) \oplus P_1 = Ann_{\mathbf{A}}(f).$$

The ideal (f^m) is formed by the last $\ell+1-m$ rows of the matrix (6.2), where, $m = 1, ..., \ell$ and $\ell \in \mathbb{Z}^{\geq 0}$. (See [14]).

$$Ann_{\mathbf{A}}(f^j) = ker(M_f^j), \quad (f^j) = Im(M_f^j).$$

Moreover, we consider the flag

$$0 \subset (f^\ell) \subset (f^{\ell-1}) \subset ... \subset (f) \subset \mathbf{A}, \tag{6.3}$$

and, if we define

$$K_m := Ann_{\mathbf{A}}(f) \cap (f^{m-1}) \subset \mathbf{A} \tag{6.4}$$

then we obtain the following flag

$$0 \subset K_{\ell+1} \subset K_\ell \subset ... \subset K_1 \subset K_0 = \mathbf{A}. \tag{6.5}$$

Similarly, let $B_0 := \frac{\mathcal{C}}{(f)}$. It is a finite dimensional vector space, $\tilde{K}_m$ is defined as the projection of the annihilator of f_1 in B_m to B_0, i.e. From(5.57), $\rho_m(Ann_{B_m}(f_1)) = \tilde{K}_m$. Namely,

$$\tilde{K}_m = \frac{(f^{m+1} : f_1)}{(f) \cap (f^{m+1} : f_1)}. \tag{6.6}$$

Where, $(f^{m+1} : f_1) := \{a \in \mathcal{C} : af_1 \in (f^{m+1})\}$, and f^{m+1} denotes f to the power $m+1$. Therefore, we have the following flag of ideals in B_0

$$0 \subset \tilde{K}_{\ell+1} \subset ... \subset \tilde{K}_1 \subset \tilde{K}_0 = B_0. \tag{6.7}$$

6.2 Stabilization of the algebraic formula

Theorem 6.1. *If we consider the flags defined earlier in (6.5) and (6.7). Then there exists a bijection between the flag defined in* (6.7) *and the flag defined in* (6.5)*. There is also an integer* ℓ *with* $dim(k_\ell) = \ell$*, where the algebraic formulas* (1.2) *are stabilized.*

Proof. If $\varphi : \tilde{K}_m \longrightarrow K_m$ is a morphism defined to be $\varphi(b) = \frac{bf_1}{f}$ and $\varphi^{-1} : K_m \longrightarrow \tilde{K}_m$ is given by $\varphi^{-1}(c) = \frac{cf}{f_1}$, then, it is an is isomorphism and

1) φ sends $(f^m : f_1) \longrightarrow (f_1 : f) \cap (f^{m-1})$

2) $\varphi(f^m) = (f^{m-1}f_1)$.

We will proof 1). If $b \in \tilde{K}_m$ *then* $bf_1 = cf^m$. Hence, $\frac{bf_1}{f} = cf^{m-1} \in (f^{m-1})$ and $f\frac{bf_1}{f} = bf_1$. Thus, $b \in (f_1 : f)$ and $\varphi(b) \in (f_1 : f) \cap (f^{m-1})$.
Similarly, if $c \in (f_1 : f) \cap (f^{m-1})$, then $\varphi^{-1}(c) = \frac{cf}{f_1}$ and $c = df^{m-1}$. Indeed, $\varphi^{-1}(df^{m-1}) = \frac{df^{m-1}f}{f_1} = \frac{df^m}{f_1}$. So $f_1\frac{df^m}{f_1} = df^m \in (f^m : f_1)$. The proof of 2) is similar.
In particular, if $m = 1$ then φ sends $Ann_{B_0}(f_1)$ in $Ann_{\mathbf{A}}(f)$.
On the other hand, since the map M_f corresponds to the Jordan blocks, and if we consider the flag defined in (6.5), then we get the following table:

.

$\mathbf{A}$	$\xrightarrow{\cdot f}$	$\mathbf{A}$
$\cup$		
$Ann_{\mathbf{A}}(f)$	$\longrightarrow$	all eigenvectors of M_f
$\cup$		
$Ann_{\mathbf{A}}(f) \cap (f)$	$\longrightarrow$	all eigenvectors coming from Jordan blocks of M_f of $size \geq 2$
$\cup$		
$Ann_{\mathbf{A}}(f) \cap (f^2)$	$\longrightarrow$	all eigenvectors coming from Jordan blocks of M_f of $size \geq 3$
$\cup$		
.		
.		
.		
$\cup$		
$Ann_{\mathbf{A}}(f) \cap (f^\ell) = 0$		

The previous table is equivalent to the following matrix

$$M_f = \left(\begin{array}{c|c|c} \begin{matrix} 0_1 & & & & \\ & 0_1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 0_1 \end{matrix} & & \\ \hline & \begin{matrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & & & \\ & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \cdot \end{matrix} & \\ \hline & & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}\right).$$

Therefore, we have a flag $0 \subset K_\ell \subset ... \subset K_3 \subset K_2 \subset K_0 = Ann_A(f) \subset \mathbf{A}$.
So, the algebraic formula stops when it reaches the maximal size of Jordan blocks of map $M_f : \mathbf{A} \to \mathbf{A}$.
Moreover, we consider the bilinear forms defined by

$$\langle\ ,\rangle : K_m \times K_m \longrightarrow \mathbb{R}$$

$$\langle a, a' \rangle = \langle \frac{a}{f^{m-1}}, a' \rangle_{L_A}$$

$$\langle\ ,\rangle : \tilde{K}_m \oplus \tilde{K}_m \longrightarrow \mathbb{R}$$

$$\langle a, a' \rangle = \langle \frac{af_1}{f^m}, a' \rangle_{L_m}.$$

And, if we define $L_0 = L_A \circ (\varphi)$, the flag in the algebra in A is carried to the flag in B_0. The algorithm stops in ℓ as well. $\square$

Example 6.2.1. Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ be a germ of real analytic function defined to be $f(x, y, z) = (x^3 + y^2)(x^2 + y^3) + z^2$, and we consider the degenerate symmetric bilinear form given by

$$\langle\ ,\rangle_m^{rel} : B_m \times B_m \dot{\to} B_m \overset{f_1}{\to} B_m \overset{L_m}{\to} \mathbb{R},$$

where B_m, is defined as in (1.2) with $m = 1, 2, 3$. So, the routine bilinearform.lib and the kanula give the following results

Results of program relative bilinearform.lib			
	size	rank	signature
B_0	15	5	$(3, 2, 10)$
B_1	30	19	$(9, 10, 11)$
B_2	45	34	$(17, 17, 11)$
B_3	60	49	$(24, 25, 11)$

.

Results of program kanulabil			
	size	an(i)	rank
B_0	15	10	5
B_1	30	1	19
B_2	45	0	34
B_3	60	0	49

.

Therefore, it is an immediate consequence of the previous tables that the signature associated with B_0 is $\tilde{\sigma}_0 = (3, 2, 10)$. It corresponds to three positive numbers, two negative numbers, and ten zeros. Furthermore, the signature associated with the algebra B_1 is given by $\tilde{\sigma}_1 = (9, 10, 11)$, and it is computed like (5.3.2). To explain this, we consider the relative bilinear form $\langle\ ,\rangle_1^{rel}$ defined to be

$$\left[\begin{array}{c|c|c|c}
0_{5\times5} & 0_{5\times10} & \begin{matrix} I_3 & \\ & -I_2 \end{matrix} & 0_{5\times10} \\ \hline
0_{10\times5} & \begin{matrix} I_4 & & \\ & -I_5 & \\ & & 0_{1\times1} \end{matrix} & 0_{10\times5} & \begin{matrix} 0_{5\times5} & \\ & 0_{5\times5} \end{matrix} \\ \hline
\begin{matrix} I_3 & \\ & -I_2 \end{matrix} & 0_{5\times10} & 0_{5\times5} & 0_{5\times10} \\ \hline
0_{10\times5} & \begin{matrix} 0_{5\times5} & \\ & 0_{5\times5} \end{matrix} & 0_{10\times15} & \begin{matrix} 0_{5\times5} & \\ & 0_{5\times5} \end{matrix}
\end{array}\right].$$

Hence, from (5.3.2) $\tilde{\sigma}_1 = (p_1, q_1, r_1) = (9, 10, 11) = 3 + 2 + 4, 2 + 3 + 5, 10 + 1$, where, $\tilde{\sigma}(E_1) = (4, 5, 1)$, the next signatures are computed in a similar form.

On the other hand, from table kanula.lib we get the flag

$$\tilde{K}_1 \subset \tilde{K}_0 \subset B_0,$$

which is equivalent to

$$\mathbb{R} \subset \mathbb{R}^{10} \subset \mathbb{R}^{15}.$$

As we can see, in this case the flag defines a 1 Jordan block of size 2, and an 8 Jordan block of size 1. i.e.

$$1(2) + 8(1) = 10.$$

Thus, 10 corresponds to the dimension of the annihilator of f_1 in the algebra B_0. i.e. $dim(Ann_{B_0}(f_1)) = 10$. If we consider the isomorphism φ, then $dim(Ann_A(f)) = 10$, and we get the stabilization of the algebraic formula (5.3.2).
Hence, we can conclude that $r_2 = r_1 = 11$, and $a_2 = a_3 = 0$, Indeed, from (5.1) we can define $p^{'}, q^{'}$ as $p^{'} = 17 - 9 = 8$, $q^{'} = 17 - 8 = 7$, therefore, $\tilde{\sigma}_2 = \tilde{\sigma}_1 + (p^{'}, q^{'}, 0) = (9 + 8, 10 + 7, 11 + 0) = (17, 17, 11)$, and so on.

Chapter 7

Applications to vector fields tangent to the Milnor fiber

In this chapter, we will describe holomorphic and real analytic vector fields. We will also give an interesting example where we exhibit the changes in the topology of the Milnor fiber.

Let $f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$ be a germ of a holomorphic function with an isolated singularity at 0. We consider, $V_t^{\mathbb{C}}(f) = f^{-1}(t)$ the Milnor fiber, and X_t a family of germs of holomorphic vector fields in $\mathbb{C}^{n+1}$, such that X_0 has an isolated singularity at 0.
If X_t is tangent to the hypersurface $V_t^{\mathbb{C}}(f)$, then

$$d(f-t)X_t = h_t(x)(f-t),$$

where $h_t(x)$ is the cofactor and it is a holomorphic function.
On the other hand, if Z is the singular set of the family of holomorhic vector fields, namely

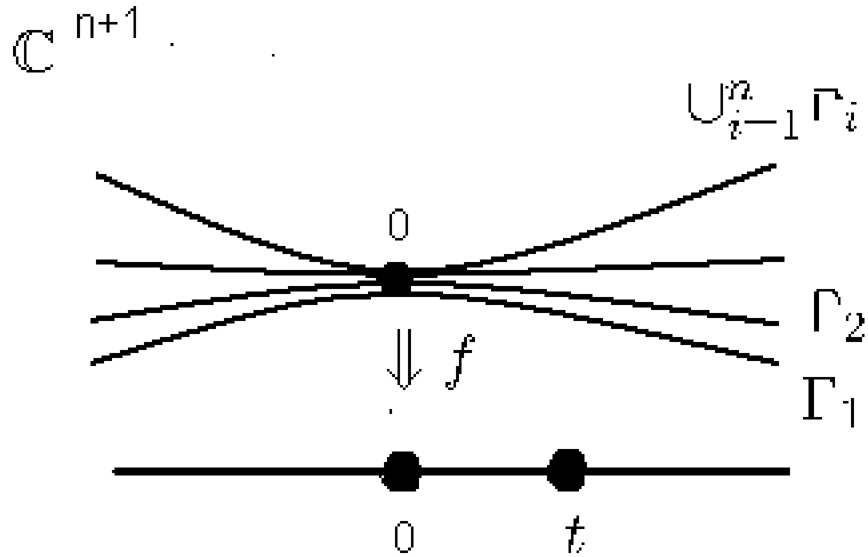
$$Z := \{(t, x) \in \mathbb{C} \times \mathbb{C}^{n+1} | X_t^0(x) = ... = X_t^n(x) = 0\}.$$

Then,

$$\mathcal{O}_Z := \frac{\mathcal{O}_{\mathbb{C}\times\mathbb{C}^{n+1}}}{(X_t^0, ..., X_t^n)}$$

is a multilocal algebra and the map $\Pi_1 : Z \to \mathbb{C}$ is a finite analytic map. The sheaf $(\Pi_1)*\mathcal{O}_Z$ is a free $\mathcal{O}_{\mathbb{C}}$ module of rank n.

Let $\{\Gamma_1 \cup \Gamma_2 \cup ... \cup \Gamma_n\}$ be the set of irreducible components of Z, each Γ_i, $(i = 1, 2, \cdot, n, n \in \mathbb{Z}^{\geq 0})$ has dimension 1, due to the hypothesis that X_0 has an isolated singularity at 0. See the following figure:



This figure represents the curves Γ_i, with $i = 1, 2, ..., n, n \in \mathbb{Z}^{\geq 0}$.

Futhermore, let $B_t^{\mathbb{C}}$ be a multilocal algebra, defined to be

$$B_t^{\mathbb{C}} = \bigoplus_{p \in Z \cap f^{-1}(\mathbf{t})} \frac{\mathcal{O}_{\mathbb{C}^{n+1},p}}{(X_t^0, X_t^1, ..., X_t^n)},$$

and $\langle\ ,\rangle_t^{rel}$ be a degenerate bilinear form, namely

$$\langle\ ,\rangle_t^{rel} : B_t^{\mathbb{C}} \times B_t^{\mathbb{C}} \stackrel{\cdot}{\longrightarrow} B_t^{\mathbb{C}} \stackrel{h_t}{\longrightarrow} B_t^{\mathbb{C}} \stackrel{L_t}{\longrightarrow} \mathbb{C}. \tag{7.1}$$

We consider, W the set defined to be $W := \{(t, x) \in \mathbb{C} \times \mathbb{C}^{n+2} | f(x) - t = 0\}$ then,

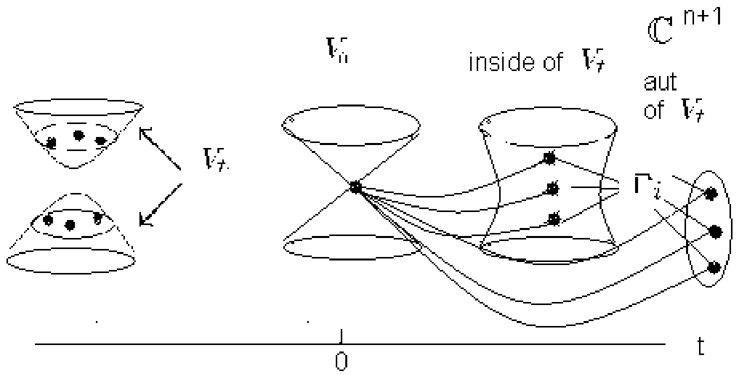
1) $\Gamma_i \not\subset W$ if and only if the map $f - t|_{\Gamma_i} \neq 0$ if and only if $V_0^{\mathbb{C}}(f) \cap \Gamma_i = \{0\}$.

2) $\Gamma_i \subset W$ if and only if $f - t|_{\Gamma_i} \equiv 0$, if and only if $\Gamma_i \cap \Pi_1^{-1}\} \subset V_t = \{p_1(t), ..., p_n(t)\}$.

Remark 7.1.

- *Similarly, for case* 1) *we have* $\{p_i(t)\} \subset \mathbb{C}^{n+1} - V_t^{\mathbb{C}}(f)$ *if and only if the points* $\{p_i(t)\}$ *are zeros of the vector field* X_t *in* $\mathbb{C}^{n+1} - V_t^{\mathbb{C}}(f)$, $t \in (\mathbb{C}, 0)$, $i = 1, 2, ..., n, n \in \mathbb{Z}^{\geq 0}$.
- *In case* 2), *the points* $\{p_i(t)\} \subset V_t^{\mathbb{C}}(f)$ *if and only if* $\{p_i(t)\}$ *are zeros of* $X_t|_{V_t^{\mathbb{C}}(f)}$.

See the next picture:



Therefore, if $\Gamma_i \not\subset W$ then $f - t|_{\Gamma_i} \neq 0$.
Since, $d(f-t)X_t = (f-t)h_t$ then $d(f-t)X_t = 0$, indeed, $(f-t)h_t = 0$ and $f-t|_{\Gamma_i} \neq 0$. So, $h_t|_{\Gamma_i} = 0$ and $Ann_{B_t}(h_t) = 0$, where, $Ann_{B_t}(h_t) = 0$ is the annihilator of h_t on B_t. In this case, we get neither a flag nor the new contribution to the signature.

Moreover, if $\Gamma_i \subset W$, then $d(f-t)X_t = 0$, and $(f-t)h_t = 0$, thus, $f|_{\Gamma_i} = 0$ and $h_t \neq 0$. So, $Ann_{B_t}(h_t) \neq 0$. Indeed, we get a flag and we focus in the real case.

Let $\Pi_{1*}\mathcal{O}_Z^+$ be a free $\mathcal{O}_{(\epsilon,\epsilon)}$-sheaf of rank s, where its sections are the fixed points of the conjugation map, and its stalk over 0 is

$$\mathcal{C} = (\Pi_{1*}O_z)_0^+ = \frac{\mathcal{O}_{\mathbb{R}^{2n+2},0}}{(X_t^1, ..., X_t^n)}. \tag{7.2}$$

We define a 1-parameter family of $\mathbb{R}$-algebras to be

$$B_{\mathbf{t}}^+ = \Pi_{1*}O_Z^+ \otimes_{\mathbb{R}} \mathbb{R}[t]_0 = [\bigoplus_{p\in Z\cap\Pi_1^{-1}(\mathbf{t})} \frac{\mathcal{O}_{\mathbb{C}^{n+1},p}}{(X_t^0, X_t^1, ..., X_t^n)}]^+. \tag{7.3}$$

It is obtained by evaluation where B_t^+ is a multilocal algebra and B_0 is a local algebra. If $\langle\ ,\rangle$ is a bilinear map defined by

$$\langle,\rangle_t^{rel} : B_t^+ \times B_t^+ \stackrel{\cdot}{\longrightarrow} B_t^+ \stackrel{h_t}{\longrightarrow} B_t^+ \stackrel{L_t}{\longrightarrow} \mathbb{R}, \tag{7.4}$$

then bilinear forms are nondegenerate for $t \neq 0$, and for $t = 0$, the relative bilinear form $\langle,\rangle_t^{rel}$, degenerates on $Ann_{B_0}([f_1]_{B_0}$.

7.1 The contact vector field, an example

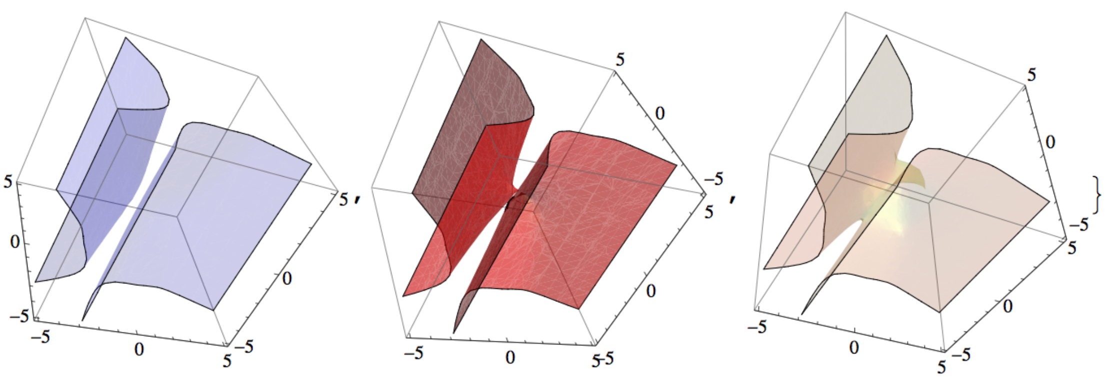
In this example, we will compute the GSV-index using the corollary (2.1), and we will exhibit the topologycal changes of the Milnor fiber.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a germ of real analytic function with isolated singularity at 0. If $X_t = (f - t, f_2, -f_3, \ldots .f_{2n}, -f_{2n+1})$ is the contact vector field, where $f_i = \frac{\partial f}{\partial x_i}$ then, $d(f - t)X_t = f_1(f - t)$, and the cofactor is f_1.

Example 7.1.1. In particular, if the germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ is defined to be

$$f = (x^3 + y^2)(x^2 + y^3) + z^2,$$

then the hypersurfaces $V_t(f)$, for $t < 0$, $t = 0$ and $t > 0$, are:



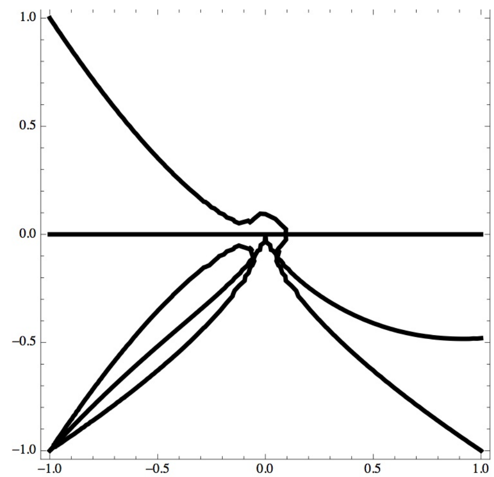
Therefore, we define the contact vector field X_t, to be

$$X_t = (f - t, f_y, -f_z) = ((x^3 + y^2)(x^2 + y^3) + z^2 - t, 2x^2y + 5y^4 + 3x^3y^2, -2z).$$

Indeed, $d(f - t) \cdot X_t = f_x(f - t) = (2xy^2 + 5x^4 + 3x^2y^3)((x^3 + y^2)(x^2 + y^3) + z^2 - t))$.

Since, the vector field is $X_t = (f - t, f_y, -f_z)$ and $f_z = -2z = 0$, then we can consider the vector field in the plane xy.

If $X_t = (f - 0.00002, f_y)$ is a vector field in the plane xy, then we have the following picture



Thus, the vector field X_t, for $t > 0$ is

$$X_{0.00002} = ((x^3 + y^2)(x^2 + y^3) + z^2 - 0.00002, 2x^2y + 5y^4 + 3x^3y^2, -2z).$$

Since $z = 0$, then the real roots of X_t, with $t = 0.00002$ are $p_1 = (0.0542741, -0.105318, 0)$, $p_2 = (0.11487, 0, 0)$, $p_3 = (-0.0547462, -0.106541, 0)$.
Hence,

$$|DX_{0.00002}| = \left| \begin{pmatrix} 2xy^2 + 5x^4 + 3x^2y^3 & 2x^2y + 5y^4 + 3x^3y^2 & -2z \\ 4xy + 9x^2y^2 & 2x^2 + 20y^3 + 6x^3y & 0 \\ 0 & 0 & 2 \end{pmatrix} \right|,$$

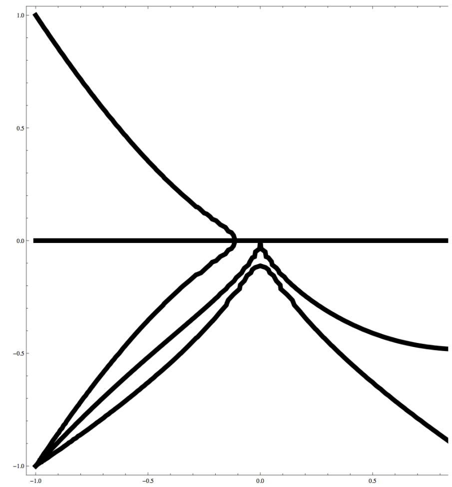
and $|DX_t| = 36x^5y^4 + 60x^7y - 54x^4y^4 + 30x^2y^6 + 200x^4y^3 + 20x^6 - 24x^3y^3 + 40xy^5 - 8x^3y^2$. Furthermore, $|DX_t(p_1)| = -0.000392937$, $|DX_t(p_2)| = 0.0000459483$, $|DX_t(p_3)| = 0.0000386401$. Since, the $Ind_{\mathbb{R}^3}(X, t)$ satisfies the conservation sign law, we have

$$Ind_{\mathbb{R}^3}(X, 0.00002) = -1 + 1 + 1 = 1.$$

Similarly, let X_t be the vector field with $t < 0$. In this case we have

$$X_{-0.00002} = ((x^3 + y^2)(x^2 + y^3) + z^2 + 0.00002, 2x^2y + 5y^4 + 3x^3y^2, -2z).$$

See the following figure,



Since $z = 0$, then we have only one point that $X_t(p) = 0$, where $t = -0.00002$ and $p = (-0.11487, 0, 0)$, thus

$$|DX_{-0.00002}| = \left| \begin{pmatrix} 2xy^2 + 5x^4 + 3x^2y^3 & 2x^2y + 5y^4 + 3x^3y^2 & -2z \\ 4xy + 9x^2y^2 & 2x^2 + 20y^3 + 6x^3y & 0 \\ 0 & 0 & 2 \end{pmatrix} \right|,$$

hence, $|DX_t| = 36x^5y^4 + 60x^7y - 54x^4y^4 + 30x^2y^6 + 200x^4y^3 + 20x^6 - 24x^3y^3 + 40xy^5 - 8x^3y^2$ and

$$|DX_t(p)| = 0.000459483.$$

So,

$$Ind_{\mathbb{R}^3}(X, -0.00002) = 1.$$

It follows that, if $t > 0$, then $p_1 = (0.0542741, -0.105318, 0)$ is the singular point of $X_0.00002$ and $f_x(p_1) = 0.0123707$. Furthermore, $f_x(p_1) > 0$, and $Ind_{\mathbb{R}^3}(Y_t, p_1) = Ind_{\mathbb{R}^3}(X_t, p_1) = -1$. Thus, we consider the point $p_2 = (0.11487, 0, 0)$, $f_x(p_2) = 0.000870556$. So, $f_x(p_2) > 0$ and $Ind_{\mathbb{R}^3}(Y_t, p_2) = Ind_{\mathbb{R}^3}(X_t, p_2) = 1$.
Since $p_3 = (-0.0547462, -0.106541, 0)$ then$f_x(p_3) = -0.0120881$. So, $f_x(p_3) < 0$ and $Ind_{\mathbb{R}^3}(Y_t, p_3) = -Ind_{\mathbb{R}^3}(X_t, p_1) = -1$. Then by conservation the sign law, the relative index is $Ind_{\mathbb{R}^3}(Y, t) = -1$.

If $t < 0$ or $t = -0.00002$ then only one real point exits, namely ,$p = (-0.11487, 0, 0)$, such that $X_t(p) = 0$, and $f_x(p) = 0.00870556 > 0$. Therefore,

$$Ind_{\mathbb{R}^3}(Y_t, p) = Ind_{\mathbb{R}3}(X_t, p) = 1.$$

Hence, we have the following conclusions

$$t > 0,\ Ind(X_1, 0) = 1, \quad Ind(Y_1, 0) = -1$$
$$t < 0,\ Ind(X_{-1}, 0) = 1, \quad Ind(Y_{-1}, 0) = 1.$$

We recall, Eisenbud-Levine proved that the signature of the bilinear form is equal to the degree of real analytic function f. Indeed, we got an algebraic formula to reconstruct the signature of degenerate relative bilinear forms with (1.2), in the real case.

Appendix A

Appendix; Singular programs

Singular Programs

The following routines compute the signature of symmetric bilinear forms, for the non-degenerate and degenerate cases. Since, $(f, f_2, \cdots, f_n)$ is a regular sequence, then we can use a routine of the singular package to compute de krull dimension of $(f, f_2, \cdots, f_n)$. In particular, we need to prove that the krull dimension of $(f, f_2, \cdots, f_n)$ is zero. It is also necessary to change the expression of the function f in the program by the expression we need to compute.
The routine **kanula.lib** constructs the flag defined in B_0.

```
    proc siggen(int iii)
{
LIB "general.lib";
LIB "PHindex.lib";
LIB "Linalg.lib";
ring r = 0, (x,y,z),ds;
int n = nvars(r);
option(redSB);
poly f = x^2 + y^2 + z^2;

    for (int m = 1; m <= iii; m = m + 1)

    {

    ideal i(m) = f**m, diff(f,y), diff(f,z);
ideal g(m) = std(i(m));
ideal kb(m) = kbase(g(m));
"Number of iteration ";m;
// "The groebner base is";
```

// g(m);

int $m1(m) = size(kb(m))$;
ideal $kb1(m) = kb(m)[m1(m)..1]$;
// "The base is kb1";
//kb1(m);

ideal $k2(m) = transpose(kb1(m)) * kb1(m)$;
ideal $k3(m) = reduce(k2(m), g(m))$;

matrix $b(m)[m1(m)][m1(m)] = k3(m)$;

// "The bililinear product is";
// print(mulq);

//Calculus of the socle
ideal $j(m) = jacob(i(m))$;
matrix $jac(m)[n][n] = j(m)$;
"The Jacobian matrix is";
print(jac(m));
poly $s(m) = det(jac(m))$;
poly $sk(m) = reduce(s(m), g(m))$;
"The socle of ring B is"; sk(m);

matrix L(m)[m1(m)][m1(m)];
poly $lc(m) = lead(sk(m))$; poly $lcb(m) = lc(m)/absValue(leadcoef(sk(m)))$;
// "The sign of socle is";
// lcb(m);
int gr(m)=degree(lc(m));
poly divis(m);

int t_2, t_3;
for $(t_2 = 1; t_2 <= m_1(m); t_2 + +)$
{
for $(t_3 = 1; t_3 <= m_1(m); t_3 + +)$
{
$divis(m) = division(jet(b(m)[t_2, t_3], gr(m)) - jet(b(m)[t_2, t_3], gr(m) - 1), lcb(m))[1][1, 1]$;
L(m)$[t_2, t_3] = divis(m)$;

}

}

// print $(L(m))$;

"The determinant of the matrix in the bilinear form de la matriz $<,>_{L_m}$ is";
$det(L(m))$;
"The signature is";
signatureL(L(m));
"The rank of matrix L is";
$mat_rk(L(m))$;

} }

Calculus of the signature in the relative case

```
proc bilinrelamod (iii)
{
LIB "general.lib";
LIB "PHindex.lib";
LIB "Linalg.lib";
ring r = 0, (x, y, z), ds;
for (int  m = 1,  m <= 3,  m = m + 1)
{
ideal i(m) = (f^m, diff(f, y), diff(f, z));
ideal g(m) = std i(m);
ideal kb(m) = kbase(g(m));
"Number of iteration"; m;
"The groebner basis is";
g(m);
"The kbasis is";
kb(m);
int m_1 = size(kb(m));
ideal kb_1(m) = kb(m)[m_1, .., 1];
"The new order in the kbasis is";
kb_1(m);
ideal k_2(m) = transpose(kb_1(m)) * kb_1(m);
ideal k_3(m) = reduce(k_2(m), g(m));
matrix mulq[m_1(m)][m_1(m)] = reduce(diff(f, x) * k_3(m), g(m));
std(i(m));
"The bilinear product is";
print(mulq);
"The socle calculus";
idealj(m) = jacob(i(m));
matrixjac(m)[n][n] = j(m);
"The Jacobian matrix is";
print(jac(m));
poly s(m) = det(jac(m));
poly sk(m) = reduce(s(m), g(m));
"The socle of ring B is";  sk(m);
matrixb(m)[m_1(m)][m_1(m)] = mulq;
matrixL(m)[m_1(m)][m_1(m)];
poly lc(m) = lead(sk(m));
poly lcb(m) = lc(m)/absValue(leadcoef(sk(m));
"The sign of socle is";
"lcb(m)";
int  gr(m) = degree(lc(m));
poly divis(m);
```

```
intt2, t3;
for (t2 = 1, t2 <= m1(m); t2++)
{
for (t3 = 1, t3 <= m1(m); t3++)
{
divis(m) = division(jet(b(m)[t2, t3], gr(m))−jet(b(m)[t2, t3], gr(m)−1), lcb(m))[1][1, 1];
Lm[t2, t3] = divis(m);
};
};
print(L(m));
"The determinant of matrix associated to the symmetric bilinear form denoted by ⟨,⟩rel_m is";
det(L(m));
"The signature is";
signatureL(L(m));
"The rank of the matrix L is";
mat_rk(L(m));
};
};
```

Calculus of a basis of a flag in the algebra B_0

```
proc Kanulabil(int iii)
{
ring r = 0, (x, y, z),ds;
option(redSB);
poly f = (x^3 + y^2) * (x^2 + y^3) + z^2;
ideal i0 = f, diff(f, y), diff(f, z);
ideal g0 = std(i0);
"An R-Basis of B0 is";
module K0 = Kbase(g0);
K0;
"We will calculate the j-flag in B0";
for (intn = 1; n <= 4; n++)
{
ideal i(n) = f**n, diff(f, y), diff(f, z);
ideal g(n) = std(i(n));
ideal q(n) = quotient(i(n), diff(f, x));
"The quotient ideal (i : fx) is"; q(n)
ideal qq(n) = q(n) + g0;
ideal h(n) = std(qq(n));
int t(n) = size(h(n));
t(n);
matrix φ(n)[1][t(n)] = h(n);
module ker(n) = syz(φ(n));
ker(n);
list d(n) = division(g0, h(n));
module b(n) = d(n)[1];
module ker2(n) = ker(n), b(n);
module ker3(n) = std(ker2(n));
ideal an(n) = reduce(φ(n) * k(n), g0);
"A basis as vector space of the corresponding flag is";
an(n);
```

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