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The Group of Homeomorphisms of a Solenoid which are Isotopic to the Identity

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Fermín Omar Reveles Gurrola

Advisors

Dr. Manuel Cruz López

Dr. Xavier Gómez–Mont Ávalos

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A mi Maestro Crúz–López que me enseñó Todo lo que Sé;
a mi Maestra, que me Enseñará todo lo que No sé;
a Ti Maestro Cantor, mi Vida, mi Muerte Chiquita.

Abstract

In this work, we present a detailed study of the connected component of the identity of the group of homeomorphisms of a solenoid $\mathrm{Homeo}_+(\mathsf{S})$. We treat the universal one–dimensional solenoid S as the universal algebraic covering of the circle $\mathbb{S}^1$; that is, as the inverse limit of all the n–fold coverings of $\mathbb{S}^1$. Moreover, this solenoid is a foliated space whose leaves are homeomorphic to $\mathbb{R}$ and a typical transversal is isomorphic to the completion of the integers $\widehat{\mathbb{Z}}$.

We are mainly interested in the homotopy type of $\mathrm{Homeo}_+(\mathsf{S})$. Using the theory of cohomology group we calculate its second cohomology groups with integer and real coefficients. In fact, we are able to calculate the associated bounded cohomology groups. That is, we found the Euler class for the universal central extension of $\mathrm{Homeo}_+(\mathsf{S})$, which is constructed via liftings to the covering space $\mathbb{R} \times \widehat{\mathbb{Z}}$ of S. We show that this is a bounded cohomology class. In particular, we find an analogue of the Poincaré rotation number.

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Introduction

At the end of the 19th century, Henri Poincaré ([Poi]) introduced an invariant of major importance for the study of the dynamics of homeomorphisms of the unit circle $\mathbb{S}^1$, this is the well known *rotation number*. There are some equivalent ways of defining this topological invariant. Our interest lies in the work of E. Ghys ([Ghy]). At the beginning of this century he found the rotation number using the language of cohomology of groups.

Consider the group $\mathrm{Homeo}_+(\mathbb{S}^1)$ of of homeomorphisms of the circle which preserves the orientation and $\widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1)$ the group of lifts with respect to the universal covering $\pi : \mathbb{R} \longrightarrow \mathbb{S}^1$. There is a surjective homomorphism

$$p : \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1) \longrightarrow \mathrm{Homeo}_+(\mathbb{S}^1)$$

with kernel isomorphic to $\mathbb{Z}$. Moreover, p is a universal covering map and this is a universal central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1) \xrightarrow{p} \mathrm{Homeo}_+(\mathbb{S}^1) \longrightarrow 1.$$

It is a consequence of a general result of Thurston ([Thu] and [Tsu]) that

$$\mathrm{H}^2(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{Z}) \simeq \mathbb{Z}$$

and a generator eu of this group is known as the Euler class for the given extension. Moreover, the last is a universal central extension and from the theory of universal central extensions (see for example [Mil]), the kernel is isomorphic to the Schur multiplier $\mathrm{H}_2(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{Z})$. Using the universal coefficient theorem we can derive the second cohomology as well.

In particular, it can be shown that the Euler class is bounded, in the sense of the cocycles being bounded maps and represents a generator for the bounded cohomology group $\mathrm{H}^2_b(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{Z})$. If $\phi : \mathbb{Z} \longrightarrow \mathrm{Homeo}_+(\mathbb{S}^1)$ is any homomorphism, the corresponding class $\phi^*(eu) \in \mathrm{H}^2_b(\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}$ is the rotation number of the homeomorphism $\phi(1)$.

The main objective of this thesis is to extend these results to the case of the group of homeomorphisms isotopic to the identity of the universal one dimensional solenoid S, which is a compact connected Abelian topological group and also a foliated space.

The universal one–dimensional solenoid can be defined as

$$\mathsf{S} := \lim_{\leftarrow} \mathbb{R}/n\mathbb{Z}.$$

S is a compact connected Abelian group, with a canonical inclusion $\mathrm{P} : \mathbb{R} \longrightarrow \mathsf{S}$, such that $\mathcal{L}_0 := \mathrm{P}(\mathbb{R})$ is the arc–connected component of the identity element in S, and $\overline{\mathcal{L}_0} \cong \mathsf{S}$.

The projection onto the first coordinate $\mathsf{S} \longrightarrow \mathbb{S}^1$ is a $\widehat{\mathbb{Z}}$–principal fiber bundle, with fiber

$$\widehat{\mathbb{Z}} := \lim_{\leftarrow} \mathbb{Z}/n\mathbb{Z};$$

where $\widehat{\mathbb{Z}}$ is the profinite completion of the integers $\mathbb{Z}$, which is a compact Abelian totally disconnected topological group. Also, it is a perfect topological space. That is, $\widehat{\mathbb{Z}}$ is isomorphic to the Cantor topological group. Moreover, S is a foliated space whose leaves are homeomorphic to $\mathbb{R}$ and every transversal is isomorphic to $\widehat{\mathbb{Z}}$.

Let $\mathrm{Homeo}_+(\mathsf{S})$ be the component of the identity of $\mathrm{Homeo}(\mathsf{S})$ and let $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})$ be the group of lifts of homeomorphisms in $\mathrm{Homeo}_+(\mathsf{S})$ via the covering $\Pi : \mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathsf{S}$, defined as the quotient map of the diagonal $\mathbb{Z}$–action

$$\gamma \cdot (x, k) \longrightarrow (x - \gamma, k + \gamma), \quad (\gamma \in \mathbb{Z}).$$

There is a surjective homomorphism

$$\mathfrak{p} : \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) \longrightarrow \mathrm{Homeo}_+(\mathsf{S}),$$

with kernel isomorphic to $\mathbb{Z}$. In fact, $\mathbb{Z}$ will be identified with the subgroup of deck transformations $\Delta(\mathbb{Z}) \subset \widetilde{\mathrm{Homeo}_+}(\mathsf{S})$.

As we said before, the main concern of this work is to emulate the theory as presented by Ghys to the case of homeomorphisms of the solenoid S. In first place, we prove that $\mathrm{Homeo}_+(\mathsf{S})$ has the homotopy type of the group of translations on the base leaf $\mathcal{L}_0 \hookrightarrow \mathrm{Homeo}_+(\mathsf{S})$. Then, we describe a central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) \longrightarrow \mathrm{Homeo}_+(\mathsf{S}) \longrightarrow 1$$

and prove that it is a universal central extension. This can be done by extending the fact that $\mathrm{Homeo}_+(\mathsf{S})$ is uniformly perfect, as shown in [AP], to the group of lifts $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})$.

Therefore we have the Schur multiplier

$$\mathrm{H}_2(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z}) \simeq \mathbb{Z}.$$

Thus, using the universal coefficient theorem, there is an Euler class

$$eu \in \mathrm{H}^2(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z}) \simeq \mathbb{Z}.$$

This class being bounded represents a generator of $\mathrm{H}^2_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z})$.

Consider the *rotation element*

$$\rho : \mathrm{Homeo}_+(\mathsf{S}) \longrightarrow \mathsf{S}$$

as introduced in the work of A. Verjovsky and M. Cruz–López (see [CV]). For $f \in \mathrm{Homeo}_+(\mathsf{S})$ take a lift of $\rho(f)$ to the covering $\mathbb{R} \times \widehat{\mathbb{Z}}$. This give us an element $\tau \in \mathbb{R} \times \widehat{\mathbb{Z}}$ and fixing a height on the covering we can describe a homogeneous quasimorphism $\mathrm{T} : \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) \longrightarrow \mathbb{R}$. Thus, we have a quasicorner $C(\mathrm{T}, \mathfrak{p})$ defined by

$$\begin{array}{ccc} \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) & \xrightarrow{\mathrm{T}} & \mathbb{R} \\ \downarrow{\scriptstyle \mathfrak{p}} & & \\ \mathrm{Homeo}_+(\mathsf{S}). & & \end{array}$$

Since the set of equivalence classes of quasicorners $\mathcal{QC}(\mathrm{Homeo}_+(\mathsf{S}))$ is in a bijective correspondence to $\mathrm{H}^2_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{R})$, the result is that the class $[C(-T, \mathfrak{p})] \in \mathcal{QC}(\mathrm{Homeo}_+(\mathsf{S}))$ is related to the class $eu_b^{\mathbb{R}} \in \mathrm{H}^2_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{R})$. Therefore $\mathrm{H}^2_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{R}) \simeq \mathbb{R}$.

Finally, if we consider the projection of $\mathrm{T} : \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) \longrightarrow \mathbb{R}$ to the circle we obtain an element

$$\varrho : \mathrm{Homeo}_+(\mathsf{S}) \longrightarrow \mathbb{S}^1,$$

and we prove that for every homomorphism $\vartheta : \mathbb{Z} \longrightarrow \mathrm{Homeo}_+(\mathsf{S})$, the class $\vartheta^*(eu_b) \in \mathrm{H}^2_b(\mathbb{Z}, \mathbb{Z})$ is the number $\varrho(\vartheta(1))$. Moreover, in view of $\mathrm{Char}(\mathsf{S}) \simeq \mathbb{Q}$ we have that $\mathrm{H}^2_b(\mathbb{Q}, \mathbb{Z}) \simeq \mathsf{S}$. Thus, for every homomorphism $\varphi : \mathbb{Q} \longrightarrow \mathrm{Homeo}_+(\mathsf{S})$, the class $\varphi^*(eu_b)$ is the rotation element of $\varphi(1)$.

In the first chapter we review the theoretical ingredients as cohomology, homology and bounded cohomology of groups. We also study the connection between the second bounded cohomology of groups and the theory of quasicorners. We put some emphasis in the case of the group $\mathrm{Homeo}_+(\mathbb{S}^1)$ and how to derive the bounded cohomology class

$$eu_b(\mathbb{S}^1) \in \mathrm{H}^2_b(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{Z}).$$

In the last section we review the work of E. Ghys ([Ghy]).

In the second chapter we present the general results of $\mathrm{Homeo}(\mathsf{S})$. As a consequence of the work of C. Odden in the two–dimensional case (see [Odd]), there is a decomposition

$$\mathrm{Homeo}(\mathsf{S}) \cong \mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S}) \times_{\mathbb{Z}} \widehat{\mathbb{Z}};$$

where $\mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S})$ is the subgroup of homeomorphisms of S which preserves the base leaf, the subgroup $\widehat{\mathbb{Z}} \hookrightarrow \mathrm{Homeo}_+(\mathsf{S})$ representing the translations on the fiber and the quotient $\mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S}) \times_{\mathbb{Z}} \widehat{\mathbb{Z}} = (\mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S}) \times \widehat{\mathbb{Z}})/\mathbb{Z}$ due to a generalized diagonal action (see section 2.3). Also, we are able to recognize the isotopy classes of elements in $\mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S})$ as

$$\mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S})/\mathrm{Homeo}_+(\mathsf{S}) \cong \mathrm{Aut}(\mathsf{S}).$$

At the end of this chapter we mention the result of $\mathrm{Homeo}_+(\mathsf{S})$ being uniformly perfect as in [AP].

The third chapter presents a complete study of the covering $\mathfrak{p} : \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) \longrightarrow \mathrm{Homeo}_+(\mathsf{S})$, we specify the construction of an universal extension for $\mathrm{Homeo}_+(\mathsf{S})$ and prove that this group has the homotopy type of the subgroup of translations over the base leaf. Also, we calculate its second cohomology groups and find an invariant of the dynamics as we mentioned before.

CONTENTS

1. HOMEOMORPHISMS OF THE CIRCLE AND THE EULER CLASS

This chapter is a collection of some important results of the theory of cohomology of groups, universal central extensions and bounded cohomology. Most of the results will be presented without proof and we will made the respective references. Our aim is to understand the universal central extension of homeomorphisms of the circle that preserves orientation; i.e.

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1) \xrightarrow{p} \mathrm{Homeo}_+(\mathbb{S}^1) \longrightarrow 1.$$

The last section presents the important results of $\mathrm{Homeo}_+(\mathbb{S}^1)$. Also, using the language of cohomology of groups we can conclude that there is a bounded cohomology class

$$eu \in \mathrm{H}^2_b(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{Z}) \simeq \mathbb{Z}.$$

This class will give us information about the Poincaré rotation number for a given $f \in \mathrm{Homeo}_+(\mathbb{S}^1)$.

1.1 Cohomology of groups

We will present the main results for the theory of cohomology of groups, for a detailed study of this theory we recommend ([Bro]).

Let G be a group and A an Abelian group, we say that a map $c : G^{n+1} \longrightarrow A$ is an n–cochain if it is an homogeneous application; that is,

$$c(gg_0, \dots, gg_n) = c(g_0, \dots, g_n), \qquad (g \in G).$$

We write $C^n(G, A)$ for the Abelian group consisting of the n–cochains.

For every $n \in \mathbb{N}$, define the coboundary map $d^n : C^n(G, A) \longrightarrow C^{n+1}(G, A)$, such that for every $c \in C^n(G, A)$,

$$d^n(c)(g_0, \dots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i c(g_0, \dots, \widehat{g_i}, \dots, g_{n+1});$$

where $(g_0, \dots, \widehat{g_i}, \dots, g_{n+1}) \in G^{n+1}$ denotes that g_i has been removed. It is a straightforward calculation to see that for each $n \in \mathbb{N}$, $d^n \circ d^{n-1} \equiv 0$; therefore $\mathrm{im}(d^{n-1}) \subset \ker(d^n)$. We call n–coboundaries to the elements of $\mathrm{im}(d^{n-1})$ and n–cocycles to the elements of $\ker(d^n)$.

Definition 1.1.1. *The* **n–*th cohomology group of* G *with coefficients in* A** *is defined by*

$$\mathrm{H}^n(G, A) = \ker(d^n)/\mathrm{im}(d^{n-1}).$$

Given an n–cochain c, we can associate to c a non–homogeneous map $\bar{c} : G^n \longrightarrow A$,

$$\bar{c}(g_1, \ldots, g_n) = c(1, g_1, g_1 g_2, \ldots, g_1 \cdots g_n).$$

Conversely, if $\bar{c} : G^n \longrightarrow A$ is a map, we associate the n–cochain $c \in C^n(G, A)$ by

$$c(g_0, \ldots, g_n) = \bar{c}(g_0^{-1} g_1, g_1^{-1} g_2, \ldots, g_{n-1}^{-1} g_n).$$

Thus, $C^n(G, A)$ is identified with the A–module of the maps $\bar{C}(G^n, A) = \{G^n \longrightarrow A\}$. Moreover, for every $n \in \mathbb{N}$, the non–homogeneous coboundary operator

$$\delta^n : \bar{C}(G^n, A) \longrightarrow \bar{C}(G^{n+1}, A)$$

it is defined by the relation

$$\delta^n(\bar{c})(g_1, \ldots, g_{n+1}) = \bar{c}(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i \bar{c}(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + \bar{c}(g_1, \ldots, g_n).$$

Then, $\mathrm{H}^n(G, A) \simeq \ker(\delta^n)/\mathrm{im}(\delta^{n-1})$.

Let us look at some particular cases:

- In degree 0, the 0–cochains are the constant functions of G in A; *i.e.* $C^0(G, A) \simeq A$ and $\mathrm{H}^0(G, A) \simeq A$.
- In degree 1, if $c \in C^1(G, A)$, then the corresponding non–homogeneous map is a map $\bar{c} : G \longrightarrow A$ and c is a 1–cocycle if and only if $\bar{c} \in \mathrm{Hom}(G, A)$. Also, $C^0(G, A) \simeq A$ implies that $d^0 \equiv 0$ and consequently
$$\mathrm{H}^1(G, A) \simeq \mathrm{Hom}(G, A).$$
- In degree 2, let E be a group defining a central extension of G by A; that is, we have an exact sequence
$$0 \longrightarrow A \stackrel{i}{\longrightarrow} E \stackrel{p}{\longrightarrow} G \longrightarrow 1,$$
with $A \simeq i(A)$ belonging to the center of E and $E/A \simeq G$. We will denote this extension by (E, p).

  We say that a map $\sigma : G \longrightarrow E$ satisfying $p \circ \sigma \equiv \mathrm{id}_G$ is a section and that the given central extension defined by E splits if there is a section σ which is a homomorphism. The next result classifies the split extensions ([Bro]).

  Proposition 1.1.2. *The following are equivalent:*

  (a) The central extension $0 \longrightarrow A \stackrel{i}{\longrightarrow} E \stackrel{p}{\longrightarrow} G \longrightarrow 1$ *splits.*

  (b) E contains a subgroup H that is mapped isomorphically to G via p; that is, H satisfies
$$E = i(A) \cdot H \quad \textit{and} \quad i(A) \cap H = \{1\}.$$

  (c) E contains a subgroup H such that every element $\tilde{g} \in E$ can be written uniquely as
$$\tilde{g} = i(a)h \qquad (a \in A, h \in H).$$

(d) The central extension $0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$ *is equivalent to the extension*

$$0 \longrightarrow A \longrightarrow A \times G \longrightarrow G \longrightarrow 1.$$

Remark 1.1.3. *Condition (d) can be rephrased to say that* E *and* $A \times G$ *are isomorphic in the sense of diagrams of exact sequences. We also remark that* A *is an Abelian group viewed as a* G*–module with trivial* G*–action.*

As a consequence, the splittings $\sigma : G \longrightarrow E$ are obviously in 1-1 correspondence with the elements of $\mathrm{Hom}(G, A) \simeq \mathrm{H}^1(G, A)$. Let $\sigma : G \longrightarrow E$ be a splitting, then $E \simeq G \times A$ via the isomorphism $(g, a) \longmapsto \sigma(g)i(a)$. Hence, in order to measure the non–triviality of a central extension we try to find an obstruction to the existence of a splitting.

Because p is surjective, we can always take a section $\sigma : G \longrightarrow E$ and define the obstruction map $\bar{c} : G^2 \longrightarrow E$ by the rule

$$\bar{c}(g_1, g_2) = \sigma(g_1 g_2)^{-1}\sigma(g_1)\sigma(g_2).$$

In particular, using that p is a homomorphism, $\bar{c}$ projects onto the identity element of G; or $\bar{c} \in \ker(p) \simeq \mathrm{im}(i)$. Therefore, $\bar{c} : G^2 \longrightarrow A$ defines a non–homogeneous 2–cochain and we take the corresponding 2–cochain $c \in C^2(G, A)$. In fact, it is easy to see that c is a cocycle called the **obstruction cocycle**.

If we choose another section $\sigma' : G \longrightarrow E$, then σ' is defined by

$$\sigma'(g) = \sigma(g)i(u(g)), \qquad (g \in G);$$

with some function $u : G \longrightarrow A$. Let $c' : G^2 \longrightarrow A$ be the obstruction cocycle associated to σ'. For every $g_1, g_2 \in G$ we have:

$$\begin{aligned}
\bar{c'}(g_1, g_2) &= \sigma'(g_1 g_2)^{-1}\sigma'(g_1)\sigma'(g_2) \\
&= (\sigma(g_1 g_2)i(u(g_1 g_2)))^{-1}\sigma(g_1)i(u(g_1))\sigma(g_2)i(u(g_2)) \\
&= i(u(g_1 g_2))^{-1}\sigma(g_1 g_2)^{-1}\sigma(g_1)i(u(g_1))\sigma(g_2)i(u(g_2)) \\
&= i(u(g_1 g_2))^{-1}\sigma(g_1 g_2)^{-1}\sigma(g_1)\sigma(g_2)i(u(g_1))i(u(g_2)) \\
&= i(u(g_1 g_2))^{-1}\bar{c}(g_1, g_2)i(u(g_1) + u(g_2)) \\
&= \bar{c}(g_1, g_2)i(u(g_2) - u(g_1 g_2) + u(g_1)) \\
&= \bar{c}(g_1, g_2)i(\delta^1(u)(g_1, g_2)).
\end{aligned}$$

Equivalently, $\bar{c'}$ and $\bar{c}$ differ by the 1–coboundary associated to u. The same is true for the cocycles c and c' and the result is that the cohomology class of c in $\mathrm{H}^2(G, A)$ does not depend on the choice of the section. We call this cohomology class the *Euler class* of the extension (E, p) and we denote it by $eu \in \mathrm{H}^2(G, A)$.

Theorem 1.1.4. *The set of equivalence classes of central extensions of* G *by* A *is in bijection with* $\mathrm{H}^2(G, A)$.

That is, two central extensions are equivalent if, and only if, they have the same Euler class. Also, every element in $\mathrm{H}^2(G, A)$ represents a central extension.

Example 1.1.5. *For the case $A = G = \mathbb{Z}$ we have that every central extension admits one splitting $\sigma : \mathbb{Z} \longrightarrow \mathbb{Z}$, because it is enough to take $\sigma(1) \in p^{-1}(1)$ and to define $\sigma(n) = \sigma(1)^n$ for every $n \in \mathbb{Z}$. Hence, $\mathrm{H}^2(\mathbb{Z}, \mathbb{Z}) = 0$ and so every central extension of $\mathbb{Z}$ by $\mathbb{Z}$ is trivial.*

In fact, this argument is valid for every free Abelian group G and $A = \mathbb{Z}$.

1.2 Homology and universal central extensions

Define the chain complex $C_*(G)$ as follows: consider the equivalent relation given by the action of G on G^{n+1} as

$$(g_0, \dots, g_n) \sim (gg_0, \dots, gg_n).$$

Then the set of equivalence classes $[g_0, \dots, g_n]$ is the $\mathbb{Z}$–basis of $C_n(G)$ and the boundary map $d_n : C_n(G) \longrightarrow C_{n-1}(G)$ is defined as

$$d_n[g_0, \dots, g_n] = \sum_{i=0}^{n} (-1)^i [g_0, \dots, \widehat{g_i}, \dots, g_n].$$

This is the so called homogeneous chain complex of G.

It is easy to see that $d_{n-1} \circ d_n \equiv 0$ and then the **n–th homology group** G **with integer coefficients** is defined as

$$\mathrm{H}_n(G, \mathbb{Z}) = \frac{\ker(d_n)}{\mathrm{im}(d_{n+1})}.$$

Note that the definition of homology is based on the standard resolution over $\mathbb{Z}$ (see [Bro]).

There is a non–homogeneous description of the previous complex. The set of non–homogeneous chains $\overline{C_n}$ is simply the set G^n with the n–th boundary operador

$$\overline{\delta_n}(g_1, \dots, g_n) = (g_2, \dots, g_n) + \sum_{i=1}^{n-1} (-1)^i (g_1, \dots, g_i g_{i+1}, \dots, g_n) + (-1)^n (g_1, \dots, g_{n-1}).$$

Thus, the n–th homology group is $\mathrm{H}_n(G, \mathbb{Z}) = \ker\left(\overline{\delta_n}\right) / \mathrm{im}\left(\overline{\delta_{n+1}}\right)$.

In low dimensions we have that $\delta_1 \equiv 0$ and $\delta_2(g, h) = h - (gh) + g$. Therefore, $\mathrm{H}_0(G, \mathbb{Z}) = 0$ and

$$\mathrm{H}_1(G, \mathbb{Z}) \simeq \frac{G}{[G, G]};$$

where $[G, G]$ denotes the commutator subgroup of G, that is , the subgroup generated by all the commutators $ghg^{-1}h^{-1}$ with $g, h \in G$.

We say that a central extension $(\tilde{G}, u)$ of G by an Abelian group A is universal if for any other central extension (E, p) of G, there is a unique morphism $\tilde{G} \longrightarrow E$ that completes a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \tilde{G} & \xrightarrow{u} & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & E & \xrightarrow{p} & G & \longrightarrow & 1. \end{array}$$

If the group has a universal central extension, then it is by definition unique up to isomorphism in the category of central extensions. For a complete treatment of this part of the theory we recommend the work of Milnor [Mil] and chapter four of [Ros].

In particular, if the group G is perfect; that is, $G = [G, G]$, we have the following main result.

Theorem 1.2.1. *G has a universal central extension if and only if G is perfect. In this case, the extension $\left(\tilde{G}, u\right)$ is universal if and only if the following two conditions hold:*

(i) $\tilde{G}$ is perfect,

(ii) all central extensions of $\tilde{G}$ are trivial.

Remark that $\mathrm{H}^2(G, A)$ is in bijection with isomorphism classes of central extensions of G by A; or equivalently, there exists an Abelian group $\mathrm{Ext}(G, A)$ that contains all the isomorphism classes of central extensions of G by A, such that every element is associated to a central extension of G and $\mathrm{H}^2(G, A) \simeq \mathrm{Ext}(G, A)$. Moreover, we have the "Universal Coefficient Theorem".

Theorem 1.2.2. *Let G be a group and A be an Abelian group. There are short exact sequences*

$$0 \longrightarrow \mathrm{Ext}(\mathrm{H}_{k-1}(G, \mathbb{Z}), A) \longrightarrow \mathrm{H}^k(G, A) \longrightarrow \mathrm{Hom}(\mathrm{H}_k(G, \mathbb{Z}), A) \longrightarrow 0$$

for every k, which split. In particular, $\mathrm{H}^2(G, A) = 0$ for all Abelian groups A if and only if $G/[G, G]$ is free (Abelian) and $\mathrm{H}_2(G, \mathbb{Z}) = 0$.

Our interest lies in the following results.

Corollary 1.2.3. *If G is a perfect group, then a central extension $\left(\tilde{G}, u\right)$ is universal if and only if*

$$\mathrm{H}_1\left(\tilde{G}, \mathbb{Z}\right) = \mathrm{H}_2\left(\tilde{G}, \mathbb{Z}\right) = 0.$$

Theorem 1.2.4. *Let G be a perfect group. Then the kernel of the universal central extension $\left(\tilde{G}, u\right)$ of G is naturally isomorphic to $A = \mathrm{H}_2(G, \mathbb{Z})$, and under the isomorphisms*

$$\mathrm{Ext}(G, A) \simeq \mathrm{H}^2(G, A) \simeq \mathrm{Hom}(\mathrm{H}_2(G, \mathbb{Z}), A)$$

the class of $\left(\tilde{G}, u\right)$ corresponds to the identity map $\mathrm{H}_2(G, \mathbb{Z}) \longrightarrow A$.

Note that in the last result the term Ext derived from the Universal Coefficient Theorem vanishes since $\mathrm{H}_1(G, \mathbb{Z}) = 0$.

1.3 Bounded cohomology of groups

Another important theory introduced by Gromov ([Gro]) is the theory of bounded cohomology of groups. Let G be a group and $A = \mathbb{Z}$ or $A = \mathbb{R}$. Given $n \in \mathbb{N}$, a map $c_b : G^{n+1} \longrightarrow A$ is a bounded n–cochain if it is a bounded homogeneous map. We denote by $C_b^n(G, A)$ the A–submodule of the bounded n–cochains. Observe that $C_b^n(G, A) \subset C^n(G, A)$ and let

$$d_b^n : C_b^n(G, A) \longrightarrow C_b^{n+1}(G, A)$$

be the n–th bounded coboundary operator defined by the restriction $d_b^n = d^n|_{C_b^n(G,A)}$. It is a well defined operator and for every $n \in \mathbb{N}$, $d_b^n \circ d_b^{n-1} \equiv 0$.

Definition 1.3.1. *The* **n–*th bounded cohomology group of* G *with coefficients in* A** *is defined by*

$$\mathrm{H}_b^n(G, A) = \ker(d_b^n)/\mathrm{im}(d_b^{n-1}).$$

We also define

$$i_A : \mathrm{H}_b^n(G, A) \longrightarrow \mathrm{H}^n(G, A)$$

as the map obtained by forgetting that the cocycle is bounded. There are some important questions involving the surjectivity or injectivity of this map. Following the notation of [Har], define

$$\mathrm{EH}_b^2(G, \mathbb{R}) := \ker(i_{\mathbb{R}}).$$

Let us review the simple cases: if $n = 0$ the bounded 0–cochains are still identified with the constant maps; that is, $\mathrm{H}_b^0(G, A) \simeq A$. For $n = 1$ as we have seen $\mathrm{H}^1(G, A) \simeq \mathrm{Hom}(G, A)$, thus $\mathrm{H}_b^1(G, A) = 0$.

Remark 1.3.2. *If A is a bounded Abelian group, the cohomology of groups coincides with the bounded cohomology of groups.*

We have the following important result due to Gromov, Brooks, or Ivanov (see [Broo], [Gro] and [Iva]).

Theorem 1.3.3. *Given an arc–connected topological space X with the homotopy type of a countable CW–complex,*

$$\mathrm{H}_b^n(X, \mathbb{R}) \simeq \mathrm{H}_b^n(\pi_1(X), \mathbb{R}), \quad \mathrm{H}_b^n(X, \mathbb{Z}) \simeq \mathrm{H}_b^n(\pi_1(X), \mathbb{Z}), \quad (n \geq 1);$$

where the left hand side denotes the bounded cohomology of topological spaces and the right hand side is the bounded cohomology of the fundamental group of X.

The bounded cohomology of amenable groups is well understood. Let G be a group and $\mathrm{B}(G)$ be the Banach space of real valued bounded functions of G, with the norm

$$||f||_\infty = \sup\{|f(g)| \,:\, g \in G\}.$$

A lineal functional $m : \mathrm{B}(G) \longrightarrow \mathbb{R}$ is called a left invariant mean if satisfies the following:

- $m(1) = 1$,
- $m(f) \geq 0$ if $f \geq 0$ and
- m is invariant under left translations; *i.e*

$$m(L_g \circ f) = m(f),$$

for every $f \in \mathrm{B}(G)$ and every $g \in G$, where $L_g : \mathrm{B}(G) \longrightarrow \mathrm{B}(G)$ is defined by

$$L_g(f)(h) = f(gh), \qquad (f \in \mathrm{B}(G), h \in G).$$

Definition 1.3.4. *A group G is said to be amenable if there is an invariant mean in* $\mathrm{B}(G)$.

Finite and Abelian groups are examples of amenable groups.

Remark 1.3.5. *Alternatively, a group G is amenable if every action on a compact space has a fixed point.*

Proposition 1.3.6 (Trauber). *If G is an amenable group, then $\mathrm{H}_b^n(G, \mathbb{R}) = 0$ for all $n \geq 1$.*

Example 1.3.7. *We calculate $\mathrm{H}_b^2(\mathbb{Z}, \mathbb{Z})$. Consider the exponential exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{S}^1 \longrightarrow 1,$$

and the associated long exact sequence in bounded cohomology

$$\begin{aligned} 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{S}^1 \longrightarrow \mathrm{H}_b^1(\mathbb{Z}, \mathbb{Z}) \longrightarrow \mathrm{H}_b^1(\mathbb{Z}, \mathbb{R}) \longrightarrow \mathrm{H}_b^1(\mathbb{Z}, \mathbb{S}^1) \longrightarrow \\ \longrightarrow \mathrm{H}_b^2(\mathbb{Z}, \mathbb{Z}) \longrightarrow \mathrm{H}_b^2(\mathbb{Z}, \mathbb{R}) \longrightarrow \mathrm{H}_b^2(\mathbb{Z}, \mathbb{S}^1) \longrightarrow \cdots . \end{aligned}$$

Using that $\mathbb{Z}$ is an amenable group, $\mathrm{H}_b^n(\mathbb{Z}, \mathbb{R}) = 0$ for every $n \geq 1$. Calculating these groups for $n = 1$ and $n = 2$ we obtain that

$$\mathrm{H}_b^2(\mathbb{Z}, \mathbb{Z}) \simeq \mathrm{H}_b^1(\mathbb{Z}, \mathbb{S}^1) \simeq \mathrm{Hom}(\mathbb{Z}, \mathbb{S}^1) \simeq \mathbb{S}^1.$$

If G is any group, the next step is to study the kernel of the comparison map $i_{\mathbb{R}}$. Let $\alpha = [c] \in \mathrm{EH}_b^2(G, \mathbb{R})$; thus, there is $f : G \longrightarrow \mathbb{R}$ such that $\delta^1 f = c$. Since

$$\sup_{g,h \in G} |f(h) - f(gh) + f(g)| = \sup_{g,h \in G} |\delta^1 f(g, h)| = \sup_{g,h \in G} |c(g, h)| < \infty,$$

we have that f is a *quasimorphism*. We say that two quasimorphisms are equivalent if

$$||f_1 - f_2||_\infty < \infty.$$

Denote by $\mathcal{Q}(G)$ the space of equivalence classes of quasimorphisms (see [Har] for details).

Proposition 1.3.8. *There is an isomorphism*

$$\mathrm{EH}_b^2(G, \mathbb{R}) \simeq \frac{\mathcal{Q}(G)}{\mathrm{Hom}(G, \mathbb{R})}.$$

The next result about canonical representatives of classes of quasimorphisms follows.

Proposition 1.3.9. *Let $f : G \longrightarrow \mathbb{R}$ be a quasimorphism. For every $g \in G$ the limit*

$$\tilde{f}(g) := \lim_{n \longrightarrow \infty} \frac{f(g^n)}{n}$$

exists and it is a quasimorphism. Moreover, it is homogeneous, i.e. $\tilde{f}(g^n) = n\tilde{f}(g)$ for all $n \geq 0$ and equivalent to f. In fact, $\tilde{f}$ is the unique homogeneous quasimorphism equivalent to f.

As a consequence, $\mathcal{Q}(G)$ can be identified with the space of homogeneous quasimorphisms on G. Also, the comparison map $i_{\mathbb{R}}$ is injective if and only if every homogeneous quasimorphism is a homomorphism. Furthermore, we have:

Corollary 1.3.10. *If G is an amenable group, then every homogeneous quasimorphism is an homomorphism.*

Our interest lies in some other properties of homogeneous quasimorphism:

Lemma 1.3.11. *Let $f : G \longrightarrow \mathbb{R}$ be a homogeneous quasimorphism. Then the following hold:*

1. $f(g^{-1}) = -f(g)$.
2. $f(g^n) = nf(g)$, $\forall n \in \mathbb{Z}$.
3. *f is invariant under conjugation.*
4. *If $N \lhd G$ is normal, $p : G \longrightarrow G/N$ is the canonical projection and $f|_N \equiv 0$, then there exists a homogeneous quasimorphism $F : G/N \longrightarrow \mathbb{R}$ with $f = F \circ p$.*
5. *f is a homomorphism if and only if $f|_{[G,G]} \equiv 0$.*

In the case of the free group generated in two letters F_2, we have that H_b^2 and EH_b^2 are infinite dimensional as $\mathbb{R}$–vector spaces (see [Broo]). In general it is not easy to calculate the bounded cohomology in second degree.

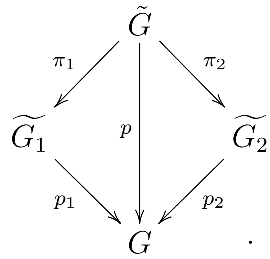
The next definition is due to T. Hartnick ([Har]), who suggested that since $\mathrm{H}^2(G, \mathbb{R})$ classifies $\mathbb{R}$–central extensions and $\mathrm{EH}_b^2(G, \mathbb{R})$ classifies quasimorphisms, then it is reasonable to expect that H_b^2 classifies a combination of the two and suggest introduced the following terminology.

Definition 1.3.12. *A* quasi–corner $C(f, p)$ *over G is a diagram*

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{f} & \mathbb{R} \\ {\scriptstyle p}\big\downarrow & & \\ G, & & \end{array}$$

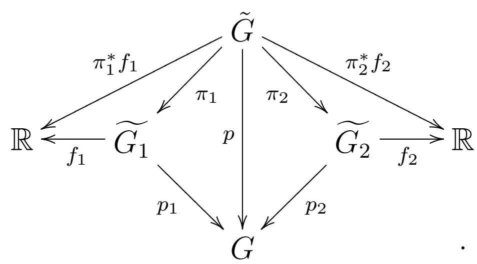
with p surjective, $\ker(p)$ *central in $\tilde{G}$ and f a homogeneous quasimorphism.*

Observe that every quasimorphism $f : G \longrightarrow \mathbb{R}$ defines a quasi–corner $C(f, \mathrm{id}_G)$. The following discussion generalizes the notion of being cohomologous for quasimorphisms. Let $C(f_1, p_1)$ and $C(f_2, p_2)$ be two quasi–corners over G. Let $p : \tilde{G} \longrightarrow G$ be the map induced by the pushout functions π_1 and π_2:



Since the kernels of p_1 and p_2 are Abelian and therefore amenable, $\ker p$ is amenable and then $\left(\tilde{G}, p\right)$ is an amenable extension.

We also have the respective quasimorphisms f_1 and f_2 and the induced ones from $\tilde{G}$:



It is said that two quasi–corners $C(f_1, p_1)$ and $C(f_2, p_2)$ are *equivalent* if

$$(\pi_1^* f_1 - \pi_2^* f_2)\,|_{[\tilde{G},\tilde{G}]} \equiv 0.$$

Observing Lemma 1.3.11, two quasi–corners as before are equivalent if the homogeneous quasimorphism

$$\pi_1^* f_1 - \pi_2^* f_2 : \tilde{G} \longrightarrow \mathbb{R}$$

is a homomorphism. Denote by $\mathcal{QC}(G)$ the set of equivalence classes of quasi–corners over G. The main result is that $\mathcal{QC}(G)$ is in bijection with $\mathrm{H}_b^2(G, \mathbb{R})$. First, an easy consequence of the so called five–term exact sequence.

Proposition 1.3.13. *For any amenable (in particular central) extension $p : \tilde{G} \longrightarrow G$ the map $p^* : \mathrm{H}_b^2(G, \mathbb{R}) \longrightarrow \mathrm{H}_b^2\left(\tilde{G}, \mathbb{R}\right)$ is an isomorphism.*

Proof. Consider the short exact sequence

$$0 \longrightarrow A \xrightarrow{\;i\;} \tilde{G} \xrightarrow{\;p\;} G \longrightarrow 1,$$

using the five–exact sequence in bounded cohomology:

$$0 \longrightarrow \mathrm{H}_b^2(G, \mathbb{R}) \longrightarrow \mathrm{H}_b^2(\tilde{G}, \mathbb{R}) \longrightarrow \mathrm{H}_b^2(A, \mathbb{R}) \longrightarrow \mathrm{H}_b^3(G, \mathbb{R}) \longrightarrow \mathrm{H}_b^3(\tilde{G}, \mathbb{R}).$$

As A is an amenable group, we have that

$$H_b^2(A, \mathbb{R}) = 0.$$

$\lrcorner$

Now, given a quasi–corner $C = C(f, p)$, define the class

$$\alpha_b(C) := (p^*)^{-1}\left(d_b^1 f\right) \in \mathrm{H}_b^2(G, \mathbb{R})$$

and say that C realizes the class $\alpha_b(C)$. Note that two equivalent quasi–corners realize the same bounded cohomology class. There is a well defined map

$$\begin{aligned} \alpha : \mathcal{QC}(G) &\longrightarrow \mathrm{H}_b^2(G, \mathbb{R}) \\ [C] &\longmapsto \alpha_b(C). \end{aligned}$$

Proposition 1.3.14. *The map* $\alpha : \mathcal{QC}(G) \longrightarrow \mathrm{H}_b^2(G, \mathbb{R})$ *is a bijection.*

Proof. Two quasi–corners $C(f_1, p_1)$ and $C(f_2, p_2)$ represent the same bounded cohomology class if and only if $[d_b^1 \pi_1^* f_1] = [d_b^1 \pi_2^* f_2]$, which means that $\pi_1^* f_1 - \pi_2^* f_2$ is a homomorphism; or by definition, the two quasi–corners are equivalent. Thus we have proved the injectivity.

Let $\alpha_b \in \mathrm{H}_b^2(G, \mathbb{R})$ and take $\alpha = i_{\mathbb{R}}(\alpha_b) \in \mathrm{H}^2(G, \mathbb{R})$. There is a result (see Corollary 2.8 of [Har]) that enables us to find a central extension $p : \tilde{G} \longrightarrow G$ along which the pullback of α vanishes. We fix such an extension

$$0 \longrightarrow C \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1,$$

where C is a subgroup of $\mathbb{R}$ and $p^*\alpha = 0$. Therefore we have

$$\widetilde{i_{\mathbb{R}}}\left(p^*(\alpha_b)\right) = p^* i_{\mathbb{R}}(\alpha_b) = p^*\alpha = 0;$$

i.e.

$$p^*\alpha_b \in \mathrm{EH}_b^2\left(\tilde{G}, \mathbb{R}\right).$$

This in fact says that there exists a quasimorphism $f : \tilde{G} \longrightarrow \mathbb{R}$, which is homogeneous and

$$d_b^1 f = p^*\alpha_b;$$

hence, $C(f, p)$ realizes the class α_b, or

$$(p^*)^{-1}(d_b^1 f) = \alpha_b.$$

$\lrcorner$

1.4 Homeomorphisms of $\mathbb{S}^1$ which preserve orientation

Consider the group of homeomorphisms of the unit circle $\mathrm{Homeo}(\mathbb{S}^1)$. It is a well known result that it has a decomposition of the form

$$\mathrm{Homeo}(\mathbb{S}^1) \simeq \mathrm{Aut}(\mathbb{S}^1) \times \mathrm{Homeo}_+(\mathbb{S}^1);$$

where the subgroup of automorphisms $\mathrm{Aut}(\mathbb{S}^1) = \{\mathrm{id}, -\mathrm{id}\} \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Homeo}_+(\mathbb{S}^1)$ is the subgroup of homeomorphisms of $\mathbb{S}^1$ that preserve orientation. Equivalently, if $f \in \mathrm{Homeo}_+(\mathbb{S}^1)$, then f is isotopic to the identity and we write $f \sim \mathrm{id}$. The mapping class group for $\mathbb{S}^1$ is given by

$$\mathrm{Homeo}(\mathbb{S}^1)/\mathrm{Homeo}_+(\mathbb{S}^1) \simeq \mathrm{Aut}(\mathbb{S}^1).$$

Let $\pi : \mathbb{R} \longrightarrow \mathbb{S}^1$ be the universal covering map and consider the group of lifts of elements in $\mathrm{Homeo}_+(\mathbb{S}^1)$:

$$\widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1) = \left\{ \tilde{f} \in \mathrm{Homeo}(\mathbb{R}) \ : \ \pi \circ \tilde{f} \equiv f \circ \pi, \ f \in \mathrm{Homeo}_+(\mathbb{S}^1) \right\}.$$

Observe that every $\tilde{f} \in \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1)$ commutes with integer translations; *i.e.* $\tilde{f}(x+1) = \tilde{f}(x)+1$ for every $x \in \mathbb{R}$. Thus, there is a surjective homomorphism

$$p : \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1) \to \mathrm{Homeo}_+(\mathbb{S}^1),$$

with kernel identified with the subgroup of integer translations $\{x \longmapsto x+\gamma \ : \ \gamma \in \mathbb{Z}\} \simeq \mathbb{Z}$. This defines a central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1) \xrightarrow{\ p\ } \mathrm{Homeo}_+(\mathbb{S}^1) \longrightarrow 1.$$

Note that both $\mathrm{Homeo}_+(\mathbb{S}^1)$ and $\widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1)$ are topological groups with the compact–open topology. An important result from [Ghy] is that, up to conjugacy $\mathrm{Homeo}_+(\mathbb{S}^1)$ admits a unique maximal compact subgroup ([Ghy]).

Proposition 1.4.1. *Up to conjugacy, the subgroup of rotations* $\mathrm{SO}(2,\mathbb{R})$ *is the maximal compact subgroup of* $\mathrm{Homeo}_+(\mathbb{S}^1)$.

The group $\mathrm{SO}(2,\mathbb{R})$ is canonically identified with $\mathbb{S}^1$ and therefore, there is a well defined inclusion $\mathbb{S}^1 \hookrightarrow \mathrm{Homeo}_+(\mathbb{S}^1)$.

Proposition 1.4.2. $\mathrm{Homeo}_+(\mathbb{S}^1)$ *is homotopically equivalent to* $\mathbb{S}^1$.

Proof. In the first place, the group of homeomorphisms of $\mathbb{R}$ that preserves orientation is a convex set because it is identified with the set of strictly increasing functions from $\mathbb{R}$ to $\mathbb{R}$ tending to $\pm\infty$ as the variable tends to $\pm\infty$.

If $\tilde{f} \in \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1)$, then $\tilde{f}$ can be written as $\tilde{f} \equiv \mathrm{id} + \phi$, with $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ a continuous $\mathbb{Z}$–periodic function. Moreover, ϕ can be decomposed in a unique way as $\phi \equiv c + \phi_0$, where $c \in \mathbb{R}$ is a constant and $\phi_0 : \mathbb{R} \longrightarrow \mathbb{R}$ is a $\mathbb{Z}$–periodic function whose average over each period vanishes. Take $\lambda \in [0,1]$ and define $\widetilde{f_\lambda} : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\widetilde{f_\lambda}(x) = x + c + (1-\lambda)\phi_0(x).$$

Using the first observation we notice that $\widetilde{f}_\lambda \in \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1)$ for every $\lambda \in \underbrace{[0,1]}$. Also $\widetilde{f}_0 \equiv \tilde{f}$ and $\widetilde{f}_1 \equiv \mathrm{id} + c$. Consequently, we obtain a continuous retraction of $\mathrm{Homeo}_+(\mathbb{S}^1)$ onto the subgroup of real translations, which is isomorphic to $\mathbb{R}$. Moreover, this retraction commutes with integer translations and therefore, there is a continuous retraction of the quotient $\mathrm{Homeo}_+(\mathbb{S}^1) \simeq \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1)/\mathbb{Z}$ onto the subgroup of rotations $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$.

⌟

The last proof shows that $\mathrm{Homeo}_+(\mathbb{S}^1)$ is homeomorphic to the product of $\mathbb{S}^1$ and a convex set. Also, note that it implies that the fundamental group $\pi_1(\mathrm{Homeo}_+(\mathbb{S}^1))$ is isomorphic to $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$.

In order to measure the non–triviality of the extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1) \xrightarrow{p} \mathrm{Homeo}_+(\mathbb{S}^1) \longrightarrow 1,$$

we calculate $\mathrm{H}^2(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{Z})$. The next result is crucial to this end (see [Thu] or [Tsu]).

Theorem 1.4.3 (Thurston–Mather)**.** *Let M be a manifold, there is an isomorphism*

$$\mathrm{H}^n(B\mathrm{Homeo}(M), \mathbb{Z}) \simeq \mathrm{H}^n(\mathrm{Homeo}(M)^\delta, \mathbb{Z}), \qquad (n \geq 1);$$

where $B\mathrm{Homeo}(M)$ denotes the classifying space of $\mathrm{Homeo}(M)$*, the left hand side is the bounded cohomology of topological spaces and the right hand side is the bounded cohomology of the abstract group* $\mathrm{Homeo}(M)$*.*

Hence, using that $\mathbb{S}^1 \hookrightarrow \mathrm{Homeo}_+(\mathbb{S}^1)$ is an homotopy equivalence, we have that

$$B\mathrm{Homeo}_+(\mathbb{S}^1) = B\mathbb{S}^1 \cong \mathbb{CP}^\infty.$$

Therefore,

$$\mathrm{H}^n(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{Z}) \simeq \mathrm{H}^n(B\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{Z}) \simeq \mathrm{H}^n(\mathbb{CP}^\infty, \mathbb{Z}) \simeq \mathbb{Z}[eu], \qquad (n \geq 1);$$

where eu is the free generator for $\mathrm{H}^2(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{Z}) \simeq \mathbb{Z}$; or equivalently, eu is the Euler class associated to the given extension.

Analogously, it can be seen that the group $\mathrm{Homeo}_+(\mathbb{S}^1)$ is uniformly perfect; that is, every element is generated by two commutators. Thus, the central extension $\left(\widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1), p\right)$ is universal. Therefore, we have that $\mathrm{H}_2(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{Z}) \simeq \mathbb{Z}$ and in consequence

$$\mathrm{H}^2(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{R}) \simeq \mathbb{R}, \qquad \mathrm{H}^2(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{Z}) \simeq \mathbb{Z},$$

by the universal coefficient theorem.

Choose a section $\sigma : \mathrm{Homeo}_+(\mathbb{S}^1) \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1)$ satisfying that $\sigma(f)(0) \in [0,1) \subset \mathbb{R}$ in $p^{-1}(f)$ for all $f \in \mathrm{Homeo}_+(\mathbb{S}^1)$. Take the non–homogeneous obstruction cocycle

$$\bar{c}(f_1, f_2) = \sigma(f_1 \circ f_2)^{-1} \circ \sigma(f_1) \circ \sigma(f_2).$$

Lemma 1.4.4. *The 2–cocycle $\bar{c}$ takes only the values 0 or 1.*

Proof. We know that $\sigma(f_1 \circ f_2)$ and $\sigma(f_1) \circ \sigma(f_2)$ are lifts of the same element $f_1 \circ f_2$. Thus, they must differ by an integer translation; *i.e*,

$$\sigma(f_1 \circ f_2) \circ \tau_\gamma \equiv \sigma(f_1) \circ \sigma(f_2),$$

and $\tau_\gamma : \mathbb{R} \longrightarrow \mathbb{R}$ is the integer translation by $\gamma \in \mathbb{Z}$. In fact, $\gamma = 0$ or $\gamma = 1$:
$\sigma(f_1 \circ f_2)(0) \in [0, 1)$, implies that $\sigma(f_1 \circ f_2)(\tau_\gamma(0)) \in [\gamma, \gamma + 1)$. However,

$$\sigma(f_1 \circ f_2)(\tau_\gamma(0)) = \sigma(f_1)(\sigma(f_2)(0)) \in [\sigma(f_1)(0), \sigma(f_1)(0) + 1) \subset [0, 2).$$

Consequently, γ is equal to 0 or 1.

⌟

Hence the 2–cocycle associated to the Euler class is bounded and therefore we can think it as $eu_b(\mathbb{S}^1) \in \mathrm{H}_b^2(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{Z}) \simeq \mathbb{Z}$. It is a well defined bounded cohomology class because changing the value of the section in 0 will only change the choice of the section by a bounded amount. We call $eu_b(\mathbb{S}^1)$ the bounded Euler class.

Denote by $eu^{\mathbb{R}}(\mathbb{S}^1)$ and $eu_b^{\mathbb{R}}(\mathbb{S}^1)$ the corresponding cohomology classes with real coefficients. Observe that under the comparison map $i_{\mathbb{R}}(eu_b^{\mathbb{R}}(\mathbb{S}^1)) = eu^{\mathbb{R}}(\mathbb{S}^1)$. Moreover, the class $eu_b^{\mathbb{R}}(\mathbb{S}^1)$ in $\mathrm{H}_b^2(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{R})$ can be translated in the quasi–corner model to

$$\begin{array}{ccc} \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1) & \xrightarrow{T} & \mathbb{R} \\ {\scriptstyle p}\big\downarrow & & \\ \mathrm{Homeo}_+(\mathbb{S}^1), & & \end{array}$$

where $T : \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1) \longrightarrow \mathbb{R}$ is uniquely determined as the Poincaré translation number. This holds because $\widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1)$ is perfect. Equivalently, for every $x \in \mathbb{R}$ the map $T_x : \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1) \longrightarrow \mathbb{R}$ given by $T_x\left(\tilde{f}\right) = \tilde{f}(x) - x$ is a quasimorphism; moreover, any two of these are at uniformly bounded distance and therefore there is a common homogenization

$$\begin{aligned} T : \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1) &\longrightarrow \mathbb{R} \\ \tilde{f} &\longmapsto \lim_{n \longrightarrow \infty} \frac{\tilde{f}^n(x) - x}{n}, \end{aligned}$$

which is independent of $x \in \mathbb{R}$.

Theorem 1.4.5. *If $p : \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1) \longrightarrow \mathrm{Homeo}_+(\mathbb{S}^1)$ is the canonical projection, then $eu_b^{\mathbb{R}}(\mathbb{S}^1)$ is represented by the quasi–corner $C(-T, p)$.*

Proof. If $\sigma : \mathrm{Homeo}_+(\mathbb{S}^1) \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1)$ is the unique section with $\sigma(f)(0) \in [0, 1)$ and eu_σ is the corresponding cocycle representative of the Euler class of the extension p, we will prove that

$$p^* eu_\sigma = -d^1 A;$$

where $A(\tilde{f}) = \lfloor \tilde{f}(0) \rfloor$.

That is, $A(\tilde{f})$ is the greatest integer such that $A(\tilde{f}) \leq \tilde{f}(0)$. Since $|A(\tilde{f}) - T_0(\tilde{f})| < 1$, this implies the statement.

The observation is that for $\tilde{f}, \tilde{g} \in \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1)$, we have

$$\lfloor (\tilde{f}\tilde{g})(0) \rfloor = -\lfloor (\tilde{f}\tilde{g})^{-1}(0) \rfloor.$$

Thus, if we denote $\bar{f} = \tilde{f} - A(\tilde{f})$, then

$$\begin{aligned}
(p^* e_\sigma)(\tilde{f}, \tilde{g}) &= i^{-1}\left(\sigma(p(\tilde{f})p(\tilde{g}))^{-1}\sigma(p(\tilde{f}))\sigma(p(\tilde{g}))\right) \\
&= \left(\overline{(fg)^{-1}}\,\overline{f}\,\overline{g}\right)(0) \\
&= \left(\overline{(fg)^{-1}}\,\overline{f}(\tilde{g} - \lfloor \tilde{g}(0) \rfloor)\right)(0) \\
&= \left(\overline{(fg)^{-1}}(\tilde{f}\tilde{g} - \lfloor \tilde{g}(0) \rfloor - \lfloor \tilde{f}(0) \rfloor)\right)(0) \\
&= 0 - \lfloor \tilde{g}(0) \rfloor - \lfloor \tilde{f}(0) \rfloor - \lfloor (\tilde{f}\tilde{g})^{-1}(0) \rfloor \\
&= \lfloor (\tilde{f}\tilde{g})(0) \rfloor - \lfloor \tilde{g}(0) \rfloor - \lfloor \tilde{f}(0) \rfloor \\
&= -d^1 A\left(\tilde{f}, \tilde{g}\right).
\end{aligned}$$

⌟

Let G be a group and $\psi : G \longrightarrow \mathrm{Homeo}_+(\mathbb{S}^1)$ a homomorphism. Consider the pull–back of the given extension; that is,

$$\tilde{G} = \left\{(g, \tilde{f}) \in G \times \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1) \;:\; \psi(g) = p(\tilde{f})\right\},$$

and the natural projection $\tilde{G} \longrightarrow G$ which gives us the central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1.$$

The Euler class of this extension is called the Euler class of the homomorphism ψ and it is denoted by $eu(\mathbb{S}^1)(\psi) \in \mathrm{H}^2(G, \mathbb{Z})$.

It is easy to see that two conjugated homomorphisms ψ_1, $\psi_2 : G \longrightarrow \mathrm{Homeo}_+(\mathbb{S}^1)$ have the same Euler class $eu(\mathbb{S}^1)(\psi_1) = eu(\mathbb{S}^1)(\psi_2) \in \mathrm{H}^2(G, \mathbb{Z})$. Also, $eu(\mathbb{S}^1)(\psi) = 0$ if and only if ψ has a lift $\tilde{\psi} : \tilde{G} \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1)$ such that $\psi \equiv p \circ \tilde{\psi}$.

We have a well defined bounded cohomology class $eu_b(\mathbb{S}^1)(\psi) \in \mathrm{H}^2_b(G, \mathbb{Z})$ called the bounded Euler class of the homomorphism ψ.

Theorem 1.4.6 (Ghys)**.** *There is a cohomology class $eu_b(\mathbb{S}^1) \in \mathrm{H}^2_b(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{Z})$ satisfying:*

(a) For every homomorphism $\psi : G \longrightarrow \mathrm{Homeo}_+(\mathbb{S}^1)$, the image of $eu(\mathbb{S}^1)_b(\psi)$ in $\mathrm{H}^2(G, \mathbb{Z})$ via the map $i_{\mathbb{Z}}$ is the Euler class of the homomorphism ψ.

(b) *If* $\psi : \mathbb{Z} \longrightarrow \mathrm{Homeo}_+(\mathbb{S}^1)$ *is an homomorphism, then* $eu_b(\mathbb{S}^1)(\psi) \in \mathrm{H}^2_b(\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{S}^1$ *is the rotation number of the homeomorphism* $\psi(1)$.

(c) *If* ψ_1 *and* ψ_2 *are two homomorphisms from* G *to* $\mathrm{Homeo}_+(\mathbb{S}^1)$ *that are conjugate by an element of* $\mathrm{Homeo}_+(\mathbb{S}^1)$*, then*

$$eu_b(\mathbb{S}^1)(\psi_1) = eu(\mathbb{S}^1)_b(\psi_2).$$

2. HOMEOMORPHISMS OF THE SOLENOID

In this chapter we will introduce the main object of our study, the solenoid S. This is a one–dimensional compact connected Abelian group, with a dense inclusion of the one parametric group $\mathbb{R}$. S is also a $\widehat{\mathbb{Z}}$–principal bundle with base space $\mathbb{S}^1$ and a foliated space whose leaves are homeomorphic to $\mathbb{R}$ and a typical fiber is isomorphic to $\widehat{\mathbb{Z}}$. The solenoid is also known as "the universal algebraic covering of the circle".

Our purpose is to investigate the isotopy classes of the group of homeomorphisms of the solenoid. In order to do that we will review some results for the continuous functions $\mathsf{S} \longrightarrow \mathsf{S}$ which preserve the zero element and important decompositions for the group of homeomorphisms of the solenoid. The most relevant subgroups are the group of automorphisms, the group of homeomorphisms that preserves the base leaf and the subgroup of translations over the fiber.

We will introduce the connected component of the identity of the group of homeomorphisms, which turns out to be open in the subgroup of homeomorphisms that preserve all the leaves of the solenoid and it is a simple and uniformly perfect group.

2.1 The solenoid

Let $\pi : \mathbb{R} \longrightarrow \mathbb{S}^1$ be the universal covering map of the circle; equivalently, we identify $\mathbb{S}^1$ as the quotient $\mathbb{R}/\mathbb{Z}$ and then $\mathbb{S}^1$ is a topological compact connected Abelian group. If for every $n \in \mathbb{N}$ we identify $\mathbb{S}^1 \cong \mathbb{R}/n\mathbb{Z}$, we have the universal covering maps $\pi_n : \mathbb{R} \longrightarrow \mathbb{S}^1$.

Consider $\mathbb{N}$ as a preordered set of indices with the relation of divisibility. Thus, for every $n, m \in \mathbb{N}$ such that $n|m$, there is only a covering map $\mathrm{p}_{nm} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ such that the following diagram is commutative:

$$\begin{array}{ccc} & & \mathbb{R} \\ & \swarrow^{\pi_n} & \downarrow{\scriptstyle \pi_m} \\ \mathbb{S}^1 & \xleftarrow{\mathrm{p}_{nm}} & \mathbb{S}^1. \end{array}$$

That is, $\{\mathbb{S}^1, p_{nm}\}_{\mathbb{N}}$ is an inverse system of circles and continuous homomorphisms.

Definition 2.1.1. *The* ***universal one–dimensional solenoid*** S *is the inverse limit of the system* $\{\mathbb{S}^1, p_{nm}\}_{\mathbb{N}}$*; that is,*

$$\mathsf{S} = \varprojlim \mathbb{S}^1 \cong \left\{ z = (z_n) \in \prod_{\mathbb{N}} \mathbb{S}^1 \ : \ n|m \Rightarrow \mathrm{p}_{nm}(z_m) = z_n \right\}.$$

S is a Hausdorff compact Abelian topological group because it is the inverse limit of Hausdorff compact Abelian topological groups and the operation is defined on the coordinates. Also, S admits a dense inclusion of $\mathbb{R}$ defined by the injective homomorphism

$$\begin{aligned} \mathrm{P} : \mathbb{R} &\longrightarrow \mathsf{S} \\ x &\longmapsto (\pi_n(x)). \end{aligned}$$

The image $\mathrm{P}(\mathbb{R})$ is the path–connected component of the zero element $0 \in \mathsf{S}$. We denote by $\mathcal{L}_0 = \mathrm{P}(\mathbb{R})$ and call it *the base leaf of* S.

Using the density of $\mathcal{L}_0$ we see that S is connected. Alternatively, we can use $\mathsf{S} \cong \mathrm{Char}(\mathbb{Q})$, where $\mathrm{Char}(\mathbb{Q})$ denotes the character group of $\mathbb{Q}$; *i.e.* the group of continuous homomorphisms

$$\mathbb{Q} \longrightarrow \mathbb{S}^1.$$

By the Pontrjagin duality theorem, $\mathrm{Char}(\mathsf{S}) \cong \mathbb{Q}$ and we can use the fact that $\mathbb{Q}$ is torsion free to conclude that S is a connected group (see [HM]).

Denote by $\mathrm{p}_n : \mathsf{S} \longrightarrow \mathbb{S}^1$ the projection onto the n–th coordinate. In particular, set $\mathrm{p} = \mathrm{p}_1$. For every $\omega \in \mathbb{S}^1$, $\mathrm{p}^{-1}(\omega) \cong \widehat{\mathbb{Z}}$, where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$; that is,

$$\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z} \cong \left\{ k = (k_n) \in \prod_{\mathbb{N}} \mathbb{Z}/n\mathbb{Z} \,:\, n|m \Rightarrow k_m \equiv k_n \mod n\mathbb{Z} \right\}.$$

$\widehat{\mathbb{Z}}$ is a topological compact Abelian totally disconnected and perfect group. Hence, $\widehat{\mathbb{Z}}$ is homeomorphic to the Cantor set and admits a dense inclusion $\mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}}$ (see [Wil] for details). For every $n \in \mathbb{N}$, let $\rho_n : \widehat{\mathbb{Z}} \to \mathbb{Z}/n\mathbb{Z}$ be the projection onto the n–th coordinate.

There is a continuous and effective action defined by

$$\begin{aligned} \widehat{\mathbb{Z}} \times \mathsf{S} &\longrightarrow \mathsf{S} \\ (k, z) &\longmapsto k \cdot z := (z_n + k_n). \end{aligned}$$

Proposition 2.1.2. $\mathrm{p} : \mathsf{S} \longrightarrow \mathbb{S}^1$ *is a* $\widehat{\mathbb{Z}}$*–principal fiber bundle.*

Proof. Let $\omega \in \mathbb{S}^1$ and U_ω be an open simply connected neighborhood of ω. For every $n \in \mathbb{N}$ the covering maps $\mathrm{p}_{1,n}$ are regular, so U_ω is totally covered by $\mathrm{p}_{1,n}$. Since $\pi : \mathbb{R} \longrightarrow \mathbb{S}^1$ is a covering, given $y \in \pi^{-1}(\omega)$ there is a unique open neighborhood U_y of y in $\mathbb{R}$, such that $\pi(U_y) \cong U_\omega$.

We have the following continuous map

$$\begin{aligned} g : U_y \times \widehat{\mathbb{Z}} &\longrightarrow \mathrm{p}^{-1}(U_\omega) \\ (x, k) &\longmapsto k \cdot \mathrm{P}(x) = (\pi_n(x) + k_n). \end{aligned}$$

Let us prove that this map is in fact a homeomorphism.

In the first place, g is surjective because $\pi(U_y) = U_\omega$ and U_ω is totally covered by every map $\mathrm{p}_{1,n}$. g is also injective because if $(\pi_n(x) + k_n) = (\pi_n(y) + l_n)$; then $\pi(x) = \pi(y)$. Hence, there is a unique $q \in \widehat{\mathbb{Z}}$ such that $q \cdot \mathrm{P}(x) = \mathrm{P}(y)$; that is,

$$g(U_y \times \widehat{\mathbb{Z}}) = \mathrm{p}^{-1}(U_\omega).$$

Therefore, g is bijective.

We will also prove that g is an open map. Let $k_n \in \mathbb{Z}/n\mathbb{Z}$ and $W = \rho_n^{-1}(k_n)$ an open subset of $\widehat{\mathbb{Z}}$. If $U \subset \mathbb{R}$ is open, then $U \times W$ is an open subset of $\mathbb{R} \times \widehat{\mathbb{Z}}$ and $g(U \times W) = \mathrm{p}_n^{-1}(\pi_n(U) + k_n)$.

Indeed, take $x \in g(U \times W)$, then $x_n \in \pi_n(U) + k_n$; *i.e.*, $x \in \mathrm{p}_n^{-1}(\pi_n(U) + k_n)$. Conversely, if $x \in \mathrm{p}_n^{-1}(\pi_n(U) + k_n)$, we have that $x_n = \pi_n(u) + k_n$ for some $u \in U$ and

$$x_1 = \mathrm{p}_{1,n}(x_n) = \mathrm{p}_{1,n}(\pi_n(u) + k_n) = \pi(u).$$

Thus, there exists only a $l \in \widehat{\mathbb{Z}}$ such that $x = g(u, l)$. As a consequence,

$$\pi_n(u) + k_n = x_n = \pi_n(u) + l_n;$$

or $l \in W$. We conclude that $x \in g(U \times W)$.

Hence $g(U \times W) = \mathrm{p}_n^{-1}(\pi_n(U) + k_n)$ implies that g is open. Consequently, g is a homeomorphism because it is a bijective and open map. In particular, the following diagram is commutative

$$\begin{array}{ccc} U_y \times \widehat{\mathbb{Z}} & \xrightarrow{\ g\ } & \mathrm{p}^{-1}(U_\omega) \\ \downarrow & & \downarrow \mathrm{p} \\ U_y & \xrightarrow{\ \pi\ } & U_\omega \end{array} .$$

Because this construction is independent of the choice of $\omega \in \mathbb{S}^1$ we get the desired fiber bundle structure.

Moreover, the bundle is locally trivial because

$$\mathrm{p}(g(x, k)) = \mathrm{p}(\pi_n(x) + k_n) = \pi(x).$$

In order to finish the proof we see that the action of $\widehat{\mathbb{Z}}$ in S is compatible with the fiber bundle structure: for every $k, l \in \widehat{\mathbb{Z}}$ and $x \in U_y$,

$$\begin{aligned} g(k \cdot (x, l)) = g(x, k + l) &= (\pi_n(x) + k_n + l_n) \\ &= k \cdot (\pi_n(x) + l_n) \\ &= k \cdot g(x, l). \end{aligned}$$

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Remark 2.1.3. *From the last proof, there is an atlas of open charts on* S *given by*

$$g_\alpha : V_\alpha \longrightarrow (a, b) \times \widehat{\mathbb{Z}},$$

where (a, b) *is an open interval of* $\mathbb{R}$. *Without loss of generality we assume that the atlas is maximal and call* $V_\alpha = g_\alpha^{-1}((a, b) \times \widehat{\mathbb{Z}})$ *an open box of* S. *We say that a collection of boxes* $\{V_i\}_{i=1}^{l}$ *in* S *is a* box decomposition *if the following two conditions are satisfied:*

- $i \neq j \Rightarrow V_i \cap V_j = \emptyset$,
- $\{\overline{V_i}\}$ *forms a cover of* S.

This notion was introduced in [BBG], where it was shown that S admits a box decomposition. Moreover, for every box cover $\{V_i\}_{i=1}^n$ of S, there exists a box decomposition $\{W_i\}_{i=1}^l$; such that if $W_i \cap V_j \neq \emptyset$ then $W_i \subset V_j$.

The solenoid also admits a product foliated structure: define the diagonal action of $\mathbb{Z}$ on the product $\mathbb{R} \times \widehat{\mathbb{Z}}$ by

$$\begin{aligned} \mathbb{Z} \times (\mathbb{R} \times \widehat{\mathbb{Z}}) &\longrightarrow \mathbb{R} \times \widehat{\mathbb{Z}} \\ (\gamma, (x, k)) &\longmapsto (x - \gamma, k + \gamma). \end{aligned}$$

Observe that in the second coordinate we are thinking of γ as the image of γ under the inclusion $\mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}}$. Obviously, $\mathbb{R} \times \widehat{\mathbb{Z}}$ has a trivial foliated structure and we also have this property for the quotient $\mathbb{R} \times_{\mathbb{Z}} \widehat{\mathbb{Z}} := (\mathbb{R} \times \widehat{\mathbb{Z}})/\mathbb{Z}$. Moreover, $\mathbb{R} \times_{\mathbb{Z}} \widehat{\mathbb{Z}}$ is a $\widehat{\mathbb{Z}}$–fiber bundle over $\mathbb{S}^1$ (see [CC]).

Next we show that S is canonically identified with $\mathbb{R} \times_{\mathbb{Z}} \widehat{\mathbb{Z}}$, the proof is analogous to the one for the bi–dimensional solenoid given in [Odd].

Theorem 2.1.4. $\mathsf{S} \cong \mathbb{R} \times_{\mathbb{Z}} \widehat{\mathbb{Z}}$.

Set $\Pi : \mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathsf{S}$ to the quotient function which is a covering map ([MR]). The next observations are plain: the space of leaves is isomorphic to $\widehat{\mathbb{Z}}/\mathbb{Z}$, all the leaves are dense on S and the base leaf $\mathcal{L}_0$ is identified with $\Pi(\mathbb{R} \times \mathbb{Z}) = \mathbb{R} \times_{\mathbb{Z}} \mathbb{Z}$.

Moreover, the identification of an open box now reads

$$V = (a, b) \times_{\mathbb{Z}} \widehat{\mathbb{Z}},$$

with $(a, b) \subset \mathbb{R}$. We will be writing $V \simeq (a, b) \times \widehat{\mathbb{Z}}$ for a box. Thus, for every $z \in V$, which reads in coordinates inside the chart (x, k), the sets $(a, b) \times \{k\}$ and $\{x\} \times \widehat{\mathbb{Z}}$ are called the slice and the vertical of z respectively.

Remark 2.1.5. *There are also boxes of* S *of the form* $W \simeq (a, b) \times C$ *with* C *a proper Cantor subset of* $\widehat{\mathbb{Z}}$*. However, for most of the details in this work, it suffices to consider boxes with a vertical isomorphic to* $\widehat{\mathbb{Z}}$*. We will mention, if necessary, the use of this kind of boxes.*

2.2 Continuous functions preserving the zero element

We recall the main results about continuous functions $\mathsf{S} \longrightarrow \mathsf{S}$ that preserve the zero element (for the complete details see [Kee]).

Denote by $\mathrm{C}(\mathsf{S})$ the space of continuous functions $\mathsf{S} \longrightarrow \mathsf{S}$ and let $\mathrm{C}_*(\mathsf{S})$ be the subspace of functions that preserve the zero element; *i.e.*, $f \in \mathrm{C}_*(\mathsf{S})$ satisfies $f(0) = 0$. If $\mathrm{Hom}(\mathsf{S})$ is the subspace of homomorphisms of the solenoid, then $\mathrm{Hom}(\mathsf{S}) \subset \mathrm{C}_*(\mathsf{S})$.

It is a consequence of a result of W. Scheffer (see Theorem 2 of [Sch]) that

$$\mathrm{C}_*(\mathsf{S}) \cong \mathrm{C}_*(\mathsf{S}, \mathrm{Hom}(\mathrm{Char}(\mathsf{S}), \mathbb{R})) \times \mathrm{Hom}(\mathsf{S}).$$

This is a topological isomorphism; that is, we are thinking of the respective spaces as topological groups, with the additive operation under evaluation and the compact–open topology.

As we mentioned before, it is well known that $\text{Char}(\mathsf{S}) \cong \mathbb{Q}$ and consequently

$$\text{Hom}(\text{Char}(\mathsf{S}), \mathbb{R}) \cong \text{Hom}(\mathbb{Q}, \mathbb{R}) \cong \mathbb{R}.$$

Thus,

$$\mathrm{C}_*(\mathsf{S}) \cong \mathrm{C}_*(\mathsf{S}, \mathbb{R}) \times \text{Hom}(\mathsf{S}).$$

Let $\mathrm{C}_0(\mathsf{S}) := \{f_0 \in \mathrm{C}_*(\mathsf{S}) \,:\, f_0 \sim 0\}$ be the subset of null–homotopic functions preserving the zero element. The main observation is that due to the lifting property $\mathrm{C}_0(\mathsf{S}) \cong \mathrm{C}_*(\mathsf{S}, \mathbb{R})$ and therefore

$$\mathrm{C}_*(\mathsf{S}) \cong \mathrm{C}_0(\mathsf{S}) \times \text{Hom}(\mathsf{S}).$$

Remark 2.2.1. *As a consequence of this result every continuous function of the solenoid which preserves the neutral element is null–homotopic or is homotopic to a homomorphism.*

2.3 The group of homeomorphisms of the solenoid

Recall that $\Pi : \mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathsf{S}$ is a covering map induced by the the diagonal action of Z. For every $\gamma \in \mathbb{Z}$ define the deck transformation associated to the action by γ as

$$\begin{aligned} \Delta_\gamma : \mathbb{R} \times \widehat{\mathbb{Z}} &\longrightarrow \mathbb{R} \times \widehat{\mathbb{Z}} \\ (x, k) &\longmapsto (x - \gamma, k + \gamma). \end{aligned}$$

We write $\Delta(\mathbb{Z}) = \{\Delta_\gamma \,:\, \gamma \in \mathbb{Z}\}$.

Definition 2.3.1. *The* ***group of homeomorphisms of the solenoid*** *is defined by*

$$\text{Homeo}(\mathsf{S}) = \{f : \mathsf{S} \longrightarrow \mathsf{S} \,:\, f \textit{ is a homeomorphism } \}.$$

The operation on $\text{Homeo}(\mathsf{S})$ is by composition and using that S is a Hausdorff compact metric space we know that $\text{Homeo}(\mathsf{S})$ is also a topological group with the compact–open topology ([Are]).

J. Keesling ([Kee]) has proved that there is a homeomorphism

$$\text{Homeo}(\mathsf{S}) \simeq \mathsf{S} \times \mathrm{H}_0(\mathsf{S}) \times \text{Aut}(\mathsf{S}),$$

where S is identified with the translation subgroup by elements in S and $\mathrm{H}_0(\mathsf{S})$ is the subspace of homeomorphisms of the form $\text{id} + \phi_0$ with $\phi_0 : \mathsf{S} \longrightarrow \mathsf{S}$ a continuous function which preserves the zero element, $\phi_0(\mathsf{S}) \subset \mathcal{L}_0$ and ϕ_0 is null–homotopic, or $\phi_0 \in \mathrm{C}_0(\mathsf{S})$. In particular, $\mathrm{H}_0(\mathsf{S})$ is a contractible space homeomorphic to the Hilbert space ℓ_2.

Finally, $\text{Aut}(\mathsf{S})$ is the subgroup of automorphisms of S as a group. Later in this chapter we will give a complete description of this object, allowing us to find an algebraic decomposition of $\text{Homeo}(\mathsf{S})$.

The next are two important examples of homeomorphisms of the solenoid.

Example 2.3.2. *Fix an element $q \in \widehat{\mathbb{Z}}$ and define*

$$\begin{aligned} S_q : \mathbb{R} \times \widehat{\mathbb{Z}} &\longrightarrow \mathbb{R} \times \widehat{\mathbb{Z}} \\ (x,k) &\longmapsto (x, k+q). \end{aligned}$$

Observe that S_q is continuous and commutes with the diagonal action in $\mathbb{R} \times \widehat{\mathbb{Z}}$; i.e.,

$$S_q \circ \Delta_\gamma \equiv \Delta_\gamma \circ S_q, \qquad (\gamma \in \mathbb{Z}).$$

Hence, S_q descends to a well defined $s_q \in \mathrm{C}(\mathsf{S})$. We call s_q the translation over the fiber by *q. Moreover, $s_q^{-1} = s_{-q} \in \mathrm{C}(\mathsf{S})$ implies that in fact $s_q \in$* Homeo(S) *and there is an injective homomorphism $\widehat{\mathbb{Z}} \hookrightarrow$* Homeo($\mathsf{S}$).

Example 2.3.3. *Given $y \in \mathbb{R}$, define the map*

$$\begin{aligned} D_y : \mathbb{R} \times \widehat{\mathbb{Z}} &\longrightarrow \mathbb{R} \times \widehat{\mathbb{Z}} \\ (x,k) &\longmapsto (x+y, k); \end{aligned}$$

which is a continuous function and as in the previous example commutes with the diagonal action. Thus, D_y descends to a continuous function $d_y : \mathsf{S} \longrightarrow \mathsf{S}$. Note that $d_y^{-1} = d_{-y} \in \mathrm{C}(\mathsf{S})$ and therefore $d_y \in$ Homeo(S). *We call d_y the* translation on the leaves by *y.*

The last example belongs to a special subgroup of Homeo(S).

Definition 2.3.4. *The **group of homeomorphisms that preserves the base leaf $\mathcal{L}_0$** is defined by*

$$\mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S}) = \{h \in \mathrm{Homeo}(\mathsf{S}) \,:\, h(\mathcal{L}_0) = \mathcal{L}_0\}.$$

The subgroup of translations over the fiber and the group of homeomorphisms that preserves $\mathcal{L}_0$ have a non–empty intersection. Indeed, take $\gamma \in \mathbb{Z}$ and $(x,k) \in \mathbb{R} \times \widehat{\mathbb{Z}}$, then

$$\begin{aligned} S_\gamma(x,k) = (x, k+\gamma) &= (x + \gamma - \gamma, k + \gamma) \\ &= \Delta_\gamma(x+\gamma, k) \\ &= \Delta_\gamma(D_\gamma(x,k)). \end{aligned}$$

Hence, S_γ and D_γ descend to the same element in Homeo(S); or equivalently, $s_\gamma \equiv d_\gamma$ in $\mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S})$. Moreover, if $h \in \mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S}) \cap \widehat{\mathbb{Z}}$, it is clear that $h = s_\gamma$ with $\gamma \in \mathbb{Z}$. Therefore,

$$\mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S}) \cap \widehat{\mathbb{Z}} = \mathbb{Z}.$$

Define an action by $\mathbb{Z}$ on the product $\mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S}) \times \widehat{\mathbb{Z}}$ as following:

$$\begin{aligned} \mathbb{Z} \times (\mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S}) \times \widehat{\mathbb{Z}}) &\longrightarrow \mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S}) \times \widehat{\mathbb{Z}} \\ (\gamma, (h, s_q)) &\longmapsto (d_{-\gamma} \circ h, s_q \circ s_\gamma) = (d_{-\gamma} \circ h, s_{q+\gamma}). \end{aligned}$$

From the ideas of C. Odden for the bi–dimensional solenoid ([Odd]) we prove that Homeo(S) is canonically identified with the orbit space of this action.

Theorem 2.3.5. $\mathrm{Homeo}(\mathsf{S}) \cong \mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S}) \times_{\mathbb{Z}} \widehat{\mathbb{Z}}$

Proof. Define the map

$$\begin{aligned} \Upsilon : \mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S}) \times \widehat{\mathbb{Z}} &\longrightarrow \mathrm{Homeo}(\mathsf{S}) \\ (h, s_q) &\longmapsto s_q \circ h. \end{aligned}$$

This map induces a well defined map on the quotient $\mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S}) \times_{\mathbb{Z}} \widehat{\mathbb{Z}}$ which turns out to be a homeomorphism.

Let $f \in \mathrm{Homeo}(\mathsf{S})$ be such that $f(\mathcal{L}_0) = \mathcal{L}_r$, with $\mathcal{L}_r$ the leaf passing through $r \in \widehat{\mathbb{Z}} \subset \mathsf{S}$. Thus, $h_1 := s_{-r} \circ f \in \mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S})$ and $\Upsilon(h_1, s_r) = s_r \circ h_1 = f$. That is, Υ is a surjective map.

Suppose that there are elements $t \in \widehat{\mathbb{Z}}$, $t \neq r$ and $h_2 \in \mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S})$, $h_2 \neq h_1$ and that they satisfy $\Upsilon(h_2, s_t) = s_t \circ h_2 = f$. Calculating

$$\begin{aligned} s_{-r+t} \equiv s_{-r} \circ s_t &\equiv s_{-r} \circ (f \circ f^{-1}) \circ s_t \\ &\equiv (s_{-r} \circ f) \circ (f^{-1} \circ s_t) \\ &\equiv h_1 \circ (h_2^{-1} \circ s_{-t} \circ s_t) \\ &\equiv h_1 \circ h_2^{-1} \end{aligned}$$

we see that $s_{-r+t} \in \mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S})$; that is, we have a $\gamma \in \mathbb{Z}$ such that $t = r + \gamma$. Consequently,

$$\begin{aligned} f \equiv s_t \circ h_2 &\equiv s_{r+\gamma} \circ h_2 \\ &\equiv s_r \circ (s_\gamma \circ h_2) \\ &\equiv s_r \circ (d_\gamma \circ h_2), \end{aligned}$$

and then

$$(h_1, s_r) \equiv (d_\gamma \circ h_2, s_{t-\gamma}) \equiv (d_\gamma \circ h_2, s_t \circ s_{-\gamma}), \qquad (\gamma \in \mathbb{Z}).$$

That is, there exists a well defined bijective map

$$\Upsilon_{\mathbb{Z}} : \mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S}) \times_{\mathbb{Z}} \widehat{\mathbb{Z}} \longrightarrow \mathrm{Homeo}(\mathsf{S}).$$

Moreover, Υ is continuous because it is defined by a composition of continuous functions, and using the properties of the quotient topology, we can also see that $\Upsilon_{\mathbb{Z}}$ is continuous and has a continuous inverse.

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2.4 Isotopy classes of Homeo(S)

We say that $f \in \mathrm{Homeo}(\mathsf{S})$ is *homotopic to the identity* if there exists a continuous map

$$\lambda : [0,1] \times \mathsf{S} \longrightarrow \mathsf{S},$$

such that $\lambda(0,*) \equiv \mathrm{id}$ and $\lambda(1,*) \equiv f$. If in addition, for every $t \in [0,1]$ we have that $\lambda(t,*)$ is a homeomorphism, f is said to be *isotopic to the identity* and we will write $f \sim id$.

Let $\text{Homeo}_{\mathcal{L}}(\mathsf{S})$ be the subgroup of homeomorphisms that preserves all the leaves of the solenoid. In particular, $\text{Homeo}_{\mathcal{L}}(\mathsf{S})$ is a subgroup of $\text{Homeo}_{\mathcal{L}_0}(\mathsf{S})$ and every $f \in \text{Homeo}_{\mathcal{L}}(\mathsf{S})$ is homotopic to the identity. Also, following J. Kwapisz ([Kwa2]) for every $f \in \text{Homeo}_{\mathcal{L}}(\mathsf{S})$, there exists a continuous map $\tilde{\phi} : \mathsf{S} \longrightarrow \mathbb{R}$ uniquely determined by

$$f \equiv \text{id} + \text{P} \circ \tilde{\phi}.$$

The function $\tilde{\phi}$ is called the *displacement* of f and if $||\tilde{\phi}||_\infty < \epsilon$, it is said that the displacement of f is smaller than ϵ.

Moreover, $\text{Homeo}_{\mathcal{L}}(\mathsf{S})$ is open in $\text{Homeo}(\mathsf{S})$ and if $\text{Homeo}^0(\mathsf{S})$ denotes the connected component of the identity in $\text{Homeo}(\mathsf{S})$, then $\text{Homeo}^0(\mathsf{S})$ is open in $\text{Homeo}(\mathsf{S})$ and

$$\text{Homeo}^0(\mathsf{S}) = \text{Homeo}^0_{\mathcal{L}}(\mathsf{S})$$

(see [AP] for the full details).

Definition 2.4.1. *The* ***group of homeomorphisms isotopic to the identity*** *is defined by*

$$\text{Homeo}_+(\mathsf{S}) = \{f \in \text{Homeo}(\mathsf{S}) \; : \; f \sim \text{id}\}.$$

It is clear that $\text{Homeo}_+(\mathsf{S})$ is a subgroup of $\text{Homeo}_{\mathcal{L}}(\mathsf{S})$; in fact,

$$\text{Homeo}_+(\mathsf{S}) = \text{Homeo}^0(\mathsf{S}) = \text{Homeo}^0_{\mathcal{L}}(\mathsf{S}).$$

Hence $\text{Homeo}_+(\mathsf{S})$ is open in $\text{Homeo}(\mathsf{S})$. Also, $\text{Homeo}_+(\mathsf{S})$ is a subgroup of $\text{Homeo}_{\mathcal{L}_0}(\mathsf{S})$ and

$$\text{Homeo}_+(\mathsf{S}) \cap \widehat{\mathbb{Z}} = \mathbb{Z};$$

that is, the translations over the fiber which are isotopic to the identity are the integer translations.

Using lifts of homeomorphisms to $\mathbb{R} \times \widehat{\mathbb{Z}}$ we will be able to describe the different isotopy classes on $\text{Homeo}(\mathsf{S})$. The idea of lifting to the covering map $\Pi : \mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathsf{S}$ goes back to the work of J. Kwapisz ([Kwa]).

Every translation along the leaves $d_y \in \text{Homeo}_{\mathcal{L}_0}(\mathsf{S})$ is isotopic to the identity because trivially for every $\lambda \in [0,1]$, $d_{(1-\lambda)y} \in \text{Homeo}_{\mathcal{L}_0}(\mathsf{S})$ defines an isotopy between d_y and id.

Proposition 2.4.2. *Let $q, r \in \widehat{\mathbb{Z}}$. Then, $s_q \sim s_r$ if and only if there is $\gamma \in \mathbb{Z}$ such that $q = r + \gamma$.*

Proof. Since $\widehat{\mathbb{Z}}$ acts transitively on itself, it is enough to consider $r = 0$; that is, we are going to prove that $s_q \sim \text{id}$ if and only if $q \in \mathbb{Z}$.

As we have seen, if $q \in \mathbb{Z}$ and s_q is the translation over the fiber by q, then $s_q \equiv d_q$; *i.e.* $s_q \sim \text{id}$.

Conversely, suppose that $s_q \sim \text{id}$ and consider the homotopy

$$\delta_\lambda : \mathsf{S} \longrightarrow \mathsf{S}, \qquad (\lambda \in [0,1]),$$

satisfying $\delta_0 \equiv s_q$ and $\delta_1 \equiv \text{id}$. For every $z \in \mathsf{S}$, $\lambda \longmapsto \delta_\lambda(z)$ define a path.

By the unique path lifting property for the covering $\Pi : \mathbb{R}\times\widehat{\mathbb{Z}} \longrightarrow \mathsf{S}$, for every $\lambda \in [0,1]$ there is $\widetilde{\delta_\lambda} : \mathbb{R}\times\widehat{\mathbb{Z}} \longrightarrow \mathbb{R}\times\widehat{\mathbb{Z}}$, such that $\widetilde{\delta_0} \equiv S_q$ and $\widetilde{\delta_1} \equiv \Delta_{\kappa(\cdot)}$, where $\kappa : \widehat{\mathbb{Z}} \longrightarrow \mathbb{Z}$ is a continuous function.

Since $\widehat{\mathbb{Z}}$ is totally disconnected, the path $\widetilde{\delta_\lambda}$ is defined over a leaf; that is, the projection onto the second coordinate $\mathrm{pr}_2 : \mathbb{R}\times\widehat{\mathbb{Z}} \longrightarrow \widehat{\mathbb{Z}}$ is constant in $\widetilde{\delta_\lambda}$ for each $\lambda \in [0,1]$. Then, for every $(x,k) \in \mathbb{R}\times\widehat{\mathbb{Z}}$:

$$\mathrm{pr}_2(\widetilde{\delta_\lambda}(x,k)) = \mathrm{pr}_2(\widetilde{\delta_0}(x,k)) = \mathrm{pr}_2(s_q(x,k)) = k+q.$$

In particular,

$$k+q = \mathrm{pr}_2(\widetilde{\delta_0}(x,k)) = \mathrm{pr}_2(\widetilde{\delta_1}(x,k)) = \mathrm{pr}_2(\Delta_{\kappa(k)}(x,k)) = k+\kappa(k).$$

Therefore, κ is constant and then $q \equiv \kappa \in \mathbb{Z}$.

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Remark 2.4.3. *From the last proof we observe that the group* $\Delta(\mathbb{Z})$ *is precisely the* group of deck transformations*; or equivalently, the identity* id $\in$ Homeo(S) *has only lifts of the form* Δ_γ *with* $\gamma \in \mathbb{Z}$.

Let $\Gamma_{\mathcal{L}_0}$ be the group of isotopy classes of homeomorphisms that preserves the base leaf; that is,

$$\Gamma_{\mathcal{L}_0} = \mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S})/\mathrm{Homeo}_+(\mathsf{S}).$$

In order to find $\Gamma_{\mathcal{L}_0}$ we calculate first the group of automorphisms of S.

Definition 2.4.4. *The* ***group of automorphisms of*** S *is defined as*

$$\mathrm{Aut}(\mathsf{S}) = \{g \in \mathrm{Homeo}(\mathsf{S}) \ : \ g \in \mathrm{Hom}(\mathsf{S})\}.$$

Observe that Aut(S) is a subgroup of $\mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S})$; that is, $h \in \mathrm{Aut}(\mathsf{S})$ is continuous and $h(0)=0$ implies that $h(\mathcal{L}_0)=\mathcal{L}_0$.

Example 2.4.5. *Define*

$$R : \mathbb{R}\times\widehat{\mathbb{Z}} \longrightarrow \mathbb{R}\times\widehat{\mathbb{Z}}$$
$$(x,k) \longmapsto (-x,-k).$$

For every $(x,k) \in \mathbb{R}\times\widehat{\mathbb{Z}}$ *and* $\gamma \in \mathbb{Z}$*:*

$$R(\Delta_\gamma(x,k)) = R(x-\gamma, k+\gamma) = (-x+\gamma, -k-\gamma) = \Delta_{-\gamma}(-x,-k) = \Delta_{-\gamma}(R(x,k)).$$

Thus, R descends to a well defined map $r : \mathsf{S} \longrightarrow \mathsf{S}$, $r(z) = -z$. Moreover, $r \in \mathrm{Aut}(\mathsf{S})$.

In order to present more examples of automorphisms we study first an important class of endomorphisms of $\widehat{\mathbb{Z}}$. Given $a \in \mathbb{N}$, $a\widehat{\mathbb{Z}}$ is a subgroup of $\widehat{\mathbb{Z}}$ with index a. Define the Frobenius endomorphism $\mathsf{F}_a : \widehat{\mathbb{Z}} \longrightarrow \widehat{\mathbb{Z}}$ by

$$\mathsf{F}_a(k) = ak.$$

Lemma 2.4.6. F_a *is an isomorphism onto its image.*

Proof. Take $k = (k_n) \in \ker(\mathsf{F}_a)$ and suppose that $k \neq 0$. Thus, there is an index $m \in \mathbb{N}$ such that k_m is not congruent to zero module m but $ak_m \equiv 0 \mod m$.

Given $l \in \mathbb{N}$ such that $m|l$, then $ak_l \equiv 0 \mod l$. In particular, $m|am$ and $ak_{am} \equiv 0 \mod am$. Hence, $k_{am} \equiv 0 \mod m$, but also using the compatibility relations, $k_{am} \equiv k_m \mod m$ which gives us a contradiction.

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Since $\widehat{\mathbb{Z}}/\mathsf{F}_a(\widehat{\mathbb{Z}}) = \widehat{\mathbb{Z}}/a\widehat{\mathbb{Z}}$ is a cyclic group of order a, for every $k \in \widehat{\mathbb{Z}}$ choose k_a between 0 and $a-1$ satisfying that $k - k_a = ak' \in \mathsf{F}_a(\widehat{\mathbb{Z}})$. Then, the Frobenius map $\mathsf{F}_{1/a} : \widehat{\mathbb{Z}} \longrightarrow \widehat{\mathbb{Z}}$ it is defined by

$$\mathsf{F}_{1/a}(k) = \mathsf{F}_a^{-1}(k - k_a) = \mathsf{F}_a^{-1}(ak') = k'.$$

Even though $\mathsf{F}_{1/a}$ defines a left inverse for F_a, $\mathsf{F}_{1/a}$ is not injective. Depending on the value of $k \in \widehat{\mathbb{Z}}$ we have the following: if $k_a < a - 1$, then $k_a + 1 = (k+1)_a$; or $\mathsf{F}_{1/a}(k+1) = \mathsf{F}_{1/a}(k)$. On the other hand, if $k_a = a - 1$, we have that $k - k_a = k - (a-1) = ak'$, and $k + 1 = a(k'+1)$; thus, $\mathsf{F}_{1/a}(k+1) = \mathsf{F}_{1/a}(k) + 1$.

Now, given $a \in \mathbb{N}$ we define the map

$$\begin{aligned} g_a : \mathsf{S} &\longrightarrow \mathsf{S} \\ z &\longmapsto az; \end{aligned}$$

which is a continuous homomorphism and has a lift defined by

$$\begin{aligned} G_a : \mathbb{R} \times \widehat{\mathbb{Z}} &\longrightarrow \mathbb{R} \times \widehat{\mathbb{Z}} \\ (x,k) &\longmapsto (ax, \mathsf{F}_a(k)). \end{aligned}$$

It is straightforward to check that G_a descends to g_a; since,

$$G_a \circ \Delta_\gamma \equiv \Delta_{a\gamma} \circ G_a, \qquad (\gamma \in \mathbb{Z}).$$

Proposition 2.4.7. *For every $a \in \mathbb{N}$, $g_a \in$ Homeo(S).*

Proof. In the first place, g_a is surjective since for every $y \in \mathbb{R}$:

$$g_a(\mathrm{P}(y)) = a(\pi_n(y)) = (a\pi_n(y)) = (\pi_n(ay)) = \mathrm{P}(ay),$$

and $\mathrm{P}(a\mathbb{R})$ is dense over S.

It is enough now to define a left inverse for the lift G_a and to verify that it commutes with the diagonal action. We define

$$\begin{aligned} G_{1/a} : \mathbb{R} \times \widehat{\mathbb{Z}} &\longrightarrow \mathbb{R} \times \widehat{\mathbb{Z}} \\ (x,k) &\longmapsto \left(\frac{x + k_a}{a}, \mathsf{F}_{1/a}(k) \right). \end{aligned}$$

Trivially, $G_{1/a}$ is continuous and $G_{1/a} \circ G_a \equiv \mathrm{Id}$. Take any $(x,k) \in \mathbb{R} \times \widehat{\mathbb{Z}}$, in order to prove that $G_{1/a}$ commutes with the diagonal action we have two cases:

- If $k_a < a-1$, then $\mathsf{F}_{1/a}(k+1) = \mathsf{F}_{1/a}(k)$; or, $(k+1)_a = k_a + 1$. Hence,

$$\begin{aligned} G_{1/a}(\Delta_1(x,k)) &= G_{1/a}(x-1,k+1) \\ &= \left(\frac{x-1+(k+1)_a}{a}, F_{1/a}(k+1)\right) \\ &= \left(\frac{x+k_a}{a}, F_{1/a}(k)\right) \\ &= G_{1/a}(x,k). \end{aligned}$$

- If $k_a = a-1$, then $\mathsf{F}_{1/a}(k+1) = \mathsf{F}_{1/a}(k) + 1$ and $(k+1)_a = 0$. Thus,

$$\begin{aligned} G_{1/a}(\Delta_1(x,k)) &= G_{1/a}(x-1,k+1) \\ &= \left(\frac{x-1}{a}, F_{1/a}(k+1)\right) \\ &= \left(\frac{x+a-1}{a} - 1, F_{1/a}(k)+1\right) \\ &= \left(\frac{x+k_a}{a} - 1, F_{1/a}(k)+1\right) \\ &= \Delta_1(G_{1/a}(x,k)). \end{aligned}$$

As a consequence, $G_{1/a}$ descends to a continuous map $g_{1/a} : \mathsf{S} \longrightarrow \mathsf{S}$. Finally, since g_a is surjective we can conclude that $g_{1/a} = g_a^{-1}$.

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Given two coprime natural numbers a and b, we define

$$\begin{aligned} G_{a/b} : \mathbb{R} \times \widehat{\mathbb{Z}} &\longrightarrow \mathbb{R} \times \widehat{\mathbb{Z}} \\ (x,k) &\longmapsto \left(\frac{a}{b}(x+k_b), \mathsf{F}_a(\mathsf{F}_{1/b}(k))\right). \end{aligned}$$

As in the previous proof it can be seen that $G_{a/b}$ commutes with the diagonal action and then descends to a continuous map $g_{a/b} : \mathsf{S} \longrightarrow \mathsf{S}$. In fact, $g_{a/b} \equiv g_a \circ g_b^{-1}$ and since g_a and g_b commute: $g_b \circ g_{a/b} \equiv g_a$; that is,

$$g_{a/b} \equiv g_b^{-1} \circ g_a \equiv g_a \circ g_b^{-1}.$$

Moreover, given any $c, d \in \mathbb{N}$, define $g_{c/d} \equiv g_{a/b}$, with a, b coprimes or a/b the reduced fraction of c/d. Consequently, given $a, b, c, d \in \mathbb{N}$ we have that

$$g_{a/b} \circ g_{c/d} \equiv g_{ac/bd}, \qquad g_{a/b}^{-1} \equiv g_{b/a}.$$

Using that Char(S) $\cong \mathbb{Q}$ and the evaluation homomorphism, it is easy to see that

$$\mathrm{Aut}(\mathsf{S}) \cong \mathrm{Aut}(\mathbb{Q}) \cong \mathbb{Q}^*.$$

Therefore,

$$\mathrm{Aut}(\mathsf{S}) = \{g_{a/b},\ r \circ g_{a/b} \ :\ a, b \in \mathbb{N}\}.$$

Proposition 2.4.8. *Given $a, b, c, d \in \mathbb{N}$, $g_{a/b} \sim g_{c/d}$ if and only if $a/b = c/d$.*

Proof. It is enough to prove that $g_{a/b} \sim \mathrm{id}$ implies $a/b = 1$.

Let $\delta_\lambda : \mathsf{S} \longrightarrow \mathsf{S}$ be an isotopy from $g_{a/b}$ to id. Take the unique lift

$$\widetilde{\delta_\lambda} : \mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathbb{R} \times \widehat{\mathbb{Z}},$$

with $\widetilde{\delta_0} \equiv G_{a/b}$. Then $\widetilde{\delta_1} \equiv \Delta_\gamma$ for some $\gamma \in \mathbb{Z}$.

Considering the projection onto the first coordinate $\mathrm{pr}_1 : \mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathbb{R}$, since S is compact the path $\lambda \longmapsto \delta_\lambda$ has a bounded displacement. Therefore, for the path lifting property and for every $(x, k) \in \mathbb{R} \times \widehat{\mathbb{Z}}$ there is a constant C such that

$$\begin{aligned} |\mathrm{pr}_1(\widetilde{\delta_0}(x,k)) - \mathrm{pr}_1(\widetilde{\delta_1}(x,k))| &= |(a/b)(x + k_b) - (x - \gamma)| \\ &= |(a/b - 1)x + (a/b)k_b + \gamma| < C. \end{aligned}$$

Since x can take every real value we conclude that $a/b = 1$.

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Corollary 2.4.9. $\Gamma_{\mathcal{L}_0} \cong \mathrm{Aut}(\mathsf{S})$.

Proof. Let $h \in \mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S})$ satisfying $h(0) = y \in \mathcal{L}_0$. Thus, $d_{-y} \circ h$ is a continuous function preserving the zero element; or equivalently, $d_{-y} \circ h \in \mathrm{C}_*(\mathsf{S})$. Therefore, using the results mentioned in the first section of the chapter, $d_{-y} \circ h \sim g_{a/b}$ for some $g_{a/b} \in \mathrm{Aut}(\mathsf{S})$.

However, $d_{-y} \sim \mathrm{id}$ implies that $h \sim g_{a/b}$.

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Finally, using the decomposition $\mathrm{Homeo}(\mathsf{S}) \cong \mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S}) \times_{\mathbb{Z}} \widehat{\mathbb{Z}}$ we can now conclude that the group of isotopy classes of homeomorphisms of S is given by

$$\Gamma = \mathrm{Homeo}(\mathsf{S})/\mathrm{Homeo}_+(\mathsf{S}) \cong \mathrm{Aut}(\mathsf{S}) \times (\widehat{\mathbb{Z}}/\mathbb{Z}).$$

2.5 Uniform perfectness of Homeo$_+$(S)

Let us recall that the group of homeomorphisms isotopic to the identity $\mathrm{Homeo}_+(\mathsf{S})$ is a simple and uniformly perfect group (see [AP]). We present the most important details of these proofs.

Using the box decomposition structure on S and the same ideas as Fischer (see [Fis]), $\mathrm{Homeo}_+(\mathsf{S})$ satisfies the *partition property.* That is, for every box cover $\{V_j\}_{j=1}^n$ of S and for any $f \in \mathrm{Homeo}_+(\mathsf{S})$, there exists a decomposition

$$f = g_1 \circ \ldots \circ g_l,$$

with $\{g_i\}_{i=1}^l \subset \mathrm{Homeo}_+(\mathsf{S})$ and $\mathrm{supp}(g_i) = \{z \in \mathsf{S} : g_i(z) \neq z\} \subset V_{j(i)}$, for every $i = 1, \ldots, l$.

Now, we mention a series of lemmas that in combination give us the property of uniform perfectness of $\mathrm{Homeo}_+(\mathsf{S})$. For all of them we start letting $f \in \mathrm{Homeo}_+(\mathsf{S})$.

The first one solves the problem of perfectness for homeomorphisms in $\mathrm{Homeo}_+(\mathsf{S})$ with support on a box.

Lemma 2.5.1. *If* $\mathrm{supp}(f)$ *is contained on a box* V *of* S*, then there exists a homeomorphism* $g \in \mathrm{Homeo}_+(\mathsf{S})$ *with* $\mathrm{supp}(g) \subset V$ *and* $gfg^{-1} = f^2$*.*

Thus, $f = [g, f]$.

Lemma 2.5.2. *Suppose that* $V_1 \simeq (\alpha_1, \beta_1) \times \widehat{\mathbb{Z}}$ *and* $V_2 \simeq (\alpha_2, \beta_2) \times \widehat{\mathbb{Z}}$ *are two boxes such that*

$$\overline{f(V_1) \cup V_1} \subset V_2.$$

Then there is a $g \in \mathrm{Homeo}_+(\mathsf{S})$ *with* $\mathrm{supp}(g) \subset V_2$ *and satisfying* $f|_{V_1} \equiv g|_{V_1}$*.*

Using this lemma we can derive the next one.

Lemma 2.5.3. *Given* $f \in \mathrm{Homeo}_+(\mathsf{S})$*, there exist two boxes* V_1 *and* V_2*,* $V_1 \subset V_2$ *and* $f_1, f_2 \in \mathrm{Homeo}_+(\mathsf{S})$ *with the properties:*

- $\mathrm{supp}(f_2) \subset V_2$,
- $f_1|_{V_1} \equiv \mathrm{id}|_{V_1}$,
- $f \equiv f_1 \circ f_2$.

We consider a topological lemma. Consider a box $(\alpha, \beta) \times \widehat{\mathbb{Z}}$ and $z \in \{\beta\} \times \widehat{\mathbb{Z}}$; the return time of z to $\{\alpha\} \times \widehat{\mathbb{Z}}$ is

$$\mathrm{t}_{\{\alpha\}\times\widehat{\mathbb{Z}}}(z) = \inf\{t > 0 \,:\, z + t \in \{\alpha\} \times \widehat{\mathbb{Z}}\}.$$

Due to the density of the base leaf $\mathcal{L}_0$ on S, this value is constant for any $z \in \{\beta\} \times \widehat{\mathbb{Z}}$; i.e. the map $\mathrm{t}_{\{\alpha\}\times\widehat{\mathbb{Z}}} : \{\beta\} \times \widehat{\mathbb{Z}} \longrightarrow \mathbb{R}$ is locally constant and hence continuous.

Lemma 2.5.4. *Let* $V \simeq (\alpha, \beta) \times \widehat{\mathbb{Z}}$ *be a box of* S *and*

$$Z = \left\{(z, t) \,:\, z \in \{\beta\} \times \widehat{\mathbb{Z}},\, 0 \le t \le \mathrm{t}_{\{\alpha\}\times\widehat{\mathbb{Z}}}(z)\right\}.$$

The following map is a homeomorphism:

$$\begin{aligned} Z &\longrightarrow \mathsf{S} \setminus V \\ (z, t) &\longmapsto z + t. \end{aligned}$$

Theorem 2.5.5. *The group* $\mathrm{Homeo}_+(\mathsf{S})$ *is uniformly perfect.*

Proof. Consider $f \in \mathrm{Homeo}_+(\mathsf{S})$. By lemma 2.5.3 there exist f_1, $f_2 \in \mathrm{Homeo}_+(\mathsf{S})$ and boxes $V_1 \subset V_2$ satisfying $\mathrm{supp}(f_2) \subset V_2$, $f|_{V_1} \equiv \mathrm{id}|_{V_1}$ and $f \equiv f_1 \circ f_2$.

Using lemma 2.5.4 for the box $V_1 \cong (\alpha, \beta) \times \widehat{\mathbb{Z}}$ and since the return time $\mathrm{t}_{\{\alpha\}\times\widehat{\mathbb{Z}}}$ is locally constant, there is a clopen partition $\{K_1, \ldots, K_l\}$ of $\widehat{\mathbb{Z}}$ with $\mathrm{t}_{\{a\}\times\widehat{\mathbb{Z}}}|_{K_i} = \mathrm{t_i}$ constant for any $i = 1, \ldots, l$ and the collection of closed boxes $\{[0, \mathrm{t_i}] \times_{\mathbb{Z}} K_i\}_{i=1}^{l}$ is a covering of $\mathsf{S} \setminus V_1$ with pairwise disjoint interiors.

Thus, f_1 preserves every box $[0, \mathrm{t_i}] \times_{\mathbb{Z}} K_i$ and can be written as $f_1 \equiv g_1 \circ \ldots \circ g_l$; where $g_i \in \mathrm{Homeo}_+(\mathsf{S})$ and $\mathrm{supp}(g_i) \subset [0, \mathrm{t_i}] \times_{\mathbb{Z}} K_i$ for every $i = 1, \ldots, l$. Hence, using lemma 2.5.1, f_2 is a commutator and any g_i is a commutator $[a_i, b_i]$, where $a_i, b_i \in \mathrm{Homeo}_+(\mathsf{S})$ satisfy $\mathrm{supp}(a_i), \mathrm{supp}(b_i) \subset [0, \mathrm{t_i}] \times_{\mathbb{Z}} K_i$. Moreover, since two homeomorphisms having disjoint supports commute, we have

$$f_1 \equiv g_1 \circ \ldots \circ g_l \equiv [a_1, b_1] \circ \ldots \circ [a_l, b_l] \equiv [a_1 \circ \ldots \circ a_l, b_1 \circ \ldots \circ b_l] .$$

Therefore, $f \equiv f_1 \circ f_2 \equiv [a_1 \circ \ldots \circ a_l, b_1 \circ \ldots \circ b_l] \circ f_2$.

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3. THE CONNECTED COMPONENT OF THE IDENTITY

In this chapter we present a complete study of the group $\mathrm{Homeo}_+(\mathsf{S})$ of homeomorphisms of the solenoid which are isotopic to the identity. Firstly, we want to know the homotopy type of $\mathrm{Homeo}_+(\mathsf{S})$ and for that we study lifts of elements in $\mathrm{Homeo}_+(\mathsf{S})$ to $\mathbb{R} \times \widehat{\mathbb{Z}}$ via the covering map $\Pi : \mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathsf{S}$.

Then, working with liftings on the covering and with the construction of a universal central extension, we will calculate the second cohomology group with coefficients in $\mathbb{Z}$. After that, we will also calculate the second bounded and related cohomology groups. The important idea is to emulate the theory previously presented for the circle, in order to find an analogue of the Thurston–Mather result (see **??**) for the particular case of the solenoid and to find an invariant of the dynamics as the rotation number for the case of $\mathbb{S}^1$.

The group $\mathrm{Homeo}_+(\mathsf{S})$ is a topological group with the compact–open topology or the k–topology of Arens ([Are]) and $\mathrm{Homeo}_+(\mathsf{S}) \cap \widehat{\mathbb{Z}} = \mathbb{Z}$; *i.e.* every integer translation over the fiber is isotopic to the identity, or

$$s_\gamma \equiv d_\gamma \qquad (\gamma \in \mathbb{Z});$$

with d_γ the translation by γ on the leaves and $d_\gamma \sim \mathrm{id}$.

Recall that $\Pi : \mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathsf{S}$ is the covering map induced by the diagonal action and the group of deck transformations $\Delta(\mathbb{Z})$ is defined as

$$\Delta(\mathbb{Z}) = \left\{ (x, k) \overset{\Delta_\gamma}{\longmapsto} (x - \gamma, k + \gamma) \, : \, \gamma \in \mathbb{Z} \right\};$$

equivalently, the maps $\Delta_\gamma \in \Delta(\mathbb{Z})$ are the only lifts that make the following diagram commutative:

$$\begin{array}{ccc} \mathbb{R} \times \widehat{\mathbb{Z}} & \xrightarrow{\Delta_\gamma} & \mathbb{R} \times \widehat{\mathbb{Z}} \\ \Pi \big\downarrow & & \big\downarrow \Pi \\ \mathsf{S} & \xrightarrow{\mathrm{id}} & \mathsf{S}. \end{array}$$

3.1 Lifts of elements in $\mathrm{Homeo}_+(\mathsf{S})$

Definition 3.1.1. *The* ***group of lifts of elements in*** $\mathrm{Homeo}_+(\mathsf{S})$ *is defined as*

$$\widetilde{\mathrm{Homeo}}_+(\mathsf{S}) = \left\{ F : \mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathbb{R} \times \widehat{\mathbb{Z}} \, : \, \Pi \circ F \equiv f \circ \Pi, \ f \in \mathrm{Homeo}_+(\mathsf{S}) \right\}.$$

It is a group by composition and $\mathbb{R}\times\widehat{\mathbb{Z}}$ is locally compact. However, $\mathbb{R}\times\widehat{\mathbb{Z}}$ is not compact and for that we need to put the g–topology defined by Arens on $\widetilde{\mathrm{Homeo}}_+(\mathsf{S})$ in order to make it a topological group (see [Are] or [Dij] for details). That is, the base of the topology for $\widetilde{\mathrm{Homeo}}_+(\mathsf{S})$ is defined by the following: consider K a closed subset and W an open subset of $\mathbb{R}\times\widehat{\mathbb{Z}}$; such that, either K or $\mathbb{R}\times\widehat{\mathbb{Z}}\setminus W$ is compact. Let

$$(K,W)=\left\{F\in\widetilde{\mathrm{Homeo}}_+(\mathsf{S})\ :\ F(K)\subset W\right\},$$

hence the elements of the basis are given by finite intersection of such sets (K,W).

Since S is compact, the g–topology associated to $\mathrm{Homeo}_+(\mathsf{S})$ coincides with the compact–open topology. Moreover, if we consider the covering $\Pi:\mathbb{R}\times\widehat{\mathbb{Z}}\longrightarrow\mathsf{S}$, then the g–topology associated to $\widetilde{\mathrm{Homeo}}_+(\mathsf{S})$ is compatible with the compact–open topology on $\mathrm{Homeo}_+(\mathsf{S})$. Therefore, the following constructions stay valid in the topological language.

Using the ideas of J. Kwapisz ([Kwa]), given a lift $F\in\widetilde{\mathrm{Homeo}}_+(\mathsf{S})$ of $f\in\mathrm{Homeo}_+(\mathsf{S})$, F can be written as:

$$\begin{aligned} F:\mathbb{R}\times\widehat{\mathbb{Z}} &\longrightarrow \mathbb{R}\times\widehat{\mathbb{Z}}\\ (x,k) &\longmapsto (x+\Phi(x,k),k+\alpha); \end{aligned}$$

where $\Phi:\mathbb{R}\times\widehat{\mathbb{Z}}\longrightarrow\mathbb{R}$ is a bounded continuous $\Delta(\mathbb{Z})$–periodic function and $\alpha\in\mathbb{Z}$ which we see as an element of $\widehat{\mathbb{Z}}$. Equivalently, the integer translation in $\widehat{\mathbb{Z}}$, $k\longmapsto k+\alpha$ is minimal and for every fixed $k\in\widehat{\mathbb{Z}}$, the function with real values

$$x\longmapsto x+\Phi_k(x)=x+\Phi(x,k),$$

has limit–periodic displacement. Moreover, this continuous map is strictly increasing and the image $x+\Phi(x,k)$ goes to $\pm\infty$ as $x\longrightarrow\pm\infty$. Thus, the set of functions of the form

$$(x,k)\longmapsto(x+\Phi(x,k),k)$$

is a convex subset of $\widetilde{\mathrm{Homeo}}_+(\mathsf{S})$.

Consider a bounded continuous $\Delta(\mathbb{Z})$–periodic function $\Phi:\mathbb{R}\times\widehat{\mathbb{Z}}\longrightarrow\mathbb{R}$. Then there exists a continuous function $\tilde{\phi}:\mathsf{S}\longrightarrow\mathbb{R}$ which makes the diagram

$$\begin{array}{ccc} \mathbb{R}\times\widehat{\mathbb{Z}} & \xrightarrow{\Phi} & \mathbb{R}\\ {\scriptstyle\Pi}\downarrow & \nearrow_{\tilde{\phi}} & \\ \mathsf{S}. & & \end{array}$$

commutative. Conversely, for every $\tilde{\phi}:\mathsf{S}\longrightarrow\mathbb{R}$, there exists such a $\Phi:\mathbb{R}\times\widehat{\mathbb{Z}}\longrightarrow\mathbb{R}$ given by $\Phi\equiv\tilde{\phi}\circ\Pi$. In particular, $\tilde{\phi}$ is the displacement function of f and there is a continuous function $\phi:\mathsf{S}\longrightarrow\mathsf{S}$, satisfying $\phi(\mathsf{S})\subset\mathcal{L}_0$ and completing the above diagram:

$$\begin{array}{ccc} \mathbb{R}\times\widehat{\mathbb{Z}} & \xrightarrow{\Phi} & \mathbb{R}\\ {\scriptstyle\Pi}\downarrow & \nearrow_{\tilde{\phi}} & \downarrow{\scriptstyle P}\\ \mathsf{S} & \xrightarrow[\phi]{} & \mathsf{S}. \end{array}$$

Therefore, we have a well defined map

$$\begin{aligned} \mathfrak{p} : \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) &\longrightarrow \mathrm{Homeo}_+(\mathsf{S}) \\ F &\longmapsto s_\alpha \circ (\mathrm{id} + \phi). \end{aligned}$$

It is plain that $\mathfrak{p}$ is a continuous homomorphism whose kernel is identified with the deck transformation group $\Delta(\mathbb{Z}) \simeq \mathbb{Z}$. For continuity it is enough to recall the descriptions on the basic sets of the topologies of $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})$ and $\mathrm{Homeo}_+(\mathsf{S})$ respectively; i.e. the compatibility with respect to the covering $\Pi : \mathbb{R} \times \widehat{\mathbb{Z}} \to \mathsf{S}$.

Consequently, there is an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) \xrightarrow{\mathfrak{p}} \mathrm{Homeo}_+(\mathsf{S}) \longrightarrow 1.$$

In fact, this sequence represents a central extension and $\Delta(\mathbb{Z})$ coincides with the center of the group $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})$; equivalently, for every $\gamma \in \mathbb{Z}$ and $F \in \widetilde{\mathrm{Homeo}_+}(\mathsf{S})$:

$$\begin{aligned} \Delta_\gamma(F(x,k)) &= \Delta_\gamma(x + \Phi(x,k), k + \alpha) \\ &= (x + \Phi(x,k) - \gamma, k + \alpha + \gamma) \\ &= (x - \gamma + \Phi(x,k), k + \gamma + \alpha) \\ &= (x - \gamma + \Phi(x - \gamma, k + \gamma), k + \gamma + \alpha) \\ &= F(x - \gamma, k + \gamma) = F(\Delta_\gamma(x,k)), \qquad ((x,k) \in \mathbb{R} \times \widehat{\mathbb{Z}}). \end{aligned}$$

3.1.1 Limit–periodic displacements

Suppose that $\Phi : \mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathbb{R}$ is a bounded continuous and $\Delta(\mathbb{Z})$–periodic function; *i.e.* it satisfies the relation

$$\Phi(x - \gamma, k + \gamma) = \Phi(x,k), \qquad (\gamma \in \mathbb{Z}).$$

Fix $k \in \widehat{\mathbb{Z}}$ and consider the continuous real valued function $\Phi_k = \Phi(\cdot, k) : \mathbb{R} \longrightarrow \mathbb{R}$. We are interested in the particular case when Φ_k is a limit–periodic function, whose convex hull is homeomorphic to S with respect to the compact–open topology in the Banach space of continuous real functions ([AS]).

Following Bohr's ideas (see [Boh] or [Bes], [Cor]), the mean value of Φ_k exists and is defined by

$$\mathrm{M}\{\Phi_k\} := \lim_{X \longrightarrow \infty} \frac{1}{X} \int_{-X/2}^{X/2} \Phi_k(x) dx.$$

Also, by the properties of the mean value we have that $\mathrm{M}\{\Phi_k \circ \mathrm{T}_y\} = \mathrm{M}\{\Phi_k\}$, where $\mathrm{T}_y : \mathbb{R} \longrightarrow \mathbb{R}$ is the translation by $y \in \mathbb{R}$, $\mathrm{T}_y(x) = x + y$. Our interest lies in the particular case of integer translations; that is,

$$\mathrm{M}\{\Phi_k \circ \mathrm{T}_\gamma\} = \mathrm{M}\{\Phi_k\}, \qquad (\gamma \in \mathbb{Z}).$$

Since, $\Phi : \mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathbb{R}$ is $\Delta(\mathbb{Z})$–periodic, Φ_k satisfies the equivariance condition

$$\Phi_k(x + \gamma) = \Phi_{k+\gamma}(x), \qquad (\gamma \in \mathbb{Z}).$$

Consequently, for every $k \in \widehat{\mathbb{Z}}$,

$$\mathrm{M}\{\Phi_{k+\gamma}\} = \mathrm{M}\{\Phi_k\} \qquad (\gamma \in \mathbb{Z}).$$

Lemma 3.1.2. *For every* $k \in \widehat{\mathbb{Z}}$, $\mathrm{M}\{\Phi_k\} = \mathrm{M}\{\Phi_0\}$.

Proof. If $\gamma \in \mathbb{Z}$, then $\mathrm{M}\{\Phi_0\} = \mathrm{M}\{\Phi_\gamma\}$. Let $k \in \widehat{\mathbb{Z}}$ be given and $\{\gamma_n\}_{n\in\mathbb{Z}} \subset \widehat{\mathbb{Z}}$ be a sequence of integers which converges uniformly to k. Hence, due to the continuity of the assignation $k \longmapsto \Phi_k$, the sequence of functions $\{\Phi_{\gamma_n}\}$ converges uniformly to the function Φ_k and therefore

$$\mathrm{M}\{\Phi_0\} = \lim \mathrm{M}\{\Phi_{\gamma_n}\} = \mathrm{M}\{\Phi_k\};$$

where the last equality is true because of the basic properties of the mean value.

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It is an immediate consequence of this result that the behaviour of the original function $\Phi : \mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathbb{R}$ is completely determined by its mean value over the marked leaf $\mathbb{R} \times \{0\}$; that is, Φ can be written in a unique way as

$$\Phi(x,k) = \mathrm{M}\{\Phi_0\} + \Psi(x,k);$$

with $\Psi : \mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathbb{R}$ a bounded continuous and $\Delta(\mathbb{Z})$–periodic function, $\Psi \equiv \Phi - \mathrm{M}\{\Phi_0\}$ and for each $k \in \widehat{\mathbb{Z}}$, $\mathrm{M}\{\Psi_k\} = 0$.

Recall that using the covering $\Pi : \mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathsf{S}$, the base leaf $\mathcal{L}_0$ of S is canonically identified with $\Pi(\mathbb{R} \times \mathbb{Z}) = \mathbb{R} \times_{\mathbb{Z}} \mathbb{Z}$.

Theorem 3.1.3. *The inclusion by translations on the base leaf*

$$\mathbb{R} \times_{\mathbb{Z}} \mathbb{Z} \hookrightarrow \mathrm{Homeo}_+(\mathsf{S})$$

is a homotopy equivalence.

Proof. Take $f \in \mathrm{Homeo}_+(\mathsf{S})$ and $F \in \widetilde{\mathrm{Homeo}_+}(\mathsf{S})$ a lift of the form

$$F(x,k) = (x + \Phi(x,k), k + \alpha), \qquad (\alpha \in \mathbb{Z}).$$

By the above remarks Φ can be written as $\Phi \equiv \mathrm{M}\{\Phi_0\} + \Psi$, where $\Psi_k : \mathbb{R} \longrightarrow \mathbb{R}$ has zero mean value for each $k \in \widehat{\mathbb{Z}}$.

We can define an homotopy through the functions

$$F_s(x,k) = (x + \mathrm{M}\{\Phi_0\} + (1-s)\Psi(x,k), k + \alpha), \quad 0 \le s \le 1.$$

That is, for every $s \in [0,1]$, we have that $F_s \in \widetilde{\mathrm{Homeo}_+}(\mathsf{S})$. Indeed, as we saw previously for any $\alpha \in \mathbb{Z}$, the subset of maps of the form

$$(x,k) \longmapsto (x + \Phi(x,k), k + \alpha)$$

is a convex subset of $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})$ and also the mean values $(1-s)\mathrm{M}\{\Psi_k\} = 0$ for every $s \in [0,1]$ by Bohr's general theory.

Observe also that $F_0 \equiv F$, $F_1(x,k) = (x+\mathrm{M}\{\Phi_0\}, k+\alpha)$ and we obtain a continuous retraction of $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})$ onto the subgroup of translations isomorphic to $\mathbb{R}\times\mathbb{Z}$. Moreover, because for each $s \in [0,1]$, F_s commutes with the elements of the group of deck transformations $\Delta(\mathbb{Z})$, we can assert that this is a continuous deformation of the quotient

$$\widetilde{\mathrm{Homeo}_+}(\mathsf{S})/\Delta(\mathbb{Z}) \simeq \mathrm{Homeo}_+(\mathsf{S})$$

onto the subgroup of translations over the base leaf $\mathcal{L}_0 \cong \mathbb{R}\times_{\mathbb{Z}}\mathbb{Z}$.

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The last argument shows that

$$\mathrm{Homeo}_+(\mathsf{S}) \simeq \mathcal{L}_0 \times \mathrm{H}_{\mathrm{id}}(\mathsf{S});$$

where $\mathrm{H}_{\mathrm{id}}(\mathsf{S})$ is a contractible convex set.

Remark 3.1.4. *Every homotopy group* $\pi_n(\mathrm{Homeo}_+(\mathsf{S}))$, $n \geq 1$ *is trivial.*

3.2 Cohomology groups of $\mathrm{Homeo}_+(\mathsf{S})$

In view of the last result, we are tempted to conclude that the groups $\mathrm{H}^n(\mathrm{Homeo}_+(\mathsf{S}),\mathbb{Z})$, for $n \geq 1$ all vanish. However, since S is not a manifold we can not use the Thurston–Mather theorem. In the first part of this section we are going to calculate the second cohomology group of $\mathrm{Homeo}_+(\mathsf{S})$ with integer coefficients. After that, using some results of bounded cohomology we will be able to calculate cohomology groups of other related groups. And we construct an invariant analogous to the Poincaré rotation number for the solenoid.

3.2.1 The second cohomology group and the Euler class

Consider the central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) \xrightarrow{\mathfrak{p}} \mathrm{Homeo}_+(\mathsf{S}) \longrightarrow 1.$$

By theorem2.5.5, $\mathrm{Homeo}_+(\mathsf{S})$ is uniformly perfect; in fact, every element of $\mathrm{Homeo}_+(\mathsf{S})$ can be written as the product of two commutators. Thus, from Theorem 1.2.1, there exists a universal central extension. Moreover, for this extension the kernel can be mapped surjectively with $\mathbb{Z}$.

For now, we focus our attention on lifts F of the form

$$(x,k) \longmapsto (x+\Phi(x,k),k).$$

As we already noted, for every fixed $k \in \widehat{\mathbb{Z}}$, the map

$$F_k(x) := x + \Phi_k(x),$$

has limit periodic displacement and is strictly increasing. Moreover, F_k is a real valued homeomorphism and using the equivariant condition of Φ_k, for every $\gamma \in \mathbb{Z}$

$$F_{k+\gamma}(x) + \gamma = F_k(x+\gamma).$$

Equivalently, F_k and $F_{k+\gamma}$ are conjugated by a translation by γ in $\mathbb{R}$.

If $k \in \widehat{\mathbb{Z}}$, consider the following subgroup of real valued homeomorphisms:

$$\widetilde{\mathrm{Homeo}_+}(\mathsf{S})_k = \left\{ F_k : \mathbb{R} \longrightarrow \mathbb{R} : F \in \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) \right\}.$$

As we will see in the proof of the next lemma, $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})_k$ is a subgroup of the compactly supported homeomorphisms group $\mathrm{Homeo}_C(\mathbb{R})$.

Lemma 3.2.1. $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})_k$ *is uniformly perfect.*

Proof. From the proof of Theorem 2.5.5, we saw that every $f \in \mathrm{Homeo}_+(\mathsf{S})$ can be written as a product of two commutators, say f_1 and f_2 in $\mathrm{Homeo}_+(\mathsf{S})$. Moreover, there is a finite collection of boxes $\{[0, \mathrm{t_i}] \times_{\mathbb{Z}} K_i\}_{i=1}^{l}$; where $\{K_i\}$ is a finite partition of the fiber $\widehat{\mathbb{Z}}$, this collection forms a cover of the support of f_2 and from the partition property

$$f_1 = g_1 \circ \ldots \circ g_l,$$

with $\mathrm{supp}(g_i) \subset [0, \mathrm{t_i}] \times_{\mathbb{Z}} K_i$.

Hence, take $F(x,k) = (x + \Phi(x,k), k)$ a lift of f. Using the diagonal action, for every fixed $k \in \widehat{\mathbb{Z}}$, there is at most a finite collection of compact sets

$$\{[\gamma_i, \mathrm{t_i} + \gamma_i] : \gamma_i \in \mathbb{Z}\}_{i=1}^{l};$$

with

$$\mathrm{supp}(F_k) \subset \bigcup_{i=1}^{l} [\gamma_i, \mathrm{t_i} + \gamma_i].$$

That is, taking the lifts of the boxes $[0, \mathrm{t_i}] \times_{\mathbb{Z}} K_i$ to $\mathbb{R} \times \widehat{\mathbb{Z}}$, we only consider the closed intervals that intersect the leaf $\mathbb{R} \times \{k\}$ as the ones that support the homeomorphism $F_k : \mathbb{R} \longrightarrow \mathbb{R}$.

Thus, for every $k \in \widehat{\mathbb{Z}}$, $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})_k$ is contained on the group of homeomorphisms of $\mathbb{R}$ with compact support. That is, as a consequence of the result in [Math]

$$\mathrm{H}_1\left(\widetilde{\mathrm{Homeo}_+}(\mathsf{S})_k, \mathbb{Z}\right) = 0;$$

or equivalently, $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})_k$ is perfect. Moreover, every $F_k \in \widetilde{\mathrm{Homeo}_+}(\mathsf{S})_k$ can be written as a product of two commutators, say

$$F_k \equiv F_{k,1} \circ F_{k,2};$$

with $F_{k,1}, F_{k,2} \in \widetilde{\mathrm{Homeo}_+}(\mathsf{S})_k$ being represented by the lifts $F_1, F_2 \in \widetilde{\mathrm{Homeo}_+}(\mathsf{S})$ of f_1, f_2.

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In particular, for every $k \in \widehat{\mathbb{Z}}$, $\mathrm{H}_1\left(\widetilde{\mathrm{Homeo}_+}(\mathsf{S})_k, \mathbb{Z}\right) = 0$. Moreover, because of every element of $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})_k$ being compactly supported, following Mather ([Math]) we have that the homology groups $\mathrm{H}_n\left(\widetilde{\mathrm{Homeo}_+}(\mathsf{S})_k, \mathbb{Z}\right)$ vanish for all $n \geq 1$.

Theorem 3.2.2. *The exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) \xrightarrow{\mathfrak{p}} \mathrm{Homeo}_+(\mathsf{S}) \longrightarrow 1,$$

is the universal central extension.

Proof. For every $k \in \widehat{\mathbb{Z}}$, let Φ_k be a bounded continuous limit periodic function of the real line. Consider the map $\widehat{\mathbb{Z}} \longrightarrow \mathrm{C}_b^{lp}(\mathbb{R})$ defined as $k \longmapsto \Phi_k$. Using the exponential map, we know that this is a continuous correspondence.

Hence, the map $\widehat{\mathbb{Z}} \longrightarrow \mathrm{Homeo}(\mathbb{R})$, $k \longmapsto F_k$ is also continuous. Moreover, F_k has constant mean value for the displacement Φ_k; that is

$$\mathrm{M}\{\Phi_k\} = \mathrm{M}\{\Phi_0\},$$

with Φ_0 the displacement of F_0.

Thus, we have a continuous map $\widehat{\mathbb{Z}} \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathsf{S})_k$ such that

$$\mathrm{H}_n\left(\widetilde{\mathrm{Homeo}_+}(\mathsf{S})_k, \mathbb{Z}\right) = 0, \qquad (n \geq 1),$$

as $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})_k$ consists only of real–valued homeomorphisms with compact support. Consequently, using also that $F_{k+\gamma}$ and F_k are conjugated via a translation by $\gamma \in \mathbb{Z}$, we can now assert that $\mathrm{H}_n\left(\widetilde{\mathrm{Homeo}_+}(\mathsf{S}), \mathbb{Z}\right) = 0$ for every $n \geq 1$.

In particular,

$$\mathrm{H}_1\left(\widetilde{\mathrm{Homeo}_+}(\mathsf{S}), \mathbb{Z}\right) = \mathrm{H}_2\left(\widetilde{\mathrm{Homeo}_+}(\mathsf{S}), \mathbb{Z}\right) = 0$$

and as a consequence of Corollary 1.2.3,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) \xrightarrow{\mathfrak{p}} \mathrm{Homeo}_+(\mathsf{S}) \longrightarrow 1$$

is the universal central extension. That is, we know that every $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})_k$ is perfect and that every central exact sequence of $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})$ by $\mathbb{Z}$ must split.

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Thus, we have the Schur multiplier

$$\mathrm{H}_2(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z}) \simeq \mathbb{Z}.$$

Moreover, using Theorem 1.2.4 we have the cohomology groups

$$\mathrm{H}^2(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z}) \simeq \mathrm{Hom}(\mathrm{H}_2(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z}), \mathbb{Z}) \simeq \mathbb{Z},$$

and $\mathrm{H}^2(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{R}) \simeq \mathbb{R}$. Let eu and $eu^{\mathbb{R}}$ be the respective Euler classes.

3.2.2 Bounded cohomology groups

Consider the long exact sequence for the bounded cohomology groups of $\mathrm{Homeo}_+(\mathsf{S})$ derived from the exponential sequence:

$$\cdots \longrightarrow \mathrm{H}^1_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z}) \longrightarrow \mathrm{H}^1_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{R}) \longrightarrow \mathrm{H}^1_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{S}^1) \longrightarrow$$
$$\longrightarrow \mathrm{H}^2_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z}) \longrightarrow \mathrm{H}^2_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{R}) \longrightarrow \cdots$$

In particular, the map

$$\mathrm{H}^2_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z}) \longrightarrow \mathrm{H}^2_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{R})$$

is injective because of its kernel is isomorphic to

$$\frac{\mathrm{H}^1(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{R})}{\mathrm{H}^1(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z})},$$

which turns out to vanish because of $\mathrm{Homeo}_+(\mathsf{S})$ being perfect (see [Ghy] for details). Moreover, using the same property of $\mathrm{Homeo}_+(\mathsf{S})$ the comparison map

$$i_{\mathbb{R}} : \mathrm{H}^2_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{R}) \longrightarrow \mathrm{H}^2(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{R}) \simeq \mathbb{R}$$

is also injective because of the absence of non–trivial quasimorphisms in $\mathrm{Homeo}_+(\mathsf{S})$.

As we would expect $\mathrm{H}^2_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{R})$ is isomorphic to $\mathbb{R}$. Indeed, consider a lift F in $\widetilde{\mathrm{Homeo}}_+(\mathsf{S})$ and without loss of generality we assume that F has the form

$$F(x,k) = (x + \Phi(x,k), k), \qquad \left((x,k) \in \mathbb{R} \times \widehat{\mathbb{Z}}\right).$$

We are considering that the lift does not move the fiber and this will be enough due to the equivariance condition

$$F_{k+\gamma}(x) + \gamma = F_k(x+\gamma) \qquad (\gamma \in \mathbb{Z}).$$

From the work of A. Verjovsky and M. Cruz–López ([CV]), we know that there is a rotation element

$$\rho : \mathrm{Homeo}_+(\mathsf{S}) \longrightarrow \mathsf{S}.$$

Given $f \in \mathrm{Homeo}_+(\mathsf{S})$ take a lift of $\rho(f)$ to the covering $\mathbb{R} \times \widehat{\mathbb{Z}}$. This give us an element $\tau \in \mathbb{R} \times \widehat{\mathbb{Z}}$. Moreover, using that the image of ρ belongs to the base leaf $\mathcal{L}_0$, we can fix a height on the covering and describe a homogeneous quasimorphism

$$\mathrm{T} : \widetilde{\mathrm{Homeo}}_+(\mathsf{S}) \longrightarrow \mathbb{R}$$

which is independent of the choice of$(x,k) \in \mathbb{R} \times \widehat{\mathbb{Z}}$.

Remark 3.2.3. *There is a way of defining this homogeneous quasimorphism by means of the displacement function*

$$F(x,k) - (x,k) = (\Phi(x,k), 0).$$

That is the content of [LR].

Therefore, we have a quasi–corner $C(\mathrm{T}, \mathfrak{p})$:

$$\begin{array}{ccc} \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) & \xrightarrow{\mathrm{T}} & \mathbb{R} \\ \big\downarrow{\scriptstyle \mathfrak{p}} & & \\ \mathrm{Homeo}_+(\mathsf{S}). & & \end{array}$$

Consider $\mathfrak{p}^* : \mathrm{H}^2_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{R}) \longrightarrow \mathrm{H}^2_b\left(\widetilde{\mathrm{Homeo}_+}(\mathsf{S}), \mathbb{R}\right)$ and let d^1_b be the first bounded coboundary operator. As we saw previously (see section 1.3), the set of classes of quasi–corners $\mathcal{QC}(\mathrm{Homeo}_+(\mathsf{S}))$ is in a bijective correspondence with $\mathrm{H}^2_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{R})$. We show that the class $[C(-\mathrm{T}, \mathfrak{p})]$ in $\mathcal{QC}(\mathrm{Homeo}_+(\mathsf{S}))$ defined by the cocycle $(\mathfrak{p}^*)^{-1}(d^1_b(-\mathrm{T}))$, represents the bounded cohomology class

$$eu^{\mathbb{R}}_b \in \mathrm{H}^2_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{R})$$

and consequently

$$\mathrm{H}^2_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{R}) \simeq \mathbb{R}.$$

Specifically, $i_{\mathbb{R}}(eu^{\mathbb{R}}_b) = eu^{\mathbb{R}} \in \mathrm{H}^2(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{R})$.

Proposition 3.2.4. *If* $\mathfrak{p} : \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) \longrightarrow \mathrm{Homeo}_+(\mathsf{S})$ *is the canonical projection, then* $eu^{\mathbb{R}}_b$ *is represented by the quasi–corner* $C(-\mathrm{T}, \mathfrak{p})$.

Proof. Define the following operator

$$\begin{aligned} \Delta : \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) &\longrightarrow \Delta(\mathbb{Z}) \\ F &\longmapsto (\gamma, -\gamma); \end{aligned}$$

where $\gamma = \lfloor \Phi(0,0) \rfloor \in \mathbb{Z}$.

Observe that

$$(\mathrm{T}_{(0,0)}(F), 0) - \Delta(F) \in [0,1) \times \widehat{\mathbb{Z}}.$$

Thus, the difference is bounded. Equivalently, we can work with $\Delta(F)$ instead of $\mathrm{T}(F)$.

Our aim is to prove that

$$\mathfrak{p}^*(eu_\sigma) = -d^1\Delta,$$

where the section σ is such that $\sigma(f)(0,0) \in [0,1) \times \widehat{\mathbb{Z}}$.

Define $\bar{F} \in \widetilde{\mathrm{Homeo}_+}(\mathsf{S})$ by

$$\bar{F} = F - \Delta(F).$$

Thus, for every two $F, G \in \widetilde{\mathrm{Homeo}_+}(\mathsf{S})$ we have that

$$
\begin{aligned}
\mathfrak{p}^*(eu_\sigma)(F,G) &= \sigma(\mathfrak{p}(F)\circ\mathfrak{p}(G))^{-1}\circ\sigma(\mathfrak{p}(F))\circ\sigma(\mathfrak{p}(G))\\
&= \left(\overline{(F\circ G)^{-1}}\circ\bar{F}\circ\bar{G}\right)(0,0)\\
&= \left((\overline{(F\circ G)^{-1}}\circ\bar{F})\circ(G-\Delta(G))\right)(0,0)\\
&= \left(\overline{(F\circ G)^{-1}}\circ(F\circ G-\Delta(G)-\Delta(F))\right)(0,0)\\
&= (0,0)-\Delta(G)-\Delta(F)-\Delta((F\circ G)^{-1})\\
&= \Delta(F\circ G)-\Delta(G)-\Delta(F)\\
&= -d^1\Delta(F,G).
\end{aligned}
$$

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Consider the obstruction cocycle

$$c(f,g)=\sigma(f\circ g)^{-1}\circ\sigma(f)\circ\sigma(g).$$

Note that the lifts $\sigma(f\circ g)$ and $\sigma(f)\circ\sigma(g)$ differ by a deck transformation $\Delta_\gamma\in\Delta(\mathbb{Z})$ for some $\gamma\in\mathbb{Z}$, because both are lifts of the same element $f\circ g\in\mathrm{Homeo}_+(\mathsf{S})$. That is,

$$\sigma(f\circ g)\circ\Delta_\gamma\equiv\sigma(f)\circ\sigma(g).$$

Define $eu:\mathrm{Homeo}_+(\mathsf{S})^2\longrightarrow\mathbb{Z}$ as $eu(f,g)=\gamma$.

Consider the specific normalized section $\sigma:\mathrm{Homeo}_+(\mathsf{S})\longrightarrow\widetilde{\mathrm{Homeo}_+}(\mathsf{S})$, such that for every $f\in\mathrm{Homeo}_+(\mathsf{S})$,

$$\sigma(f)(0,0)\in[0,1)\times\widehat{\mathbb{Z}}.$$

In particular,

$$\sigma(f\circ g)(\gamma,-\gamma)\in[\gamma,\gamma+1)\times\widehat{\mathbb{Z}}.$$

Thus, $\sigma(g)(0,0)\in[0,1)\times\widehat{\mathbb{Z}}$ and because in particular $\sigma(f)(1,-1)\in[1,2)\times\widehat{\mathbb{Z}}$, we conclude that

$$\sigma(f)\circ\sigma(g)(0,0)\in[0,2)\times\widehat{\mathbb{Z}}.$$

Therefore γ is equal to 0 or 1.

Consequently, the Euler class $eu\in\mathrm{H}^2(\mathrm{Homeo}_+(\mathsf{S}),\mathbb{Z})$ associated to $c(f,g)$ is bounded and

$$\mathrm{H}^2_b(\mathrm{Homeo}_+(\mathsf{S}),\mathbb{Z})\simeq\mathbb{Z}.$$

Define this class as $eu_b=[c(f,g)]\in\mathrm{H}^2_b(\mathrm{Homeo}_+(\mathsf{S}),\mathbb{Z})$.

Consider $f\in\mathrm{Homeo}_+(\mathsf{S})$ and $F\in\widetilde{\mathrm{Homeo}_+}(\mathsf{S})$ a lift. Let $\varrho:\mathrm{Homeo}_+(\mathsf{S})\longrightarrow\mathbb{S}^1$ be defined by the canonical projection of $\mathrm{T}(F)$ onto $\mathbb{S}^1$. We just proved the following.

Theorem 3.2.5. *Let $\vartheta:\mathbb{Z}\longrightarrow\mathrm{Homeo}_+(\mathsf{S})$ be a homomorphism. The class $\vartheta^*(eu_b)\in\mathrm{H}^2_b(\mathbb{Z},\mathbb{Z})$ is the number $\varrho(\vartheta(1))$.*

Moreover, the rotation element $\rho:\mathrm{Homeo}_+(\mathsf{S})\longrightarrow\mathsf{S}$ is founded in the group $\mathrm{H}^2_b(\mathbb{Q},\mathbb{Z})$.

Lemma 3.2.6. $\mathrm{H}_b^2(\mathbb{Q},\mathbb{Z}) \simeq \mathsf{S}$.

Proof. Let

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{S}^1 \longrightarrow 1$$

be the exponential exact sequence and construct the associated long exact sequence in bounded cohomology

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{S}^1 \longrightarrow \mathrm{H}_b^1(\mathbb{Q},\mathbb{Z}) \longrightarrow \mathrm{H}_b^1(\mathbb{Q},\mathbb{R}) \longrightarrow \mathrm{H}_b^1(\mathbb{Q},\mathbb{S}^1) \longrightarrow$$
$$\longrightarrow \mathrm{H}_b^2(\mathbb{Q},\mathbb{Z}) \longrightarrow \mathrm{H}_b^2(\mathbb{Q},\mathbb{R}) \longrightarrow \mathrm{H}_b^2(\mathbb{Q},\mathbb{S}^1) \longrightarrow \cdots .$$

Using that $\mathbb{Q}$ is Abelian and therefore is an amenable group, $\mathrm{H}_b^n(\mathbb{Q},\mathbb{R}) = 0$ for every $n \geq 1$. Calculating these groups for $n = 1$ and $n = 2$ we obtain that

$$\mathrm{H}_b^2(\mathbb{Q},\mathbb{Z}) \simeq \mathrm{H}_b^1(\mathbb{Q},\mathbb{S}^1) \simeq \mathrm{Hom}(\mathbb{Q},\mathbb{S}^1) \simeq \mathsf{S}.$$

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Theorem 3.2.7. *Let* $\varphi : \mathbb{Q} \longrightarrow \mathrm{Homeo}_+(\mathsf{S})$ *be a homomorphism. The class* $\varphi^*(eu_b) \in \mathrm{H}_b^2(\mathbb{Q},\mathbb{Z})$ *is the rotation element of* $\varphi(1)$.

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