Centro de Investigación en Matemáticas A.C

CIMAT

PREDICTING THE LAST ZERO OF A SPECTRALLY NEGATIVE LÉVY PROCESS

T E S I S

QUE PARA OBTENER EL GRADO DE:
MAESTRO EN CIENCIAS CON ESPECIALIDAD EN PROBABILIDAD Y ESTADÍSTICA

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Introduction

Every decision we make in daily life is risky business. Then selecting the best time to stop and act is crucial. A decision maker observes a process evolving in time that involves some randomness. Based only on what is known, we must make a decision on how to maximise reward or minimise cost.

More formally, the theory of optimal stopping is concerned with the problem of choosing a time to take a given action based on sequentially observed random variables in order to maximise an expected pay-off or to minimise an expected cost. Problems of this type have applications in particular in the following areas:

1. Statistics: The action may to be to test a hypothesis or to find a parameter as quickly and accurately as possible.
2. Quickest detection problem: When a natural phenomenon threatens to destroy a town, one needs to decide when to send out an alarm to avoid disaster based on observable data.
3. Operation research: Decide when it is optimal to replace a machine, hire a secretary, or reorder stock.
4. Finance: Establish the non-arbitrage price of an American option.

Lévy processes are the continuous time version of random walks and form a wide class of stochastic processes. Their applications appear in many areas of classical and modern stochastic processes, including storage models, renewal processes, insurance risk models, optimal stopping problems and mathematical finance. In particular, a special class of Lévy process called spectrally negative Lévy processes, which are Lévy processes with only negative jumps, plays a central role in risk theory and degradation models.

Consider the classical risk process (also known as the Cramér–Lundberg process) which consists of a deterministic, positive drift plus a compound Poisson process with only negative jumps (see Figure 1). This process models the capital of an insurance company. The drift may be viewed as a premium rate which is continuously collected and the compound Poisson process represents the claims made to the insurance company. A quantity of interest is the moment of ruin, i.e. the first time that the company has negative capital. Instead of going bankrupt when the risk process becomes negative, suppose that the company has funds to support the negative capital for a while. Then another quantity of interest is the last time that the process is below the level zero. This approach can be extended to a general spectrally negative Lévy process.

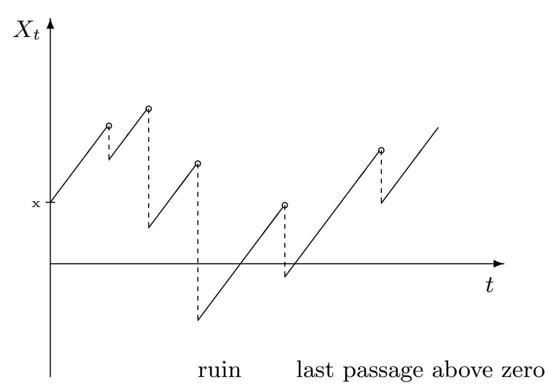

Figure 1: Cramér–Lundberg process.

For several decades, degradation data have been used to understand ageing of a device, instead of only failure data. Lévy processes turn out to be a useful tool for degradation models (see Figure 2). In particular there are three models that are mainly used: Brownian motion with positive drift, gamma process and compound Poisson process (see Park and Padgett (2005)). More generally we may consider a spectrally positive Lévy process. The failure time of a component or system can traditionally be derived from a degradation model by considering the first hitting time of a critical level. Recently a new approach is considered as a failure time (see Barker and Newby (2009)), by considering the last passage of the degradation process any critical level previously established.

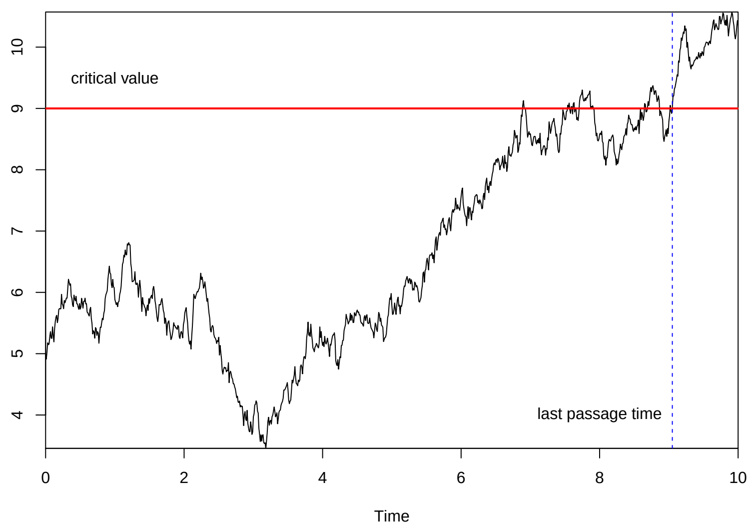

Figure 2: Degradation model.

The examples that we mentioned above show that last passage times play an important role in applications of spectrally negative Lévy processes. To know the value of a last passage time it is necessary to be able to observe the whole process.

Stopping times are random times such that the decision whether to stop or not depends only on the past and present information. Due to this fact, the aim of this work is to predict the last zero by a stopping time. This is, find a stopping time which is as close as possible to the last time that a spectrally negative Lévy process is equal to zero.

In the first chapter we present the general theory of Lévy processes. First, we mention the relation between infinitely divisible distributions and Lévy processes. Next, we explain briefly the Lévy–Itô decomposition. Then, we give general properties of general Lévy process and some results in the spectrally negative case. We define the scale functions for spectrally negative Lévy process and give important properties which will be useful in Chapter 3.

In the second chapter we give some aspects of the theory of optimal stopping. We present the concept of essential supremum. With this, we define the Snell envelope which is the fundamental tool for the martingale approach for solving optimal stopping problems. We then move to the Markovian approach. In this part we take advantage of the Markov property and the martingale approach to give more general results.

In the third chapter the optimal prediction problem, which is the main object of study of this work, is formulated and solved. We prove that the problem can be reduced to a standard optimal stopping problem which can be solved using a direct method with the help of the general theory of optimal stopping given in Chapter 2. To the best of our knowledge, this optimal prediction problem has not been studied before for a spectrally negative Lévy process.

In the appendix we present some classical results concerning martingales, Markov processes and Poisson point processes.

Chapter 1

Lévy Processes

Lévy processes can be thought of as random walks in continuous time, that is they are stochastic processes with independent and stationary increments. The best known and most important examples are the Brownian motion, Poisson process and compound Poisson process. Lévy processes concern many aspects of probability theory and its applications; they are used as models in the study of queues, insurance risks, degradation and mathematical finance. From the viewpoint of functional analysis, these appear in connection with potential theory of convolution semigroups.

The content of this chapter is mainly based on the work of Kyprianou (2014) and we used Bertoin (1998) and Sato (1999) as a secondary bibliography.

1.1 Lévy Processes and Infinite Divisibility

In this section we present the definition of Lévy processes and their connection with infinitely divisible distributions, then the main examples of these processes are presented. In the sequel we work with $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space, i.e. a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ which is the natural enlargement[1] of the sigma-algebra generated by $\{X_s, s \leq t\}$ (see Definition 1.3.38 of Bichteler (2002)). According to Kyprianou (2014) we present the formal definition of a Lévy process.

Definition 1.1.1 (Lévy Process)**.** *A process $X = \{X_t : t \geq 0\}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in $\mathbb{R}$, is said to be a Lévy process if it satisfies the following properties:*

i) The paths of X are $\mathbb{P}$-almost surely càdlàg[2] (right-continuous with left limits).

ii) $\mathbb{P}(X_0 = 1)$.

iii) For $s, t \geq 0$, $X_{t+s} - X_s$ is equal in distribution to X_t.

iv) For $s, t \geq 0$, $X_{t+s} - X_t$ is independent of $\{X_u, u \leq t\}$.

If a process X satisfies the condition *iii)* we say that X has stationary increments and if condition *iv)* holds then we say that X has independent increments. Now we give a sufficient condition to check that a process has stationary and independent increments.

Remark 1.1.2. *For proving that a process $X = \{X_t, t \geq 0\}$ has stationary an independent increments it suffices to prove that for all $n \in \mathbb{N}$ and $0 \leq s_1 \leq t_1 \leq \cdots \leq s_n \leq t_n < \infty$ and $\theta_1, \ldots, \theta_n \in \mathbb{R}$,*

$$\mathbb{E}\left[\prod_{j=1}^{n} e^{i\theta_j (X_{t_j} - X_{s_j})}\right] = \prod_{j=1}^{n} \mathbb{E}(e^{i\theta_j X_{t_j - s_j}}).$$

[1] Many authors assume that the filtration $\mathbb{F}$ satisfies the usual conditions. This can cause some problems, for example, using change of measures with Girsanov's theorem (see Warning 1.3.39 of Bichteler (2002)).

[2] Abbreviating the French phrase continues à droite, limites à gauche

Now we introduce the notion of infinitely divisible distributions and then we will show that this concept is intimately related with Lévy processes.

Definition 1.1.3. *We say that a real-valued random variable, Θ, has an infinitely divisible distribution if, for each $n = 1, 2, \dots,$ there exists a sequence of i.i.d. random variables $\Theta_{i,n}$ $i = 1, \dots, n$ such that*

$$\Theta \stackrel{d}{=} \Theta_{1,n} + \cdots + \Theta_{n,n},$$

where $\stackrel{d}{=}$ is equality in distribution.

Definition 1.1.4. *Let Θ a real-valued random variable. We define the characteristic exponent of Θ as*

$$\Psi(u) = -\log\left(\mathbb{E}(e^{iu\Theta})\right),$$

for all $u \in \mathbb{R}$.

Remark 1.1.5. *i) If Θ has a probability distribution μ we could say that μ (and hence Θ) is infinitely divisible if for any positive integer n, there exists a probability measure μ_n such that $\mu = \mu_n^{*n}$, where μ_n^{*n} denotes the n-fold convolution of μ_n. We can write the above condition as $\widehat{\mu} = (\widehat{\mu}_n)^n$ where $\widehat{\mu}$ and $\widehat{\mu}_n$ are the characteristic function of μ and μ_n respectively, i.e. for all $\lambda \in \mathbb{R}$,*

$$\widehat{\mu}_n(\lambda) = \int_{\mathbb{R}} e^{i\lambda x}\mu_n(dx) \qquad \textit{and} \qquad \widehat{\mu}(\lambda) = \int_{\mathbb{R}} e^{i\lambda x}\mu(dx).$$

ii) In view of i) then we can establish when a random variable Θ has an infinitely divisible distribution via its characteristic function. Then a quantity of interest is the characteristic exponent defined by $\Psi(u) = -\log(\mathbb{E}(e^{iu\Theta}))$ for all $u \in \mathbb{R}$. Hence Θ has an infinitely divisible distribution if, for all $n \geq 1$, there exists a characteristic exponent of a probability distribution, say Ψ_n, such that $\Psi(u) = n\Psi_n(u)$ for all $u \in \mathbb{R}$.

An important result which characterises the infinitely divisible laws in terms of its characteristic exponent is the famous Lévy–Khintchine formula.

Theorem 1.1.6 (Lévy–Khintchine formula)**.** *A probability law, μ, of a real-valued random variable is infinitely divisible with characteristic exponent Ψ,*

$$\int_{\mathbb{R}} e^{i\theta x}\mu(dx) = e^{-\Psi(\theta)}, \qquad \textit{for}\ \theta \in \mathbb{R},$$

if and only if there exists a triple (a, σ, Π), where $a \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and Π is a measure concentrated on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$, such that

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}}(1 - e^{i\theta x} + i\theta x\mathbb{I}_{\{|x|<1\}})\Pi(dx),$$

for every $\theta \in \mathbb{R}$. Moreover, the triple (a, σ^2, Π) is unique.

Proof. See Sato (1999) (Theorem 8.1). □

Definition 1.1.7. *The measure Π is called the Lévy (characteristic) measure.*

Remark 1.1.8. *Note that the condition $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$ implies that $\Pi(A) < \infty$ for all Borel A such that 0 is in the interior of A^c. Indeed, since $x^2 \wedge \varepsilon \leq x^2 \wedge 1$ for all $0 < \varepsilon \leq 1$,*

$$\infty > \int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) \geq \int_{\mathbb{R}}(\varepsilon \wedge x^2)\Pi(dx) = \varepsilon\int_{(-\sqrt{\varepsilon},\sqrt{\varepsilon})^c}\Pi(dx) + \int_{(-\sqrt{\varepsilon},\sqrt{\varepsilon})} x^2\Pi(dx).$$

Then $\Pi((-\varepsilon, \varepsilon)^c) < \infty$ for all $\varepsilon > 0$.

Let us now state the relationship between infinitely divisible distribution and Lévy processes.

Proposition 1.1.9. *Let $X = \{X_t, t \geq 0\}$ be a Lévy process. Then for any $t > 0$, X_t is a random variable belonging to the class of infinitely divisible distributions. Moreover, the characteristic exponent of X_t defined by $\Psi_t(\theta) = -\log(\mathbb{E}(e^{i\theta X_t}))$ for all $t \geq 0$ satisfies*

$$\Psi_t(\theta) = t\Psi_1(\theta). \tag{1.1}$$

Proof. Let $t > 0$ and $n \geq 1$, note that we can write

$$X_t = \sum_{k=1}^{n}(X_{kt/n} - X_{(k-1)t/n}) = X_{t/n} + (X_{2t/n} - X_{t/n}) + \cdots + (X_t - X_{(n-1)t/n}). \tag{1.2}$$

Together with the facts that X has stationary independent increments and that $X_0 = 0$ we have that X_t has an infinitely divisible distribution.

From the definition of Ψ_t and using (1.2), we have, for any positive integer m

$$\Psi_m(\theta) = -\log\left(\mathbb{E}(e^{i\theta X_m})\right) = -\log\left(\mathbb{E}(e^{i\theta X_1})^m\right) = m\Psi_1(\theta)$$

and for any positive integer n

$$\Psi_m(\theta) = -\log\left(\mathbb{E}(e^{i\theta X_m})\right) = -\log\left(\mathbb{E}(e^{i\theta X_{m/n}})^n\right) = n\Psi_{m/n}(\theta).$$

Then for every m, n positive integers,

$$\Psi_{m/n}(\theta) = \frac{m}{n}\Psi_1(\theta).$$

Hence, for any rational $t > 0$,

$$\Psi_t(\theta) = t\Psi(\theta).$$

If t is an irrational number, then we can choose a decreasing sequence of rational $\{t_n : n \geq 1\}$ such that $t_n \downarrow t$ as n tends to infinity. Note that $|e^{i\theta X_t}| \leq 1$ so by the dominated convergence theorem and the almost sure right-continuity of X we have

$$\begin{aligned}
\Psi_t(\theta) &= -\log\left(\mathbb{E}(e^{i\theta X_t})\right) \\
&= -\log\left(\mathbb{E}(\lim_{n\to\infty} e^{i\theta X_{t_n}})\right) \\
&= \lim_{n\to\infty} -\log\left(\mathbb{E}(e^{i\theta X_{t_n}})\right) \\
&= \lim_{n\to\infty} \Psi_{t_n}(\theta) \\
&= \lim_{n\to\infty} t_n\Psi_1(\theta) \\
&= t\Psi_1(\theta).
\end{aligned}$$

Therefore $\Psi_t(\theta) = t\Psi_1(\theta)$ holds for all $t \geq 0$.

□

The above proposition tells us that any Lévy process has the property that, for all $t \geq 0$,

$$\mathbb{E}(e^{i\theta X_t}) = e^{-t\Psi(\theta)}, \tag{1.3}$$

where $\Psi(\theta) := \Psi_1(\theta)$ is the characteristic exponent of X_1.

Definition 1.1.10. *In the sequel, we shall also refer to $\Psi(\theta)$ as the characteristic exponent of the Lévy process.*

An important result in the theory of Lévy processes is that any infinitely distribution μ can be viewed as the distribution of a Lévy process evaluated at time 1.

Theorem 1.1.11 (Lévy–Khintchine formula for Lévy processes)**.** *Suppose that $a \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and Π is a measure concentrated on $\mathbb{R} \setminus \{0\}$ such that $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$. From this triple, define for each $\theta \in \mathbb{R}$,*

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}}(1 - e^{i\theta x} + i\theta x \mathbb{I}_{\{|x|<1\}})\Pi(dx).$$

Then there exists a probability space, $(\Omega, \mathcal{F}, \mathbb{P})$, on which a Lévy process is defined having characteristic exponent Ψ.

The proof of the Lévy–Khintichine formula for Lévy processes, which can be found in Bertoin (1998), provides an explicit construction of this Lévy process and sheds a probabilistic light on the Lévy–Khintchine formula. In section 1.2 we give an idea of this construction which is called the Lévy–Itô decomposition.

Some Examples of Lévy Processes

Poisson Processes

Definition 1.1.12. *A process valued on the non-negative integers, $N = \{N_t, t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is said to be a Poisson process with intensity $\lambda > 0$ if the following hold:*

1. *The paths of N are $\mathbb{P}$-almost surely right-continuous with left limits.*
2. *$\mathbb{P}(N_0 = 0) = 1$.*
3. *For $s, t \geq 0$, $N_{t+s} - N_t$ is equal in distribution to N_t.*
4. *For $s, t \geq 0$, $N_{t+s} - N_t$ is independent of $\{N_u : u \leq t\}$.*
5. *For each $t > 0$, N_t is equal in distribution to a Poisson random variable with parameter λt*

Clearly, the Poisson process is a Lévy process such that N_1 has a Poisson distribution. Now we check that N satisfies the Lévy–Khintchine formula.

For each $\lambda > 0$, consider the Poisson distribution, i.e. take a measure μ_λ which is concentrated on $k = 0, 1, 2, \ldots$ such that $\mu_\lambda(\{k\}) = e^{-\lambda}\lambda^k/k!$. It is well known that the characteristic function $\widehat{\mu}_\lambda(\theta)$ is given by

$$\widehat{\mu}_\lambda(\theta) = e^{-\lambda(1-e^{i\theta})} = [e^{-\frac{\lambda}{n}(1-e^{i\theta})}]^n.$$

The right-hand side is the characteristic function of the sum of n independent Poisson variables, each of which has parameter λ/n. In the Lévy–Khintchine formula, we see that $a = \sigma = 0$ and $\Pi = \lambda\delta_1$, where δ_1 is the Dirac measure supported on $\{1\}$. From the above calculations, we have

$$\mathbb{E}(e^{i\theta N_t}) = e^{-\lambda t(1-e^{i\theta})}$$

and hence its characteristic exponent is given by $\Psi(\theta) = \lambda(1 - e^{i\theta})$, for $\theta \in \mathbb{R}$.

Compound Poisson Processes

Suppose now that N is a Poisson random variable with parameter $\lambda > 0$ and that $\{\xi_i, i \geq 1\}$ is a sequence of i.i.d. random variables (independent of N) with common law F which has no atom at zero. By first conditioning on N, we have for $\theta \in \mathbb{R}$,

$$
\begin{aligned}
\mathbb{E}(e^{i\theta \sum_{i=1}^{N} \xi_i}) &= \sum_{n=0}^{\infty} \mathbb{E}(e^{i\theta \sum_{i=1}^{n} \xi_i}) e^{-\lambda} \frac{\lambda^n}{n!} \\
&= \sum_{n=0}^{\infty} (\mathbb{E}(e^{i\theta \xi_1}))^n e^{-\lambda} \frac{\lambda^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\int_{\mathbb{R}} e^{i\theta x} F(dx) \right)^n e^{-\lambda} \frac{\lambda^n}{n!} \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\left(\lambda \int_{\mathbb{R}} e^{i\theta x} F(dx)\right)^n}{n!} \\
&= e^{-\lambda} e^{\lambda \int_{\mathbb{R}} e^{i\theta x} F(dx)} \\
&= e^{-\lambda \int_{\mathbb{R}} (1 - e^{i\theta x}) F(dx)}.
\end{aligned}
$$

We see from the Lévy Khintchine formula (Theorem 1.1.6) that distributions of the form $\sum_{i=1}^{N} \xi_i$ are infinitely divisible with triple $a = -\lambda \int_{\{0<|x|<1\}} xF(dx)$, $\sigma = 0$ and $\Pi(dx) = \lambda F(dx)$, for $x \neq 0$.

Suppose now that $\{N_t, t \geq 0\}$ is a Poisson process with intensity $\lambda > 0$ and consider a compound Poisson process $\{X_t, t \geq 0\}$ defined by

$$X_t = \sum_{i=1}^{N_t} \xi_i, \qquad t \geq 0$$

Using the fact that N has stationary independent increments together with the mutual independence of the random variables $\{\xi_i, i \geq 1\}$, by writing

$$X_{t+s} = X_t + \sum_{N_t+1}^{N_{t+s}} \xi_i,$$

for $s, t \geq 0$, it is clear that X_{t+s} is the sum of X_t and an independent copy of X_s. Right-continuity and left limits of the process $\{N_t, t \geq 0\}$ also ensure right-continuity and left limits of X. In conclusion, compound Poisson processes are Lévy processes. From the calculations in the previous paragraph we have that the Lévy–Khintchine formula for a compound Poisson process takes the form $\Psi(\theta) = \lambda \int_{\mathbb{R}} (1 - e^{i\theta x}) F(dx)$.

Linear Brownian Motion

Definition 1.1.13. *A real-valued process, $B = \{B_t, t \geq 0\}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Brownian motion if the following hold:*

1. *The paths of B are $\mathbb{P}$-almost surely continuous.*
2. *$\mathbb{P}(B_0 = 0) = 1$.*
3. *For $s, t \geq 0$, $B_{t+s} - B_t$ is equal in distribution to B_t.*
4. *For $s, t \geq 0$, $B_{t+s} - B_t$ is independent of $\{B_u : u \leq t\}$.*
5. *For each $t > 0$, B_t is equal in distribution to a normal random variable with zero mean and variance t*

Clearly a Brownian motion is a continuous Lévy process where B_1 has a Gaussian distribution. Take the probability law of a Gaussian distribution with mean $\gamma \in \mathbb{R}$ and variance $s^2 > 0$,

$$\mu_{s,\gamma}(dx) := \frac{1}{\sqrt{2\pi s^2}} e^{-(x-\gamma)^2/2s^2} dx.$$

It is well known that its characteristic function $\widehat{\mu}_{s,\gamma}(\theta)$ is given by

$$\widehat{\mu}_{s,\gamma}(\theta) = e^{-s^2\theta^2/2+i\theta\gamma} = \left[e^{-\left(s/\sqrt{n}\right)^2\theta^2/2+i\theta\gamma/n}\right]^n,$$

showing, again, that it is an infinitely divisible distribution, this time with $a = -\gamma$, $\sigma = s$ and $\Pi = 0$.

We immediately recognise the characteristic exponent $\Psi(\theta) = s^2\theta^2/2 - i\theta\gamma$ as that of a scaled Brownian motion with linear drift (otherwise referred to as linear Brownian motion),

$$X_t := sB_t + \gamma t, \qquad t \geq 0.$$

1.2 The Lévy–Itô Decomposition

The Lévy–Itô decomposition describes the structure of a general Lévy process in terms of three independent auxiliary Lévy processes, each with a different type of path behaviour. For a better understanding of this decomposition it is necessary to have a brief overview of the general theory of Poisson random measures (see Appendix A.3).

Theorem 1.2.1 (Lévy–Itô decomposition)**.** *Let $a \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and a measure Π concentrated on $\mathbb{R} \setminus \{0\}$ satisfying*

$$\int_{\mathbb{R}} (1 \wedge x^2)\Pi(dx) < \infty.$$

Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which three independent Lévy processes exist, $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$, where $X^{(1)}$ is a linear Brownian motion given by (1.4), $X^{(2)}$ is a compound Poisson process given by (1.5) and $X^{(3)}$ is a square-integrable martingale with an almost surely countable number of path discontinuities (or jumps) on each finite interval, which are of magnitude less than unity, and with characteristic exponent given by $\Psi^{(3)}$. Moreover, by taking $X = X^{(1)} + X^{(2)} + X^{(3)}$, the conclusion of Theorem 1.1.11 holds, namely that there exists a probability space on which a Lévy process is defined with characteristic exponent

$$\Psi(\theta) = ai\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbb{I}_{\{|x|<1\}})\Pi(dx),$$

for $\theta \in \mathbb{R}$, and path, or Lévy–Itô decomposition

$$X_t = \sigma B_t - at + \int_0^t \int_{\{|x|\geq 1\}} xN(ds, dx) + \int_0^t \int_{\{|x|<1\}} x(N(ds, dx) - ds\Pi(dx)),$$

where N is a Poisson random measure with intensity $\eta(dt \times dx) = dt \times \Pi(dx)$

Proof. See chapter 4 of Sato (1999) or chapter 2 of Kyprianou (2014). □

The Lévy–Itô decomposition is a hard mathematical result to prove. We give a rough sketch of the proof because it gives some ideas about the structure of the paths of a Lévy process. According to Theorem 1.1.11, any characteristic exponent Ψ of a Lévy process can be written in the form

$$\Psi(\theta) = \Psi^{(1)}(\theta) + \Psi^{(2)}(\theta) + \Psi^{(3)}(\theta),$$

where

$$\begin{aligned}
\Psi^{(1)}(\theta) &= ia\theta + \frac{1}{2}\sigma^2\theta^2, \\
\Psi^{(2)}(\theta) &= \Pi(\mathbb{R} \setminus (-1, 1)) \int_{\{|x|\geq 1\}} (1 - e^{i\theta x}) \frac{\Pi(dx)}{\Pi(\mathbb{R} \setminus (-1, 1))}, \\
\Psi^{(3)}(\theta) &= \int_{\{0<|x|<1\}} (1 - e^{i\theta x} + i\theta x)\Pi(dx),
\end{aligned}$$

for all $\theta \in \mathbb{R}$, where $a \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and Π is a measure on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$. In the case that $\Pi(\mathbb{R} \setminus (-1,1)) = 0$, one should think that $\Psi^{(2)} \equiv 0$. The key of the proof of the Lévy–Itô decomposition is to recognise $\Psi^{(1)}$, $\Psi^{(2)}$ and $\Psi^{(3)}$ as the characteristic exponent of three independent Lévy processes, where each Lévy process has a particular path behaviour. As we have already seen in Section 1.1, $\Psi^{(1)}$ and $\Psi^{(2)}$ correspond, respectively, to a linear Brownian motion, say, $X^{(1)} = \{X_t^{(1)}, t \geq 0\}$, where

$$X_t^{(1)} = \sigma B_t - at, \qquad t \geq 0, \tag{1.4}$$

and an independent compound Poisson process, say $X^{(2)} = \{X_t^{(2)}, t \geq 0\}$, where,

$$X_t^{(2)} = \sum_{i=1}^{N_t} \xi_i, \qquad t \geq 0, \tag{1.5}$$

$\{N_t, t \geq 0\}$ is a Poisson process with rate $\Pi(\mathbb{R} \setminus (-1,1))$ and $\{\xi_i, i \geq 1\}$ are independent and identically distributed with common distribution $\Pi(dx)/\Pi(\mathbb{R} \setminus (-1,1))$ concentrated on $\{x : |x| \geq 1\}$ (unless $\Pi(\mathbb{R} \setminus (-1,1)) = 0$ in which case $X^{(2)}$ is the process which is identically zero). Using the notation of Poisson random measure we can express the process $X^{(2)}$ as

$$X_t^{(2)} = \int_0^t \int_{\{|x| \geq 1\}} xN(ds, dx), \qquad t \geq 0,$$

where N is a Poisson random measure with intensity $dt \times \Pi(dx)$, defined as in Appendix A.3. As a direct consequence of Lemma A.3.5 we have that $X^{(2)}$ is indeed a compound Poisson process.

The proof of existence of a Lévy process with characteristic exponent Ψ is reduced to showing the existence of a Lévy process, $X^{(3)}$, whose characteristic exponent is given by $\Psi^{(3)}$. Note that the characteristic exponent $\Psi^{(3)}$ appears to be the characteristic exponent of a compensated compound Poisson process. However, due to the fact that Π is not necessarily finite in the set $(-1,1) \setminus \{0\}$, this might not be case.

We give a brief outline of the construction of $X^{(3)}$. For $\varepsilon > 0$ define the compensated compound Poisson process $X^{(3,\varepsilon)} = \{X_t^{(3,\varepsilon)}, t \geq 0\}$ where

$$X_t^{(3,\varepsilon)} = \int_{[0,t]} \int_{\{\varepsilon \leq |x| < 1\}} xN(ds, dx) - t \int_{\{\varepsilon \leq |x| < 1\}} x\Pi(dx), \qquad t \geq 0.$$

The process $X_t^{(3)}$ is a square-integrable martingale with characteristic exponent

$$\Psi^{(3,\varepsilon)}(\theta) = \int_{\{\varepsilon \leq |x| < 1\}} (1 - e^{i\theta x} + i\theta x)\Pi(dx).$$

Moreover, it can be shown that (see Kyprianou (2014)) there exists a Lévy process $X^{(3)}$ which is also a square-integrable martingale to which $X^{(3,\varepsilon)}$ converges uniformly to $X^{(3)}$ on $[0,T]$ as $\varepsilon \downarrow 0$ and its characteristic exponent is

$$\Psi^{(3)}(\theta) = \int_{\{|x| < 1\}} (1 - e^{i\theta x} + i\theta x)\Pi(dx).$$

Then, the process $X = \{X_t, t \geq 0\}$ where

$$X_t = X_t^{(1)} + X_t^{(2)} + X_t^{(3)}, \qquad t \geq 0$$

has stationary independent increments, has paths that are right-continuous with left limits and has characteristic exponent

$$\begin{aligned}\Psi(\theta) &= \Psi^{(1)}(\theta) + \Psi^{(2)}(\theta) + \Psi^{(3)}(\theta)\\ &= ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}}(1 - e^{i\theta x} + i\theta x\mathbb{I}_{\{|x|<1\}})\Pi(dx).\end{aligned}$$

Remark 1.2.2. *To summarise, a Lévy process can be seen as a linear Brownian motion with jumps that are determined by a Poisson point process. The process $X^{(2)}$ model the "big jumps" while the process $X^{(3)}$ handles the "small jumps". Using the properties of the Poisson random measure, we can see that*

$$\mathbb{E}(N([0,1], A)) = \Pi(A),$$

for every set $A \in \mathbb{B}(\mathbb{R} \setminus \{0\})$. This tells us that the Lévy measure describes the expected number of jumps of a certain height in a time interval of length one. On the other hand we have that

$$\mathbb{E}\left(\int_0^1 \int_A xN(ds, dx)\right) = \int_A x\Pi(dx),$$

then the integral of the right-hand side of the above expression is the expected sum of jumps of a certain height in a time interval of length one.

Remark 1.2.3. *The relation between Poisson random measures and Lévy measures allows us to draw the following conclusion about the sample paths of Lévy processes based on their Lévy measure: the Lévy measure has no mass at the origin, thus a Lévy process can have an infinite number of "small" jumps. Moreover, the mass away from the origin is bounded, hence only a finite number of "big" jumps can occur.*

1.3 General Properties of Lévy Processes

The Lévy–Itô decomposition gives us tools to analyse the behaviour of a general Lévy process as a sum of three independent Lévy processes. It turns out that the Lévy measure is responsible for the richness of the class of Lévy processes. The behaviour of the sample paths of a Lévy process, as well as many other properties, e.g. existence of moments, smoothness of densities, etc, can be completely characterised based on the Lévy measure and the presence or absence of a Brownian component.

In this section we will study some properties of Lévy processes. We will study further a particular subclass of Lévy processes called spectrally negative Lévy process which will be key in Chapter 3. This section is based in Kyprianou (2014), but for a more complete summary of properties we refer to Sato (1999).

Path Properties

Now we study some properties of the paths of Lévy process, in particular, when they have finite or finite variation.

Path variation

First, we analyse the variation of the paths of a Lévy process. We start recalling the definition of variation of a real function and then we give some elementary properties. Then we define this concept for a general stochastic process according to Klebaner et al. (2005).

For a function $f : \mathbb{R} \mapsto \mathbb{R}$, its variation over the interval $[a, b]$ is defined as

$$V_f([a, b]) = \sup_{\pi} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|, \tag{1.6}$$

where $\pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of the interval $[a, b]$. If $V_f([a, b])$ is finite then f is said to be a function of finite variation on $[a, b]$. Otherwise, f is of infinite variation on $[a, b]$. The variation function of f as a function of t is defined by

$$V_f(t) = V_f([0, t]).$$

We say that f is of finite variation if $V_f(t) < \infty$ for all $t \geq 0$ and f is of bounded variation if $\sup_{t\geq 0} V_f(t) < \infty$, in other words, if for all $t \geq 0$, $V_f(t) < C$, where C is a constant independent of t. For example, it follows from the definition that

1. If $f(t)$ is a increasing function then
$$V_f(t) = f(t) - f(0).$$

2. If $f(t)$ is decreasing then
$$V_f(t) = f(0) - f(t).$$

3. If f is a càdlàg function and is a pure jump function (changes only by jumps), i.e. it is of the form $f(t) = \sum_{0\leq s\leq t} \Delta f(s) = \sum_{0\leq s\leq t}[f(s+) - f(s-)]$ then we have
$$V_f(t) = \sum_{0\leq s\leq t} \Delta|f(s)| = \sum_{0\leq s\leq t} |f(s+) - f(s-)|.$$

Now, we give a theorem which links the variation and the discontinuities of a function.

Theorem 1.3.1. *A finite variation function can have no more than countably many discontinuities. Moreover, all discontinuities are jumps.*

Proof. See Klebaner et al. (2005) (Theorem 1.7). □

The following lemma gives a lower bound of the variation of a function in terms of the sum of the sizes of its jumps. This lemma will be useful later to analyse the variation of a Lévy process.

Lemma 1.3.2. *If $f : \mathbb{R} \mapsto \mathbb{R}$ is càdlàg and has finite variation on $[a, b]$ then*
$$V_f([a, b]) \geq \sum_{t\in[a,b]} |\Delta f(t)|.$$

Proof. See Applebaum (2009) (Theorem 2.314). □

Now we define the variation of a stochastic process.

Definition 1.3.3. *A stochastic process $X = \{X_t, t \geq 0\}$ has finite variation if the paths $\{X_t(\omega), t \geq 0\}$ have finite variation for almost all $\omega \in \Omega$. Otherwise, the process has infinite variation.*

It is well known that the Brownian motion has paths of infinite variation (see Klebaner et al. (2005) or Revuz and Yor (1999)). On the other hand since the compound Poisson process is a pure jump process and in the interval $[0, t]$ we have only a finite number of jumps, we have that its variation is given by the finite sum of the size of all jumps which is also finite. Therefore we have that the compound Poisson process has paths of finite variation.

Now we give necessary and sufficient conditions for a Lévy process to have paths of finite variation in terms of its Lévy triple (a, σ, Π).

Lemma 1.3.4. *A Lévy process with Lévy–Khintchine exponent corresponding to the triple (a, σ, Π) has paths of finite variation if and only if*
$$\sigma = 0 \qquad \textit{and} \qquad \int_{\mathbb{R}} (1 \wedge |x|)\Pi(dx) < \infty. \tag{1.7}$$

Proof. We know that the Brownian motion has paths of infinite variation. From the Lévy–Itô decomposition the presence of Brownian motion ($\sigma > 0$) implies that the paths of the Lévy process have infinite variation. On the other hand the process $X^{(2)}$ from the Lévy Itô decomposition, which is a compound Poisson process, has paths of finite variation. Hence we just need to analyse whether the process $X^{(3)}$ has finite variation.

First suppose that $\int_{\mathbb{R}}(1\wedge|x|)\Pi(dx) < \infty$. Then Theorem A.3.4 *i*) ensures that $\int_0^t \int_{|x|\leq 1} |x|N(ds,dx) < \infty$ a.s. Let $\pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ a partition of the interval $[0,t]$, then

$$\begin{aligned}\sum_{i=1}^n |X^{(3)}_{t_i} - X^{(3)}_{t_{i-1}}| &= \sum_{i=1}^n \left| \int_{t_{i-1}}^{t_i} \int_{\{|x|\leq 1\}} x(N(ds,dx) - \Pi(dx)ds) \right| \\ &\leq \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_{\{|x|\leq 1\}} |x|(N(ds,dx) - \Pi(dx)ds) \\ &= \int_0^t \int_{\{|x|\leq 1\}} |x|(N(ds,dx) - \Pi(dx)ds).\end{aligned}$$

Since the last double integral does not depend on π and from the finiteness of the integral with respect the Poisson random measure we have

$$\sup_\pi \sum_{i=1}^n |X^{(3)}_{t_i} - X^{(3)}_{t_{i-1}}| \leq \int_0^t \int_{\{|x|\leq 1\}} |x|N(ds,dx) - t\int_{\{|x|\leq 1\}} |x|\Pi(dx) < \infty \qquad \mathbb{P}\text{-}a.s.$$

thus $X^{(3)}$ has finite variation.

Conversely, suppose that $X^{(3)}$ has finite variation. Then by Lemma 1.3.2 we have that

$$\infty > \sup_\pi \sum_{i=1}^n |X^{(3)}_{t_i} - X^{(3)}_{t_{i-1}}| \geq \sum_{t\in[0,t]} |\Delta X^{(3)}_t| = \int_0^t \int_{\{|x|\leq 1\}} |x|N(ds,dx).$$

Then Theorem A.3.4 implies that $\int_{\{|x|\leq 1\}} x\Pi(dx) < \infty$ and hence

$$\int_{\mathbb{R}} (1\wedge|x|)\Pi(dx) < \infty.$$

In conclusion we have that $X^{(3)}$ is a process of finite variation if only if $\int_{\mathbb{R}}(1\wedge|x|)\Pi(dx) < \infty$ and the result follows. □

Remark 1.3.5. *Note that if a Lévy process $\{X_t, t\geq 0\}$ is of finite variation then (1.7) holds, and we have that its Lévy Khintchine exponent can be rewritten as*

$$\begin{aligned}\Psi(\theta) &= ai\theta + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x\mathbb{I}_{\{|x|<1\}})\Pi(dx) \\ &= \left(a + \int_{\{|x|<1\}} x\Pi(dx)\right) i\theta + \int_{\mathbb{R}} (1 - e^{i\theta x})\Pi(dx) \\ &= -id\theta + \int_{\mathbb{R}} (1 - e^{i\theta x})\Pi(dx),\end{aligned}$$

where $d = -\left(a + \int_{\{|x|<1\}} x\Pi(dx)\right)$. Therefore the process X_t has the form

$$X_t = dt + \int_{[0,t]} \int_{\mathbb{R}} xN(ds,dx), \qquad t\geq 0 \tag{1.8}$$

Note that the above expression corresponds to a compound Poisson process with drift d when $\Pi(\mathbb{R}) < \infty$.

Subordinators

Now we introduce an important class of Lévy processes called subordinators. These processes play an important role in the theory and applications of Lévy processes in various fields, as they constitute a stochastic model for the evolution of time.

Definition 1.3.6. *A process X is called subordinator if it is a Lévy process with a.s. non-decreasing paths.*

As the jumps of a Lévy process are handled by the Lévy measure Π we have that $\Pi(-\infty,0)=0$ if and only X has only positive jumps, noting that in presence of a Brownian motion we cannot have monotone paths, using Lemma 1.3.4 and Remark 1.3.5, we have the following result.

Lemma 1.3.7. *A Lévy process X is a subordinator if and only if $\Pi(-\infty,0)=0$, $\int_{(0,\infty)}(1\wedge x)\Pi(dx)<\infty$, $\sigma=0$ and $d=-(a+\int_{(0,1)}x\Pi(dx))\geq 0$.*

The subordinators are very important as they are useful to construct new processes via time change. It can be shown that a Lévy process time changed by an independent subordinator is again a Lévy process.

One-Sided Jumps

Another class of Lévy processes are the spectrally positive Lévy processes and spectrally negative Lévy processes which are processes that only have one-sided jumps. The spectrally negative Lévy processes are of vital importance in this work and they will be studied deeper in the last section of this chapter.

Definition 1.3.8. *Let X be a Lévy process. If $\Pi(-\infty,0)=0$ and X does not have monotone paths, then X is referred to as a spectrally positive Lévy process. A Lévy process, X, will then be referred to as a spectrally negative Lévy process if $-X$ is spectrally positive. Together, these two classes of processes are called spectrally one-sided.*

Remark 1.3.9. *i) It is important to emphasise that subordinators are excluded from the definition of spectrally positive Lévy process. This tells us that compound Poisson processes are not spectrally positive nor spectrally negative Lévy processes.*

ii) Spectrally one-sided Lévy processes may be of finite or infinite variation and, in the latter case, may or may not possess a Gaussian component. If a spectrally negative Lévy process has finite variation, then it must take the form

$$X_t = dt - S_t, \qquad t\geq 0, \tag{1.9}$$

where $\{S_t, t\geq 0\}$ is a pure jump subordinator and $d>0$ is called the drift of the process. Note the above decomposition implies that if $\mathbb{E}(X_1)\geq 0$, then $\mathbb{E}(S_1)<\infty$, and if $\mathbb{E}(X_1)<0$ it is possible that $\mathbb{E}(S_1)=\infty$.

A special feature of spectrally negative Lévy processes is that, if $\tau_x^+=\inf\{t>0 : X_t>x\}$ where $x>0$, then $\mathbb{P}(\tau_x^+<\infty)>0$. Hence, as there are no upwards jumps,

$$\mathbb{P}(X_{\tau_x^+}=x|\tau_x^+<\infty)=1.$$

The Strong Markov Property

The Markov property (see Appendix A.2 for a more detailed review of Markov processes) is satisfied for all Lévy process. Moreover, Lévy processes satisfy the stronger condition that the law of $X_{t+s}-X_t$ is independent of $\mathcal{F}_t$, for all $s,t\geq 0$. Some of the proofs of the results corresponding to this subsection are quite technical so we omit them, these can be found in Kyprianou (2014).

Theorem 1.3.10. *Suppose that τ is a stopping time. Define on $\{\tau<\infty\}$ the process $\widetilde{X}=\{X_t, t\geq 0\}$ where*

$$\widetilde{X}_t = X_{\tau+t}-X_\tau, \qquad t\geq 0.$$

Then, on the event $\{\tau<\infty\}$, the process $\widetilde{X}$ is independent of $\mathcal{F}_\tau$, has the same law as X and hence in particular is a Lévy process.

In the following chapter we use stochastic processes which are left-continuous over increasing sequences of stopping times. Lévy processes are not the exception as we state in the following lemma, Lévy processes are left-continuous over stopping times.

Lemma 1.3.11 (Quasi-Left-Continuity). *If T is a stopping time and $(T_n, n \geq 1)$ is an increasing sequence of stopping times such that $\lim_{n\to\infty} T_n = T$ a.s., then $\lim_{n\to\infty} X_{T_n} = X_T$ on $\{T < \infty\}$. Hence, if $T_n < T$ a.s. for each $n \geq 1$, then X is left-continuous at T on $\{T < \infty\}$.*

Examples of stopping times which are useful in the present work are those of the first-entrance time and first-hitting time of a given or closed set $B \subset \mathbb{R}$. They are defined as

$$T^B = \inf\{t \geq 0 : X_t \in B\} \qquad \text{and} \qquad \tau^B = \inf\{t > 0 : X_t \in B\},$$

where we take the usual convention that $\inf \emptyset = \infty$. In the following result we state that the above random times are indeed stopping times no matter if the set B is open or closed.

Theorem 1.3.12. *Suppose that B is open or closed. Then,*

1. *T^B is a stopping time and $X_{T^B} \in \bar{B}$ on $\{T^B < \infty\}$ and*
2. *τ^B is a stopping time and $X_{\tau^B} \in \bar{B}$ on $\{\tau^B < \infty\}$.*

Moments and Martingales

In this subsection we put our attention to the moments of a Lévy process. In particular, we analyse the expectation of a special class of transformations of a Lévy processes called submultiplicative. It is well known that the Brownian motion has finite moments of all orders. Note that if $X = \{X_t, t \geq 0\}$ is a compound Poisson, i.e. is of the form

$$X_t = \sum_{i=1}^{N_t} \xi_i, \qquad t \geq 0,$$

where $N = \{N_t, t \geq 0\}$ is a Poisson process and $\{\xi_i, i \geq 1\}$ are independent and identically distributed. We know that

$$\mathbb{E}(X_t) = \lambda t \mathbb{E}(\xi_1).$$

If ξ_1 has infinite first moment it follows that $\mathbb{E}(X_1) = \infty$. As a consequence of the Lévy–Itô decomposition and the above observations me may suspect that the moments of a Lévy process and the Lévy measure are closely related. Indeed, in what follows we will give some conditions under which the expectation of a large class of functions of Lévy processes has finite expectation. These functions are called submultiplicative.

Definition 1.3.13. *A measurable function, $g : \mathbb{R} \mapsto [0, \infty)$, is called submultiplicative if there exists a constant $a > 0$ such that $g(x + y) \leq a g(x) g(y)$ for all $x, y \in \mathbb{R}$.*

We state a useful result that links the submultiplicative functions with the exponential function.

Lemma 1.3.14. *If g is a submultiplicative function which is bounded on compacts, then there exist constants $b_g > 0$ and $c_g > 0$ such that*

$$g(x) \leq b_g \exp(c_g |x|).$$

Proof. Since g is submultiplicative then there exists a constant $a > 0$ such that $g(x + y) \leq a g(x) g(y)$ for all $x, y \in \mathbb{R}$. As g is locally bounded we may choose b_g in a such way that $\sup_{|x|\leq 1} g(x) \leq b_g$ and $a b_g > 1$. If $n - 1 < |x| \leq n$, then using the submultiplicative property of g we have

$$g(x) = g\left(\sum_{i=1}^{n} \frac{1}{n} x\right) \leq a^{n-1} g(\frac{1}{n} x)^n \leq a^{n-1} b_g^n = b_g (a b_g)^{n-1} \leq b_g (a b_g)^{|x|} = b_g \exp(\log(a b_g)|x|).$$

Letting $c := \log(a b_g)$ the result is then proved. □

The following result tells us when g-moments of a Lévy process exist. In Kyprianou (2014) only the case of the exponential function is considered. Thus the proof is mainly based in the ideas of Sato (1999).

Theorem 1.3.15. *Suppose that g is submultiplicative and bounded on compacts. Then $\mathbb{E}(g(X_t)) < \infty$ for all $t > 0$ if and only if $\int_{\{|x|\geq 1\}} g(x)\Pi(dx) < \infty$.*

Proof. Recall $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$ given in the Lévy–Itô decomposition. Note, in particular, that $X^{(2)}$ is a compound Poisson process with arrival rate $\lambda = \Pi(\mathbb{R} \setminus (-1,1))$ and jump distribution $F(dx) := \mathbb{I}_{\{|x|\geq 1\}}\Pi(dx)/\Pi(\mathbb{R} \setminus (-1,1))$ and $X^{(1)} + X^{(3)}$ is a Lévy process with Lévy measure $\mathbb{I}_{\{|x|<1\}}\Pi(dx)$. First suppose that $\mathbb{E}(g(X_t)) < \infty$ for all $t > 0$. Since

$$\infty > \mathbb{E}(g(X_t)) = \int_{\mathbb{R}}\int_{\mathbb{R}} g(x+y)\mathbb{P}(X_t^{(2)} \in dx)\mathbb{P}(X_t^{(1)} + X_t^{(3)} \in dy),$$

we have that $\mathbb{E}(g(X_t^{(2)} + y)) = \int_{\mathbb{R}} g(x+y)\mathbb{P}(X_t^{(2)} \in dx) < \infty$ for some $y \in \mathbb{R}$. Hence, as $X^{(2)}$ is a compound Poisson process,

$$\begin{aligned}
\mathbb{E}(g(X_t^{(2)} + y)) &= e^{-\lambda t}\sum_{k\geq 0}\frac{(\lambda t)^k}{k!}\int_{\mathbb{R}} g(x+y)F^{*k}(dx) \\
&= e^{-\Pi(\mathbb{R}\setminus(-1,1))t}\sum_{k\geq 0}\frac{t^k}{k!}\int_{\mathbb{R}} g(x+y)(\Pi|_{\mathbb{R}\setminus(-1,1)})^{*k}(dx) < \infty,
\end{aligned}$$

where F^{*n} and $(\Pi|_{\mathbb{R}\setminus(-1,1)})^{*n}$ are the n-fold convolution of F and $\Pi|_{\mathbb{R}\setminus(-1,1)}$, the restriction of Π to $\mathbb{R}\setminus(-1,1)$, respectively. Since g is a multiplicative function and by Lemma 1.3.14, there exist constants b_g and c_g such that,

$$g(x) \leq ag(-y)g(x+y) \leq ab_g\exp(c_g|y|)g(x+y).$$

Then we get

$$\sum_{k\geq 0}\frac{t^k}{k!}\int_{\mathbb{R}} g(x)(\Pi|_{\mathbb{R}\setminus(-1,1)})^{*k}(dx) < \infty.$$

It follows that for all $k \geq 0$, $\int_{\mathbb{R}} g(x)(\Pi|_{\mathbb{R}\setminus(-1,1)})^{*k}(dx) < \infty$. In particular if we take $k = 1$ we obtain that

$$\int_{\{|x|\geq 1\}} g(x)\Pi(dx) < \infty.$$

For the opposite implication, suppose that $\int_{\{|x|\geq 1\}} g(x)\Pi(dx) < \infty$. By the submultiplicativity property of g,

$$\begin{aligned}
\int_{\mathbb{R}} g(x)(\Pi|_{\mathbb{R}\setminus(-1,1)})^{*n}(dx) &= \int_{\mathbb{R}}\cdots\int_{\mathbb{R}} g(x_1+\cdots+x_n)\Pi|_{\mathbb{R}\setminus(-1,1)}(dx_1)\cdots\Pi|_{\mathbb{R}\setminus(-1,1)}(dx_n) \\
&\leq a^{n-1}\left(\int_{|x|\geq 1} g(x)\Pi(dx)\right)^n \\
&< \infty.
\end{aligned}$$

Then $\sum_{k\geq 0} t^k/k!\int_{\mathbb{R}} g(x)(\Pi|_{\mathbb{R}\setminus(-1,1)})^{*k}(dx) < \infty$ and hence $g(X_t^{(2)})$ has finite moment for every $t \geq 0$. Since g is submultiplicative and from Lemma 1.3.14,

$$\mathbb{E}(g(X_t)) = \mathbb{E}(g(X_t^{(2)} + X_t^{(1)} + X_t^{(3)})) \le a\mathbb{E}(g(X_t^{(1)}) + X_t^{(3)})\mathbb{E}(g(X_t^{(2)})) \le ab_g\mathbb{E}(e^{c_g|X_t^{(1)}+X_t^{(3)}|})\mathbb{E}(g(X_t^{(2)})). \tag{1.10}$$

Then it remains to prove that $\mathbb{E}(e^{c_g|X_t^{(1)}+X_t^{(3)}|}) < \infty$.

However, since $X^{(1)} + X^{(3)}$ is a Lévy process whose Lévy measure has bounded support, it follows that its characteristic exponent

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-1,1)} (1 - e^{i\theta x} + i\theta x)\Pi(dx), \qquad \theta \in \mathbb{R}$$

can be extended to an analytic function on $\mathbb{C}$. To see why note that using the Taylor expansion of $e^{i\theta x}$ we obtain

$$\int_{(-1,1)} (1 - e^{i\theta x} + i\theta x)\Pi(dx) = -\int_{(-1,1)} \sum_{k\ge 0} \frac{(i\theta x)^{k+2}}{(k+2)!}\Pi(dx).$$

The sum and the integral on the above expression may be exchanged using Fubini's theorem and the estimate

$$\sum_{k\ge 0}\int_{(-1,1)} \frac{|\theta x|^{k+2}}{(k+2)!}\Pi(dx) \le \sum_{k\ge 0}\frac{|\theta|^{k+2}}{(k+2)!}\int_{(-1,1)} x^2\Pi(dx) < \infty.$$

Therefore, Ψ can be extended to an analytic function to the whole complex plane $\mathbb{C}$. Define $\mu_t(dx) = \mathbb{P}(X_t^{(1)} + X_t^{(3)} \in dx)$. The above guarantees that

$$\widehat{\mu}_t(\theta) = e^{-\Psi(\theta)t} = \int_{\mathbb{R}} e^{i\theta x}\mu_t(dx)$$

is also an analytic function. In consequence, all moments of μ_t exists and we can write

$$\widehat{\mu}_t(\theta) = \sum_{n\ge 0}\frac{1}{n!} i^n m_n(t)\theta^n, \qquad \theta \in \mathbb{C},$$

where $m_n(t) = \int_{\mathbb{R}} x^n \mu_t(dx)$ for every $n \in \mathbb{N}$. Since $\widehat{\mu}_t$ is analytic then the above sum is absolutely convergent for all $\theta \in \mathbb{C}$.

Now define $a_n(t) = \int_{\mathbb{R}} |x|^n\mu_t(dx)$ for every $n \in \mathbb{N}$. Notice that $a_{2k} = m_{2k}(t)$ and $a_{2k+1}(t) \le \frac{1}{2}(m_{2k}(t) + m_{2k+2}(t))$ where the latter follows on account of the fact

$$|x|^{2k+1} \le \frac{1}{2}(x^{2k} + x^{2k+2}), \qquad x \in \mathbb{R}.$$

We thus have that

$$\mathbb{E}(e^{c_g|X_1^{(1)}+X_t^{(3)}|}) = \int_{\mathbb{R}} e^{c_g|x|}\mu_t(dx) = \sum_{n\ge 0}\frac{1}{n!}a_n(t)c_g^n < \infty,$$

where the final equality is justified by writing $e^{\beta|x|}$ as a power series and then using Fubini's theorem, the estimates for $a_n(t)$ and the absolute convergence of $\widehat{\mu}_t$. In conclusion $\mathbb{E}(e^{c_g|X_1^{(1)}+X_t^{(3)}|}) < \infty$ and then by (1.10) we have that $\mathbb{E}(g(X_t)) < \infty$ for all $t \ge 0$.

□

We list some properties of submultiplicative functions.

Proposition 1.3.16. *Let f, g be submultiplicative functions then:*

i) The product fg is also submultiplicative.

ii) For all $\alpha > 0$, $c \in \mathbb{R}$ and $\gamma \in \mathbb{R}$ the function $g(cx+\gamma)^\alpha$ is submultiplicative.

Here are some applications to g-moments of Lévy processes.

Corollary 1.3.17. *Let X be a Lévy process with triple (a, σ, Π) then:*

i) For every $\beta \in \mathbb{R}$, $\mathbb{E}(e^{\beta X_t}) < \infty$ for all $t \geq 0$ if and only if $\int_{\{|x|\geq 1\}} e^{\beta x}\Pi(dx) < \infty$.

ii) For every $p \geq 0$, $\mathbb{E}(|X_t|^p) < \infty$ for all $t \geq 0$ if and only if $\int_{\{|x|\geq 1\}} |x|^p\Pi(dx) < \infty$.

Proof. The functions $f(x) = \exp(\beta x)$ and $g(x) = |x|^p \vee 1$ are submultiplicative functions. □

Remark 1.3.18. *As we have seen already, a Lévy process has finite first moment if and only if $\int_{\{|x|\geq 1\}} |x|\Pi(dx) < \infty$. Therefore, we can also compensate the big jumps to form a martingale, hence the Lévy–Ito decomposition of X takes the form*

$$X_t = \sigma B_t - a' t + \int_0^t \int_{\mathbb{R}} x(N(ds, dx) - \Pi(dx)ds). \tag{1.11}$$

And the characteristic exponent takes the form

$$\Psi(\theta) = a' i\theta + \frac{1}{2}\sigma^2\theta + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x)\Pi(dx),$$

where $a' = a - \int_{\{|x|\geq 1\}} x\Pi(dx)$.

Definition 1.3.19. *Let $\beta \in \mathbb{R}$ define the Laplace exponent as*

$$\psi(\beta) = \frac{1}{t}\log\left(\mathbb{E}(e^{\beta X_t})\right) = -\Psi(-i\beta), \tag{1.12}$$

whenever it exists.

We know that the Laplace exponent is finite if and only if $\int_{\{|x|\geq 1\}} e^{\beta x}\Pi(dx) < \infty$.

Proposition 1.3.20. *Let X be a spectrally negative Lévy process. Then $\mathbb{E}(e^{\beta X_t}) < \infty$ for any $t \geq 0$ and $\beta \in \mathbb{R}_+$. The function $\psi : [0, \infty) \mapsto \mathbb{R}$ is zero at zero and tends to infinity at infinity. Further, it is infinitely differentiable and strictly convex on $(0, \infty)$. In particular, $\psi'(0+) = \mathbb{E}(X_1) \in [-\infty, \infty)$.*

Proof. Suppose that X has Lévy triple (a, σ, Π). Since X has no positive jumps, $\Pi((0, \infty)) = 0$ and thus for any $\beta \in \mathbb{R}_+$

$$\int_{\{|x|\geq 1\}} e^{\beta x}\Pi(dx) = \int_{\{x \leq -1\}} e^{\beta x}\Pi(dx) \leq \Pi((-\infty, -1]) < \infty.$$

From Corollary 1.3.17 *i)* follows that $\mathbb{E}(e^{\beta X_t}) < \infty$ for all $t \geq 0$ and $\beta \in \mathbb{R}_+$. This implies that the characteristic exponent Ψ of X can be extended to complex β with negative imaginary part, and hence the Laplace exponent can be written as

$$\psi(\beta) = -\Psi(-i\beta) = -a\beta + \frac{1}{2}\sigma^2\beta^2 + \int_{\{x\leq -1\}} (e^{\beta x} - 1)\Pi(dx) + \int_{\{0>x>-1\}} (e^{\beta x} - 1 - \beta x)\Pi(dx), \qquad \beta > 0.$$

Note that for $x < -1$, $|e^{\beta x} - 1| \le 1$ and $0 > x > -1$ we have that,

$$|e^{\beta x} - 1 - \beta x| = \left| \sum_{k \ge 2} \frac{\beta^k}{k!} x^k \right| \le \sum_{k \ge 2} \frac{\beta^k}{k!} |x|^k \le e^{\beta} x^2.$$

Hence using dominated convergence theorem we obtain

$$\psi'(\beta) = -a + \sigma^2 \beta + \int_{\{x \le -1\}} x e^{\beta x} \Pi(dx) + \int_{\{0 > x > -1\}} (x e^{\beta x} - x) \Pi(dx). \tag{1.13}$$

Using similar arguments one may use dominated convergence and the integrability condition on Π to deduce that for $n \ge 2$,

$$\frac{\partial^n}{\partial \beta^n} \psi(\beta) = \sigma^2 \mathbb{I}_{\{n=2\}} + \int_{(-\infty,0)} x^n e^{\beta x} \Pi(dx). \tag{1.14}$$

Then ψ is infinitely differentiable in $(0, \infty)$. In particular we can see that $\psi''(\beta) > 0$ implying that ψ is strictly convex on $(0, \infty)$.

From the definition of ψ we have that

$$\psi(0) = \frac{1}{t} \log \left(\mathbb{E}(e^{(0) X_t}) \right) = 0.$$

To show that ψ is infinity at infinity note that

$$e^{\psi(\beta)} = \mathbb{E}(e^{\beta X_1}) \ge \mathbb{E}(e^{\beta X_1} \mathbb{I}_{\{X_1 > 0\}})$$

and using the monotone convergence Theorem we obtain that $\mathbb{E}(e^{\beta X_1} \mathbb{I}_{\{X_1 > 0\}}) \to \infty$ as $\beta \to \infty$ since in the spectrally negative case $\mathbb{P}(X_1 > 0) > 0$. Finally, we know that

$$\mathbb{E}(X_1) = \frac{d}{d\beta} \mathbb{E}(e^{\beta X_1}) \Big|_{\beta=0} = \frac{d}{d\beta} e^{\psi(\beta)} \Big|_{\beta=0} = \psi'(0+).$$

Using dominated convergence theorem in (1.13) we obtain

$$\psi'(0+) = -a + \int_{\{x \le -1\}} x \Pi(dx).$$

Note that $\int_{\{x \le -1\}} x \Pi(dx) \in [-\infty, 0]$ and therefore $\psi'(0+) \in [-\infty, \infty)$.

□

Now we introduce some martingales that are driven by Lévy processes.

Proposition 1.3.21. *Let X be a Lévy process with Lévy triple (a, σ, Π), characteristic exponent Ψ and Laplace exponent ψ.*

i) If $\int_{\{|x| \ge 1\}} |x| \Pi(dx) < \infty$, then X is a martingale if and only if

$$a - \int_{\{|x| \ge 1\}} x \Pi(dx) = 0.$$

ii) If $\int_{\{|x| \ge 1\}} |x| \Pi(dx) < \infty$, then $\{X_t - \mathbb{E}(X_t), t \ge 0\}$ is a martingale.

iii) *If $\int_{\{|x|\geq 1\}} e^{\beta x}\Pi(dx) < \infty$, for some $\beta \in \mathbb{R}$, then $M = \{M_t, t \geq 0\}$ is a unit mean martingale, where*

$$M_t = e^{\beta X_t - t\psi(\beta)}, \qquad t \geq 0.$$

iv) *The process $N = \{N_t, t \geq 0\}$ is a martingale, where*

$$N_t = e^{i\theta X_t + t\Psi(\theta)}, \qquad t \geq 0.$$

Proof. By Theorem 1.3.15 and the integrability conditions of Π we have that $\mathbb{E}(|X_t|) < \infty$ and $\mathbb{E}(e^{\beta X_t}) < \infty$. Since X_t is an infinitely divisible distribution and by an argument similar to the one used in Proposition 1.1.9 we can see that $\mathbb{E}(X_t) = t\mathbb{E}(X_1)$ and $\mathbb{E}(e^{\beta X_t}) = e^{\psi(\beta)t}$ for all $t \geq 0$. Using Remark 1.3.18, more specifically (1.11), we have that

$$\mathbb{E}(X_1) = a - \int_{\{|x|\geq 1\}} x\Pi(dx).$$

i) Let $s, t \geq 0$ then using the stationary and independent increments of X we have

$$\mathbb{E}(X_{t+s}|\mathcal{F}_t) = \mathbb{E}(X_{t+s} - X_t) + X_t = \mathbb{E}(X_s) + X_t$$

Then X is a martingale if and only if $\mathbb{E}(X_t) = 0$ for all $t \geq 0$ which happens if and only if $a - \int_{\{|x|\geq 1\}} x\Pi(dx) = 0$.

ii) Notice that $X_t - \mathbb{E}(X_t) = X_t - t\mathbb{E}(X_1)$ is a Lévy process with mean 0. Then by *i)*, $\{X_t - \mathbb{E}(X_t), t \geq 0\}$ is martingale.

iii) Let $s, t \geq 0$ then using the stationary and independent increments of X we have

$$\begin{aligned}
\mathbb{E}(M_{t+s}|\mathcal{F}_t) &= \mathbb{E}(e^{\beta X_{t+s}}|\mathcal{F}_t)e^{-(t+s)\psi(\beta)} \\
&= \mathbb{E}(e^{\beta(X_{t+s}-X_t)})e^{\beta X_t}e^{-(t+s)\psi(\beta)} \\
&= \mathbb{E}(e^{\beta X_s})e^{\beta X_t}e^{-(t+s)\psi(\beta)} \\
&= e^{s\psi(\beta)}e^{\beta X_t}e^{-(t+s)\psi(\beta)} \\
&= M_t.
\end{aligned}$$

Therefore M is a martingale.

iv) Note that $\mathbb{E}(|e^{i\theta X_t}|) \leq 1$ for all $\theta \in \mathbb{R}$ and hence N_t is integrable for all $t \geq 0$. Let $s, t \geq 0$ then using the stationary and independent increments of X we have

$$\begin{aligned}
\mathbb{E}(N_{t+s}|\mathcal{F}_t) &= \mathbb{E}(e^{i\theta X_{t+s}}|\mathcal{F}_t)e^{(t+s)\Psi(\theta)} \\
&= \mathbb{E}(e^{i\theta(X_{t+s}-X_t)})e^{i\theta X_t}e^{(t+s)\Psi(\theta)} \\
&= \mathbb{E}(e^{i\theta X_s})e^{i\theta X_t}e^{(t+s)\Psi(\theta)} \\
&= e^{-s\Psi(\beta)}e^{i\theta X_t}e^{(t+s)\Psi(\beta)} \\
&= N_t.
\end{aligned}$$

Therefore N is a martingale.

$\square$

Roughly speaking, a change of measure transforms the initial probability $\mathbb{P}$ into another probability measure $\widetilde{\mathbb{P}}$ (which is equivalent to $\mathbb{P}$) in such a way that the process X is a simpler process under the measure $\widetilde{\mathbb{P}}$. Theorem 1.3.21 *iii*) gives a criterion under which we can perform an exponential change of measure.

Definition 1.3.22. *Let $\beta \in \mathbb{R}$ and suppose that $\psi(\beta) < \infty$. Define the process $\mathcal{E}(\beta) = \{\mathcal{E}_t(\beta), t \geq 0\}$ where*

$$\mathcal{E}_t(\beta) = e^{\beta X_t - \psi(\beta)t}, \qquad t \geq 0.$$

Since $\mathcal{E}(\beta)$ is a unit mean martingale, it may be used to perform a change of measure via

$$\left.\frac{d\mathbb{P}^\beta}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}_t(\beta), \qquad t \geq 0. \tag{1.15}$$

The process $\mathcal{E}(\beta)$ is known as the Esscher transform. An important result related to the Esscher transform is that the process X is again a Lévy process under the new measure $\mathbb{P}^\beta$.

Theorem 1.3.23. *Suppose that X is a Lévy process with characteristic triple (a, σ, Π), and that $\beta \in \mathbb{R}$ is such that*

$$\int_{\{|x|\geq 1\}} e^{\beta x}\Pi(dx) < \infty.$$

Under the change of measure $\mathbb{P}^\beta$, the process X is still a Lévy process with characteristic triple (a^*, σ^*, Π^*), where

$$a^* = a - \beta\sigma^2 + \int_{\{|x|<1\}} (1 - e^{\beta x})x\Pi(dx), \qquad \sigma^* = \sigma \qquad \textit{and} \qquad \Pi^*(dx) = e^{\beta x}\Pi(dx).$$

Proof. It is easy to show that under $\mathbb{P}^\beta$, X is again a Lévy process (see Kyprianou (2014)). For purposes of this work we only deduce the Lévy triplet of X under $\mathbb{P}^\beta$. Let $t > 0$, then

$$\begin{aligned} e^{-\Psi_\beta(\theta)t} &= \mathbb{E}^\beta(e^{i\theta(X_t)}) \\ &= \mathbb{E}(e^{(i\theta+\beta)(X_t) - \psi(\beta)t}) \\ &= \exp(\psi(i\theta + \beta)t - \psi(\beta)t) \\ &= \exp(-\Psi(\theta - i\beta)t + \Psi(-i\beta)t), \end{aligned}$$

where $\mathbb{E}^\beta$ is the expectation under the measure $\mathbb{P}^\beta$. Thus

$$\Psi_\beta(\theta) = \Psi(\theta - i\beta) - \Psi(-i\beta).$$

By writing out the characteristic exponent Ψ_β in terms of the triple (a, σ, Π) associated with X under $\mathbb{P}$, we see that for all $\theta \in \mathbb{R}$,

$$\begin{aligned} \Psi_\beta(\theta) &= ai(\theta - i\beta) + \frac{1}{2}\sigma^2(\theta - i\beta)^2 + \int_{\mathbb{R}} (1 - e^{i(\theta - i\beta)x} + i(\theta - i\beta)x\mathbb{I}_{\{|x|<1\}})\Pi(dx) \\ &\quad - \left(ai(-i\beta) + \frac{1}{2}\sigma^2(-i\beta)^2 + \int_{\mathbb{R}} (1 - e^{i(-i\beta)x} + i(-i\beta)x\mathbb{I}_{\{|x|<1\}})\Pi(dx)\right) \\ &= i\theta\left(a - \beta\sigma^2 + \int_{\{|x|<1\}} (1 - e^{\beta x})x\Pi(dx)\right) + \frac{1}{2}\theta^2\sigma^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x\mathbb{I}_{\{|x|<1\}})e^{\beta x}\Pi(dx). \end{aligned}$$

Therefore the triple (a^*, σ^2, Π^*) is given as the statement of the theorem.

□

As a consequence, we get for the spectrally negative case the following result.

Corollary 1.3.24. *The Esscher transform may be applied for all $\beta \geq 0$ when X is a spectrally negative Lévy process. Further, under $\mathbb{P}^\beta$, X remains within the class of spectrally negative Lévy processes. The Laplace exponent, ψ_β, of X under $\mathbb{P}^\beta$ satisfies*

$$\psi_\beta(\theta) = \psi(\theta + \beta) - \psi(\beta),$$

for all $\theta \geq -\beta$.

Proof. Note that the change measure using the Esscher transform does not change the support of the Lévy measure, i.e. $\Pi^* = e^{\beta x}\Pi(dx)$ has support on $(-\infty, 0)$. Using the identity between Ψ and ψ we obtain that for all $\theta \geq -\beta$,

$$\psi_\beta(\theta) = -\Psi_\beta(-i\theta) = -\Psi(-i\theta - i\beta) + \Psi(-i\beta) = \psi(\theta + \beta) - \psi(\beta).$$

□

Corollary 1.3.25. *Suppose that X is a Lévy process with chracteristic triple (a, σ, Π) and β is such that*

$$\int_{\{|x|\geq 1\}} e^{\beta x}\Pi(dx) < \infty.$$

If τ is a stopping time, then

$$\left.\frac{d\mathbb{P}^\beta}{d\mathbb{P}}\right|_{\mathcal{F}_\tau} = \mathcal{E}_\tau(\beta) \text{ on } \{\tau < \infty\}.$$

Proof. Suppose that $A \in \mathcal{F}_\tau$ then by definition of $\mathcal{F}_\tau$, given in the Appendix A.1, we have that $A \cap \{\tau \leq t\}$. Hence conditioning with respect to $\mathcal{F}_\tau$

$$\begin{aligned}
\mathbb{P}^\beta(A \cap \{\tau \leq t\}) &= \mathbb{E}(\mathcal{E}_t(\beta)\mathbb{I}_{A\cap\{\tau\leq t\}}) \\
&= \mathbb{E}(\mathbb{E}(\mathcal{E}_t(\beta)\mathbb{I}_{A\cap\{\tau\leq t\}}|\mathcal{F}_\tau)) \\
&= \mathbb{E}(\mathbb{I}_{A\cap\{\tau\leq t\}}\mathbb{E}(\mathcal{E}_t(\beta)|\mathcal{F}_\tau)) \\
&= \mathbb{E}(\mathbb{I}_{A\cap\{\tau\leq t\}}e^{-\psi(\beta)t}e^{\beta X_\tau}\mathbb{E}(e^{\beta(X_t - X_\tau)}|\mathcal{F}_\tau)) \\
&= \mathbb{E}(\mathbb{I}_{A\cap\{\tau\leq t\}}e^{-\psi(\beta)t}e^{\beta X_\tau}\mathbb{E}(e^{\beta(X_{t-\tau})}) \\
&= \mathbb{E}(\mathbb{I}_{A\cap\{\tau\leq t\}}\mathcal{E}_\tau(\beta)\mathbb{E}(\mathcal{E}_{t-\tau}(\beta))) \\
&= \mathbb{E}(\mathbb{I}_{A\cap\{\tau\leq t\}}\mathcal{E}_\tau(\beta))
\end{aligned}$$

where the fifth and the last equality follows by the strong Markov property as well as the martingale property of $\mathcal{E}$. Now taking limits, as $t \uparrow \infty$ and using the monotone convergence Theorem,

$$\mathbb{P}^\beta(A) = \mathbb{E}(\mathcal{E}_\tau(\beta)\mathbb{I}_A)$$

for all $A \in \mathcal{F}_\tau$ and the theorem is now proved. □

Now we give some applications of the Esscher transform $\mathcal{E}(\beta)$ when X is a spectrally negative Lévy process. For this purpose define the right inverse of the function ψ as

$$\Phi(q) = \sup\{\lambda \geq 0 : \psi(\lambda) = q\} \tag{1.16}$$

for each $q \geq 0$.

Note that from Proposition 1.3.20 we know that, on $[0,\infty)$, ψ is infinitely differentiable, strictly convex and that $\psi(0)=0$ whilst $\psi(\infty)=\infty$ and as consequence of these facts, $\mathbb{E}(X_1)=\psi'(0+)\in[-\infty,\infty)$. In the case that $\mathbb{E}(X_1)\geq 0$, $\Phi(q)$ is the unique solution to $\psi(\theta)=q$ in $[0,\infty)$, in particular $\Phi(0)=0$. When $\mathbb{E}(X_1)<0$ the previous statement is true only when $q>0$. If $\mathbb{E}(X_1)<0$ and $q=0$, then there are two roots to the equation $\psi(\theta)=0$, one of them being $\theta=0$ and the other being $\Phi(0)>0$.

Let τ_x^+ the first-passage time above the level $x\geq 0$, i.e.

$$\tau_x^+=\inf\{t>0: X_t>x\}. \tag{1.17}$$

Theorem 1.3.12 implies that τ_x^+ is a stopping time. In the following theorem we give an expression for the Laplace transform of τ_x^+.

Theorem 1.3.26. *For any spectrally negative Lévy process,*

$$\mathbb{E}(e^{-q\tau_x^+}\mathbb{I}_{\{\tau_x^+<\infty\}})=e^{-\Phi(q)x}$$

where $q\geq 0$, $x\geq 0$ and $\Phi(q)$ is the largest root of the equation $\psi(\theta)=q$.

Proof. Fix $q>0$. From the fact that X is a spectrally negative Lévy process we deduce that the process $\{\mathcal{E}_t(\Phi(q)), t\geq 0\}$ is martingale. Using the Doob's stopping time theorem (Theorem A.1.28) we have that $\{\mathcal{E}_{t\wedge\tau_x^+}(\Phi(q)), t\geq 0\}$ is also a martingale, then

$$\mathbb{E}(e^{\Phi(q)X_{t\wedge\tau_x^+}-q(t\wedge\tau_x^+)})=1.$$

Note that if $t<\tau_x^+$ then $X_t\leq x$ and since X is spectrally negative then it creeps upwards, i.e. when $\tau_x^+<\infty$ we have that $X_{\tau_x^+}=x$, then $X_{t\wedge\tau_x^+}\leq x$ under the set $\{\tau_x^+<\infty\}$ which implies that

$$e^{\Phi(q)X_{t\wedge\tau_x^+}-q(t\wedge\tau_x^+)}\leq e^{\Phi(q)X_{t\wedge\tau_x^+}}\leq e^{\Phi(q)x}.$$

Hence, using the dominated convergence theorem we have

$$\mathbb{E}(e^{\Phi(q)x-q\tau_x^+}\mathbb{I}_{\{\tau_x^+<\infty\}})=\mathbb{E}(e^{\Phi(q)X_{\tau_x^+}-q\tau_x^+}\mathbb{I}_{\{\tau_x^+<\infty\}})=\lim_{t\to\infty}\mathbb{E}(e^{\Phi(q)X_{t\wedge\tau_x^+}-q(t\wedge\tau_x^+)})=1.$$

It follows that

$$\mathbb{E}(e^{-q\tau_x^+}\mathbb{I}_{\{\tau_x^+<\infty\}})=e^{-\Phi(q)x}.$$

For the case $q=0$ we may take limits and use the monotone convergence theorem.

□

Corollary 1.3.27. *From the previous theorem, we have that for $x\geq 0$, $\mathbb{P}(\tau_x^+<\infty)=e^{-\Phi(0)x}$. In particular, if $x>0$ we have that $\mathbb{P}(\tau_x^+<\infty)=1$ if and only if $\Phi(0)=0$, if and only if $\psi'(0+)\geq 0$, if and only if $\mathbb{E}(X_1)\geq 0$.*

The following corollary gives a sufficient condition for τ_x^+ to have finite expectation. This result will be useful in the development of the third chapter.

Corollary 1.3.28. *Supose that $\psi'(0+)>0$. Then $\mathbb{E}(\tau_x^+)<\infty$ for all $x\geq 0$.*

Proof. We know from Theorem 1.3.26 and the previous corollary that

$$\mathcal{L}(q)=\mathbb{E}(e^{-q\tau_x^+})=e^{-\Phi(q)x}.$$

Calculating the first derivative of $\mathcal{L}$ we have that

$$\mathcal{L}'(q)=-\Phi'(q)xe^{-\Phi(q)x}.$$

Using the well-known result which links the moments and derivatives of the Laplace transform (see Feller (1971) (section XIII.2)), the expectation of τ_x^+ is given by

$$\mathbb{E}(\tau_x^+) = -\mathcal{L}'(0) = \Phi'(0)xe^{-\Phi(0)x} = \Phi'(0)x.$$

Hence τ_r^+ has finite moment if $\Phi'(0)$ is finite, thus we calculate the term $\Phi'(0)$. Using the strict convexity of ψ and the inverse function theorem we obtain that

$$\Phi'(0) = \frac{1}{\psi'(\Phi(0)+)} = \frac{1}{\psi'(0+)} < \infty.$$

In conclusion τ_x^+ has finite moment for all $x \geq 0$. □

Wiener–Hopf Factorisation

A fundamental part within the theory of Lévy processes is the Wiener–Hopf factorisation, which was first given in the theory of random walks. The Wiener–Hopf factorisation has many applications, and it serves as a tool to extract some generic results concerning coarse and fine path properties of Lévy processes. For a detailed study of this factorisation it is required to know some aspects of the theory of excursions, local time and the maximum and ladder processes. As all these issues are beyond the scope of this work, we only mention a small part of all the set of conclusions concerning to the Wiener–Hopf factorisation.

Before we state the Wiener–Hopf factorisation, we give some tools that will help us to give a more explicit expression in the case of spectrally negative Lévy processes. In the following lemma we discuss what happens when we take a process resulting of a time reversal of a Lévy process. It turns out that the new process is equal in law to the negative of the original process.

Lemma 1.3.29 (Duality Lemma)**.** *For each fixed $t > 0$, define the reversed process*

$$\{X_{(t-s)^-} - X_t, 0 \leq s \leq t\}$$

and the dual process,

$$\{-X_s, 0 \leq s \leq t\}.$$

Then the two processes have the same law under $\mathbb{P}$.

Proof. See Kyprianou (2014) (Lemma 3.4). □

One interesting consequence of the Duality Lemma is the relationship between the running supremum, the running infimum, the process reflected in its supremum and the process reflected in its infimum.

Lemma 1.3.30. *For each fixed $t > 0$, the pairs $(\overline{X}_t, \overline{X}_t - X_t)$ and $(X_t - \underline{X}_t, -\underline{X}_t)$ have the same distribution under $\mathbb{P}$. Here $\overline{X}_t = \sup_{0\leq s\leq t} X_t$ and $\underline{X}_t = \inf_{0\leq s\leq t} X_t$ are the running supremum and the running infimum, respectively.*

Proof. Define $\widetilde{X} = \{\widetilde{X}_s, 0 \leq s \leq t\}$ where $\widetilde{X}_s = X_t - X_{(t-s)-}$ and write $\underline{\widetilde{X}}_t = \inf_{0\leq s\leq t} \widetilde{X}_s$. Using right-continuity and left limits of the paths, we have

$$-\underline{\widetilde{X}}_t = -\inf_{0\leq s\leq t}(X_t - X_{(t-s)-}) = \sup_{0\leq s\leq t} X_s - X_t = \overline{X}_t - X_t.$$

Therefore

$$(\overline{X}_t, \overline{X}_t - X_t) = (\widetilde{X}_t - \underline{\widetilde{X}}_t, -\underline{\widetilde{X}}_t) \qquad \mathbb{P}\text{-a.s.}$$

Now from Duality Lemma, we know that $\{\widetilde{X}_s, 0 \leq s \leq t\}$ is equal in law to $\{X_s, 0 \leq s \leq t\}$ under $\mathbb{P}$ and the result follows. □

We need to introduce some additional notation before give the main statement of the Wiener–Hopf factorisation. For $p > 0$ we shall understand $\mathbf{e}_p$ to be an independent random variable which is exponentially distributed with mean $1/p$. Further, we define the last time that a process reaches its minimum and its maximum before time t, respectively, as follows

$$\overline{G}_t = \sup\{s < t : \overline{X}_s = X_s\} \qquad \text{and} \qquad \underline{G}_t = \sup\{s < t : \underline{X}_s = X_s\}.$$

Theorem 1.3.31 (The Wiener–Hopf factorisation)**.** *Suppose that X is any Lévy process other than a compound Poisson process. Denote by $\mathbf{e}_p$ an independent and exponentially distributed random variable with parameter $p > 0$.*

i) The pairs

$$(\overline{G}_{\mathbf{e}_p}, \overline{X}_{\mathbf{e}_p}) \qquad \textit{and} \qquad (\mathbf{e}_p - \overline{G}_{\mathbf{e}_p}, \overline{X}_{\mathbf{e}_p} - X_{\mathbf{e}_p})$$

are independent and infinitely divisible, yielding the factorisation

$$\frac{p}{p - \vartheta + \Psi(\theta)} = \Psi_p^+(\vartheta, \theta) \cdot \Psi_p^-(\vartheta, \theta), \tag{1.18}$$

where $\theta, \vartheta \in \mathbb{R}$,

$$\Psi_p^+(\vartheta, \theta) = \mathbb{E}(e^{i\vartheta \overline{G}_{\mathbf{e}_p} + i\theta \overline{X}_{\mathbf{e}_p}}) \qquad \textit{and} \qquad \Psi_p^-(\vartheta, \theta) = \mathbb{E}(e^{i\vartheta \underline{G}_{\mathbf{e}_p} + i\theta \underline{X}_{\mathbf{e}_p}}).$$

Here, the pair $\Psi_p^+(\vartheta, \theta)$ and $\Psi_p^-(\vartheta, \theta)$ are called the Wiener–Hopf factors.

Proof. See Kyprianou (2014) (Theorem 6.15). □

In the case of spectrally negative Lévy processes we obtain a more explicit factorisation.

Corollary 1.3.32. *Suppose that X be a spectrally negative Lévy process with characteristic exponent Ψ. Let $\mathbf{e}_p$ an independent and exponentially distributed random variable with parameter $p > 0$. Then $\overline{X}_{\mathbf{e}_p}$ is exponentially distributed with parameter $\Phi(p)$ and*

$$\mathbb{E}(e^{i\theta \underline{X}_{\mathbf{e}_p}}) = \frac{p}{\Phi(p)} \frac{\Phi(p) - i\theta}{p + \Psi(\theta)}. \tag{1.19}$$

Proof. From the above theorem we have that the variables $X_{\mathbf{e}_p} - \underline{X}_{\mathbf{e}_p}$ and $-\underline{X}_{\mathbf{e}_p}$ are independent. Moreover, from Lemma 1.3.30 we have that $X_t - \underline{X}_t$ is equal in distribution to $\overline{X}_t$. Then for all $\theta \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}(e^{i\theta X_{\mathbf{e}_p}}) &= \mathbb{E}(e^{i\theta(X_{\mathbf{e}_p} - \underline{X}_{\mathbf{e}_p})} e^{i\theta \underline{X}_{\mathbf{e}_p}}) \\
&= \mathbb{E}(e^{i\theta \underline{X}_{\mathbf{e}_p}}) \mathbb{E}(e^{i\theta(X_{\mathbf{e}_p} - \underline{X}_{\mathbf{e}_p})}) \\
&= \mathbb{E}(e^{i\theta \underline{X}_{\mathbf{e}_p}}) \int_0^\infty p e^{-pt} \mathbb{E}(e^{i\theta(X_t - \underline{X}_t)}) dt \\
&= \mathbb{E}(e^{i\theta \underline{X}_{\mathbf{e}_p}}) \int_0^\infty p e^{-pt} \mathbb{E}(e^{i\theta \overline{X}_t}) dt \\
&= \mathbb{E}(e^{i\theta \underline{X}_{\mathbf{e}_p}}) \mathbb{E}(e^{i\theta \overline{X}_{\mathbf{e}_p}}).
\end{aligned}$$

Then for all $\theta \in \mathbb{R}$,

$$\mathbb{E}(e^{i\theta X_{\mathbf{e}_p}}) = \mathbb{E}(e^{i\theta \underline{X}_{\mathbf{e}_p}}) \mathbb{E}(e^{i\theta \overline{X}_{\mathbf{e}_p}}). \tag{1.20}$$

On the other hand, note that $\{\overline{X}_t > a\} = \{\tau_a^+ < t\}$ for all $t \geq 0$, where $\tau_a^+ = \inf\{t > 0 : X_t > a\}$. Then, Theorem 1.3.26 and Fubini's theorem implies,

$$\begin{aligned}
\mathbb{P}(\overline{X}_{\mathbf{e}_p} > a) &= \mathbb{P}(\tau_a^+ < \mathbf{e}_p) \\
&= \int_0^\infty pe^{-pt}\mathbb{P}(\tau_a^+ < t)dt \\
&= \int_0^\infty pe^{-pt}\int_0^t \mathbb{P}(\tau_a^+ \in dy)dt \\
&= \int_0^\infty \mathbb{P}(\tau_a^+ \in dy)\int_y^\infty pe^{-pt}dt \\
&= \int_0^\infty e^{-py}\mathbb{P}(\tau_a^+ \in dy) \\
&= \mathbb{E}(e^{-p\tau_a^+}\mathbb{I}_{\{\tau_a^+ < \infty\}}) \\
&= e^{-\Phi(p)a}.
\end{aligned}$$

Thus $\overline{X}_{\mathbf{e}_p}$ is exponentially distributed with parameter $\Phi(p)$ and therefore for all $\theta \in \mathbb{R}$,

$$\mathbb{E}(e^{i\theta\overline{X}_{\mathbf{e}_p}}) = \frac{\Phi(p)}{\Phi(p) - i\theta}.$$

It follows from (1.20) that,

$$\begin{aligned}
\mathbb{E}(e^{i\theta\underline{X}_{\mathbf{e}_p}}) &= \frac{\mathbb{E}(e^{i\theta X_{\mathbf{e}_p}})}{\mathbb{E}(e^{i\theta\overline{X}_{\mathbf{e}_p}})} \\
&= \frac{\Phi(p) - i\theta}{\Phi(p)}\int_0^\infty pe^{-pt}\mathbb{E}(e^{i\theta X_t})dt \\
&= \frac{\Phi(p) - i\theta}{\Phi(p)}\int_0^\infty pe^{-pt}e^{-\Psi(\theta)t}dt \\
&= \frac{\Phi(p) - i\theta}{\Phi(p)}\frac{p}{p + \Psi(\theta)}.
\end{aligned}$$

□

Drifting and Oscillating

We are interested in the limit behaviour of Lévy processes. It can be shown that a Lévy process present only three options at infinity (see Kyprianou (2014)):

Definition 1.3.33. *i) We say that X drifts to infinity if*

$$\lim_{t\to\infty} X_t = \infty, \qquad \mathbb{P}\text{-a.s.}$$

ii) We say that X drifts to minus infinity if

$$\lim_{t\to\infty} X_t = -\infty, \qquad \mathbb{P}\text{-a.s.}$$

iii) We say that X oscillates if

$$\limsup_{t\to\infty} X_t = -\liminf_{t\to\infty} X_t = \infty, \qquad \mathbb{P}\text{-a.s.}$$

Due to the objectives stated in this work we only analyse the behaviour at infinity of spectrally negative Lévy processes. For more results about general Lévy processes see Kyprianou (2014) or Bertoin (1998).

Proposition 1.3.34. *Suppose that X is a spectrally negative Lévy process. Then we have the following:*

i) *X drifts to infinity if and only if $\psi'(0+) > 0$.*

ii) *X drifts to minus infinity if and only if $\psi'(0+) < 0$.*

iii) *X oscillates if and only if $\psi'(0+) = 0$.*

Proof. Thanks to the finiteness and convexity of its Laplace exponent, $\psi(\lambda) = \log(\mathbb{E}(e^{\lambda X_1}))$ on $\lambda \geq 0$, one always has that $\mathbb{E}(X_1) \in [-\infty, \infty)$. Note that from the definition of limit, if X drifts to infinity we have that $\mathbb{E}(X_t) > 0$ for some $t > 0$. Thus from the infinite divisibility property of X we deduce that $\psi'(0+) = \mathbb{E}(X_1) > 0$. In the same way, if X drifts to minus infinity then $\psi'(0+) = \mathbb{E}(X_1) < 0$.

Recall from the strict convexity of ψ it follows that $\psi'(0+) \geq 0$ if and only if $\Phi(0) = 0$ and hence

$$\lim_{q \downarrow 0} \frac{q}{\Phi(q)} = \begin{cases} \psi'(0+) & \text{if } \psi'(0+) \geq 0 \\ 0 & \text{if } \psi'(0+) < 0. \end{cases}$$

From Corollary 1.3.32 we know that if $\mathbf{e}_p$ is and exponentially distributed random variable with parameter $p > 0$ independent of X then

$$\mathbb{E}(e^{\beta \underline{X}_{\mathbf{e}_p}}) = \frac{p}{\Phi(p)} \frac{\Phi(p) - \beta}{p - \psi(\theta)} \qquad \text{and} \qquad \mathbb{E}(e^{-\beta \overline{X}_{\mathbf{e}_p}}) = \frac{\Phi(p)}{\Phi(p) + \beta}.$$

By taking $p \downarrow 0$ we obtain,

$$\mathbb{E}(e^{\beta \underline{X}_\infty}) = \begin{cases} \psi'(0+)\beta/\psi(\beta) & \text{if } \psi'(0+) \geq 0 \\ 0 & \text{if } \psi'(0+) < 0 \end{cases} \qquad \text{and} \qquad \mathbb{E}(e^{-\beta \overline{X}_{\mathbf{e}_p}}) = \begin{cases} 0 & \text{if } \psi'(0+) \geq 0 \\ \Phi(0)/(\Phi(0) + \beta) & \text{if } \psi'(0+) < 0. \end{cases}$$

This tells us that when $\psi'(0+) > 0$ then $\underline{X}_\infty > -\infty$ and $\overline{X}_\infty = \infty$ $\mathbb{P}$-a.s. Therefore

$$\liminf_{t \to \infty} X_t = \lim_{t \to \infty} \inf_{s > t} X_s \geq \underline{X}_\infty > -\infty \qquad \mathbb{P}\text{-a.s.}$$

Thus by trichotomy we have that X drifts to infinity. When $\psi'(0) < 0$ we obtain that $\underline{X}_\infty = \infty$ and $\overline{X}_\infty$ $\mathbb{P}$-a.s. Thus

$$\limsup_{t \to \infty} X_t = \lim_{t \to \infty} \sup_{s > t} X_s \leq \overline{X}_\infty < \infty \qquad \mathbb{P}\text{-a.s.}$$

Thus by trichotomy we have that X drifts to minus infinity. The statement *iii)* follows directly from *i)* and *ii)*. □

1.4 Spectrally Negative Lévy Process and Scale Functions

In this section we study in more detail the case of spectrally negative Lévy processes. Recall that this process only have negative jumps, i.e. $\Pi((0, \infty) = 0$. This condition turns out that to offer a significant advantage for many calculations. Specifically, we consider a special class of functions called scale functions. These functions are intimately related to some fluctuation identities which are semi-explicit in terms of the scale functions.

Existence of Scale Functions

Let us now turn our attention to the definition of scale functions.

Definition 1.4.1. *For a given spectrally negative Lévy process, X, with Laplace exponent ψ, we define a family of functions indexed by $q \geq 0$, $W^{(q)} : \mathbb{R} \mapsto [0, \infty)$, as follows. For each given $q \geq 0$, we have $W^{(q)}(x) = 0$ when $x < 0$ and otherwise on $[0, \infty)$, $W^{(q)}$ is the unique right continuous function whose Laplace transform is given by*

$$\int_0^\infty e^{-\beta x} W^{(q)}(x)dx = \frac{1}{\psi(\beta) - q}, \qquad \beta > \Phi(q).$$

For convenience we shall always write W in place of $W^{(0)}$. Typically we shall refer to the functions $W^{(q)}$ as q-scale functions, but shall also refer to W as just the scale function.

We are now ready to show the existence of scale functions. That is to say, we will show that the function $(\psi(\beta) - q)^{-1}$ is indeed a Laplace transform in β.

Theorem 1.4.2. *For a spectrally negative Lévy process, q-scale functions exist for $q \geq 0$.*

Proof. First assume that $\psi'(0+) > 0$, define the function

$$W(x) = \frac{1}{\psi'(0+)} \mathbb{P}_x(\underline{X}_\infty \geq 0). \tag{1.21}$$

From this definition we can see that $W(x) = 0$ for $x < 0$ and W is a non-decreasing and right-continuous function. Via analytical extension from (1.19) we may deduce that for $\beta \geq 0$,

$$\mathbb{E}(e^{\beta \underline{X}_{\mathbf{e}_p}}) = \frac{p}{\Phi(p)} \frac{\Phi(p) - \beta}{p - \psi(\beta)}.$$

Taking limits in the above expression as $p \downarrow 0$ we obtain

$$\mathbb{E}(e^{\beta \underline{X}_\infty}) = \lim_{p \downarrow 0} \frac{p}{\Phi(p)} \frac{\Phi(p) - \beta}{p - \psi(\beta)} = \psi'(0+) \frac{\beta}{\psi(\beta)}, \qquad \beta \geq 0, \tag{1.22}$$

where the last equality holds since $\Phi(0) = 0$ (see discussion above of (1.16)) and from the fact that $\psi(0) = 0$ we obtain

$$\psi'(0+) = \lim_{\theta \downarrow 0} \frac{\psi(\theta)}{\theta} = \lim_{q \downarrow 0} \frac{\psi(\Phi(q))}{\Phi(q)} = \lim_{q \downarrow 0} \frac{q}{\Phi(q)}.$$

Using Fubini's theorem and (1.22), we also see that for $\beta > 0 = \Phi(0)$,

$$\begin{aligned}
\int_0^\infty e^{-\beta x} W(x)dx &= \frac{1}{\psi'(0+)} \int_0^\infty e^{-\beta x} \mathbb{P}(-\underline{X}_\infty \leq x)dx \\
&= \frac{1}{\psi'(0+)} \int_0^\infty e^{-\beta x} \int_{[0,x]} \mathbb{P}(-\underline{X}_\infty \in dy)dx \\
&= \frac{1}{\psi'(0+)} \int_{[0,\infty)} \mathbb{P}(-\underline{X}_\infty \in dy) \int_y^\infty e^{-\beta x}dx \\
&= \frac{1}{\psi'(0+)\beta} \int_{[0,\infty)} e^{-\beta y} \mathbb{P}(-\underline{X}_\infty \in dy) \\
&= \frac{1}{\psi'(0+)\beta} \mathbb{E}(e^{\beta \underline{X}_\infty}) \\
&= \frac{1}{\psi'(0+)\beta} \psi'(0+) \frac{\beta}{\psi(\beta)} \\
&= \frac{1}{\psi(\beta)}.
\end{aligned}$$

Then the statement has been proved for $q = 0$ and $\psi'(0+) > 0$.

Now we deal with the case where $q > 0$ or $q = 0$ and $\psi'(0+) < 0$. Note that this assumption implies that $\Phi(q) > 0$. Using the change of measure $\mathbb{P}^{\Phi(q)}$ given in (1.15) and by Corollary 1.3.24 we have that under $\mathbb{P}^{\Phi(q)}$, X is again a spectrally negative Lévy process which Laplace exponent $\psi_{\Phi(q)}$ satisfies,

$$\psi_{\Phi(q)}(\theta) = \psi(\theta + \Phi(q)) - \psi(\Phi(q)) = \psi(\theta + \Phi(q)) - q. \tag{1.23}$$

Thus $\psi'_{\Phi(q)}(0+) = \psi'(\Phi(q)) > 0$ on account of the strict convexity of ψ. Define for all $x \in \mathbb{R}$,

$$W^{(q)}(x) = e^{\phi(q)x} W_{\Phi(q)}(x), \tag{1.24}$$

where $W_{\Phi(q)}(x) = \frac{1}{\psi'_{\Phi(q)}(0+)} \mathbb{P}_x^{\Phi(q)}(\underline{X}_\infty \geq 0)$. Clearly $W^{(q)}(x) = 0$ for $x < 0$ and is a non-decreasing function. Moreover, using the conclusion from the previous paragraph we have that for all $\beta > \Phi(q)$,

$$\begin{aligned}
\int_0^\infty e^{-\beta x} W^{(q)}(x)dx &= \int_0^\infty e^{-(\beta - \Phi(q))x} W_{\Phi(q)}(x)dx \\
&= \frac{1}{\psi_{\Phi(q)}(\beta - \Phi(q))} \\
&= \frac{1}{\psi(\beta) - q}.
\end{aligned}$$

Thus completing the proof for the case $q > 0$ or $q = 0$ and $\psi'(0+) < 0$. Finally, the case that $q = 0$ and $\psi'(0+) = 0$ can be dealt with as follows. Define for $p > 0$ the function $W_{\Phi(p)}(x)$ as before. Since $W_{\Phi(p)}$ is an non-decreasing function, we may also treat it as a distribution function of a measure which we also, as an abuse of notation, call $W^{\Phi(p)}$. Using Fubini's theorem we obtain for all $\beta > 0$,

$$\begin{aligned}
\int_{[0,\infty)} e^{-\beta x} W_{\Phi(p)}(dx) &= \int_{[0,\infty)} \int_x^\infty \beta e^{-\beta y} dy W_{\Phi(p)}(dx) \\
&= \int_0^\infty \beta e^{-\beta y} dy \int_{[0,y]} W_{\Phi(p)}(dx) \\
&= \int_0^\infty \beta e^{-\beta y} W_{\Phi(p)}(y)dy \\
&= \frac{\beta}{\psi_{\Phi(p)}(\beta)}.
\end{aligned}$$

Note that the assumption $\psi'(0+) = 0$, implies that $\Phi(0) = 0$, and hence for $\theta \geq 0$, $\lim_{p \downarrow 0} \psi_{\Phi(p)}(\theta) = \lim_{p \downarrow 0}[\psi(\theta + \Phi(p)) - p] = \psi(\theta)$. If follows that

$$\lim_{p \downarrow 0} \int_{[0,\infty)} e^{-\beta x} W_{\Phi(p)}(dx) = \frac{\beta}{\psi(\beta)}.$$

Appealing to the Extended Continuity Theorem for Laplace transforms, (see Feller (1971), Theorem XIII.1.2a) there exists a measure W^* such that

$$\int_{[0,\infty)} W^*(dx) = \frac{\beta}{\psi(\beta)}.$$

Define $W(x) := W^*[0, x]$ then W satisfies

$$\int_0^\infty e^{-\beta x} W(x) dx = \frac{1}{\psi(\beta)}$$

for $\beta > 0$ as required.

□

Corollary 1.4.3. *Let X be a spectrally negative Lévy process and $\mathbf{e}_q$ an independent and exponentially distributed random variable with parameter $q > 0$. For $x \geq 0$,*

$$\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \in dx) = \frac{q}{\Phi(q)} W^{(q)}(dx) - qW^{(q)}(x)dx. \tag{1.25}$$

Proof. We know that for $\beta \geq 0$,

$$\int_0^\infty e^{-\beta x} \mathbb{P}(-\underline{X}_{\mathbf{e}_q} \in dx) = \mathbb{E}(e^{\beta \underline{X}_{\mathbf{e}_q}}) = \frac{q}{\Phi(q)} \frac{\Phi(q) - \beta}{q - \psi(\beta)}.$$

On the other hand, for $\beta > \Phi(q)$ and using the same arguments of the above theorem we have,

$$\begin{aligned}
\int_0^\infty e^{-\beta x} \left(\frac{q}{\Phi(q)} W^{(q)}(dx) - qW^{(q)} dx \right) &= \frac{q}{\Phi(q)} \int_0^\infty e^{-\beta x} W^{(q)}(dx) - q \int_0^\infty e^{-\beta x} W^{(q)}(x) dx \\
&= \frac{q}{\Phi(q)} \frac{\beta}{\psi(\beta) - q} - \frac{q}{\psi(\beta) - q} \\
&= \frac{q}{\Phi(q)} \frac{\beta - \Phi(q)}{\psi(\beta) - q}.
\end{aligned}$$

Then the result follows.

□

From the definition alone, we cannot account for the omnipresence of the scale functions in the theory of Lévy processes. The following result gives us an important link between the two-sided exit problem and scale functions. For this, let us define the first-passage time above and below a level $x \in \mathbb{R}$, respectively as

$$\tau_x^+ = \inf\{t > 0 : X_t > x\} \qquad \text{and} \qquad \tau_x^- = \inf\{t > 0 : X_t < x\}.$$

Theorem 1.4.4. *Let X be a spectrally negative Lévy process and $W^{(q)}$ for $q \geq 0$ the family of scale functions from Definition 1.4.1.*

i) For any $x \in \mathbb{R}$,

$$\mathbb{P}_x(\tau_0^- < \infty) = \begin{cases} 1 - \psi'(0+)W(x) & \text{if } \psi'(0+) > 0 \\ 1 & \text{if } \psi'(0+) \leq 0. \end{cases} \tag{1.26}$$

ii) For any $x \leq a$ and $q \geq 0$,

$$\mathbb{E}_x(e^{-q\tau_a^+} \mathbb{I}_{\{\tau_0^- > \tau_a^+\}}) = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{1.27}$$

Proof of i). Denote by $\mathbf{e}_q$ an independent and exponentially distributed random variable with mean $1/q$. Let $x \geq 0$ and $q > 0$, noting that $\{\tau_0^- < \mathbf{e}_q\} = \{\underline{X}_{\mathbf{e}_q} < 0\}$ and using Fubini's theorem we have

$$\begin{aligned}
\mathbb{E}_x(e^{-q\tau_0^-}\mathbb{I}_{\{\tau_0^-<\infty\}}) &= \int_0^\infty e^{-qy}\mathbb{P}_x(\tau_0^- \in dy) \\
&= \int_0^\infty \int_y^\infty qe^{-qt}\mathbb{P}_x(\tau_0^- \in dy)dt \\
&= \int_0^\infty qe^{-qt} \int_0^t \mathbb{P}_x(\tau_0^- \in dy) \\
&= \int_0^\infty qe^{-qt}\mathbb{P}_x(\tau_0^- < t) \\
&= \mathbb{P}_x(\tau_0^- < \mathbf{e}_q) \\
&= \mathbb{P}_x(\underline{X}_{\mathbf{e}_q} < 0) \\
&= \mathbb{P}(-\underline{X}_{\mathbf{e}_q} > x) \\
&= 1 - \mathbb{P}(-\underline{X}_{\mathbf{e}_q} \le x) \\
&= 1 - \int_{[0,x]} \mathbb{P}(-\underline{X}_{\mathbf{e}_q} \in du).
\end{aligned}$$

Using the density of $\underline{X}_{\mathbf{e}_q}$ given in (1.25) we get

$$\begin{aligned}
\mathbb{E}_x(e^{-q\tau_0^-}\mathbb{I}_{\{\tau_0^-<\infty\}}) &= 1 - \int_{[0,x]} \mathbb{P}_x(-\underline{X}_{\mathbf{e}_q} \in du) \\
&= 1 + q\int_0^x W^{(q)}(u)du - \frac{q}{\Phi(q)} \int_{[0,x]} W^{(q)}(du).
\end{aligned}$$

Suppose that $\psi'(0+) \ge 0$ then $\Phi(0) = 0$ and $\lim_{q\downarrow 0} \frac{q}{\Phi(q)} = \psi'(0+)$ and thus

$$\mathbb{P}_x(\tau_0^- < \infty) = \lim_{q\downarrow 0} \mathbb{E}_x(e^{-q\tau_0^-}\mathbb{I}_{\{\tau_0^-<\infty\}}) = 1 - \psi'(0+)W(x).$$

Otherwise, $\Phi(0) > 0$ and therefore

$$\mathbb{P}_x(\tau_0^- < \infty) = 1.$$

Proof of ii). First we prove (1.27) for the case $\psi'(0+) > 0$ and $q = 0$. Recall that we define

$$W(x) = \frac{1}{\psi'(0+)}\mathbb{P}_x(\underline{X}_\infty \ge 0).$$

As we are supposing that $\psi'(0+) > 0$ we have that $\Phi(0) = 0$ which implies that $\tau_a^+ < \infty$ $\mathbb{P}$-a.s. Note that

$$\{\underline{X}_t \ge 0\} = \{\text{for all } s \le t,\, X_s \ge 0\} = \{\tau_0^- > t\}.$$

From the above observation and from the strong Markov property we have that for all $x \in [0, a]$,

$$\begin{aligned}
\mathbb{P}_x(\underline{X}_\infty \ge 0) &= \mathbb{E}_x(\mathbb{P}_x(\underline{X}_\infty \ge 0|\mathcal{F}_{\tau_a^+})) \\
&= \mathbb{E}_x(\mathbb{P}_x(\underline{X}_{\tau_a^+} \ge 0,\, \inf_{t\ge\tau_a^+} X_t \ge 0|\mathcal{F}_{\tau_a^+})) \\
&= \mathbb{E}_x(\mathbb{I}_{\{\underline{X}_{\tau_a^+}\ge 0\}}\mathbb{P}_x(\inf_{t\ge\tau_a^+} X_t \ge 0|\mathcal{F}_{\tau_a^+})) \\
&= \mathbb{E}_x(\mathbb{I}_{\{\tau_0^->\tau_a^+\}}\mathbb{P}_x(\inf_{t\ge 0} X_{t+\tau_a^+} \ge 0|\mathcal{F}_{\tau_a^+})) \\
&= \mathbb{E}_x(\mathbb{I}_{\{\tau_0^->\tau_a^+\}}\mathbb{P}_{X_{\tau_a^+}}(\underline{X}_\infty \ge 0)) \\
&= \mathbb{E}_x(\mathbb{I}_{\{\tau_0^->\tau_a^+\}}\mathbb{P}_a(\underline{X}_\infty \ge 0)) \\
&= \mathbb{P}_a(\underline{X}_\infty \ge 0)\mathbb{P}_x(\tau_a^+ < \tau_0^-).
\end{aligned}$$

We now have for $0 \leq x \leq a$,

$$\mathbb{P}_x(\tau_a^+ < \tau_0^-) = \frac{\mathbb{P}_x(\underline{X}_\infty \geq 0)}{\mathbb{P}_a(\underline{X}_\infty \geq 0)} = \frac{W(x)}{W(a)}.$$

Clearly the same equality holds even when $x < 0$ as both left and right hand side are identically equal to zero.

Next we deal with the case $q > 0$, using the change of measure $\mathbb{P}^{\Phi(q)}$ in a similar way as in the proof of Theorem 1.4.2. As X drifts to infinity under $\mathbb{P}^{\Phi(q)}$ then

$$\mathbb{P}_x^{\Phi(q)}(\tau_a^+ < \tau_0^-) = \frac{W_{\Phi(q)}(x)}{W_{\Phi(q)}(a)},$$

where $W_{\Phi(q)}(x) = \frac{1}{\psi'_{\Phi(q)}(0+)}\mathbb{P}_x^{\Phi(q)}(\underline{X}_\infty \geq 0)$. However, by definition of $\mathbb{P}^{\Phi(q)}$ and from Corollary 1.3.25,

$$\begin{aligned}\mathbb{P}_x^{\Phi(q)}(\tau_a^+ < \tau_0^-) &= \mathbb{E}_x(e^{\Phi(q)(X_{\tau_a^+}-x)-q\tau_a^+}\mathbb{I}_{\{\tau_a^+<\tau_0^-\}}) \\ &= e^{\Phi(q)(a-x)}\mathbb{E}_x(e^{-q\tau_a^+}\mathbb{I}_{\{\tau_a^+<\tau_0^-\}}).\end{aligned}$$

Therefore, by definition of $W^{(q)}$ given in (1.24) we get

$$\mathbb{E}_x(e^{-q\tau_a^+}\mathbb{I}_{\{\tau_a^+<\tau_0^-\}}) = e^{-\Phi(q)(a-x)}\frac{W_{\Phi(q)}(x)}{W_{\Phi(q)}(a)} = \frac{W^{(q)}(x)}{W^{(q)}(a)}.$$

Finally, to deal with the case that $q = 0$ and $\psi'(0+) = 0$, one needs only to take limits as $q \downarrow 0$ in the above identity, making use of monotone convergence on the left hand side and continuity in q on the right hand side thanks to the Continuity Theorem for Laplace transforms.

□

The identity (1.27) is the justification for the name "scale functions". Indeed, (1.27) has some mathematical similarities with an analogous identity which holds for a large class of one-dimensional diffusions and which involves so-called scale functions for diffusions; see for example Proposition VII, 3.2 of Revuz and Yor (1999). Moreover, note that if we choose $q = 0$ in (1.27) then we are calculating the probability that X exits an interval $[0, a]$ (where $a > 0$) into (a, ∞) before exiting into $(-\infty, 0)$ when issued at $x \in [0, a]$.

Analytical Properties of Scale Functions

Let us explore a little further some analytical properties of the functions $W^{(q)}$. For a more extensive review of properties of scale functions the reader can check the work of Kuznetsov et al. (2011).

In the following two lemmas we state properties of continuity and differentiability of the scale functions. The proofs of these results are omitted since these require some aspects of the theory of excursions of Lévy processes. As an abuse of notation, let us write $W^{(q)} \in C^1(0, \infty)$ to mean the restriction of $W^{(q)}$ to $(0, \infty)$ belongs to $C^1(0, \infty)$.

Lemma 1.4.5. *For any $q \geq 0$, the scale function $W^{(q)}$ is continuous and strictly increasing on $[0, \infty)$.*

Proof. See Kyprianou (2014) (Theorem 8.1). □

In the Chapter 3 we will suppose without loss of generality that the function $W \in C^1(0, \infty)$. Now we give sufficient conditions which guarantees that $W^{(q)} \in C^1(0, \infty)$.

Lemma 1.4.6. *For all $q \geq 0$, the function $W^{(q)}$ has left and right derivatives on $(0, \infty)$. Moreover, $W^{(q)}$ belongs to $C^1(0, \infty)$ if and only if at least one of the following three criteria holds,* *i)* $\sigma \neq 0$.

ii) $\int_{(-1,0)} |x|\Pi(dx) = \infty$.

iii) $\overline{\Pi}(x) := \Pi(-\infty, -x)$ *is continuous.*

Proof. See Kuznetsov et al. (2011) (Lemma 2.4). □

As we need it in the third chapter, we check that the function $W^{(q)}$ can be extended analytically in the parameter q.

Lemma 1.4.7. *For each $x \geq 0$, the function $q \mapsto W^{(q)}(x)$ may be analytically extended in q to $\mathbb{C}$.*

Proof. Fix $q > 0$ and choose $\beta > \Phi(q)$ such that $0 < q/\psi(\beta) < 1$. Then from the definition of the scale function,

$$\begin{aligned}\int_0^\infty e^{-\beta x} W^{(q)} dx &= \frac{1}{\psi(\beta) - q} \\ &= \frac{1}{\psi(\beta)} \frac{1}{1 - q/\psi(\beta)} \\ &= \frac{1}{\psi(\beta)} \sum_{k \geq 0} q^k \frac{1}{\psi(\beta)^k} \\ &= \sum_{k \geq 0} q^k \int_0^\infty e^{-\beta x} W^{*(k+1)}(x) dx,\end{aligned}$$

where $W^{*(k+1)}$ is the $(k+1)$-th convolution of W with itself. The last equality holds by the well-known property that the Laplace transform of the k-th convolution of W is the Laplace transform of W raised to the k-th power. Then using Fubini's theorem we have

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \int_0^\infty e^{-\beta x} \sum_{k \geq 0} q^k W^{*(k+1)}(x) dx.$$

Thanks to continuity of W and $W^{(q)}$, we get that

$$W^{(q)}(x) = \sum_{k \geq 0} q^k W^{*(k+1)}(x). \tag{1.28}$$

Now we claim that $\sum_{k \geq 0} q^k W^{*(k+1)}(x)$ converges for each $x \geq 0$ and $q \geq 0$. For this, we will use the following estimation for $k \geq 0$ and $x \geq 0$,

$$W^{*(k+1)}(x) \leq \frac{x^k}{k!} W(x)^{k+1}. \tag{1.29}$$

Indeed, we prove it by induction. Trivially (1.29) holds for $k = 0$, now suppose that (1.29) holds for $k \geq 0$, then since W is non-decreasing,

$$\begin{aligned}W^{*(k+1)}(x) &= \int_0^x W^{*k}(y) W(x-y) dy \\ &\leq \int_0^x \frac{y^{k-1}}{(k-1)!} W(y)^k W(x-y) dy \\ &\leq \frac{1}{(k-1)!} W(x)^{k+1} \int_0^x y^{k-1} dy \\ &= \frac{x^k}{k!} W(x)^{k+1}.\end{aligned}$$

Therefore

$$W^{(q)}(x) = \sum_{k\geq 0} q^k W^{*(k+1)}(x) \leq \sum_{k\geq 0} q^k \frac{x^k}{k!} W(x)^{k+1}.$$

Now noting that $\sum_{k\geq 0} q^k \frac{x^k}{k!} W(x)^{k+1}$ converges for all $x \geq 0$ and $q \in \mathbb{C}$, we may extend the definition of $W^{(q)}$ for each fixed $x \geq 0$.

□

For each $c \geq 0$, we call $W_c^{(q)}$ the function fulfilling the definitions of scale function (see Definition 1.4.1) but with respect to the measure $\mathbb{P}^c$. We establish the relationship for $W_c^{(q)}$ and $W^{(q)}$ (the scale function under the original measure $\mathbb{P}$).

Lemma 1.4.8. *For any $q \in \mathbb{C}$ and $c \in \mathbb{R}$ such that $\psi(c) < \infty$, we have*

$$W^{(q)}(x) = e^{cx} W_c^{(q-\psi(c))}(x) \tag{1.30}$$

for all $x \geq 0$.

Proof. Let $c \in \mathbb{R}$, such that $\psi(c) < \infty$. From the definition of scale functions we have that (1.30) holds for $q - \psi(c) \geq 0$. By Lemma 1.4.7 both sides in (1.30) are analytic in q for each $x \geq 0$. The Identity Theorem for analytic functions thus implies that they are equal for all $q \in \mathbb{C}$. □

We are interested in the behaviour of scale functions at the origin and to infinity. In order to state the results more precisely, we recall that from Remark 1.3.9 *ii*), when X is of finite variation, we can write $X_t = dt - S_t$ for $t \geq 0$, where $\{S_t, t \geq 0\}$ is a pure jump subordinator and $d > 0$.

The following lemma shows that a discontinuity at zero may occur even when $W^{(q)}$ belongs to $C^1(0, \infty)$.

Lemma 1.4.9. *For all $q \geq 0$, $W^{(q)}(0) = 0$ if and only if X has infinite variation. Otherwise, when X has finite variation, it is equal to $1/d$, where $d > 0$ is the drift.*

Proof. Denote as $W^{(q)}(dx)$ the measure induced by $W^{(q)}$. Using integration by parts one may deduce easily that

$$\int_0^\infty e^{-\beta x} W^{(q)}(dx) = \frac{\beta}{\psi(\beta) - q}.$$

Then for $q > 0$,

$$\begin{aligned} W^{(q)}(0) &= \lim_{\beta\to\infty} \int_{[0,\infty)} e^{-\beta x} W^{(q)}(dx) \\ &= \lim_{\beta\to\infty} \frac{\beta}{\psi(\beta) - q} \\ &= \lim_{\beta\to\infty} \frac{\beta - \Phi(q)}{\psi(\beta) - q}. \end{aligned}$$

From Corollary 1.19 one may deduce that

$$\mathbb{E}(e^{\beta \underline{X}_{\mathbf{e}_q}}) = \frac{q}{\Phi(q)} \frac{\beta - \Phi(q)}{\psi(\beta) - q}.$$

Then

$$W^{(q)}(0) = \frac{\Phi(q)}{q} \lim_{\beta\to\infty} \mathbb{E}(e^{\beta \underline{X}_{\mathbf{e}_q}}) = \frac{\Phi(q)}{q} \mathbb{P}(\underline{X}_{\mathbf{e}_q} = 0).$$

Now, it can be proven that $\mathbb{P}(\underline{X}_{\mathbf{e}_q} = 0) > 0$ if and only if X is of finite variation. From the same calculation we can also obtain

$$W^{(q)}(0) = \lim_{\beta\to\infty} \frac{\beta}{\psi(\beta) - q} = \lim_{\beta\to\infty} \frac{\beta}{\psi(\beta)}.$$

Recall that, in the case that X is of finite variation we can write

$$\psi(\beta) = d\beta - \int_{(-\infty,0)} (1 - e^{\beta x})\Pi(dx).$$

Therefore

$$W^{(q)}(0) = \lim_{\beta\to\infty} \frac{\beta}{\psi(\beta)} = \lim_{\beta\to\infty} \frac{1}{d - \int_{(-\infty,0)}(1 - e^{\beta x})\Pi(dx)/\beta} = \frac{1}{d}.$$

For the case $q = 0$ note that from (1.28) we have that for $p > 0$

$$W^{(p)}(0) = W(0) + \sum_{k\geq 1} p^k W^{*(k+1)}(0) = W(0),$$

where the last equality holds since for $n \geq 2$, $W^{*(n)}(0) = 0$ by definition of the convolution.

□

Next we look at the asymptotic behaviour of the scale function at infinity.

Lemma 1.4.10. *For $q \geq 0$ we have,*

$$\lim_{x\to\infty} e^{-\Phi(q)x} W^{(q)}(x) = \frac{1}{\psi'(\Phi(q))}. \tag{1.31}$$

In addition the following hold for $q \geq 0$,

$$\lim_{a\to\infty} \frac{W^{(q)}(a - x)}{W^{(q)}(a)} = e^{-\Phi(q)x}. \tag{1.32}$$

Proof of (1.31). Assume that $\psi'(0+) > 0$ and $q = 0$, then $\Phi(0) = 0$ and X drifts to infinity, recall that from (1.21) the definition of W is given by

$$W(x) = \frac{1}{\psi'(0+)} \mathbb{P}_x(\underline{X}_\infty \geq 0).$$

Then

$$\lim_{x\to\infty} W(x) = \frac{1}{\psi'(0+)} \lim_{x\to\infty} \mathbb{P}(\underline{X}_\infty \geq -x) = \frac{1}{\psi'(0+)}.$$

In the case that $q > 0$ or $q = 0$ and $\psi'(0+) < 0$, the definition of $W^{(q)}$ is given in (1.24) by

$$W^{(q)}(x) = e^{\Phi(q)x} \frac{1}{\psi'_{\Phi(q)}(0+)} \mathbb{P}_x^{\Phi(q)}(\underline{X}_\infty \geq 0).$$

Appealing to (1.23) we note that $\psi'_{\Phi(q)}(0+) = \psi'(\Phi(q)) > 0$ and hence X drifts to infinity under the measure $\mathbb{P}^{\Phi(q)}$, using the previous case we now have

$$\lim_{x\to\infty} e^{-\Phi(q)x}W^{(q)}(x) = \frac{1}{\psi'(\Phi(q))} \lim_{x\to\infty} \mathbb{P}_x^{\Phi(q)}(\underline{X}_\infty \geq 0) = \frac{1}{\psi'(\Phi(q))}.$$

Now suppose that $q = 0$ and $\psi'(0+) = 0$ then $\Phi(0) = 0$, then

$$\begin{aligned}\lim_{x\to\infty} W(x) &= \lim_{\beta\downarrow 0} \beta \int_0^\infty e^{-\beta x} W(x)dx \\ &= \lim_{\beta\downarrow 0} \frac{\beta}{\psi(\beta)} \\ &= \frac{1}{\psi'(0+)}\end{aligned}$$

Proof of (1.32). Using (1.31) we obtain

$$\lim_{a\to\infty} \frac{W^{(q)}(a-x)}{W^{(q)}(a)} = e^{-\Phi(q)x} \lim_{a\to\infty} \frac{e^{-\Phi(q)(a-x)}W^{(q)}(a-x)}{e^{-\Phi(q)a}W^{(q)}(a)} = e^{-\Phi(q)x}.$$

□

Potential Measures

Let us define two q-potential measures of X which will be useful later. First, denote as $U^{(q(}(a, x, dy)$ as the q-potential measure of X killed on exiting $[0, a]$ for $q \geq 0$ which is given by

$$U^{(q)}(a, x, dy) := \int_0^\infty e^{-qt}\mathbb{P}_x(X_t \in dy, \tau_a^+ > t, \tau_0^- > t).$$

On the other hand, define the q-potential measure of X killed on exiting $(-\infty, a)$ for $q \geq 0$ as follows

$$R^{(q)}(a, x, dy) := \int_0^\infty e^{-qt}\mathbb{P}_x(X_t \in dy, \tau_a^+ > t).$$

In particular, the potential measure $R^{(0)}$ will be needed in the third chapter. In the following theorems we give a semi-explicit expression of the q-potential measures defined above in terms of scale functions.

Theorem 1.4.11. *The potential measure $U^{(q)}$ has a density $u^{(q)}(a, x, y)$ given by*

$$u^{(q)}(a, x, y) = \frac{W^{(q)}(x)W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y). \tag{1.33}$$

Proof. See Kyprianou (2014). □

Theorem 1.4.12. *The potential measure $R^{(q)}$ has a density $r^{(q)}(a, x, y)$ given by*

$$r^{(q)}(a, x, y) = e^{-\Phi(q)(a-x)}W^{(q)}(a-y) - W^{(q)}(x-y). \tag{1.34}$$

Proof. From Theorem 1.4.11 we know that $U^{(q)}$ has a density $u^{(q)}(a, x, y)$ given by

$$u^{(q)}(a, x, y) = \frac{W^{(q)}(x)W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y).$$

The result is obtained moving the killing barrier below from an arbitrary large distance from the initial point. Formally, with the help of spatial homogeneity,

$$\begin{aligned}
\lim_{z\to\infty} U^{(q)}(a+z,x+z,dy+z) &= \lim_{z\to\infty}\int_0^\infty e^{-qt}\mathbb{P}_{x+z}(X_t\in dy+z,\tau^+_{a+z}>t,\tau^-_0>t)\\
&= \lim_{z\to\infty}\int_0^\infty e^{-qt}\mathbb{P}_{x}(X_t\in dy,\tau^+_{a}>t,\tau^-_{-z}>t)\\
&= \int_0^\infty e^{-qt}\mathbb{P}_{x}(X_t\in dy,\tau^+_{a}>t)\\
&= R^{(q)}(a,x,dy),
\end{aligned}$$

where the third equality holds since $\tau^-_{-z}\xrightarrow{z\to\infty}\infty$. On the other hand we have

$$\begin{aligned}
\lim_{z\to\infty} u^{(q)}(a+z,x+z,y+z) &= \lim_{z\to\infty}\left[\frac{W^{(q)}(x+z)W^{(q)}(a+z-y-z)}{W^{(q)}(a+z)} - W^{(q)}(x+z-y-z)\right]\\
&= \lim_{z\to\infty}\left[\frac{W^{(q)}(x+z)W^{(q)}(a-y)}{W^{(q)}(a+z)} - W^{(q)}(x-y)\right]\\
&= W^{(q)}(a-y)\lim_{z\to\infty}\frac{W^{(q)}(x+z)}{W^{(q)}(a+z)} - W^{(q)}(x-y)\\
&= W^{(q)}(a-y)\lim_{z\to\infty}\frac{W^{(q)}(a+z-(a-x))}{W^{(q)}(a+z)} - W^{(q)}(x-y).
\end{aligned}$$

Using Lemma 1.4.10, specifically (1.32), we conclude that

$$\lim_{z\to\infty} u^{(q)}(a+z,x+z,y+z) = e^{-\Phi(q)(a-x)}W^{(q)}(a-y) - W^{(q)}(x-y).$$

Thus, using monotone convergence theorem we have that for all $A\in\mathbb{B}(\mathbb{R})$

$$\begin{aligned}
R^{(q)}(a,x,A) &= \lim_{z\to\infty} U^{(q)}(a+z,x+z,A+z)\\
&= \lim_{z\to\infty}\int_A u^{(q)}(a+z,x+z,y+z)dy\\
&= \int_A \lim_{z\to\infty} u^{(q)}(a+z,x+z,y+z)dy\\
&= \int_A [e^{-\Phi(q)(a-x)}W^{(q)}(a-y) - W^{(q)}(x-y)]dy.
\end{aligned}$$

Therefore $R^{(q)}$ has density given by

$$r^{(q)}(a,x,y) = e^{-\Phi(q)(a-x)}W^{(q)}(a-y) - W^{(q)}(x-y).$$

□

Examples

In this subsection we present some known examples of scale functions. In the work of Hubalek and Kyprianou (2010) is given a general methodology for generating new families of scale functions. We list a few here.

Brownian motion with drift

Let $X=\{X_t,t\geq 0\}$ a Brownian motion with drift, i.e.

$$X_t = \sigma B_t + \mu t, \qquad t \geq 0,$$

where $\sigma > 0$ and $\mu \in \mathbb{R}$. It is well known that its Laplace exponent is given by

$$\psi(\theta) = \frac{\sigma^2}{2}\theta^2 + \mu\theta.$$

Then the scale function associated to this process is for $q \geq 0$ and $x \geq 0$,

$$W^{(q)}(x) = \frac{2}{\sqrt{2q\sigma^2 + \mu}} e^{-\mu x/\sigma^2} \sinh\left(\frac{x}{\sigma^2}\sqrt{2q\sigma^2 + \mu}\right),$$

where in the case $q = 0 = \mu$ the above expression is taken in the limiting sense.

Spectrally negative stable process with stability parameter $\beta \in (1, 2)$

Suppose that X is a Lévy process with

$$\psi(\theta) = \theta^\beta,$$

where $\beta \in (1, 2)$. We have for $x \geq 0$ and $q \geq 0$

$$W^{(q)}(x) = \beta x^{\beta-1} E'_{\beta,1}(qx^\beta),$$

where $E_{\beta,1}$ is the Mittag–Leffler function given by

$$E_{\beta,1}(z) = \sum_{k\geq 0} \frac{z^k}{\Gamma(1 + \beta k)}.$$

There also exists an expression for the scale function of aforementioned β-stable process but now with a strictly positive drift $c > 0$ (but only for the case $q = 0$). For $x \geq 0$ its scale function is given by

$$W(x) = \frac{1}{c}(1 - E_{\beta-1,1}(-cx^{\beta-1})).$$

Spectrally negative Lévy process of finite variation drifting to infinity

Suppose that X is of the form

$$X_t = ct - S_t,$$

where $S = \{S_t, t \geq 0\}$ is a subordinator with jump measure Π and no drift. The measure $W(dx)$ induced by the scale-function $W(x)$ is given by

$$W(dx) = \frac{1}{c}\sum_{n\geq 0} \nu^{*n}(dx),$$

where $\nu(dx) = c^{-1}\Pi(x, \infty)dx$ and ν^{*n} is the n-convolution of ν with itself.

As a special case of the latter, consider the Cramér–Lundberg risk process with exponentially distributed jumps. This process has the following expression

$$X_t = ct - \sum_{i=1}^{N_t} \xi_i,$$

where $c > 0$, $N = \{N_t, t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ and $\{\xi_i, i \geq 1\}$ is an independent sequence of exponentially distributed random variables with parameter $\mu > 0$. Suppose that $c - \lambda/\mu > 0$, this condition guarantees that X drifts to infinity. Its Laplace exponent is given by

$$\psi(\theta) = c\theta - \lambda\left(1 - \frac{\mu}{\mu + \theta}\right).$$

Then the scale function is given by

$$W(x) = \frac{1}{c}\left(1 + \frac{\lambda}{c\mu - \lambda}(1 - e^{-(\mu - c^{-1}\lambda)x})\right).$$

Another example is the following, consider a spectrally negative compound Poisson process whose jumps are exactly of size $\alpha \in (0, \infty)$, with arrival rate $\lambda > 0$ and with positive drift $c > 0$ such that $c - \lambda\alpha > 0$. In that case,

$$\psi(\theta) = c\theta - \lambda(1 - e^{-\alpha\theta}).$$

For $x \geq 0$ we have,

$$W(x) = \frac{1}{c} \sum_{n=1}^{\lfloor x/\alpha \rfloor} e^{-\lambda(\alpha n - x)/c} \frac{1}{n!} \left(\frac{\lambda}{c}\right)^n (\alpha n - x)^n,$$

where $\lfloor x/\alpha \rfloor$ is the integer part of x/α.

A Spectrally Negative Lévy Process with no Gaussian Component

Consider a spectrally negative Lévy process X with $\sigma = 0$ and with Lévy measure

$$\Pi(dy) = \frac{e^{(\beta-1)y}}{(e^y - 1)^{\beta+1}} dy, \qquad y < 0,$$

where $\beta \in (1, 2)$ and whose Laplace transform takes the form

$$\psi(\theta) = \frac{\Gamma(\theta - 1 + \beta)}{\Gamma(\theta - 1)\Gamma(\beta)}.$$

Note that $\psi'(0+) < 0$ and hence the process drifts to minus infinity. In that case it was found that for $x \geq 0$,

$$W(x) = (1 - e^{-x})^{\beta - 1} e^x.$$

Another example is the scale function associated with the aforementioned Lévy process when conditioned to drift to infinity. It follows that there is still no Gaussian component and the Lévy measure takes the form

$$\Pi(dy) = \frac{e^{\beta y}}{(e^y - 1)^{\beta+1}} dy, \qquad y < 0.$$

The associated Laplace exponent is given by

$$\psi(\theta) = \frac{\Gamma(\theta + \beta)}{\Gamma(\theta)\Gamma(\beta)},$$

and the scale function is then given for $x \geq 0$ by

$$W(x) = (1 - e^{-x})^{\beta - 1}.$$

Chapter 2

Optimal Stopping

The theory of optimal stopping is concerned with the problem of choosing a time to take a given action based on sequentially observed random variables in order to maximise an expected payoff or to minimise an expected cost. Problems of this type are found in the area of statistics, where the action taken may be to test a hypothesis or to estimate a parameter, in the area of operations research, where the action may be to replace a machine, hire a secretary, or reorder stock and in applications to finance, valuation of American options.

The aim of the present chapter is to introduce basic results of general theory of optimal stopping. First we study the martingale approach in continuous time and then the Markovian approach, both only in an infinite horizon of time. This chapter is mainly based on Peskir and Shiryaev (2006).

2.1 Essential Supremum

Recall that if we take the supremum over an uncountable set of random variables then this does not necessarily defines a measurable function. To overcome this difficulty the concept of essential supremum proves to be useful. This section is entirely devoted to the following theorem which gives us the definition and functionality of the concept of an essential supreumum of random variables.

Theorem 2.1.1. *Let $\{Z_\alpha, \alpha \in I\}$ be a collection of real-valued random variables in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with I an arbitrary index set. Then there exists a countable subset $J \subseteq I$ such that the random variable $Z^* : \Omega \mapsto \mathbb{R} \cup \{-\infty, \infty\}$ defined by*

$$Z^* = \sup_{\alpha \in J} Z_\alpha, \tag{2.1}$$

satisfies

i) $\mathbb{P}(Z_\alpha \leq Z^*) = 1$ *for all* $\alpha \in I$.

ii) *If* $Y : \Omega \mapsto \mathbb{R} \cup \{-\infty, \infty\}$ *is another random variable satisfying i) then,*

$$\mathbb{P}(Z^* \leq Y) = 1.$$

We call Z^* the essential supremum of $\{Z_\alpha, \alpha \in I\}$, and write

$$Z^* = \operatorname*{ess\,sup}_{\alpha \in I} Z_\alpha.$$

It is defined uniquely $\mathbb{P}$-almost surely.

Proof. Since the function $f(x) = (2/\pi)\arctan(x)$ is bijection between the sets $[-\infty, \infty]$ and $[-1, 1]$ we may assume without loss of generality that $|Z_\alpha| \leq 1$ for all $\alpha \in I$. Let $\mathcal{C}$ be the set of countable subsets of I and define

$$a := \sup_{C \in \mathcal{C}} \mathbb{E}\left(\sup_{\alpha \in C} Z_\alpha\right).$$

Then we have that $|a| \leq 1$. Choose an increasing sequence $\{C_n, n \geq 1\} \subseteq \mathcal{C}$ such that

$$a = \sup_{n \geq 1} \mathbb{E}\left(\sup_{\alpha \in C_n} Z_\alpha\right).$$

Let $J := \bigcup_{n \geq 1} C_n$, then J is a countable subset of I and define

$$Z^* = \sup_{\alpha \in J} Z_\alpha.$$

We claim that Z^* satisfies the properties $i)$ and $ii)$. First, we verify the condition $i)$, if we take $\alpha \in J$ then by definition, $\mathbb{P}(Z_\alpha \leq Z^*) = 1$. Now take $\beta \in I \setminus J$ and suppose $\mathbb{P}(Z_\beta > Z^*) > 0$, this implies with the help of the dominated convergence theorem that,

$$\begin{aligned}
\mathbb{E}(Z_\beta \vee Z^*) &> \mathbb{E}(Z^*) \\
&= \mathbb{E}\left(\lim_{n \to \infty} \sup_{\alpha \in C_n} Z_\alpha\right) \\
&= \lim_{n \to \infty} \mathbb{E}\left(\sup_{\alpha \in C_n} Z_\alpha\right) \\
&= a.
\end{aligned}$$

Hence, $\mathbb{E}(Z_\beta \vee Z^*) > a$ and on the other hand,

$$\mathbb{E}(Z_\beta \vee Z^*) = \mathbb{E}\left(\sup_{\alpha \in J \cup \{\beta\}} Z_\alpha\right) \leq \sup_{C \in \mathcal{C}} \mathbb{E}\left(\sup_{\alpha \in C} Z_\alpha\right) = a,$$

where the inequality holds since the set $J \cup \{\beta\}$ is countable which is a contradiction and hence $\mathbb{P}(Z_\beta \leq Z^*) = 1$ and the property $i)$ is then proved. For the property $ii)$ suppose that $Z_\alpha \leq Y$ a.s. for all $\alpha \in I$, then in particular for all $\beta \in J$ and hence by the countability of J,

$$\mathbb{P}(Z^* > Y) = \mathbb{P}\left(\sup_{\beta \in J} Z_\beta > Y\right) = \mathbb{P}\left(\bigcup_{\beta \in J} \{Z_\beta > Y\}\right) \leq \sum_{\beta \in J} \mathbb{P}(Z_\beta > Y) = 0.$$

Therefore $Z^* \leq Y$ a.s. □

Definition 2.1.2. *A family of random variables $\{Z_\alpha, \alpha \in I\}$ has the lattice property if for any α, β in I there exists $\gamma \in I$ such that $Z_\alpha \vee Z_\beta \leq Z_\gamma$ $\mathbb{P}$-a.s.*

Corollary 2.1.3. *If the family $\{Z_\alpha, \alpha \in I\}$ has the lattice property then the countable subset $J = \{\alpha_0, \alpha_1, \alpha_2, \cdots\}$ may be chosen so that*

$$\operatorname{ess\,sup}_{\alpha \in I} Z_\alpha = \lim_{n \uparrow \infty} Z_{\alpha_n},$$

where $Z_{\alpha_0} \leq Z_{\alpha_1} \leq Z_{\alpha_2} \leq \cdots$ $\mathbb{P}$-a.s.

Proof. Suppose that the countable set in the proof of the previous theorem is $J = \{\alpha_0, \alpha_1, \alpha_2, \dots\}$ then it can be replaced by a new countable set $J^* = \{\alpha_0^*, \alpha_1^*, \alpha_2^*, \dots\}$ where $\alpha_0^* = \alpha_0$ and α_{n+1}^* is such that $Z_{\alpha_{n+1}^*} \geq Z_{\alpha_n^*} \vee Z_{\alpha_n}$ $\mathbb{P}$-a.s., note that this can be done thanks to the lattice property.

By construction we have that $Z_{\alpha_{n+1}^*} \geq Z_{\alpha_n}$ $\mathbb{P}$-a.s. and $\{Z_{\alpha_n^*}\}$ is an increasing sequence whose elements are almost surely bounded by $\operatorname{ess\,sup}_{\alpha \in I} Z_\alpha$. For this reason,

$$\operatorname*{ess\,sup}_{\alpha \in I} Z_\alpha \geq \lim_{n \uparrow \infty} Z_{\alpha_n^*} = \sup_{\alpha \in J^*} Z_\alpha \geq \sup_{\alpha \in J} Z_\alpha = \operatorname*{ess\,sup}_{\alpha \in I} Z_\alpha.$$

Therefore $\operatorname{ess\,sup}_{\alpha \in I} Z_\alpha = \lim_{n \to \infty} Z_{\alpha_n^*}$ as claimed. □

2.2 Martingale Approach

Let $G = \{G_t, t \geq 0\}$ a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ is a filtration of $\mathcal{F}$. Suppose that the filtration $\mathbb{F}$ satisfies the natural conditions (see Definition 1.3.38 of Bichteler (2002)), also assume that G is adapted to the filtration $\mathcal{F}$. We interpret G_t as the gain if the observation of G is stopped at time t.

From here on we will assume that the process G is right-continuous and left-continuous over stopping times (if τ_n and τ are stopping times such that $\tau_n \to \tau$ as $n \to \infty$ then $G_{\tau_n} \to G_\tau$ $\mathbb{P}$-a.s. as $n \to \infty$). We will also assume that the following condition is satisfied,

$$\mathbb{E}\left(\sup_{t \geq 0} |G_t|\right) < \infty. \tag{2.2}$$

Define for all $t \geq 0$,

$$\mathcal{T}_t = \{\tau \geq t : \tau \text{ is stopping time}\},$$

the set of all stopping times greater or equal to t. For simplicity we only write $\mathcal{T}$ instead of $\mathcal{T}_0$, i.e. we denote by $\mathcal{T}$ the set of all stopping times.

Consider the optimal stopping problem

$$V_t = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}(G_\tau). \tag{2.3}$$

To solve the problem (2.3), consider the process $S = \{S_t, t \geq 0\}$ defined as follows:

$$S_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}(G_\tau | \mathcal{F}_t), \tag{2.4}$$

the process S is often called the Snell envelope of G. Note that by the definition of S_t we have that if we take $\tau = t$ then $S_t \geq G_t$ $\mathbb{P}$-a.s.

Consider the following stopping time for $t \geq 0$

$$\tau_t = \inf\{s \geq t : S_s = G_s\},$$

where we define $\inf \emptyset = \infty$. Now we state a useful result which will help us to prove that τ_t is an optimal stopping time for (2.3).

Lemma 2.2.1. *The process $\{S_t, t \geq 0\}$ defined in (2.4) is a supermartingale and admits a càdlàg modification. Moreover, the following relation holds,*

$$\mathbb{E}(S_t) = V_t. \tag{2.5}$$

Proof. Note that S_t is $\mathcal{F}_t$-measurable by definition, and

$$\mathbb{E}\left(|S_t|\right) \leq \mathbb{E}\left(\sup_{t\geq 0}|G_t|\right) < \infty$$

Fix $t \geq 0$ we first show that the family

$$\{\mathbb{E}(G_\tau|\mathcal{F}_t),\, \tau \in \mathcal{T}_t\}$$

has the lattice property. Indeed, if we take $\sigma_1, \sigma_2 \geq t$ stopping times and we set $\sigma_3 = \sigma_1 \mathbb{I}_A + \sigma_2 \mathbb{I}_{A^c}$ where $A = \{\mathbb{E}(G_{\sigma_1}|\mathcal{F}_t) \geq \mathbb{E}(G_{\sigma_2}|\mathcal{F}_t)\}$ then σ_3 is a stopping time with $\sigma_3 \geq t$ since $A, A^c \in \mathcal{F}_t$ and for all $s \geq 0$

$$\{\sigma_1 \mathbb{I}_A + \sigma_2 \mathbb{I}_{A^c} \leq s\} = \begin{cases} \emptyset & s < t \\ (A \cap \{\sigma_1 \leq s\}) \cup (A^c \cap \{\sigma_2 \leq s\}) & s \geq t \end{cases} \in \mathcal{F}_s.$$

Then, we have

$$\begin{aligned} \mathbb{E}(G_{\sigma_3}|\mathcal{F}_t) &= \mathbb{E}(G_{\sigma_1 \mathbb{I}_A + \sigma_2 \mathbb{I}_{A^c}}|\mathcal{F}_t) \\ &= \mathbb{E}(G_{\sigma_1}\mathbb{I}_A + G_{\sigma_2}\mathbb{I}_{A^c}|\mathcal{F}_t) \\ &= \mathbb{I}_A \mathbb{E}(G_{\sigma_1}|\mathcal{F}_t) + \mathbb{I}_{A^c}\mathbb{E}(G_{\sigma_2}|\mathcal{F}_t) \\ &= \mathbb{E}(G_{\sigma_1}|\mathcal{F}_t) \vee \mathbb{E}(G_{\sigma_2}|\mathcal{F}_t). \end{aligned}$$

Therefore $\{\mathbb{E}(G_\tau|\mathcal{F}_t),\, \tau \in \mathcal{T}_t\}$ has the lattice property. Hence by Corollary 2.1.3 there exists a sequence of stopping times $\{\tau_k, k \geq 1\}$ such that $\tau_k \geq t$ and

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}(G_\tau|\mathcal{F}_t) = \lim_{k\to\infty} \mathbb{E}(G_{\tau_k}|\mathcal{F}_t),$$

where $\mathbb{E}(G_{\tau_1}|\mathcal{F}_t) \leq \mathbb{E}(G_{\tau_2}|\mathcal{F}_t) \leq \cdots$ $\mathbb{P}$-a.s. Then using the monotone convergence theorem for conditional expectation we have that for any $s \leq t$,

$$\begin{aligned} \mathbb{E}(S_t|\mathcal{F}_s) &= \mathbb{E}(\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}(G_\tau|\mathcal{F}_t)|\mathcal{F}_s) \\ &= \mathbb{E}(\lim_{k\to\infty} \mathbb{E}(G_{\tau_k}|\mathcal{F}_t)|\mathcal{F}_s) \\ &= \lim_{k\to\infty} \mathbb{E}(\mathbb{E}(G_{\tau_k}|\mathcal{F}_t)|\mathcal{F}_s) \\ &= \lim_{k\to\infty} \mathbb{E}(G_{\tau_k}|\mathcal{F}_s) \\ &\leq \operatorname*{ess\,sup}_{\tau\geq s} \mathbb{E}(G_\tau|\mathcal{F}_s) \\ &= S_s. \end{aligned}$$

This shows that $\{S_t, \geq 0\}$ is a supermartingale. Note that by definition of S_t we have that $S_t \geq \mathbb{E}(G_\tau|\mathcal{F}_t)$ for all stopping times $\tau \in \mathcal{T}_t$, this implies that

$$\mathbb{E}(S_t) \geq \sup_{\tau\in\mathcal{T}_t} \mathbb{E}(G_\tau) = V_t.$$

Also, using the monotone convergence theorem we have

$$
\begin{aligned}
\mathbb{E}(S_t) &= \mathbb{E}(\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}(G_\tau|\mathcal{F}_t)) \\
&= \mathbb{E}(\lim_{k\to\infty} \mathbb{E}(G_{\tau_k}|\mathcal{F}_t)) \\
&= \lim_{k\to\infty} \mathbb{E}(\mathbb{E}(G_{\tau_k}|\mathcal{F}_t)) \\
&= \lim_{k\to\infty} \mathbb{E}(G_{\tau_k}) \\
&= \sup_{k\geq 1} \mathbb{E}(G_{\tau_k}) \\
&\leq \sup_{\tau\in\mathcal{T}_t} \mathbb{E}(G_\tau),
\end{aligned}
$$

where the last inequality holds since $\{\tau_1, \tau_2, \ldots\} \subset \mathcal{T}_t$. Therefore,

$$
\mathbb{E}(S_t) = \sup_{\tau\in\mathcal{T}_t} \mathbb{E}(G_\tau) = V_t.
$$

Now we check that the supermartingale $\{S_t, t \geq 0\}$ admits a càdlàg modification. Using Theorem A.1.24 we only have to check that $t \mapsto \mathbb{E}(S_t)$ is right-continuous. To verify this note that by the supermartingale property we have for all sequence $\{t_n, n \geq 1\}$ such that $t_n \downarrow t$ we have $\mathbb{E}(S_t) \geq \cdots \geq \mathbb{E}(S_{t_n}) \geq \cdots \geq \mathbb{E}(S_{t_1})$, and hence $L_t := \lim_{n\to\infty} \mathbb{E}(S_{t_n})$ exists and $\mathbb{E}(S_t) \geq L$. Now we prove the reverse inequality, from the fact that $\mathbb{E}(S_t) = \sup_{\tau\in\mathcal{T}_t} \mathbb{E}(G_\tau)$, then for all $\varepsilon > 0$ we may choose a stopping time $\sigma_\varepsilon \geq t$ such that

$$
\mathbb{E}(G_{\sigma_\varepsilon}) \geq \mathbb{E}(S_t) - \varepsilon.
$$

Fix $\delta > 0$ and suppose that $t_n \in [t, t+\delta]$ for all $n \geq 1$. Define the sequence of stopping times $\{\sigma_n, n \geq 1\}$ where for $n \geq 1$

$$
\sigma_n = \begin{cases} \sigma_\varepsilon & \sigma_\varepsilon > t_n, \\ t+\delta & \sigma_\varepsilon \leq t_n. \end{cases}
$$

Note that $\sigma_n \geq t_n$ and for all $s \geq 0$,

$$
\{\sigma_n \leq s\} = \begin{cases} \emptyset & s < t_n \\ \{t_n < \sigma_\varepsilon \leq s\} \cup [\{t+\delta \leq s\} \cap \{\sigma_\varepsilon \leq t_n\}] & s \geq t_n \end{cases} \in \mathcal{F}_s.
$$

Hence, σ_n is a stopping time such that $\sigma_n \geq t_n$ for all $n \geq 0$. Then using that $\mathbb{E}(S_{t_n}) = \sup_{\tau\in\mathcal{T}_n} \mathbb{E}(S_\tau)$ we have that,

$$
\mathbb{E}(G_{\sigma_\varepsilon} \mathbb{I}_{\{\sigma_\varepsilon > t_n\}}) + \mathbb{E}(G_{t+\delta} \mathbb{I}_{\{\sigma_\varepsilon \leq t_n\}}) = \mathbb{E}(G_{\sigma_n}) \leq \mathbb{E}(S_{t_n}).
$$

Letting $n \to \infty$ and using the dominated convergence theorem we have

$$
\mathbb{E}(G_{\sigma_\varepsilon} \mathbb{I}_{\{\sigma_\varepsilon > t\}}) + \mathbb{E}(G_{t+\delta} \mathbb{I}_{\{\sigma_\varepsilon = t\}}) \leq L_t
$$

for all $\delta > 0$. Letting $\delta \downarrow 0$ and from the fact that $\{G_t, t \geq 0\}$ is right-continuous we have

$$
\mathbb{E}(G_{\sigma_\varepsilon} \mathbb{I}_{\{\sigma_\varepsilon > t\}}) + \mathbb{E}(G_t \mathbb{I}_{\{\sigma_\varepsilon = t\}}) = \mathbb{E}(G_{\sigma_\varepsilon}) \leq L_t.
$$

Therefore

$$
L_t \geq \mathbb{E}(G_{\sigma_\varepsilon}) \geq \mathbb{E}(S_t) - \varepsilon
$$

for all $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ we obtain that $\mathbb{E}(S_t) \leq L_t$ and thus $\mathbb{E}(S_t) = L_t$ proving that S admits a right-continuous modification. □

Theorem 2.2.2. *Consider the optimal stopping problem* (2.3) *upon assuming that condition* (2.2) *holds. Assume moreover when required below that*

$$\mathbb{P}(\tau_t < \infty) = 1, \tag{2.6}$$

where $t \geq 0$*. Then for all* $t \geq 0$ *we have,*

$$S_t \geq \mathbb{E}(G_\tau|\mathcal{F}_t) \qquad \text{for each stopping time } \tau \in \mathcal{T}_t \tag{2.7}$$
$$S_t = \mathbb{E}(G_{\tau_t}|\mathcal{F}_t). \tag{2.8}$$

Moreover, if $t \geq 0$ *is given and fixed, we have:*

i) The stopping time τ_t *is optimal in* (2.3)*.*

ii) If τ_* *is an optimal stopping time in* (2.3) *then* $\tau_t \leq \tau_*$ $\mathbb{P}$*-a.s.*

iii) The process $\{S_s, s \geq t\}$ *is the smallest right-continuous supermartingale which dominates* $\{G_s, s \geq t\}$ *.*

iv) The stopped process $\{S_{s\wedge\tau_t}, s \geq t\}$ *is a right-continuous martingale.*

Proof of (2.7). By Doob's optimal sampling theorem (Theorem A.1.28) we have that for all stopping time τ the process $\{S_{s\wedge\tau}, s \geq t\}$ is also a supermartingale. If we take any stopping time $\tau \in \mathcal{T}$ we have

$$S_t = S_{t\wedge\tau} \geq \mathbb{E}(S_\tau|\mathcal{F}_t).$$

Proof of iii). Let $\tilde{S} = \{\tilde{S}_s, s \geq t\}$ be another right continuous supermartingale which dominates $G = \{G_s, s \geq t\}$. Then by Doob's optimal sampling theorem (Theorem A.1.28) we have that for all stopping time τ the process $\{\tilde{S}_{s\wedge\tau}, s \geq t\}$ is also a supermartingale. If we take any stopping time $\sigma \geq s$ we have

$$\tilde{S}_s \geq \mathbb{E}(\tilde{S}_\sigma|\mathcal{F}_s) \geq \mathbb{E}(G_\sigma|\mathcal{F}_s),$$

where the last inequality holds since $\tilde{S}$ dominates G. Hence by the definition of S_s given by

$$S_s = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_s} \mathbb{E}(G_\tau|\mathcal{F}_s),$$

we have that $\tilde{S}_s \geq S_s$ $\mathbb{P}$-a.s. for all $s \geq t$. By the right-continuity of S and $\tilde{S}$ this further implies that $\mathbb{P}(S_s \leq \tilde{S}_s \text{ for all } s \geq t) = 1$.

Proof of (2.8). The proof of this result is rather technical and then we omit it. For a detailed proof see Peskir and Shiryaev (2006) page 31.

Proof of i) and ii). Taking expectations in (2.8) and using (2.5) we have that

$$V_t = \sup_{\tau\in\mathcal{T}_t} \mathbb{E}(G_\tau) = \mathbb{E}(S_t) = \mathbb{E}(G_{\tau_t}).$$

Therefore, the stopping time τ_t is an optimal stopping time for V_t. Now suppose that τ^* is another optimal stopping time. We will show that

$$S_{\tau^*} = G_{\tau^*} \qquad \mathbb{P}\text{-a.s.}$$

Indeed, if we suppose that $\mathbb{P}(S_{\tau^*} > G_{\tau^*}) > 0$ and hence

$$\mathbb{E}(G_{\tau^*}) < \mathbb{E}(S_{\tau^*}) \leq \mathbb{E}(S_t) = V_t,$$

where the second inequality follow from Doob's optional sampling theorem and the supermartingale property of S. Note that the above inequality contradicts the fact that τ^* is optimal and hence $S_{\tau^*} = G_{\tau^*}$. Then, by definition of τ_t, we have that $\tau_t \leq \tau^*$.

Proof of iv) and v). By the optional sampling theorem we have $\{S_{s\wedge\tau_t^*}, s \geq t\}$ is also a supermartingale, then we have for all $s \geq t$,

$$\mathbb{E}(S_{s\wedge\tau_t}) \leq \mathbb{E}(S_t).$$

On the other hand, from (2.8) and from the fact that $s \wedge \tau_t \leq \tau_t$ we have,

$$\mathbb{E}(S_t) = \mathbb{E}(G_{\tau_t}) = \mathbb{E}(S_{\tau_t}) \leq \mathbb{E}(S_{s\wedge\tau_t}).$$

Hence $\mathbb{E}(S_{s\wedge\tau_t}) = \mathbb{E}(S_t)$. Then we have a supermartingale for which the mapping $s \mapsto \mathbb{E}(S_{s\wedge\tau_t})$ is constant, therefore the process $\{S_{s\wedge\tau_t^*}, s \geq t\}$ is a martingale. □

2.3 Markovian Approach

In this section we will consider a strong Markov process $X = \{X_t, t \geq 0\}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}_x)$ and taking values in $(E, \mathcal{B}) = (\mathbb{R}, \mathbb{B}(\mathbb{R}))$. It is assumed that the process X starts at x under the probability measure $\mathbb{P}_x$ for $x \in \mathbb{R}$ and the sample paths of X are right-continuous and left-continuous over stopping times. It is also assumed that the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ satisfies the natural conditions. In addition, it is assumed that the mapping $x \mapsto \mathbb{P}_x(F)$ is measurable for each $F \in \mathcal{F}$. Finally, without loss of generality we will assume that $(\Omega, \mathcal{F})$ is equal to the canonical space $(E^{[0,\infty)}), \mathcal{B}^{[0,\infty)})$ so that the shift operator $\theta_t : \Omega \mapsto \Omega$ is well defined by $\theta_t(\omega)(s) = \omega(t+s)$ for $\omega = \{\omega(s), s \geq 0\} \in \Omega$ and $s, t \geq 0$.

Suppose that $G : E \mapsto \mathbb{R}$ is a measurable function which satisfies the condition

$$\mathbb{E}_x\left(\sup_{t\geq 0} |G(X_t)|\right) < \infty, \tag{2.9}$$

where $\mathbb{E}_x$ is the expectation under the measure $\mathbb{P}_x$ and $x \in \mathbb{E}$. We consider the optimal stopping problem

$$V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x(G(X_\tau)), \tag{2.10}$$

where $x \in E$ and $\mathcal{T}$ is the set of all stopping times of $\mathbb{F}$. The function V is called the value function and G is called the gain function. Solving the optimal stopping problem (2.10) means two things. Firstly, we need to find an optimal stopping time, i.e. a stopping time τ_* at which the supremum is attained. Secondly, we need to compute the value $V(x)$ for $x \in E$ as explicitly as possible.

Note that if we take $\tau \equiv 0$ we have that from definition of V given in (2.10),

$$V(x) \geq \mathbb{E}_x(G(X_0)) = G(x) \tag{2.11}$$

The Markovian structure of X means that the process always starts afresh. Then for a fixed sample path we shall be able to decide whether to continue with the observation or to stop it. Thinking in this way we split the set E into two disjoint subsets, the continuation set C and the stopping set $D = E \setminus C$. It follows that as soon as the process enters into D, the observation should be stopped and an optimal stopping time is obtained.

It turns out that the continuation set is given by

$$C = \{x \in E : V(x) > G(x)\} \tag{2.12}$$

and the stopping set

$$D = \{x \in E : V(x) = G(x)\}. \tag{2.13}$$

Formally, we define the process $\{G_t, t \geq 0\}$ where

$$G_t = G(X_t), \qquad t \geq 0.$$

Then the Snell envelope process of $\{G_t, t \geq 0\}$ under the measure $\mathbb{P}_x$ for $x \in E$ is given by $\{S_t, t \geq 0\}$ where

$$\begin{aligned} S_t &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}_x(G_\tau | \mathcal{F}_t) \\ &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}_x(G(X_\tau) | \mathcal{F}_t) \\ &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}} \mathbb{E}_x(G(X_{\tau+t}) | \mathcal{F}_t) \\ &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}} \mathbb{E}_{X_t}(G(X_\tau)) \\ &= V(X_t). \end{aligned}$$

Hence an optimal stopping time is given by

$$\begin{aligned} \tau_0^* &= \inf\{t \geq 0 : S_t = G_t\} \\ &= \inf\{t \geq 0 : V(X_t) = G(X_t)\} \\ &= \inf\{t \geq 0 : X_t \in D\}. \end{aligned}$$

Proving that we have to stop when the process enters for the first time into the set D and continue otherwise.

Definition 2.3.1. *Let $f : E \mapsto \mathbb{R}$ be a function and take $c \in E$. The function f is said to be upper semi-continuous at a point c when*

$$f(c) \geq \limsup_{x \to c} f(x).$$

It is said to be upper semi-continuous (usc) on E if it is upper semi-continuous at every point of E. In a similar way, f is said to be lower semi-continuous at a point c when

$$f(c) \leq \liminf_{x \to c} f(x).$$

It is said to be lower semi-continuous (lsc) on E if it is lower semi-continuous at every point of E.

When $E = \mathbb{R}$ upper semi-continuity in $c \in E$ can be written in the following way. For all $\varepsilon > 0$ there exists $\delta > 0$ such that for all x such that $|x - c| < \delta$ then $f(x) \leq f(c) + \varepsilon$. Lower semi-continuity can be written, for all $\varepsilon > 0$ exists $\delta > 0$ such that for all x such that $|x - c| < \delta$ then $f(x) \geq f(c) - \varepsilon$.

It can be shown that if V is lower semi-continuous and G upper semi-continuous then C is open and D is closed. Introduce the first entry time τ_D of X into D by setting

$$\tau_D = \inf\{t \geq 0 : X_t \in D\}. \tag{2.14}$$

Definition 2.3.2. *A measurable function $F : E \mapsto \mathbb{R}$ is said to be superharmonic related to X if*

$$\mathbb{E}_x(F(X_\sigma)) \leq F(x)$$

for all stopping times σ and all $x \in E$.

The following theorem presents necessary conditions for the existence of an optimal stopping time.

Theorem 2.3.3. *Let us assume that there exists an optimal stopping time τ_* in (2.10), i.e.,*

$$V(x) = \mathbb{E}_x(G(X_{\tau_*}))$$

for all $x \in E$. Then we have

i) The value function V is the smallest superharmonic function which dominates the gain function G on E.

If we suppose that V is lsc and G is usc, then

ii) The process $\{V(X_t), t \geq 0\}$ is a right-continuous supermartingale.

iii) The stopping time τ_D satisfies $\tau_D \leq \tau_*$ $\mathbb{P}_x$-a.s. for all $x \in E$ and is optimal in (2.10).

iv) The stopped process $\{V(X_{t\wedge\tau_D}), t \geq 0\}$ is a right-continuous martingale under $\mathbb{P}_x$ for every $x \in E$.

Proof of i). To show that V is superharmonic note that by the strong Markov property we have that for all stopping times σ and all $x \in E$,

$$\begin{aligned}
\mathbb{E}_x(V(X_\sigma)) &= \mathbb{E}_x(\mathbb{E}_{X_\sigma}(G(X_{\tau_*}))) \\
&= \mathbb{E}_x(\mathbb{E}_x(G(X_{\tau_*}) \circ \theta_\sigma | \mathcal{F}_\sigma)) \\
&= \mathbb{E}_x(\mathbb{E}_x(G(X_{\sigma+\tau_*\circ\theta_\sigma}) | \mathcal{F}_\sigma)) \\
&= \mathbb{E}_x(G(X_{\sigma+\tau_*\circ\theta_\sigma})) \\
&\leq \sup_{\tau\in\mathcal{T}} \mathbb{E}_x(G(X_\tau)) \\
&= V(x).
\end{aligned}$$

This proves that V is superharmonic, now suppose that F is another superharmonic function which dominates G on E, then for all τ stopping time and $x \in E$

$$\mathbb{E}_x(G(X_\tau)) \leq \mathbb{E}_x(F(X_\tau)) \leq F(x),$$

where the first inequality holds since F dominates G and the second since F is superharmonic. Taking the supremum over all $\tau \in \mathcal{T}$ we have

$$V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x(G(X_\tau)) \leq F(x).$$

Therefore V is the smallest superharmonic function which dominates G.

Proof of ii). Note that (2.9) guarantees that $\mathbb{E}(|V(X_t)|) < \infty$ and clearly $V(X_t)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$. From the Markov property and the fact that V is superharmonic (taking $\sigma = s$) we have

$$\mathbb{E}_x(V(X_{t+s})|\mathcal{F}_t) = \mathbb{E}_{X_t}(V(X_s)) \leq V(X_t)$$

for all $s, t \geq 0$ and all $x \in E$. This show that the process $\{V(X_t), t \geq 0\}$ is a supermartingale under $\mathbb{P}_x$ for each $x \in E$. The right-continuity of $\{V(X_t), t \geq 0\}$ follows from the fact that V is lsc, since the proof is rather technical we omit it, for details see Peskir and Shiryaev (2006) page 39.

Proof of iii). If we have that $V(X_{\tau_*}) = G(X_{\tau_*})$ $\mathbb{P}_x$-a.s. for all $x \in E$ then by definition of τ_D given in (2.14) we obtain that $\tau_D \leq \tau_*$ $\mathbb{P}_x$-a.s. Then we only need to prove that $V(X_{\tau_*}) = G(X_{\tau_*})$.

Recall from (2.11) that $V(x) \geq G(x)$ for all $x \in E$, then $V(X_{\tau_*}) \geq G(X_{\tau_*})$. Suppose that $\mathbb{P}_x(V(X_{\tau_*}) > G(X_{\tau_*})) > 0$ for some $x \in E$, then

$$\mathbb{E}_x(G(X_{\tau_*})) < \mathbb{E}_x(V(X_{\tau_*})) \leq V(x),$$

where the second inequality holds since V is superharmonic. Note that the above inequality contradicts the fact that τ_* is optimal in (2.10). Therefore $\tau_D \leq \tau_*$.

By *ii*) we have that $\{V(X_t), t \geq 0\}$ is a right-continuous supermartingale, thus by the optional sampling theorem, we see that for all stopping times σ and τ such that $\sigma \leq \tau$ $\mathbb{P}_x$-a.s. with $x \in E$.

$$\mathbb{E}_x(V(X_\tau)) \leq \mathbb{E}_x(V(X_\sigma)).$$

In particular if we take $\tau_D \leq \tau_*$ we get for $x \in E$,

$$V(x) = \mathbb{E}_x(G(X_{\tau_*})) = \mathbb{E}_x(V(X_{\tau_*})) \leq \mathbb{E}_x(V(X_{\tau_D})) = \mathbb{E}_x(G_{\tau_D}) \leq V(x),$$

where we used the fact that $V(X_{\tau_D}) = G(X_{\tau_D})$ as V is lsc and G is usc. This shows that τ_D is optimal and concludes the proof.

Proof of iv). By the strong Markov property we have that for all $s \leq t$ and all $x \in E$,

$$\begin{aligned}
\mathbb{E}_x(V(X_{t\wedge\tau_D})|\mathcal{F}_{s\wedge\tau_D}) &= \mathbb{E}_x(\mathbb{E}_{X_{t\wedge\tau_D}}(G(X_{\tau_D}))|\mathcal{F}_{s\wedge\tau_D}) \\
&= \mathbb{E}_x(\mathbb{E}_x(G(X_{\tau_D}) \circ \theta_{t\wedge\tau_D}|\mathcal{F}_{t\wedge\tau_D})|\mathcal{F}_{s\wedge\tau_D}) \\
&= \mathbb{E}_x(\mathbb{E}_x(G(X_{t\wedge\tau_D+\tau_D\circ\theta_{t\wedge\tau_D}})|\mathcal{F}_{t\wedge\tau_D})|\mathcal{F}_{s\wedge\tau_D}) \\
&= \mathbb{E}_x(\mathbb{E}_x(G(X_{\tau_D})|\mathcal{F}_{t\wedge\tau_D})|\mathcal{F}_{s\wedge\tau_D}) \\
&= \mathbb{E}_x(G(X_{\tau_D})|\mathcal{F}_{s\wedge\tau_D}) \\
&= \mathbb{E}_x(G(X_{s\wedge\tau_D+\tau_D\circ\theta_{s\wedge\tau_D}})|\mathcal{F}_{s\wedge\tau_D}) \\
&= \mathbb{E}_{X_{s\wedge\tau_D}}(G_{\tau_D}) \\
&= V(X_{s\wedge\tau_D}),
\end{aligned}$$

where the fourth equality holds since $t \wedge \tau_D \leq \tau_D$ and

$$\begin{aligned}
\tau_D \circ \theta_{t\wedge\tau_D} &= \inf\{s \geq 0 : X_{s+t\wedge\tau_D} \in D\} \\
&= \inf\{s \geq t \wedge \tau_D : X_s \in D\} - t \wedge \tau_D \\
&= \tau_D - t \wedge \tau_D.
\end{aligned}$$

Hence the martingale property is already proved for the filtration $\{\mathcal{F}_{t\wedge\tau_D}, t \geq 0\}$ and automatically is satisfied for $\{\mathcal{F}_t, t \geq 0\}$. The right-continuity of $\{V(X_{t\wedge\tau_D}), t \geq 0\}$ follow from the right-continuity of $\{V(X_t), t \geq 0\}$ and the proof is complete.

□

The following theorem provides sufficient conditions for the existence of an optimal stopping time.

Theorem 2.3.4. *Consider the optimal stopping problem* (2.10) *upon assuming that the condition* (2.9) *is satisfied. Let us assume that there exists the smallest superharmonic function* $\widehat{V}$ *which dominates the gain function* G *on* E*. Let us assume that* $\widehat{V}$ *is lsc and* G *is usc. Set* $D = \{x \in E : \widehat{V}(x) = G(x)\}$ *and let* τ_D *defined by* (2.14).

Proof. We only consider the case when G is bounded, for the general case see Peskir and Shiryaev (2006) (Theorem 2.7). Since $\widehat{V}$ is superharmonic and dominates G, we have

$$\mathbb{E}_x(G(X_\tau)) \leq \mathbb{E}_x(\widehat{V}(X_\tau)) \leq \widehat{V}(x)$$

for all stopping times τ and all $x \in E$. Taking supremum over all τ we find that for all $x \in E$,

$$V(x) \leq \widehat{V}(x). \tag{2.15}$$

Now we will proof the reverse inequality, for this purpose let $\varepsilon > 0$ and consider the sets

$$\begin{aligned} C_\varepsilon &= \{x \in E : \widehat{V}(x) > G(x) + \varepsilon\}, \\ D_\varepsilon &= \{x \in E : \widehat{V}(x) \leq G(x) + \varepsilon\} \end{aligned}$$

Since $\widehat{V}$ is lsc and G is usc then C_ε is open and D_ε is closed. Moreover, it holds that $C_\varepsilon \uparrow C$ and $D_\varepsilon \downarrow D$ as $\varepsilon \downarrow 0$ where C and D are defined by

$$\begin{aligned} C &= \{x \in E : \widehat{V}(x) > G(x)\}, \\ D &= \{x \in E : \widehat{V}(x) = G(x)\}. \end{aligned}$$

Define the stopping time

$$\tau_{D_\varepsilon} = \inf\{t \geq 0 : X_t \in D_\varepsilon\}.$$

Suppose that $\mathbb{P}_x(\tau_D < \infty) = 1$ for all $x \in E$, since $D \subseteq D_\varepsilon$ thus $\mathbb{P}_x(\tau_{D_\varepsilon} < \infty) = 1$ for all $x \in E$.

If we show that for all $x \in E$,

$$\mathbb{E}_x(\widehat{V}(X_{\tau_{D_\varepsilon}})) = \widehat{V}(x) \tag{2.16}$$

we then get,

$$\widehat{V}(x) = \mathbb{E}_x(\widehat{V}(X_{\tau_{D_\varepsilon}})) \leq \mathbb{E}_x(G(X_{\tau_{D_\varepsilon}})) + \varepsilon \leq V(x) + \varepsilon \tag{2.17}$$

where the first and second inequalities holds by the definition of τ_{D_ε} and V respectively using that $\widehat{V}$ is lsc and G is usc. Letting $\varepsilon \downarrow 0$ we see that $\widehat{V}(x) \leq V(x)$ for all $x \in E$ and therefore $\widehat{V} = V$. From (2.17) we also have that

$$V(x) \leq \mathbb{E}_x(G(X_{\tau_{D_\varepsilon}})) + \varepsilon.$$

Letting $\varepsilon \downarrow 0$ and using that $D_\varepsilon \downarrow D$ it can be proved that $\tau_{D_\varepsilon} \uparrow \tau_D$ (see Peskir and Shiryaev (2006) page 43). Then applying Fatou's lemma, we get

$$\begin{aligned} V(x) &\leq \limsup_{\varepsilon \downarrow 0} \mathbb{E}_x(G(X_{\tau_{D_\varepsilon}})) \\ &\leq \mathbb{E}_x(\limsup_{\varepsilon \downarrow 0} G(X_{\tau_{D_\varepsilon}})) \\ &\leq \mathbb{E}_x(G(\limsup_{\varepsilon \downarrow 0} X_{\tau_{D_\varepsilon}})) \\ &= \mathbb{E}_x(G(X_{\tau_D})). \end{aligned}$$

Thus $V(x) \leq \mathbb{E}_x(G(X_{\tau_D}))$, applying the definition of V we have the reverse inequality and hence $V(x) = \mathbb{E}_x(G(X_{\tau_D}))$ which implies that τ_D is optimal. It only remains to prove (2.16), for this we will first show that for all $x \in E$

$$G(x) \leq \mathbb{E}_x(\widehat{V}(X_{\tau_{D_\varepsilon}})).$$

For this define

$$c = \sup_{x\in E}(G(x) - \mathbb{E}_x(\widehat{V}(X_{\tau_{D_\varepsilon}}))).$$

Note that

$$G(x) \leq c + \mathbb{E}_x(\widehat{V}(X_{\tau_{D_\varepsilon}}))$$

for all $x \in E$. By the strong Markov property we find that for all stopping times σ and all $x \in E$

$$\begin{aligned}
\mathbb{E}_x(\mathbb{E}_{X_\sigma}(\widehat{V}(X_{\tau_{D_\varepsilon}}))) &= \mathbb{E}_x(\mathbb{E}_x(\widehat{V}(X_{\tau_{D_\varepsilon}}) \circ \theta_\sigma | \mathcal{F}_\sigma)) \\
&= \mathbb{E}_x(\mathbb{E}_x(\widehat{V}(X_{\sigma + \tau_{D_\varepsilon} \circ \theta_\sigma}) | \mathcal{F}_\sigma)) \\
&= \mathbb{E}_x(\widehat{V}(X_{\sigma + \tau_{D_\varepsilon} \circ \theta_\sigma})) \\
&\leq \mathbb{E}_x(\widehat{V}(X_{\tau_{D_\varepsilon}})),
\end{aligned}$$

where the inequality holds using that $\{\widehat{V}(X_t), t \geq 0\}$ is a supermartingale as $\widehat{V}$ is superharmonic and lsc (see proof of Theorem 2.3.3 *ii*)), and $\sigma + \tau_{D_\varepsilon} \circ \theta_\sigma \geq \tau_{D_\varepsilon}$ since

$$\begin{aligned}
\tau_{D_\varepsilon} \circ \theta_\sigma &= \inf\{t \geq 0 : X_t \circ \theta_\sigma \in D_\varepsilon\} \\
&= \inf\{t \geq 0 : X_{t+\sigma} \in D_\varepsilon\} \\
&= \inf\{t \geq \sigma : X_t \in D_\varepsilon\} - \sigma \\
&\geq \tau_{D_\varepsilon} - \sigma.
\end{aligned}$$

The above shows that

$$x \mapsto \mathbb{E}_x(\widehat{V}(X_{\tau_{D_\varepsilon}}))$$

is a superharmonic function from E to $\mathbb{R}$. The latter implies that $c + \mathbb{E}_x(\widehat{V}(X_{\tau_{D_\varepsilon}}))$ is also superharmonic and then by assumption of $\widehat{V}$ we can conclude that

$$\widehat{V}(x) \leq c + \mathbb{E}_x(\widehat{V}(X_{\tau_{D_\varepsilon}})) \tag{2.18}$$

for all $x \in E$. Given $0 < \delta \leq \varepsilon$ then by definition of c there exists $x_\delta \in E$ such that

$$G(x_\delta) - \mathbb{E}_{x_\delta}\widehat{V}(X_{\tau_{D_\varepsilon}}) \geq c - \delta \tag{2.19}$$

then by (2.18) and the above inequality

$$\widehat{V}(x_\delta) \leq c + \mathbb{E}_{x_\delta}(\widehat{V}(X_{\tau_{D_\varepsilon}})) \leq G(x_\delta) + \delta \leq G(x_\delta) + \varepsilon.$$

This shows that $x_\delta \in D_\varepsilon$ and thus $\tau_{D_\varepsilon} = 0$ under $\mathbb{P}_{x_\delta}$. Then equation (2.19) becomes

$$c - \delta \leq G(x_\delta) - \widehat{V}(x_\delta) \leq 0$$

Letting $\delta \downarrow 0$ we see that $c \leq 0$ and hence

$$G(x) \leq \mathbb{E}_x(\widehat{V}(X_{\tau_{D_\varepsilon}}))$$

Using the definition of $\widehat{V}$ and the fact that $x \mapsto \mathbb{E}_x(\widehat{V}(X_{\tau_{D_\varepsilon}}))$ is a superharmonic function we see that

$$\widehat{V}(x) \leq \mathbb{E}_x(\widehat{V}(X_{\tau_{D_\varepsilon}}))$$

for all $x \in E$. The definition of V and (2.15) implies that

$$\mathbb{E}_x(\widehat{V}(X_{\tau_{D_\varepsilon}})) \leq V(x) \leq \widehat{V}(x).$$

Therefore $\widehat{V}(x) = \mathbb{E}_x(\widehat{V}(X_{\tau_{D_\varepsilon}}))$ for all $x \in E$. Then $\widehat{V} = V$ and τ_D is optimal.

In the case that $\mathbb{P}_x(\tau_D < \infty) < 1$ for some $x \in E$. Suppose that exists an stopping time τ_* in (2.10), then from Theorem 2.3.3 we have that $V = \widehat{V}$ and from part *iii*), $\tau_D \leq \tau_*$ so $\mathbb{P}(\tau_* < \infty) < 1$ and there is no optimal stopping time with probability 1.

□

The following corollary is an elegant tool for tackling the optimal stopping problem in the case when one can prove directly from the definition of V that V is lsc.

Corollary 2.3.5. *Consider the optimal stopping problem* (2.10) *upon assuming that the condition* (2.9) *is satisfied. Suppose that V is lsc and G is usc. If $\mathbb{P}_x(\tau_D < \infty) = 1$ for all $x \in E$, then τ_D is optimal in* (2.10).

Proof. We will show that V is superharmonic. From the strong Markov property we have

$$\begin{aligned} V(X_\sigma) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}_{X_\sigma}(G(X_\tau)) \\ &= \sup_{\tau \geq 0} \mathbb{E}_x(G(X_\tau) \circ \theta_\sigma | \mathcal{F}_\sigma) \\ &= \sup_{\tau \geq 0} \mathbb{E}_x(G(X_{\sigma + \tau \circ \theta_\sigma}) | \mathcal{F}_\sigma) \end{aligned}$$

Note that V is measurable since it is lsc and thus so is $V(X_\sigma)$ then

$$V(X_\sigma) = \operatorname*{ess\,sup}_{\tau \geq 0} \mathbb{E}_x(G(X_{\sigma + \tau \circ \theta_\sigma}) | \mathcal{F}_\sigma)$$

for all $x \in E$. Next, we claim that the family

$$\{\mathbb{E}_x(X_{\sigma + \tau \circ \theta_\sigma}) : \tau \in \mathcal{T}\}$$

has the lattice property. Indeed, suppose that τ_1 and τ_2 are stopping times given and fixed, set $\rho_1 = \sigma + \tau_1 \circ \theta_\sigma$ and $\rho_2 = \sigma + \tau_2 \circ \theta_\sigma$, and define

$$B = \{\mathbb{E}_x(X_{\rho_1} | \mathcal{F}_\sigma) \geq \mathbb{E}_x(X_{\rho_2 | \mathcal{F}_\sigma})\}.$$

Then $B \in \mathcal{F}_\sigma$ and the random time

$$\rho = \rho_1 \mathbb{I}_B + \rho_2 \mathbb{I}_{B^c}$$

is a stopping time since

$$\{\rho \leq t\} = (\{\rho_1 \leq t\} \cap B) \cup (\{\rho_2 \leq t\} \cap B^c) \in \mathcal{F}_t$$

due to the fact that B and B^c belong to $\mathcal{F}_\sigma$ and the claim is proved. Finally, we have

$$\begin{aligned}
\mathbb{E}(X_\rho|\mathcal{F}_\sigma) &= \mathbb{E}(X_{\rho_1 \mathbb{I}_B + \rho_2 \mathbb{I}_{B^c}}|\mathcal{F}_\sigma) \\
&= \mathbb{E}(X_{\rho_1}|\mathcal{F}_\sigma)\mathbb{I}_B + \mathbb{E}(X_{\rho_2}|\mathcal{F}_\sigma)\mathbb{I}_{B^c} \\
&= \mathbb{E}(X_{\rho_1}|\mathcal{F}_\sigma) \vee \mathbb{E}(X_{\rho_2}|\mathcal{F}_\sigma),
\end{aligned}$$

proving that the family $\{\mathbb{E}_x(X_{\sigma+\tau\circ\theta_\sigma}) : \tau \in \mathcal{T}\}$ has the lattice property. From Corollary 2.1.3 we know that there exists a sequence of stopping times $\{\tau_n : n \geq 1\}$ such that

$$V(X_\sigma) = \lim_{n\to\infty} \mathbb{E}_x(G(X_{\sigma+\tau_n\circ\theta_\sigma})|\mathcal{F}_\sigma),$$

where the sequence $\{\mathbb{E}_x(G(X_{\sigma+\tau_n\circ\theta_\sigma})|\mathcal{F}_\sigma), n \geq 1\}$ is increasing $\mathbb{P}_x$-a.s. By the monotone convergence theorem we can therefore conclude that for all stopping times σ and all $x \in E$,

$$\begin{aligned}
\mathbb{E}_x(V(X_\sigma)) &= \mathbb{E}_x\left(\lim_{n\to\infty} \mathbb{E}_x(G(X_{\sigma+\tau_n\circ\theta_\sigma})|\mathcal{F}_\sigma)\right) \\
&= \lim_{n\to\infty} \mathbb{E}_x\left(\mathbb{E}_x(G(X_{\sigma+\tau_n\circ\theta_\sigma})|\mathcal{F}_\sigma)\right) \\
&= \lim_{n\to\infty} \mathbb{E}_x(G(X_{\sigma+\tau_n\circ\theta_\sigma})) \\
&= \sup_{n\geq 1} \mathbb{E}_x(G(X_{\sigma+\tau_n\circ\theta_\sigma})) \\
&\leq V(x).
\end{aligned}$$

Therefore V is a superharmonic function which dominates G. Suppose that F is another superharmonic function which dominates G then for all stopping times τ

$$\mathbb{E}_x(G(X_\tau)) \leq \mathbb{E}_x(F(X_\tau)) \leq F(x).$$

Taking the supremum over all stopping times we have for all $x \in E$,

$$V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x(G(X_\tau)) \leq F(x).$$

Hence V is the smallest superharmonic function which dominates G. Therefore all the claims follow directly from Theorem 2.3.4.

□

Remark 2.3.6. *In this Chapter we only consider optimal stopping problems of the form*

$$V_t = \sup_{\tau\in\mathcal{T}_t} \mathbb{E}(G_\tau).$$

It is important to emphasise that we may also consider optimal stopping problems of the form

$$V_t = \inf_{\tau\in\mathcal{T}_t} \mathbb{E}(G_\tau).$$

The theory studied in this chapter also applies for these problems. We only have to consider the process $G' = \{G'_t, t \geq 0\}$ *where* $G'_t = -G_t$ *for all* $t \geq 0$.

Chapter 3

Predicting the Last Zero of a Spectrally Negative Lévy Process

3.1 Last Exit Times and Optimal Prediction Problems

In recent years last exit times have been studied in several areas of applied probability, e.g. in risk theory (see Chiu et al. (2005)). Consider the Cramér–Lundberg process, which is a process consisting of a deterministic drift plus a compound Poisson process which has only negative jumps (see Figure 3.1) which typically models the capital of an insurance company. A quantity of interest is the moment of ruin τ_0, i.e. the first moment that the process becomes negative. Let us suppose the insurance company has funds to endure negative capital for a while. Then another quantity of interest is the last time g that the process is below zero. In a more general setting we may consider a spectrally negative Lévy process instead of the classical risk process. Baurdoux (2009) and Chiu et al. (2005) found the Laplace transform of the last time before an exponential time a spectrally negative Lévy process is below some level.

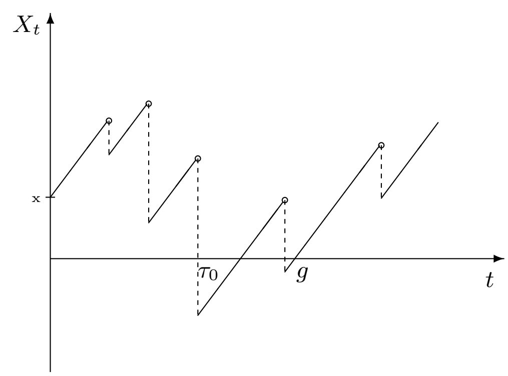

Figure 3.1: Cramér–Lundberg process with τ_0 the moment of ruin and g the last zero.

Last passage times have also played an incresing role in financial modeling, Madan et al. (2008a,b) showed that the price of a European put and call option can be modelled by non-negative and continous martingales that vanish at infinity, can be expressed in terms of the probability distributions of some last passage times.

Another application is in degradation models. Paroissin and Rabehasaina (2013) proposed a spectrally positive Lévy process as a degradation model. They consider a subordinator perturbed by an independent Brownian motion. One motivation of consider this model is that the presence of a Brownian motion can model small repairs of the component or system and the jumps represents major deterioration. Classically, the failure time of a component or system is defined as the first hitting time of a critical level b which represents a failure or a bad performance of the component or system. Another approach is to consider instead the last time that the process is under b. Indeed, for this process the paths are not necessarily monotone and hence when the process is above the level b it can return back below.

The aim of this work is to predict the last time a spectrally negative Lévy process is below zero. We refer to predict as to find a stopping time that is closest (in L^1 sense) to the above random time. This is an example of an optimal prediction problem. Recently, these problems have received considerable attention, for example, Bernyk et al. (2011) predicted the time at which a stable spectrally negative Lévy process attains its ultimate supremum in a finite horizon of time. A few years later Baurdoux and Van Schaik (2014) did the same but now for a general Lévy process in infinite horizon of time. Glover et al. (2013) predicted the time of its ultimate minimum for a transient diffusion processes. Du Toit et al. (2008) predicted the last zero of a linear Brownian motion. It turns out that the problems just mentioned are equivalent to an optimal stopping problem, in other words, optimal prediction problems and optimal stopping problems are intimately related.

3.2 Prerequistes and Formulation of the Problem

Formally, let X be a spectrally negative Lévy process drifting to infinity (see Proposition 1.3.34) starting from 0 defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ is the filtration generated by X which is naturally enlarged (see Definition 1.3.38 in Bichteler (2002)). Suppose that X has Lévy triple (c, σ, Π) where $c \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure (Lévy measure) concentrated on $(-\infty, 0)$ satisfying $\int_{(-\infty,0)} (1 \wedge x^2)\Pi(dx) < \infty$. Let W the scale function defined in Section 1.4. Recall that W is such that $W(x) = 0$ for $x < 0$, and is characterised on $[0, \infty)$ as a strictly increasing and continuous function whose Laplace transform satisfies

$$\int_0^\infty e^{-\beta x} W(x)dx = \frac{1}{\psi(\beta)} \qquad \text{for } \beta > \Phi(0),$$

where ψ and Φ are the Laplace exponent and its right inverse given in (1.12) and (1.16), respectively. We know from Lemma 1.4.6 that the right and left derivatives of W exist. Nevertheless, for ease of notation we shall assume that Π has no atoms when X is of finite variation, which guarantees that $W \in C^1(0, \infty)$, since all the proofs presented below remain valid using the left and right derivatives of W.

From Remark 1.3.5 we have that if X is of finite variation we may write

$$\psi(\lambda) = d\lambda - \int_{(-\infty,0)} (1 - e^{\lambda y})\Pi(dy),$$

where necessarily

$$d = -b - \int_{(-1,0)} x\Pi(dx) > 0.$$

With this notation, from the fact that $0 \leq 1 - e^{\lambda y} \leq 1$ for $y \leq 0$ and using the dominated convergence theorem we have that

$$\psi'(0+) = d + \int_{(-\infty,0)} x\Pi(dx). \tag{3.1}$$

Lemma 1.4.9 tells us that the value of W at zero depends on the path variation of X: in the case that X is of infinite variation we have that $W(0) = 0$, otherwise

$$W(0) = \frac{1}{d}. \tag{3.2}$$

Let g_r be the last passage time below $r \geq 0$, i.e.

$$g_r = \sup\{t \geq 0 : X_t \leq r\}. \tag{3.3}$$

When $r = 0$ we simply write $g_0 = g$.

Remark 3.2.1. *Note that from the fact that X drifts to infinity we have that $g_r < \infty$ $\mathbb{P}$-a.s. Moreover, as we are supposing that X is a spectrally negative Lévy process, and hence the case of a compound Poisson process is excluded, the only way of exiting the set $(-\infty, r]$ is by creeping upwards. This tells us that $X_{g_r-} = r$ and that $g_r = \sup\{t \geq 0 : X_t < r\}$ $\mathbb{P}$-a.s.*

Clearly, up to any time $t \geq 0$ the value of g is unknown (unless X is trivial), and it is only with the realisation of the whole process that we know that the last passage time below 0 has occurred. However, this is often too late: typically one would like to know how close X is to g at any time $t \geq 0$ and then take some action based on this information. We search for a stopping time τ_* of X that is as "close" as possible to g. Consider the optimal prediction problem

$$V_* = \inf_{\tau \in \mathcal{T}} \mathbb{E}(|g - \tau|), \tag{3.4}$$

where $\mathcal{T}$ is the set of all stopping times.

Before giving an equivalence between the optimal prediction problem (3.4) and an optimal stopping problem we prove that the random times g_r for $r \geq 0$ have finite mean. For this purpose, recall from Section 1.3 that for $x \geq 0$ the first passage time above x is denoted by

$$\tau_x^+ = \inf\{t > 0 : X_t > x\}.$$

Lemma 3.2.2. *Let X be a spectrally negative Lévy process drifting to infinity with Lévy measure Π such that*

$$\int_{(-\infty,-1)} x^2 \Pi(dx) < \infty. \tag{3.5}$$

Then $\mathbb{E}_x(g_r) < \infty$ for every $x, r \in \mathbb{R}$.

Proof. Note that by the spatial homogeneity of Lévy processes we have to prove that for all $x, r \in \mathbb{R}$.

$$\mathbb{E}_x(g_r) = \mathbb{E}_{x-r}(g) < \infty.$$

Then it suffices to take $r = 0$. From Baurdoux (2009) (Theorem 1) or Chiu et al. (2005) (Theorem 3.1) we know that for a spectrally negative Lévy process such that $\psi'(0+) > 0$ the Laplace transform of g for $q \geq 0$ and $x \in \mathbb{R}$ is given by

$$\mathbb{E}_x(e^{-qg}) = e^{\Phi(q)x}\Phi'(q)\psi'(0+) + \psi'(0+)(W(x) - W^{(q)}(x)).$$

Then, from the well-known result which links the moments and derivatives of the Laplace transform (see Feller (1971) (section XIII.2)), the expectation of g is given by

$$\begin{aligned}
\mathbb{E}_x(g) &= -\frac{\partial}{\partial q}\mathbb{E}_x(e^{-qg})\Big|_{q=0+} \\
&= \psi'(0+)\frac{\partial}{\partial q}W^{(q)}(x)\Big|_{q=0+} - \psi'(0+)[\Phi''(q)e^{\Phi(q)x} + x\Phi'(q)^2 e^{\Phi(q)x}]\Big|_{q=0+} \\
&= \psi'(0+)\frac{\partial}{\partial q}W^{(q)}(x)\Big|_{q=0+} - \psi'(0+)[\Phi''(0) + x\Phi'(0)^2]
\end{aligned}$$

We know from Lemma 1.4.7 that for any $x \in \mathbb{R}$ the function $q \mapsto W^{(q)}$ is analytic, therefore the first term in the last expression is finite. Then g has finite second moment if $\Phi'(0)$ and $\Phi''(0)$ are finite. Now we calculate the terms $\Phi'(0)$ and $\Phi''(0)$. Recall from Proposition 1.3.20 that the function $\psi : [0,\infty) \mapsto \mathbb{R}$ is zero at zero and tends to infinity at infinity. Further, it is infinitely differentiable and strictly convex on $(0,\infty)$. As we are supposing that X drifts to infinity we have that $\psi'(0+) > 0$ thus $\psi(\lambda) \geq 0$ for all $\lambda > 0$ and hence using the definition of strict convexity we have that for all $t \in (0,1)$

$$\psi(tx + (1-t)y) < t\psi(x) + (1-t)\psi(y).$$

Taking $y = 0$ we have that $\psi(tx) < t\psi(x) < \psi(x)$, now if we have $\lambda_1 < \lambda_2$ and we take $t = \lambda_1/\lambda_2 < 1$, $x = \lambda_2$ we have $\psi(\lambda_1) < \psi(\lambda_2)$. In conclusion ψ is strictly increasing in $[0,\infty)$ and the right inverse $\Phi(q)$ is the usual inverse for ψ. From the fact that ψ is strictly convex we have that $\psi''(x) > 0$ for all $x > 0$.

Now we state a useful result of calculus. Let f be a continuous one to one function defined on an interval, and suppose that f is differentiable at $f^{-1}(b)$, with derivative $f'(f^{-1}(b)) \neq 0$. Then f^{-1} is differentiable at b, and

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}.$$

Using the above result we have that

$$\Phi'(0) = \frac{1}{\psi'(\Phi(0)+)} = \frac{1}{\psi'(0+)} < \infty$$

and

$$\Phi''(0) = [\Phi'(q)]'\Big|_{q=0} = \left[\frac{1}{\psi'(\Phi(q))}\right]'\Bigg|_{q=0} = -\frac{\psi''(\Phi(q)+)\Phi'(q)}{\psi'(\Phi(q)+)^2}\Bigg|_{q=0} = -\frac{\psi''(0+)}{\psi'(0+)^3}.$$

Note that from (1.14) we have that

$$\psi''(0+) = \sigma^2 + \int_{(-\infty,0)} x^2\Pi(dx) = \sigma^2 + \int_{(-\infty,-1)} x^2\Pi(dx) + \int_{(-1,0)} x^2\Pi(dx) < \infty,$$

where the last inequality holds by assumption (3.5) and from the fact that $\int_{(-1,0)} x^2\Pi(dx) < \infty$ since Π is a Lévy measure. Then we have that $\Phi''(0) > -\infty$ and hence $\mathbb{E}_x(g) < \infty$ for all $x \in \mathbb{R}$. The conclusion of the Lemma follows. □

Now we are ready to state the equivalence between the optimal prediction problem and an optimal stopping problem mentioned earlier. This equivalence is mainly based on the work of Urusov (2005).

Lemma 3.2.3. *Suppose that $\{X_t, t \geq 0\}$ is a spectrally negative Lévy process which drifts to infinity with Lévy measure Π satisfying (3.5). Let g be the last time that X is below the level zero, as defined in (3.3). Let us consider the standard optimal stopping problem given by*

$$V = \inf_{\tau \in \mathcal{T}} \mathbb{E}\left(\int_0^\tau G(X_s)ds\right), \tag{3.6}$$

where the function G is given by $G(x) = 2\psi'(0+)W(x) - 1$ for all $x \in \mathbb{R}$. Then the stopping time which minimises (3.4) is the same which minimises (3.6). In particular,

$$V_* = V + \mathbb{E}(g). \tag{3.7}$$

Proof. Fix any stopping time of $\mathbb{F}$. We then have

$$
\begin{aligned}
|g-\tau| &= (\tau-g)^+ + (\tau-g)^- \\
&= (\tau-g)^+ + g - (\tau\wedge g) \\
&= \int_0^\tau \mathbb{I}_{\{g\le s\}} ds + g - \int_0^\tau \mathbb{I}_{\{g>s\}} ds \\
&= \int_0^\tau \mathbb{I}_{\{g\le s\}} ds + g - \int_0^\tau [1-\mathbb{I}_{\{g\le s\}}] ds \\
&= g + \int_0^\tau [2\mathbb{I}_{\{g\le s\}} - 1] ds.
\end{aligned}
$$

From Fubini's theorem we have

$$
\begin{aligned}
\mathbb{E}\left[\int_0^\tau \mathbb{I}_{\{g\le s\}} ds\right] &= \mathbb{E}\left[\int_0^\infty \mathbb{I}_{\{s<\tau\}}\mathbb{I}_{\{g\le s\}} ds\right] \\
&= \int_0^\infty \mathbb{E}[\mathbb{I}_{\{s<\tau\}}\mathbb{I}_{\{g\le s\}}] ds \\
&= \int_0^\infty \mathbb{E}[\mathbb{E}[\mathbb{I}_{\{s<\tau\}}\mathbb{I}_{\{g\le s\}}|\mathcal{F}_s]] ds \\
&= \int_0^\infty \mathbb{E}[\mathbb{I}_{\{s<\tau\}}\mathbb{E}[\mathbb{I}_{\{g\le s\}}|\mathcal{F}_s]] ds \\
&= \mathbb{E}\left[\int_0^\infty \mathbb{I}_{\{s<\tau\}}\mathbb{E}[\mathbb{I}_{\{g\le s\}}|\mathcal{F}_s] ds\right] \\
&= \mathbb{E}\left[\int_0^\tau \mathbb{P}(g\le s|\mathcal{F}_s) ds\right].
\end{aligned}
$$

Note that in consequence of Remark 3.2.1 the event $\{g \le s\}$ is equal to $\{X_u \ge 0 \text{ for all } u \in [s,\infty)\}$ (up to a $\mathbb{P}$-null set). Hence, since X_s is $\mathcal{F}_s$-measurable,

$$
\begin{aligned}
\mathbb{P}(g\le s|\mathcal{F}_s) &= \mathbb{P}(X_u > 0 \text{ for all } u\in[s,\infty)|\mathcal{F}_s) \\
&= \mathbb{P}\left(\inf_{u\ge s} X_u \ge 0|\mathcal{F}_s\right) \\
&= \mathbb{P}\left(\inf_{u\ge s} (X_u - X_s) \ge -X_s|\mathcal{F}_s\right) \\
&= \mathbb{P}\left(\inf_{u\ge 0} \widetilde{X}_u \ge -X_s|\mathcal{F}_s\right),
\end{aligned}
$$

where $\widetilde{X}_u = X_{s+u} - X_s$ for $u \ge 0$. From the Markov property for Lévy process (see Theorem 1.3.10) we have that $\widetilde{X} = (\widetilde{X}_u, u \ge 0)$ is Lévy process with the same law as X, independent of $\mathcal{F}_s$. From the above and the fact that X_s is $\mathcal{F}_s$-measurable we have

$$
\mathbb{P}(g\le s|\mathcal{F}_s) = h(X_s),
$$

where $h(b) = \mathbb{P}(\inf_{u\ge 0} X_u \ge -b)$. Note that the event $\{\inf_{u\ge 0} X_u \ge 0\}$ is equal to $\{\tau_0^- = \infty\}$ where $\tau_0^- = \inf\{s > 0 : X_s < 0\}$. Hence, by the spatial homogeneity of Lévy processes

$$
\begin{aligned}
h(b) &= \mathbb{P}(\inf_{u\ge 0} X_u \ge -b) \\
&= \mathbb{P}_b(\inf_{u\ge 0} X_u \ge 0) \\
&= \mathbb{P}_b(\tau_0^- = \infty) \\
&= [1 - \mathbb{P}_b(\tau_0^- < \infty)] \\
&= \psi'(0+)W(b),
\end{aligned}
$$

where the last equality holds by Theorem 1.4.4 i) and the fact that $\psi'(0+) > 0$ (since X drifts to infinity). Therefore,

$$\begin{aligned}
V_* &= \inf_{\tau \in \mathcal{T}} \mathbb{E}(|g-\tau|) \\
&= \mathbb{E}(g) + \inf_{\tau \in \mathcal{T}} \left\{ 2\mathbb{E}\left(\int_0^\tau \mathbb{I}_{\{g \leq s\}} ds \right) - \mathbb{E}(\tau) \right\} \\
&= \mathbb{E}(g) + \inf_{\tau \in \mathcal{T}} \left\{ 2\mathbb{E}\left(\int_0^\tau \mathbb{P}(g \leq s | \mathcal{F}_s) ds \right) - \mathbb{E}(\tau) \right\} \\
&= \mathbb{E}(g) + \inf_{\tau \in \mathcal{T}} \left\{ 2\mathbb{E}\left(\int_0^\tau h(X_s) ds \right) - \mathbb{E}(\tau) \right\} \\
&= \mathbb{E}(g) + \inf_{\tau \in \mathcal{T}} \left\{ \mathbb{E}\left(\int_0^\tau [2h(X_s) - 1] ds \right) \right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
V_* &= \mathbb{E}(g) + \inf_{\tau \in \mathcal{T}} \left\{ \mathbb{E}\left(\int_0^\tau [2\psi'(0+)W(X_s) - 1] ds \right) \right\} \\
&= \mathbb{E}(g) + \inf_{\tau \in \mathcal{T}} \left\{ \mathbb{E}\left(\int_0^\tau G(X_s) ds \right) \right\}.
\end{aligned}$$

□

We define the function $V : \mathbb{R} \mapsto \mathbb{R}$ as

$$V(x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}_x \left(\int_0^\tau G(X_s) ds \right). \tag{3.8}$$

Thus,

$$V_* = V(0) + \mathbb{E}(g).$$

Now we give some intuition about the function G. For this define x_0 as the lower value x such that $G(x) \geq 0$, i.e.

$$x_0 = \inf\{x \in \mathbb{R} : G(x) \geq 0\}. \tag{3.9}$$

From Lemma 1.4.5 we know that W is continuous and strictly increasing on $[0,\infty)$ and vanishes on $(-\infty, 0)$. Moreover, from Lemma 1.4.10 we have $\lim_{x\to\infty} W(x) = 1/\psi'(0+)$. As a consequence we have that G is a strictly increasing and continuous function on $[0,\infty)$ such that $G(x) = -1$ for $x < 0$ and $G(x) \xrightarrow{x\to\infty} 1$. In the same way as W, G may have a discontinuity at zero depending of the path variation of X (see Figure 3.2). From the fact that $G(x) = -1$ for $x < 0$ and the definition of x_0 given in (3.9) we have that $x_0 \geq 0$.

The above observations tell us that, to solve the optimal stopping problem (3.8), we are interested in a stopping time such that before stopping, the process X spent most of the time in those values of which G is negative, taking into account that X can pass some time in the set $\{x \in \mathbb{R} : G(x) > 0\}$ and then return back to the set $\{x \in \mathbb{R} : G(x) \leq 0\}$.

From the above observations concerning the function G it seems reasonable to thinking that a stopping time which attains the infimum in (3.8) is of the form,

$$\tau_a^+ = \inf\{t > 0 : X_t > a\}$$

for some $a \in \mathbb{R}$.

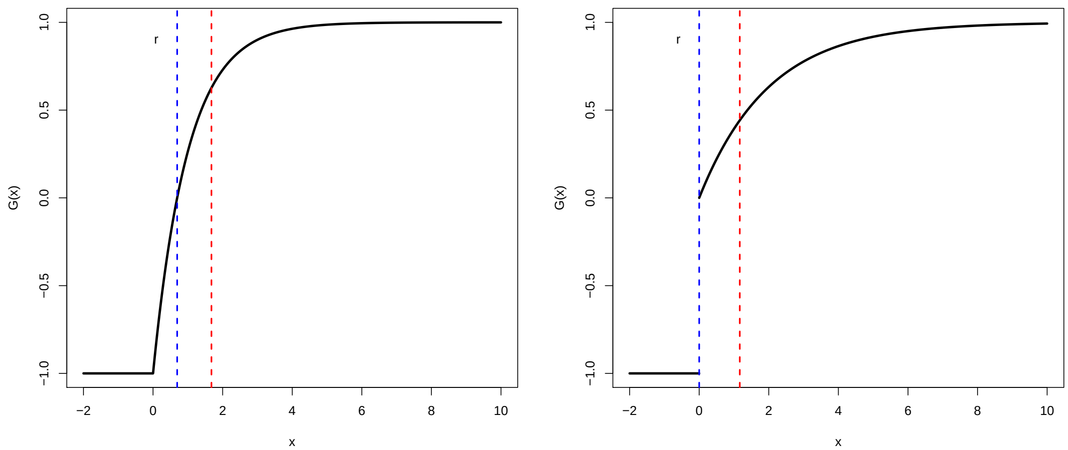

Figure 3.2: Left side: $\Pi(dx) = e^{2x}(e^x - 1)^{-3}dx$, $x > 0$ without Gaussian component. Right side: Cramér–Lundberg model with $c = 2$, $\lambda = 1$ and $\xi \sim \exp(1)$.

The following theorem is the main result of this work. It gives us a stopping time which attains the infimum in (3.8) and hence in (3.4). As well as expressing the value function in terms of scale functions.

Theorem 3.2.4. *Suppose that X is a spectrally negative Lévy process drifting to infinity with Lévy measure Π satisfying*

$$\int_{(-\infty,-1)} x^2 \Pi(dx) < \infty.$$

Then there exists a value $a^* \in [x_0, \infty)$ such that an optimal stopping time in (3.8) is given by

$$\tau^* = \inf\{t \geq 0 : V(X_t) = 0\} = \inf\{t \geq 0 : X_t \geq a^*\}.$$

Furthermore V is a non-decreasing, continuous function satisfying the following:

i) If X is of infinite variation or finite variation with

$$\rho := \frac{\int_{(-\infty,0)} x\Pi(dx)}{d} < \frac{1}{\sqrt{2}} - 1, \tag{3.10}$$

then $a^* > 0$ is the unique value which satisfies the following equation

$$2\psi'(0+)\left[W(0)W(a) + \int_0^a W(y)W'(a-y)dy\right] = \frac{1}{\psi'(0+)}. \tag{3.11}$$

The value function is given by

$$V(x) = \left(2\psi'(0+)\int_0^{a^*} W(y)W(a^*-y)dy - 2\psi'(0+)\int_0^x W(y)W(x-y)dy - \frac{a^*-x}{\psi'(0+)}\right)\mathbb{I}_{\{x \leq a^*\}}. \tag{3.12}$$

Moreover, there is smooth fit at a^* i.e. $V'(a^*-) = 0 = V'(a^*+)$.

ii) If X is of finite variation with $\rho \geq \frac{1}{\sqrt{2}} - 1$ then $a^* = 0$ and

$$V(x) = \frac{x}{\psi'(0+)} \mathbb{I}_{\{x \leq 0\}}.$$

In particular, there is continous fit at $a^* = 0$ i.e. $V(0-) = 0$ and there is no smooth fit at a^* i.e. $V'(a^*-) > 0$.

Remark 3.2.5. *i) Note that in the case that X is of finite variation the value ρ is always negative since $\int_{(-\infty,0)} x\Pi(dx) \leq 0$ so the condition given in ii): $0 \geq \rho \geq 1/\sqrt{2} - 1$ tells us that the drift d is much larger than the average size of the jumps. This implies that the process drifts quickly to infinity and then we have to stop the first time that the process X is above zero. In this case, concerning the optimal prediction problem, the stopping time which is nearest (in the L^1 sense) to the last time that the process is below zero is the first time that the process is above the level zero.*

ii) If X is of finite variation with $\rho \leq 1/\sqrt{2} - 1 < 0$ we have that the average of size of the jumps of X are sufficiently large such that when the process crosses above the level zero the process is more likely (than in case i)) that the process X jumps again below and spend more time in the region where G is negative. The condition for ρ also tells us that the process X drifts a little slower to infinity that in the case i). The stopping time which is nearest (in the L^1 sense) to the last time that the process is below zero is the first time that the process is above the level a^*.

As the proof of Theorem 3.2.4 is rather long, we break it into a number of lemmas which we prove in the following section.

3.3 Proof of Main Result

In this section we use the general theory of optimal stopping discussed in Section 2.2 to get a direct proof of Theorem 3.2.4. First, using the Snell envelope defined in (2.4), we will show that an optimal stopping time for (3.8) is the first time that the process enters to a stopping set D, defined in terms of the value function V. Recall the set

$$\mathcal{T}_t = \{\tau \geq t : \tau \text{ is a stopping time}\}.$$

At it has been used before we simply write $\mathcal{T} = \mathcal{T}_0$ as the set of all stopping times.

Lemma 3.3.1. *Denoting by $D = \{x \in \mathbb{R} : V(x) = 0\}$ the stopping set, we have that for any $x \in \mathbb{R}$ the stopping time*

$$\tau_D = \inf\{t \geq 0 : X_t \in D\}$$

attains the infimum in $V(x)$, i.e. $V(x) = \mathbb{E}_x \left(\int_0^{\tau_D} G(X_s) ds \right)$.

Proof. From the general theory of optimal stopping given in Section 2.2, consider the Snell envelope defined as

$$S_t^x = \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_t} \mathbb{E} \left(\int_0^{\tau} G(X_s + x) ds \middle| \mathcal{F}_t \right)$$

and define the stopping time

$$\tau_x^* = \inf \left\{ t \geq 0 : S_t^x = \int_0^t G(X_s + x) ds \right\}.$$

Then from Theorem 2.2.2 the stopping time is τ_x^* is optimal for

$$\inf_{\tau \in \mathcal{T}} \mathbb{E}\left(\int_0^\tau G(X_s + x)ds\right). \tag{3.13}$$

On account of the Markov property we have

$$\begin{aligned}
S_t^x &= \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_t} \mathbb{E}\left(\int_0^\tau G(X_s+x)ds \middle| \mathcal{F}_t\right) \\
&= \int_0^t G(X_s+x)ds + \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_t} \mathbb{E}\left(\int_0^\tau G(X_s+x)ds - \int_0^t G(X_s+x)ds \middle| \mathcal{F}_t\right) \\
&= \int_0^t G(X_s+x)ds + \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_t} \mathbb{E}\left(\int_t^\tau G(X_s+x)ds \middle| \mathcal{F}_t\right) \\
&= \int_0^t G(X_s+x)ds + \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_t} \mathbb{E}\left(\int_0^{\tau-t} G(X_{s+t}+x)ds \middle| \mathcal{F}_t\right) \\
&= \int_0^t G(X_s+x)ds + \operatorname*{ess\,inf}_{\tau \in \mathcal{T}} \mathbb{E}_{X_t}\left(\int_0^\tau G(X_s+x)ds\right) \\
&= \int_0^t G(X_s+x)ds + V(X_t+x),
\end{aligned}$$

where the last equality follows from the spatial homogeneity of Lévy processes and from definition of V. Therefore $\tau_x^* = \inf\{t \geq 0 : V(X_t+x) = 0\}$. So we have

$$\tau_x^* = \inf\{t \geq 0 : X_t + x \in D\}$$

Thus

$$\begin{aligned}
V(x) &= \inf_{\tau \in \mathcal{T}} \mathbb{E}_x\left(\int_0^\tau G(X_t)dt\right) \\
&= \inf_{\tau \in \mathcal{T}} \mathbb{E}\left(\int_0^\tau G(X_t+x)dt\right) \\
&= \mathbb{E}\left(\int_0^{\tau_x^*} G(X_t+x)dt\right) \\
&= \mathbb{E}_x\left(\int_0^{\tau_D} G(X_t)dt\right),
\end{aligned}$$

where the third equality holds since τ_x^* is optimal for (3.13) and the fourth follows from the spatial homogeneity of Lévy processes. Therefore the stopping time τ_D is the optimal stopping time for $V(x)$ for all $x \in \mathbb{R}$. □

Next, we will prove that $V(x)$ is finite for all $x \in \mathbb{R}$ which implies that there exists a stopping time τ_* such that the infimum in (3.8) is attained.

Lemma 3.3.2. *The function V is non-decreasing with $V(x) \in (-\infty, 0]$ for all $x \in \mathbb{R}$. In particular, $V(x) < 0$ for any $x \in (-\infty, x_0)$.*

Proof. From the spatial homogeneity of Lévy processes,

$$V(x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}\left(\int_0^\tau G(X_s+x)ds\right).$$

Then, if $x_1 \leq x_2$ we have $G(X_s+x_1) \leq G(X_s+x_2)$ since G is a non-decreasing function (see discussion before Theorem 3.2.4). This implies that $V(x_1) \leq V(x_2)$ and V is non-decreasing as claimed. If we take the stopping time $\tau \equiv 0$, then for any $x \in \mathbb{R}$ we have $V(x) \leq 0$. Let $x < x_0$ and let $y_0 \in (x, x_0)$ then $G(x) \leq G(y_0) < 0$ then from the fact that for all $s < \tau_{y_0}^+$, $X_s \leq y_0$ we have

$$V(x) \leq \mathbb{E}_x \left(\int_0^{\tau_{y_0}^+} G(X_s) ds \right) \leq \mathbb{E}_x \left(\int_0^{\tau_{y_0}^+} G(y_0) ds \right) = G(y_0) \mathbb{E}_x(\tau_{y_0}^+) < 0,$$

where the last inequality holds due to $\mathbb{P}_x(\tau_{y_0}^+ > 0) > 0$ and then $\mathbb{E}_x(\tau_{y_0}^+) > 0$.

Now we will see that $V(x) > -\infty$ for all $x \in \mathbb{R}$. Note that $G(x) \geq -\mathbb{I}_{\{x \leq x_0\}}$ holds for all $x \in \mathbb{R}$ and thus

$$\begin{aligned} V(x) &= \inf_{\tau \in \mathcal{T}} \mathbb{E}_x \left(\int_0^\tau G(X_s) ds \right) \\ &\geq \inf_{\tau \in \mathcal{T}} \mathbb{E}_x \left(\int_0^\tau -\mathbb{I}_{\{X_s \leq x_0\}} ds \right) \\ &= -\sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left(\int_0^\tau \mathbb{I}_{\{X_s \leq x_0\}} ds \right). \end{aligned}$$

The indicator function inside the last integral is always greater than or equal to zero so for all stopping times τ, $\int_0^\infty \mathbb{I}_{\{X_s \leq x_0\}} ds \geq \int_0^\tau \mathbb{I}_{\{X_s \leq x_0\}} ds$. Therefore

$$\mathbb{E}_x \left(\int_0^\infty \mathbb{I}_{\{X_s \leq x_0\}} ds \right) \geq \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left(\int_0^\tau \mathbb{I}_{\{X_s \leq x_0\}} ds \right).$$

Hence

$$V(x) \geq -\mathbb{E}_x \left(\int_0^\infty \mathbb{I}_{\{X_s \leq x_0\}} ds \right) \geq -\mathbb{E}_x(g_{x_0}),$$

where the last inequality holds since if $s > g_{x_0}$ then $\mathbb{I}_{\{X_s \leq x_0\}} = 0$. So we can rewrite

$$\int_0^\infty \mathbb{I}_{\{X_s \leq x_0\}} ds = \int_0^{g_{x_0}} \mathbb{I}_{\{X_s \leq x_0\}} ds \leq g_{x_0}.$$

From Lemma 3.2.2 we have that $\mathbb{E}_x(g_{x_0}) < \infty$. Hence for all $x < x_0$ we have $V(x) \geq -\mathbb{E}_x(g_{x_0}) > -\infty$ and due to the monotonicity of V, $V(x) > -\infty$ for all $x \in \mathbb{R}$.

□

Now, we derive some properties of V which will be useful to find the form of the set D.

Lemma 3.3.3. *The set D is non-empty. Moreover, there exists an $\widetilde{x}$ (sufficiently large) such that*

$$V(x) = 0 \qquad \textit{for all } x \geq \widetilde{x}$$

Proof. Suppose that $D = \emptyset$ then by Lemma 3.3.1 the optimal stopping time for (3.8) is $\tau_D = \infty$. This implies that

$$V(x) = \mathbb{E}_x \left(\int_0^\infty G(X_t) dt \right).$$

Let m be the median of G, i.e.

$$m = \inf\{x \in \mathbb{R} : G(x) = 1/2\}$$

and let g_m the last time that the process is below the level m defined in (3.3). Then

$$\mathbb{E}_x\left(\int_0^\infty G(X_t)dt\right) = \mathbb{E}_x\left(\int_0^{g_m} G(X_t)dt\right) + \mathbb{E}_x\left(\int_{g_m}^\infty G(X_t)dt\right). \tag{3.14}$$

Note that from the fact that G is finite and g_m has finite expectation (see Lemma 3.2.2)) the the first term on the right-hand side of (3.14) is finite. Now we analyse the second term in the right-hand side of (3.14). Let $n \in \mathbb{N}$, since $G(X_t)$ is non-negative for all $t \geq g_m$ we have

$$\begin{aligned}
\mathbb{E}_x\left(\int_{g_m}^\infty G(X_t)dt\right) &= \mathbb{E}_x\left(\mathbb{I}_{\{g_m<n\}}\int_{g_m}^\infty G(X_t)dt\right) + \mathbb{E}_x\left(\mathbb{I}_{\{g_m\geq n\}}\int_{g_m}^\infty G(X_t)dt\right)\\
&\geq \mathbb{E}_x\left(\mathbb{I}_{\{g_m<n\}}\int_{g_m}^n G(X_t)dt\right)\\
&\geq \frac{1}{2}\mathbb{E}_x\left(\mathbb{I}_{\{g_m<n\}}(n-g_m)\right).
\end{aligned}$$

Then letting $n \to \infty$ and using the monotone convergence theorem we deduce that $V(x) = \infty$ which leads to a contradiction. Then D must be a non-empty subset. From the fact that V is a non-decreasing function and the set $D \neq \emptyset$ we have that there exists a $\widetilde{x}$ sufficiently large such that $V(x) = 0$ for all $x \geq \widetilde{x}$. □

Lemma 3.3.4. *The function V is continuous.*

Proof. From the above lemma we know that there exists an $\tilde{x}$ such that $V(x) = 0$ for all $x \geq \widetilde{x}$. As X is a spectrally negative Lévy process drifting to infinity we have that $X_{\tau^+_{\widetilde{x}}} = \widetilde{x}$ $\mathbb{P}$-a.s. then we have

$$\begin{aligned}
V(x) &= \inf_{\tau\in\mathcal{T}}\mathbb{E}_x\left(\int_0^\tau G(X_t)dt\right)\\
&= \inf_{\tau\in\mathcal{T}}\mathbb{E}_x\left(\mathbb{I}_{\{\tau<\tau^+_{\widetilde{x}}\}}\int_0^\tau G(X_t)dt + \mathbb{I}_{\{\tau\geq\tau^+_{\widetilde{x}}\}}\int_0^{\tau^+_{\widetilde{x}}} G(X_t)dt + \mathbb{I}_{\{\tau\geq\tau^+_{\widetilde{x}}\}}\int_{\tau^+_{\widetilde{x}}}^\tau G(X_t)dt\right)\\
&= \inf_{\tau\in\mathcal{T}}\mathbb{E}_x\left(\int_0^{\tau\wedge\tau^+_{\widetilde{x}}} G(X_t)dt + \mathbb{I}_{\{\tau\geq\tau^+_{\widetilde{x}}\}}\int_0^{\tau-\tau^+_{\widetilde{x}}} G(X_{t+\tau^+_{\widetilde{x}}})dt\right)\\
&= \inf_{\tau\in\mathcal{T}}\mathbb{E}_x\left(\mathbb{E}_x\left(\int_0^{\tau\wedge\tau^+_{\widetilde{x}}} G(X_t)dt + \mathbb{I}_{\{\tau\geq\tau^+_{\widetilde{x}}\}}\int_0^{\tau-\tau^+_{\widetilde{x}}} G(X_{t+\tau^+_{\widetilde{x}}})dt\middle|\mathcal{F}_{\tau^+_{\widetilde{x}}}\right)\right)
\end{aligned}$$

Using the strong Markov property of X and the fact that $V(\widetilde{x}) = 0$ we have

$$\begin{aligned}
V(x) &= \inf_{\tau\in\mathcal{T}}\mathbb{E}_x\left(\int_0^{\tau\wedge\tau^+_{\widetilde{x}}} G(X_t)dt + \mathbb{I}_{\{\tau\geq\tau^+_{\widetilde{x}}\}}\mathbb{E}_{X_{\tau^+_{\widetilde{x}}}}\left(\int_0^\tau G(X_{t+\tau^+_{\widetilde{x}}})dt\right)\right)\\
&= \inf_{\tau\in\mathcal{T}}\mathbb{E}_x\left(\int_0^{\tau\wedge\tau^+_{\widetilde{x}}} G(X_t)dt + \mathbb{I}_{\{\tau\geq\tau^+_{\widetilde{x}}\}}V(\widetilde{x})\right)\\
&= \inf_{\tau\in\mathcal{T}}\mathbb{E}_x\left(\int_0^{\tau\wedge\tau^+_{\widetilde{x}}} G(X_t)dt\right).
\end{aligned}$$

Note that the process $\left\{\int_0^t G(X_s)ds, t\geq 0\right\}$ is continuous. From the fact that $\mathbb{E}(\tau^+_{\widetilde{x}}) < \infty$ (see Corollary

1.3.28) we have,

$$\begin{aligned}
\mathbb{E}_x\left(\sup_{t\geq 0}\left|\int_0^{t\wedge\tau_{\widetilde{x}}^+} G(X_s)ds\right|\right) &\leq \mathbb{E}_x\left(\sup_{t\geq 0}\int_0^{t\wedge\tau_{\widetilde{x}}^+}|G(X_s)|ds\right)\\
&\leq \mathbb{E}_x\left(\sup_{t\geq 0} t\wedge\tau_{\widetilde{x}}^+\right)\\
&= \mathbb{E}_x(\tau_{\widetilde{x}}^+)\\
&< \infty.
\end{aligned}$$

Then from the general theory of optimal stopping time we have that the infimum in

$$V(x) = \inf_{\tau\in\mathcal{T}}\mathbb{E}_x\left(\int_0^{\tau\wedge\tau_{\widetilde{x}}^+} G(X_t)dt\right) \tag{3.15}$$

is attained, say $\tau_x^* = \inf\{t\geq 0 : X_t + x \in D\}$. Note that from the definition of τ_x^* and $\tau_{\widetilde{x}}^+$ we have that $\tau_x^* \leq \tau_{\widetilde{x}}^+$. Now we check the continuity of V. As W is continuous in $[0,\infty)$ we have that W is uniformly continuous in the interval $[0,\widetilde{x}]$. Now take $\varepsilon > 0$, then there exists $\delta > 0$ such that for all $x, y \in [0,\widetilde{x}]$ it holds $|W(x) - W(y)| < \varepsilon$ when $|x-y| < \delta$. Then we have

$$\begin{aligned}
V(x+\delta) - V(x) &\leq \mathbb{E}\left(\int_0^{\tau_x^*} G(X_t+x+\delta)dt\right) - \mathbb{E}\left(\int_0^{\tau_x^*} G(X_t+x)dt\right)\\
&= 2\psi'(0)\mathbb{E}\left(\int_0^{\tau_x^*}[W(X_t+x+\delta) - W(X_t+x)]dt\right)\\
&\leq 2\psi'(0)\mathbb{E}\left(\int_0^{\tau_{\widetilde{x}}^+}[W(X_t+x+\delta) - W(X_t+x)]dt\right),
\end{aligned}$$

where the first inequality holds since τ_x^* is not necessarily optimal for $V(x+\delta)$ and the last inequality follows since $W(X_t+x+\delta) - W(X_t+x)$ is always positive and from $\tau_x^* \leq \tau_{\widetilde{x}}^+$.

Recall that we have a possible discontinuity for W in zero, and $W(x) = 0$ for $x < 0$, this implies that

$$\begin{aligned}
V(x+\delta) - V(x) &\leq 2\psi'(0)\mathbb{E}\left(\int_0^{\tau_{\widetilde{x}}^+}[W(X_t+x+\delta) - W(X_t+x)]dt\right)\\
&= 2\psi'(0)\mathbb{E}\left(\int_0^{\tau_{\widetilde{x}}^+}\mathbb{I}_{\{X_t+x+\delta<0\}}[W(X_t+x+\delta) - W(X_t+x)]dt\right.\\
&\qquad + \int_0^{\tau_{\widetilde{x}}^+}\mathbb{I}_{\{X_t+x+\delta\geq 0, X_t+x<0\}}[W(X_t+x+\delta) - W(X_t+x)]dt\\
&\qquad \left. + \int_0^{\tau_{\widetilde{x}}^+}\mathbb{I}_{\{X_t+x\geq 0\}}[W(X_t+x+\delta) - W(X_t+x)]dt\right).
\end{aligned} \tag{3.16}$$

Note that the first term in (3.16) is zero. Now we analyse the second term. Using the monotonicity of W, Fubini's theorem and Theorem 1.4.12 we have

$$\begin{aligned}
&\mathbb{E}\left(\int_0^{\tau_{\widetilde{x}}^+} \mathbb{I}_{\{X_t+x+\delta\geq 0, X_t+x<0\}}[W(X_t+x+\delta)-W(X_t+x)]dt\right)\\
&\leq W(\widetilde{x}+x+\delta)\mathbb{E}\left(\int_0^{\tau_{\widetilde{x}}^+} \mathbb{I}_{\{X_t+x+\delta\geq 0, X_t+x<0\}}dt\right)\\
&= W(\widetilde{x}+x+\delta)\int_0^\infty \mathbb{P}_x(-\delta\leq X_t<0, \tau_{\widetilde{x}}^+>t)dt\\
&= W(\widetilde{x}+x+\delta)\int_{-\delta}^0 [W(\widetilde{x}-y)-W(x-y)]dy\\
&\leq W(\infty)^2\delta\\
&<\infty.
\end{aligned}$$

Finally we inspect the third term in (3.16). Using the finiteness of the moment of $\tau_{\widetilde{x}}^+$ we obtain

$$\mathbb{E}\left(\int_0^{\tau_{\widetilde{x}}^+} \mathbb{I}_{\{X_t+x\geq 0\}}[W(X_t+x+\delta)-W(X_t+x)]dt\right) < \varepsilon\mathbb{E}\left(\int_0^{\tau_{\widetilde{x}}^+} \mathbb{I}_{\{X_t+x\geq 0\}}dt\right) \leq \varepsilon\mathbb{E}(\tau_{\widetilde{x}}^+) < \infty.$$

Hence

$$V(x+\delta)-V(x) < W(\infty)^2\delta + \varepsilon\mathbb{E}(\tau_{\widetilde{x}}^+)$$

and the continuity holds. □

From Lemmas 3.3.3 and 3.3.4 we have that the set $D=\{x : V(x)=0\}=[a,\infty)$ for some $a\in\mathbb{R}_+$. From Lemma 3.3.1 we know that for some $a\in\mathbb{R}_+$

$$\tau_D = \inf\{t>0 : X_t\in[a,\infty] = \{t>0 : X_t\geq a\}$$

attains the infimum in $V(x)$. As X is a spectrally negative Lévy process we have that $\tau_D=\tau_a^+$ $\mathbb{P}$-a.s. and hence τ_a^+ is an optimal stopping time for (3.8) for some $a\in\mathbb{R}_+$. Then we just have to find the value of a which minimises the right hand side of the above expression. So in what follows we will analyse the function

$$\mathcal{V}(x,a) = \mathbb{E}_x\left(\int_0^{\tau_a^+} G(X_t)dt\right). \tag{3.17}$$

and find some value a^* which minimises the function $a\mapsto\mathcal{V}(x,a)$ for a fixed $x\in\mathbb{R}$. And then conclude that $\mathcal{V}(x,a^*)=V(x)$ and $\tau_{a^*}^+$ is optimal.

Using Theorem 1.4.12 we find an a explicit form of (3.17) in terms of scale functions.

Lemma 3.3.5. *For $x\geq a$, $\mathcal{V}(x,a)=0$ and for $x<a$,*

$$\begin{aligned}
\mathcal{V}(x,a) &= \int_{-\infty}^a [2\psi'(0+)W(y)-1][W(a-y)-W(x-y)]dy && (3.18)\\
&= 2\psi'(0+)\int_0^a W(y)W(a-y)dy - 2\psi'(0+)\int_0^x W(y)W(x-y)dy - \frac{a-x}{\psi'(0+)}. && (3.19)
\end{aligned}$$

Proof. It is clear that $\mathcal{V}(x,a)=0$ for $x\geq a$, since if the process begins above the level a, then the first passage time above a is zero and the integral inside of the expectation in (3.17) is again zero. Now suppose that $x<a$, then using Fubini's theorem twice,

$$\begin{aligned}
\mathcal{V}(x,a) &= \mathbb{E}_x\left(\int_0^{\tau_a^+} G(X_t)dt\right) \\
&= \mathbb{E}_x\left(\int_0^{\infty} G(X_t)\mathbb{I}_{\{\tau_a^+>t\}}dt\right) \\
&= \int_0^{\infty} \mathbb{E}_x(G(X_t)\mathbb{I}_{\{\tau_a^+>t\}})dt \\
&= \int_0^{\infty}\int_{-\infty}^{a} G(y)\mathbb{P}_x(X_t\in dy,\tau_a^+>t)dt \\
&= \int_{-\infty}^{a} G(y)\int_0^{\infty}\mathbb{P}_x(X_t\in dy,\tau_a^+>t)dt \\
&= \int_{-\infty}^{a} G(y)r^{(0)}(a,x,y)dy.
\end{aligned}$$

Using the fact that $\psi'(0)>0$ implies that $\Phi(q)=0$ and from Theorem 1.4.12 we have,

$$\mathcal{V}(x,a)=\int_{-\infty}^{a}[2\psi'(0+)W(y)-1][W(a-y)-W(x-y)]dy$$

which proves the equality (3.18). Now we derive the equation (3.19). Using Fubini's theorem once again we have,

$$\begin{aligned}
\mathcal{V}(x,a) &= \int_{-\infty}^{a}[2\psi'(0+)W(y)-1][W(a-y)-W(x-y)]dy \\
&= 2\psi'(0+)\int_0^a W(y)[W(a-y)-W(x-y)]dy-\int_{-\infty}^{a}[W(a-y)-W(x-y)]dy \\
&= 2\psi'(0+)\int_0^a W(y)[W(a-y)-W(x-y)]dy-\int_{-\infty}^{a}\int_{(x-y,a-y]}W(dz)dy \\
&= 2\psi'(0+)\int_0^a W(y)[W(a-y)-W(x-y)]dy-\int_{(x-a,\infty)}W(dz)\int_{x-z}^{a-z}dy \\
&= 2\psi'(0+)\int_0^a W(y)[W(a-y)-W(x-y)]dy-(a-x)[W(\infty)-W(x-a)] \\
&= 2\psi'(0+)\int_0^a W(y)W(a-y)dy-2\psi'(0+)\int_0^x W(y)W(x-y)dy-\frac{a-x}{\psi'(0+)},
\end{aligned}$$

where the last equality holds since $W(\infty)=1/\psi'(0+)$ (see Lemma 1.4.10) and $W(x-a)=0$ as $x<a$. □

Now we characterise the value at which the function $a\mapsto\mathcal{V}(x,a)$ achieves its minimum value. Recall that x_0 is the first time that the function G is positive, i.e.

$$x_0=\inf\{x\in\mathbb{R}:G(x)\geq 0\}$$

Lemma 3.3.6. *For all $x\in\mathbb{R}$ the function $a\mapsto\mathcal{V}(x,a)$ achieves its minimum value in $a^*\geq x_0$ which does not depend on the value of x. The value a^* is characterised as in Theorem 3.2.4.*

Proof. Define the function $H:\mathbb{R}\mapsto\mathbb{R}$ by

$$H(x)=2\psi'(0+)\int_0^x W(y)W(x-y)dy-\frac{x}{\psi'(0+)}.$$

Then, $\mathcal{V}(x,a) = (H(a) - H(x))\mathbb{I}_{\{x\leq a\}}$. Now, we calculate the derivative of H,

$$H'(z) = 2\psi'(0+)[W(0)W(z) + \int_0^z W(y)W'(z-y)dy] - \frac{1}{\psi'(0+)}. \tag{3.20}$$

From the fact that $W(z)$ is increasing we have that $W'(z) > 0$ for all $z \geq 0$ and $W'(z) = 0$ for $z < 0$. If we take $z_2 \geq z_1$ we have

$$\begin{aligned}
\int_0^{z_2} W(y)W'(z_2-y)dy &= \int_0^{z_2} W(z_2-y)W'(y)dy \\
&> \int_0^{z_2} W(z_1-y)W'(y)dy \\
&= \int_0^{z_1} W(z_1-y)W'(y)dy + \int_{z_1}^{z_2} W(z_1-y)W'(y)dy \\
&> \int_0^{z_1} W(z_1-y)W'(y)dy
\end{aligned}$$

where the last inequality holds due to $W(z_1 - y) = 0$ for all $y \in (z_1, z_2]$. This implies that H' is a strictly increasing and continuous function in $[0,\infty)$. If we take $z < 0$ we have that $W(z) = 0$, then $H'(z) = -\frac{1}{\psi'(0+)} < 0$.

Now we analyse the behaviour of H' at infinity, for this purpose recall that W is a strictly increasing function. Hence using monotone convergence theorem and the fact that $W(\infty) = 1/\psi'(0+)$,

$$\begin{aligned}
\lim_{a\to\infty} \int_0^a W(y)W'(a-y)dy &= \lim_{a\to\infty} \int_0^a W(a-y)W'(y)dy \\
&= \lim_{a\to\infty} \int_0^\infty W(a-y)W'(y)dy \\
&= \int_0^\infty \lim_{a\to\infty} W(a-y)W'(y)dy \\
&= W(\infty)[W(\infty) - W(0)] \\
&= \frac{1}{\psi'(0+)^2} - \frac{W(0)}{\psi'(0+)}.
\end{aligned}$$

Thus

$$\lim_{z\to\infty} H'(z) = 2\psi'(0+)\left[\frac{W(0)}{\psi'(0+)} + \frac{1}{\psi'(0+)^2} - \frac{W(0)}{\psi'(0+)}\right] - \frac{1}{\psi'(0+)} = \frac{1}{\psi'(0+)} > 0.$$

Note that the function H' may have a discontinuity in zero. In the case that X is of infinite variation the function W is continuous on $\mathbb{R}$ and hence H' is continuous on $\mathbb{R}$ with

$$H'(0+) = -\frac{1}{\psi'(0+)} < 0,$$

when X is of finite variation we have that

$$H'(0+) = 2\psi'(0+)W(0)^2 - \frac{1}{\psi'(0+)}.$$

Now we check when $H'(0+) < 0$. Using the expressions of $\psi'(0+)$ and $W(0)$ given in (3.1) and (3.2) respectively we have

$$
\begin{aligned}
& H'(0+) < 0 \\
\Leftrightarrow\quad & \frac{2\psi'(0+)}{d^2} < \frac{1}{\psi'(0+)} \\
\Leftrightarrow\quad & \frac{2\psi'(0+)^2}{d^2} < 1 \\
\Leftrightarrow\quad & \frac{\sqrt{2}\psi'(0+)}{d} < 1 \\
\Leftrightarrow\quad & \frac{\psi'(0+)}{d} < \frac{1}{\sqrt{2}} \\
\Leftrightarrow\quad & \frac{\int_{(-\infty,0)} x\Pi(dx)}{d} < \frac{1}{\sqrt{2}} - 1 \\
\Leftrightarrow\quad & \rho < \frac{1}{\sqrt{2}} - 1,
\end{aligned}
$$

where ρ is defined in Theorem 3.2.4. The above implies that when X is of infinite variation or finite variation with $\rho < \frac{1}{\sqrt{2}} - 1$ then $H'(0-) < 0$ and $H'(0+) < 0$ and then by continuity and from the fact that $\lim_{z\to\infty} H'(z) > 0$ there exists a unique value a^* which satisfies the equation (3.11). When X is of finite variation and $\rho \geq \frac{1}{\sqrt{2}} - 1$ then $H'(0-) < 0$ and $H'(0+) > 0$ and in this case we let $a^* = 0$.

Therefore we have the following: there exists a value $a^* \geq 0$ such that for $x < a^*$ we have $H'(x) < 0$ and for $x > a^*$ it holds that $H'(x) > 0$. This implies that the behaviour of H is as follows: for $x < a^*$, $H(x)$ is a decreasing function, and for $x > a^*$, $H(x)$ is increasing. Consequently H reaches its minimum value uniquely at $x = a^*$.

Note that

$$\frac{\partial}{\partial a}\mathcal{V}(x,a) = H'(a)\mathbb{I}\{x \leq a\}.$$

Then without loss of generality we can choose $x < 0$ and hence the same conclusions given for H are also valid for $a \mapsto \mathcal{V}(x,a)$. Therefore, $a \mapsto \mathcal{V}(x,a)$ reaches its minimum value uniquely at $a = a^*$. Moreover, since a^* is the unique value at which H reaches its minimum, we have that for all $x \leq a^*$, $H(a^*) \leq H(x)$ and then

$$\mathcal{V}(x,a^*) = (H(a^*) - H(x))\mathbb{I}_{\{x \leq a^*\}} \leq 0.$$

It only remains to prove that $a^* \geq x_0$. From the definition of a^* we have that

$$
\begin{aligned}
0 &\leq 2\psi'(0+)\left[W(0)W(a^*) + \int_0^{a^*} W(y)W'(a^*-y)dy\right] - \frac{1}{\psi'(0+)} \\
&= 2\psi'(0+)\left[W(0)W(a^*) + \int_0^{a^*} W(a^*-y)W'(y)dy\right] - \frac{1}{\psi'(0+)} \\
&\leq 2\psi'(0+)\left[W(0)W(a^*) + W(a^*)\int_0^{a^*} W'(y)dy\right] - \frac{1}{\psi'(0+)} \\
&\leq 2\psi'(0+)(W(a^*))^2 - \frac{1}{\psi'(0+)}.
\end{aligned}
$$

This implies that $G(a^*) = 2\psi'(0+)W(a^*) - 1 \geq \sqrt{2} - 1 > 0$ and hence $a^* \geq x_0$.

□

In conclusion we have that for all $x \in \mathbb{R}$,

$$V(x) = \mathbb{E}_x\left(\int_0^{\tau_{a^*}^+} G(X_t)dt\right),$$

where a^* is characterised in Theorem 3.2.4. To conclude we check when there is smooth fit at a^*.

Lemma 3.3.7. *We have the following:*

i) If X is of infinite variation or finite variation with (3.10) then there is smooth fit at a^* i.e. $V'(a^*-) = 0$.

ii) If X is of finite variation and (3.10) does not hold then there is continous fit at $a^* = 0$ i.e. $V(0-) = 0$. There is no smooth fit at a^* i.e. $V'(a^*-) > 0$.

Proof. From Lemma 3.3.5 we know that $V(x) = 0$ for $x \geq a^*$ and for $x \leq a^*$

$$V(x) = 2\psi'(0+)\int_0^{a^*} W(y)W(a^*-y)dy - 2\psi'(0+)\int_0^x W(y)W(x-y)dy - \frac{a^*-x}{\psi'(0+)}.$$

Note that when X is of finite variation with

$$\frac{\int_{(-\infty,0)} x\Pi(dx)}{d} \geq \frac{1}{\sqrt{2}} - 1$$

we have $a^* = 0$ and hence

$$V(x) = \frac{x}{\psi'(0+)}\mathbb{I}_{\{x\leq 0\}}$$

and hence $V(0-) = 0 = V(0+)$. Its left and right derivatives at 0 are given by

$$V'(0-) = \frac{1}{\psi'(0+)} \qquad \text{and} \qquad V'(0+) = 0.$$

Therefore in this case only the continuous fit at 0 is satisfied. If X is of infinite variation or finite variation with

$$\frac{\int_{(-\infty,0)} x\Pi(dx)}{d} < \frac{1}{\sqrt{2}} - 1$$

we have from Lemma 3.3.5 that $a^* > 0$. Calculating its derivative we have

$$V'(x) = \left(2\psi'(0+)\left[W(0)W(x) + \int_0^x W(y)W'(x-y)dy\right] - \frac{1}{\psi'(0+)}\right)\mathbb{I}_{\{x\leq a^*\}}.$$

Since a^* satisfies the equation (3.11) we have that

$$V'(a^*-) = 0 = V'(a^*+)$$

Thus we have smooth fit at a^*.

□

Example 3.3.8. *We calculate numerically using the statistical software R Core Team (2015) the value function $x \mapsto \mathcal{V}(x,a)$ for some values of $a \in \mathbb{R}$. The models used were the Cramér–Lundber risk process with $c = 2$, $\lambda = 1$ and $\xi \sim \exp(1)$ and a spectrally negative Lévy process with no Gaussian component and Lévy measure given by $\Pi(dx) = e^{2x}(e^x - 1)^{-3}dx, x > 0$ (see examples of Section 1.4).*

Note that the value a^* is the unique value for which the function $x \mapsto \mathcal{V}(x,a)$ exhibits smooth fit (or continuous fit) at a^* see Figure 3.3. When we choose $a_2 > a^*$, the function $x \mapsto \mathcal{V}(x,a_2)$ it is not differentiable at a_2. Moreover, there exists some x such that $\mathcal{V}(x,a_2) > 0$. Similarly, If $a_1 < a^*$ the function $x \mapsto \mathcal{V}(x,a_2)$ is also not differentiable at a_1.

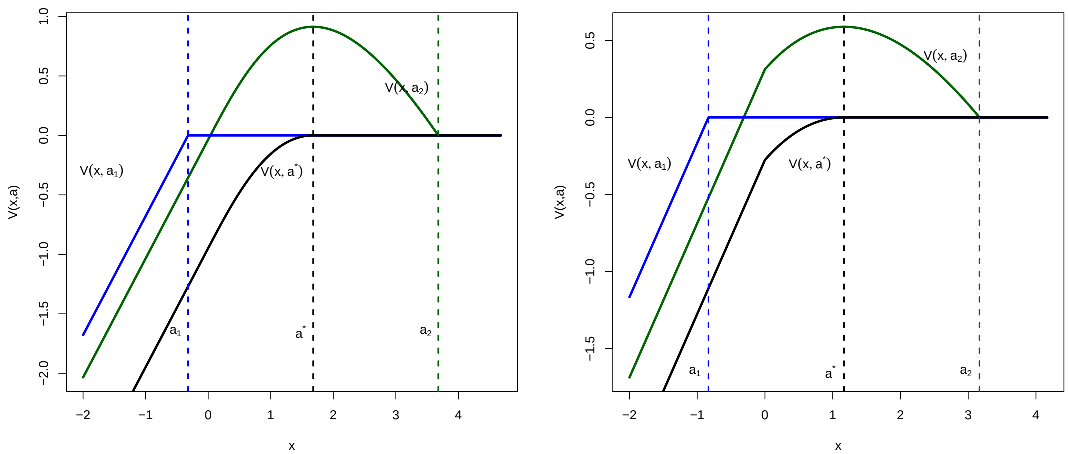

Figure 3.3: Left side: $\Pi(dx) = e^{2x}(e^x - 1)^{-3}dx$, $x > 0$ without Gaussian component. Right side: Cramér–Lundberg model with $c = 2$, $\lambda = 1$ and $\xi \sim \exp(1)$.

Appendix A

Stochastic Processes: General Facts

We review here some known facts about stochastic processes that are used in the main body. In the first section, we give basic definitions and classic results about martingale theory which can be found in Revuz and Yor (1999). Then we give a brief overview of the theory of Markov processes which can also be found in Revuz and Yor (1999). Finally, we give a short review of Poisson random measures (see Chapter 2 of Kyprianou (2014)).

A.1 Martingale Theory

Definition A.1.1. *If $X = \{X_t, t \geq 0\}$ is a family of random variables defined on $(\Omega, \mathcal{F})$ taking values in some measurable space $(E, \mathcal{E})$ (i.e. such that E-valued variables $X_t = X_t(\omega)$ are $\mathcal{F}/\mathcal{E}$-measurable for each $t \geq 0$) then one says that X is a* stochastic process *with values in E.*

Definition A.1.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. A filtration is a family $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \leq t$. The system $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called a* filtered probability space.

We interpret the σ-algebra $\mathcal{F}_t$ as the "information" (a family of events) obtained during the time interval $[0, t]$.

Let $X = \{X_t, t \geq 0\}$ a stochastic process defined in $(\Omega, \mathcal{F}, \mathbb{P})$. The natural filtration associated to the process X is $\mathcal{F}_t = \sigma(\{X_s, s \leq t\})$. Let us denote for all $t \geq 0$

$$\mathcal{F}_{t-} = \sigma\left(\bigcup_{s<t} \mathcal{F}_s\right),$$
$$\mathcal{F}_{t+} = \bigcap_{s \geq t} \mathcal{F}_s.$$

It is clear that $\mathcal{F}_{t-} \subseteq \mathcal{F}_t \subseteq \mathcal{F}_{t+}$.

Definition A.1.3. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space and $X = \{X_t, t \geq 0\}$ a stochastic process defined in $(\Omega, \mathcal{F}, \mathbb{P})$. We say that the process process X is adapted to the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ if for all $t \geq 0$ the random variable $X_t = X_t(\omega)$ is $\mathcal{F}_t$-measurable.*

Definition A.1.4. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space and let $X = \{X_t, t \geq 0\}$ a stochastic process taking values in $(E, \mathcal{C})$ is* progressively measurable *with respect to the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ if for all $t \geq 0$ the map,*

$$[0, t] \times \Omega \to E,$$
$$(s, \omega) \mapsto X_s(\omega)$$

is $\mathbb{B}([0, t]) \otimes \mathcal{F}_t$-measurable. A subset Γ of $\mathcal{F}_+ \times \Omega$ is progressive if the process $X = \mathbb{I}_\Gamma$ is progressively measureable.

The family of progressive sets is a σ-algebra on $\mathbb{R}_+ \times \Omega$ called the progressive σ-algebra.

Proposition A.1.5. *Let $X = \{X_t, t \geq 0\}$ a stochastic process taking values in $(E, \mathbb{B}(E))$, where E is a metric space, adapted to $\mathbb{F}$ and right or left continuous. Then X is progressively measurable.*

Definition A.1.6. *A filtration $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$ is called right-continuous if $\mathcal{F}_t = \mathcal{F}_{t+}$ for all $t \geq 0$. Furthermore, is called $\mathbb{P}$-complete if $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets.*

Definition A.1.7. *We say that the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ satisfies the* usual *conditions if it is right-continuous and complete.*

In this work we suppose that the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ satisfies the natural conditions (see Definition 1.3.38 of Bichteler (2002)). Many authors assume that the filtration satisfies the usual conditions. This can cause some problems, for example, using change of measures with Girsanov's theorem (see Warning 1.3.39 of Bichteler (2002)).

Stopping Times

Definition A.1.8. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space. A random variable $\tau : \Omega \mapsto [0, \infty]$ is called* stopping time *with respect to $\mathbb{F}$ if*

i) τ is $\mathcal{F}_\infty = \sigma(\mathcal{F}_t, t \in [0, \infty))$-measurable.

ii) The set $\{\tau \leq t\}$ if $\mathcal{F}_t$-measurable for all $t \geq 0$.

The property *ii)* has clear meaning: for each $t \geq 0$ a decision "to stop or not to stop" depends only on the "past and present information" $\mathcal{F}_t$ obtained on the interval $[0, t]$ and not depending on the "future".

Proposition A.1.9. *If the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ is right-continuous then τ is a stopping time if and only if $\{\tau < t\}$ is $\mathcal{F}_t$-measurable for all $t \geq 0$.*

Definition A.1.10. *Let us define the first* first-entrance time *and* first-hitting time *of a given open or closed set $B \subseteq \mathbb{R}$ as*

$$T^B = \inf\{t \geq 0 : X_t \in B\} \qquad \textit{and} \qquad \tau^B = \inf\{t > 0 : X_t \in B\},$$

where $\inf \emptyset = \infty$.

Proposition A.1.11. *Let $X = \{X_t, t \geq 0\}$ be a stochastic process adapted to the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ taking values in $(E, \mathbb{B}(E))$, where E is a metric space and let $A \in \mathbb{B}(E)$,*

i) If X and $\mathbb{F}$ are right-continuous and A is open, then T^A is a $\mathbb{F}$-stopping time.

ii) If X is continuous and A is closed then T^A is an $\mathbb{F}$-stopping time.

Definition A.1.12. *Let $X = \{X_t, t \geq 0\}$ be a stochastic process. If τ is an $\mathbb{F}$-stopping time and $\{\tau_n, n \geq 1\}$ is an increasing sequence of $\mathbb{F}$-stopping times such that $\tau_n < \tau$ for all $n \geq 0$ and $\lim_{n\to\infty} \tau_n = \tau$ a.s. We say that X is* left-continuous *at τ on $\{\tau < \infty\}$ if $\lim_{n\to\infty} X_{\tau_n} = X_\tau$ on $\{\tau < \infty\}$.*

Definition A.1.13. *Let $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ a filtration and τ a $\mathbb{F}$-stopping time. The σ-algebra of events before τ is given by*

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \textit{ for all } t \geq 0\}.$$

Lemma A.1.14. *Let θ and τ $\mathbb{F}$-stopping times, then $\theta \wedge \tau$ and $\theta \vee \tau$ are $\mathbb{F}$-stopping times.*

Lemma A.1.15. *Let $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ a filtration and $\{\tau_n, n \geq 0\}$ a sequence of $\mathbb{F}$-stopping times. Then $\sup_{n\geq 0} \tau_n$ is $\mathbb{F}$-stopping time. Moreover, if $\mathbb{F}$ is right-continuous then*

$$\inf_{n\geq 1} \tau_n, \qquad \liminf_{n\to\infty} \tau_n \qquad \textit{and} \qquad \limsup_{n\to\infty} \tau_n$$

are all $\mathbb{F}$-stopping times.

Lemma A.1.16. *Let θ and τ $\mathbb{F}$-stopping times and let $A \in \mathcal{F}_\theta$ then $A \cap \{\theta \leq \tau\} \in \mathcal{F}_\tau \cap \mathcal{F}_\theta$. In particular, if $\theta \leq \tau$ then $\mathcal{F}_\theta \subset \mathcal{F}_\tau$.*

Lemma A.1.17. *Let τ a $\mathbb{F}$-stopping time and θ a function $\mathcal{F}_\tau$-measurable such that $\theta \geq \tau$, then θ is a $\mathcal{F}$-stopping time.*

Lemma A.1.18. *Let θ and τ $\mathbb{F}$-stopping times then $\theta + \tau$ is a $\mathbb{F}$-stopping time.*

Definition A.1.19. *Let $X = \{X_t, t \geq 0\}$ a stochastic process and τ a $\mathbb{F}$-stopping time let us define the function X_τ under $\{\tau < \infty\}$ by*

$$X_\tau(\omega) = X_{\tau(\omega)}(\omega).$$

Theorem A.1.20. *Let $X = \{X_t, t \geq 0\}$ a progressively measurable stochastic process with respect to $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ and let τ a $\mathbb{F}$-stopping time. Then*

i) The random variable X_τ defined on the set $\{\tau < \infty\}$ is $\mathcal{F}_\tau$-measurable.

ii) The "stopped process" $X^\tau = \{X_{t\wedge\tau}, t \geq 0\}$ is progressively measurable with respect to $\mathbb{F}$.

iii) The stopped proces X^τ is adapted to the filtration $\{\mathcal{F}_{t\wedge\tau}, t \geq 0\}$.

Continuous Time Martingales

Definition A.1.21. *Let $X = \{X_t, t \geq 0\}$ a stochastic process adapted to the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$*

i) We say that X is a submartingale if

  a) $\mathbb{E}(X_t^+) < \infty$ for all $t \geq 0$.

  b) For all $0 \leq s < t < \infty$, $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$.

ii) The process X is supermartingale if the process $-X = \{-X_t, t \geq 0\}$ is submartingale.

iii) Finally, we say that X is martingale if it is both submartingale and supermartingale.

Proposition A.1.22. *Let $X = \{X_t, t \geq 0\}$ a stochastic process adapted to the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$.*

i) If X is a martingale and f is a convex function such that $\mathbb{E}(f(X_t)) < \infty$ for all $t \geq 0$. Then $\{f(X_t), t \geq 0\}$ is a submartingale.

ii) If X is a submartingale and f is an increasing convex function such that $\mathbb{E}(f(X_t)) < \infty$ for all $t \geq 0$. Then $\{f(X_t), t \geq 0\}$ is a submartingale.

Theorem A.1.23 (Doob's L^p-inequality)**.** *If $X = \{X_t, 0 \leq t \leq T\}$ is right-continuous martingale or positive submartingale. Then, for $\lambda > 0$ and for $p \geq 1$,*

$$\lambda^p \mathbb{P}[\sup_{0\leq t\leq T} |X_t| \geq \lambda] \leq \sup_{0\leq t\leq T} \mathbb{E}[|X_t|^p].$$

For $p > 1$,

$$\mathbb{E}((\sup_{0\leq t\leq T} |X_t|)^p) \leq \left(\frac{p}{p-1}\right)^p \sup_{0\leq t\leq T} \mathbb{E}(|X_t|^p).$$

Theorem A.1.24. *Let $\{X_t, t \geq 0\}$ a submartingale (martingale) with respect to $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ which satisfies the usual conditions. If the function $t \mapsto \mathbb{E}(X_t)$ is right-continuous then the procces $\{X_t, t \geq 0\}$ has a modification which is càdlàg.*

Theorem A.1.25 (Doob's convergence theorem). *Let $X = \{X_t, t \geq 0\}$ a right-continuous submartingale such that* $\sup_{t\geq 0} \mathbb{E}[X_t^+] < \infty$*. Then there exists an integrable random variable X_∞ such that $X_t \mapsto X_\infty$ a.s. as $t \to \infty$.*

Definition A.1.26. *A family $\{X_t, t \in T\}$ of L_1 random variables indexed by T is uniformly integrable if*

$$\sup_{t\in T} \mathbb{E}\left(|X_t| \mathbb{I}_{|X_t|>a}\right) = \sup_{t\in T} \int_{\{|X_t|>a\}} |X| d\mathbb{P} \to 0$$

as $a \to \infty$. That is,

$$\int_{\{|X_t|>a\}} |X_t| d\mathbb{P} \to 0$$

as $a \to \infty$, uniformly in $t \in T$.

Theorem A.1.27. *Let $X = \{X_t, t \geq 0\}$ a martingale. The following three conditions are equivalent,*

i) $\lim_{t\to\infty} X_t$ *exists in the L^1 sense.*

ii) There exists a random variable X_∞ in L^1 such that $X_t = \mathbb{E}(X_\infty|\mathcal{F}_t)$.

iii) The family $\{X_t, t \geq 0\}$ is uniformly integrable.

If these conditions hold, then $X_\infty = \lim_{t\to\infty} X_t$ a.s. Moreover, if for some $p > 1$, $\sup_{t\geq 0} \mathbb{E}(|X_t|^p) < \infty$, then the equivalent conditions above are satisfied and the convergence holds in the L^p-sense.

Theorem A.1.28 (Doob's stopping time theorem). *Suppose that $X = \{X_t, t \geq 0\}$ is a submartingale (martingale) with respect to the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ and τ is a stopping time. Then the stopped process $X^\tau = \{X_{t\wedge\tau}, t \geq 0\}$ is also a submartingale (martingale) with respect to $\mathbb{F}$.*

Theorem A.1.29 (Hunt's stopping time theorem). *Let $\{X_t, t \geq 0\}$ a submartingale with respect to $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$. Assume that σ y τ are bounded stopping times and $\sigma \leq \tau$. Then*

$$X_\sigma \leq \mathbb{E}(X_\tau|\mathcal{F}_\sigma) \qquad \mathbb{P} \text{ a.s.}$$

The statements of these theorems remain valid also for unbounded stopping times under the additional assumption that the family of random variables $\{X_t, t \geq 0\}$ is uniformly integrable.

Theorem A.1.30. *If $X = \{X_t, t \geq 0\}$ is a positive right-continuous supermartingale and if we set $X_\infty = 0$, for any pair S, T of stopping times with $S \leq T$, then*

$$X_S \geq \mathbb{E}(X_T|\mathcal{F}_S).$$

Proposition A.1.31. *A càdlàg process $X = \{X_t, t \geq 0\}$ is a martingale if and only if for every bounded stopping time T, the random variable X_T is in L^1 and*

$$\mathbb{E}(X_T) = \mathbb{E}(X_0)$$

A.2 Markov Processes

Intuitively speaking, a process X with state space $(E, \mathcal{E})$ is a Markov process if, to make prediction at any time s on what is going to happen in the future, it is not necessary to know anything more about the whole past up to time s than the present state X_s.

Definition A.2.1. *Let $(E, \mathcal{E})$ be a measurable space. A* transition probability *P on E is a map from $E \times \mathcal{E}$ into $\mathbb{R}_+ \cup \{+\infty\}$ such that*

i) *$P(x, E) = 1$ for every $x \in E$.*

ii) *For every $x \in E$, the map $A \mapsto P(x, A)$ is a positive measure on $\mathcal{E}$.*

iii) *For every $A \in \mathcal{E}$, the map $x \mapsto P(x, A)$ is $\mathcal{E}$-measurable.*

If f is measurable bounded or positive function and if P is a transition probability, we define a function $P\,f$ on E by

$$P\,f(x) = \int_E P(x, dy) f(y).$$

Definition A.2.2. *A transition function on $(E, \mathcal{E})$ is a family $P_{s,t}$, $0 \leq s < t$ of transition probabilities on $(E, \mathcal{E})$ such that for every three real numbers $s < t < v$, we have*

$$\int_E P_{s,t}(x, dy) P_{t,v}(y, A) = P_{s,v}(x, A)$$

for every $x \in E$ and $A \in \mathcal{E}$. This relation is known as the Chapman–Kolmogorov equation. The transition function is said to be homogeneous if $P_{s,t}$ depends on s and t only through the difference $t - s$. In that case, we write P_t instead of $P_{0,t}$ and the Chapman-Kolmogorov equation reads

$$P_{t+s}(x, A) = \int_E P_s(x, dy) P_t(y, A)$$

for every $s, t \geq 0$.

Definition A.2.3. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. An adapted process is a* Markov process *with respect to $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$, with transition function $P_{s,t}$ if for any f bounded or positive and any pair $s < t$,*

$$\mathbb{E}(f(X_t)|\mathcal{F}_s) = P_{s,t}\,f(X_s) \qquad \mathbb{P} - a.s.$$

The probability measure $\mathbb{P}(X_0 \in \cdot)$ is called the initial distribution *of X. The process is said to be homogeneous if the transition function is homogeneous in which case the above equality reads*

$$\mathbb{E}(f(X_t)|\mathcal{F}_s) = P_{t-s}\,f(X_s).$$

From now on, we now work in the canonical space Ω. We set $\Omega = E^{\mathbb{R}_+}$, $\mathcal{F} = \mathcal{E}^{\mathbb{R}_+}$ and $\mathcal{F}_t = \sigma(X_u, u \leq t)$ where X is the coordinate process, i.e. if $\omega \in \Omega$ then we write $\omega = \{x_s, s \geq 0\}$ and $X_s(\omega) = x_s$.

Theorem A.2.4. *Given a transition function $P_{s,t}$ on $(E, \mathcal{E})$, for any probability measure ν on $(E, \mathcal{E})$ there is a unique probability measure $\mathbb{P}_\nu$ on $(\Omega, \mathcal{F})$ such that X is Markov with respect to $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ with transition function $P_{s,t}$ and initial measure ν.*

From now on, we will consider only homogeneous transition functions and processes. For each $x \in E$, we have a probability measure $\mathbb{P}_{\delta_x}$ which we will denote simply by $\mathbb{P}_x$ and satisfies

$$\mathbb{P}_x(X_0 = x) = 1.$$

If Z is an $\mathcal{F}$-measurable function, its mathematical expectation with respect to $\mathbb{P}_x$ (resp. $\mathbb{P}_\nu$) will be denoted by $\mathbb{E}_x(Z)$ (resp. $\mathbb{E}_\nu(Z)$). If in particular $Z = \mathbb{I}_{X_t \in A}$ for some $A \in \mathcal{E}$, we get

$$\mathbb{P}_x(X_t \in A) = P_t(x, A).$$

This reads: the probability that the process started at x is in A at time t is given by the value $P_t(x, A)$ of the transition function. It proves that in particular that $x \mapsto \mathbb{P}_x(X_t \in A)$ is measurable. More generally we have

Theorem A.2.5. *If Z is $\mathcal{F}$ measurable and positive or bounded, the map $x \mapsto \mathbb{E}_x(Z)$ is $\mathcal{E}$-measurable and*

$$\mathbb{E}_\nu(Z) = \int_E \nu(dx)\mathbb{E}_x(Z).$$

Now we define a family of transformations $\{\theta_t, t \geq 0\}$ where θ_t acts on $\omega = \{x_s, s \geq 0\}$ in the following way, $\theta_t(\omega) = \omega'$ where $\omega' = \{x_{t+s}, s \geq 0\}$. The operator θ_t is known as the shift operator.

The notions introduced above imply that the composition $X_s \circ \theta_t(\omega) = X_s(\theta_t(\omega)) = X_{s+t}(\omega)$ and thus the Markov property takes the form

Theorem A.2.6 (Markov property)**.** *If Z is $\mathcal{F}$-measurable function positive or bounded, for every $t > 0$ and starting measure ν,*

$$\mathbb{E}_\nu(Z \circ \theta_t | \mathcal{F}_t) = \mathbb{E}_{X_t}(Z) \qquad \mathbb{P}_\nu - a.s.$$

If in particular we take $Z = \mathbb{I}_{\{X_s \in A\}}$, the above formula reads

$$\mathbb{P}_\nu(X_{t+s} \in A | \mathcal{F}_t) = \mathbb{P}_{X_t}(X_s \in A) = P_s(X_t, A).$$

In the general theory of Markov processes an important role is played by those processes which, in addition to the Markov property, have the following strong Markov property

Definition A.2.7. *The process $X = \{X_t, t \geq 0\}$ possesses the* strong Markov property *with respect to $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ if for each stopping time τ*

$$\mathbb{P}_x(X_{\tau+s} \in B | \mathcal{F}_\tau) = P_s(X_\tau, B), \qquad \mathbb{P}_x\text{-}a.s.$$

Using the shift operator θ_t the strong Markov property takes the form: for any stopping time τ

$$\mathbb{P}_x(X_s \circ \theta_\tau \in B | \mathcal{F}_\tau) = \mathbb{P}_{X_\tau}(X_s \in B), \qquad \mathbb{P}_x\text{-a.s.}$$

for every $x \in E$ and $B \in \mathcal{E}$ where $\theta_\tau(\omega)$ by definition equals $\theta_{\tau(\omega)}(\omega)$ if $\tau(\omega) < \infty$. In other words if $\tau = \tau(\omega)$ is a stopping time such that $\tau < \infty$ then the operator $\theta_\tau(\omega) = \theta_t(\omega)$ for all ω such that $\tau(\omega) = t$.

If Z is a $\mathcal{F}$-measurable function we denote by $Z \circ \theta_\tau$ the function

$$(Z \circ \theta_\tau)(\omega) = Z(\theta_\tau(\omega))$$

for all $t \geq 0$.

The following useful property can be deduced from the strong Markov property

Theorem A.2.8. *If Z is a $\mathcal{F}$-measurable function positive or bounded, for everty $t > 0$ and starting measure ν we have,*

$$\mathbb{E}_\nu(Z \circ \theta_\tau | \mathcal{F}_\tau) = \mathbb{E}_{X_\tau}(Z), \qquad \mathbb{P}_\nu\text{-}a.s.$$

Now we derive some properties about the shift operator θ_t. Let σ, τ finite stopping times, then

$$X_\tau \circ \theta_\sigma = X_{\tau\circ\theta_\sigma+\sigma}.$$

Suppose that $B \in \mathcal{E}$ and

$$T^B = \inf\{t \geq 0 : X_t \in B\} \qquad \text{and} \qquad \tau^B = \inf\{t > 0 : X_t \in B\}$$

are finite stopping times. Let γ another stopping time, then

$$\gamma + T^B \circ \theta_\gamma = \inf\{t \geq \gamma : X_t \in B\},$$
$$\gamma + \tau^B \circ \theta_\gamma = \inf\{t > \gamma : X_t \in B\}$$

are stopping times. In particular if $\gamma \leq T^B$ then we get the following formula

$$\gamma + T^B \circ \theta_\gamma = T^B.$$

A.3 Poisson Random Measures

Definition A.3.1 (Poisson random measure)**.** *Let $(S, \mathcal{S}, \eta)$ be a σ-finite measure space and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. Let $N : \Omega \times \mathcal{S} \mapsto \{1, 2, \dots\} \cup \{\infty\}$ so that the mapping $\omega \mapsto N(\omega, A)$ is a random variable for all $A \in \mathcal{S}$. N is called a Poisson random measure on S with intensity η if*

i) For mutually disjoint $A_1, \dots, A_n$ in $\mathcal{S}$, the random variables $N(A_1), \dots, N(A_n)$ are independent.

ii) For each $A \in \mathcal{S}$, $N(A)$ is Poisson distributed with parameter $\eta(A)$

iii) The mapping $A \mapsto N(A)$ is a measure $\mathbb{P}$-a.s.

Theorem A.3.2. *Suppose that N is a Poisson random measure on $(S, \mathcal{S}, \eta)$. Then for each $A \in \mathcal{S}$, the mapping $B \mapsto N(B \cap A)$ is a Poisson random measure on $(S \cap A, \mathcal{S} \cap A, \eta(\cdot \cap A))$. Further, if $A, B \in \mathcal{S}$ and $A \cap B = \emptyset$, then $N(\cdot \cap A)$ and $N(\cdot \cap B)$ are independent.*

Theorem A.3.3. *Suppose that N is a Poisson random measure on $(S, \mathcal{S}, \eta)$, then the support of N is $\mathbb{P}$-a.s. countable. If, in addition, η is a finite measure, then the support is $\mathbb{P}$-a.s. finite.*

If N is a Poisson random measure then for a fixed ω, we have that $N(\cdot)$ is $\mathbb{P}$-a.s. a measure on the space $(S, \mathcal{S}, \mathbb{P})$ and hence we can define

$$\int_S f(x) N(dx)$$

for any measurable function $f : S \mapsto [0, \infty]$. It is easy to prove that the above integral is a random variable.

Theorem A.3.4. *Suppose that N is a Poisson random measure on $(S, \mathcal{S}, \eta)$. Let $f : S \mapsto \mathbb{R}$ a measurable function.*

i) Then

$$X = \int_S f(x) N(dx)$$

is almost surely convergent if and only if

$$\int_S (1 \wedge |f(x)|) \eta(dx) < \infty. \tag{A.1}$$

ii) *When condition* (A.1) *holds, then*

$$\mathbb{E}(e^{i\beta X}) = \exp\left\{-\int_S (1 - e^{i\beta f(x)})\eta(dx)\right\}$$

for any $\beta \in \mathbb{R}$.

iii) *Further,*

$$\mathbb{E}(X) = \int_S f(x)\eta(dx) \qquad \textit{when} \ \int_S |f(x)|\eta(dx) < \infty$$

and

$$\mathbb{E}(X^2) = \int_S f(x)^2\eta(dx) + \left(\int_S f(x)\eta(dx)\right)^2$$

when

$$\int_S f(x)^2\eta(dx) < \infty \qquad \textit{and} \qquad \int_S |f(x)|\eta(dx) < \infty.$$

In the theory of Lévy process is of vital importance a Poisson random measure which is related directly to the jumps of a Lévy process. Specifically we will work on the measure space $(\mathbb{R}_+ \times \mathbb{R}, \mathbb{B}(\mathbb{R}_+) \times \mathbb{B}(\mathbb{R}), dt \times \Pi(dx))$, where Π is a measure concentrated on $\mathbb{R} \setminus \{0\}$. We are interested in integrals of the form

$$\int_{[0,t]} \int_B xN(ds, dx)$$

Lemma A.3.5. *Suppose that* N *is a Poisson random measure on* $(\mathbb{R}_+ \times \mathbb{R}, \mathbb{B}(\mathbb{R}_+) \times \mathbb{B}(\mathbb{R}), dt \times \Pi(dx))$ *where* Π *is a measure concentrated on* $\mathbb{R} \setminus \{0\}$. *Let* $B \in \mathbb{B}(\mathbb{R})$ *such that* $0 < \Pi(B) < \infty$, *then the process* $X = \{X_t, t \geq 0\}$ *where*

$$X_t := \int_{[0,t]} \int_B xN(du, dx), \qquad t \geq 0,$$

is a compound Poisson process with arrival rate $\Pi(B)$ *and jump distribution* $\Pi(dx \cap B)/\Pi(B)$.

Suppose that $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ is the filtration generated by X satisfying the usual conditions.

Lemma A.3.6. *Suppose that* N *is a Poisson random measure on* $(\mathbb{R}_+ \times \mathbb{R}, \mathbb{B}(\mathbb{R}_+) \times \mathbb{B}(\mathbb{R}), dt \times \Pi(dx))$ *where* Π *is a measure concentrated on* $\mathbb{R} \setminus \{0\}$. *Let* $B \in \mathbb{B}(\mathbb{R})$ *such that* $\int_B |x|\Pi(dx) < \infty$.

i) *The compound Poisson process with drift*

$$M_t := \int_{[0,t]} \int_B xN(ds, dx) - t\int_B x\Pi(dx), \qquad t \geq 0$$

is a $\mathbb{P}$*-martingale with respect to the filtration* $\mathbb{F}$.

ii) *If further,* $\int_B x^2\Pi(dx) < \infty$ *then it is a square-integrable martingale.*

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