Hodge Theory of Isolated Hypersurface Singularities

(A Study of Asymptotic Polarization)

by

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Abstract

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a germ of hypersurface with isolated singularity. The variation of mixed Hodge structure associated to the Milnor fibration of f is polarized and can be extended over the puncture using the gluing data of the local system of the variation of mixed Hodge structure associated to f. The extended fiber can be explained as Ω_f the module of relative differentials of f which is also isomorphic to the Jacobi algebra associated to f. A MHS structure is also defined on the new fiber Ω_f. We show that the polarization on the extended fiber is a modification of the Grothendieck residue product. In this way the Grothendieck residue induces a set of forms $\{\widehat{Res_k}\}$ which define polarization on the primitive subspaces of pure Hodge structures $Gr_k^W \Omega_f$. The extension procedure using some gluing isomorphisms always defines an opposite filtration to the limit Hodge filtration by the works of M. Saito, P. Deligne and G. Pearlstein. The above form polarizes the complex variation of Hodge structure defined by G. Pearlstein et. al. According to this we formulate the Riemann-Hodge bilinear relations for the Grothendieck residue on Ω_f (and the same for the Jacobi algebra of f). The formulation of Riemann-Hodge relations allows to associate a signature to the modified Grothendieck residue pairing. We generalize these results for any admissible normal crossing PVMHS with quasi-unipotent monodromy. An application of this to extension of Neron models of pure Hodge structure is also given. We provide a proof of semi-definiteness of Hochster Theta pairing as another application.

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Chapter 1

Introduction

1.0.1 Classical Hodge theory

It was first observed by W. Hodge that the cohomologies of a smooth complex projective variety X (more generally any compact Kahler manifold) admits an additional structure as:

$$\forall k \in \mathbb{N}, H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \qquad H^{q,p} = \overline{H^{p,q}} \tag{1.1}$$

where $H^{p,q}(X) = H^q(X, \Omega_X^p)$ is the Dolbeault cohomology. Such a decomposition property for a vector space is classically called a pure Hodge structure. There is an important point to be mentioned. That is the vector space $H^k(X, \mathbb{C})$ has a structure defined over $\mathbb{Q}$ (or even $\mathbb{Z}$) coming from $H^k(X, \mathbb{Q})$. Another point in (1.1) is the integer k, called the weight of the Hodge structure, which plays an important role. The decomposition in (1.1) can also be equivalently explained by the existence of a decreasing filtration $F^p := \oplus_{r \geq p} H^{r,s}$ on $H^k(X, \mathbb{C})$, in which case $H^{p,q} = F^p \cap \bar{F}^q$.

The cohomology of a non-compact Kahler manifold, or a non-projective smooth quasi-projective variety does not usually satisfy this decomposition. Singularities of varieties also prevent such a decomposition (although the $\mathbb{Q}$-structure still exists). A more complicated structure holds in the latter cases, which is referred to as a mixed Hodge structure. In which case the vector space has an increasing filtration $W_\bullet$ defined over $\mathbb{Q}$ whose graded pieces $Gr_l^W H^k(X, \mathbb{C})$ satisfy (1.1), for all k, l. We do not have a decomposition for the whole $H^k(X, \mathbb{C})$ in the mixed case. Mixed Hodge structures (MHS) fit into a category by considering morphisms which preserve the filtrations upto a shift.

A. Grothendieck in his comparison theorem, shows that

$$H^i_{dR}(X/\mathbb{C}) := \mathbb{H}^i(X, \Omega^\bullet_X) \cong H^i(X^{an}, \mathbb{C}) \tag{1.2}$$

where $\mathbb{H}^i(X, \Omega^\bullet_X)$ stands for hypercohomology of the complex of sheaves of holomorphic differential forms on X. This isomorphism is given by integration of algebraic differential forms along homology cycles, and is the base knowledge to define periods. The comparison theorem enables us to calculate the cohomology of the complement to a hyper-surface (or more generally a normal crossing divisor) in projective space (a projective manifold) by means of the cohomology classes generated by rational differential forms. Using a spectral sequence argument Deligne [D1] shows that the cohomologies of the complement of a normal crossing divisor has also a mixed Hodge structure.

A polarization for a Hodge structure $(H_{\mathbb{Q}}, F^\bullet)$ of weight k, consists of a bilinear form S (also denoted by Q) on $H_{\mathbb{C}}$ defined over $\mathbb{Q}$ and which is symmetric for k even, skew symmetric for k odd, such that

$$S(H^{p,q}, H^{r,s}) = 0, \qquad \text{unless} \;\; p = s, \;\; q = r$$

$$i^{p-q} S(v, \bar{v}) > 0, \qquad v \neq 0$$

where $H^{p,q} := F^p \cap \bar{F}^q$. If $N \in End(H_{\mathbb{Q}})$ is a nilpotent transformation, it defines a filtration $W_\bullet$ called weight filtration according to a theorem by N. Jacobson. Then the primitive subspaces w.r.t N are defined by

$$PGr_l^W := \ker N^{l+1} : Gr_l^W H_{\mathbb{Q}} \to Gr_{-l-2}^W H_{\mathbb{Q}}.$$

The non-degenerate form S determines a set of non-degenerate forms

$$S_l : PGr_l^W \otimes PGr_l^W \to \mathbb{C}, \qquad S_l := S \circ (id \otimes N^l)$$

which are polarization forms for PGr_l^W, see Theorem 3.1.2.

J. Steenbrink establishes a mixed Hodge structure (limit MHS) on the cohomologies of fibers of a projective map $f : X \to S$, by using Hironaka resolution of singularities, followed by a spectral sequence (of a double complex) associated to a natural stratification of the normal crossing divisor. There exists a canonical isomorphism $\mathcal{H}^i_{dR}(X/S) \cong R^i f_*^{an}(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{S^{an}}$. The presence of the locally constant sheaf $\mathcal{H} = R^i f_*(\mathbb{C})$ provides a flat (Gauss-Manin) connection on $\mathcal{H}^i_{dR}(X/S)$. When S is one dimensional, and locally like the disc and the family of varieties X_t defined by the morphism f degenerates at the point $0 \in S$, then the Gauss-Manin connection has a singularity at the point 0. This singularity is regular (in the sense we explain in Chap. 6). The monodromy on the vector space $H^i(X_t, \mathbb{C})$ is quasi-unipotent, and one of the important facts we have is the Griffiths transversality

$$\nabla F^p \subset F^{p-1} \otimes \Omega^1_S.$$

The regular singular connections can be explained via Deligne canonical extension. The Deligne extension is by definition the vector bundle whose sections are,

$$s(A_i)(z) = \exp(N \log(z)).A_i(z) \tag{1.3}$$

where $(A_1(z), ..., A_n(z))$ is a multi-valued flat frame of $\mathcal{H}$, and $N = \frac{-1}{2\pi i} \log M_u$, in which case $M = M_s.M_u$ is the Jordan decomposition of a monodromy. It is isomorphic to $\mathcal{H}$ and its sections have moderate growth near the boundary points. It is well-known that the connection ∇ has an extension over sections $s(A_i)$, in which case it has poles of order at most one at the singularity.

By considering the Hodge filtration $F^p H^i(X_t)$ only, we can associate a flag in $H_t = H^i(X_t)$ to every point t in S,

$$H_t = F_t^0 \supset F_t^1 \supset ... \supset F_t^{n+1} = 0 \tag{1.4}$$

and hence obtain a point $F_t^\bullet$ in a flag manifold. We obtain a period map

$$\Phi : S \to D. \tag{1.5}$$

In order to have the map Φ well-defined, we have to consider a map $\tilde{\Phi} : H \to D$ of the universal cover of S, or a map Φ from S to a quotient of D

by some group. Therefore, a period map is of the form

$$\Phi : S \to D/\Gamma$$

where Γ is the monodromy group. In coordinates the period map is given by periods of integrals. The domain D parametrizes the Hodge filtrations with the same Hodge numbers and polarization, and is referred to as the period domain. To inherit a complex structure on D, it is convenient to consider a period domain D as an open subset of a complex manifold $\check{D}$ namely its compact dual.

The extension of the map Φ is the major content of asymptotic Hodge theory in the study of degeneration of Hodge structure. For local systems of Hodge structures on manifolds, locally a period map looks like

$$\Phi : (T')^r \times T^{n-r} \to D/\Gamma \tag{1.6}$$

and is equivalent to its lifting,

$$\tilde{\Phi} : H^r \times T^{n-r} \to D \tag{1.7}$$

where T is a disc, $T' = T \setminus 0$ and H is the upper half plane. Assuming $r = n$ for simplicity and the monodromies are quasi-unipotent, the map $\Psi : H^r \to \check{D}$ defined by

$$\Psi(z_1, ..., z_r) := \exp(-\sum_{j=1}^{r} z_j N_j)\tilde{\Phi}(z_1, ..., z_r) \tag{1.8}$$

where, $N_j = \log M_{j,u}$ is called the un-twisted period map and is the lifting of a holomorphic map $\psi : T'^r \to D$;

$$\psi(t_1, ..., t_r) = \Psi(\frac{\log t_1}{2\pi i}, ..., \frac{\log t_r}{2\pi i}). \tag{1.9}$$

A fundamental result is the following,

Theorem 1.0.1. *(Nilpotent Orbit Theorem - W. Schmid) ([SCH] Theorem 4.9 and 4.12) Let* $\Phi : T'^r \times T^{n-r} \to D$ *be a period map, and let* $N_1, ..., N_r$

be monodromy logarithms. Let

$$\psi : (T')^r \times T^{n-r} \to \check{D} \tag{1.10}$$

be as above; then

- *The map ψ extends holomorphically to $T^r \times T^{n-r}$.*
- *For each $w \in T^{n-r}$, the map $\theta : \mathbb{C}^r \times T^{n-r} \to \check{D}$ given by*

  $$\theta(z, w) = \exp(\sum z_j N_j).\psi(0, w)$$

  is a nilpotent orbit. Moreover, for $w \in C$ a compact subset, there always exists $\alpha > 0$ such that $\theta(z, w) \in D$ for $Im(z_j) > \alpha$.
- *For any G-invariant distance on D, there exists positive constants β, K such that for $Im(z_j) > \alpha$,*

  $$d(\Phi(z, w), \theta(z, w)) \leq K \sum_j (Im(z_j))^\beta e^{-2\pi Im(z_j)}. \tag{1.11}$$

  Moreover, the constants α, β, K depend only on the choice of the metric d and the weight and Hodge numbers used to define D. They may be chosen uniformly for w in a compact subset.

A candidate for the metric on D in the theorem is given by the polarization form Q. This is explained by a classical fact about the tangent spaces of flag manifolds, cf. [SCH]. In case of taking the metric induced by polarization, the (Lie) group G in the last item is the automorphism group of Q. The estimate obtained in the last item in the theorem is of interest in asymptotic Hodge theory. The nilpotent orbit theorem guarantees the existence of a limit for the map ψ at 0 using some distance estimates on natural metrics on $\check{D}$ (induced by polarization). This limit is called the limit Hodge filtration, which plays an important role for us. It is not in general unique, because it depends on the choice of coordinates. In several variables (i.e. over a base of higher dimension) one may study the afore-mentioned limit of (1.9) along different positive cones of nilpotent transformations on the Hodge structure. The definition of period domains and the nilpotent orbit theorem can be generalized to mixed Hodge structures, where a similar theorem can be obtained for admissible MHS, [P1].

Another important theorem in asymptotic Hodge theory is the $\mathfrak{sl}_2$-orbit theorem,

Theorem 1.0.2. *($\mathfrak{sl}_2$-orbit Theorem - W. Schmid) ([SCH] Theorem 5.3)Let $z \to \exp(z.N).F$ be a nilpotent orbit. Then there exists,*

- *A filtration $F_{\sqrt{-1}} := \exp(iN).F_0$ lies in D.*
- *A homomorphism $\rho : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}$, Hodge at $F_{\sqrt{-1}}$.*
- *$N = \rho(X_-)$.*
- *A real analytic $G_{\mathbb{R}}$-valued function $g(y)$, such that;*
- *For $y >> 0$, $\exp(iy.N).F = g(y)\exp(iyN).F_0$, where $F_0 = \exp(-iN).F_{\sqrt{-1}}$.*
- *Both $g(y)$ and $g(y)^{-1}$ have convergent power series expansion at $y = \infty$ of the form $1 + \sum A_n y^{-n}$ with*

$$A_n \in W_{n-1}\mathfrak{g} \cap \ker(adN)^{n+1}. \tag{1.12}$$

The different aspects of items in this theorem are explained in 3.5. The theorem mainly asserts that from any nilpotent orbit (i.e. the orbit of $F \in \check{D}$, lies in D from some point on) one can withdraw some distinguished orbit (the role of the function g in the theorem) which is real split, i.e can be written as direct sum of pure Hodge structures (by the representation ρ). The concepts of the theorem should be understood as a matter of representation theory, and can be applied to general period maps. Exploring the $\mathfrak{sl}_2$-triples for Hodge structure is a basic tool to study their real splittings. In section 8.7 we apply this idea to the mixed Hodge structure of isolated hypersurface singularities and their extensions to 0.

Our strategy after mentioning the limit MHS is to express its polarization and the corresponding Riemann-Hodge bilinear relations. Assume we are given a projective map $f : X \to T$ (i.e. the fibers are projective varieties) over the disc T which is holomorphic over $T' = T - 0$. Set $X_\infty := X \times_{T'} H$ where H is the upper half plane. Equip $H^k(X_\infty, \mathbb{C})$ with the limit mixed Hodge structure (F_∞, W_L), where L is the Kahler class. On the primitive subspaces $P^k(X_\infty)$ consider the bilinear form

$$Q(x, y) = \int_{X_t} (-1)^{k(k-1)/2} i^{n-k} \psi_t^{-1}(x \wedge y). \tag{1.13}$$

where $\psi_t : H^k(X_t, \mathbb{C}) \cong H^k(X_\infty, \mathbb{C})$ is fixed. Then Q does not depend on the choice of t. Denote

$$P_{k,r}(X_\infty) = \ker(N^{r+1} : Gr^W_{k+r} P^k(X_\infty) \to Gr^W_{k-r} P^k(X_\infty)). \tag{1.14}$$

Then $P_{k,r}(X_\infty)$ carries a Hodge structure of weight $k + r$. Let

$$P_{k,r}(X_\infty) = \bigoplus_{a+b=k+r} P^{a,b}_{k,r}(X_\infty) \tag{1.15}$$

be its Hodge decomposition. Denote Q_r the bilinear form on $P_{k,r}(X_\infty)$ defined by $Q_r(x, y) = Q(\tilde{x}, N^r \tilde{y})$.

Theorem 1.0.3. *(J. Steenbrink-W. Schmid)(see [JS2]) Assume $f : X \to S$ be a family of projective manifolds. Equip the variation of Hodge structure $H^k(X_\infty, \mathbb{C})$ with the W. Schmid limit MHS (F_∞, W_L). Let $P_k, P_{k,r}$ be the primitive subspaces as defined above. Then the following holds,*

- $Q_r(x, y) = 0$ *if* $x \in P^{a,b}_{k,r}(X_\infty), y \in P^{c,d}_{k,r}(X_\infty)$ *and* $(a, b) \neq (c, d)$
- $i^{a-b} Q_r(x, \bar{x}) > 0$ *if* $x \in P^{a,b}_{k,r}(X_\infty), x \neq 0.$

Theorem 1.0.3 is important to us stating explicit form of Riemann-Hodge bilinear relations for polarized variation of MHS in projective fibrations. In Chapter 3 we try to approach a proof of 1.0.3. In Chapters 4 and 5 we explain how this generalizes to the affine (local) fibrations with isolated singularity.

1.0.2 Isolated hypersurface singularities

Assume $f : X \to T$ is the Milnor fibration associated to an isolated hypersurface singularity. With a suitable coordinate change one can embed the fibration into a projective one $f_Y : \mathbb{P}^{n+1} \to \mathbb{C}$ by possibly inserting a singular fiber such that the degree of f_Y is as large as we like. If f is a polynomial of sufficiently high degree s.t the properties above are satisfied. Then, the mapping $i^* : P^n(Y_\infty) \to H^n(X_\infty)$ is surjective and the kernel is $\ker(i^*) = \ker(M_Y - id)$, where M_Y is the monodromy for the fibration f_Y. There is a short exact sequence of mixed Hodge structures

$$0 \to \ker(M_Y - id) \to P^n(Y_\infty) \to H^n(X_\infty) \to 0. \tag{1.16}$$

There is a $(-1)^n$-symmetric non-degenerate intersection form I_Y^{coh} on $P^n(Y_t, \mathbb{Q})$. We set $S_Y = (-1)^{n(n-1)/2} I_Y^{coh}$. The pure Hodge structures on $P^n(Y_t, \mathbb{Q})$ are polarized by S_Y and give a variation of Hodge structure. By 1.0.1 the limit Hodge filtration $F_\infty^\bullet$ on $P^n(Y_t, \mathbb{C})$ is well defined. The nilpotent transformation $N_Y = \frac{1}{2\pi i}\log(M_{Y,u})$ defines a weight filtration $W_\bullet$ on $P^n(Y_\infty, \mathbb{Q})$. Then, $S_Y, W_\bullet, F_\infty^\bullet$ give a canonical polarized mixed Hodge structure on $P^n(Y_\infty)$. There exists a unique mixed Hodge structure on $H^n(X_\infty, \mathbb{C})$ which makes the sequence (1.16) a short exact sequence in the category of mixed Hodge structures. It is called the Steenbrink limit MHS. The Steenbrink limit MHS is polarized as follows. Let

$$H^n = H^n_{\neq 1} \oplus H^n_1$$

be the decomposition into generalized eigenspaces of monodromy. Then one shows that the bilinear form

$$S(a,b) = \begin{cases} S_Y(i^*a, i^*b) & a,b \in H_{\neq 1} \\ S_Y(i^*a, N_Y . i^*b) & a,b \in H_1 \end{cases}$$

defines a polarization for $H^n(X_\infty, \mathbb{C})$.

Theorem 1.0.4. *[H1] Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is an isolated hypersurface singularity. The Steenbrink MHS and S yields a PMHS of weight n on $H^n(X_\infty, \mathbb{Q})_{\neq 1}$ and PMHS of weight $n+1$ on $H^n(X_\infty, \mathbb{Q})_1$.*

Theorem 1.0.4 is repeatedly used in our text in order to generalize the criteria on polarization from projective case to the local case i.e subsets of the affine space, $\mathbb{C}^n$. It plays a role in the proof of the main contribution 8.6.1, and its applications in 9.1 and 9.4.

There exists a second method to define limit Hodge filtration. For this we follow E. Brieskorn to consider the $\mathcal{O}_T$-modules

$$H'' = f_* \frac{\Omega_X^{n+1}}{df \wedge d(\Omega_X^{n-1})}; \qquad H' = f_* \frac{\Omega_X^{n+1}}{d\Omega_X^n + df \wedge \Omega_X^n} \tag{1.17}$$

of rank μ, such that

$$H'|_{T'} = H''|_{T'} = \mathcal{H}. \tag{1.18}$$

The Gauss-Manin system associated to the Milnor fibration of f has a canonical filtration due to Malgrange-Kashiwara, namely V-filtration indexed by $\alpha \in \mathbb{Q}$, cf. Chap. 6. It is characterized by the properties; $t.V^\alpha \subset V^{\alpha+1}$, $\partial_t.V^\alpha \subset V^{\alpha-1}$ and the operator $t\partial_t - \alpha$ is nilpotent on Gr_V^α. Using the V-filtration on the Gauss-Manin module; we may define two Hodge filtrations on $H^n(X_\infty, \mathbb{C})$ by

$$F_{St}^p H^n(X_\infty, \mathbb{C})_\lambda = \psi_\alpha^{-1}(\frac{V^\alpha \cap \partial_t^{n-p} H_0''}{V^{>\alpha}}), \qquad \alpha \in (-1, 0] \tag{1.19}$$

$$F_{Va}^p H^n(X_\infty, \mathbb{C})_\lambda = \psi_\alpha^{-1}(\frac{V^\alpha \cap t^{-(n-p)} H_0''}{V^{>\alpha}}), \qquad \alpha \in (-1, 0] \tag{1.20}$$

namely Steenbrink-Scherk and Varchenko Hodge filtrations respectively. The maps ψ_α, where α's are logarithms of eigenvalues of the monodromy; are the nearby maps, cf. Chap 6 sec. 1, introduced by P. Deligne. Each of the two filtrations together with the weight filtration $W_\bullet$ defines a Hodge structure on $H^n(X_\infty, \mathbb{C})$, [H3]. A. Varchenko proved that the two Hodge filtrations agree for curve singularities and quasi-homogeneous singularities. In the general case he showed that the two Hodge filtrations agree on $Gr_l^W H^n(X_\infty)$, [SC1], [H3].

Theorem 1.0.5. *[H1] The filtration defined in (1.9) is the Steenbrink limit Hodge filtration.*

1.0.3 Main results

For a holomorphic germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated critical point; the local residue

$$g \longmapsto Res_0 \left[\frac{gdx}{\frac{\partial f}{\partial x_0} \cdots \frac{\partial f}{\partial x_n}} \right] := \frac{1}{(2\pi i)^{n+1}} \int_{\Gamma_\varepsilon} \frac{gdx}{\frac{\partial f}{\partial x_0} \cdots \frac{\partial f}{\partial x_n}}$$

induces a bilinear form $Res_{f,0}$ on

$$\Omega_f := \Omega^{n+1}_{\mathbb{C}^{n+1},0} / df \wedge \Omega^n_{\mathbb{C}^{n+1},0}$$

$$res_{f,0} : \Omega_f \times \Omega_f \to \mathbb{C}$$

$$(g_1 dx, g_2 dx) \longmapsto Res_0 \left[\frac{g_1 g_2 dx}{\frac{\partial f}{\partial x_0} \cdots \frac{\partial f}{\partial x_n}} \right],$$

defines a symmetric bilinear pairing (Grothendieck pairing), which is non-degenerate (proved by A. Grothendieck). If ω and η are $(n+1)$-differential forms, after division by df each of the forms ω and η define a middle dimensional cohomology class of every local level hyper-surface of the function f.

Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a germ of isolated singularity. Suppose,

$$H^n(X_\infty, \mathbb{C}) = \bigoplus_{p,q,\lambda} (I^{p,q})_\lambda \tag{1.21}$$

is the Deligne-Hodge C^∞-splitting, and generalized eigenspaces. Consider the isomorphism obtained by composing the three maps,

$$\Phi^{p,q}_\lambda : I^{p,q}_\lambda \xrightarrow{\hat{\Phi}_\lambda} Gr_V^{\alpha+n-p} H'' \xrightarrow{pr} Gr_V^\bullet H'' / \partial_t^{-1} H'' \cong \Omega_f \tag{1.22}$$

where

$$\hat{\Phi}_\lambda^{p,q} := \partial_t^{p-n} \circ \psi_\alpha | I_\lambda^{p,q}$$
$$\Phi = \bigoplus\nolimits_{p,q,\lambda} \Phi_\lambda^{p,q}, \qquad \Phi_\lambda^{p,q} = pr \circ \hat{\Phi}_\lambda^{p,q}$$

and ∂_t is the Gauss-Manin connection, and ψ_α is defined in section 6.1. In Chapter 8 we extend the Gauss-Manin system associated to a hypersurface with isolated singularity over the origin, with the new fiber be Ω_f. Specifically, in section 8.5 we define a MHS on Ω_f by this isomorphism. The main contribution is;

Theorem 1.0.6. *[1] Assume $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, is a holomorphic germ with isolated singularity at 0. Then, the isomorphism Φ makes the following diagram commutative up to a complex constant;*

$$\begin{array}{ccc} \widehat{Res}_{f,0} : \Omega_f \times \Omega_f & \longrightarrow & \mathbb{C} \\ \Big\downarrow {\scriptstyle (\Phi^{-1}, \Phi^{-1})} & & \Big\downarrow {\scriptstyle \times *} \\ S : H^n(X_\infty) \times H^n(X_\infty) & \longrightarrow & \mathbb{C} \end{array} \qquad * \neq 0 \tag{1.23}$$

where,

$$\widehat{Res}_{f,0} = res_{f,0}\ (\bullet, \tilde{C}\ \bullet)$$

If $J^{p,q} = \Phi^{-1} I^{p,q}$ is the corresponding subspace of Ω_f, then

$$\Omega_f = \bigoplus_{p,q} J^{p,q} \qquad \tilde{C}|_{J^{p,q}} := (-1)^p. \tag{1.24}$$

In other words;

$$S(\Phi^{-1}(\omega), \Phi^{-1}(\eta)) = * \times \ res_{f,0}(\omega, \tilde{C}.\eta), \qquad 0 \neq * \in \mathbb{C}. \tag{1.25}$$

Let $\mathfrak{f}$ be the nilpotent operator on Ω_f corresponding to $N = \log(M_u)$ via the isomorphism Φ. Define the primitive components;

$$PGr_l^W := \ker(\mathfrak{f}^{l+1} : Gr_l^W \Omega_f \to Gr_{-l-2}^W \Omega_f).$$

The induced form on W-graded pieces;

$$\widehat{Res}_l : PGr_l^W \Omega_f \otimes_{\mathbb{C}} PGr_l^W \Omega_f \to \mathbb{C} \tag{1.26}$$

is non-degenerate and according to Lefschetz decomposition we will obtain a set of positive definite bilinear forms,

$$\widehat{Res} \circ (id \otimes \mathfrak{f}^l) : PGr_l^W \Omega_f \otimes_{\mathbb{C}} PGr_l^W \Omega_f \to \mathbb{C}. \tag{1.27}$$

Theorem 1.0.7. *Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a holomorphic isolated singularity germ. The modified Grothendieck residue $\widehat{Res}$ provides a polarization for the extended fiber Ω_f, via the isomorphism Φ. Moreover, there exists a set of forms $\{Res_k\}$ polarizing the primitive subspaces of $Gr_k^W \Omega_f$ providing a graded polarization for Ω_f.*

Corollary 1.0.8. *The polarization S of $H^n(X_\infty)$ will always define a polarization of Ω_f, via the isomorphism Φ. In other words S is also a polarization in the extension, i.e. of Ω_f.*

Using this corollary and summing up all the results obtained above, we can give the following picture for the extension of PVMHS associated to isolated hypersurface singularity.

Theorem 1.0.9. *Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a holomorphic hypersurface germ with isolated singularity at $0 \in \mathbb{C}^{n+1}$. Then the variation of mixed Hodge structure defined in 4.2 is polarized by 5.2.2. This VMHS can be extended to the puncture with the extended fiber isomorphic to Ω_f in the sense of 8.4 and 8.5, and it is polarized by 8.7.4. The Hodge filtration on the new fiber Ω_f correspond to an opposite Hodge filtration on $H^n(X_\infty, \mathbb{C})$ in the way explained in 8.5.3.*

Corollary 1.0.10. *(Riemann-Hodge bilinear relations for Grothendieck residue on Ω_f) Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a holomorphic germ with an isolated singularity. Suppose $\mathfrak{f}$ is the corresponding map to N on $H^n(X_\infty)$, via the isomorphism Φ. Define*

$$P_l = PGr_l^W := \ker(\mathfrak{f}^{l+1} : Gr_l^W \Omega_f \to Gr_{-l-2}^W \Omega_f)$$

Going to W-graded pieces;

$$\widehat{Res}_l : PGr_l^W \Omega_f \otimes_{\mathbb{C}} PGr_l^W \Omega_f \to \mathbb{C} \tag{1.28}$$

is non-degenerate and according to Lefschetz decomposition

$$Gr_l^W \Omega_f = \bigoplus_r \mathfrak{f}^r P_{l-2r}$$

we will obtain a set of non-degenerate bilinear forms,

$$\widehat{Res}_l \circ (id \otimes \mathfrak{f}^l) : PGr_l^W \Omega_f \otimes_{\mathbb{C}} PGr_l^W \Omega_f \to \mathbb{C}, \tag{1.29}$$

$$\widehat{Res}_l = res_{f,0}\ (id \otimes \tilde{C}.\ \mathfrak{f}^l) \tag{1.30}$$

where $\tilde{C}$ is as in 8.6.1, such that the corresponding hermitian form associated to these bilinear forms is positive definite. In other words,

- $\widehat{Res}_l(x, y) = 0, \qquad x \in P_r,\ y \in P_s,\ r \neq s$
- *If $x \neq 0$ in P_l,*

  $$Const \times res_{f,0}\ (C_l x, \tilde{C}.\ \mathfrak{f}^l.\bar{x}) > 0$$

  where C_l is the corresponding Weil operator.

An application of a theorem of E. Cattani-A. Kaplan-W. Schmid in [CKS], deduces the existence of a nilpotent transformation $\delta \in \mathfrak{g} = \mathfrak{gl}(H_{\mathbb{C}})$, such that the operator

$$\tilde{C}_1 := Ad(e^{-i.\delta}).\tilde{C} = Ad(e^{i.\delta}).\overline{\tilde{C}}, \qquad Ad(g) : X \mapsto gXg^{-1},\ Ad : G \to Gl(\mathfrak{g})$$

is a real transformation (notation of Theorem 8.6.1).

Proposition 1.0.11. *The bi-grading $J_1^{p,q}$ defined by $J_1^{p,q} := e^{-i.\delta}.J^{p,q}$ is split over $\mathbb{R}$. The operator $\tilde{C}_1 = e^{-i.\delta}.\tilde{C} : \Omega_f \to \Omega_f$ defines a real structure on Ω_f.*

This says if $\Omega_{f,1} = \oplus_{p<q} J_1^{p,q}$ then

$$\Omega_f = \Omega_{f,1} \oplus \overline{\Omega_{f,1}} \oplus \bigoplus_p J_1^{p,p}, \qquad \overline{J_1^{p,p}} = J_1^{p,p}.$$

The statement of Theorem 1.0.4 is valid when the operator $\tilde{C}$ is replaced with $\tilde{C}_1$;

$$S(\Phi^{-1}(\omega), \Phi^{-1}(\eta)) = * \times \ \mathrm{res}_{f,0}(\omega, \tilde{C}_1.\eta), \qquad 0 \neq * \in \mathbb{C} \tag{1.31}$$

and this equality is defined over $\mathbb{R}$. The concept of this result should be understood as a matter of representation theory in relation with $\mathfrak{sl}_2$-orbit theorem. It provides an $\mathfrak{sl}_2$-triple for Ω_f, cf. sec. 8.8. The above form also polarizes the complex variation of Hodge structure studied by G. Pearlstein and J. Fernandez, cf. [P2], see also Chap 9 sec. 3.

Theorem 1.0.12. *Let $\mathcal{V}$ be an admissible variation of polarized mixed Hodge structure associated to a holomorphic germ of an isolated hyper-surface singularity. Set*

$$U' = \overline{F_\infty^\vee} * W. \tag{1.32}$$

Then U' extends to a filtration $\underline{U'}$ of $\mathcal{V}$ by flat sub-bundles, which pairs with the limit Hodge filtration $\mathcal{F}$ of $\mathcal{V}$, to define a polarized $\mathbb{C}$-variation of mixed Hodge structure, on a neighborhood of the origin. As a consequence, the mixed Hodge structure on the extended fiber Ω_f defined by (1.23) can be identified with

$$\Phi(U' = \overline{F_\infty^\vee} * W)$$

where $A * B = \sum_{r+s=q} A_r \cap B_s$ for two filtrations A and B.

We apply the above result to the extensions of Jacobian bundle associated to a projective family of curves. For this we define;

$$\mathcal{A}_s = J^1(H_s^1) = H^1_{s,\mathbb{Z}} \setminus H^1_{s,\mathbb{C}}/F^0 H^1_{s,\mathbb{C}}$$

$$J(\mathcal{H}) = \bigcup_{s \in S^*} J^1(H_s)$$

to be the family of Jacobians associated to the variation of Hodge structure in a projective degenerate family of algebraic curves (here we have assumed the Hodge structures have weight -1).
The extended Jacobian simply is

$$J_0 = J^1(\Omega_f) = \Omega_{f,\mathbb{Z}} \setminus \Omega_f / F^0 \Omega_f.$$

Theorem 1.0.13. *The extension of a degenerate 1-parameter holomorphic family of Θ-divisors polarizing the Jacobian of curves in a projective fibration, is a Θ-divisor polarizing the extended Jacobian.*

In order to generalize theorem 1.0.7 to general (admissible) variation of mixed Hodge structure on a quasi-projective manifold we formulate the following theorem, which is a version of the work of C. Sabbah in [SA4] on hermitian duality of quasi-unipotent D-modules.

Theorem 1.0.14. *[SA4] Assume $(\mathcal{G}, F, W, H, S)$ is a polarized MHM (hence regular holonomic) with quasi-unipotent underlying variation of mixed Hodge structure H, defined on a Zariski dense open subset $U = X \setminus Z$ of an algebraic manifold X, where Z is a smooth projective hypersurface. Then, the Gauss-Manin system $\mathcal{G}$ has a smooth extension to all of X and the extended MHM is also polarized. The polarization on the fibers can be described by residues of the Mellin transform of a formal extension of the polarization S over the elementary sections, by the two formulas*

$$\psi_\lambda S\langle \sum_{l=0}^{p} m_l \otimes e_{\alpha,l}, \overline{\sum_{l=0}^{p} m_l \otimes e_{\alpha,l}} \rangle = *.\ Res_{s=\alpha} \langle \tilde{S}, |t|^{2s} dt \wedge d\bar{t} \rangle, \qquad * \neq 0,\ \alpha \neq 0$$

$$\phi_1 S(\bullet, \bar{\bullet}) = *.\ Res_{t=-1} \langle \tilde{S}, |t|^{2s} \mathcal{F}_{loc} dt \wedge d\bar{t} \rangle, \qquad * \neq 0.$$

Summarizing Theorems 1.0.2, 1.0.3, 1.0.4 and 1.0.5 we conclude with the following theorem.

Theorem 1.0.15. *Assume $(\mathcal{G}, F, W, H)$ be a polarized MHM with underlying admissible variation of mixed Hodge structure H, defined on a Zariski dense open subset U of an algebraic manifold X. Assume $X \setminus U = D$ is a normal crossing divisor defined by a holomorphic germ f. Then the extended MHM is polarized and in a neighborhood of D, the polarization of the extension of H is given either by a sign modification of the Grothendieck residue associated to the holomorphic germ f locally defining the normal crossing divisor or the usual residues of moderate extension of polarization as Theorem 1.0.5. Moreover, the Hodge filtration on the extended fibers are opposite to the limit Hodge filtration on H. These Hodge filtrations pair together to constitute a polarized complex variation of HS.*

Theorem 1.0.12 generalizes as follows.

Theorem 1.0.16. *The limit of the Poincar̀e product on the canonical fibers of the Neron model of a degenerate admissible variation of Hodge structure $\mathcal{H}$ is given by the modification of the residue pairing or induced by the residues as in 1.0.12. The extension describes the limit Jacobians as the Jacobian of the Opposite Hodge filtration on $\mathcal{H}$.*

We provide with an application to positivity results in algebraic geometry of singular varieties in section 9.4. By a hyper-surface ring we mean a ring of the form $R := P/(f)$, where P is an arbitrary ring and f a non-zero divisor. Localizing we may assume P is a local ring of dimension $n + 1$. Assume $P = \mathbb{C}\{x_0, ..., x_n\}$ and f a holomorphic germ, or $P = \mathbb{C}[x_0, ..., x_n]$ and then f is a polynomial with isolated singularity. We shall assume f has an isolated singularity at $0 \in \mathbb{C}^{n+1}$. It is a basic fact, discovered by D. Eisenbud, that the R-modules have a minimal resolution that is eventually 2-periodic. Specifically, In a free resolution of such a module M, we see that after n-steps we have an exact sequence of the following form,

$$0 \to M' \to F_{n-1} \to F_{n-2} \to ... \to F_0 \to M \to 0 \tag{1.33}$$

where the F_i are free R-modules of finite rank and $depth_R(M') = n$.

M. Hochster in his study of direct summand conjecture defined the following invariant namely Θ-invariant. The theta pairing of two R-modules M and N over a hyper-surface ring $R/(f)$ is

$$\Theta(M,N) := l(Tor_{2k}^{R}(M,N)) - l(Tor_{2k+1}^{R}(M,N)), \qquad k >> 0$$

This definition makes sense as soon as the lengths appearing are finite. This certainly happens if R has an isolated singular point.

Theorem 1.0.17. *Let S be an isolated hypersurface singularity of dimension n over $\mathbb{C}$. If n is odd, then $(-1)^{(n+1)/2}\Theta$ is positive semi-definite on $G(R)_{\mathbb{Q}}$, i.e $(-1)^{(n+1)/2}\Theta(M,M) \geq 0$.*

1.0.4 Organization of the text

Chapter 1 is an introduction to the whole text and contains some historical remarks. The main results of the thesis have been briefly listed.

Chapter 2 contains definitions and basic properties of Hodge structures and their variation. We provide a step by step explanation of the fundamental tools in Hodge theory and provide elementary examples. In successive sections we explains local systems, Gauss-Manin connections and Deligne intermediate extension. A brief explanation of the global and local invariant cycle theorem is also given. We end this chapter with the theorem of Lefschetz on $(1,1)$ classes used in application of the main contribution in 9.4.

In Chapter 3 the limit mixed Hodge structure is introduced following W. Schmid in [SCH]. We state the nilpotent orbit theorem in one variable of W. Schmid and propose an outline of fundamental theorems on the proof of Riemann-Hodge bilinear relations (Theorem 3.2.1). The second part is devoted to the Deligne-Hodge bigrading of a mixed Hodge structure with its basic properties. We briefly express the $\mathfrak{sl}_2$-orbit theorem of W. Schmid and Mixed Hodge Metric theorem. The chapter ends with a description on Higgs bundles and their relation with the variation of Hodge structures.

In Chapter 4 we explain the special case of variation of mixed Hodge structure associated to isolated hypersurface singularities. It concerns the major points important to us in the frontier sections. We give the first definition of Steenbrink limit MHS on vanishing cohomology of these fibrations. In chapter 5 we explain basic bilinear forms relevant to Milnor fibrations and specially to that of isolated singularities. Some examples of these forms are also given for convenience.

Chapter 6 is fundamental in the text. We develop the needed tools in order to present the main contributions in Chapter 8. In this part the V-filtration, Brieskorn lattice, the weight and Hodge filtrations of Gauss-Manin system and the spectral pairs are defined. We give another definition of Steenbrink limit mixed Hodge structure equivalent to that presented in Chapter 4.

In Chapter 7 the case of quasi-homogeneous fibrations is discussed as an example to what has been introduced before. Some basic examples are also listed for more convenience. The Steenbrink method for the proof of the mixed Hodge structure is also classified at the end.

Chapters 8 and 9 contain the main contributions namely Theorems 8.6.1 and 8.7.1 with successive corollaries on a formulation of Riemann-Hodge bilinear relations for Grothendieck residue pairing. Specifically, we give a standard method to calculate a signature for this form in section 9.1. Chapter 9 mainly investigates the applications of Theorems 8.6.1 and 8.7.1 and explains their relations with other known facts.

In Chapter 10 we give more comments for further studies. In section 10.1 a discussion on primitive elements is given. We have provided a summary of the work of K. Saito on higher residue pairing. Section 10.3 presents some generalizations of the result for arbitrary admissible PVMHS.

Chapter 2

Basics on Hodge theory

This chapter is devoted with the basic definitions in Hodge theory. We provide the definition and first examples of Hodge structure (HS) and mixed Hodge structure (MHS), and their variations. The material we mention are organized with respect to what we need in the later chapters.

2.1 Hodge Theory of Compact Riemann Surface

For a compact connected Riemann surface X of genus g, the cohomology group $H^1(X,\mathbb{Q}) = \mathbb{Q}^{2g}$ admits a new structure as follows. First, Poincarè duality induces a skew symmetric non-degenerate bilinear form

$$(\bullet,\bullet) : H^1(X,\mathbb{Q}) \times H^1(X,\mathbb{Q}) \to H^2(X,\mathbb{Q}) \overset{\int_{[X]}}{\to} \mathbb{Q}. \tag{2.1}$$

We have $\dim H^1(X,\mathcal{O}_X) = g$. The Serre duality,

$$H^1(X,\mathcal{O}_X) \simeq H^0(X,\Omega^1_X)^\vee$$

also gives $\dim H^0(X,\Omega^1_X) = g$. That implies the Hodge decomposition

$$H^1(X,\mathbb{C}) = H^{0,1}(X) \oplus H^{1,0}(X)$$
$$H^{0,1}(X) = H^1(X,\mathcal{O}_X), \qquad H^{1,0}(X) = H^0(X,\Omega^1_X).$$

If we regard Serre duality as a pairing

$$H^{1,0} \otimes H^{0,1} \overset{\bullet\wedge\bullet}{\longrightarrow} H^{1,1} \overset{\int}{\to} \mathbb{C}, \tag{2.2}$$

it is equivalent to the complexified Poincarè duality

$$(\bullet,\bullet)_{\mathbb{C}} : H^1(X,\mathbb{C}) \otimes_{\mathbb{C}} H^1(X,\mathbb{C}) \to \mathbb{C}. \tag{2.3}$$

Since $(H^{1,0}, H^{1,0}) = 0$ and $(H^{0,1}, H^{0,1}) = 0$.

Thus, the matrix of this pairing with respect to some normalized basis becomes

$$(\bullet, \bullet)_{\mathbb{C}} \leftrightarrow \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

With respect to the real structure $H^1(X, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} H^1(X, \mathbb{R})$, $H^{1,0}$ is conjugate to $H^{0,1}$, and using Serre duality we get a sesquilinear pairing

$$k : H^{1,0} \otimes_{\mathbb{C}} H^{1,0} \to \mathbb{C}, \qquad (\alpha, \beta) \mapsto \alpha \wedge \bar{\beta}. \tag{2.4}$$

Then the Hodge-Riemann bilinear relations assert that $h = \sqrt{-1}.k$ is a positive definite hermitian form, [SA1]. The definition of a Hodge structure is an abstract version of this example.

2.2 Hodge Structures

In the following sections, the basic definitions and examples of Hodge theory concepts are given, [G2].

Definition 2.2.1. A Hodge structure of weight m is given by a data $(H_{\mathbb{Q}}, F^p)$ where $H_{\mathbb{Q}}$ is a finitely generated $\mathbb{Q}$-vector space, and $F^p, p = (0, ..., m)$ is a decreasing filtration on the complexification $H = H_{\mathbb{Q}} \otimes \mathbb{C}$ such that $F^p \oplus \overline{F^{m-p+1}} \cong H$, for all p.

Setting $H^{p,q} = F^p \cap \bar{F}^q$, the condition is equivalent to

$$H = \bigoplus_{p+q=m} H^{p,q}, \qquad H^{p,q} = \bar{H}^{q,p}. \tag{2.5}$$

The relation between them is $F^p = \oplus_{p' \geq p} H^{p',m-p'}$. We shall use the abbreviation HS for Hodge structures. A sub-Hodge structure is given by a sub-vector space H' such that $F'^p = H' \cap F^p$, (H', F'^p) is again a Hodge structure.

Example 2.2.2. (1) Let $H_{\mathbb{C}} = \mathbb{C}^2 = \mathbb{C}e_1 + \mathbb{C}e_2$ and set $H^{1,0} = \mathbb{C}(e_1 - ie_2)$ and $H^{0,1} = \mathbb{C}(e_1 + ie_2)$. This gives a Hodge structure of weight 1.

(2) $\mathfrak{sl}(\mathbf{2}, \mathbb{C})$: The lie algebra $\mathfrak{sl}(2, \mathbb{C})$ carries a HS of weight 0.

The elements

$$Z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; X_+ = \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, \; X_- = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$$

span $\mathfrak{sl}_2$. They satisfy

$$[Z, X_+] = 2X_+, \qquad [Z, X_-] = -2X_-, \qquad [X_+, X_-] = Z$$
$$\bar{Z} = -Z, \qquad \bar{X}_+ = X_-.$$

Setting,

$$\mathfrak{sl}(2, \mathbb{C})^{-1,1} = \overline{\mathfrak{sl}(2, \mathbb{C})^{1,-1}} = \mathbb{C}(iZ + X_- + X_+)$$
$$\mathfrak{sl}(2, \mathbb{C})^{0,0} = \mathbb{C}(X_+ - X_-)$$

provides a HS.

(3) The one dimensional complex vector space $\mathbb{C}$, with the obvious real structure carries a unique Hodge structure of weight -2, where $F^{-1} = \mathbb{C}$. Deligne denoted it by $\mathbb{Q}(1)$, and called it the Hodge structure of Tate. For $n \geq 0; \mathbb{Q}(n)$, will be the n-th symmetric power of $\mathbb{Q}(1)$, and $\mathbb{Q}(-n)$ the dual of $\mathbb{Q}(n)$.

(4) Let e_1 and e_2 be the standard basis of $\mathbb{C}^2$. For $p \neq q$, we define a Hodge structure $E(p, q)$ of weight $p + q$ on $\mathbb{C}^2$, with natural real structure by requiring that $v_+ = e_1 + ie_2$ of type (q, p), and $v_- = e_1 - ie_2$ of type (p, q).

Definition 2.2.3. A polarized Hodge structure of weight m is given by the data $(H_{\mathbb{Q}}, F^p, Q)$ where $(H_{\mathbb{Q}}, F^p)$ is a Hodge structure of weight m and

$$Q : H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \to \mathbb{Q} \tag{2.6}$$

is a bilinear form satisfying the conditions

- $Q(u, v) = (-1)^m Q(v, u)$
- $Q(F^p, F^{m-p+1}) = 0$
- The Hermitian form $Q(Cu, \bar{v})$ is positive definite where $Cu = (\sqrt{-1})^{p-q}u,\ u \in H^{p,q}$.

A sub-Hodge structure of a polarized Hodge structure is again polarized as its orthogonal w.r.t Q, and the original HS is a direct sum of the sub-Hodge structure and its orthogonal.

Example 2.2.4. In Example (1) defining $(e_1, e_2) = 1,\ (e_i, e_i) = 0,\ i = 1, 2$; provides a polarization on the Hodge structure.

Example 2.2.5. $\mathbb{Q}(n)$ has a natural polarization. The bilinear form Q on $\mathbb{C}^2$ defined by

$$Q(v_+, v_+) = 0,\ Q(v_-, v_-) = 0,\ Q(v_+, v_-) = 2i^{p-q}$$

polarizes $E(p, q)$.

Theorem 2.2.6. *(W. Hodge) (see [PS] Theorem 1.8) The cohomology group $H^m(X, \mathbb{Q})$ of a compact Kaḧler manifold has a canonical Hodge structure of weight m.*

By definition this means we have decomposition

$$H^m(X, \mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}, \qquad H^{q,p} = \overline{H^{p,q}}$$

for all $0 \leq m \leq \dim(X)$. Assume $\omega \in H^2(X, \mathbb{C})$ is the Kaḧler class of X.

Theorem 2.2.7. *(Hard Lefschetz Theorem)(see [PS] Theorem 1.30) For any Kaḧler manifold (X,ω), cup product with the Kaḧler class ω induces isomorphisms*

$$. \wedge \omega^{n-m} : H^m \to H^{2n-m}, \qquad m \leq n$$

$$. \wedge \omega^{n-(p+q)} : H^{p,q} \to H^{n-q,n-p}, \qquad p+q \leq n.$$

For $m \leq n = \dim(X)$ the primitive subspace is defined by

$$PH^m(X,\mathbb{Q}) = \ker(H^m(X,\mathbb{Q}) \stackrel{\wedge\omega^{n-m+1}}{\longrightarrow} H^{2n-m+2}(X,\mathbb{Q})) \tag{2.7}$$

and it has an induced Hodge decomposition

$$PH^m = \bigoplus_{p+q=m} PH^{p,q}, \qquad PH^{q,p} = \overline{PH^{p,q}}.$$

The primitive decomposition theorem says

$$H^m(X,\mathbb{Q}) = \bigoplus_k \omega^k.PH^{m-2k}(X,\mathbb{Q})$$

$$H^{p,q}(X,\mathbb{Q}) = \bigoplus_k \omega^k.PH^{p-k,q-k}(X,\mathbb{Q}).$$

Definition 2.2.8. The operator

$$C : H^m = \bigoplus_{p+q=m} H^{p,q} \to H^m = \bigoplus_{p+q=m} H^{p,q}, \qquad C|H^{p,q} = i^{p-q}$$

is called the Weil-operator.

Theorem 2.2.9. *(Riemann-Hodge bilinear relations for the cohomology of Kaḧler manifolds)(see [PS] Theorem 1.33) Assume (X,ω) is a Kaḧler manifold. Then the Hodge structure on the cohomologies $H^m(X,\mathbb{Q})$ is polarized by*

$$Q(u,v) = (-1)^{m(m-1)/2} \int_X u \wedge v \wedge \omega^{n-m} \tag{2.8}$$

and satisfies the following properties;

- $Q(H^{p,q}, H^{r,s}) = 0 \qquad$ *if* $(r, s) \neq (p, q)$
- $Q(C.u, \bar{u}) > 0 \qquad$ *if* $u \neq 0$

where C is the Weil-operator.

The two conditions in the theorem are called Riemann-Hodge bilinear relations. The primitive cohomologies are the basic building blocks of the cohomology of a smooth projective variety.

2.3 Mixed Hodge structures

Definition 2.3.1. A mixed Hodge structure is given by the data $(H_{\mathbb{Q}}, W_m, F^p)$ where $\{W_m\}$ is increasing (namely weight) filtration and F^p, a decreasing (called Hodge) filtration;

$$... \subseteq W_m \subseteq W_{m+1} \subseteq ... \subset H_{\mathbb{Q}}$$

defined over $\mathbb{Q}$, and the Hodge filtration

$$... \subseteq F^p \subseteq F^{p-1} \subseteq ... \subseteq H_{\mathbb{C}}$$

defined on $H = H_{\mathbb{Q}} \otimes \mathbb{C}$ where the filtrations $F^P(Gr_m^W)$ induced by F^p give a Hodge structure of weight m. If the pure Hodge structures on $Gr_m^W H$ are polarized by Q_m, as in 2.2.3, for all m, we say the mixed Hodge structure is polarized. It is also called graded polarized MHS.

General examples of mixed Hodge structure are given by the cohomologies $H^k(X, \mathbb{C})$ of a complex variety.

Theorem 2.3.2. *(P. Deligne) [D1] The complex cohomology groups of a complex quasi-projective variety carry mixed Hodge structures which are functorial. In case of a non-singular projective variety these mixed Hodge structures reduce to the ordinary HS of pure weight.*

Remark 2.3.3. The weight in the Theorem satisfies $0 \leq m \leq 2k$. For X smooth but possibly not complete the weight satisfies $k \leq m \leq 2k$, while for X complete but possibly singular they satisfy $0 \leq m \leq k$. These mixed Hodge structures are functorial.

A morphism of MHS's of type (r, r) between $(H_{\mathbb{Q}}, W_m, F^p)$ and $(H'_{\mathbb{Q}}, W'_l, F'^q)$ is a linear map

$$L : H_{\mathbb{Q}} \to H'_{\mathbb{Q}}$$

satisfying $L(W_m) \subseteq W'_{m+2r}, L(F^p) \subseteq F'^{p+r}$, and such a morphism is then strict in the sense that $L(F^p) = F'^{p+r} \cap Im(L)$ and similarly for the weight filtration. For a pair $Y \subseteq X$ of complex algebraic varieties, the relative cohomologies $H^k(X, Y)$ have mixed Hodge structures and the long exact sequence of the pair will be an exact sequence of MHS's. The category of mixed Hodge structures is abelian, Moreover, it is closed under the operations of direct sum, tensor product, and dual.

Definition 2.3.4. We say a mixed Hodge structure (H, W, F); of weight $m \in \mathbb{Z}$ is polarized by N, where N is a $(-1, -1)$-morphism $N \in \mathfrak{g} \cap \mathfrak{gl}(H_{\mathbb{Q}})$, $\mathfrak{g} = End(H_{\mathbb{C}})$, If it is equipped with a non-degenerate rational bilinear form Q such that:

- $N^{m+1} = 0$
- $W = W(N)[-m]$, where $W[-k]_l = W_{l-k}$
- $Q(F^p, F^{m-p+1}) = 0$
- The Hodge structure of weight $m + l$ induced by F on $\ker N^{l+1} : Gr^W_{m+l} \to Gr^W_{m-l-2}$ is polarized by $Q(., N^l.)$.

Remark 2.3.5. By $W(N)$ we refer to the corresponding nilpotent operator N. It refers to the next proposition.

Proposition 2.3.6. *(Jacobson-Morosov)(see [SA1] page 12) Let $H_{\mathbb{Q}}$ be a vector space with a nilpotent transformation N. There exists a unique increasing filtration of $H_{\mathbb{Q}}$ indexed by $\mathbb{Z}$, called the monodromy filtration relative to N and denoted by $W(N)$ satisfying the following properties;*

- *For any $l \in \mathbb{Z}$, $N(W_l) \subset W_{l-2}$*
- *For any $l \geq 1$, N^l induces an isomorphism $Gr^W_l H_{\mathbb{Q}} \xrightarrow{\cong} Gr^W_{-l} H_{\mathbb{Q}}$.*

We provide a more complete form to 2.3.6 in 3.1.2. However, for the purpose of this chapter we gave the above elementary version to handle some basic definitions.

Example 2.3.7. **X singular and compact:** [DU] Let $X = X_1 \cup X_2$ with $X_1,\ X_2$ two projective non-singular varieties intersecting transversely. In the Meyer-Vietoris sequence;

$$\stackrel{\beta_{m-1}}{\to} H^{m-1}(X_1 \cap X_2) \stackrel{\delta}{\to} H^m(X) \stackrel{\alpha}{\to} H^m(X_1) \oplus H^m(X_2) \stackrel{\beta_m}{\to} H^m(X_1 \cap X_2) \to \tag{2.9}$$

where δ is the connecting homomorphism, and all maps are morphisms of Hodge structures. The first term has pure weight $m-1$ and the last two terms have pure weight m. Define the weight filtration on $H^m(X)$ by;

$$W_{m-2} = 0, \quad W_{m-1} = im(\delta), \quad W_m = H^m(X). \tag{2.10}$$

Then, $W_{m-1}/W_{m-2} \cong im(\delta) \cong coker(\beta_{m-1})$ has a Hodge structure of weight $m-1$, and $W_m/W_{m-1} \cong \ker(\beta_m)$ has a Hodge structure of weight m, since the kernel and cokernel of a map of Hodge structures have a Hodge structure.

The cohomology of X can be computed by forms on the disjoint union of X_1 and X_2 which agree on $X_1 \cap X_2$. Let the Hodge filtration on $H^m(X)$ be induced by the usual filtration on this complex of forms. For example, when $\dim(X_1) = \dim(X_2) = 1$, the exact sequence is

$$0 \to \tilde{H}^0(X_1 \cap X_2) \to H^1(X) \to H^1(X_1) \oplus H^1(X_2) \to 0 \tag{2.11}$$

where $\tilde{H}$ is the reduced cohomology, and $H^1(X)$ has classes of two types: those of weight 1, which lie in one of the X_i's and naturally have types $(1,0)$ and $(0,1)$, and those of weight 0 and type $(0,0)$, which come from the intersection of X_1 and X_2 via the Meyer-Vietoris sequence.

Example 2.3.8. X **open and smooth:** [DU] Let Z be a smooth projective variety and $D \subset Z$ is a smooth co-dimension one sub-variety. We will find the mixed Hodge structure on the cohomology of the open smooth space $X = Z - D$. The cohomology of X can be computed using the de Rham complex of smooth forms, So let $F^p \subset H^m(X)$ be those cohomology classes which can be represented by forms with p or more dz's. Now, let us find the weight filtration. Let $i : X \subset Z$ be the inclusion. In the Gysin sequence

$$\to H^{m-2}(D) \overset{\gamma_m}{\to} H^m(Z) \overset{i^*}{\to} H^m(X) \overset{R}{\to} H^{m-1}(D) \overset{\gamma_{m+1}}{\to} H^{m+1}(Z) \to \quad (2.12)$$

The group $H^{m-2}(D)$ has a pure Hodge structure of weight $m-2$ and the group $H^m(Z)$ has a Hodge structure of weight m. Furthermore, the Gysin map γ_m takes a form of type (p,q) to a form of type $(p+1, q+1)$, the restriction i^* preserves the Hodge filtration, and the residue map R has the property that $R(F^p) \subset F^{p-1}$ since it removes a factor of dz/z. Now change the Hodge filtration on $H^{m-2}(D)$ by refining a class of type (p,q) to be of type $(p+1, q+1)$. Then $H^{m-2}(D)$ has pure HS of weight m, and γ_m is a morphism of HS, and R now takes F^p to F^p. We can now define the weight filtration on $H^m(X)$ by

$$W_{m-1} = 0, \quad W_m = im(i^*), \quad W_{m+1} = H^m(X). \quad (2.13)$$

Then, $W_m/W_{m-1} = im(i^*) = coker(\gamma_m)$, has a Hodge structure of weight m since $H^m(Z)$ does, and $W_{m+1}/W_m = \ker(\gamma_{m+1})$ has a Hodge structure of weight $m+1$ since $H^{m-1}(D)$ does. Furthermore, both these Hodge structures are the same as the ones induced by the Hodge filtration on $H^m(X)$.

For example, when Z is a smooth connected curve and $D = \{p_1, ..., p_k\}$ is a collection of points, the sequence is

$$0 \to H^1(Z) \to H^1(X) \to \tilde{H}^0(D) \to 0. \quad (2.14)$$

The classes of weight 1 (type $(1,0)$ or $(0,1)$) in $H^1(X)$ are restrictions of the classes in $H^1(Z)$. The classes of weight 2 are represented by linear combinations of the forms $dz/(z-p_1), ..., dz/(z-p_k)$ and have type $(1,1)$.

Example 2.3.9. Logarithmic de Rham Complex: (P. Deligne) [D1] Let $D = \cup D_i$ be a normal crossing divisor in a smooth proper algebraic variety X, and $U = X - D$. The de Rham complex with logarithmic singularities along D namely; $\Omega_X^\bullet(\log(D)) \subset i_*\Omega_U^\bullet$, where $i : U \hookrightarrow X$ is the inclusion, is defined as follows. Assume D is given locally by an equation $z_1....z_r = 0$. Locally, $\Omega_X^k(\log(D))|_U$ is the free $\mathcal{O}_U$-module generated by $\frac{dz_{i_1}}{z_{i_1}} \wedge ... \wedge \frac{dz_{i_l}}{z_{i_l}} \wedge dz_{j_1} \wedge ... \wedge dz_{j_r}$, where $l + r = k$.
The weight filtration W is given by

$$W_m\Omega_X^p(\log D) = \begin{cases} 0 & \text{for } m < 0 \\ \Omega_X^p(\log D) & \text{for } m \geq p \\ \Omega_X^{p-m} \wedge \Omega_X^m(\log D) & \text{if } 0 \leq m \leq p \end{cases}.$$

The Hodge filtration is given by the truncations of the logarithmic de Rham complex

$$F^p\Omega_X^\bullet(\log D) := \Omega_X^{\bullet \geq p}(\log D).$$

Theorem 2.3.10. *(P. Deligne) [D1] The following are true,*

- $$\mathbb{H}^k(X, \Omega_X^\bullet(\log(D))) \cong H^k(U, \mathbb{C})$$
- *The two filtrations $W_\bullet$ and $F^\bullet$ defined by*

$$W_mH^k(U,\mathbb{C}) = Im(\mathbb{H}^k(X, W_{m-k}\Omega_X^\bullet(\log D)) \to H^k(U,\mathbb{C}))$$
$$F^pH^k(U,\mathbb{C}) = Im(\mathbb{H}^k(X, F^p\Omega_X^\bullet(\log D)) \to H^k(U,\mathbb{C}))$$

put a mixed Hodge structure on $H^k(U)$.

Example 2.3.11. $\mathfrak{gl}(\mathbf{V})$: Given a MHS on $V_{\mathbb{C}}$, we may define a MHS on $\mathfrak{gl}(V_{\mathbb{C}})$ by :

$$W_a\mathfrak{gl} := \{X \in \mathfrak{gl} : X(W_l) \subset W_{l+a}\}$$
$$F^b\mathfrak{gl} := \{X \in \mathfrak{gl} : X(F^p) \subset F^{p+b}\}.$$

An element $T \in (W_{2a} \cap F^a\mathfrak{gl}) \cap \mathfrak{gl}(V_{\mathbb{Q}})$ is called an (a, a)-morphism.

2.4 Variation of Hodge Structure

Definition 2.4.1. A variation of Hodge structure (VHS) is given by the data $(S, \mathcal{H}_{\mathbb{Q}}, \mathcal{F}^p, \nabla)$ where

- S is a smooth complex algebraic variety.
- $\mathcal{H}_{\mathbb{Q}}$ is a local system of $\mathbb{Q}$-vector spaces on S.
- $\mathcal{H} = \mathcal{H}_{\mathbb{Q}} \otimes \mathcal{O}_S$ is a holomorphic vector bundle with a filtration $\mathcal{F}^p$ by holomorphic sub-bundles.
- $\nabla : \mathcal{H} \to \Omega^1_S \otimes \mathcal{H}$ is an integrable connection.

  and where the conditions

- $\nabla\mathcal{H}_{\mathbb{Q}} = 0$
- For each $s \in S$, on each fiber the induced data $(\mathcal{H}_{\mathbb{Q},s}, \mathcal{F}^p_s)$ gives a Hodge structure of weight m, and
- The transversality conditions

$$\nabla\mathcal{F}^p \subseteq \Omega^1_S \otimes \mathcal{F}^{p-1} \tag{2.15}$$

  are satisfied.

There is also the notion of polarized variation of Hodge structure given by the additional data of

$$Q : \mathcal{H}_{\mathbb{Q}} \otimes \mathcal{H}_{\mathbb{Q}} \to \mathbb{Q} \tag{2.16}$$

satisfying $\nabla Q = 0$ and inducing polarized Hodge structure on each fiber.

The basic example of a variation of Hodge structure is the cohomology along the fibers $X_s = \pi^{-1}(s)$ of a smooth family $X \to S$ of compact complex manifolds, where X is Kaḧler and where $\mathcal{H}_{\mathbb{Q}} = R^m \pi_* \mathbb{Q}$, $F_s^p = F^p H^m(X_s, \mathbb{C})$.

Consider the 1-parameter degenerating family

$$\begin{array}{ccc} X & \longrightarrow & \bar{X} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \bar{S} \end{array} \tag{2.17}$$

where $\bar{S} = |s| < 1$ is a disc, and $S = \bar{S} - 0$ a punctured disc, $X, \bar{X}, X_s$ are smooth for $s \neq 0$. Then there is a monodromy transformation $M : H_{\mathbb{Z}} \to H_{\mathbb{Z}}, H_{\mathbb{Z}} = H^m(X_{s_0}, \mathbb{Z})$. It is known that, after possibly passing to a finite covering of S, we have

$$(M - I)^{m+1} = 0. \tag{2.18}$$

Setting $N = \log(M_u)$ where $M = M_s.M_u$ is the Jordan decomposition; a weight filtration can be defined on $H_{\mathbb{Z}}$ by the conditions

- $N : W_k \to W_{k-2}$
- $N^k : Gr^W_{m+k} \to Gr^W_{m-k}$ is an isomorphism.

Letting $s = \exp(\sqrt{-1}.t)$ where $Im(t) > 0$; in the early 1970's, W. Schmid proved that

$$\lim_{Im(t) \to \infty} \exp(-tN) F_s^p =: F_\infty^p$$

exists and that $(H_{\mathbb{Q}}, W_k, F_\infty^p)$ gives a polarized MHS on $H^m(X_{s_0}, \mathbb{C})$, relative to which N is a morphism of type $(-1, -1)$, [SCH].

2.5 Invariant cycle theorem

Consider the surjective algebraic map $\pi : X \to S$ where X is a projective variety and S a smooth projective curve. It possibly has a finite number of critical values. Let

$$\begin{aligned} S^* &= S - \{ \text{ critical values of } \pi\}, \\ X^* &= \pi^{-1}(S^*), \\ X_t &= \pi^{-1}(t), \qquad \text{for } t \neq 0, \end{aligned}$$

and let $i : X_t \hookrightarrow X^*$, $j : X^* \hookrightarrow X$ be the inclusions. Consider the map

$$H^m(X) \xrightarrow{j^*} H^m(X^*) \xrightarrow{i^*} H^m(X_t).$$

The fundamental group $\pi_1(S^*)$ acts on $H^m(X_t)$. Let $H^m(X_t)^{\pi_1(S^*)}$ denote the elements invariant under this action. It lies in the image of i^* in $H^m(X_t)$.

Theorem 2.5.1. *(Global invariant cycle theorem) (see [PS] cor. 1.40) $H^m(X_t)^{\pi_1(S^*)}$ is the image of $i^* \circ j^*$. In other words, for all $t \in S$, the invariants in $H^m(X_t, \mathbb{C})$ under the monodromy action come from restriction of global sections on X.*

Note that a class in $H^m(X_t)^{\pi_1(S^*)}$ is of pure weight m, and hence pulls back under i^* to a class of pure weight m (i^* strictly preserves the weight filtration). As X is a compactification of X^*, all classes of weight m on X^* are restrictions of classes on X, [DU], [PS] cor. 1.40.

Localizing the above situation near a degenerate point $0 \in S$ and using Hironaka theorem for resolution of singularities, one has the following commutative diagram,

$$\begin{array}{ccccccc} X_\infty & \longrightarrow & U & \longrightarrow & X & \longleftarrow & E \\ {\scriptstyle f_\infty}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow \\ H & \xrightarrow{e} & \Delta^* & \longrightarrow & \Delta & \longleftarrow & 0 \end{array} \tag{2.19}$$

where H is the upper half plane, e is the exponential map and $X_\infty := X \times_{\Delta^*} H$. E is a normal crossing divisor.

Theorem 2.5.2. *(Local invariant cycle theorem) (see [PS] Theorem 11.43) There is an exact sequence for all k,*

$$H^k(E,\mathbb{Q}) \to H^k(X_\infty,\mathbb{Q}) \overset{M-I}{\longrightarrow} H^k(X_\infty,\mathbb{Q}).$$

In other words, the invariant cycles in the generic fiber X^∞ are the classes in the image of the first map.

2.6 Local systems

A local system of $\mathbb{K}$-vector spaces, over a complex manifold S is sheaf $\mathcal{L}$ locally isomorphic to a constant sheaf with stack $\mathbb{K}^n$ for a fixed n, where $\mathbb{K}$ is an arbitrary field. We only consider local systems over the fields $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$. Fix a point $s_0 \in S$, then for any curve $\gamma : [0,1] \to B, \gamma(0) = s_0, \gamma(1) = s_1$, the pull back $\gamma^*(\mathcal{L})$ to $[0,1]$ is locally constant. Thus, we get a $\mathbb{C}$-vector space isomorphism

$$\tau^\gamma : \mathcal{L}_{s_1} \to \mathcal{L}_{s_0}. \tag{2.20}$$

It depends only on the homotopy class of the path γ. Taking closed loops based on s_0, we obtain a representation:

$$\rho : \pi_1(B, s_0) \to Gl(\mathcal{L}_{s_0}) \cong Gl(n,\mathbb{C}). \tag{2.21}$$

If S is connected this construction is independent of the base point s_0, up to conjugation.

Conversely, if we begin from the representation ρ, and let $\tilde{S}$ be the universal cover of S, then define the vector bundle $\mathcal{H} \to S$ by

$$\mathcal{H} := \tilde{S} \times \mathbb{C}^n / \sim,$$
$$(\tilde{s}, v) \sim (\sigma(\tilde{s}), \rho(\sigma^{-1})(v)),\ \sigma \in \pi_1(S)$$

where σ acts as a covering deck transformation. The sheaf of constant local sections of $\mathcal{H}$ gives $\mathcal{L}$. In this way there is a 1-1 correspondence between the local systems on S and representations of $\pi_1(S, *)$, [C1].

Example 2.6.1. [C1]

Let $S = T' := \{z \in \mathbb{C} : 0 < |z| < 1 < r\}$. For $t_0 = 1 \in T'$, we have $\pi_1(T', t_0) \simeq \mathbb{Z}$ where we choose as generator a simple loop c oriented clockwise. Let

$$\rho : \pi_1(T', t_0) \cong \mathbb{Z} \to Gl(2, \mathbb{C}) \tag{2.22}$$

where

$$\rho(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

Recalling that the upper half plane $H = \{z = x + ix \in \mathbb{C} : y > 0\}$ is the universal cover of T' with projection $z \to \exp(2\pi i z)$, we have a commutative diagram

$$\begin{array}{ccc} H \times \mathbb{C}^2 & \longrightarrow & \mathcal{H} \cong H \times \mathbb{C}^2/\sim \\ \Big\downarrow{\scriptstyle pr_1} & & \Big\downarrow \\ H & \xrightarrow{\exp(2\pi i \bullet)} & T'. \end{array} \tag{2.23}$$

Let N be the nilpotent transformation

$$N = \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}.$$

Then for any $v \in \mathbb{C}^2$, the map $\tilde{v} : T' \to \mathcal{H}$ defined by

$$\tilde{v}(t) := [\frac{\log t}{2\pi i}, \exp(\frac{\log t}{2\pi i} N).v] \in H \times \mathbb{C}^2/\sim \tag{2.24}$$

is a section of a vector bundle $\mathcal{H}$. We can write

$$\tilde{v}(t) = \exp((\log t/2\pi i)N).\tilde{v}, \tag{2.25}$$

where $\hat{v}(t)$ is the constant section defined on a neighborhood of t. This example may be generalized to an arbitrary nilpotent transformation $N \in$

$\mathfrak{gl}(H)$ if we define

$$\rho : \pi_1(T', t_0) \to Gl(H) \tag{2.26}$$

by $\rho(c) = \exp(N)$, where c is again a simple clockwise loop. And also may be generalized to higher dimensions when, $\{N_1, ..., N_r\} \in \mathfrak{gl}(H)$ are commuting nilpotent transformations, by considering $S = (T')^r$ and

$$\rho : \pi_1((T')^r, t_0) \cong \mathbb{Z}^r \to Gl(H) \tag{2.27}$$

the representation that maps the j-th standard generator of $\mathbb{Z}^r$ to $M_j = \exp N_j$.

A general example is the cohomology group of the generic fiber of a proper smooth map $f : X \to S$ of complex manifolds, namely $H^k(X_t)$ as in 2.4. We denote this local system by $H^k(X/S) = R^k f_* \mathbb{C}_X$. According to some fundamental theorems this local system underlies a variation of mixed Hodge structure. We will explain these theorems in the next chapters. Such variations are called geometric.

2.7 Gauss-Manin connection

The concept of connection on analytic manifolds is a generalization of a system of n linear first order differential equations.

Definition 2.7.1. Let E be a holomorphic vector bundle on a complex manifold S. A connection on E is a $\mathbb{C}$-linear map

$$\nabla : E \to \Omega^1_S \otimes_{\mathbb{C}} E \tag{2.28}$$

satisfying;

$$\nabla(f.\phi) = df \otimes \phi + f \nabla \phi \tag{2.29}$$

for all sections f of $\mathcal{O}_S$ and ϕ of E, known as Leibnitz condition.

Similarly, ∇ can be defined on differential forms in degree p as a $\mathbb{C}$-linear map

$$\nabla^p : E \otimes \Omega_S^p \to E \otimes \Omega_S^{p+1}$$

satisfying the Leibnitz rule. The connection is said to be integrable if the curvature $\nabla^1 \circ \nabla^0 = 0$. In this case the de Rham complex associated to ∇, is

$$DR(E) := (\Omega_S^\bullet \otimes E, \nabla) : E \xrightarrow{\nabla} \Omega_S^1 \otimes E \to ... \to \Omega_S^p \otimes E \xrightarrow{\nabla^p} ... \to \Omega_S^n \otimes E.$$

Proposition 2.7.2. *The horizontal sections E^∇ of a connection ∇ on E are defined as the solutions of the differential equation on X*

$$E^\nabla = \{\phi : \nabla(\phi) = 0\}.$$

When the connection is integrable, E^∇ is a local system of dimension $rank(E)$.

Locally on an open chart U for E, the connection is given by;

$$\nabla_U = d + A_U \wedge$$

where A_U is a matrix of differential forms $(A_{ij})_{i,j \in [1,m]}$ called the connection matrix. This may be proved by choosing a frame of E over U. If $(x_1, ..., x_n)$ is a coordinate system on U, then one may write

$$\omega_{i,j} = \sum_{k \in [1,n]} \Gamma_{ij}^k(x) dx_k \ .$$

So that the equation of coordinates of the horizontal sections is given by the linear partial differential equations

$$\frac{\partial y_i}{\partial x_k} + \sum_{j \in [1,m]} \Gamma_{ij}^k(x) y_j = 0 \ .$$

The solutions form a local system of dimension m (the Frobenius condition is satisfied), [BZ], [PS] sec. 10.4.

Theorem 2.7.3. *(Riemann-Hilbert Correspondence) (see [PS] Theorem 11.7)*

The functor $(E, \nabla) \to E^{\nabla}$ is an equivalence between the category of integrable connections on vector bundles on a manifold S, and the category of complex local systems on S.

Remark 2.7.4. A section s of the vector bundle E is called a flat section of the connection ∇, if $\nabla s = 0$. The connection ∇ is called flat if there is a trivialization of E, for which the corresponding frame consists of flat sections.

Theorem 2.7.3 says that, there is a 1-1 correspondence between the local systems on S and flat connections on vector bundles over S, as is the same with finite dimensional representations of $\pi_1(S, *)$.

Let $\mathcal{L}$ be a local system on S and $E = \mathcal{L} \otimes \mathcal{O}_S$ a locally free sheaf (vector bundle) of holomorphic sections of $\mathcal{L}$. By Theorem 2.7.3 there is a unique connection $\nabla : E \to \Omega^1_S \otimes \mathcal{O}_S$ such that $\mathcal{L}$ is its kernel, i.e. $\nabla(\mathcal{L}) = 0$ called Gauss-Manin connection. It is simply defined by

$$\nabla(g.s) = dg.s, \qquad g \in \mathcal{O}_S, \quad s \in \Gamma(E). \tag{2.30}$$

An example of this is $H^k_{DR}(X/S) = R^k f_*(\Omega^\bullet_{X/S})$ associated to the local system encountered at the end of the previous section. In this case the Gauss-Manin connection satisfies

$$\nabla(F^p H^k) \subset \Omega^1_S \otimes F^{p-1} H^k$$

called Griffiths transversality. Such Gauss-Manin systems are also called Hodge modules. A Hodge module always carries a weight and a Hodge filtration with the appropriate property mentioned.

2.8 Period map

Consider a family $\pi : X \to S$ such that $X \subset \mathbb{P}^N$, that each fiber X_t is a smooth projective variety. The chern class of the hyper-plane bundle restricted to X gives integral Kahler classes $\omega_t \in H^{1,1}(X_t) \cap H^2(X_t, \mathbb{Z})$ which fit together to define a section of the local system $R^2\pi_*\mathbb{Z}$ over S. On each fiber X_t we have a Hodge decomposition:

$$H^k(X_t, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X_t) \tag{2.31}$$

where $H^{p,q}(X_t) \cong H^{p,q}_{\bar{\partial}}(X_t) \cong H^q(X_t, \Omega^p_{X_t})$ is the space of Dolbeault cohomology classes. The Hodge numbers $h^{p,q}(X_t) = \dim H^{p,q}(X_t)$ are constant. We have

$$F^p(X_t) := \bigoplus_{a \geq p} H^{a,k-a}(X_t) \tag{2.32}$$

which satisfies the condition $H^k(X_t, \mathbb{C}) = F^p(X_t) \oplus \overline{F^{k-p+1}(X_t)}$. Therefore, we obtain a local system which also defines a holomorphic vector bundle $\mathcal{H}^k$ over S.

Set $f^p = \sum_{a \geq p} h^{a,k-a}$. Assume that S is contractible and X is C^∞ trivial over S. Fix also a $t_0 \in S$. Then we have diffeomorphisms $g_t : X_{t_0} \to X_t$

$$g_t^* : H^k(X_t, \mathbb{C}) \to H^k(X_{t_0}, \mathbb{C}). \tag{2.33}$$

This allows us to define a map

$$\Phi^p : S \to Grass(f^p, H^k(X_{t_0}, \mathbb{C})), \qquad \Phi^p(t) = g_t^*(F^p(X_t)). \tag{2.34}$$

Theorem 2.8.1. *(K. Kodaira) (see [C1]) The period map Φ^p is holomorphic.*

By the following theorem the above construction defines a variation of Hodge structure on S, cf. def. 2.4.1.

Theorem 2.8.2. *(Griffiths Transversality) (see [C1], [SCH] Theorem 2.13) Let $\pi : X \to S$ be a projective family and let $(\mathcal{H}^k, \nabla)$ denote the holomorphic vector bundle with the flat Gauss-Manin connection. Let $\sigma \in \Gamma(S, \mathcal{F}^p)$ be a smooth section of the holomorphic subbundle $\mathcal{F} \subseteq \mathbb{H}^k$, then for any $(1,0)$-vector field V on S,*

$$\nabla_V(\sigma) \in \Gamma(S, \mathcal{F}^{p-1}). \tag{2.35}$$

The classifying space D, consists of all decreasing filtrations F of H such that (H, F) is a Hodge structure, polarized by $\mathcal{Q}$ and

$$\dim_{\mathbb{C}} F^p = \sum_{r \geq p} h^{r,r-k} \tag{2.36}$$

To obtain a complex structure on D, one may regard it as an open subset of 'compact dual' $\check{D}$. Assume $G_{\mathbb{R}} := Aut(Q, \mathbb{R})$. It is a basic linear algebra that $G_{\mathbb{R}}$ acts transitively on D. If we define $\check{D}$ as those filtrations that only satisfy $\mathcal{Q}(F^p, F^{k-p+1}) = 0$, then naturally $D \hookrightarrow \check{D}$ as an open submanifold of the complex manifold $\check{D}$. The equation $h_F(u, v) := \mathcal{S}(C.u, \bar{v})$ defines a hermitian metric on D which is $G_{\mathbb{R}}$-invariant, called Hodge metric ,cf. [C1], [P1], [P3].

Families of algebraic manifolds usually have singular fibers. By Hironaka a suitable modification turns the ambient space into a manifold, and the sub-variety in question into a divisor with at most normal crossings. Thus, localizing the problem one can assume; the period map is defined on a product of puncture discs and discs. The mapping, by its very definition takes values in the quotient of a classifying space for Hodge structures, modulo a discrete group of automorphisms [C1], [SCH], see also 1.0.1.

Thus, we consider period maps

$$\Phi : T'^r \times T^{n-r} \to D/\Gamma \tag{2.37}$$

or their liftings

$$\tilde{\Phi} : H^r \times T^{n-r} \to D \tag{2.38}$$

where T is the disc, $T' = T \setminus 0$, and H is the upper half plane. By the monodromy theorem (Theorem 3.1.1), the monodromy transformations $M_j,\ j = 1, ..., r$ are quasi-unipotent, that is, there exist integers ν_j such that $(M_j^{\nu_j} - id)$ is nilpotent (see Theorem 3.1.1). Set $M_{j,u} = e^{N_j}\ :\ j = 1, ..., r,$

where N_j are nilpotent. For simplicity assume $r = n$. We then have, [C1],

$$\tilde{\Phi}(z_1, ..., z_j + 1, ..., z_r) = \exp(N_j)\tilde{\Phi}(z_1, ..., z_{j+1}, ..., z_r) \tag{2.39}$$

and the map $\Psi : H^r \to \check{D}$ defined by

$$\Psi(z_1, ..., z_r) := \exp(-\sum_{j=1}^{r} z_j N_j)\tilde{\Phi}(z_1, ..., z_r) \tag{2.40}$$

is the period map and is the lifting of a holomorphic map $\psi : T'^r \to D$;

$$\psi(t_1, ..., t_r) = \Psi(\frac{\log t_1}{2\pi i}, ..., \frac{\log t_r}{2\pi i}). \tag{2.41}$$

Example 2.8.3. [KU]
Take

$$H_0 = \mathbb{Q}^2 = \mathbb{Q}e_1 + \mathbb{Q}e_2.$$

Let W be the increasing filtration

$$W_{-2} = 0 \subset W_{-1} = H_0.$$

Let $(e_1, e_2)_{-1} = 1$, where the sub-index means the corresponding W-graded levels, and let

$$h^{-1,0} = h^{0,-1} = 1, \text{ and let } h^{p,q} = 0 \text{ for all other } (p, q).$$

For $\tau \in \mathbb{C}$ let

$$F(\tau)^1 = 0 \subset F(\tau)^0 = \mathbb{C}(\tau e_1 + e_2) \subset F(\tau)^{-1} = H_{0,\mathbb{C}}.$$

Then $D = H$ the upper half plane, where τ is corresponded to $F(\tau)$.

A map $\theta : \mathbb{C}^r \to \check{D}$ of the form

$$\theta(z) = \exp(\sum z_i N_i).F$$

where $F \in \check{D}$, and N_i a commuting set of nilpotent elements, such that there exists an $\alpha \in \mathbb{R}$ with $\theta(z) \in D$ for $Im(z_i) > \alpha$ is called a nilpotent orbit.

Theorem 2.8.4. *(Nilpotent Orbit Theorem - W. Schmid) ([SCH] Theorem 4.9 and 4.12)*

Let $\Phi : T'^r \times T^{n-r} \to D$ *be a period map, and let* $N_1, ..., N_r$ *be monodromy logarithms. Let*

$$\psi : T'^r \times T^{n-r} \to \check{D} \tag{2.42}$$

be as above; then

- *The map* ψ *extends holomorphically to* $T^r \times T^{n-r}$.
- *For each* $w \in T^{n-r}$, *the map* $\theta : \mathbb{C}^r \times T^{n-r} \to \check{D}$ *given by*

$$\theta(z, w) = \exp(\sum z_j N_j).\psi(0, w)$$

  is a nilpotent orbit. Moreover, for $w \in C$ *a compact subset, there always exists* $\alpha > 0$ *such that* $\theta(z, w) \in D$ *for* $Im(z_j) > \alpha$.

- *For any* G*-invariant distance on* D, *there exists positive constants* β, K *such that for* $Im(z_j) > \alpha$,

$$d(\Phi(z, w), \theta(z, w)) \leq K \sum_j (Im(z_j))^\beta e^{-2\pi Im(z_j)}. \tag{2.43}$$

  Moreover, the constants α, β, K *depend only on the choice of the metric* d *and the weight and Hodge numbers used to define* D. *They may be chosen uniformly for* w *in a compact subset.*

Given a period map $\Phi : T'^r \to D/\Gamma$, we will call the value

$$F_{\lim} := \psi(0) \in \check{D} \tag{2.44}$$

the limiting Hodge filtration. $F_{\lim}$ depends on the choice of coordinates on T'^r. Indeed a change of coordinates compatible with divisor structure, must be after relabeling if necessary, of the form $(\hat{t}_1, ..., \hat{t}_r) = (t_1 f_1(t), ..., t_r f_r(t))$ where f_j are holomorphic around $0 \in T^r$, $f_j(0) \neq 0$. Then after letting $t \to 0$, [C1],

$$\hat{F}_{\lim} = -\frac{1}{2\pi i}\sum_j \log(f_j(0))N_j.F_{\lim}. \tag{2.45}$$

Theorem 2.8.5. *Let* $\theta(z) = \exp(\sum_{j=1}^{r} z_j N_j).F$ *be a nilpotent orbit. Then,*

- *Every element in the cone*

$$\mathcal{C} := \{N = \sum_{j=1}^{r} \lambda_j.N_j; \lambda_j \in \mathbb{R}_{>0}\} \subset \mathfrak{g}$$

  defines the same weight filtration $W(\mathcal{C})$*.*

- *The pair* $(W(\mathcal{C}[-k], F)$ *defines a MHS polarized by every* $N \in \mathcal{C}$*.*

Example 2.8.6. [C1]

(1) A Hodge structure of weight 1 is just a complex structure on $H_{\mathbb{R}}$, that is a decomposition $H_{\mathbb{C}} = H^{1,0} \oplus \overline{H^{1,0}}$. The polarization Q is a non-degenerate alternating form and the polarization conditions reduce to:

$$Q(H^{1,0}, H^{1,0}) = 0; \quad iQ(u, \bar{u}) > 0, \text{ if } 0 \neq u \in H^{1,0}.$$

It follows that there exists a basis $\{w_1, ..., w_{2n}\}$ of $H_{\mathbb{C}}$ such that $\{w_1, ..., w_n\}$ is a basis of $H^{1,0}$, and in this basis the form Q is written in the form:

$$Q = \begin{pmatrix} 0 & -iI_n \\ iI_n & 0 \end{pmatrix}.$$

The subgroup of invertible transformations on H preserving such bilinear form is by definition the Symplectic group $Sp(n, \mathbb{C})$. On the other hand we can choose our basis so that $w_{n+i} = \bar{w}_i$ and consequently, the group of real transformations $G = Sp(n, \mathbb{R})$ acts transitively on our classifying space D. The isotropy group at some point $\Omega_0 \in D$, consists of real transformations in $Gl(H_{\mathbb{R}})$ which preserve a complex structure and hermitian form in the resulting n-dimensional complex vector space. Hence, $Stab(\Omega_0) = U(n)$ and

$$D = Sp(n, \mathbb{R})/U(n).$$

Geometrically, the weight 1 case corresponds to the Hodge structure on the cohomology $H^1(X, \mathbb{C})$ of a smooth algebraic curve X. Hence, the classifying space for Hodge structures of weight 1 is the Siegel upper half space.

(2) In the weight 2 case, $\dim(H) = 2h^{2,0} + h^{1,1}$, Q is a non-degenerate symmetric form defined over $\mathbb{R}$. So, we get the complex Lie group $G = O(2h^{2,0} + h^{1,1}, \mathbb{C})$. Given a reference polarized Hodge structure

$$H_{\mathbb{C}} = H_0^{2,0} \oplus H_0^{1,1} \oplus H_0^{0,2}; \qquad H_0^{0,2} = \overline{H_0^{2,0}}.$$

The real vector space decomposes as

$$H_{\mathbb{R}} = (H_0^{2,0} \oplus H_0^{1,1}) \cap H_{\mathbb{R}}) \oplus (H_0^{0,2} \cap H_{\mathbb{R}})$$

and the form Q is positive definite on first summand and negative definite on the second. Hence, $G = O(2h^{2,0}, h^{1,1})$. On the other hand the elements that fix the reference Hodge structure must preserve each summand of the second decomposition. Hence, we get $Stab = U(h^{2,0}) \times O(h^{1,1})$, and

$$D = O(2h^{2,0}, h^{1,1})/U(h^{2,0}) \times O(h^{1,1}).$$

2.9 Deligne Canonical extention

The nilpotent orbit theorem guarantees that the holomorphic bundle arising from a VHS can extend to the singular locus. Assume $(\mathcal{H}, \nabla)$ is an analytic vector bundle with an integrable connection on a complex manifold S, with $S \hookrightarrow \bar{S}$ as a Zariski open dense submanifold. Choose a multivalued flat frame $(A_1(z), ..., A_n(z))$ for $\mathcal{H}_p$ over a small neighborhood of $p \in \bar{S} - S$. Let $M = M_s.M_u$ be the Jordan decomposition of a monodromy around p, where $M_s = diag(d_k)$ is the semi-simple and M_u is unipotent upper-triangular. Set $N = \frac{-1}{2\pi i} \log M_u$. Let

$$s_i(A)(z) = \exp(N \log(z)).A_i(z)$$

$s_i(A)(z)$ is a single valued section of $\mathcal{H}$ over U. Furthermore, $\{s_i(A)\}_{1\leq i\leq n}$ provide a frame for a holomorphic extension $\bar{\mathcal{H}}$ of the bundle. For the same reason when S is higher dimensional, and $D = \bar{S} - S = \bigcup D_j$ is a normal crossing divisor only, and when all the monodromy transformations around D_j are quasi-unipotent, then the local system has a canonical extension to D, which is compatible with integral structures. It is characterized by the fact that, in a local basis around 0, the matrix of 1-forms defining the connection has logarithmic poles along $s_j = 0$, with nilpotent residues. The canonical extension depends on the choice of log branch. we summarize all of these in the following theorem;

Theorem 2.9.1. *([SA1] sec 22.b) If we are given* $(\mathcal{H}, \nabla)$ *on* S *with* $D = \bar{S} - S$ *normal crossing, there exists a unique meromorphic extension, called Deligne meromorphic extension, of the bundle* $\mathcal{H}$ *to a meromorphic bundle* $\tilde{\mathcal{H}}$ *(that is a free sheaf of* $\mathcal{O}_{\bar{S}}[\log(D)]$*-modules), equipped with a connection (by Riemann-Hilbert correspondence) such that the coefficients of any multi-valued horizontal section of* $\tilde{\mathcal{H}}$ *are multi-valued functions on* S *with moderate growth on* D*.*

2.10 The Lefschetz theorem on (1,1)-classes

Assume X is a compact Kaḧler manifold. We have the Hodge decomposition

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$$
$$H^n(X, \mathbb{R}) = (\bigoplus_{\substack{p+q=n \\ p\leq q}} H^{p,q}(X) \oplus H^{q,p}) \cap H^n(X, \mathbb{R}).$$

A natural question is whether we can characterize geometrically the classes in homology that are Poincarѐ dual to classes in one of these factors. For instance, consider a homology class $\Gamma \in H_{2p}(X, \mathbb{Z})$ that is a rational linear combination of fundamental classes of analytic sub-varieties of dim $= p$ of X and denote its Poincarѐ dual by η_Γ. If ψ is any differential form, then

$$\int_\Gamma \psi = \int_\Gamma \psi^{n-p,n-p}.$$

Thus, if η is the harmonic form on X representing the Poincarè dual η_Γ, and ψ is any harmonic form,

$$\int_\Gamma \psi \wedge \eta = \int_\Gamma \psi = \int_\Gamma \psi^{n-p,n-p} = \int_\Gamma \psi \wedge \eta^{p,p}$$

that is $\eta = \eta^{p,p}$, and we see that any cohomology class of degree $2p$ is of pure type (p,p). The famous Hodge conjecture asserts that the converse is also true: On $X \subset \mathbb{P}^N$ a rational cohomology class of type (p,p) is Poincarè dual to some rational divisor. The only case which this conjecture has been proved in general is the case $p = 1$.

Theorem 2.10.1. *(Lefschetz theorem on (1,1)-classes) ([G3] page 163) For $X \subset \mathbb{P}^N$ a sub-manifold, every cohomology class*

$$\gamma \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$$

is Poincarè dual of some rational divisor;

$$\gamma = \eta_D.$$

When X is a complex projective variety, then the cycle class map is

$$c : CH_k(X) \to H^{2n-2k}(X),$$

$$\Gamma \longmapsto \eta_\Gamma.$$

The cycle class is easily seen to be a Hodge class; that is to belong to $H^{2k}(X) \cap H^{k,k}(X)$. The famous Hodge conjecture asserts that $H^{2k}(X) \cap H^{k,k}(X)$ is equal to $im(c) \otimes \mathbb{Q}$. The Lefschetz theorem 2.10.1 is the only case of Hodge conjecture that is proved generally.

Chapter 3

Limit Mixed Hodge structure

As mentioned in section 2.8 a family of polarized algebraic manifolds given by a holomorphic projective map $f : X \to T$ gives rise to a family of Hodge structures F_t on $H^n(X_t, \mathbb{C})$, and a period map $\phi : T \to D/\Gamma$, where Γ is the monodromy group.

The idea of limit mixed Hodge structure is to replace all the Hodge structures F_t with a canonical one namely the limit Hodge structure, denoted F_{lim}, or F_∞. There exist three equivalent ways to define F_∞ in the Hodge theory literature. The first which is due to W. Schmid is based on the nilpotent orbit theorem. The second method belongs to J. Steenbrink that uses the Hironaka resolution of singularities, and we explain this in 7.4. The third method that is the content of the technical part of this text is explained in Chap. 6 and is developed in Chap. 8.

Through all of this chapter unless otherwise stated, we assume $f : X \to T$ is a family of projective algebraic manifolds, and T is the disc, and $X' = X \setminus X_0 \to T' = T \setminus 0$ is a C^∞-fibration. This assumption is specifically fixed through the whole sections 3.1 and 3.2. We propose to give a proof of 3.2.1.

3.1 Limit Hodge filtration

Suppose $f : X \to T$ is a family of projective varieties, where $f : X' \to T'$ is smooth. Then, $\pi_1(T')$ acts on the cohomology group of a general fiber $X_t = f^{-1}(t)$. Denote by M a monodromy i.e. the action of some generator of this group. Let $M = M_s M_u$ be the Jordan decomposition into semi-simple and unipotent part of monodromy. M_s is a diagonal matrix whose diagonal entries are eigenvalues of M. M_u is has only eigenvalue 1, and is upper triangular obtained from dividing each Jordan block by its appropriate eigenvalue.

Theorem 3.1.1. *(Monodromy theorem) (see [PS] Theorem 11.8, [SCH] Theorem 6.1) The eigenvalues of M are $m-th$ root of unity, for a suitable integer m, so that $M_s^m = 1$. Let l be the largest number of successive non-zero Hodge subspaces of $H^k(X_t, \mathbb{C})$. In other words, l is the largest integer that for some p, $H^{i,k-i}(X_t, \mathbb{C}) \neq 0$, if $p \leq i \leq p+l$. Then $(M_u - 1)^l = 0$ and hence $(M^m - 1)^l = 0$. Specifically, $(M^m - 1)^l = 0$, for $l \leq min(k, 2n-k)+1$.*

Assume $N : H \to H$ is a nilpotent transformation, $N^{k+1} = 0$. The following theorem is crucial in the study of existence of polarization for VHS.

Proposition 3.1.2. *([SCH] p. 247) There exists a unique filtration*

$$0 \subseteq W_0 \subseteq W_1 \subset ..... \subseteq W_{2n-1} \subseteq W_{2n} = H \tag{3.1}$$

such that $N(W_l) \subseteq W_{l-2}$, and such that

$$N^l : Gr_{k+l} W_* \to Gr_{k-l} W_* \tag{3.2}$$

is an isomorphism , for each $l \geq 0, Gr_l W_* = \frac{W_l}{W_{l-1}}$. If $l \geq k$ let $P_l \subseteq Gr(W_*)$ be the kernel of

$$N^{l-k+1} : Gr_l W_* \to Gr_{2k-l-2} W_* \tag{3.3}$$

and set $P_l = 0, l < k$, then one has the decomposition

$$Gr_l W_* = \bigoplus N^i(P_{l+2i}), \qquad i \geq max(k-l, 0). \tag{3.4}$$

If N is an infinitesimal isometry of a non-degenerate symmetric or skew symmetric form S on H. i.e if $S(Nu, v) + S(u, Nv) = 0$ for all $u, v \in H$, the filtration 3.1 becomes self dual, in the sense that each W_l is the orthogonal complement of W_{2k-l-1}. In this situation, the spaces GrW_* carry non-degenerate bilinear form S_l which are uniquely determined by the following requirements.
If $l \geq k$ and if $u, v \in W_l$ represent $\tilde{u}, \tilde{v} \in Gr_l W^*, S_l(\tilde{u}, \tilde{v}) = S(u, N^{l-k} v)$, if $l < k$, and N^{l-k} is to be an isometry from $Gr_{2k-l} W_*$ to $Gr_l W_*$. The

decomposition (3.4) then becomes orthogonal w.r.t S_l. Whenever S is symmetric and $k-l$ even, or S skew-symmetric and $k-l$ odd; S_l is symmetric. S_l is skew-symmetric in the remaining cases.
Finally, if ψ is a representation of a 3-dim algebra $\mathfrak{g}$ with generators X_+, X_-, Z on H, with $\psi(X_+) = N$. Each W_l coincides with the linear sum of the eigenspaces of $\psi(Z)$ which belong to eigenvalues less or equal than $l-k$ and P_l is the isomorphic image in GrW_* of the kernel of $\psi(X_-)^{l-k+1}$ on the $(l-k)$-subspace of $\psi(Z)$.

The proof is based on the following fact on representation theory of $\mathfrak{sl}_2$. Let $\mathfrak{g}$ a three dimensional complex Lie algebra with generators Z, X_+, X_- satisfying

$$[Z, X_+] = 2X_+, \qquad [Z, X_-] = 2X_-, \qquad [X_+, X_-] = Z. \tag{3.5}$$

Lemma 3.1.3. *([SCH] p. 247) Every finite dimensional representation of $\mathfrak{g}$ is fully reducible. Let $\psi : \mathfrak{g} \to End(H)$ be an irreducible representation of $\mathfrak{g}$ on an $(n+1)$-dimensional vector space H. Then, $\psi(Z)$ acts semi-simply, with eigenvalues $n, n-2, n-4, ..., -n$, each with multiplicity one. The l-eigenspace of $\psi(Z)$ maps onto the $(l+2)$-eigenspace By $\psi(X_+)$, except for $l = -n-2$. Similarly, for $l \neq n+2$, $\psi(X_-)$ maps the l-eigenspace onto $(l-2)$-eigenspace.*

Such a representation always exists. If ψ is a representation of $\mathfrak{g}$ on H with $\psi(X_-) = N$, the last statement of the lemma suggests how the filtration W_l should be constructed. An example of this case is the cohomology group of a compact Kähler manifold, where the Kähler operator L is adjoint to Λ and their commutator $B = [L, \Lambda]$ satisfies

$$[B, L] = 2L, \qquad [B, \Lambda] = -2\Lambda, \qquad [L, \Lambda] = B \tag{3.6}$$

and these three operators span a lie algebra isomorphic to $\mathfrak{g}$.

Let $\sigma : H \to T'$ be the universal covering of the punctured disk. $H^k_{\mathbb{C}}$ pulls back to a trivial bundle on H with fiber $H^k_{\mathbb{C}}$. For each $z \in H$, there is a natural identification between $H^k_{\mathbb{C}}$ and $H^k(X_t, \mathbb{C})$ with $t = \sigma(z)$. By transforming the Hodge filtration via this identification one obtains Hodge filtrations

$$H_{\mathbb{C}} = F^0_z \supseteq F^1_z \supseteq .... \supseteq F^{p-1}_z \supseteq ... \supseteq 0. \tag{3.7}$$

Let $N = \log(M_u)$. Then

$$z \longmapsto \exp(-zN)F^p_z \tag{3.8}$$

considered as a mapping of H to an appropriate Grassmann variety is invariant under the transformation $z \to z + m$. Because the Lefschetz decomposition is always compatible with action of monodromy, the nilpotent orbit theorem guarantees the existence of the limit

$$F^{\infty}_z = \lim_{Im(z)\to\infty} exp(-zN)F^p_z \tag{3.9}$$

uniformly in $Re(z)$. The resulting filtration

$$H_{\mathbb{C}} = F^0_{\infty} \supseteq F^1_{\infty} \supseteq .... \supseteq F^{p-1}_{\infty} \supseteq ... \supseteq 0 \tag{3.10}$$

is called the limit Hodge filtration. It should be pointed out that the filtration $F^{\bullet}_{\infty}$ depends on the choice of the coordinate t on the disk T. Passing from one local coordinate to another has the effect of replacing the filtration F^p_{∞} by $\exp(\lambda N)F^p_{\infty}$. However, the filtrations induced on the kernel and cokernel of N and quotients of weight filtrations are canonical. If $L : H^k_{\mathbb{C}} \to H^{k+2}_{\mathbb{C}}$ is the Kaḧler operator, then it commutes with M and hence, also with N. Since it maps F^p_z to F^p_{z+1} for any $z \in U$, it raises the index of weight filtration by 2 and the index of $F^{\bullet}_{\infty}$ by one.

Theorem 3.1.4. *([SCH] p. 255) The two filtrations* $\{W_L\}$ *and* $\{F^p_{\infty}\}$ *determine a MHS on* $H^k_{\mathbb{C}}$ *w.r.t which* N *is a morphism of type* $(-1,-1)$*, and the Kaḧler operator* L *is a morphism of type* $(1,1)$*. In particular the MHS of* $H^k_{\mathbb{C}}$ *restricts to the primitive part* $P^k_{\mathbb{C}} \subseteq H^k_{\mathbb{C}}$*. The induced Hodge structure of weight* l *on* $Gr_l(P^k_{\mathbb{C}} \cap W_*)$ *further restricts to a Hodge structure on*

$P_l \subseteq P^k_{\mathbb{C}} \cap W_*$ which are polarized with respect to the non-degenerate bilinear form S_l on P_l.

The next theorem shows the compatibility of the limiting MHS with that induced on the primitive components.

Theorem 3.1.5. *([SCH] p. 256) Let $P^k_{\mathbb{C}} \subset H^k_{\mathbb{C}}$, be the primitive part and suppose that, the classifying space for the Hodge structures on $P^k_{\mathbb{C}}$ happens to be hermitian symmetric (i.e. a connected hermitian manifold with a symmetry s_P fixing a point P, that is $s_P^2 = 1$). Then, for every $z \in H$, the upper half plane, with sufficiently large imaginary part, the two filtrations $\{F^p_z \cap P^k_{\mathbb{C}}\}$, and $\{W_l \cap P^k_{\mathbb{C}}\}$, determine a mixed Hodge structure on $P^k_{\mathbb{C}}$. The resulting Hodge structure of pure weight l on $Gr_l(W \cap P^k_{\mathbb{C}})$, viewed as a function of z, has a limit as $Im(z) \to \infty$. The limit coincides with the Hodge structure of weight l induced by the filtration $\{F^p_\infty \cap P^k_{\mathbb{C}}\}$.*

3.2 Polarization for Projective family

In this part we use the theorems of W. Schmid mentioned in the previous section in order to prove the Riemann-Hodge bilinear relations in the variation of hodge structure associated to a projective family, [JS1], [JS2].

Consider the projective family X_t defined by a hyper-surface germ. Let L be the cohomology class of a hyperplane section of X_t, $t \neq 0$. Set $X_\infty := X \times_{T'} H$ where H is the upper half plane. Equip $H^k(X_\infty, \mathbb{C})$ with the limit mixed Hodge structure (F_∞, W_L) as in the previous section. We consider L as an element of $H^2(X_\infty)$ by means of the natural map $\psi_t : H^2(X_t) \to H^2(X_\infty)$. On the primitive subspaces $P^k(X_\infty)$ consider the bilinear form

$$Q(x,y) = \int_{X_t} (-1)^{k(k-1)/2} i^{n-k} \psi_t^{-1}(x \wedge y). \tag{3.11}$$

Then Q does not depend on the choice of t. Because L is an M-invariant class, $P^k(X_\infty) \subset H^k(X_\infty)$ also carries a mixed Hodge structure, and for all $r \geq 0$ one has

$$Gr^W_{k+r} P^k(X_\infty) \cong Gr^W_{k-r} P^k(X_\infty). \tag{3.12}$$

Denote

$$P_{k,r}(X_\infty) = \ker(N^{r+1} : Gr^W_{k+r} P^k(X_\infty) \to Gr^W_{k-r} P^k(X_\infty)). \quad (3.13)$$

Then $P_{k,r}(X_\infty)$ carries a Hodge structure of weight $k + r$. Let

$$P_{k,r}(X_\infty) = \bigoplus_{a+b=k+r} P^{a,b}_{k,r}(X_\infty) \quad (3.14)$$

be its Hodge decomposition. Denote Q_r the bilinear form on $P_{k,r}(X_\infty)$ defined by $Q_r(x, y) = Q(\tilde{x}, N^r \tilde{y})$, where $x, y \in P_{k,r}(X_\infty)$ and $\tilde{x}, \tilde{y}$ are elements of $W_{k+r}P^k(X_\infty)$ whose classes mod W_{k+r-1} are x, y respectively. The fact that N is an infinitesimal isometry implies that Q_r is well-defined.

Theorem 3.2.1. *(J. Steenbrink-W. Schmid)(see [JS2]) Assume $f : X \to S$ is a family of projective manifolds. Equip the variation of Hodge structure $H^k(X_\infty, \mathbb{C})$ with the W. Schmid limit MHS (F_∞, W_L). Let $P_k, P_{k,r}$ be the primitive subspaces as defined above. Then the following holds,*

- $Q_r(x, y) = 0$ *if* $x \in P^{a,b}_{k,r}(X_\infty), y \in P^{c,d}_{k,r}(X_\infty)$ *and* $(a, b) \neq (c, d)$
- $i^{a-b} Q_r(x, \bar{x}) > 0$ *if* $x \in P^{a,b}_{k,r}(X_\infty), x \neq 0$.

Proof. This follows from Theorems 3.1.4, 3.1.5 and prop. 3.1.2. □

3.3 Deligne-Hodge decomposition of MHS's

In [P1], G. Pearlstein develops the ideas of W. Schmid, [SCH] into VMHS's, generalizing Schmid nilpotent orbit theorem and classifying space for MHS's.

Definition 3.3.1. A graded polarization of a mixed Hodge structure (F, W) consists of a choice of polarization S_k for each non-trivial layer Gr^W_k.

Example 3.3.2. [P1]

Let S be a finite number of points on a compact Riemann surface M. Let $\Omega^1_M(\log(S))$ be the space of meromorphic 1-forms on M which have at worst simple poles along S. Then the mixed Hodge structure (F, W) attached to $H^1(M - S, \mathbb{C})$ is given by the following

$$W_0 = 0,\ W_1 = H^1(M, \mathbb{C}),\ W_2 = H^1(M - S, \mathbb{C})$$
$$F^2 = 0,\ F^1 = \Omega^1_M(\log(S)),\ F^0 = H^1(M - S, \mathbb{C}).$$

The two bilinear forms

$$S_2(\alpha, \beta) = 4\pi^2 \textstyle\sum_{p \in S} res_p(\alpha) res_p(\beta), \qquad S_1(\alpha, \beta) = \int_M \alpha \wedge \beta$$

defined on Gr^W_2 , and Gr^W_1 respectively, provide a graded polarization of $H^1(M - S, \mathbb{C})$.

Definition 3.3.3. (P. Deligne) A bi-grading of a mixed Hodge structure (F, W), is a direct sum decomposition $V = \oplus_{p,q} I^{p,q}$ of the underlying complex vector space which has the following two properties,

$$F^p = \bigoplus_{r \geq p, s} I^{r,s}$$
$$W_k = \bigoplus_{r+s \leq k} I^{r,s}.$$

Lemma 3.3.4. *(P. Deligne) Let (F, W) be a mixed Hodge structure. Then there exists a unique bi-grading $\{I^{p,q}\}$ of (F, W) with the following additional properties.*

$$I^{p,q} = \bar{I}^{q,p} \qquad \mod \bigoplus_{\substack{r<p \\ s<q}} I^{r,s}. \tag{3.15}$$

The choice of a MHS (F, W) on a space $V = V_{\mathbb{Q}} \otimes \mathbb{C}$ induces a mixed Hodge structure on $\mathfrak{gl}(V)$ via the bi-grading.

$$\mathfrak{gl}(V)^{r,s} = \{\alpha \in gl(V) | \alpha : I^{p,q} \to I^{p+r,q+s}, \quad \forall p, q\} \tag{3.16}$$

Example 3.3.5. [P1] In the case of finitely punctured Riemann surface $M - S$ considered in Ex. 3.3.2, the Deligne-Hodge decomposition (F, W) is given by the following subspaces of $H^1(M - S, \mathbb{C})$:

$$I^{1,1} = F^1 \cap \overline{F^1}, \quad I^{1,0} = H^{1,0}(M), \quad I^{0,1} = H^{0,1}(M).$$

Definition 3.3.6. A mixed Hodge structure is said to split over $\mathbb{R}$, if it admits a bigrading $\{I^{p,q}\}$ such that

$$I^{p,q} = \overline{I^{q,p}}. \tag{3.17}$$

In this case

$$V_{\mathbb{C}} = \bigoplus_k \bigoplus_{p+q=k} I^{p,q} \tag{3.18}$$

is a decomposition of $V_{\mathbb{C}}$ into direct sum of HS's.

Example 3.3.7. [C1] The basic example of mixed Hodge structure split over $\mathbb{R}$ is the Hodge decomposition on the cohomology of a compact Kahler manifold X. Let

$$V_{\mathbb{Q}} = H^*(X, \mathbb{Q}) = \bigoplus_{k=0}^{2n} H^k(X, \mathbb{Q})$$

and set

$$I^{p,q} = H^{n-p,n-q}(X).$$

Thus

$$W_l = \bigoplus_{d \geq 2n-l} H^d(X, \mathbb{C}), \quad F^p = \bigoplus_s \bigoplus_{r \leq n-p} H^{r,s}(X).$$

With this choice of indexing the operators L_ω where ω is a Kahler class, are $(-1,-1)$-morphisms of MHS. Using Hard Lefschetz Theorem, the Riemann-Hodge bilinear relations state that mixed Hodge structure is polarized of weight n, by the rational bilinear form Q defined by

$$Q(\alpha, \beta) = (-1)^{r(r+1)/2} \int_X \alpha \wedge \beta; \quad \alpha \in H^r, \beta \in H^{n-r}.$$

3.4 Mixed Hodge metric

Mixed Hodge Metric is a theorem of A. Kaplan, explaining that the $I^{p,q}$-decomposition of a polarized MHS is split over $\mathbb{R}$. If (F, W) is a mixed Hodge structure, then there exists a unique functorial bi-grading,

$$V = \bigoplus_{p,q} I^{p,q} \tag{3.19}$$

of the underlying vector-space $V_{\mathbb{C}}$, such that

- $F^p = \bigoplus_{a \geq p} I^{a,b}$
- $W_k = \bigoplus_{a+b \leq k} I^{a,b}$
- $\overline{I^{p,q}} = I^{q,p} \mod \bigoplus_{r<q, s<p} I^{r,s}$.

Proposition 3.4.1. *(A. Kaplan) [K] Let $(F, W, \mathcal{S})$ be a graded polarized mixed Hodge structure, with underlying vector space $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$. Then, there exists a unique positive definite hermitian inner product h on V with following two properties:*

(a) The bi-grading $V = \oplus_{p,q} I^{p,q}$ is orthogonal with respect to h.

(b) If u, v are elements of $I^{p,q}$, then $h(u, v) = i^{p-q}\mathcal{S}([u], [\bar{v}])$.

The associated mixed Hodge metric h is the unique hermitian inner product on $V_{\mathbb{C}}$, which makes the associated bi-grading (3.19) orthogonal and satisfies

$$h(u, v) = i^{p-q}\langle Gr^W_{p+q} u, Gr^W_{p+q} \bar{v}\rangle_{p+q}, \qquad u, v \in I^{p,q}. \tag{3.20}$$

A rapid consequence of mixed Hodge metric theorem is the uniqueness of the polarization when exists for a mixed Hodge structure. The concepts of the period map and period domain of pure polarized Hodge structure discussed in 1.0.1 and 2.8, can be generalized for mixed Hodge structures.

This naturally raises the question of possibility to generalize the nilpotent orbit theorem of W. Schmid to mixed Hodge structure. Let $\mathcal{M}$ be this generalized period domain. $\mathcal{M}$ can be regarded as a subspace of a flag variety. The afore-mentioned bilinear form defines a metric on the period domain of the mixed Hodge structure, i.e. a hermitian form on its tangent space. This means the trivial bundle $V \times \mathcal{M} \to \mathcal{M}$ inherits a hermitian structure, and this structure is via this identification. This metric is $G_{\mathbb{R}} = Aut(Q, \mathbb{R})$-invariant, cf. [P1].

3.5 $\mathfrak{sl}_2$-orbit Theorem for VHS's

Let (W, F_0) be a MHS on $H_{\mathbb{C}}$, split over $\mathbb{R}$ and polarized by $F_0^{-1}\mathfrak{g} \cap \mathfrak{gl}(H_{\mathbb{Q}})$. Since $W = W(N)[-k]$, the subspaces

$$H_l = \sum_{p+q=k+l} I^{p,q}(W, F_0), \qquad -k \leq l \leq k$$

constitute a bi-grading defined over $\mathbb{R}$. Let Y be the linear map of multiplication by l on H_l. Since $NH_l = H_{l-2}$,

$$[Y, N] = -2N.$$

Then there would exist an $N_+ \in \mathfrak{g} = \mathfrak{gl}(H)$, such that $[Y, N_+] = 2N_+,\ [N_+, N] = Y$. One may define a homomorphism $\rho : \mathfrak{sl}_2 \to \mathfrak{gl}(H)$, such that

$$\rho(X_-) = N,\ \rho(X_+) = N_+,\ \rho(Z) = Y.$$

Such a homomorphism is called Hodge at $F \in D$, if it is a morphism of Hodge structures when $\mathfrak{g}$ is equipped with filtration F.

Theorem 3.5.1. *($\mathfrak{sl}_2$-orbit Theorem - W. Schmid) ([SCH] Theorem 5.3)Let $z \to \exp(z.N).F$ be a nilpotent orbit. Then there exists,*

- *A filtration $F_{\sqrt{-1}} := \exp(iN).F_0$ lies in D.*
- *A homomorphism $\rho : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}$, Hodge at $F_{\sqrt{-1}}$.*
- $N = \rho(X_-)$

- *A real analytic $G_{\mathbb{R}}$-valued function $g(y)$, such that;*
- *For $y >> 0$, $\exp(iy.N).F = g(y)\exp(iyN).F_0$, where $F_0 = \exp(-iN).F_{\sqrt{-1}}$.*
- *Both $g(y)$ and $g(y)^{-1}$ have convergent power series expansion at $y = \infty$ of the form $1 + \sum A_n y^{-n}$ with*

$$A_n \in W_{n-1}\mathfrak{g} \cap \ker(adN)^{n+1}. \tag{3.21}$$

The $\mathfrak{sl}_2$-orbit theorem expresses some additional data that can be taken out from a nilpotent orbit. The choice of the function g characterizes a distinguished orbit of polarized Hodge structures which are real split, i.e can be written as a direct sum of pure Hodge structures (this is by the representation ρ). The concepts of this theorem should be understood as a matter of representation theory, and can be applied to general period maps. In section 8.7 we apply some ideas relevant to this theorem to the mixed Hodge structure of isolated hypersurface singularities.

3.6 Variation of Polarized Mixed Hodge Structures

In this section we try to generalize some of the previous concepts to Variation of MHS's.

Definition 3.6.1. [P1] A variation of graded polarized mixed Hodge structure (VGPMHS) $\mathcal{H} \to S$ consists of a $\mathbb{Q}$-local system $\mathcal{H}_{\mathbb{Q}}$ over S equipped with

- A rational increasing weight filtration

$$0 \subset ...\mathcal{W}_k \subset \mathcal{W}_{k+1} \subset ... \subset \mathcal{H}_{\mathbb{Q}}$$

  by sub-local systems.
- A decreasing Hodge filtration

$$0 \subset ...\mathcal{F}^p \subset \mathcal{F}^{p-1} \subset ... \subset \mathcal{H} = \mathcal{H}_{\mathbb{C}} \otimes \mathcal{O}_S$$

by holomorphic sub-bundles.

- Relative to the Gauss-Manin connection of $\mathcal{V}$:

$$\nabla \mathcal{F}^p \subset \Omega_S^1 \otimes \mathcal{F}^{p-1}. \tag{3.22}$$

- A collection of rational non-degenerate bilinear forms

$$\mathcal{S}_k : Gr_k^{\mathcal{W}}(\mathcal{H}_{\mathbb{Q}}) \otimes Gr_k^{\mathcal{W}}(\mathcal{H}_{\mathbb{Q}}) \to \mathbb{Q} \tag{3.23}$$

such that,

- For each integer k the triple $(Gr_k^{\mathcal{W}}(\mathcal{H}_{\mathbb{Q}}), \mathcal{F}Gr_k^{\mathcal{W}}(\mathcal{H}_{\mathbb{Q}}), \mathcal{S}_k)$ define a variation of pure polarized Hodge structure of weight k.

The data of such variation may be effectively encoded in its monodromy representation;

$$\rho : \pi_1(S, s_0) \to Gl(\mathcal{H}_{s_0}), \qquad Image(\rho) = \Gamma \tag{3.24}$$

and its period map

$$\phi : S \to D/\Gamma. \tag{3.25}$$

To obtain such a reformulation, we may assume S to be simply-connected. Trivialization of $\mathcal{H}$ relative to a fixed reference fiber $H := \mathcal{H}_{s_0}$ via parallel transform will determine the following data

- A rational structure $H_{\mathbb{Q}}$ on H.
- A rational weight filtration W on H.
- A variable Hodge filtration $F(s)$ on H.

- A collection of rational non-degenerate bilinear forms

$$\mathcal{S}_k : Gr_k^W \otimes Gr_k^W \to \mathbb{C} \tag{3.26}$$

of alternating parity $(-1)^k$.

subject to the conditions

- The Hodge filtration $F(s)$ is holomorphic and horizontal i.e.

$$\frac{\partial}{\partial(\bar{s}_j)} F^p(s) \subset F^p(s), \qquad \frac{\partial}{\partial(s_j)} F^p(s) \subset F^{p-1}(s) \tag{3.27}$$

relative to any choice of holomorphic coordinates.

- Each pair $(F(s), W)$ is a mixed Hodge structure, graded polarized by the bilinear form $\mathcal{S}$.

Conversely, the above properties determine a VGPMHS. To extract a classifying space, note that by the properties, the graded Hodge numbers $h^{p,q}$ of H are constant. Therefore, the classifying space consists of all decreasing filtrations F of H such that (F, W) is a MHS, graded polarized by $\mathcal{S}$ and

$$\dim_{\mathbb{C}} F^p Gr_k^W = \sum_{r \geq p} h^{r,r-k}. \tag{3.28}$$

To obtain a complex structure on D, one may regard it as an open subset of 'compact dual' $\check{D}$. More precisely one starts with a flag variety $\check{\mathfrak{F}}$ consisting of all decreasing filtrations F of H such that

$$\dim F^p = f^p, \qquad f^p = \sum_{r \geq p, s} h^{r,s}. \tag{3.29}$$

To take account the filtration W, define D as the sub-manifold of $\mathcal{F}$ consisting of all filtration F with additional property

$$\dim F^p Gr_k^W = \sum_{r \geq p} h^{r,k-r}. \tag{3.30}$$

The equation $h_F(u,v) := \mathcal{S}(C.u, \bar{v})$ defines a hermitian metric on D called Hodge metric, In our case Mixed Hodge metric, [P1].

Given a VGPMHS; $\mathcal{V} \to S$ we may apply Deligne decomposition pointwise, to get a C^∞-decomposition

$$\mathcal{V} = \bigoplus_{p,q} \mathcal{I}^{p,q}. \tag{3.31}$$

Example 3.6.2. [KU] Take

$$H_0 = \mathbb{Q}^4 = \mathbb{Q}e_1 + ... + \mathbb{Q}e_4.$$

Let

$$W_{-3} = 0 \subset W_{-2} = \mathbb{R}e_1 \subset W_{-1} = W_{-2} + \mathbb{R}e_2 + \mathbb{R}e_3 \subset W_0 = H_{0,\mathbb{R}}$$

and

$$(e_4, e_4)_0 = 1, (e_1, e_1)_{-2} = 1, (e_3, e_2)_{-1} = 1$$

where the sub-index correspond to W-grading. Choose,

$$h^{0,0} = h^{0,-1} = h^{-1,0} = h^{-1,-1} = 1, h^{p,q} = 0 \text{ for all other } p, q \ .$$

as Hodge numbers. Define the F-filtration is

$$F^{-1} = H_{0,\mathbb{C}} \supset F^0 = \mathbb{C}(z_1e_1 + z_2e_2 + e_3) + \mathbb{C}(z_2e_1 + z_3e_2 + e_4) \supset 0.$$

Then we have isomorphism of complex analytic manifolds

$$D = H \times \mathbb{C}^3, \qquad D(gr^W) = D(gr^W_{-1}) = H$$

where H is the upper half plane.

3.7 Hodge sub-bundles and complex structure

We begin by the definition of a Higgs bundle,

Definition 3.7.1. A Higgs bundle $(E, \bar{\partial} + \theta)$ consists of a holomorphic vector bundle $(E, \bar{\partial})$ endowed with an endomorphism valued 1-form

$$\theta : \mathcal{E}^0(E) \to \mathcal{E}^{0,1}(E) \tag{3.32}$$

which is both holomorphic and symmetric ($\bar{\partial}(\theta) = 0,\ \theta \wedge \theta = 0$).

Let $\mathcal{V}$ be a variation of pure polarized Hodge structure arising from the cohomology of a family of smooth non-singular projective varieties $f : Y \to X$. Then by virtue of the C^∞-decomposition

$$\mathcal{V} = \oplus_{p+q=k} \mathcal{H}^{p,q}$$

underlying smooth vector bundle $E = \mathcal{V} \otimes \mathcal{E}^0_X$ of $\mathcal{V}$ inherits an integrable complex structure $\bar{\partial}$ via the isomorphism

$$\mathcal{H}^{p,q} = \frac{\mathcal{F}^p}{\mathcal{F}^{p+1}}$$

and the holomorphic structure of $\mathcal{F}^p$. Likewise, the Kodaira-Spencer map

$$\kappa_p : T_p(X) \to H^1(Y_p, \Theta(Y_p))$$

defines a symmetric endomorphism valued 1-form θ on E via the rule $\theta(\xi)(\sigma) = \kappa_p(\xi) \cup \sigma$. To prove that $(E, \bar{\partial} + \theta)$ is indeed a Higgs bundle, observe that we may write the Gauss-Manin connection as

$$\nabla = \tau + \bar{\partial} + \partial + \theta,$$

relative to a pair of differential operators,

$$\bar{\partial} : \mathcal{E}^0(E) \to \partial^{0,1}(E), \qquad \partial : \mathcal{E}^0(E) \to \partial^{1,0}(E),$$

preserving the Hodge decomposition, and a pair of tensor fields

$$\tau : \mathcal{H}^{p,q} \to \mathcal{E}^{0,1} \otimes \mathcal{H}^{p+1,q-1}$$
$$\theta : \mathcal{H}^{p,q} \to \mathcal{E}^{1,0} \otimes \mathcal{H}^{p-1,q+1}.$$

Expanding out integrability condition $\nabla^2 = 0$ it follows that

$$\bar{\partial}^2 = 0, \qquad \bar{\partial}\theta = 0, \qquad \theta \wedge \theta = 0.$$

This proves $(E, \bar{\partial} + \theta)$ is a Higgs bundle. Moreover, given any element $\lambda \in \mathbb{C}^*$, the map

$$f : \mathcal{V} \to \mathcal{V} \qquad f|_{\mathcal{H}^{p,q}} = \lambda^p$$

defines a bundle isomorphism

$$(E, \bar{\partial} + \theta) = (E, \bar{\partial} + \lambda.\theta).$$

Consequently, the isomorphism class of a Higgs bundle is the fixed point of the $\mathbb{C}^*$-action

$$\lambda : (E, \bar{\partial} + \theta) \to (E, \bar{\partial} + \lambda.\theta)$$

cf. [P1]. This shows the following fact;

Proposition 3.7.2. *[P1] Every complex variation of pure Hodge structure carries a natural Higgs bundle structure $\bar{\partial}+\theta$, invariant under the $\mathbb{C}^*$ action.*

By a complex variation we mean to forget about the real structure. More generally we have the following

Proposition 3.7.3. *[P1] A Higgs bundle defined over a compact complex manifold X admits a decomposition into a system of Hodge bundles if and only if*

$$(E, \bar{\partial} + \theta) = (E, \bar{\partial} + \lambda\theta)$$

for each element $\lambda \in \mathbb{C}^*$.

3.8 Example

Consider the PVHS over T' the punctured disc in $\mathbb{C}$, of weight 1 on H of dim $= 2n$. Denote by Q the polarization form, and let

$$\Phi : T' \to D/Sp(H_{\mathbb{Z}}, Q)$$

be the corresponding period map. The monodromy logarithm satisfies, $N^2 = 0$, and its weight filtration is

$$W_{-1} = Im(N),\ W_0 = \ker(N).$$

Let $F_{\lim}$, be the limiting Hodge filtration. We have a bi-grading on $H_{\mathbb{C}}$;

$$H_{\mathbb{C}} = I^{0,0} \oplus I^{0,1} \oplus I^{1,0} \oplus I^{1,1}$$

defined by $(W(N)[-1], F_{\lim})$. The nilpotent transformation N maps $I^{1,1}$ isomorphically onto $I^{0,0}$, and vanishes on the other summands. The form $Q(., N.)$ polarizes the Hodge structure on gr_2^W, and hence defines a positive definite hermitian form on $I^{1,1}$. Similarly, Q polarizes the Hodge decomposition on $I^{0,1} \oplus I^{1,0}$. Thus, there is a basis such that

$$N = \begin{pmatrix} 0 & 0 & I_v & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & 0 & -I_v & 0 \\ 0 & 0 & 0 & -I_{n-v} \\ I_v & 0 & 0 & 0 \\ 0 & I_{n-v} & 0 & 0 \end{pmatrix}$$

where $v = \dim I^{1,1}$. The limit Hodge filtration on $I^{1,0} \oplus I^{0,1}$ is given by the subspaces spanned by the columns of $2n \times n$ matrix

$$F_{\lim} = \begin{pmatrix} 0 & 0 \\ 0 & iI_{n-v} \\ 0 & 0 \\ 0 & I_{n-v} \end{pmatrix}.$$

The period map can be written as

$$\Phi(t) = \exp(\frac{\log t}{2\pi i}.N).\exp(\Gamma(t)).F_{\text{lim}}$$

$$\Phi(t) = \begin{pmatrix} W(t) \\ I_n \end{pmatrix}$$

where

$$W(t) = \begin{pmatrix} \frac{\log t}{2\pi i} I_v + A_{11}(t) & A_{12}(t) \\ A_{12}^T(t) & A_{22}(t) \end{pmatrix}$$

with $A_{11}(T), A_{22}(t)$ symmetric, cf. [C1].

Chapter 4

Isolated hypersurface singularities

A series of examples for variation of mixed Hodge structure is given by families of analytic manifolds given by a germ of holomorphic function on $(\mathbb{C}^{n+1}, 0)$. An interesting case where the associated VMHS is polarized is when this germ has a unique isolated singular point at 0. Such a germ far from the singular fiber defines a C^∞-fibration, according to a theorem of Milnor. By a well-known fact namely Finite Determinacy Theorem, ([SCHU] page 12), there always exists a coordinate change such that our germ becomes a polynomial with degree as large as we like. We use this fact through the text, without mentioning it.

4.1 Milnor fibration

Consider the isolated singularity germ,

$$f : \mathbb{C}^{n+1} \to \mathbb{C}, \qquad f(0) = 0. \tag{4.1}$$

We take a sufficiently small ball $B(0, \epsilon)$ such that the spheres $\partial B(0, \epsilon'),\ \epsilon' < \epsilon$ are all transverse to $f^{-1}(t),\ t < \epsilon$. Then we put $X = B(0, \epsilon) \cap f^{-1}(\triangle_\eta = T)$, where η are taken sufficiently small. The fiber X_0 has an isolated singularity at 0. The restriction

$$f : X' = X \setminus X_0 \to T' = T \setminus 0 \tag{4.2}$$

is a locally trivial fibration namely Milnor fibration. The Milnor fibers $f^{-1}(t)$ have the homotopy type of the wedge of μ spheres, where

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_0, ..., x_n\}}{J(f)}, \qquad J(f) = (\partial_0 f, ..., \partial_n f).$$

The singular fiber is homeomorphic to the cone over the manifold $L := S \cap X_0 = \partial X_0$, the link of the singularity, which is homotopy equivalent to the complex manifold $X_0 - 0$. By the triviality of the Milnor fibration near the boundary, one can identify L with the boundary ∂X for $t \in \Delta^*$. From the homology sequence

$$0 \to H_n(L) \to H_n(X_t) \to H_n(X_t, \partial X_t) \to H_{n-1}(L) \to 0 \tag{4.3}$$

of the pair $(X_t, \partial X_t)$, we see that L has non-trivial homology only in degrees $n-1$, n and that these are put in duality by the intersection product. We use the notation $X_\infty := X_t \times_{T'} H$, namely the canonical fiber, where $H := \{z \in \mathbb{C} | \ Im(z) > 0\}$ is the upper half plane.

We construct the cohomology bundle

$$\mathcal{H} := \bigcup_{t \in T'} H^n(X_t, \mathbb{C}) \tag{4.4}$$

which is a complex vector bundle admitting an integrable connection

$$\nabla_{d/dt} = \partial_t : \mathcal{H} \to \mathcal{H} \tag{4.5}$$

which is holomorphic. One has $\mathcal{H} \cong (\mathcal{O}_{T'})^\mu$ and hence, $(i_*\mathcal{H})_0 \cong (i_*\mathcal{O}_{T'})^\mu$. The connection $\nabla_{d/dt}$ induces a differential operator

$$\partial_t : (i_*\mathcal{H})_0 \to (i_*\mathcal{H})_0. \tag{4.6}$$

P. Deligne introduced a $\mathbb{C}\{t\}[t^{-1}]$-vector space $\mathcal{G} \subset (i_*\mathcal{H})_0$ of dimension μ, which is ∂_t invariant and is a regular singular $\mathbb{C}\{t\}[t^{-1}]$-module as follows.

For $\omega \in \Omega_X^{n+1}$, the Leray residue provides a holomorphic section

$$s[\omega](t) = [\frac{\omega}{df}|_{X_t}] = res \frac{\omega}{(f-t)} \in \mathcal{H}_t \tag{4.7}$$

of $\mathcal{H}$. When no ambiguity arises we omit the restriction symbol and simply write ω/df through the text. Then, one can define

$$C^{\alpha} := \ker(t\partial_t - \alpha)^r \subset \mathcal{G}, \qquad r >> 0. \tag{4.8}$$

and

$$H^n(X_\infty, \mathbb{C})_\lambda = \ker(M_s - \lambda)^r \subset H^n(X_\infty, \mathbb{C}) \tag{4.9}$$

to be the generalized eigne-spaces, where M is the monodromy. Let M_u be the unipotent part of monodromy and $N = \log(M_u)$. Then for $A \in H^n(X_\infty, \mathbb{C})_\lambda$ and $\alpha \in \mathbb{Q}$ with $e^{-2\pi i\alpha} = \lambda$, the section

$$s(A, \alpha)(t) = t^{\alpha} \exp(\log t.\frac{-N}{2\pi i}).A(t) \tag{4.10}$$

is well-defined, namely elementary sections of $\mathcal{H}$. In this way we build a map $\psi_\alpha : H^n(X_\infty, \mathbb{C}) \to (i_*\mathcal{H})_0$, with $\psi_\alpha(A) := (i_* s(A, \alpha))_0$. It gives an isomorphism

$$\psi_\alpha : H^n(X_\infty, \mathbb{C})_\lambda \to C^{\alpha} \subset \mathcal{G} \tag{4.11}$$

where C^α is the image of $H^n(X_\infty, \mathbb{C})_\lambda$. The map ψ_α fulfills the properties; $(t\partial_t - \alpha) \circ \psi_\alpha = \psi_\alpha \circ (-N/2\pi i)$, $t \circ \psi_\alpha = \psi_{\alpha+1}$. They build up the isomorphism

$$\psi = \bigoplus_{-1<\alpha\leq 0} \psi_\alpha : H^n(X_\infty, \mathbb{C}) = \bigoplus_{-1<\alpha\leq 0} H_{\mathbb{C}}^{e^{-2\pi i\alpha}} \to \bigoplus_{-1<\alpha\leq 0} C^{\alpha} \tag{4.12}$$

such that the monodromy M on $H^n(X_\infty, \mathbb{C})$ corresponds to $exp(-2\pi i.t\partial_t)$ on $\bigoplus_{-1<\alpha\leq 0} C^\alpha$. It holds that $t : C^\alpha \to C^{\alpha+1}$ is always bijective and $\partial_t : C^\alpha \to C^{\alpha-1}$ is bijective for $\alpha \neq 0$. Then we have

$$\mathcal{G} = \bigoplus_{-1<\alpha\leq 0} \mathbb{C}\{t\}[t^{-1}]C^{\alpha}. \tag{4.13}$$

Theorem 4.1.1. *([SC2] cor. 3.8) The connection $\partial_t : \mathcal{G} \to \mathcal{G}$ is regular* *singular at* 0*, i.e. has a pole of order at most 1 at* 0*.*

4.2 MHS on Cohomology of Milnor fiber

Assume $f : X \to T$ is a Milnor fibration. By the finite determinacy theorem, with a suitable coordinate change one can embed the fibration into a projective one $\pi_f = f_Y : \mathbb{P}^{n+1} \to \mathbb{C}$ such that:

- f is a polynomial of sufficiently high degree, say $d = \deg f$
- Zero is the only singular point of the closure Y_0 of $f_Y^{-1}(0)$ in $P^{n+1}(\mathbb{C})$
- The closure Y_t of $f^{-1}(t)$ in $P^{n+1}(\mathbb{C})$ is smooth for $t \in T'$.

Remark 4.2.1. The 0 mentioned in the second item is different from the the origin in $\mathbb{C}^{n+1}$. It would lie in a hyperplane in $\mathbb{P}^{n+1}$.

We obtain a locally trivial C^∞-fibration $\pi_f : Y' \to T'$ with

$$F(z_0, ..., z_{n+1}) = z_{n+1}^d f(z_0/z_{n+1}, ..., z_n/z_{n+1}),$$
$$Y = \{(z,t) \in \mathbb{P}^{n+1}(\mathbb{C}) \times T \mid F(z) - tz_{n+1}^d = 0\}.$$

The map π_f is the projection on the second factor. The monodromy M_Y on the primitive part $P^n(Y_t, \mathbb{Q}), t \in T'$ of the middle cohomology of a regular fiber is quasi-unipotent. $M_{Y,s}, M_{Y,u}$ and N_Y can be defined similarly and they satisfy similar relations as the local case. There is a $(-1)^n$-symmetric nondegenerate intersection form I_Y^* on $P^n(Y_t, \mathbb{Q})$. We set $S_Y = (-1)^{n(n-1)/2} I_Y^{coh}$. Also set $Y_\infty = Y' \times_{T'} H$. The pure Hodge structures on $P^n(Y_t, \mathbb{Q})$ are polarized by S_Y. By the nilpotent orbit theorem, the limit filtration

$$F_\infty^\bullet = \lim_{Im(t) \to \infty} \exp(N_Y.t) F_{u(t)}^\bullet \qquad (4.14)$$

on $P^n(Y_t, \mathbb{C})$ is well defined. This means that we equip $H^n(Y_\infty, \mathbb{C})$ with the limit Hodge filtration as in (2.44) or the same in (3.10).

Theorem 4.2.2. *[H1]*

$S_Y, N_Y, W_\bullet, F_\infty^\bullet$ *give a polarized mixed Hodge structure on* $P^n(Y_\infty)$. *It is invariant w.r.t* $M_{Y,s}$.

Remark 4.2.3. There exists an exact sequence

$$0 \to H^n(Y_0) \to H^n(Y_\infty) \to H^n(X_\infty) \to H^{n+1}(Y_0) \to H^{n+1}(Y_\infty) \to 0. \tag{4.15}$$

Theorem 4.2.4. *(Invariant Cycle Theorem)[H1] If f is a polynomial of sufficiently high degree s.t the properties above are satisfied. Then the mapping $i^* : P^n(Y_\infty) \to H^n(X_\infty)$ is surjective and the kernel is $\ker(i^*) = \ker(M_Y - id)$. Moreover, there exists a unique MHS on $H^n(X_\infty)$ namely Steenbrink MHS, which makes the following short exact sequence an exact sequence of mixed Hodge structures*

$$0 \to \ker(M_Y - id) \to P^n(Y_\infty) \to H^n(X_\infty) \to 0. \tag{4.16}$$

The MHS's are invariant w.r.t the semi-simple part of the monodromy.

The aforementioned MHS on $H^n(X_\infty, \mathbb{C})$ is called Steenbrink MHS, which is also polarized cf. sec. 5.2, see also section 6.3.

4.3 Twisted de Rham complex

Assume $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is a hypersurface germ with isolated critical point. The formula for the dimension of cohomology of the Milnor fiber X_t is

$$\dim(H^n(X_t, \mathbb{C})) = \dim(\frac{\mathbb{C}[[x_0, ..., x_n]]}{J(f)}). \tag{4.17}$$

There are various proofs of this fact. One of them consists of deforming f by adding a generic linear form and counting the number of simple critical points of the deformed function.

Following Brieskorn one may prove it in this way. The right hand side of equality is the $(n+1)$-cohomology of

$$0 \to \mathcal{O}_{\mathbb{C}^{n+1},0} \xrightarrow{df} \Omega^1_{\mathbb{C}^{n+1},0} \xrightarrow{df} ... \xrightarrow{df} \Omega^{n+1}_{\mathbb{C}^{n+1},0} \to 0$$

known to have no other non-zero cohomologies. If we twist it with a new variable z, we get the following

$$0 \to \mathcal{O}_{\mathbb{C}^{n+1},0}[[z]] \overset{zd-df\wedge}{\to} \Omega^1_{\mathbb{C}^{n+1},0}[[z]] \overset{zd-df\wedge}{\to} ... \to \Omega^{n+1}_{\mathbb{C}^{n+1},0}[[z]] \to 0.$$

It has non-zero cohomology at most in degree $n+1$, that is a free $\mathbb{C}[[z]]$-module of rank $\dim(H^n(X_t, \mathbb{C}))$. The cohomologies of the second complex (f isolated or not) are called local Gauss-Manin systems. One can use a polynomial version of the complex rather than a power series one. Then, we obtains the following well-known result.

Proposition 4.3.1. *[SA1] For each k, $G_0^k := H^k(X, \Omega_X^\bullet[z], zd - df\wedge)$ is a free $\mathbb{C}[z]$-module of finite rank. Moreover, the two modules*

$$H^k(X, \Omega_X^\bullet, df) = \frac{G_0^k}{z.G_0^k}, \qquad H^k(X, \Omega_X^\bullet, d - df) = \frac{G_0^k}{(z-1).G_0^k}$$

have the same finite dimensions. The theorem is true if $\Omega_X^\bullet$ is replaced by $\Omega_X^\bullet(\log D)$.

The $\mathbb{C}[[z]]$-module

$$\mathcal{H}^{(0)} = H^{n+1}(\Omega^\bullet_{\mathbb{C}^{n+1},0}[[z]], zd - df\wedge) \tag{4.18}$$

is called the Brieskorn lattice. In this setting the role of the polarization in ordinary Hodge structures is played by the so called higher residue pairings,

$$K = \sum_{k \geq 0} z^k K^k : \mathcal{H}^{(0)} \otimes \mathcal{H}^{(0)} \to \mathbb{C}[[z]]. \tag{4.19}$$

or by the generating function

$$K = \sum_{k \geq 0} z^k K^k : \mathcal{H}^{(0)} \otimes \mathcal{H}^{(0)} \to \mathbb{C}[[z]]. \tag{4.20}$$

It is a sesqui-linear, flat skew hermitian pairing. We have the following iomorphism,

$$\frac{\mathcal{H}^{(0)}}{z.\mathcal{H}^{(0)}} = \frac{\Omega^{\bullet}_{\mathbb{C}^{n+1},0}[[z]]}{df \wedge \Omega^{\bullet}_{\mathbb{C}^{n+1},0}[[z]]}. \tag{4.21}$$

Chapter 5

Bilinear forms for singularities

In this chapter different bilinear forms on the middle cohomology of Milnor fibers are studied. We also provide some examples for more convenience. We assume the holomorphic germ $f : \mathbb{C}^{n+1} \to \mathbb{C}$ to have an isolated singularity at $0 \in \mathbb{C}^{n+1}$ throughout this chapter. When concerned with the projective compactification we specify as f_X for the local fibration in affine space and f_Y for the projective one as in 4.2. In section 5.3 by using the hermitian form of D. Barlet, we give another proof for the existence of polarization, namely prop. 5.3.4 which corresponds to 3.2.1.

5.1 Intersection form

Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a milnor fibration with isolated singularity, with the monodromy $M \in \pi_1(T')$. The intersection form

$$I : H_n(X_t, \mathbb{Z}) \times H_n(X_t, \mathbb{Z}) \to \mathbb{Z}$$

is $(-1)^n$-symmetric and M-invariant. Its kernel is

$$RadI = Ker(M - id) \subset H_n(X_t, \mathbb{Z}).$$

The long exact sequence of the pair $(X_t, \partial X_t)$ becomes

$$0 \to H_n(\partial X_t, \mathbb{Z}) \to H_n(X_t, \mathbb{Z}) \to H_n(X_t, \partial X_t, \mathbb{Z}) \to H_{n-1}(\partial X_t, \mathbb{Z}) \to 0$$

and $H_n(\partial X_t, \mathbb{Z}) = ker(M - id) \subset H_n(X_t, \mathbb{Z})$. The canonical map

$$can : H_n(X_t, \mathbb{Z}) \to H^n(X_t, \mathbb{Z}) = H_n(X_t, \partial X_t, \mathbb{Z})$$

and I are related by $I(a, b) = \langle can(a), b \rangle$, where $\langle , \rangle$ is evaluation given by duality. The canonical isomorphism $H^n(X_t, \mathbb{Z}) = H_n(X_t, \partial X_t, \mathbb{Z})$ gives

$$Var : H^n(X_t, \mathbb{Z}) \to H_n(X_t, \mathbb{Z}), \qquad Var(\gamma) = [M(\gamma) - \gamma]$$

where $[\gamma]$ is the representative in $H_n(X_t, \partial X_t, \mathbb{Z})$. A form $I^{coh}(A, B)$ can be defined by

$$I^{coh}(A, B) = I(Can^{-1}A, Can^{-1}B).$$

The Milnor fibration f defines a usual fibration on S^1 given by $(\frac{f}{|f|})^{-1}(t/|t|)$. The form

$$L : H_n(X_t, \mathbb{Z}) \times H_n(X_t, \mathbb{Z}) \to \mathbb{Z} \tag{5.1}$$

defined by $L(a, b) = \langle Var^{-1}a, b \rangle$, is called the Seifert form. It can also be expressed by $L(a, b) = lk(\tilde{a}, \tilde{b})$, where '$\tilde{a}$' means lift of the cycle by the fibration $(\frac{f}{|f|})^{-1}(t/|t|)$ and lk is the linking form on S^{2n+1}. Classically, it is equal to the intersection number of $(A, \tilde{b})$ where A is a cycle on S^{2n+1} which $\partial A = \tilde{a}$, [H2].

Example 5.1.1. [A] Consider the family $f_\lambda = x^3 + y^3 + \lambda x^2 y$. For any fixed λ, the function f_λ has an isolated singularity at 0. $\mu = 4$ is constant in the family and a basis of Jacobi algebra is given by $1, x, x^2, y$. Set $\omega = xdy - ydx$, easy computation gives

$$d\omega = \frac{2}{3}\frac{df_\lambda}{f_\lambda} \wedge \omega, \qquad d(x\omega) = \frac{df_\lambda}{f_\lambda} \wedge x\omega, \qquad d(y\omega) = \frac{df_\lambda}{f_\lambda} \wedge y\omega,$$
$$d(x^2\omega) = \frac{4}{3}\frac{df_\lambda}{f_\lambda} \wedge x^2\omega.$$

According to [A] page 134 and [BA], the set $\mathcal{B} = \{[f_\lambda^{-2/3}\omega], [f_\lambda^{-1}x\omega], [f_\lambda^{-1}y\omega], [f_\lambda^{-4/3}x^2\omega]\}$ provides a multivalued horizontal basis of the Gauss-Manin bundle. The matrix of the intersection form takes the form, ([A] page 135)

$$A = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

5.2 Polarization form S

Consider the Milnor fibration $f : X \to T$ embedded into a compactified (projective) fibration $f_Y : Y \to T$ such that the fiber Y_t sits in $\mathbb{P}^{n+1}$ for $t \neq 0$ with only a unique singularity at $0 \in Y_0$ over $t = 0$, and also there exists a short exact sequence

$$0 \to H^n(Y_0, \mathbb{Q}) \to H^n(Y_t, \mathbb{Q}) \to H^n(X_t, \mathbb{Q}) \to 0, \qquad t \neq 0. \tag{5.2}$$

We have $H^n(Y_0, \mathbb{Q}) = \ker(M_Y - id)$, by the invariant cycle theorem, where M_Y is the monodromy of f_Y. The form $S_Y := (-1)^{n(n-1)/2} I_Y^{coh} : H^n(Y_t, \mathbb{Q}) \times H^n(Y_t, \mathbb{Q}) \to \mathbb{Q}$ is the polarization form of pure Hodge structure on $H^n(Y_t, \mathbb{C})$, $t \in T'$. W. Schmid has defined a canonical MHS on $H^n(Y_t, \mathbb{Q})$ namely limit MHS, which makes the above sequence an exact sequence of MHS's. In the short exact sequence, the map i^* is an isomorphism on $H^n(Y_t, \mathbb{Q})_{\neq 1} \to H^n(X_t, \mathbb{Q})_{\neq 1}$ giving $S = (-1)^{n(n-1)/2} I^{coh} = (-1)^{n(n-1)/2} I_Y^{coh} = S_Y$ on $H^n(X_t, \mathbb{Q})_{\neq 1}$. The above short exact sequence restricts to the following,

$$0 \to \ker\{N_Y : H^n(Y_t, \mathbb{Q})_1 \to H^n(Y_t, \mathbb{Q})_1\} \to H^n(Y_t, \mathbb{Q})_1 \to H^n(X_t, \mathbb{Q})_1 \to 0. \tag{5.3}$$

So $a, b \in H^n(X_t, \mathbb{Q})_1$ have pre-images $a_Y, b_Y \in H^n(Y_t, \mathbb{Q})_1$ and

$$S(a, b) = S_Y(a_Y, (-N_Y)b_Y) \tag{5.4}$$

is independent of the lifts of a_Y, b_Y, by the fact that N_Y is an infinitesimal isometry for S_Y. The equation 5.3 defines the desired polarization on $H^n(X_t, \mathbb{Q})_1$. The polarization form S is M-invariant, non-degenerate, $(-1)^n$-symmetric on $H^n(X_t, \mathbb{Q})_{\neq 1}$ and $(-1)^{n+1}$-symmetric on $H^n(X_t, \mathbb{Q})_1$, [H1].

Lemma 5.2.1. *[H1] The bilinear form S on $H^n(X_\infty, \mathbb{Q})$ defined by*

$$S(a,b) = \begin{cases} S_Y(a_Y, b_Y) & a, b \in H_{\neq 1} \\ S_Y(a_Y, (-N_Y)b_Y) & a, b \in H_1 \end{cases} \tag{5.5}$$

is non-degenerate and invariant with respect to the monodromy.

Theorem 5.2.2. *[H1] Steenbrink MHS and S yields a PMHS of weight n on $H^n(X_\infty, \mathbb{Q})_{\neq 1}$ and PMHS of weight $n+1$ on $H^n(X_\infty, \mathbb{Q})_1$.*

Example 5.2.3. [HS] Consider the following topological data: Let $H^\infty_{\mathbb{R}}$ be a 3-dimensional real vector space , $H^\infty_{\mathbb{C}} = H^\infty_{\mathbb{R}} \otimes \mathbb{C}$ its complexification and choose a basis $H^\infty = \oplus_{i=1}^3 \mathbb{C}A_i$ such that $\overline{A_1} = A_3$ and $A_2 \in H^\infty_{\mathbb{R}}$. Moreover, choose a real number $\alpha_1 \in (-3/2, -1)$ and put $\alpha_2 = 0,\ \alpha_3 := -\alpha_1$, and let $M \in Aut(H^\infty_{\mathbb{C}})$ be given by $M(\underline{A}) = \underline{A}.diag(\lambda_1, \lambda_2, \lambda_3)$ where $\underline{A} = (A_1, A_2, A_3)$ and $\lambda_i = exp(-2\pi i \alpha_i)$. Putting

$$0 = F_0^2 \subset F_0^1 = \mathbb{C}A_1 \subset F_0^0 = \mathbb{C}A_1 \oplus \mathbb{C}A_2 = F_0^{-1} \subset F_0^{-2} = H^\infty$$

defines a sum of pure Hodge structures of weights 0 and -1 on $H^\infty_{=1}$ and $H^\infty_{\neq 1}$. A polarization form is defined by

$$S(A^{tr}, A) = \begin{pmatrix} 0 & 0 & \gamma \\ 0 & 1 & 0 \\ -\gamma & 0 & 0 \end{pmatrix}$$

where $\gamma = \frac{-1}{2\pi i}\Gamma(\alpha_1 + 2)\Gamma(\alpha_3 - 1)$. In particular we have for $p = 1$,

$$i^{p-(-1-p)} S(A_1, A_3) = (-1).i.\ S(A_1, A_3) = \tfrac{\Gamma(\alpha_1+2)\Gamma(\alpha_3-1)}{2\pi} > 0$$

and for $p = 0$

$$i^{p-(-p)}S(A_2, A_2) = S(A_2, A_2) > 0.$$

So that $F_0^\bullet$ indeed induces a pure polarized Hodge structure of weight -1 on $H^\infty_{\neq 1} = \mathbb{C}A_1 \oplus \mathbb{C}A_2$ and a pure polarized Hodge structure of weight 0 on $H^\infty_{=1} = \mathbb{C}A_2$. M is semi-simple and its eigenspaces are one dimensional.

Remark 5.2.4. A Hermitian form can be associated to the polarization form S:

$$h : H^n(X_t, \mathbb{C}) \times H^n(X_t, \mathbb{C}) \to \mathbb{C}$$

$$h(a, b) = (-1)^{n(n-1)/2} \tfrac{1}{(2\pi i)^n} S(a, \bar{b}), \qquad \textit{on } H^n(X_t, \mathbb{C})_{\neq 1}$$

$$h(a, b) = (-1)^{n(n-1)/2} \tfrac{1}{(2\pi i)^{n+1}} S(a, \bar{b}) \qquad \textit{on } H^n(X_t, \mathbb{C})_1.$$

5.3 Hermitian form

D. Barlet, [BV], [L], defines the Hermitian form

$$\mathcal{B} : \Omega^{n+1}_{\mathbb{C}^{n+1},0} \times \Omega^{n+1}_{\mathbb{C}^{n+1},0} \to \mathcal{N}$$

$$(\omega, \omega') \to \tfrac{1}{(2\pi i)^n} \int_{X_t} \rho \frac{\omega}{df} \wedge \overline{\frac{\omega'}{df}}$$

where $\mathcal{N} = \bigoplus_{\alpha \in \mathbb{Q}, k \in \mathbb{N}} \mathbb{C}[[t, \bar{t}]]|t|^{2\alpha} \log(t\bar{t})^k / \mathbb{C}[[t, \bar{t}]]$, and ρ is a bump function on a sufficiently small neighbourhood of $0 \in \mathbb{C}$.

Theorem 5.3.1. *([SC2], [L], [V] page 38) If $d = \deg(f)$ is sufficiently large then the form $\omega \in \Omega^{n+1}_X$ can be prolonged to $\mathbb{P}^{n+1}$ such that the its Leray residue $\omega/df := Res\omega/(f-t)$ extends to the Leray residue of the prolongation form on Y. Moreover, the extension can be such that its Jordan blocks decomposition remain similar.*

The residue $\dfrac{\omega}{df}|_{Y(s)}$ is expanded as Laurent series expansions in terms of powers of $\log(s)$ in the following form,

$$\frac{\omega}{df} = s^k(a_0^Y + a_1^Y \log(s) + ... + a_{n-1}^Y \log(s)^{n-1} + ...)$$

$$\frac{\omega'}{df} = s^{k'}(b_0^Y + b_1^Y \log(s) + ... + b_{n-1}^Y \log(s)^{n-1} + ...)$$

where a_i^Y, b_i^Y are multi-valued horizontal sections of the Gauss-Manin system of $s \to P^n(Y(s))$ and $N_Y a_i^Y = a_{i+1}^Y$, and similar for $b's$.

$$F(s) = \frac{1}{(2\pi i)^n} \int_{f=s} \rho \frac{\omega}{df} \wedge \overline{\frac{\omega'}{df}}. \tag{5.6}$$

The difference between the function F and a similar one for projective fibration Y namely,

$$G(s) = \frac{1}{(2\pi i)^n} \int_{f_Y=s} \rho \frac{\eta}{df} \wedge \overline{\frac{\eta'}{df}} \tag{5.7}$$

is C^∞, where η is the prolongation of ω as in 5.3.1. That is because of the fiber of f_X (resp. f_Y) are transversal to ∂X (resp. $\partial(Y)$) and f_Y has no critical point in $Y - X$. Let $h(a,b)$ is the coefficient of $s^k \bar{s}^{k'} \log|s|^2$ in the expansion at $s = 0$ of G and F both, [L]. By replacing the Laurent series expansions we write

$$(2\pi i)^n G(s) = s^k \bar{s}^{k'} \Big(\int_{f_Y=s} a_0^Y \wedge \overline{b_0^Y} + \log(s) \int_{f_Y=s} N_Y a_0^Y \wedge \overline{b_0^Y} + \log(\bar{s}) \int_{f_Y=s} a_0^Y \wedge \overline{N_Y b_0^Y} + \ldots \Big). \tag{5.8}$$

On the other hand expanding the form $\mathcal{B}$ gives,

$$\sum_{\alpha,\beta \notin \mathbb{N}} t^\alpha . \bar{t}^\beta \frac{(\log t.\bar{t})^k}{k!} h(N^k.s_\alpha(\omega), s_\beta(\eta)) + \sum_{\alpha,\beta \in \mathbb{N}} t^\alpha . \bar{t}^\beta \frac{(\log t.\bar{t})^{k+1}}{(k+1)!} h(N^k.s_\alpha(\omega), s_\beta(\eta)) \tag{5.9}$$

When the section belongs to the eigenspace $H_{\neq 1}$, the intersection or polarization form for Milnor fibration of f agrees with that of f_Y. A comparison of coefficients in (5.8) and (5.9) provides the following theorem.

Theorem 5.3.2. *[L]*

(1) h is non degenerate.

(2) If Q is the cup product on $H_{\neq 1}$,

$$\forall x, y \in H_{\neq 1} \times H_{\neq 1}, \qquad h(x,y) = \frac{1}{(2\pi i)^n} Q(x, \bar{y}). \tag{5.10}$$

[The coefficient of $|t|^{2m} \log(t\bar{t})^l$ in the first sum is $\frac{1}{(l)!} Q'(N_Y^l U, \bar{U})$. On the other hand, this coefficient is $\frac{1}{(l)!} h(N^l u, \bar{u})$ [L].]
The embedding $X_t \to Y_t$ can be in a way that the restriction $r_t : H^n(Y_t, \mathbb{C}) \to H^n(X_t, \mathbb{C}),\ t \neq 0$ is surjective. Then we have the following short exact sequence for the eigenspace H_1 of these spaces.

$$0 \to I \to H'_1(Y_t) \to H_1(X_t) \to 0$$

where I being the kernel. We write the residue in the form

$$[R(\tilde{\omega}(t)] = \sum_{j<m} t^j V_j + t^m t^{N_Y} U + \sum_{\alpha > m} t^\alpha t^{N_Y} U_\alpha$$

where $r(U) = u,\ V_j \in I,\ U_\alpha \in H'$. In this way

$$\int_{Y_t} R(\tilde{\omega}(t)) \wedge \overline{R(\tilde{\omega}(t))} =$$

$$P(|t|^2) + |t|^{2m} (\sum_{r=0}^{n} \sum_{s=0}^{n} Q'(\frac{(\log t)^r}{r!} {N_Y}^r U, \frac{\overline{(\log t)^s}}{s!} \overline{{N_Y}^s U}) + o(|t|^{2m})$$

$$= P(|t|^2) + |t|^{2m} (\sum_{l=0}^{n} \frac{(\log t\bar{t})^l}{l!} Q'({N_Y}^l U, \bar{U}) + o(|t|^{2m})).$$

A similar argument of comparison of coefficients in the last form with that of (5.8) yields,

Theorem 5.3.3. *[L]*
$\forall x, y \in H_1 \times H_1, \qquad h(x,y) = \frac{1}{(2\pi i)^n} Q'(N_Y \tilde{x}, \tilde{y}).$

It is easy to reprove Riemann-Hodge bilinear relations in the variation of mixed Hodge structure associated to a projective fibration. If $F^\bullet, W_\bullet$ be the Hodge and weight filtration defined by Steenbrink, and $P^{k+n} := \{u \in Gr_{k+n}^W | N^{k+1} u = 0\}$ the primitive components, it has a pure Hodge structure of weight $k+n$, so

$$P^{k+n} = \bigoplus_{p+q=k+n} P^{p,q}.$$

Let $P_1^{p,q} = H_1 \cap P^{p,q}$, $P_{\neq 1}^{p,q} = H_{\neq 1} \cap P^{p,q}$. Then the Riemann-Hodge bilinear relations on the mixed Hodge structure of $H^n(X_\infty, \mathbb{C})$ have the following description.

Proposition 5.3.4. *[L] The following holds;*

1) If $(u,v) \in P_1^{p,q} \times P_1^{r,s}$, $\quad p+q = r+s = n+k$, *then*

- $h(N^{k-1}u, v) = 0$, *if* $\quad (p,q) \neq (r,s)$.
- *If* $u \neq 0$, $\quad (-1)^{n(n-1)/2+k+p} h(N^{k-1}u, u) > 0$.

2) If $(u,v) \in P_{\neq 1}^{p,q} \times P_{\neq 1}^{r,s}$, $\quad p+q = r+s = n+k$, *then*

- $h(N^k u, v) = 0$ *if* $\quad (p,q) \neq (s,r)$.
- $u \neq 0$, $\quad (-1)^{n(n-1)/2+p+k} h(N^k u, v) > 0$.

Proof. This follows from 5.2.4, 5.3.2 and 5.3.3. □

Example 5.3.5. [BA] Take $f = x^3 + y^3 + z^3$. A monomial basis of Jacobi algebra is given by,

$$1,\ x,\ y,\ z,\ xy,\ yz,\ zx,\ xyz.$$

Hence $\mu = 8$. Set $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$. Then by easy calculation we have,

$$d\omega = \frac{df}{f} \wedge \omega \ ,$$

$$d(x\omega) = \frac{4}{3}\frac{df}{f} \wedge x\omega, \qquad d(y\omega) = \frac{4}{3}\frac{df}{f} \wedge y\omega, \qquad d(z\omega) = \frac{4}{3}\frac{df}{f} \wedge z\omega \ ,$$

$$d(xy\omega) = \frac{5}{3}\frac{df}{f} \wedge xy\omega, \quad d(yz\omega) = \frac{5}{3}\frac{df}{f} \wedge yz\omega, \quad d(zx\omega) = \frac{5}{3}\frac{df}{f} \wedge zx\omega \ ,$$

$$d(xyz\omega) = 2\frac{df}{f} \wedge xyz\omega.$$

The forms

$$\{[\frac{\omega}{f}], [\frac{x\omega}{f^{4/3}}], [\frac{y\omega}{f^{4/3}}], [\frac{z\omega}{f^{4/3}}], [\frac{xy\omega}{f^{5/3}}], [\frac{yz\omega}{f^{5/3}}], [\frac{zx\omega}{f^{5/3}}], [\frac{xyz\omega}{f^2}]\},$$

give a basis of horizontal multivalued sections of the Gauss-Manin system. In this basis the monodromy is given by the matrix

$$\begin{pmatrix} 1 & & & & & & & \\ & exp(4\pi i/3) & & & & & & \\ & & exp(4\pi i/3) & & & & & \\ & & & exp(4\pi i/3) & & & & \\ & & & & exp(2\pi i/3) & & & \\ & & & & & exp(2\pi i/3) & & \\ & & & & & & exp(2\pi i/3) & \\ & & & & & & & 1 \end{pmatrix}$$

The forms

$$\{\omega,\ x\omega,\ y\omega,\ z\omega,\ xy\omega,\ yz\omega,\ zx\omega,\ xyz\omega\}$$

also provide a basis for the Gauss-Manin module. If we denote this bases as $\{m_\alpha\}$ for suitable bump function ρ,

$$\int_{f=s} \rho\omega \wedge \bar{\omega} = c_1 |s|^2 . \log |s|$$

$$\int_{f=s} \rho |m_i|^2 \omega \wedge \bar{\omega} = c_1 |s|^2 . |s|^{8/3}, \quad i = 2, 3, 4$$

$$\int_{f=s} \rho |m_i|^2 \omega \wedge \bar{\omega} = c_i |s|^2 . |s|^{10/3}, \quad i = 5, 6, 7$$

$$\int_{f=s} \rho\omega \wedge \bar{\omega} = c_8 |s|^2 . |s|^4 \log |s|$$

where c_i are real.

So the matrix of the hermitian form with respect to bases of multivalued forms above is given by

$$\frac{-1}{4\pi^2}\begin{pmatrix} c_1 & & & & & & & \\ & c_2 & & & & & & \\ & & c_3 & & & & & \\ & & & c_4 & & & & \\ & & & & c_5 & & & \\ & & & & & c_6 & & \\ & & & & & & c_7 & \\ & & & & & & & c_8 \end{pmatrix}$$

5.4 Grothendieck Residue Pairing

The Grothendieck residue is a linear form on Jacobi algebra defined by

$$A_f = \frac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{(\partial f/\partial x_0, ..., \partial f/\partial x_n)} \to \mathbb{C}$$

$$g \longmapsto Res_0\left[\frac{gdx}{\frac{\partial f}{\partial x_0}\cdots\frac{\partial f}{\partial x_n}}\right] := \frac{1}{(2\pi i)^{n+1}}\int_{\Gamma_\varepsilon}\frac{gdx}{\frac{\partial f}{\partial x_0}\cdots\frac{\partial f}{\partial x_n}}.$$

It does not depend on ε , but does depend on coordinates $x_0, ..., x_n$. It induces a bilinear form $Res_{f,0}$ on

$$\Omega_f := \Omega^{n+1}_{\mathbb{C}^{n+1},0}/df \wedge \Omega^n_{\mathbb{C}^{n+1},0}$$

$$Res_{f,0} : \Omega_f \times \Omega_f \to \mathbb{C}$$

$$(g_1dx, g_2dx) \longmapsto Res_0\left[\frac{g_1g_2dx}{\frac{\partial f}{\partial x_0}\cdots\frac{\partial f}{\partial x_n}}\right],$$

which is independent of the coordinates $x_0, ..., x_n$. The form $Res_{f,0}$ is symmetric and non-degenerate (proved by Grothendieck), and it is equal to the sum of local residues at each critical point.

Example 5.4.1. [PV] Take $f = x^3 + xy^2$, Then the Milnor algebra for f is

$$A_f = \mathbb{C}\{x, y\}/(3x^2 + y^2, 2xy).$$

For $\omega = 2ydx \wedge dy$, compute $res_f(\omega, \omega)$:

$$Res_{f,0}(\omega, \omega) = Res[\frac{4y^2 dx \wedge dy}{(3x^2 + y^2).2xy}].$$

To compute this residue we change the variable to $u = \sqrt{3}x + y$ and $v = \sqrt{3}x - y$, and observe that $u^2 = (3x^2 + y^2) + \sqrt{3}(2xy)$ and $v^2 = (3x^2 + y^2) - \sqrt{3}(2xy)$. Therefore, the above expression is equal to

$$Res_{f,0} \frac{-4(\frac{u-v}{2})^2.du \wedge dv}{u^2.v^2} = 2.$$

Example 5.4.2. Take $f = x^4$ in one variable only. The forms $x^m dx,\ m = 0, 1, 2$ give a basis of G_0^n. Then

$$K_f(\omega_m, \omega_{m'}) = \frac{z}{2\pi i} \int \frac{x^{m+m'}}{4x^3} = \frac{1}{4} \delta_{m,2-m'}.z$$

is the higher residue pairing of K. Saito. The Grothendieck residue is easy to calculate in this basis

$$Res_{f,0}(x^m dx, x^{m'} dx) = \frac{1}{4} \delta_{m,2-m'}.$$

Remark 5.4.3. The residue pairing can be defined in a more general context [G3], for a regular sequence $\{f_0, ..., f_n\}$ of holomorphic germs defining an isolated singularity at $0 \in \mathbb{C}^{n+1}$. For a function g, first set

$$\omega = \frac{g dz_0 \wedge ... \wedge dz_n}{f_0 ... f_n} \tag{5.11}$$

then define the residue as

$$Res_0 \omega = (\frac{1}{2\pi i})^n \int_\Gamma \omega \tag{5.12}$$

where $\Gamma = \{z;\ |f_i(z)| = \epsilon\}$, is oriented by

$$d(\arg f_0) \wedge ... \wedge d(\arg f_n) > 0.$$

This generalizes the previous definition. Residue depends only on the homology class of Γ and the cohomology class of ω for trivial reasons. Also,

$$Res_0 = \frac{g(0)}{\mathcal{J}_f(0)}, \qquad \mathcal{J}_f(0) = |\frac{\partial(f_0, ..., f_n)}{\partial(z_0, ..., z_n)}(0)| \neq 0.$$

The residue pairing in this way just depends on the ideal generated by $J(f) = (f_0, ..., f_n)$, and not to the generators chosen. Moreover, for germs already in this ideal the residue degenerates. In this way it induces a form

$$res_f : \mathcal{O}/J(f) \otimes \mathcal{O}/J(f) \to \mathbb{C}.$$

The last pairing is known to be non-degenerate, namely local duality theorem.

Remark 5.4.4. The image of 1 or the integral

$$\int_\Gamma \frac{df_0}{f_0} \wedge ... \wedge \frac{df_n}{f_n}$$

evaluates the intersection number of divisors $D_i := \{f_i = 0\}$ as $\deg(f)$. It also shows that this number is locally constant. In case, $f_i = \partial f/\partial x_i$ this number is equal to the Milnor number of f that is $\dim A_f$, the Jacobi or Milnor algebra. Residue also satisfies a type of continuity principle, meaning it remains constant in continues deformations.

Chapter 6

Hodge theory of Brieskorn lattice

In this chapter we enter a systematic analysis of the Gauss-Manin system of hypersurface fibrations, specially when they have isolated singularities. Most of the definitions and concepts can also be applied to the non-isolated case. The main tools is the configuration of different lattices inside the Gauss-Manin system $\mathcal{G}$. Using the lattice structure on $\mathcal{G}$ we give another definition of Steenbrink limit mixed Hodge structure. The V-filtration, Brieskorn lattice, and spectral pairs of the isolated singularities have been defined in this part. These are the major tools to be used in Chapter 8 for the main contributions.

6.1 Elementary sections

Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a holomorphic germ having isolated singularity at origin. Let X be the intersection of a closed ball in $\mathbb{C}^{n+1}$ centered at $0 \in \mathbb{C}$ with the pre-image under f of an open disk T in $\mathbb{C}$ centred at $0 \in \mathbb{C}$. By appropriate choice of X and T, the restriction

$$X' = X \setminus f^{-1}(0) \xrightarrow{f} T \setminus 0 = T' \tag{6.1}$$

is a C^{∞} fiber bundle, and the restriction of f to ∂X is a trivial fiber bundle. Let $M = M_s.M_u$ be the decomposition of M into semi-simple and unipotent parts and

$$N = -\frac{\log M_u}{2\pi i} \tag{6.2}$$

the logarithm of the unipotent part. Then by the monodromy theorem (Theorem 3.1.1) $N^{n+1} = 0$ and $N^n = 0$ on the generalized 1-eigenspace. The eigenvalues of M_s are the root of unity,

$$-2\pi i N \in End(H_{\mathbb{Q}}) \tag{6.3}$$

is defined over $\mathbb{Q}$. We consider the pull back of the cohomology bundle under the universal covering $\sigma : H \to T'$; where we denote it by $\mathcal{H}_\infty$. The sections of $\mathcal{H}_\infty$ are called multivalued sections of $\mathcal{H}$. We use the same symbol $\mathcal{H}$ for the sheaf of multivalued sections, when no ambiguity. Let

$$C^\alpha = \ker(t\partial_t - \alpha)^r \subset (i_*\mathcal{H})_0, \qquad \lambda = \log(2\pi i\alpha), -1 \leq \alpha < 0 \tag{6.4}$$

for the greatest r such that the transformation $(t\partial_t - \alpha)^r \neq 0$. The sections

$$s_\alpha(A) := t^\alpha exp(N \log t)A(t) \qquad -1 \leq \alpha < 0. \tag{6.5}$$

Where $A \in H_{\mathbb{C}}^{\lambda_\alpha}$ is a multivalued section of $\mathcal{H}$, define a single valued sections called the elementary section associated to A.
The local elementary sections of the cohomology bundle at the critical value 0 generate a regular $\mathbb{C}\{t\}[t^{-1}]$-module $\mathcal{G}$, namely the local Gauss-Manin system defined by;

$$\mathcal{G} = \sum_{-1 \leq \alpha < 0} \mathbb{C}\{t\}[t^{-1}]C^\alpha. \tag{6.6}$$

$\mathcal{G}$ is a μ-dimensional $\mathbb{C}\{t\}[t^{-1}]$-subvector space of $(i_*\mathcal{H})_0$.

Proposition 6.1.1. *([SC2] prop. 3.5)* $\partial_t : \mathcal{G} \to \mathcal{G}$ *is invertible.*

Definition 6.1.2. We denote $s := \partial_t^{-1}$. Then, $\partial_s := \partial_t^2.t$.

The identities $[t, \partial_t] = [s, \partial_s] = 1,\ [t, s] = s^2,\ t = s^2.\partial_s$ are straight-forward.

Definition 6.1.3. We call the maximal $\mathbb{C}\{s\}$-module

$$\tilde{\mathcal{G}} := \mathbb{C}\{t\}C^0 \oplus \bigoplus_{-1<\alpha<0} \mathbb{C}\{t\}[t^{-1}]C^\alpha$$

in $\mathcal{G}$, the reduced local Gauss-Manin connection. The μ-dimensional $\mathbb{C}\{\{s\}\}[s^{-1}]$-vector space $\tilde{\mathcal{G}} \otimes_{\mathbb{C}\{\{s\}\}} \mathbb{C}\{s\}[s^{-1}]$ is called the Gauss-Manin system.

6.2 V-filtration

The V-filtration on $\mathcal{G}$ is a decreasing filtration indexed by rational numbers $\alpha \in \mathbb{Q}$. It is also called Kashiwara-Malgrange filtration.

Definition 6.2.1. (V-filtration)

The (Kashiwara-Malgrange) V-filtration on $\mathcal{G}$ is a decreasing filtration of $\mathbb{C}[[t]]$-modules $V = (V^\alpha)_{\alpha\in\mathbb{Q}}$ defined by,

$$V^\alpha := \sum_{\alpha\leq\beta} \mathbb{C}\{t\}C^\beta = \oplus_{\alpha\leq\beta<\alpha+1}\mathbb{C}\{t\}C^\beta$$

$$V^{>\alpha} := \sum_{\alpha<\beta} \mathbb{C}\{t\}C^\beta = \oplus_{\alpha<\beta\leq\alpha+1}\mathbb{C}\{t\}C^\beta$$

The V-filtration can be characterized by the following properties,

- $t.V^\alpha \subset V^{\alpha+1}$,
- $\partial_t.V^\alpha \subset V^{\alpha-1}$
- $t^i\partial_t^j V^\alpha \subset V^\alpha$ for all $i > j$.
- The operator $t\partial_t - \alpha$ is nilpotent on Gr_V^α.

The definition of V^α and $V^{>\alpha}$ is independent of choice of t, and

$$C^\alpha = \frac{V^\alpha}{V^{>\alpha}}$$

Then the isomorphism 4.12 becomes

$$\psi := \bigoplus_{-1<\alpha\leq 0} \psi^\alpha : \bigoplus_{-1<\alpha\leq 0} H_{\mathbb{C}}^{\lambda_\alpha} \to \bigoplus_{-1<\alpha\leq 0} C^\alpha = \frac{V^{>-1}}{t.V^{>-1}} \cong \frac{V^{>-1}}{s.V^{>-1}}. \quad (6.7)$$

where the last isomorphism is for trivial reasons.

Proposition 6.2.2. *(see [SCHU] sec 1.6)*
V^α and $V^{>\alpha}$ are free $\mathbb{C}\{t\}$-modules of rank $= \mu$.

A straight forward calculation shows

$$\mathcal{G} = \mathbb{C}\{t\}[\partial_t] \bigoplus \bigoplus_{\lambda} \bigoplus_{j=1}^{m_\lambda} \frac{\mathbb{C}\{t\}[\partial_t]}{\mathbb{C}\{t\}[\partial_t](t\partial_t - \alpha_\lambda)^{n_{\lambda,j}}} = \mathbb{C}\{t\}[t^{-1}]V^{-1}. \tag{6.8}$$

where

$$V^{-1} = \bigoplus_{\lambda} \bigoplus_{j=1}^{m_\lambda} \mathbb{C}\{t\} C^{\alpha_\lambda, j}$$
$$C^{\alpha' j} := \psi_\alpha(H_{\mathbb{C}}^{\lambda,j})$$

The above direct sum is an application of the Jordan-Holder structure theorem.

Remark 6.2.3. On the $\mathbb{C}\{t\}[\partial_t]$-module $\tilde{\mathcal{G}}$ (Gauss-Manin system) ∂_t is invertible. Note that this is true when f defines an isolated singularity germ. The subspaces $\tilde{C}_\alpha, \tilde{V}^\alpha, \tilde{V}^{>\alpha}$ for $\tilde{\mathcal{G}} = (\int_f^{n+1} \mathcal{O}_X)$ can be defined similarly.

Definition 6.2.4. The ring

$$R = \mathbb{C}\{\{\partial_t^{-1}\}\} = \{\sum_{i \geq 0} a_i \partial_t^{-i} \mid \sum_{i \geq 0} a_i t^i / i! \in \mathbb{C}\{t\}\}$$

is called the ring of micro-differential operators with constant coefficients.

Theorem 6.2.5. *[H3]*

(1) $\tilde{\mathcal{G}} = \bigoplus_{-1 < \alpha \leq 0} R[\partial_t]\tilde{C}_\alpha$ is an $R[\partial_t]$-vector space of dimension μ. $\tilde{V}^\alpha, \tilde{V}^{>\alpha}$ are R-modules of rank μ.

(2) $\mathcal{G}$ is canonically isomorphic to $\tilde{\mathcal{G}} \otimes_{\mathbb{C}\{t\}} \mathbb{C}\{t\}[t^{-1}]$ as $\mathbb{C}\{t\}[\partial_t]$-module.

(3) The $\mathbb{C}\{t\}[\partial_t]$-module homomorphism $\tilde{\mathcal{G}} \to \mathcal{G}$ induces and an R-module isomorphism $\tilde{V}^{>-1} \to V^{>-1}$.

(4) $H_0'' \subset \tilde{V}^{>-1} \subset \tilde{\mathcal{G}}$ are R-modules of rank μ.

6.3 Mixed Hodge structure on the vanishing cohomology

We begin with the definition of the Brieskorn lattice, due to E. Brieskorn himself.

Definition 6.3.1. The $\mathcal{O}_T$-module

$$\mathcal{H}^{(0)} := f_*\Omega_X^{n+1}/df \wedge d(f_*\Omega_X^{n-1})$$

is called Brieskorn lattice. We call its stack

$$H'' := \mathcal{H}_0^{(0)}$$

at $0 \in T$ the local Brieskorn lattice.

We use the two notations $\mathcal{H}^{(0)}$ and H'' equally through the text for Brieskorn lattice, when no ambiguity arises. We have two other sub-$\mathcal{O}_T$-module of H'' defined by,

$$H = f_*df \wedge \Omega_X^n/df \wedge d(\Omega_X^{n-1})$$
$$H' = f_*\Omega_X^{n+1}/d\Omega_X^n + df \wedge (\Omega_X^n)$$

of rank μ, such that

- $H \hookrightarrow H' \overset{df}{\hookrightarrow} H''$
- $H'|_{T'} = H''|_{T'} = \mathcal{H}$, cf. def. (4.4).
- $H''/H' \cong \Omega_X^{n+1}/df \wedge \Omega_X^n =: \Omega_f \cong H''/H'$.
- $H \overset{\cong}{\to} H' \overset{\cong}{\to} H''$, given by,

$$[\eta] \mapsto [\frac{d\eta}{df}], \qquad [df \wedge \eta] \mapsto [d\eta]$$

cf. [SCHU] sec. 1.4, Theorems 1.4.5, 1.4.6 and 1.4.8.

The $\mathcal{H}_0^{(0)}$ is a $\mathbb{C}\{t\}$-module with a regular connection ∇ equipped with an action of ∂_t^{-1}. The module $\mathcal{G}$ is a localization of $\mathcal{H}^{(0)}$ by the action ∂_t^{-1}. We have the following important relation,

$$\frac{\mathcal{H}^{(0)}}{\partial_t^{-1}.\mathcal{H}^{(0)}} = \frac{\Omega_X^{n+1}/df \wedge d(f_*\Omega_X^{n-1})}{df \wedge \Omega_X^n/df \wedge d(f_*\Omega_X^{n-1})} = \Omega_f \tag{6.9}$$

which follows from the identity $\partial_t^{-1} d\eta = df \wedge \eta$, where $\eta \in \Omega_X^n$.

By definition the operator $t\partial_t - \alpha$ is nilpotent on C^α and thus Jacobson-Morosov theorem (cf. [SA1] page 12 or by 3.1.2), we obtain a unique weight filtration W on C^α centered at $-n$, cf. [SCHU] sec 1.7. Similar statement holds for the linear map $M - id$ on H^1 and $M - \lambda$ on H^λ which provide us weight filtrations centered at $-n-1$ and $-n$, respectively, cf. [SCHU], def. 1.7.5. By the monodromy theorem both of these weight filtrations have length at most $2n$. This suggests the following definitions.

Definition 6.3.2. 1) The increasing weight filtration $W = (W_k)_{k\in\mathbb{Z}}$ on $\mathcal{G}$ is defined by

$$W := \oplus_{-1<\alpha\leq 0}\mathbb{C}\{\{t\}\}WC^\alpha \text{ , by } \mathbb{C}\{\{t\}\}\text{-vector spaces.}$$

2) The filtration W on $H_\mathbb{Q}$ is defined by

$$W := WH_\mathbb{Q}^1 \oplus WH_\mathbb{Q}^{\neq 1}, \text{ by } \mathbb{Q}\text{-vector spaces}$$

We have that, $gr_V gr^W \mathcal{G} = gr^W gr_V \mathcal{G}$, cf. [SCHU] page 48.

Definition 6.3.3. Two Hodge filtrations can be defined on $H^n(X_\infty, \mathbb{C})$ by

$$F_{St}^p H^n(X_\infty, \mathbb{C})_\lambda = \psi_\alpha^{-1}\left(\frac{V^\alpha \cap \partial_t^{n-p} H''}{V^{>\alpha}}\right), \qquad \alpha \in (-1, 0], \tag{6.10}$$

$$F_{Va}^p H^n(X_\infty, \mathbb{C})_\lambda = \psi_\alpha^{-1}\left(\frac{V^\alpha \cap t^{-(n-p)} H''}{V^{>\alpha}}\right), \qquad \alpha \in (-1, 0], \tag{6.11}$$

namely Steenbrink-Scherk and Varchenko Hodge filtrations respectively (knowing that $V^{-1} \supset H_0''$, and $0 = F_{St}^{n+1} = F_{Va}^{n+1}$). These two filtrations together with the weight filtration W define two Hodge structures on $H^n(X_\infty, \mathbb{C})$.

The above definition agrees with that given in 4.2.4, cf. [H1] proposition 4.6 by

Theorem 6.3.4. *([H1] prop. 4.6) The Hodge filtration defined in (6.10) is the Steenbrink Hodge filtration (see Theorem 4.2.4).*

The filtrations

$$F^k_{St}\mathcal{G} := \partial_t^k H'' \tag{6.12}$$

and

$$F^k_{Va}\mathcal{G} := t^{-k} H'', \tag{6.13}$$

on $\mathcal{G}$ are also called the Steenbrink and Varchenko Hodge filtrations.

Theorem 6.3.5. *([H3], [SCHU] prop 1.7.9)*

$F^\bullet$ and $F^\bullet_{Va}$ are different on $H^n(X_\infty, \mathbb{C})$ in general, however the induced filtrations on the $Gr^W_l H^n$ coincide, i.e.

$$F^p_{Va} Gr^W_l H^n(X_\infty, \mathbb{C}) = F^p_{St} Gr^W_l H^n(X_\infty, \mathbb{C})$$

.

Lemma 6.3.6. *([SAI6] lemma 2.4 of sec. 2) Let f be a germ of isolated singularity as before, such that $f : X' \to T'$ is smooth. Define $A^k :=$ $\ker(\wedge df) : \Omega^k_X \to \Omega^{k+1}_X$. Then $\wedge df$ induces an isomorphism $\Omega^\bullet_{X'/T'} \to A[1]$, and the natural inclusion $A^\bullet \hookrightarrow \Omega^\bullet_X[t, t^{-1}]$ is a filtered quasi-isomorphism.*

There is an alternative definition for Gauss-Manin system due to Brieskorn, which we pointed out in 4.3 in brief. We end up with the following theorem;

Theorem 6.3.7. *([PS] Theorems 10.26 and 10.27, [SCHU] sec. 1.5 cor. 1.5.5 and prop. 1.5.6, [S1], [SA8] Lec. 3, see also [SA5]) Assume f : $\mathbb{C}^{n+1} \to \mathbb{C}$ is a holomorphic map with isolated singularity, inducing the Milnor fibration $f : X' \to T'$. Then we have the following isomorphisms*

$$\mathcal{G} = R^n f_* \mathbb{C} \otimes \mathcal{O}_{T'} = R^n f_* \Omega_{X'/T'} = \frac{\Omega^{n+1}[t, t^{-1}]}{(d - t df \wedge)\Omega^{n+1}[t, t^{-1}]}$$

where t is a variable.

The relative de Rham complex $\Omega_{X'/T'}$ in the theorem is

$$0 \to f^{-1}\mathcal{O}_{T'} \to \mathcal{O}_{X'} \to \Omega_{X'/T'} \to ... \to \Omega^{n+1}_{X'/T'} \to 0$$

where

$$\Omega^k_{X'/T'} = \Omega^k_X / f^*\Omega^1_{T'} \wedge \Omega^{k-1}_{X'}$$

and the differential is induced by the usual differential of the de Rham complex for X'.

Example 6.3.8. ([KUL] page 109, [KUL] 7.3.5) Let $f = x^p + y^q + z^r + axyz,\ a \neq 0, 1/p + 1/q + 1/r < 1$. Set

$$\omega = dx \wedge dy \wedge dz.$$

By a method due to J. Scherk, [SC3], the forms

$$\omega, \qquad t\partial_t\omega$$

$$\partial_t(x^k\omega), (0 < k < p), \qquad \partial_t(y^k\omega), 0 < k < q, \qquad \partial_t(z^k\omega), (0 < k < r)$$

provide a basis for the canonical lattice $\mathcal{L} = V^{>-1}$. The operator $t\partial_t$ on $C = \mathcal{L}/t\mathcal{L}$ has the following form,

$$t\partial_t(t\partial_t\omega) = 0$$
$$t\partial_t(\partial_t x^k\omega) = \frac{k-p}{p}\partial_t x^k\omega$$
$$t\partial_t(\partial_t y^k\omega) = \frac{k-q}{p}\partial_t y^k\omega$$
$$t\partial_t(\partial_t z^k\omega) = \frac{k-r}{p}\partial_t z^k\omega$$

This basis is a Jordan basis for the operator $t\partial_t$. Decompose $C = \oplus_{-1<\alpha\leq 0}C_\alpha = C_0 \oplus C_{\neq 0}$ where C_0 is the subspace generated by ω and $t\partial_t\omega$. The weight filtration on C, is defined as follows. On $C_{\neq 0}$ the operator $N = 0$ and $W_1 = 0, W_2 = C_{\neq 0}$. On C_0 the operator $N \neq 0, N^2 = 0$ and we have

$$W_1 = 0,\ W_2 = W_3 = \mathbb{C}t\partial_t,\ W_4 = C_0.$$

Therefore, the weight filtration is as follows

$$0 \subseteq W_2 = W_3 \subseteq W_4$$
$$W_2 = C_{\neq 0} \oplus \mathbb{C} t \partial_t \omega.$$

The Hodge filtration is defined by,

$$C = F^0 \supseteq F^1 \supseteq F^2 = 0$$

where F^1 is the subspace generated by ω.

6.4 Hodge numbers and Spectral pairs

Assume the holomorphic germ $f : \mathbb{C}^{n+1} \to \mathbb{C}$ has an isolated singularity at 0. In the previous section we defined the three filtrations F, W and V on $\mathcal{G}$.

Definition 6.4.1. We define the following invariants of the Gauss-Manin system $\mathcal{G}$;

- Hodge numbers are defined by

$$h_\lambda^{p,l-p} := \dim_{\mathbb{C}}(gr_F^p gr_l^W H_{\mathbb{C}}^\lambda).$$

- Spectral numbers are $\alpha \in \mathbb{Q}$, such that

$$d^\alpha := \dim_{\mathbb{C}} gr_V^\alpha gr_0^F \mathcal{G} > 0$$
$$Sp(f) := (d^\alpha)_{\alpha \in \mathbb{Q}}.$$

- Spectral pairs are the pairs $(\alpha, l) \in \mathbb{Q} \times \mathbb{Z}$, such that

$$d_l^\alpha = \dim_{\mathbb{C}} gr_l^W gr_V^\alpha gr_0^F \mathcal{G} > 0$$
$$Spp(f) := (d_l^\alpha)_{(\alpha,l) \in \mathbb{Q} \times \mathbb{Z}} \in \mathbb{N}^{\mathbb{Q} \times \mathbb{Z}}.$$

Remark 6.4.2. $d^\alpha = \sum_l d_l^\alpha$.

The symmetries between Hodge numbers implies the following relations for the multiplicities of spectral pairs of an isolated hypersurface singularity.

Lemma 6.4.3. *([SCHU] sec. 1.8) We have the following relations between Hodge numbers and spectral numbers.*

$$d_l^{\alpha+p} = h_{\lambda_\alpha}^{n-p,l-n+p}, \qquad -1 < \alpha \leq 0$$
$$d_l^{\alpha} = h_1^{n-p,l+1-n+p}, \qquad -1 < \alpha \leq 0.$$

We have the following duality relations;

$$d_l^{\alpha} = d_l^{2n-l-1-\alpha},$$
$$d_l^{\alpha} = d_{2n-l}^{\alpha-n+l},$$
$$d_l^{\alpha} = d_{2n-l}^{n-1-\alpha},$$
$$d^{\alpha} = d^{n-1-\alpha}.$$

Corollary 6.4.4. *([H1] or [SCHU] page 56, cor. 1.8.6) If $\alpha \notin (-1, n)$, or $l \notin [0, 2n]$ or ($\alpha \in \mathbb{Z}$ and $l \notin [1, 2n-1]$) then $d_l^{\alpha} = 0$. In particular $V^{>-1} \supseteq H'' \supseteq V^{n-1}$.*

The first inclusion in corollary 6.4.4 is explained as follows. Let $\omega \in \Omega_X^{n+1}$, holomorphic $(n+1)$-form. Then $\frac{\omega}{df}|_{X_t}$ gives a section $s[\omega](t)$ of cohomology bundle $\mathcal{H}$. The kernel of the map

$$s : \Omega_X^{n+1} \to V^{>-1}, \qquad \omega \to s[\omega] = (\omega/df_{|X_t}) \tag{6.14}$$

is $df \wedge d\Omega^{n-1}$ cf. [H1] page 17. Therefore,

$$H'' = \frac{\Omega_{X,0}^{n+1}}{df \wedge d\Omega_{X,0}^{n-1}}$$

is identified with its image in $V^{>-1}$. By this corollary we sometimes consider H'' as a subset of $V^{>-1}$. This fact has been used in the definition of the Steenbrink limit Hodge filtration (6.10).

Theorem 6.4.5. *([SCHU], Theorem 1.8.2) For $p \in \mathbb{Z}$, multiplication by t^p induces a $\mathbb{C}$-isomorphism*

$$Gr_F^p Gr_V^{\alpha} Gr_l^W \mathcal{G} \xrightarrow{\times t^p} Gr_V^{\alpha+p} Gr_l^W Gr_F^0 \mathcal{G}$$

Theorem 6.4.5 is quite crucial for us in Chapter 8, and is the base of some gluing data between lattices inside the Gauss-manin system $\mathcal{G}$. It provides a base in order to explain the extension of the Gauss-manin system of isolated hypersurface singularities, in 8.4.

Theorem 6.4.6. *(Thom-Sebastiani) ([H3] ,[KUL] sec. 8.7) If* $f \in \mathbb{C}\{x_0, ..., x_n\},\ g \in \mathbb{C}\{y_0, ..., y_m\}, Sp(f) = (\alpha_1, ..., \alpha_{\mu(f)}), Sp(g) = (\beta_1, ..., \beta_{\mu(g)})$. *Then*

$$Sp(f+g) = (\alpha_i + \beta_j + 1 \mid i = 1, ..., \mu(f), j = 1, ..., \mu(g)).$$

For instance the spectrum of the zero dimensional singularity $g(y) = y^2$ consists of one number $\{-1/2\}$. Thus, if the holomorphic isolated singularity $f : \mathbb{C}^{n+1} \to \mathbb{C}$ has spectrum $\{\alpha_i\}$, then the singularity $f(x)+y^2$ has spectrum $\{\alpha_i + 1/2\}$.

Theorem 6.4.7. *([SCHU] Theorem 1.8.10) The spectral pairs and numbers are constant in a μ-constant deformation, where μ is the rank of Brieskorn lattice.*

Example 6.4.8. Consider $f = x^2y^2 + x^5 + y^5$.[SCHU]

A monomial basis for the Jacobi algebra $A_f = \mathbb{C}\{x, y\}/(\partial_x f, \partial_y f)$ is given by

$$B = \{1,\ x,\ x^2,\ x^3,\ x^4,\ y^2,\ y^3,\ y^4,\ y^5,\ xy\}.$$

The spectral pairs are

$$(-1/2, 2), (-3/10, 1), (-3/10, 1), (-1/10, 1), (-1/10, 1),$$
$$(0, 1),$$
$$(1/10, 1), (1/10, 1), (3/10, 1), (3/10, 1), (1/2, 0).$$

They are symmetric around $(0, 1)$. By the isomorphism

$$\frac{H''}{sH''} \cong \frac{\Omega^{n+1}}{df \wedge \Omega^n} \cong \frac{\mathbb{C}\{x, y\}}{(\partial_x(f), \partial_y(f))}$$

$\{gdx \wedge dy, g \in B\}$ can be considered as a vector space basis of H''/sH''. The multiplicities d_i appear as the dimension of (α_i, l_i)-graded part of the weight filtration on H''/sH'', i.e.,

$$d_i = \dim_{\mathbb{C}} Gr_V^{\alpha_i} Gr_{l_i}^W (H''/sH'').$$

In this example the spectral numbers are

$$-1/2, -3/10, -3/10, -1/10, -1/10, 0, 1/10, 1/10, 3/10, 3/10, 1/2$$

and all the Jordan blocks of monodromy are of size 1, except one for eigenvalue -1, which is of size two.

6.5 The form of K. Saito (duality on Gauss-Manin module)

In this subsection we provide a definition of K. Saito higher residue pairing on V^{-1}. Later in Chapter 9 we generalize this definition. We define a non-degenerate bilinear form P_S on $V^{>-1}$ originally due to K. Saito, [S1] and [S2].

Definition 6.5.1. [H1], [S1] Define the bilinear form; $P_S : V^{>-1} \times V^{>-1} \to \mathbb{C}\{\{\partial_t^{-1}\}\}\partial_t^{-1}$, as follows;

- $P_S(a,b) = 0, \quad \alpha + \beta \notin \mathbb{Z}$
- $P_S(a,b) = \frac{1}{(2\pi i)^n} S(\psi^{-1}a, \psi^{-1}b)\partial_t^{-1}, \qquad \alpha + \beta = -1.$
- $P_S(a,b) = \frac{1}{(2\pi i)^{n+1}} S(\psi^{-1}a, \psi^{-1}b)\partial_t^{-2}, \qquad \alpha = \beta = 0$
- $P_S(g_1(\partial_t^{-1})a, g_2(\partial_t^{-1})b) = g_1(\partial_t^{-1}) g_2(-\partial_t^{-1}) P_S(a,b)$
- $P_S(a,b) = \sum_{l \geq 1} P_S^{(-l)}, \quad P_S^{(-l)} \in \mathbb{C}\partial_t^{-l}.$

Proposition 6.5.2. *[H1]*

(1) $P_S(H_0'', H_0'') \subseteq \mathbb{C}\{\{\partial_t^{-1}\}\}\partial_t^{-n-1}$ *that is* $P_S^{(-l)}(H_0'', H_0'') = 0, \; 1 \leq l \leq n$

(2) $P_S^{(-n-1)}(s[\omega_1], s[\omega_2]) = Res_f(\omega_1, \omega_2)\partial_t^{-n-1}$. $\omega_1, \omega_2 \in \Omega_{X,0}^{n+1}$

Corollary 6.5.3. *[H1]*

- *H_0'' is isotropic of maximal size w.r.t the anti-symmetric bilinear form $P_S^{(-n)}$.*

- $H_0'' \supseteq V^{n-1}, \ \dim\big(\frac{H_0''}{V^{n-1}}\big) = \frac{1}{2}\dim\big(\frac{V^{>-1}}{V^{n-1}}\big).$

  We have the following orthogonality relations for the form K_f;

- $P_S : C^\alpha \times C^\beta \to 0, \qquad \alpha > -1, \beta > -1$

- $P_S : C^\alpha \times C^\beta \to \mathbb{C}\partial_t^{-\alpha-\beta-2}, \qquad \alpha + \beta \in \mathbb{Z}.$

The last form is non-degenerate and $(-1)^{\alpha+\beta+n+1}$- symmetric, [H2].

If $A \in H_{e^{-2\pi i\alpha}}, B \in H_{e^{-2\pi i\beta}}, \alpha, \beta \in]-1, 0[$, then

$$P_S(s(A,\alpha), s(B,\beta)) = \frac{1}{(2\pi i)^n} S(A,B).\partial_t{}^{-1}, \qquad \alpha + \beta = -1$$
$$P_S(s(A,\alpha), s(B,\beta)) = \frac{1}{(2\pi i)^{n+1}} S(A,B).\partial_t{}^{-1}, \qquad \alpha = \beta = 0.$$

Chapter 7

Quasi-homogeneous Fibrations

This chapter is devoted to the case of homogeneous fibrations or more general the quasi-homogeneous polynomials. It may be considered as an example to many definitions already stated. Specifically we express the Griffiths-Steenbrink method to determine the limit Hodge filtration of quasi-homogeous polynomial fibrations. The last section describes the Steenbrink method of proving mixed Hodge structure of projective fibrations, using resolution of singularities.

7.1 Weighted Projective Space

We assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a quasi homogeneous polynomial of type $(w_0, ..., w_n)$ and $V \subseteq \mathbb{C}^{n+1}$ be defined by $f = 1$. Set

$$w_i = u_i/v_i, \qquad (u_i, v_i) = 1, \qquad d = lcm(v_0, ..., v_n), \qquad b_i = d.w_i$$

Then

$$f(z_0, ..., z_n) - z_{n+1}^d \tag{7.1}$$

would be quasi-homogeneous of type $(w_0, ..., w_n, 1/d)$. Let M be the weighted projective space of type $(w_0, ..., w_n, 1/d)$,

$$M = Proj\mathbb{C}[z_0, ..., z_{n+1}] \tag{7.2}$$

with $\deg(z_i) = b_i,\ i = 0, ...n,\ \deg(z_{n+1}) = 1$.

M is a compactification of $\mathbb{C}^{n+1}$ by putting $Z_i = z_i/z_{n+1}^{b_i}$. Moreover, the hyper-surface in M with equation

$$f(z_0, ..., z_n) - z_{n+1}^d = 0$$

is a compactification $\bar{V}$ of V. Denote $M_\infty = M - \mathbb{C}^{n+1}, V_\infty = \bar{V} - V = \bar{V} \cap M_\infty$. Then M_∞ is isomorphic to the weighted projective space of type $(w_0, ..., w_n)$ and $V_\infty \subseteq M_\infty$ is given by $f(z_0, ..., z_n) = 0$.

7.2 MHS for Quasi-homogeneous f

From now on assume f has an isolated singularity at 0. We briefly review the work of J. Steenbrink. $H^i(\bar{V}), H^i(V_\infty), i \geq 0$ carry Hodge structures which are purely of weight i. The canonical mixed Hodge structure on $H^i(V), i \geq 0$ can be computed using the logarithmic complex $\bar{\Omega}^\bullet_{\bar{V}}(\log V_\infty)$ which sits in the short exact sequence

$$0 \to \bar{\Omega}^\bullet_{\bar{V}} \to \bar{\Omega}^\bullet_{\bar{V}}(\log V_\infty) \to \bar{\Omega}^{\bullet -1}_{V_\infty} \to 0. \tag{7.3}$$

The long exact sequence associated;

$$.... \to H^i(\bar{V}) \to H^i(V) \to H^{i-1}(V_\infty)(-1) \to H^{i+1}(\bar{V}) \to . \tag{7.4}$$

Let f be given by the formula $f = \sum a_\beta z^\beta,\ \beta = (\beta_0, ..., \beta_n)$. We compute the mixed Hodge structure on $H^n(V)$ in terms of invariants of the Artinian ring

$$A_f = \mathbb{C}\{z_0, ..., z_n\}/(\partial f/\partial z_0, ..., \partial f/\partial z_n).$$

Let $\{z^\alpha | \alpha \in I \subset \mathbb{N}\}$ be a set of monomials in $\mathbb{C}\{z_0, ..., z_n\}$ whose residue classes form a basis of A_f. For $\alpha \in I$, let

$$l(\alpha) = \sum_{i=0}^{n} (\alpha_i + 1) w_i.$$

Define a rational $(n+1)$-form ω_α on $\mathbb{C}^{n+1}$ by

$$\omega_\alpha = z^\alpha (f(z) - 1)^{[-l(\alpha)]} dz_0 \wedge ... \wedge dz_n.$$

The long exact sequence (7.4) gives

$$Gr_n^W H^n(V) \cong H^n(\bar{V})_0$$
$$Gr_{n+1}^W H^n(V) \cong H^{n-1}(V_\infty)(-1)_0$$

where

$$H^n(\bar{V})_0 = Coker(H^{n-2}(V_\infty)(-1) \to H^n(\bar{V})$$
$$H^{n-1}(V_\infty)(-1)_0 = \ker(H^{n-1}(V_\infty)(-1) \to H^{n+1}(\bar{V}).$$

Assume N is a hyper-surface defined by a quasi-homogeneous polynomial in M. By using Bott's vanishing theorem repeatedly, one obtains identities of the form;

$$F^p H^{n-1}(N, \mathbb{C})_0 \cong \frac{H^0(M, \Omega_M^n((n-p)N)}{dH^0(M, \Omega_M^{n-1}((n-p-1)N)}. \tag{7.5}$$

Taking N to be $\bar{V}$ and V_∞ respectively, the following theorems are straight forward coordinate calculations, [JS7].

Proposition 7.2.1. *(P. Griffiths-J. Steenbrink) [JS7] If $l(\alpha) \notin \mathbb{Z}$, then the forms ω_α with $k < l(\alpha) < k+1$ map to a basis of*

$$\frac{H^0(M, \Omega_M^n(k\bar{V})}{H^0(M, \Omega_M^{n+1}(k\bar{V})) + dH^0(M, \Omega_M^n(k\bar{V})}.$$

Proposition 7.2.2. *[JS7] (P. Griffiths-J Steenbrink) If $l(\alpha) \in \mathbb{Z}$, define $\eta_\alpha = res_{M_\infty}\omega_\alpha$. Then the forms η_α with $l(\alpha) = k$ map to a basis of*

$$\frac{H^0(M_\infty, \Omega_{M_\infty}^n(kV_\infty)}{H^0(M_\infty, \Omega_{M_\infty}((k-1)V_\infty)) + dH^0(M_\infty, \Omega_{M_\infty}^{n-1}((k-1)V_\infty)}.$$

Theorem 7.2.3. *[JS7] (P. Griffiths, J. Steenbrink) Denote W and F the weight and Hodge filtration on $H^n(V, \mathbb{C})$. Then $Gr_k^W H^n(V) = 0$, $for\ k \neq n,\ n+1$. The forms η_α with $p < l(\alpha) < p+1$ form a basis for $Gr_F^p Gr_n^W H^n(V, \mathbb{C})$. The forms η_α with $\alpha = p$ form a basis for $Gr_F^p Gr_{n+1}^W H^n(V, \mathbb{C})$.*

The theorem 7.2.3 is a consequence of Propositions 7.2.1 and 7.2.2.

Remark 7.2.4. The filtration

$$0 \subset H^0(V, \hat{\Omega}^n) \subset H^1(V, \hat{\Omega}^{n-1}) \subset ... \subset H^{n-1}(V, \hat{\Omega}^1) \subset H^n(V, \hat{\Omega}^0 = \mathbb{C})$$

where $\hat{\Omega}^i$ is the subgroup of closed forms, is exactly the Hodge filtration, [G5].

The MHS on $H_c^n(V)$ is dual to $H^n(V)$, therefore

$$Gr_k^W H_c^n(V) = 0, \qquad k \neq n,\ n-1$$
$$W_{n-1} H_c^n(V) = \{\ \omega \in H_c^n(V) |\ \langle \omega,\ \eta \rangle = 0,\ \forall \eta \in W_n H^n(V)\ \}.$$

If $i : V \hookrightarrow \bar{V}$ is the inclusion, we have

$$\begin{array}{ccc} H_c^n(V) & \xrightarrow{i_*} & H_c^n(\bar{V}) \\ j \downarrow & & \bar{j} \downarrow \cong \\ H^n(V) & \xrightarrow{i^*} & H^n(\bar{V}) \end{array}$$

where j is the natural map that is also a morphism of Hodge structures. The bilinear (intersection) form on $H_c^n(V)$ is given by $S(x, y) = \langle x, j(y) \rangle$. It follows that $S(x, y) = 0$ if x or $y \in W_{n-1} H_c^n(V)$. We also have $S(y, x) = (-1)^{n(n-1)/2} S(x, y)$. Moreover, i_* identifies $Gr_n^W H_n^W(V)$ with the primitive part of $H_c^n(\bar{V})$, and hence S is described as follows on $Gr_n^W H_c^n(V)$. Denote

$$Gr_n^W H_c^n(V) = \oplus_{p+q=n} H^{p,q}(V)$$

the Hodge decomposition then,

(a) $S(x, y) = 0,\ x \in H^{p,q},\ y \in H^{r,s},\ (p, q) \neq (r, s)$.

(b) If $x \in H^{p,q},\ x \neq 0$, then $(-1)^{n(n-1)/2}.i^{p-q} S(x, \bar{x}) > 0$.

cf. [JS7]. This Riemann-Hodge bilinear relation is the same as 3.2.1 and 5.3.4 proved before. We summarize it in the following theorem, not stated in [JS7].

Theorem 7.2.5. *(Riemann-Hodge bilinear relations) The Riemann-Hodge bilinear relations of polarised MHS on $H^n = H^n(X_\infty, \mathbb{C})$ where $X_\infty = f^{-1}(1)$ with f a quasi-homogeneous polynomial, can be explained via the isomorphisms $Gr^W_{n-1}H^n_c \cong Gr^W_{n+1}H^n$, $Gr^W_n H^n_c \cong Gr^W_n H^n$ by (a) and (b), where $S(x,y) := S(x, j(y))$, for $x, y \in H^n_c$, and $j : H^n_c \to H^n$ the natural map.*

Corollary 7.2.6. *(A. Varchenko) Suppose that n is even. Then*

$$\mu_+ = \sum_{\substack{q \ even, \\ p+q=n}} \dim H^{p,q}, \qquad \mu_- = \sum_{\substack{q \ odd, \\ p+q=n}} \dim H^{p,q} \ ,$$

$$\mu_0 = \dim Gr^W_{n+1} H^n(V)$$

where μ_+, μ_- and μ_0 are the number of positive, negative and zero eigenvalues of the intersection form.

Remark 7.2.7. [JS7] The $l(\alpha)$'s are also the eigenvalues of the Gauss-Manin connection, namely $\bigtriangledown z^\alpha = l(\alpha) z^\alpha$.

7.3 Examples

(1) [JS7], $f_a = x^3 + y^3 + z^3 + 3axyz,\ a^3 \neq 1$

The following monomials form a basis of the Jacobi algebra

$$1 \qquad x \qquad y \qquad z \qquad xy \qquad xz \qquad yz \qquad xyz,$$

and the corresponding weights for the forms are

$$l(\alpha) : 1 \qquad 4/3 \qquad 4/3 \qquad 4/3 \qquad 5/3 \qquad 5/3 \qquad 5/3 \qquad 2.$$

Using Theorem 7.2.3 we get

$$h^{2,0} = h^{0,2} = 0,\ h^{1,1} = 6,\ h^{1,2} = h^{2,1} = 1.$$

So

$$\mu_+ = 0,\ \mu_- = 6,\ \mu_0 = 2.$$

(2) [KUL] When f is quasi-homogeneous of weight $(w_0, ..., w_n)$ we have the Euler relation,

$$f = \sum_{i=0}^{n} w_i.z_i \partial_i(f).$$

There exists a form η such that $f.dx = df \wedge \eta$. We have the explicit form

$$\eta = \sum_{i=0}^{n} (-1)^i w_i.z_i d\underline{z}.$$

Second, the image of the inclusion $H' \overset{df}{\hookrightarrow} \mathcal{H}^{(0)}$ coincides with $f\mathcal{H}^{(0)} = t\mathcal{H}^{(0)}$, that is a sub-module of $\mathcal{H}^{(0)}$ generated by the maximal ideal $(t) \subset \mathcal{O}_T$. So by Nakayama's Lemma, in order to find a basis $\omega_1, ..., \omega_\mu$ of the $\mathcal{O}_T$ module $\mathcal{H}^{(0)}$, it is enough to find a basis in a vector space $f_*\Omega_f = A_f$. So if $\{z^\alpha\}$ is a monomial basis for A_f, then $\omega_\alpha = z^\alpha d\underline{z}$, represents a basis of the $\mathcal{O}_T$-module Ω_f. From $f.dx = df \wedge \eta$, we obtain $f.\omega_\alpha = df \wedge z^\alpha \eta$. This implies that

$$\partial_t \omega_\alpha = \frac{1}{f}[-\omega_\alpha + d(z^\alpha \eta)] = \frac{1}{f}\{-\omega_\alpha + [\sum_{i=0}^{n} w_i(m_i + 1)]\omega_\alpha\}.$$

Putting

$$l(\omega_\alpha) = \sum_{i=0}^{n} w_i(m_i + 1)$$

we then obtain

$$t\partial_t(\omega_\alpha) = [l(\alpha) - 1]\omega_\alpha.$$

Thus, the monodromy of a quasi-homogeneous fibration is semi-simple with eigenvalues $\lambda_\alpha = e^{-2\pi i.l(\alpha)}$, where α coming from a monomial basis of A_f, the Jacobi algebra. By the isomorphism

$$H''/sH'' \cong \Omega^{n+1}/df \wedge \Omega^n \cong \mathbb{C}\{z\}/(\partial(f))$$

$\{[\omega_\alpha = z^\alpha d\underline{z}]\}$ gives a vector space basis of H''/sH''.

(3) Suppose Y is a projective hypersurface of dimension n. Then

$$H^n(Y, \mathbb{C}) = \bigoplus_{0 \leq p \leq n/2}^{\perp} (H^{p,n-p} \oplus H^{n-p,p})$$

with respect to the cup product,

$$S = \oplus^{\perp} S_p : \bigoplus_p (H^{p,n-p} \otimes H^{n-p,p}) \to \mathbb{C}$$

each of the forms S_p induce a definite hermitian form on $H^{n,n-p}$.

If we are involved with a family of projective varieties parametrized by $t \in T$, then we will obtain a family of structures as above. For each t we can always choose a basis $\{\epsilon_i\}$ of W_k such that $\nabla(\epsilon_i) = 0$ and also another basis for F^p satisfying Griffiths transversality. It is always possible to express each of these bases in terms of the other one with coefficients being multi-valued functions of t. However, the total expression for ϵ_i is always uni-valued. We apply the limit definition of W. Schmid to obtain a canonical Hodge structure compatible with usual short exact sequences as in 4.2.

7.4 MHS via resolution of singularities

The limit Hodge structure in a projective fibration can be explained using a Hironaka resolution of singularities argument as follows. Assume the fibration is explained by the diagram

$$\begin{array}{ccccccc} X_\infty & \longrightarrow & U & \longrightarrow & X & \longleftarrow & D \\ {\scriptstyle f_\infty}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow \\ H & \xrightarrow{e} & \Delta^* & \longrightarrow & \Delta & \longleftarrow & 0 \end{array} \tag{7.6}$$

as in 2.9, where $D = \bigcup_i^m D_i$ is a normal crossing divisor.

Lemma 7.4.1. *[JS3] The spectral sequence of the double complex*

$$A_D^{pq} := (a_{q+1})_* \Omega^p_{D^{(q+1)}}, \quad D^{(q+1)} = \amalg D_{i_1} \cap ... \cap D_{i_q}$$

with horizontal arrows to be the de Rham differentials, and vertical arrows $\sum_j (-1)^{q+j} \delta_j$*, induced by the inclusions* $\amalg D_{i_1} \cap ... \cap D_{i_q} \hookrightarrow \amalg D_{i_1} \cap .. \hat{D_{i_j}} .. \cap D_{i_q}$ *obtained by possible omitting the indices, on* A_D^{pq}*, degenerates at* E_2*, with*

$$E_1^{pq} = H^q(D^{(p+1)}, \mathbb{C}) \Rightarrow H^{p+q}(D, \mathbb{C})$$

and computes the cohomologies of D*. Moreover, the two filtrations*

$$F^p A_D^{\cdot\cdot} = \bigoplus_{r \geq p} A_D^{r\cdot}, \qquad W_q A_D^{\cdot\cdot} = \bigoplus_{s \geq -q} A_D^{\cdot s}$$

induce the Hodge and the weight filtrations on $H^k(D, \mathbb{C})$ *for each* k*, to define a mixed Hodge structure.*

The above lemma is a generalization of 2.3.7, and can be obtained by an inductive argument as well.

Proposition 7.4.2. *[JS2] The spectral sequence of* $B^{pq} := A^{pq}/W_q$ *degenerates at* E_2 *term with*

$$E_1^{-r,q+r} = \bigoplus_{k \geq 0, r} H^{q-r-2k}(D^{(2k+r+1)}, \mathbb{C})(-r-k) \Rightarrow H^q(X_\infty, \mathbb{C})$$

and equips $H^q(X_\infty, \mathbb{C})$ *with a mixed Hodge structure.*

These MHS's fit in the Clemens-Schmid exact sequence ([PS] page 285),

$$... \to H_{2n+2-m}(X_0) \xrightarrow{\alpha} H^m(X_0) \xrightarrow{i_t^*} H^m(X_t) \xrightarrow{N} H^m(X_t) \xrightarrow{\beta} H_{2n-2m}(X_0) \to ... \tag{7.7}$$

with $X_0 = D$ and $X_t = f^{-1}(t) \cong X_\infty$, and where α is induced by Poincaré duality followed by projection, and β is by inclusion followed by Poincaré duality. The monodromy weight filtration on $H^m(X_0)$ can be described using the hypercover structure obtained by intersections of NC divisors. Then the weight filtration of $H^m(X_t)$ can be computed via the induced filtration on $\ker(N) =: K_t^m$, with N is the logarithm of monodromy, and satisfies;

$$Gr_k^W H^m(X_t) \cong \begin{cases} Gr_k K_t^m \oplus Gr_{m-2} K_t^m \oplus ... \oplus Gr_{k-2[k/2]} K_t^m, & k \leq m \\ Gr_{2m-k} H^m(X_t), & k > m \end{cases}$$

The relations between the weight filtrations can be explained by the following.

- i_t^* induces
$$Gr_k H^m(X_0) \cong Gr_k K_t^m$$
- The following sequence is exact
$$0 \to Gr_{m-2} K_t^{m-2} \to Gr_{m-2n-2} H_{2n+2-m}(X_0) \stackrel{\alpha}{\to} Gr_m H^m(X_0) \to Gr_m K_t^m \to 0.$$

Then we have the long exact sequence

$$... \to H^{m-1}(X \setminus X_0) \to H^m(X, X \setminus X_0) \to H^m(X) \to H^m(X \setminus X_0) \to ...$$

where the morphisms are of MHS, and the isomorphism

$$H^{2n+2-m}(X, X \setminus X_0) \cong H_c^{2n+2-m}(X) \cong H^m(X)^\vee$$

computes $H^m(X \setminus X_0)$ as MHS, [MO]. The reader should convince himself that the mixed Hodge structure defined in this section is the same as the one in 7.2.3, and the Riemann-Hodge bilinear relations are as 3.2.1 or the same as in 5.3.4.

Chapter 8

Polarization of extended fiber

This chapter concerns the main contributions of the text. Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a holomorphic germ with isolated singularity at $0 \in \mathbb{C}^{n+1}$. Up to now we studied the limit mixed Hodge structure on the cohomology of the canonical fiber of f namely vanishing cohomology. We extend the Gauss-Manin system of f to the whole disc T in a way that the stack on 0 of the extended Gauss-Manin module can be naturally identified with the module of relative $(n+1)$-differentials Ω_f. Next, we equip this fiber with a mixed Hodge structure. Our major task is to relate the polarization forms. We show that a modification of Grothendieck residue provides the polarization for the MHS of Ω_f.

8.1 Steenbrink limit Hodge filtration (review)

Suppose we have an isolated singularity holomorphic germ $f : \mathbb{C}^{n+1} \to \mathbb{C}$. By the Milnor fibration Theorem we can always associate to f a C^∞-fiber bundle over a small punctured disc T'. The corresponding cohomology bundle $\mathcal{H}$, constructed from the middle cohomologies of the fibers defines a variation of mixed Hodge structure on T'. The Brieskorn lattice is defined by,

$$\mathcal{H}^{(0)} = H'' = f_* \frac{\Omega^{n+1}_{X,0}}{df \wedge d\Omega^{n-1}_{X,0}}.$$

The Brieskorn lattice is the stack at 0 of a locally free $\mathcal{O}_T$-module $\mathcal{H}''$ of rank μ with $\mathcal{H}''_{T'} \cong \mathcal{H}$, and hence $H'' \subset (i_*\mathcal{H})_0$, with $i : T' \hookrightarrow T$. The regularity of the Gauss-Manin connection proved by Brieskorn and Malgrange implies that $H'' \subset \mathcal{G}$, cf. 6.1.

Theorem 8.1.1. *(Malgrange) (see [SCHU] 1.4.10)* $H'' \subset V^{-1}$

The embedding is via the map s defined in (6.14). The Leray residue formula can be used to express the action of ∂_t in terms of differential forms by

$$\partial_t^{-1} s[d\omega] = s[df \wedge \omega]$$

.

where, $\omega \in \Omega_X^n$ cf. notation in 4.1. In particular , $\partial_t^{-1}.H'' \subset H''$, and

$$\frac{H''}{\partial_t^{-1}.H''} \cong \frac{\Omega_{X,0}^{n+1}}{df \wedge \Omega_{X,0}^n} \cong \frac{\mathbb{C}\{\underline{z}\}}{(\partial(f))}. \tag{8.1}$$

we keep the notation

$$\Omega_f \cong \frac{\Omega_{X,0}^{n+1}}{df \wedge \Omega_{X,0}^n} \tag{8.2}$$

for the module of relative differentials of the map f. The Hodge filtration on $H^n(X_\infty, \mathbb{C})$ is defined by

$$F^p H^n(X_\infty)_\lambda = \psi_\alpha^{-1} \partial_t^{n-p} Gr_V^{\alpha+n-p} H'' \tag{8.3}$$

cf. 6.3, where ψ_α was defined in 4.1 and 6.2. Therefore,

$$Gr_F^p H^n(X_\infty, \mathbb{C})_\lambda = Gr_V^{\alpha+n-p} \Omega_f \tag{8.4}$$

where Gr_V^β is defined as follows.

Definition 8.1.2. (cf. [KUL] page 110) The V-filtration on Ω_f is defined by

$$V^\alpha \Omega_f = pr(V^\alpha \cap H'') \tag{8.5}$$

Clearly $V^\alpha \Omega_f = \oplus_{\beta \geq \alpha} \Omega_f^\beta$ and $\Omega_f \cong \oplus Gr_V^\alpha \Omega_f$ hold.

8.2 Theorem of Varchenko on multiplication by f

A theorem of A. Varchenko, shows the relation between the operator N, on vanishing cohomology and multiplication by f on Ω_f. A feature of this theorem appears in Theorems 8.7.1 and 8.7.5.

Theorem 8.2.1. *[SC2] The maps $Gr(f)$ and $N = \log M_u \in End(H^n(X_\infty, \mathbb{C}))$ have the same Jordan normal forms.*

Proof. The map N is a morphism of mixed Hodge structures of type $(-1,-1)$. Hence, all the powers of N are strictly compatible with the filtration F (with the appropriate shift). This implies the existence of a splitting of the Hodge filtration, i.e a grading of $H^n(X_\infty, \mathbb{C})$ which has F as its associated filtration, such that N becomes a graded morphism of degree -1. In particular, one concludes that N and its induced endomorphism $Gr_F N$ of degree -1 of $Gr_F H^n(X_\infty, \mathbb{C})$, have the same Jordan normal forms.

We have a canonical isomorphism

$$Gr_F H^n(X_\infty, \mathbb{C}) = \bigoplus_{-1<\alpha\leq 0} Gr_F C^\alpha$$

and the corresponding endomorphism

$$N_{F,\alpha} : Gr_F^p C^\alpha \to Gr_F^{p-1} C^\alpha$$

are given by

$$N_{p,\alpha}(x) = -2\pi i(t\partial_t - \alpha)x \cong -2\pi i.t\partial_t x \qquad (\mod F^p).$$

On the other hand, it is immediately seen that for $\beta \in \mathbb{Q}$, $\beta = n - p + \alpha$ with $p \in \mathbb{Z}$ and $-1 < \alpha \leq 0$, the map

$$\partial_t^{n-p} : V^\beta \cap F^n \mathcal{H}_{X,0} \to V^\alpha / V^{>\alpha} = C^\alpha$$

induces an isomorphism from $Gr_\beta^V \Omega_f \to Gr_F^p C^\alpha$, and the diagram

$$\begin{array}{ccc} Gr^V_\beta \Omega_f & \xrightarrow{Gr(f)} & Gr^V_{\beta+1}\Omega_f \\ \partial_t^{n-p}\Big\downarrow & & \partial_t^{n-p+1}\Big\downarrow \\ Gr^p_F C^\alpha & \xrightarrow{N_{p,a}} & Gr^{p-1}_F C^\alpha \end{array}$$

commutes up to a factor of $-2\pi i$. Hence $Gr(f)$ and $Gr_F N$ have the same Jordan normal form.

□

8.3 Integrals along Lefschetz thimbles

The brief of this section involves with a technical issue applied in the proof of the main contribution in section 8.6. Consider the function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ with isolated singularity at 0, and a holomorphic differential $(n+1)$-form ω given in a neighborhood of the critical point. We shall study the asymptotic behavior of the integral,

$$\int_\Gamma e^{\tau f}\omega \tag{8.6}$$

for large values of the parameter τ, namely a complex oscillatory integral. In the long exact homology sequence of the pair (X, X_t) where X is a tubular neighborhood of the singular fiber X_0 in the Milnor ball. We have

$$... \to H_{n+1}(X) \to H_{n+1}(X, X_t) \stackrel{\partial_t}{\to} H_n(X_t) \to H_n(X) \to .... \tag{8.7}$$

Because X is contractible, we get an isomorphism $\partial_t : H_{n+1}(X, X_t) \cong H_n(X_t)$, and similar in cohomologies, i.e. $H^{n+1}(X, X_t) \cong H^n(X_t)$. Now if ω is a holomorphic differential $(n+1)$-form on X, and let $\Gamma \in H_{n+1}(X, X_t)$, we have the following.

Proposition 8.3.1. *(cf. [AGV] Theorems 8.6, 8.7, 8.8, 11.2)*
Assume $\omega \in \Omega^{n+1}$, *and let* $\Gamma \in H_n(X, X_t)$. *Then*

$$\int_{\Gamma} e^{-\tau f}\omega = \int_0^{\infty} e^{-t\tau} \int_{\Gamma\cap\{f=t\}} \frac{\omega}{df}_{|X_t} dt = e^{\tau . f(0)} \int_{\Gamma\cap\{f=t\}} \frac{\omega}{df}_{|X_t} \qquad (8.8)$$

for $Re(\tau)$ large, and in this way can also be expressed as $\sum \tau^{\alpha} \log \tau^{k} A_{\alpha,k}$ in that range.

By Theorem 8.3.1, we identify the cohomology classes $\int_{\Gamma} e^{-\tau f}\omega$ and $\int_{\Gamma\cap\{f=t\}} \frac{\omega}{df}_{|X_t}$ via integration on the corresponding homology cycles. We can also choose Γ such that its intersection with each Milnor fiber has compact support, and its image under f is the positive real line, [PH]. We use these facts in the proof of Theorem 8.6.1.

The asymptotic integral

$$I(\tau) = \int e^{\tau f} \phi dx_0 ... dx_n, \qquad \tau \to \pm\infty$$

satisfies

$$\frac{d^p}{d\tau^p} I = \int e^{\tau f} f^p \phi dx_0 ... dx_n.$$

In case f is analytic then it has an asymptotic expansion

$$I(\tau) = \sum_{\alpha,p,q} c_{\alpha,p,q}(f) \tau^{\alpha-p} (\log \tau)^q, \qquad \tau \to +\infty$$

for a finite number of rational numbers $\alpha < 0,\ p \in \mathbb{N},\ 0 \leq q \leq n-1$. Then $\phi \to c_{\alpha,p,q}(\phi)$ is a distribution with support contained in the support of f, [MA].

Remark 8.3.2. (see [PH] page 27) We have the formula;

$$I(\tau) = (2\pi)^{n/2} (Hessf)^{-1/2} f(0) \tau^{-n/2} [1 + O(1/\tau)].$$

Theorem 8.3.1 says, the form $e^{-\tau f}\omega$ (for τ large enough) and the form $\frac{\omega}{df} |_{X_t}$, define the same cohomology classes via integration on cycles. We will use this fact together with the following proposition in the proof of Theorem 8.6.1.

Proposition 8.3.3. *([AGV] lemma 11.4, 12.2, and its corollary) There exists a basis* $\omega_1, ..., \omega_\mu$ *of* Ω_f *such that the corresponding Leray residues* $\omega_1/df, ..., \omega_\mu/df$ *define a basis for the sections of vanishing cohomology.*

8.4 Extension of the Gauss-Manin system

As mentioned in the introduction the Gauss-Manin system of an isolated hypersurface singularity can be extended over the puncture by a process of gluing of vector bundles. The gluing is done by some comparison between V-lattices and the Brieskorn lattices in $\mathcal{G}$. By this we mean to glue the Gauss-Manin system $\mathcal{G}$ defined before with another one defined in a chart around 0. We are interested to understand the fiber $\mathcal{G}$ on 0 after the extension.

The Gauss-Manin system $\mathcal{G} := R^n f_* \mathbb{C}_{X'}$ of a polynomial or holomorphic map $f : X' \to T'$ is a module over the ring $\mathbb{C}[\tau, \tau^{-1}]$, where τ is a new variable, and comes equipped with a connection, that we view as a $\mathbb{C}$-linear morphism $\partial_\tau : \mathcal{G} \to \mathcal{G}$ satisfying Leibnitz rule

$$\partial_\tau(\phi.g) = \frac{\partial \phi}{\partial \tau}.g + \phi \partial_\tau(g).$$

We put $\tau = t^{-1}$, and consider (τ, t) as coordinates on $\mathbb{P}^1$. Then $\mathcal{G}$ is a $\mathbb{C}[t, t^{-1}]$-module with connection and $\partial_\tau = -t^2 \partial_t$, [SA6], [SA5], [SA8].

Let $\Omega^{n+1}[\tau, \tau^{-1}]$ be the space of Laurent polynomials with coefficients in Ω^{n+1}. According to its very definition (cf. Theorem 6.3.7, [PS] sec. 10.4, [SAI6] lemma 2.4), the Gauss-Manin System is given by;

$$\mathcal{G} = \frac{\Omega^{n+1}[\tau, \tau^{-1}]}{(d - \tau df \wedge)\Omega^{n+1}[\tau, \tau^{-1}]},$$

$$(d - \tau df \wedge) \sum_k \eta_k \tau^k = \sum_k (d\eta_k - df \wedge \eta_{k-1})\tau^k.$$

The action of the connection ∇_τ on $\mathcal{G}$ i.e. the $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module structure on G, is first defined on the image of Ω^{n+1} by

$$\partial_\tau[\omega] = [f\omega]$$

and then extended to $\mathcal{G}$ using the Leibnitz rule

$$\partial_\tau(\tau^p[\omega]) = p\tau^{p-1}[\omega] + \tau^p[f\omega].$$

In order to extend it as a rank μ-vector bundle on $\mathbb{P}^1$, one is led to study lattices i.e. $\mathbb{C}[\tau]$, and $\mathbb{C}[t]$-submodules which are free of rank μ.

In the chart t, the Brieskorn lattice

$$\mathcal{G}_0 = image(\Omega^{n+1}[\tau^{-1}] \to \mathcal{G}) = \frac{\Omega^{n+1}[t]}{(td - df\wedge)\Omega^{n+1}[t]}$$

is a free $\mathbb{C}[t]$ module of rank μ. It is stable by the action of $\partial_\tau = -t^2\partial_t$. Therefore ∂_t is a connection on $\mathcal{G}$ with a pole of order 2. We consider the increasing exhaustive filtration $\mathcal{G}_p := \tau^p\mathcal{G}_0$ of $\mathcal{G}$.

In the chart τ, there are various natural lattices indexed by $\mathbb{Q}$, we denote them by V^α, with $V^{\alpha-1} = \tau V^\alpha$. On the quotient space $C_\alpha = V^\alpha/V^{>\alpha}$ there exists a nilpotent endomorphism $(\tau\partial_\tau - \alpha)$.

The space $\oplus_{\alpha\in[0,1[}C_\alpha$ is isomorphic to $H^n(X_\infty, \mathbb{C})$ cf. def. 4.1, or the same 6.1, and $\oplus_{\alpha\in[0,1[}F^pC_\alpha$ is the limit MHS on $H^n(X_\infty, \mathbb{C})$, cf. 8.1 and 6.3. A basic isomorphism can be constructed cf. Theorem 6.4.5, as

$$\begin{array}{c} \dfrac{\mathcal{G}_p \cap V^\alpha}{\mathcal{G}_{p-1}\cap V^\alpha + \mathcal{G}_p \cap V^{>\alpha}} = Gr_F^{n-p}(C_\alpha) \\ \tau^p \Big\downarrow \cong \\ \dfrac{V^{\alpha+p}\cap \mathcal{G}_0}{V^\alpha \cap \mathcal{G}_{-1} + V^{>\alpha}\cap\mathcal{G}_0} = Gr^V_{\alpha+p}(\mathcal{G}_0/\mathcal{G}_{-1}) \end{array}.$$

Thus, the gluing is done via the isomorphisms,

$$Gr_F^{n-p}(H_\lambda) \cong Gr_{\alpha+p}^V(\mathcal{H}^{(0)}/\tau^{-1}.\mathcal{H}^{(0)}), \qquad \mathcal{H}^{(0)} = \mathcal{G}_0$$

where $\lambda = exp(2\pi i\alpha)$ and we have chosen $-1 \leq \alpha < 0$ (cf. [SA3], [SA5], [SA6]). We have

$$\frac{\mathcal{H}^{(0)}}{\tau^{-1}.\mathcal{H}^{(0)}} = \frac{\Omega^{n+1}}{df \wedge \Omega^n} = \Omega_f \tag{8.9}$$

canonically. We conclude that;

Theorem 8.4.1. *The identity (8.9) defines the extension fiber of the Gauss-Manin system of the isolated hypersurface singularity* $f : \mathbb{C}^{n+1} \to \mathbb{C}$.

The same conclusion can be obtained when f is a holomorphic germ, However one needs to consider the completions of the modules involved, (see [MA] page 422 or [S1]). In this way for $f : \mathbb{C}^{n+1} \to \mathbb{C}$ we have;

$$\frac{\widehat{H}''}{\tau^{-1}\widehat{H}''} \cong \Omega_f.$$

By identifying the sections with those of relative cohomology, via section 8.3, this formula is a direct consequence of the formula

$$\int_\Gamma e^{-\tau f} d\omega = \tau \int_\Gamma e^{-\tau f} df \wedge \omega, \qquad \omega \in \Omega_X^n.$$

We refer to [MA] page 422 for details on this (see also section 10.2).

8.5 MHS on the extended fiber

Asuume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a germ of isolated singularity. As previously mentioned, the Gauss-Manin system of f can be extended to the whole disc T using standard isomorphisms between V-lattices and Bieskorn lattices. This method identified the new fiber above the degenerate point 0 by that of Brieskorn lattice namely Ω_f. In this section we build up an isomorphism

$$\Phi : H^n(X_\infty, \mathbb{C}) \to \Omega_f$$

which allows us to equip a mixed Hodge structure on Ω_f. This also motivates the definition of opposite filtrations. It is based on the following theorem.

Proposition 8.5.1. *([H1] prop. 5.1) Assume $\{(\alpha_i, d_i)\}$ is the spectrum of a germ of isolated singularity $f : \mathbb{C}^{n+1} \to \mathbb{C}$. There exists elements $s_i \in C^{\alpha_i}$ with the properties*

(1) $s_1, ..., s_\mu$ project onto a $\mathbb{C}$-basis of $\bigoplus_{-1<\alpha<n} Gr_V^\alpha H''/Gr_V^\alpha \partial_t^{-1} H''$.

(2) $s_{\mu+1} := 0$; there exists a map $\nu : \{1, ..., \mu\} \to \{1, ..., \mu, \mu + 1\}$ with $(t - (\alpha_i + 1)\partial_t^{-1})s_i = s_{\nu(i)}$.

(3) There exists an involution $\kappa : \{1, ..., \mu\} \to \{1, ..., \mu\}$ with $\kappa(i) = \mu+1-i$ if $\alpha_i \neq \frac{1}{2}(n-1)$ and $\kappa(i) = \mu + 1 - i$ or $\kappa(i) = i$ if $\alpha_i = \frac{1}{2}(n-1)$, and

$$P_S(s_i, s_j) = \pm\delta_{(\mu+1-i)j}.\partial_t^{-1-n}.$$

where P_S is the higher residue pairing of K. Saito.

Condition (1) implies

$$Gr_V^\alpha \partial_t^q H'' = \bigoplus_{\alpha_i - p = \alpha,\ p \leq q} \mathbb{C}.\partial_t^p.s_i$$

Condition (2) can be replaced by

[(2')] $(t - (\alpha_i + 1)\partial_t^{-1})s_i \in \bigoplus_{\alpha_j = \alpha_i + 1} \mathbb{C}.s_j$

in which case the involution $\kappa(i) = \mu - i + 1$ for any i, [H1]. The proof of the proposition 8.5.1 concerns with the construction of a $\mathbb{C}$-linear isomorphism as follows. Suppose,

$$H^n(X_\infty, \mathbb{C}) = \bigoplus_{p,q,\lambda} I_\lambda^{p,q}$$

is the Deligne-Hodge bigrading, and generalized eigenspaces of vanishing cohomology cf. 3.3.3, and also $\lambda = \exp(-2\pi i\alpha)$ with $\alpha \in (-1, 0]$. Consider the isomorphism obtained by composing the three maps,

$$\Phi_\lambda^{p,q} : I_\lambda^{p,q} \xrightarrow{\hat{\Phi}_\lambda} Gr_V^{\alpha+n-p} H'' \xrightarrow{pr} Gr_V^{\bullet} H''/\partial_t^{-1} H'' \xrightarrow{\cong} \Omega_f \tag{8.10}$$

where

$$\hat{\Phi}_\lambda^{p,q} := \partial_t^{p-n} \circ \psi_\alpha | I_\lambda^{p,q}$$

$$\Phi = \bigoplus\nolimits_{p,q,\lambda} \Phi_\lambda^{p,q}, \qquad \Phi_\lambda^{p,q} = pr \circ \hat{\Phi}_\lambda^{p,q}$$

ψ_α is the isomorphism defined in section 6.1.

Lemma: The map Φ is a well-defined $\mathbb{C}$-linear isomorphism.

We list some of the properties of the map ϕ as follows;

- $\hat{\Phi}_\lambda^{p,q}$ takes values in $C^{\alpha+n-p}$. By the formula $F^p = \oplus_{r \geq p} I^{r,s}$, any cohomology class in $I_\lambda^{p,q}$, is of the form $\psi_\alpha^{-1}[\partial_t^{n-p} h'' + V^{>\alpha}] = \psi_\alpha^{-1} \partial_t^{n-p} [h'' + V^{>\alpha+n-p}]$, for $h'' \in H''$, cf. def. 6.3.3. By substituting in the formula it explains the image of $\hat{\Phi}_\lambda^{p,q}$.

- Taking two different representatives $\omega_1, \omega_2 \in \Omega_X^{n+1}$ for h'' has no effect on the class $h'' + V^{>\alpha+n-p}$. Because by identifying H'' with its image in V^{-1}, the difference $\omega_1 - \omega_2$ belongs to $V^{>\alpha+n-p}$.

- The map Φ is obviously a $\mathbb{C}$-linear isomorphism because both of the ψ_α and ∂_t^{-1} are $\mathbb{C}$-linear isomorphisms on the appropriate domains, and

$$\Phi(I_\lambda^{p,q}) \subset \Phi(F^p H^n(X_\infty)_\lambda) \subset V^\alpha \cap H''/V^{\alpha+1} \hookrightarrow gr_V^\alpha H''/\partial_t^{-1} H''.$$

- The definition of Φ concerns with an isomorphism $Gr_V^{\bullet} \Omega_f \cong \Omega_f$. On the eigenspace H_λ this corresponds to a choice of sections of

$$Gr_V^\alpha [V^\alpha \cap H''] \to Gr_V^\alpha [H''/\partial_t^{-1} H'']$$

  for $-1 \leq \alpha < 0$.

Definition 8.5.2. (MHS on Ω_f) The mixed Hodge structure on Ω_f is defined by using the isomorphism Φ. This means that

$$W_k(\Omega_f) = \Phi W_k H^n(X_\infty, \mathbb{Q}), \qquad F^p(\Omega_f) = \Phi F^p H^n(X_\infty, \mathbb{C})$$

and all the data of the Steenbrink MHS on $H^n(X_\infty, \mathbb{C})$ such as the $\mathbb{Q}$ or $\mathbb{R}$-structure is transformed via the isomorphism Φ to that of Ω_f. Specifically; in this way we also obtain a conjugation map

$$\bar{.} : \Omega_{f,\mathbb{Q}} \otimes \mathbb{C} \to \Omega_{f,\mathbb{Q}} \otimes \mathbb{C}, \qquad \Omega_{f,\mathbb{Q}} := \Phi H^n(X_\infty, \mathbb{Q}) \tag{8.11}$$

defined from the conjugation on $H^n(X_\infty, \mathbb{C})$ via this isomorphism.

The basis discussed in 8.5.2 is usually called a good basis. The condition (1) corresponds to the notion of opposite filtration. Two filtrations F and U on H are called opposite (cf. [SAI6] sec. 3) if

$$Gr_p^F Gr_U^q H = 0, \qquad \text{for } p \neq q.$$

When one of the filtrations is decreasing say $\{F^p\}$ and the other increasing say $\{U_q\}$ then this is equivalent to

$$H = F^p \oplus U_{p-1}, \qquad \forall p. \tag{8.12}$$

Two decreasing filtrations F and U are said to be opposite if F is opposite to the increasing filtration $U'_q := U^{k-q}$, [P2].

Proposition 8.5.3. *([SAI6] prop. 3.5) The filtration*

$$U^p C^\alpha := C^\alpha \cap V^{\alpha+p} H''$$

is opposite to the Hodge filtration F on $\mathcal{G}$.

By this theorem the two filtrations F^p and

$$U'_q := U^{n-q} = \psi^{-1}\{\oplus_\alpha C^\alpha \cap V^{\alpha+n-q} H''\} = \psi^{-1}\{\oplus_\alpha Gr_V^\alpha [V^{\alpha+n-q} H'']\}$$

are two opposite filtrations on $H^n(X_\infty, \mathbb{C})$. We also have

$$F^p H^n(X_\infty, \mathbb{C})_\lambda \cong U'_p H^n(X_\infty, \mathbb{C}).$$

A standard example is when the variation of MHS namely $\mathcal{H}$ is mixed Tate (also called Hodge-Tate). By definition a mixed Tate Hodge structure H is when $Gr^W_{2l-1} H = 0$ and $Gr^W_{2l} H = \oplus_i \mathbb{Q}(-n_i)$. One easily shows the Deligne-Hodge decomposition becomes

$$\bigoplus_p (W_{2p} \cap F^p) H = H_\mathbb{C}$$

and the two filtrations F and W are opposite. In a pure Hodge structure H of weight n one has

$$Gr_p^F Gr_q^{\overline{F}} H = 0 \qquad \textit{unless} \;\; p + q = n$$

In other words, the two filtrations $F^\bullet$ and $\overline{F}^{n-\bullet}$ are opposite in this case.

Proposition 8.5.4. *([SAI6] Theorem 3.6) There is a 1-1 correspondence between the opposite filtrations and the sections* $s : H''/\partial_t^{-1} H'' \to H''$ *compatible with the conditions of Theorem 8.5.1.*

In our situation this amounts to a choice of a section $s : H''/\partial_t^{-1} H'' \to H''$ of the projection $pr : H'' \to H''/\partial_t^{-1} H''$ such that the submodule generated by $Image(s)$ is $\oplus_\alpha (H'' \cap C^\alpha)$. Note also that $V^\alpha H''$ is the submodule generated by $s(V^\alpha \Omega_f)$.

The data of an opposite filtration in a VMHS is equivalent to give a linear subspace $\mathcal{L} \subset \mathcal{G}$ such that:

- $\mathcal{G} = \mathcal{H}^{(0)} \oplus \mathcal{L}$.
- $t^{-1} : \mathcal{L} \to \mathcal{L}$.
- $t\partial_t : \mathcal{L} \to \mathcal{L}$.

Example 8.5.5. This example is taken from [SAI6]. If n is even, the duality S on $H^4(X_\infty,\mathbb{C})_1$ is anti-symmetric. Assume $H^n(X_\infty,\mathbb{C})_1 = H' \oplus H''$ as a direct sum of MHS, compatible with S and N, where

$$H' = \oplus_{0\leq i\leq 3} H'_i, \qquad H'_i = \mathbb{Q}(-i-(n-2)/2),$$
$$NH'_i = H'_{i-1}(i>0)$$
$$H'' = \oplus_{0\leq i\leq 2} H''_i, \qquad H''_i = \mathbb{Q}(-i-(n-2)/2),$$
$$NH''_2 = H''_1,\ NH''_1 = NH''_0 = 0$$

and

$$S(H', H'') = 0$$
$$S(H'_i, H'_j) \neq 0, \qquad \text{only when } i+j=3$$
$$S(H''_i, H''_j) \neq 0, \qquad \text{only when } i+j=2.$$

Then we have $H'_i = F^pW_{2p}H'$ for $p = i+(n-2)/2$ and we obtain a filtration U opposite to F on $H' \oplus H''$ compatible with S. If we choose generators as $H'_i = < e_i >$, $H''_i = < f_i >$ such that $S(e_0, e_3) = S(f_1, f_2)$. then the splitting F^pW_{-2p}, $p = i+(n-2)/2$ is generated by $< e_3 >$ $(i=3)$, $< e_2, f_2 + e_3 >$, $(i=2)$ and $< e_1, f_1 >$ $(i=1)$, $< e_0 - f_1 >$, $(i=0)$. By this the aforementioned Deligne-Hodge decomposition becomes,

$$H^4(X_\infty,\mathbb{C})_1 = < e_3 > \oplus < e_2, f_2 + e_3 > \oplus < e_1, f_1 > \oplus < e_0 - f_1 > .$$

For the corresponding section we have

$$P_S(Im(v), Im(v)) \subset \mathbb{C}\partial_t^{-1-n}.$$

The situation explained in this example appears for the singularity $f = x^{10} + y^{10} + z^{10} + w^{10} + (xyzw)^2 + v^2$, cf. [SAI6] page 60.

Remark 8.5.6. ([SAI6] page 42) By definition we have the isomorphism

$$H^n(X_\infty,\mathbb{C})_\lambda \cong gr_V^\alpha H'', \qquad -1 \leq \alpha < 0.$$

It is compatible with the Hodge filtration;

$$F^p H^n(X_\infty, \mathbb{C})_\lambda \cong \partial_t^{n-p} gr_V^\alpha H''.$$

In general

$$H'' \cap V^\alpha \mathcal{G} \neq (H'' \cap V^\alpha) + (H'' \cap V^{>\alpha}).$$

This is why we have to take $Gr_V^\alpha \cong C^\alpha$.

Remark 8.5.7. The complex structure defined on Ω_f via $\Phi : H^n(X_\infty) \cong \Omega_f$ is not unique, and it depends on the good basis chosen, or the section of $H'' \to H''/\partial_t^{-1}H''$. However, it does not affect the polarization, which we discuss in the next section.

8.6 Polarization form on extension

This section concerns the main contribution. Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a germ of isolated singularity. In 8.4 we described the extension of MHS associated to f over the degenerate point 0. In 8.5 we defined a MHS on the new fiber appearing over 0, namely Ω_f. The MHS was defined by the specific isomorphism Φ in (8.10). We use the isomorphism $\Phi : H^n(X_\infty, \mathbb{C}) \to \Omega_f$ introduced in the previous section to express a correspondence between polarization form on vanishing cohomology and the Grothendieck pairing on Ω_f.

Theorem 8.6.1. *[1] Assume $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, is a holomorphic germ with isolated singularity at 0. Then, the isomorphism Φ makes the following diagram commutative up to a complex constant;*

$$\begin{array}{ccc} \widehat{Res}_{f,0} : \Omega_f \times \Omega_f & \longrightarrow & \mathbb{C} \\ \big\downarrow {\scriptstyle (\Phi^{-1},\Phi^{-1})} & & \big\downarrow {\scriptstyle \times *} \\ S : H^n(X_\infty) \times H^n(X_\infty) & \longrightarrow & \mathbb{C} \end{array} \qquad * \neq 0 \qquad (8.13)$$

where X_∞ and S are as in section 5.2, Lemma 5.2.1 and

$$\widehat{Res}_{f,0} = res_{f,0}\ (\bullet, \tilde{C}\ \bullet)$$

and $\tilde{C}$ is defined relative to the Deligne decomposition of Ω_f, via the isomorphism Φ. If $J^{p,q} = \Phi^{-1} I^{p,q}$ is the corresponding subspace of Ω_f, then

$$\Omega_f = \bigoplus_{p,q} J^{p,q} \qquad \tilde{C}|_{J^{p,q}} = (-1)^p. \tag{8.14}$$

In other words;

$$S(\Phi^{-1}(\omega), \Phi^{-1}(\eta)) = * \times\ \ res_{f,0}(\omega, \tilde{C}.\eta), \qquad 0 \neq *. \in \mathbb{C} \tag{8.15}$$

Part of this proof is given in [CIR] for homogeneous fibrations in the context of mirror symmetry, see also [PH].

Proof. Before starting the proof, let us mention that the map Φ is classically used to correspond the mixed Hodge structures on $H^n(X_\infty, \mathbb{C})$ and Ω_f. We only prove the correspondence on polarizations.

Step 1: Choose a $\mathbb{C}$-basis of the module $\Omega(f)$, namely $\{\phi_1, ..., \phi_\mu\}$, where $\phi_i = f_i.d\underline{x}$. We identify the class $[e^{-f/t}\phi_i]$ with a cohomology class in $H(X_t)$. We may also choose the basis $\{\phi_i\}$ so that the forms $\{\eta_i = e^{(-f/t)}\phi_i\}$ correspond to a basis of vanishing cohomology, by the formula

$$\int_\Gamma e^{-\tau f}\omega = \int_0^\infty e^{-t\tau} \int_{\Gamma \cap X_t} \frac{\omega}{df}\ |_{X_t}. \tag{8.16}$$

Step 2: In this step, we restrict the cup product to $H^n(X_{infty}, \mathbb{C})_{\neq 1}$, and assume the Poincarè product is non-degenerate. By this assumption and Theorems 4.3.1, 4.3.2, we may also assume f is homogeneous of degree d and ϕ_i's are chosen by homogeneous basis of Ω_f, via Theorem 7.2.3. Consider the Morse deformation

$$f_s = f + \sum_{i=0}^{n} s_i x_i$$

and set

$$S_{ij}(s,z) := \langle [e^{-f_s/z}\phi_i], [e^{+f_s/z}\phi_j]\rangle.$$

The cup product is the one on the relative cohomology, and we may consider it in the projective space $\mathbb{P}^{n+1}$. The perturbation f_s and also the Saito form S_{ij} are weighted homogeneous. This can be seen by choosing new weights, $\deg(x_i) = 1/d,\ \deg(s_i) = 1 - 1/d$, and $\deg(z) = 1$ then the invariance of the product with respect to the change of variable $x \to \lambda^{1/d}x,\ z \to \lambda z$, shows that $S_{ij}(s,z)$ is weighted homogeneous. We show that,

$$S_{ij}(s,z) := (-1)^{n(n+1)/2}(2\pi i z)^{n+1}(Res_f(\phi_i,\phi_j) + O(z)).$$

Then the fact that S_{ij} is some multiple of $\widehat{Res}_{f,0}$ follows from homogeneity. Suppose that s is generic so that $x \to Re(f_s/z)$ is a Morse function. Let $\Gamma_1^+, ..., \Gamma_\mu^+$, (resp. $\Gamma_1^-, ..., \Gamma_\mu^-$) denote the Lefschetz thimbles emanating from the critical points $\sigma_1, ..., \sigma_\mu$ of $Re(f_s/t)$ given by the upward gradient flow (resp. downward). Choose an orientation so that $\Gamma_r^+.\Gamma_s^- = \delta_{rs}$. We have

$$S_{ij}(s,z) = \sum_{r=1}^{\mu}(\int_{\Gamma_r^+} e^{-f_s/z}\phi_i)(\int_{\Gamma_r^-} e^{f_s/z}\phi_j).$$

For a fixed argument of z we have the stationary phase expansion as $z \to 0$.

$$(\int_{\Gamma_r^+} e^{-f_s/z}\phi_i) \cong \pm\frac{(2\pi z)^{(n+1)/2}}{\sqrt{Hess f_s(\sigma_r)}}(f_i(\sigma_r) + O(z))$$

where $\phi_i = f_i(x)d\underline{x}$. Therefore,

$$S_{ij}(s,z) = (-1)^{n(n+1)/2}(2\pi i z)^{n+1}\sum_{r=1}^{\mu}(\frac{f_i(\sigma_r)f_j(\sigma_r)}{Hess(f_s)(\sigma_r)} + O(z))$$

where the lowest order term in the right hand side equals the Grothendieck residue. As this holds for an arbitrary argument of z, and S_{ij} is holomorphic for $z \in \mathbb{C}^*$; the conclusion follows for generic s. By analytic continuation the same holds for all s. By homogeneity we get,

$$S_{ij}(0,z) = (-1)^{n(n+1)/2}(2\pi i z)^{n+1}Res_f(\phi_i,\phi_j). \tag{8.17}$$

Note that there appears a sign according to the orientations chosen for the integrals. However, this only modifies the constant in the theorem. Thus, we have;

$$S_{ij}(0,1) = (-1)^{n(n+1)/2}(2\pi i)^{n+1} Res_f(\phi_i, \phi_j). \tag{8.18}$$

Step 3: The sign appearing in residue pairing is caused by compairing the two products

$$(e^{-f}\phi_i, e^{-f}\phi_j), \qquad (e^{-f}\phi_i, e^{+f}\phi_j). \tag{8.19}$$

Assume we embed the fibration in a projective one as before, replacing f with a homogeneous polynomial germ of degree d. We can consider a change of variable as $I : z \to e^{\pi i/d} z$ which changes f by $-f$. Thus, this map is an involution on the value of f. By Theorem 7.2.3 we have $Gr_F^p Gr_{n+1}^W H^n \subset I^{p,q}$. We will assume $\phi_i = f_i dz$, with f_i monomials and chosen as the method described in Theorem 7.2.3. This shows the cohomology class $e^{-f}\phi_j$ after the this change of variable is replaced by $c_p.e^{+f}\phi_j$ where $c_p \in \mathbb{C}$ only depends to the Hodge filtration (defined by degree of forms). We obtain;

$$(e^{-f}\phi_i, I^* e^{-f}\phi_j) = (e^{-f}\phi_i, (-1)^{\deg[\phi_j]/d} e^{+f}\phi_j). \tag{8.20}$$

Because $I^d = id$, if we iterate I^*, d times we obtain;

$$(e^{-f}\phi_i, e^{-f}\phi_j) = res_{f,0}(a\ , (-1)^{(d-1)\deg[\phi_j]/d}.b).$$

The Riemann-Hodge bilinear relations in $H_{\neq 1}$ implies that, the products of the forms under consideration are non-zero except when the degrees of ϕ_i and ϕ_j sum to n, cf. 7.3. This explains the formula in $H_{\neq 1}$. The above argument will still hold when the form is replaced by $(\bullet, N_Y \bullet)$, by the linearity of N_Y. Thus, we still have the same result on $H_{\neq 1}$.

Step 4: In case the Poincarè product is degenerate, we still assume f is homogeneous but we change the cup product by applying N_Y on one component. The relation

$$S_Y(\Phi^{-1}(\omega), \Phi^{-1}(\eta)) = *. \ \widehat{Res}(a\ , \tilde{C}.b), \qquad * \neq 0,\ a, b \in \Omega_{f,\neq 1}$$

proved on $H^n(X_\infty)_{\neq 1}$ or more generally when the cup product of the cohomology is non-degenerate is generic. The same relation can be proved between the level form $S_Y(\bullet, N_Y \bullet)$, and the corresponding local residue when the form is non-degenerate i.e,

$$S_Y(\Phi^{-1}(\omega), N_Y.\Phi^{-1}(\eta)) = *. \ \widehat{Res}(a\ , \mathfrak{f}.\tilde{C}.b), \qquad * \neq 0, \quad a, b \in \Omega_{f,1}$$

where $\mathfrak{f}$ is the nilpotent transformation corresponding to N_Y via Φ.

Step 5: By theorem 5.3.1 and continuity of Grothendieck residue, [G3] page 657, and the corollary in [V] sec. 3 page 37, after embedding of the Milnor fibration of a general isolated singularity f_X into that of f_Y by 4.2, the Grothendieck pairing for f_Y is the prolongation of that of f.

Step 6: Until now we have proved the relation

$$S(\Phi^{-1}(\omega), \Phi^{-1}(\eta)) = * \times \ \text{res}_{f_Y,0}(\omega, (-1)^{p(d-1)/d}.\eta), \qquad 0 \neq * \in \mathbb{C}. \quad (8.21)$$

For some d and, d can be as large as we like. Because the left hand side is independent of d, if we let $d \to \infty$ then by Step 5 we obtain (8.16). □

Remark 8.6.2. ([PH] page 37) Setting

$$\psi_s^i(\omega, \tau) = \int_{\Gamma(i)} e^{-\tau f} \omega$$

$$\bar{\psi}_s^i(\omega', \tau) = \int_{\Gamma'(i)} e^{+\tau f} \omega'$$

with $\zeta = \frac{\omega}{df}$, $\zeta' = \frac{\omega'}{df}$, the expression (which is the same as in the proof)

$$P_s([\zeta],[\zeta'])(\tau) = \sum_{i=1}^{\mu} \psi_s^i(\tau,\omega)\bar{\psi}_s^i(\tau,\omega') = \sum_{r=0}^{\infty} P_s^r([\zeta],[\zeta'])(\tau).\tau^{-n-r} \qquad (8.22)$$

is a presentation of K. Saito higher residue pairing.

Corollary 8.6.3. *Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is an isolated singularity germ. The polarization form of the MHS of vanishing cohomology and the modified residue pairing on the extended fiber Ω_f are given by the same matrix in corresponding bases.*

Example 8.6.4. We try to explain the situation of Theorem 8.6.1 and its proof in the quasi-homogeneous case. We keep the notation of Chapter 7 for the MHS on $H^n(X_\infty,\mathbb{C})$, i.e the Hodge filtration given by the degree of forms in the weighted projective space. The weight filtration is

$$0 = W_{n-1} \subset W_n \subset W_{n+1} = H^n(X_\infty,\mathbb{C})$$

where X_∞ is explained as the generic fiber $f^{-1}(1)$ of a quasi homogeneous polynomial f in weighted degrees $(w_1,...,w_n)$. Assume $\{\phi_1,...,\phi_\mu\}$ is a basis for Ω_f as in 8.3.3 which is the same as the proof of 8.6.1, Step 1. Then by Theorem 7.2.3 (and the same situation in Steps 2 and 3), we consider the corresponding Leray residues

$$\eta_i = c_i.Res_{f=1}\left(\frac{\phi_i}{(f-1)^{l(i)}}\right).$$

Here $c_i \in \mathbb{C}$ is a normalizing constant. It can be calculated according to the oscillatory integration formulas in 8.3, or the following lemma.

Lemma 8.6.5. *([CIR] page 59) Under the isomorphism $H^{n+1}(X,X_t) \cong H^n(X_t)$ explained in 8.3, the class representing $e^{-f}\phi_i$ corresponds to η_i, defined above.*

As in Step 1, let Γ be a Lefschetz thimble for f, i.e. a homology cycle in $H_{n+1}(X)$ which projects on the positive real line under f (We may also assume the intersection of Γ with any fiber of f has compact support, and this is the situation explained in [PH]). Assume Γ corresponds to $C \in H_n(X_\infty)$ under the dual isomorphism $H_{n+1}(X, X_t) \cong H_n(X_t)$. Then,

$$\int_\Gamma e^{-f}\phi_i = \int_0^\infty e^t P(t)dt$$

where

$$P(t) = \int_{\Gamma \cap X_t} \frac{\phi_i}{df} = \frac{1}{2\pi i}\int_T \frac{\phi_i}{f-t}$$

and T is a circle bundle over $\Gamma \cap X_t$. Using the homogenity, under the coordinate change $x_i \to t^{-w_i/d}x_i$, we get $P(t) = t^{l(i)-1}P(1)$. Therefore,

$$\int_\Gamma e^{-f}\phi_i = \Gamma(l(i))P(1) =: c_i.$$

By differentiating the defining equation for $P(t)$ and setting $t = 1$, one obtains

$$c_i = \int_C \eta_i.$$

This proves the lemma. Then, what we said in the step 3, says

$$S(\eta_i, \eta_j) = * \times res_{f,0}(\phi_i, \tilde{C}\phi_j).$$

According to the above description the isomorphism Φ is as follows,

$$\Phi^{-1} : [z^i dz] \longmapsto c_i.[res_{f=1}(z^i dz/(f-1)^{[l(i)]})]$$

with $c_i \in \mathbb{C}$, and z^i in the basis mentioned above (see [CIR], Appendix A). For instance by taking $f = x^3 + y^4$, then as basis for Jacobi ring, we choose

$$z^i: \ 1, \ y, \ x, \ y^2, \ xy, \ xy^2$$

which correspond to the top forms with degrees

$$l(i): \ 7/12, \ 10/12, \ 11/12, \ 13/12, \ 14/12, \ 17/12$$

respectively. The above basis projects onto a basis

$$\bigoplus_{-1<\alpha=l(i)-1<n} Gr_\alpha^V H'' \twoheadrightarrow Gr_V \Omega_f$$

as in Theorem 8.5.1. The Hodge filtration is explained as follows. First, we have $h^{1,0} = h^{0,1} = 3$. Therefore, because Φ is an isomorphism.

$$< 1.\omega,\ y.\omega,\ x.\omega >= \Omega_f^{0,1}, \qquad < y^2.\omega,\ xy.\omega,\ xy^2.\omega >= \Omega_f^{1,0}$$

where $\omega = dx \wedge dy$, and the Hodge structure is pure, because $Gr_2^W H^n(X_\infty) = 0$, by 7.2.3.

$$\overline{< 1.dx \wedge dy,\ y.dx \wedge dy,\ x.dx \wedge dy >} = \\ < c_1.xy^2.dx \wedge dy,\ xy.dx \wedge dy,\ y^2.dx \wedge dy >.$$

In general in order to be able to understand the conjugation operator, one needs to understand how it applies to elementary sections of Deligne extension (see the discussion in 10.1, and the example there).

8.7 Riemann-Hodge bilinear relations for Grothendieck pairing on Ω_f

The isomorphism $\Phi : H^n(X_\infty, \mathbb{C}) \to \Omega_f$ transforms the mixed Hodge structures already defined for $H^n(X_\infty)$ to Ω_f. It makes a correspondence between the Deligne-Hodge decompositions and also the Lefschetz decompositions. We use this to organize the polarization on the fiber Ω_f.

Theorem 8.7.1. *Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a holomorphic isolated singularity germ. The modified Grothendieck residue provides a polarization for the extended fiber Ω_f, via the aforementioned isomorphism Φ. Moreover, there exists a unique set of forms $\{\widehat{Res}_k\}$ polarizing the primitive subspaces of $Gr_k^W \Omega_f$ providing a graded polarization for Ω_f.*

Proof. Because $H^n(X_\infty)$ is graded polarized, hence using theorem 8.6.1 Ω_f is also graded polarized via the isomorphism Φ. By the Mixed Hodge Metric theorem 3.4, the Deligne-Hodge decomposition;

$$\Omega_f = \bigoplus_{p,q} J^{p,q} \tag{8.23}$$

is graded polarized and there exists a unique hermitian form; $\mathcal{R}$ with,

$$i^{p-q}\mathcal{R}(v,\bar{v}) > 0, \qquad v \in J^{p,q} \tag{8.24}$$

and the decomposition is orthogonal with respect to $\mathcal{R}$. Here the conjugation is that in 8.10. This shows that the polarization forms $\{\widehat{Res}_l\}$ are unique. Let $N := \log M_u$ be the logarithm of the unipotent part of the monodromy for the Milnor fibration defined by f. We have

$$Gr_l^W H^n(X_\infty) = \bigoplus_r N^r P_{l-2r}, \qquad P_l := \ker N^{l+1} : Gr_l^W H^n \to Gr_{-l-2}^W H^n$$

and the level forms

$$S_l : P_l \otimes P_l \to \mathbb{C}, \qquad S_l(u,v) := S(u, N^l v)$$

polarize the primitive subspaces P_l cf. 3.1.2. By using the isomorphism Φ, a similar type of decomposition exists for Ω_f. That is the isomorphic image $P_l' := \Phi^{-1} P_l$ satisfies

$$Gr_l^W \Omega_f = \bigoplus_r N^r P'_{l-2r}, \qquad P'_l := \ker \mathfrak{f}^{l+1} : Gr_l^W \Omega_f \to Gr_{-l-2}^W \Omega_f$$

and the level forms

$$\widehat{Res}_l : P'_l \otimes P'_l \to \mathbb{C}, \qquad \widehat{Res}_l := \widehat{Res}(u, \mathfrak{f}^l v)$$

polarize the primitive subspaces P'_l, where $\mathfrak{f}$ is the map induced from multiplication by f on $Gr_l^W \Omega_f$. Specifically, this shows

- $\widehat{Res}_l(x,y) = 0, \qquad x \in P'_r, y \in P'_s, r \neq s.$

- $Const \times \widehat{Res}_l(C_l x, \mathfrak{f}^l \bar{x}) > 0, \qquad 0 \neq x \in P_l'.$

  where C_l is the corresponding Weil operator cf. 2.2.8.

□

Remark 8.7.2. Let $\mathcal{G}$ be the Gauss-Manin system associated to a polarized variation of Hodge structure $(\mathcal{L}_{\mathbb{Q}}, \nabla, F, S)$ of weight n, with $S : \mathcal{L}_{\mathbb{Q}} \otimes \mathcal{L}_{\mathbb{Q}} \to \mathbb{Q}(-n)$ the polarization. Then, we have the isomorphism

$$\bigoplus_{k\in\mathbb{Z}} Gr_F^k \mathcal{G} \to \bigoplus_{k\in\mathbb{Z}} Hom_{\mathcal{O}_X}(Gr_F^{n-k}\mathcal{G}, \mathcal{O}_X) \tag{8.25}$$

given by (up to a sign factor) $\lambda \to S(\lambda, -)$, for $\lambda \in Gr_F^k \mathcal{G}$.

Remark 8.7.3. The contribution behind Theorems 8.6.1 and 8.7.1 is more than what one directly obtains from 10.2.6 in property 5. Drawing out the polarization form S from the K. Saito form P_S (or K_f) to obtain the positivity part of Riemann-Hodge bilinear relations is not direct. This is because the variable t or ∂_t in mentioned Theorem is quite twisted.

The following corollary is easily obtained in the course of the proof of Theorem 8.7.1.

Corollary 8.7.4. *The polarization S of $H^n(X_\infty)$ will always define a polarization of Ω_f, via the isomorphism Φ. In other words S is also a polarization in the extension, i.e. of Ω_f.*

Using this corollary and summing up all the material in 8.5, 8.6 and 8.7, we can give the following picture for the extension of PVMHS associated to isolated hypersurface singularity.

Theorem 8.7.5. *Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a holomorphic hypersurface germ with isolated singularity at $0 \in \mathbb{C}^{n+1}$. Then, the variation of mixed Hodge structure defined in 4.2 is polarized by 5.2.2. This VMHS can be extended to the puncture with the extended fiber isomorphic to Ω_f in the sense of 8.4 and 8.5, and it is polarized by 8.4.7. The Hodge filtration on the new fiber Ω_f corresponds to an opposite Hodge filtration on $H^n(X_\infty, \mathbb{C})$ in the way explained in 8.5.3.*

One formally can formulate the following R-H bilinear relations for $\widehat{Res}$, cf. prop. 3.1.2, 3.1.4 and 3.1.5.

Corollary 8.7.6. *(Riemann-Hodge bilinear relations for Ω_f) Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a holomorphic germ with isolated singularity. Suppose $\mathfrak{f}$ is the corresponding map to N on $H^n(X_\infty)$, via the isomorphism Φ. Define*

$$P_l = PGr_l^W := \ker(\mathfrak{f}^{l+1} : Gr_l^W \Omega_f \to Gr_{-l-2}^W \Omega_f).$$

Going to W-graded pieces;

$$\widehat{Res}_l : PGr_l^W \Omega_f \otimes_{\mathbb{C}} PGr_l^W \Omega_f \to \mathbb{C} \tag{8.26}$$

is non-degenerate and according to Lefschetz decomposition

$$Gr_l^W \Omega_f = \bigoplus_r \mathfrak{f}^r P_{l-2r}$$

we will obtain a set of non-degenerate bilinear forms,

$$\widehat{Res}_l \circ (id \otimes \mathfrak{f}^l) : PGr_l^W \Omega_f \otimes_{\mathbb{C}} PGr_l^W \Omega_f \to \mathbb{C}, \tag{8.27}$$

$$\widehat{Res}_l = res_{f,0}\ (id \otimes \tilde{C}.\ \mathfrak{f}^l) \tag{8.28}$$

where $\tilde{C}$ is as in 8.6.1, such that the corresponding hermitian form associated to these bilinear forms is positive definite. In other words,

- $\widehat{Res}_l(x, y) = 0, \qquad x \in P_r,\ y \in P_s,\ r \neq s$
- *If $x \neq 0$ in P_l,*

$$Const \times res_{f,0}\ (C_l x, \tilde{C}.\ \mathfrak{f}^l.\bar{x}) > 0$$

where C_l is the corresponding Weil operator, cf. 2.2.8, and the conjugation is as in 8.10.

Proof. This follows directly from 3.1.2, 3.1.4, 3.1.5, 8.6.1 and 8.7.1. □

Note that the map

$$A_f = \frac{\mathcal{O}_X}{\partial f} \to \Omega_f, \qquad f \mapsto f dx_0...dx_n$$

is an isomorphism. Thus, the above corollary would state similarly for A_f.

Example 8.7.7. According to Example 8.6.4, for $f = x^3 + y^4$ for instance $\widehat{Res}(x.\omega, xy.\omega) = 0$, but

$$* \times \widehat{Res}(1.\omega, \overline{1.\omega}) > 0$$

$$* \times \widehat{Res}(x.\omega, \overline{x.\omega}) > 0$$

$$* \times \widehat{Res}(y.\omega, \overline{y.\omega}) > 0$$

simultaneously for one $* \in \mathbb{C}$.

8.8 Real Structure vs real splitting

In this section we show the possibility to modify the Hodge filtration in the commutative diagram of Theorem 8.6.1 in a way to obtain a real split MHS cf. def. 3.3.6. This is interesting from the representation theory point of view relevant to $\mathfrak{sl}_2$-orbit theorem of W. Schmid in 3.5. In the following we work with a MHS (H, F, W) and $\mathfrak{g} = \mathfrak{gl}(H) = End_{\mathbb{C}}(H)$, where $\mathfrak{g}^{r,s}$ as in 3.16. We begin by the following theorem.

Theorem 8.8.1. *([CKS] sec. 2) Given a mixed Hodge structure (W, F), there exists a unique $\delta \in \mathfrak{g}_{\mathbb{R}}^{-1,-1}(W, F)$ s.t. $(W, e^{-i\delta}.F)$ is a mixed Hodge structure which splits over $\mathbb{R}$.*

In the course to prove Theorem 8.8.1 one shows the existence of a unique $Z \in \mathfrak{g}^{-1,-1}$ (nilpotent) such that

$$\overline{J^{p,q}} = e^Z.J^{p,q}, \qquad \bar{Z} = -Z.$$

The operation Z obviously preserves the weight filtration. We write $Z = -2i\delta$. Define another Hodge filtration by setting

$$\tilde{F} := e^{i.\delta}.F.$$

Since $\delta \in \mathfrak{g}_{\mathbb{R}}^{-1,-1} \subset W_{-2}^{\mathfrak{gl}}$, this element leaves W invariant and acts trivially on the quotient Gr_l^W. Therefore, both $F, \tilde{F}$ induce the same filtrations on $Gr_l^W H$. Now it is clear that

$$e^{-i.\delta}.J^{p,q} = e^{i.\delta}.\overline{J^{p,q}}$$

gives a real splitting for H.

This non-trivial fact specifically applies to the mixed Hodge structure of $H^n(X_\infty)$ and Ω_f. It means that a modification of Hodge filtration of both MHS provides a real splitting in the Theorem 8.6.1. Another

$$\tilde{C}_1 := Ad(e^{-i.\delta}).\tilde{C} = Ad(e^{i.\delta}).\overline{\tilde{C}}, \qquad Ad(g) : X \mapsto gXg^{-1},\ Ad : G \to Gl(\mathfrak{g})$$

is a real transformation (notation of Theorem 8.6.1).

Proposition 8.8.2. *The bigrading $J_1^{p,q}$ defined by $J_1^{p,q} := e^{-i.\delta}.J^{p,q}$ is split over $\mathbb{R}$. The operator $\tilde{C}_1 = Ad(e^Z).\tilde{C} : \Omega_f \to \Omega_f$ defines a real structure on Ω_f.*

This says if $\Omega_{f,1} = \oplus_{p<q} J_1^{p,q}$ then

$$\Omega_f = \Omega_{f,1} \oplus \overline{\Omega_{f,1}} \oplus \bigoplus_p J_1^{p,p}, \qquad \overline{J_1^{p,p}} = J_1^{p,p}.$$

The statement of theorem 8.6.1 is valid when the operator $\tilde{C}$ is replaced with $\tilde{C}_1$;

$$S(\Phi^{-1}(\omega), \Phi^{-1}(\eta)) = * \times \ \mathrm{res}_{f,0}(\omega, \tilde{C}_1.\eta), \qquad 0 \neq * \in \mathbb{C} \tag{8.29}$$

and this equality is defined over $\mathbb{R}$. The content of Theorem 8.8.1 is related to the $\mathfrak{sl}_2$-orbit theorem, cf. 3.5. The real splitting $I_1^{q,p} = \overline{I_1^{p,q}}$ corresponds to a semisimple transformation $Y_1.v = (p+q-k).v$ for $v \in I_1^{p,q}$. Then the pair $\{Y_1, N\}$ can be completed to a $\mathfrak{sl}_2$-triple $\{N_1^+, Y_1, N\}$. N_1^+ is real and $\delta, Y_1, N_1^+ \in \mathfrak{g}_{\mathbb{R}}$ are infinitesimal isometries of the polarization [CKS] page 477. This shows that Ω_f can be equipped with a MHS that is real split and is a sum of pure Hodge structures, cf. (3.18).

$$\Omega_f = \bigoplus_k \bigoplus_{p+q=k} J_1^{p,q}$$

and also a $\mathfrak{sl}_2$-triple $\{\mathfrak{f}_1^+, Y_1', \mathfrak{f}\}$ as infinitesimal isometries of the bilinear form $\widehat{Res}_{f,0}$ which are morphisms of the MHS (F_1', W) explained above and are of types $(1,1), (0,0), (-1,-1)$ respectively.

Example 8.8.3. [GGK1] We provide an example of real splitting by the orbit of a nilpotent transformation. Consider $H_{\mathbb{Q}} = \oplus_0^4 \mathbb{Q}e_i$, where,

$$e_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with the bilinear form

$$Q = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and the nilpotent operator

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad F^p = \{e_3, ..., e_p\}(3 \geq p \geq 0).$$

Then, N and F^p, define a nilpotent orbit where the limit mixed Hodge structure $(\hat{F}, W(N))$ is $\mathbb{R}$-split. The $\mathfrak{sl}_2$-triple associated to this orbit is

$$H = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \qquad N^+ = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let

$$X = \begin{pmatrix} -3 & 3i & 0 & 0 \\ i & -1 & 4i & 0 \\ 0 & i & 1 & -3i \\ 0 & 0 & -i & 3 \end{pmatrix}, \qquad X^3 \neq 0,\ X^4 = 0.$$

Then, define:

$$u_3 := \frac{\sqrt{3}}{2} \exp(iN) e_3 = \frac{\sqrt{3}}{2} \begin{pmatrix} 6 \\ 6i \\ -3 \\ i \end{pmatrix}$$

$$X.u_3 = \frac{\sqrt{3}}{4} \begin{pmatrix} -6 \\ -6i \\ -3 \\ i \end{pmatrix}, \qquad X^2.u_3 = \frac{\sqrt{3}}{2} \begin{pmatrix} 6 \\ -2i \\ 1 \\ i \end{pmatrix}, \qquad X^3.u_3 = \frac{\sqrt{3}}{2} \begin{pmatrix} -6 \\ 6i \\ 3 \\ i \end{pmatrix}.$$

Thus, if $u_2 := \dfrac{\sqrt{3}}{2} X.u_3$, then

$$\{u_3,\ u_2,\ \bar{u}_3,\ \bar{u}_2\}$$

defines the desired real splitting.

Chapter 9

Applications

In this chapter, we give several applications of the Theorems 8.6.1 and 8.7.1 to other problems relevant to Hodge theory.

9.1 Hodge index for Grothendieck residue

Hodge theory assigns to any polarized Hodge structure (H, F, S) a signature which is the signature of the hermitian form $S(C\bullet, \bar{\bullet})$, where C is the Weil operator, cf. 2.2.8. In case of a polarized mixed Hodge structure $(H, F, W(N), S)$, where N is a nilpotent operator this signature is defined to be the sum of the signatures of the hermitian forms associated to the graded polarizations $S_l : PGr_l^W H \times PGr_l^W H \to \mathbb{C}$, i.e signatures of $h_l := S_l(C_l\bullet, N^l\bar{\bullet})$ for all l. A basic example of this is the signature associated to mixed Hodge structure on the total cohomology of a compact Kähler manifold, namely Hodge index theorem. In this case the MHS is polarized by

$$S(u, v) = (-1)^{m(m-1)/2} \int_X u \wedge v \wedge \omega^{n-m}, \qquad u, v \in H^m$$

where ω is the Kähler class. The signature associated to the polarization S is calculated by W. Hodge in this case.

Theorem 9.1.1. *(W. Hodge) The signature associated to the polarized mixed Hodge structure of an even dimensional compact Kähler manifold is* $\sum_{p,q}(-1)^q h^{p,q}$, *where the sum runs over all the Hodge numbers,* $h^{p,q}$. *This signature is* 0 *when the dimension is odd.*

Similar definitions can be applied to polarized variation of mixed Hodge structure, according to the invariance of Hodge numbers in a variation of

MHS. In the special case of isolated hypersurface sigularities, the polarization form is given by $S_{\neq 1} \oplus S_1$ where

$$S_{\neq 1}(a,b) = S_Y(i^*a, i^*b), \qquad a, b \in H^n(X_\infty)_{\neq 1}$$
$$S_1(a,b) = S_Y(i^*a, i^*N_Y b), \qquad a, b \in H^n(X_\infty)_1.$$

A repeated application of Theorem 9.1.1 to the situation of 5.2 gives the following,

Theorem 9.1.2. *[JS2] The signature associated to the polarized variation of mixed Hodge structure of an isolated hypersurface singularity with even dimensional fibers is given by*

$$\sigma = \sum_{p+q=n+2} (-1)^q h_1^{pq} + 2 \sum_{p+q \geq n+3} (-1)^q h_1^{pq} + \sum (-1)^q h_{\neq 1}^{pq} \tag{9.1}$$

where $h_1 = \dim H^n(X_\infty)_1, h_{\neq 1} = \dim H^n(X_\infty)_{\neq 1}$ *are the corresponding Hodge numbers. This signature is* 0 *when the fibers have odd dimensions.*

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of analytic function having an isolated singularity at the origin. Consider

$$A_f = \frac{\mathbb{C}[[x_0, ..., x_n]]}{(\partial_0 f, ..., \partial_n f)}.$$

By Jacobson-Morosov Theorem in 2.3.6 or 3.1.2, there exists a unique increasing filtration W_l on A_f (or A) such that

$$\times \bar{f} : Gr_l^W A \to Gr_{l-2}^W A, \qquad \times \bar{f}^l : Gr_l^W A \cong Gr_{-l}^W A.$$

Define the primitive components $P_l = PGr_l^W A_f := \ker \bar{f}^{l+1} : Gr_l^W A \to Gr_{-l}^W A$. Then, we obtain a set of non-degenerate forms

$$Q_m : PGr_m^W A \times PGr_m^W A \to \mathbb{C}.$$

The mixed Hodge structure defined on Ω_f. We defined a MHS on Ω_f in 8.5 and saw in sections 8.6 and 8.7 that it is polarized by the form $\widehat{Res_f}$ in a way that the map Φ is an isomorphism of polarizations.

Theorem 9.1.3. *The signature associated to the modified Grothendieck pairing $\widehat{Res}_{f,0}$ associated to an isolated hypersurface singularity germ f; is equal to the signature of the polarization form associated to the MHS of the vanishing cohomology, given by (9.1).*

Proof: Trivial by Theorems 8.6.1, 8.7.1.

In the Hodge terminology this index can also be associated to the real hypersurface $f : \mathbb{R}^{n+1} \to \mathbb{R}$ when the isolated singularity is algebraic.

9.2 Asymptotic Hodge theory and Geometry of Deligne Decomposition

The concept of opposite filtrations plays an important role in the study of asymptotic behaviour of a VMHS and Mirror symmetry. In this section we compare Theorems 8.6.1 and 8.7.1 with some results in asymptotic Hodge theory due to G. Pearlstein and J. Fernandez, [P2] developing some works of P. Deligne, [D2]. We begin by the following definition;

Definition 9.2.1. A pure, polarized $\mathbb{C}$-Hodge structure of weight k over S consists of, a local system of finite dimensional $\mathbb{C}$-vector spaces $\mathcal{V}_{\mathbb{C}}$ over S equipped with a decreasing Hodge filtration $\mathcal{F}$ of $\mathcal{V} = \mathcal{V}_{\mathbb{C}} \otimes \mathcal{O}_S$ by holomorphic sub-bundles, and a flat $(-1)^k$-symmetric bilinear form $Q : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ such that

- $\mathcal{F}$ and $\bar{\mathcal{F}}$ are opposite filtrations.
- $\mathcal{F}$ is horizontal, i.e. $\nabla(\mathcal{F}) \subset \mathcal{F} \otimes \Omega_S^1$
- Q polarizes each fiber of $\mathcal{V}$.

A variation of graded polarized $\mathbb{C}$-mixed Hodge structure is defined analogously having horizontality for F, and a collection of $(Gr_k^{\mathcal{W}}, \mathcal{F}Gr_k^{\mathcal{W}}, Q_k)$ of pure polarized $\mathbb{C}$-Hodge structures.

Theorem 9.2.2. *(P. Deligne) Let $\mathcal{V} \to \triangle^{*n}$ be a variation of pure polarized Hodge structure of weight k, for which the associated limiting mixed Hodge structure is Hodge-Tate. Then the Hodge filtration $\mathcal{F}$ pairs with the shifted monodromy weight filtration $\mathcal{W}[-k]$, of $\mathcal{V}$, to define a Hodge-Tate variation over a neighborhood of 0 in $\triangle^{*n}$.*

There exists a generalization of Deligne's theorem to MHS as follows,

Theorem 9.2.3. *([P2] Theorem 3.28) Let $\mathcal{V}$ be a variation of mixed Hodge structure, and*

$$\mathcal{V} = \bigoplus_{p,q} I^{p,q}$$

denotes the C^∞-decomposition of $\mathcal{V}$ to the sum of C^∞-subbundles, defined by point-wise application of Deligne theorem. Then the Hodge filtration $\mathcal{F}$ of $\mathcal{V}$ pairs with the increasing filtration

$$\bar{U}_q = \sum_k \bar{\mathcal{F}}^{k-q} \cap \mathcal{W}_k \tag{9.2}$$

to define an un-polarized $\mathbb{C}VHS$.

Remark 9.2.4. Given a pair of increasing filtrations A and B of a vector space V one can define the convolution $A * B$ to be the increasing filtration

$$A * B = \sum_{r+s=q} A_r \cap B_s. \tag{9.3}$$

In particular for any F setting $F_r^\vee = F^{-r}$, then the increasing filtration $\bar{U}$ is given by the formula

$$\bar{U} = \overline{\mathcal{F}^\vee} * \mathcal{W} \tag{9.4}$$

Theorem 9.2.5. *(G. Pearlstein-J. Fernandez)[P2] Let $\mathcal{V}$ be an admissible variation of graded polarized mixed Hodge structures with quasi-unipotent monodromy, and $\mathcal{V} = \oplus I^{p,q}$ the decomposition relative to the limiting mixed Hodge structure. Define*

$$U'_p = \bigoplus_{a \leq p} I^{p,q} \tag{9.5}$$

and $\mathfrak{g}_- = \{\alpha \in \mathfrak{g}_{\mathbb{C}} | \alpha(U'_p) \subset U'_{p-1}\}$*, then;*

(a) U' *is opposite to* F_∞*. Moreover, relative to the decomposition*

$$\mathfrak{g} = \bigoplus_{r,s} \mathfrak{g}^{r,s} \tag{9.6}$$

(b) If $\psi(s) : \triangle^{*n} \to \check{D}$ *is the associated untwisted period map, then in a neighborhood of the origin it admits a unique representation of the form*

$$\psi(s) = e^{\Gamma(s)}.F_\infty \tag{9.7}$$

where $\Gamma(s)$ *is a* $\mathfrak{g}_-$*-valued function.*

(c) U' *is independent of the coordinate chosen for* F_∞*. Moreover,*

$$U' = \overline{F_{nilp}^\vee} * W = \overline{F_\infty^\vee} * W. \tag{9.8}$$

Here above F_{nilp} is an arbitrary element in the nilpotent orbit of the limit Hodge filtration corresponding to the nilpotent cone on the logarithms of the generators of the monodromy group, i.e $F_{nilp} = exp(z_1 N_1 + ... + z_r N_r)F_\infty$ where N_k are logarithms of different local monodromies, cf. [P2].

Theorem 9.2.6. *Let* $\mathcal{V}$ *be an admissible variation of polarized mixed Hodge structure associated to a holomorphic germ of an isolated hyper-surface singularity. Set*

$$U' = \overline{F_\infty^\vee} * W. \tag{9.9}$$

Then U' *extends to a filtration* $\underline{U'}$ *of* $\mathcal{V}$ *by flat sub-bundles, which pairs with the limit Hodge filtration* $\mathcal{F}$ *of* $\mathcal{V}$*, to define a polarized* $\mathbb{C}$*-variation of Hodge structure, on a neighborhood of the origin.*

Proof. The first part of the theorem that the two filtrations F_∞ and U' pair opposite together in a neighborhood of 0 was shown in sec. 8.5, see also Theorem 9.2.3. The way that it is polarized is the content of Theorems 8.6.1, 8.7.1 and 8.7.4. □

Remark 9.2.7. [P2] Associated to a MHS (F, W) the inclusion

$$\bigoplus_{s \leq q} I^{r,s} \subset \bigoplus_{k} W_k \cap F^{k-q} = \bigoplus_{s \leq q} \bar{I}^{r,s} \tag{9.10}$$

is easily verified. For VMHS, $\mathcal{V}$, the Griffiths transversality for $\mathcal{F}$ induces a similar one for the increasing filtration $U_\bullet$;

$$\nabla U_q \subset \Omega^1 \otimes U_{q+1}. \tag{9.11}$$

To the C^∞-vector bundle

$$E = \bigoplus_{p} \mathcal{V}^p, \qquad \mathcal{V}^p = \bigoplus_{q} I^{p,q} \tag{9.12}$$

$\mathcal{F}, \bar{U}$ are the two filtrations associated. Then, Griffiths transversality is equivalent to saying that the decomposition defines a complex variation of Hodge structure.

The polarization of a complex variation of Hodge structure will probably be interpreted to mean a parallel hermitian form which makes the system of Hodge bundles $\mathcal{V}^p$ orthogonal, and becomes positive definite on multiplying the form by $(-1)^p$ on $\mathcal{V}^p$. Suppose that in the situation of Theorem 8.6 there is any such hermitian form $\mathcal{R}$. Then, on one hand since $\mathcal{R}$ and U' are flat, so is the orthogonal complement of U'_{p-1} in U'_p. On the other hand, the way things have been setup, the orthogonal complement of U'_{p-1} in U'_p is exactly

$$\mathcal{V}^p = U'_p \cap F^p. \tag{9.13}$$

But this is the system of Hodge bundles, and so the Hodge filtration is also flat. The theorem of Pearlstein and Fernandez characterizes the opposite filtration as in item (c). The above discussion also proves the following,

Corollary 9.2.8. *The mixed Hodge structure on the extended fiber Ω_f defined in 8.5, can be identified with*

$$\Phi(U' = \overline{F_\infty^\vee} * W)$$

where Φ *is as in 8.5.*

Proof. This follows from 9.2.6, and 9.2.1 (c). □

The corollary characterizes the MHS on Ω_f more specific than in section 8.5. This result can also be obtained from the M. Saito theorem 8.5.3.

9.3 Family of curve Jacobians

Let V be a complex vector space and Λ a discrete lattice of maximal rank. Let $\Pi = (\pi_{ij})$ be the $2n \times n$ matrix such that

$$dx_i = \sum_{\alpha} \pi_{i\alpha} dz_\alpha + \bar{\pi}_{i\alpha} d\bar{z}_\alpha$$

$(\Pi, \bar{\Pi})$ is the matrix of the change of basis from $\{dz_\alpha, d\bar{z}_\alpha\}$. A necessary and sufficient condition for the complex torus $M = V/\Lambda$ to be an abelian variety is given by the well-known Riemann conditions. M is an abelian variety iff there exists an integral skew symmetric matrix Q such that

$${}^t\Pi.Q\Pi = 0$$

and

$$-\sqrt{-1}^t\Pi.Q\bar{\Pi} > 0$$

In terms of the matrix $\tilde{\Pi} = (\Pi, \bar{\Pi})$

$$-\sqrt{-1}^t\Pi.Q\bar{\Pi} = \begin{pmatrix} H & 0 \\ 0 & -{}^tH \end{pmatrix}$$

where $H > 0$. These conditions can also be written in terms of the inverse matrix $\tilde{\Omega} = \begin{pmatrix} \Omega \\ \bar{\Omega} \end{pmatrix}$ similarly. There exists a basis for Λ such that the matrix of Q in this basis is of the form

$$Q = \begin{pmatrix} 0 & \Delta_\delta \\ -\Delta_\delta & 0 \end{pmatrix}, \qquad \Delta_\delta = \begin{pmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{pmatrix}, \quad \delta_i \in \mathbb{Z}.$$

One can modify this process more to choose a complex basis $e_1, ..., e_n$ such that $\Omega = (\delta_\delta, Z)$ with Z symmetric and $\text{Im}Z > 0$, [G3].

Theorem 9.3.1. *[G3] $M = V/\Lambda$ is an abelian variety iff there exists an integral basis for Λ and a complex basis for V such that,*

$$\Omega = (\Delta_\delta, Z)$$

with Z symmetric and $ImZ > 0$.

Then the form

$$\omega = \sum_{i=0}^{n} \delta_i \; dx_i \wedge dx_{n+i}$$

namely the (invariant harmonic) Hodge form is non-degenerate and (some power of that) provides an embedding of M in projective space. The form ω is also called a polarization of M, and δ_i's are called elementary divisors of ω. When $\delta_\alpha = 1$ the abelian variety is called principally polarized.

The basic example of a principally polarized abelian variety is the Jacobian variety of a complex Riemann surface S of genus g. It is given by the choice of a basis $\delta_1, ..., \delta_{2g}$ for $H_1(S, \mathbb{Z})$ and a basis $\omega_1, ..., \omega_g$ for $H^0(S, \Omega^1)$, we have

$$\mathcal{J}(S) = \frac{\mathcal{C}^g}{\mathbb{Z}\{\lambda_1, ..., \lambda_{2g}\}}$$

where λ_i are the columns of the matrix

$$\lambda_i =^t \Big(\int_{\delta_i} \omega_1, ..., \int_{\delta_g} \omega_g\Big).$$

We may choose the bases such that

$$\int_{\lambda_i} \omega_\alpha = \delta_{i\alpha}, \qquad 1 \leq i, \alpha \leq g.$$

Then, the period matrix is of the form

$$\Omega = (I, Z).$$

Thus, $\mathcal{J}(S)$ is an abelian variety principally polarized given in terms of the basis $\{dx_i\}$ for $H^1(\mathcal{J}(S), \mathbb{Z})$ dual to the basis $\{\lambda_i\} \in H_1(\mathcal{J}(S), \mathbb{Z})$, by

$$\omega = \sum dx_\alpha \wedge dx_{n+\alpha}$$

Geometrically $\mathcal{J}(S) = H^0(S, \Omega^1)^\vee / H_1(S, \mathbb{Z})$, where $H_1(S, \mathbb{Z})$ is embedded in $H^0(S, \Omega^1)^\vee$ by integration. Then the polarization form $\omega \in H^2(\mathcal{J}(S), \mathbb{Z}) = Hom_{\mathbb{Z}}(\bigwedge^2 H_1(S, \mathbb{Z}), \mathbb{Z})$ is the skew symmetric bilinear form

$$H_1(S, \mathbb{Z}) \otimes H_1(S, \mathbb{Z}) \to \mathbb{Z}$$

given by intersection of cycles, [G3]. Thus we have shown the following important fact;

Theorem 9.3.2. *([G3] page 307) Let C be a smooth projective curve over the field $\mathbb{C}$, and $J(C)$ its Jacobian. Then, the Poincarè duality of $H^1(C, \mathbb{C})$ is identified with the polarization of $J(C)$, given by the Θ-divisor.*

This theorem simply says that the cup product of H^1 defines a well-defined bilinear map on Jacobian of the curve. We want to consider this situation in a family parametrized by a 1-dimensional variety S. Suppose that

$$J^1(H_s^1) = H^1_{s,\mathbb{Z}} \setminus H^1_{s,\mathbb{C}} / F^0 H^1_{s,\mathbb{C}}$$

$$J(\mathcal{H}) = \bigcup_{s \in S^*} J^1(H_s)$$

is the family of Jacobians associated to the variation of Hodge structure in a projective degenerate family of algebraic curves (here we have assumed the Hodge structures have weight -1), and $\dim(S) = 1$. Then the fibers of this model are principally polarized abelian varieties. The polarization of each fiber is given by the Poincarè product of the middle cohomology of the curves, via a holomorphic family of Θ-divisors. We are going to apply the construction in 8.4 and 8.5 to the variation of Jacobians. To extend $J(\mathcal{H})$ to a space over S, we let $\mathcal{G}$ be the Gauss-Manin system on S^*, obtained from the variation $\mathcal{H}$ as in Chap. 6. On S^* we have an extension of integral local classes

$$0 \to \mathcal{H}_s \to \mathcal{J}_s \to \mathbb{Z}_s \to 0$$

On the Gauss-Manin systems we get

$$0 \to M \to N \to \mathbb{Q}_{S^*}^H[n] \to 0$$

with $\mathbb{Q}_S^H[n]$ is the trivial module with sheaf of sections $\mathcal{O}_{S^*}$.

Remark 9.3.3. A holomorphic section of $\bar{J}$ is called quasi-horizontal if it has a lift to a horizontal section of $\mathcal{G}$. In our case this condition is not necessary, but it is crucial in higher dimensions.

Similar to the sections 8.4 and 8.5, the first and the last objects in the short exact sequences extend to the punctures in a way that the extended fiber is polarized by modified Grothendieck pairing. The extended fiber of the Jacobian bundle is the Jacobian of the opposite Hodge filtration. In this way the extended fiber is an abelian variety and principally polarized, with some Θ-divisor. Note that we do not obtain any curve on the puncture whose Jacobian gives the fiber. The extended Jacobian simply is

$$X_0 = J^1(\Omega_f) = \Omega_{f,\mathbb{Z}} \setminus \Omega_f / F^0\Omega_f.$$

Theorem 9.3.4. *The extension of a degenerate 1-parameter holomorphic family of Θ-divisors polarizing the Jacobian of curves in a projective fibration, is a Θ-divisor polarizing the extended Jacobian, i.e the Jacobian associated to the pure Hodge structure in the extension.*

Proof. At the level of local systems (Hodge structures) we have the diagram of flat pairings,

$$\begin{array}{rcccccc} \kappa: & \mathcal{H} & \otimes & \mathcal{H} & \to & \mathbb{C} \\ & \downarrow & & \downarrow & & \\ \kappa_J: & J & \otimes & J & \to & \mathbb{C} \\ & \downarrow & & \downarrow & & \\ \times: & \mathbb{Q} & \otimes & \mathbb{Q} & \to & \mathbb{C} \end{array} \tag{9.14}$$

The extension of the first and the last provides an extension of the middle line. Similar non-degenerate bilinear forms can be defined on the Gauss-Manin modules, where the above diagram is its reduction on fibers;

$$\begin{array}{rcccccc} K: & \mathcal{G} & \otimes & \mathcal{G} & \to & \mathbb{C}[t,t^{-1}] \\ & \downarrow & & \downarrow & & \\ K_J: & N & \otimes & N & \to & \mathbb{C}[t,t^{-1}] \\ & \downarrow & & \downarrow & & \\ \times: & \mathbb{Q}_S^H & \otimes & \mathbb{Q}_S^H & \to & \mathbb{C}[t,t^{-1}] \end{array} \tag{9.15}$$

where the map in the first line is the K. Saito higher residue pairing.

□

9.4 Modules over Hypersurface rings

A hyper-surface ring is a ring of the form $R := P/(f)$, where P is an arbitrary ring and f a non-zero divisor. Localizing we may assume P is a local ring of dimension $n+1$. As according to the title we assume $P = \mathbb{C}\{x_0, ..., x_n\}$ and f a holomorphic germ, or $P = \mathbb{C}[x_0, ..., x_n]$ and then f would be a polynomial. Then we are mainly interested to study finitely generated modules over these rings. Consider $f : \mathbb{C}^{n+1} \to \mathbb{C}$ in this form, and choose a representative for the Milnor fibration as $f : X \to T$, where T is the disc around 0.

Then, through all the rest of this section we assume $0 \in \mathbb{C}^{n+1}$ is the only singularity of f.

A matrix factorization of f in P is a pair of matrices A and B such that $AB = BA = f.\, id$. It is equivalent to the data of a pair of finitely generated free P-modules,

$$d_0 : X^0 \leftrightarrows X^1 : d_1, \qquad d_0 d_1 = d_1 d_0 = f.\, id$$

It is a basic fact, discovered by D. Eisenbud, that the R-modules have a minimal resolution that is eventually 2-periodic. Specifically, in a free resolution of such a module M, we see that after n-steps we have an exact sequence of the following form.

$$0 \to M' \to F_{n-1} \to F_{n-2} \to ... \to F_0 \to M \to 0 \tag{9.16}$$

where the F_i are free R-modules of finite rank and $depth_R(M') = n$. If $M' = 0$ then M has a free resolution of finite length., If $M' \neq 0$, then M' is a maximal Cohen-Macaulay module, that is $depth_R(M') = n$. So "up to free modules" any R-module can be replaced by a maximal Cohen-Macaulay module. If M is a maximal Cohen-Macaulay R-module that is minimally generated by p elements, its resolution as P-module has the form

$$\begin{array}{ccccccccc} 0 & \to & P^p & \overset{A}{\to} & P^p & \to & M & \to & 0 \\ & & \downarrow & \overset{B}{\swarrow} & \downarrow & & \downarrow 0 & & \\ 0 & \to & P^p & \overset{A}{\to} & P^p & \to & M & \to & 0 \end{array}$$

where A is some $p \times p$ matrix with $det(A) = f^q$. The fact that multiplication by f acts as 0 on M produces a matrix B such that $A.B = B.A = f.I$, where I is the identity matrix. In other words we find a matrix factorization (A, B) of f determined uniquely up to base change in the free module P^p, by M. This matrix factorization not only determines M but also a resolution of M as R-module.

$$.... \to R^p \to R^p \to R^p \to M \to 0.$$

So a minimal resolution of M looks in general as follows

$$... \to G \to F \to G \to F_{n-1} \to ... \to F_0 \to M \to 0.$$

As a consequence all the homological invariants like $Tor_k^R(M,N), Ext_R^k(M,n)$ are 2-priodic, [BVS], [EP].

The category of matrix factorizations of f over R, namely $MF(R,f)$; is defined to be the differential $\mathbb{Z}/2$-graded category, whose objects are pairs (X,d), where $X = X^0 \oplus X^1$ is a free $\mathbb{Z}/2$-graded R-module of finite rank equipped with an R-linear map d of odd degree satisfying $d^2 = f.\ id_X$. Here the degree is calculated in $\mathbb{Z}/2$. Regarding the first definition

$$d = \begin{pmatrix} 0 & d_0 \\ d_1 & 0 \end{pmatrix}, \qquad d^2 = f.\ id$$

The morphisms $MF(X,X')$ are given by $\mathbb{Z}/2$-graded R-module maps from X to X' (or equivalent between the components X^0 and X^1) provided that the differential is given by

$$d(f) = d_{X'} \circ f - (-1)^{|f|} f \circ d_X. \tag{9.17}$$

Here d_X or d'_X may be considered as the matrix given above or to be separately d_0 and d_1, and also it is evident that $d(f)^2 = 0$. By choosing bases for X^0 and X^1 we reach to the former definition, [EP].

M. Hochster in his study of direct summand conjecture defined the following invariant namely Θ-invariant.

Definition 9.4.1. (Hochster Theta pairing) The theta pairing of two R-modules M and N over a hyper-surface ring $R/(f)$ is

$$\Theta(M,N) := l(Tor_{2k}^R(M,N)) - l(Tor_{2k+1}^R(M,N)), \qquad k >> 0.$$

This definition makes sense as soon as the lengths appearing are finite. This certainly happens if R has an isolated singular point.

Example 9.4.2. [BVS] Take $f = xy - z^2, M = \mathbb{C}[[xyz]]/(x,y)$. A matrix factorization (A,B) associated to M is given by

$$A = \begin{pmatrix} y & -z \\ -z & x \end{pmatrix}, \qquad B = \begin{pmatrix} x & z \\ z & y \end{pmatrix}.$$

The $Tor_k^R(M,M)$ is the homology of the complex

$$..... \to \mathbb{C}[[y]]^2 \to \mathbb{C}[[y]]^2 \to \mathbb{C}[[y]] \to 0$$

where

$$\alpha = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}.$$

So we find that $\Theta(M, M) = 0$.

Hochster theta pairing is additive on short exact sequences in each argument, and thus determines a $\mathbb{Z}$-valued pairing on $G(R)$, the Grothendieck group of finitely generated R-modules. One loses no information by tensoring with $\mathbb{Q}$ and so often theta is interpreted as a symmetric bilinear form on the rational vector space $G(R)_{\mathbb{Q}}$. It is basic that Theta would vanish if either M or N be Artinian or have finite projective dimension [MPSW], [BVS]. The Θ-invariant has different interpretations as intersection multiplicity in the singular category.

Theorem 9.4.3. *[BVS] When $M = \mathcal{O}_Y = R/I, N = \mathcal{O}_Z = R/J$, where $Y, Z \subseteq X_0$ are the sub-varieties defined by the ideals I, J respectively, then*

$$\Theta(\mathcal{O}_Y, \mathcal{O}_Z) = i(0; Y, Z)$$

in case that $Y \cap Z = 0$. Here $i(0; ,)$ is the ordinary intersection multiplicity in $\mathbb{C}^{n+1}$.

By additivity over short exact sequences and the fact that any module admits a finite filtration with sub-quotients of the form R/I, knowing $\Theta(\mathcal{O}_Y, \mathcal{O}_Z)$ determines $\Theta(M, N)$ for all modules M, N.

Theorem 9.4.4. *[BVS] Assume $f \in \mathbb{C}[[x_1, ..., x_{2m+2}]$ is a homogeneous polynomial of degree d, and $X_0 = f^{-1}(0) \in \mathbb{C}^{2m+2}$ and $T = V(f) \in \mathbb{P}^{2m+1}$ the associated projective cone of degree d. Let Y and Z be also co-dimension m cycles in T. If Y, Z intersect transversely, then*

$$\Theta(\mathcal{O}_Y, \mathcal{O}_Z) = -\tfrac{1}{d}[[Y]].[[Z]]$$

where $[[Y]] := d[Y] - deg(Y).h^m$ *is the primitive class of* $[Y]$*, with* $h \in H^1(T)$ *the hyperplane class.*

The primitive class of a cycle Y is the projection of its fundamental class $[Y] \in H^m(T)$ into the orthogonal complement to h^m with respect to the intersection pairing into $H^{2m}(T) = \mathbb{C}$. As $h^m.h^m = d = deg(T)$ and $[Y].h^m = deg(Y)$ the description of the primitive class follows. Substituting the claim can be reformulated

$$\Theta(\mathcal{O}_Y, \mathcal{O}_Z) = -\tfrac{1}{d}[[Y]].[[Z]] = -d[Y].[Z] + deg(Y)deg(Z)$$

Where $[Y].[Z]$ denotes the intersection form on the cohomology of the projective space, [BVS].

When f in consideration is a homogeneous polynomial of degree d, such that $X := Proj(R)$ is a smooth k-variety, the Theta pairing is induced, via chern character map, from the pairing on the primitive part of de Rham cohomology

$$\frac{H^{(n-1)/2}(X,\mathbb{C})}{\mathbb{C}.\gamma^{(n-1)/2}} \times \frac{H^{(n-1)/2}(X,\mathbb{C})}{\mathbb{C}.\gamma^{(n-1)/2}} \to \mathbb{C}$$

given by

$$(a,b) \to (\textstyle\int_X a \cup \gamma^{(n-1)/2})(\int_X a \cup \gamma^{(n-1)/2}) - d(\int_X a \cup b)$$

where γ is the class of a hyperplane section and Theta would vanish for rings of this type having even dimensions. When $n = 1$ by γ^0 we mean $1 \in H^0(X, \mathbb{C})$, [MPSW].

Theorem 9.4.5. *[MPSW] For* R *and* X *as above, if* n *is odd there is a commutative diagram*

$$\begin{array}{ccc} G(R)_{\mathbb{Q}}^{\otimes 2} & \longleftarrow & (\frac{K(X)_{\mathbb{Q}}}{\alpha})^{\otimes 2} \\ \Theta \big\downarrow & & \big\downarrow (ch^{n-1/2})^{\otimes 2} \\ \mathbb{C} & \underset{\theta}{\longleftarrow} & (\frac{H^{(n-1)/2}(X,\mathbb{C})}{\mathbb{C}.\gamma^{(n-1)/2}})^{\otimes 2}. \end{array} \tag{9.18}$$

Theorem 9.4.6. *[MPSW] For R and X as above and n odd the restriction of the pairing $(-1)^{(n+1)/2}\Theta$ to*

$$im(ch^{\frac{n-1}{2}}) : K(X)_{\mathbb{Q}}/\alpha \to \frac{H^{(n-1)/2}(X,\mathbb{C})}{\mathbb{C}.\gamma^{\frac{n-1}{2}}}$$

is positive definite. i.e. $(-1)^{(n+1)/2}\Theta(v,v) \geq 0$ with equality holding if and only if $v = 0$. In this way θ is semi-definite on $G(R)$.

Proof. [MPSW] Define

$$W = H^{n-1}(X(\mathbb{C}),\mathbb{Q}) \cap H^{\frac{n-1}{2},\frac{n-1}{2}}(X(\mathbb{C})).$$

It is classical that the image of $ch^{(n-1)/2}$ is contained in W. Define $e : W/\mathbb{Q}.\gamma^{(n-1)/2} \hookrightarrow H^{n-1}(X,\mathbb{Q})$ by

$$e(a) = a - \frac{\int_X a \cup \gamma^{(n-1)/2}}{d}.\gamma^{(n-1)/2} \in W.$$

We know that $\theta(a,b) = -d.I^{coh}(e(a),e(b))$ Now the theorem follows from the polarization properties of cup product on cohomology of projective varieties.

□

The Hochschild chain complex of $MF(R,f)$ is quasi-isomorphic to the Koszul complex of the regular sequence $\partial_0 f, ..., \partial_n f$. In particular the Hochschild homology (and also the Hochschild cohomology) of 2-periodic dg-category $MF(R,f)$ is isomorphic to the module of relative differentials or the Jacobi ring of f, [D].

Theorem 9.4.7. *(T. Dykerhoff) [D], [PV] The canonical bilinear form on the Hochschild homology of category of matrix factorizations $\mathcal{C} = MF(P,f)$ of f, after the identification*

$$HH_*MF(P,f) \cong A_f \otimes dx[n] \tag{9.19}$$

coincides with

$$\langle g \otimes dx, h \otimes dx \rangle = (-1)^{n(n-1)/2} res_{f,0}(g,h). \tag{9.20}$$

The chern character or Denis trace map is a ring homomorphism

$$ch : K_0'(X) \to HH_0(X) \cong \Omega_f \tag{9.21}$$

where K' is the free abelian group on the isomorphism classes of finitely generated modules modulo relations obtained from short exact sequences. The construction of the chern character map or chern classes is functorial w.r.t flat pull back. In the special case of $i : X \hookrightarrow Y$ the compactification, the following diagram commutes,

$$\begin{array}{ccccc} K_0'(Y_0) & \xrightarrow{ch_Y} & HH_0(Y_0) \cong \Omega_f^Y & \xrightarrow{\Phi_Y^{-1}} & H^n(Y_\infty) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\ K_0'(X_0) & \xrightarrow[ch_X]{} & HH_0(X_0) \cong \Omega_f^X & \xrightarrow[\Phi_X^{-1}]{} & H^n(X_\infty). \end{array} \tag{9.22}$$

Given a matrix factorization (A, B) for a maximal Cohen-Macaulay M, one can find de Rham representatives for the chern classes. Consider $\mathbb{C}[[x_0, ..., x_n]]$ as a $\mathbb{C}[[t]]$-module with t acting as multiplication by f. Denote by Ω^p the module of germs of p-forms on $\mathbb{C}^{n+1}$, and let $\Omega_f^p = \Omega^p/(df \wedge \Omega^{p-1})$. One puts $\omega(M) = dA \wedge dB$. The components of the chern character

$$ch_M := tr(\exp(\omega(M))) = \sum_i \frac{1}{i!}\omega^i(M) \tag{9.23}$$

are well-defined classes

$$\omega^i(M) = tr((dA \wedge dB)^i) \in \Omega_f^{2i}/(df \wedge \Omega^{2i-1}). \tag{9.24}$$

There are also odd degree classes

$$\eta^i(M) := tr(AdB(dA \wedge dB)^i) \in \Omega_f^{2i+1}/\Omega_f^{2i}.$$

The group $\Omega_f^{2i+1}/d\Omega_f^{2i}$ can be identified with the cyclic homology $HC_i(P/\mathbb{C}\{t\})$. They fit into the following short exact sequence such that $d\eta^{i-1} = \omega^i(M)$.

$$0 \to \Omega_f^{2i-1}/\Omega_f^{2i-2} \to \Omega^{2i}/(df \wedge \Omega^{2i-1}) \to \Omega^{2i}/\Omega^{2i-1} \to 0.$$

If the number of variables $n+1$ is even, then a top degree form sits in the Brieskorn module

$$\mathcal{H}_f^{(0)} = \Omega^n/(df \wedge d\Omega^{n-1})$$

a free $\mathbb{C}[[t]]$-module of rank μ. The higher residue pairing

$$K : \mathcal{H}_f^{(0)} \times \mathcal{H}_f^{(0)} \to \mathbb{C}[t, t^{-1}]$$

of K. Saito can be seen as the de Rham realization of the Seifert form of the singularity, [BVS]. The following theorem is conjectured in [MPSW].

Theorem 9.4.8. *Let S be an isolated hypersurface singularity of dimension n over $\mathbb{C}$. If n is odd, then $(-1)^{(n+1)/2}\Theta$ is positive semi-definite on $G(R)_{\mathbb{Q}}$, i.e $(-1)^{(n+1)/2}\Theta(M,M) \geq 0$.*

Proof. By additivity of Θ on each variable, we may replace M, N by maximal Cohen-Macaulay modules. According to Theorem 9.4.7 and 8.7.5, the determination of the sign of Θ amounts to understanding how the image of chern classes look like in the MHS of Ω_f. By theorem 8.6.1 it amounts to the same things for the image in $H^n(X_\infty)$ under the isomorphism Φ. The following diagram is commutative by the functorial properties of chern character.

$$\begin{array}{ccc} K_0'(Y_0) & \xrightarrow{\Phi_Y^{-1} \circ ch_Y} & H^n(Y_\infty) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ K_0'(X_0) & \xrightarrow[\Phi_X^{-1} \circ ch_X]{} & H^n(X_\infty). \end{array} \tag{9.25}$$

We are assuming that i^* is surjective. By what was said, the chern class we are concerned with is a Hodge cycle. The commutativity of the above diagram allows us to replace the pre-image of the chern character for X, with similar cycle upstairs. The polarization form S_X was defined via that of S_Y. Therefore, if

$$H^n(Y_\infty) = \oplus_{p+q=n} H^{p,q}$$

is the Hodge decomposition, the only non trivial contribution in the cup product will be for the $H^{n/2,n/2}$, and the polarization form is evidently definite on this subspace (Hodge cycles). The map N_Y is of type $(-1,-1)$ for the Hodge structure of $H^n(Y_\infty)$ and the polarization $S_Y(H^{n/2,n/2}, H^{n/2-1,n/2-1}) = 0$ for obvious reasons. Thus, the corresponding chern class should lie in $H^n_{\neq 1}$. In this way, one only needs to prove the positivity statement for Hochster Θ when the chern character is in $H_{Y,\neq 1}$, and this is the content of Theorem 2.8. □

9.5 Fourier-Laplace Transform of Polarization

The extensions of PVMHS can be explained by Fourier-Laplace transform of sheaves. For the set up, let $\mathcal{G}$ be the Gauss-Manin system associated to the VMHS $(\mathcal{H}, F, W)$ as before. We consider $\mathcal{G}(*\infty) = \mathcal{G} \otimes D_{\mathbb{P}^1}(*\infty)$ and define its Fourier-Laplace transform by

$$\widehat{\mathcal{G}} := q_+(p^+\mathcal{G}(*\infty)) \otimes \mathcal{E}^{-t\tau}), \qquad \mathcal{E}^{-t\tau} = (\mathcal{O}_{\mathbb{P}^1 \times \mathbb{C}}, \nabla = d - \tau dt - t d\tau).$$

Here $p : \mathbb{P}^1 \times \mathbb{C} \to \mathbb{P}^1$ and $q : \mathbb{P}^1 \times \mathbb{C} \to \mathbb{C}$ are projections and upper (resp. lower) $+$ denote the pull back (resp. pushforward) in the category of D-modules. By a D-module over a complex manifold X we mean an $\mathcal{O}_X$ module (i.e. sheaf of $\mathcal{O}_X$-module as in algebraic geometry) together with an action of flat connection ∇ on that. This is equivalent to define a D_X-module as a sheaf on T_X^* the co-tangent bundle of X. The Fourier-Laplace transform of $\mathcal{G}$ can also be defined as, (cf. [SA7])

$$\widehat{\mathcal{G}} = \text{coker}(\mathbb{C}[\tau] \otimes \mathcal{G} \stackrel{\nabla_t - \tau dt}{\longrightarrow} \mathbb{C}[\tau] \otimes \mathcal{G}), \qquad \tau.m := \partial_t.m.$$

If we have a polarization as

$$K : \mathcal{H}' \otimes_{\mathcal{O}} \overline{\mathcal{H}} \to \mathcal{L}^{\mathbb{R}-an}$$

where $\mathcal{L}^{\mathbb{R}-an}$ is the set of elements (distributions) of the form, (cf. [SA4])

$$\sum_{\alpha,p} \mathbb{C}\{t\}[t^{-1}]\mathbb{C}\{\bar{t}\}[\bar{t}^{-1}](\log|t|)^p.$$

The above bilinear form carries over

$$\widehat{K} : \widehat{\mathcal{H}'} \otimes_{\mathcal{O}} \imath^+ \overline{\widehat{\mathcal{H}}} \to \mathcal{L}^{\mathbb{R}-an},$$

Here $\imath : \mathbb{P}^1 = \mathbb{C} \cap \infty \to \mathbb{P}^1$ is $z \mapsto -z$ and $\imath^+$ is necessary for we use $\exp(\overline{t\tau})$ not $\exp(-\overline{t\tau})$, in a way that the distribution on the integral is twisted by $\exp(-\overline{t\tau}).\exp(t\tau)$. The product after Fourier transform is

$$(\sum \tau^i m_i)dt \otimes \overline{(\sum \tau^i n_i)dt} \mapsto [\psi \to \sum_{i,j} k(m_i, n_j)\tau^i \bar{\tau}^j e^{-\overline{t\tau}}.e^{t\tau}\psi dt \wedge d\bar{t}]$$

up to a complex constant, [SA7].

Example 9.5.1. [SA7]

- $\mathcal{G} = \mathbb{C}[t]\langle \partial_t \rangle/(t-c) \Longrightarrow K(m, \bar{m}) = \delta_c,\ \widehat{K}(m, \bar{m}) = i/2\pi \exp(\overline{c\tau} - c\tau)$
- $\mathcal{G} = \mathbb{C}[t]\langle \partial_t \rangle/(t\partial_t - \alpha) \Rightarrow K(m, \bar{m}) = |t|^{2\alpha},\ \widehat{K}(m, \bar{m}) = \Gamma(\alpha+1)/\Gamma(-\alpha)|\tau|^{-2(\alpha+1)}$

Theorem 9.5.2. *[SA4], [SA7] Assume $(\mathcal{G}, F, W, H, S)$ be a polarized MHM (hence regular holonomic) with quasi-unipotent underlying variation of mixed Hodge structure K, defined on a Zariski dense open subset U of an algebraic manifold X. Then, $\mathcal{G}$ has a smooth extension to all of X Given by the Fourier-Laplace transform of $\mathcal{G}$, and similar for the perverse sheaf H. The extended MHM (resp. perverse solution) is also polarized. The polarizations on the fibers can be described by the Fourier-Laplace transform of the polarization of $\mathcal{G}$ and H.*

Theorem 9.5.3. *([DW], page 53, 54, prop. 2.6 - [SA7] sec. 5) Assume $\mathcal{H}' = R^n f_* \mathbb{C}_{X'}$ be the local system associated to a holomorphic isolated singularity f. Consider the map*

$$F : \Omega_X^{n+1} \to i_* \bigcup_z Hom(H_n(X, f^{-1}(\eta.\frac{z}{|z|}), \mathbb{Z}) \cong \oplus_i \mathbb{Z}\Gamma_i, \mathbb{C})$$

$$\omega \mapsto [z \to (\Gamma_i \to \int_{\tilde{\Gamma}_i} e^{-t/z}\omega)],$$

and define

$$\mathcal{H} := Im(F)$$

where Γ_i *are the classes of Lefschetz thimbles, and* $\tilde{\Gamma}_i$ *is the extension to infinity. Then the vector bundle* $\mathcal{H}$ *is exactly the Fourier-Laplace transform of the cohomology bundle* $R^n f_* \mathbb{C}_{X'} = \cup_t H^n(X_t, \mathbb{C})$*, equipped with a connection with poles of order at most two at* ∞*.*

$$(\mathcal{H}', \nabla') \leftrightarrows (\mathcal{H}, \nabla)$$

Comparing this with Theorem 8.6.1 , 8.7.1 and 3.4.1 we obtain the following important corollary.

Corollary 9.5.4. *In case of the PVMHS associated to the Milnor fibration of an isolated hypersurface singularity* f*, the modified Grothendieck residue*

$$\widehat{Res}_{f,0} = res_{f,0}(\bullet, \hat{C}\bullet)$$

where $\hat{C}$ *is defined relative to the Deligne-Hodge decomposition of* Ω_f *as before, is the Fourier-Laplace transform of the polarization* S *on* $H^n(X_\infty, \mathbb{C})$*, that is*

$$\widehat{Res} = \ *. \ ^F S, \qquad * \neq 0.$$

Proof. The corollary follows from 8.7.5, 9.5.2, 9.5.3 and the uniqueness of the polarization form of mixed Hodge structure, namely mixed Hodge metric theorem, 3.4.1. □

Remark 9.5.5. Another important fact is that, a polarization of the form

$$K : \mathcal{H}' \otimes_{\mathcal{O}} \overline{\mathcal{H}} \to \mathbb{C}[t, t^{-1}]$$

induces an isomorphism

$$\mathcal{H}'^{\vee} \cong_{\mathcal{O}} \overline{\mathcal{H}}.$$

We can glue the above bundles by this isomorphism obtained from the polarization. Thus, the process of gluing is equivalent to polarization. Therefore, in the situation of 8.4 and 8.5, we have

$$\mathcal{H}^{(0)\vee} \cong \overline{\mathcal{G}}, \qquad \Rightarrow \qquad \Omega_f^{\vee} \cong \overline{H^n(X_\infty, \mathbb{C})}$$

as PVMHS, and PMHS respectively. The corresponding connections are given by

$$\nabla' : \mathcal{H}' \to \frac{1}{z}\Omega^1 \otimes \mathcal{H}', \qquad \nabla : \overline{\mathcal{H}} \to z\Omega^1 \otimes \overline{\mathcal{H}}$$

respectively, [DW] exp. 1, C. Sabbah, pages 12, 13.

Chapter 10

Further Studies

10.1 Primitive elements

In this section we explain primitive elements as a basis for primitive subspaces of vanishing cohomology, and try to explain the conjugation map on vanishing cohomology of an isolated hypersurface singularity, via elementary sections.

Assume $\mathcal{G}$ the associated Gauss-Manin system of the isolated singularity $f : \mathbb{C}^{n+1} \to \mathbb{C}$ and $\phi_1, ..., \phi_\mu$, a frame basis for $\mathcal{G}$ and (α, s_α) are the spectral pairs of f. According to [SA2] it is possible to choose the basis in a way that we have the following recursive relations;

$$\phi_{s(i)+k} = \partial_t^{-k} \prod_\alpha (t\partial_t - \alpha)^j \phi_{s(i)}, \qquad 1 \leq i \leq r,\ 1 \leq k \leq k_i. \tag{10.1}$$

for specific numbers $0 \leq s(1), ..., s(r) \leq \mu$. In this way we reach a set of forms $\phi_{s(i)}$ indexed by spectral numbers which produce other basis elements by applying the operators $t\partial_t - \alpha$ successively. They also describe $Gr_l^W Gr_\alpha^V \mathcal{G}$ concretely. These forms are called primitive elements relative to the nilpotent operator induced by $t\partial_t - \alpha$ on C_α. They provide information about the Jordan blocks structure in $\mathcal{G}$. If we denote the Jordan block as

$$B_k := \langle N^j[\omega_{k_0}] \mid j = 0, ..., \nu_k \rangle,$$

then it holds that;

$$\overline{B_{\alpha,l}} = \begin{cases} B_{1-\alpha,\nu_k - l}, & \alpha \in]0,1[\\ B_{0,\nu_k - l}, & \alpha = 0 \end{cases} \tag{10.2}$$

See [SA2] for the proof.

Proposition 10.1.1. *[SA5], [SA2] There is a 1-1 correspondence between opposite filtrations on* $H^n(X_\infty)_\mathbb{C}$ *and free, rank* μ, $\mathbb{C}[t]$*-submodules* $\mathcal{G}_\infty$ *on which the connection is logarithmic where* $\mathcal{G}_0, \mathcal{G}_\infty$ *define a trivial vector bundle on* $\mathbb{P}^1$.

The submodule $\mathcal{G}_\infty$ in Proposition 10.1.1 is given by;

$$\mathcal{G}_\infty = \mathbb{C}[t]\langle[\omega_0], ..., \partial_t^{-s_0}[\omega_0], ..., \partial_t^{-s_r}[\omega_r]\rangle.$$

The primitive elements provide the good bases of the Brieskorn module. They also prove the existence of a solution to The Poincarè-Birkhoff problem. In such a basis the matrix of the operator t has the form;

$$t = A_0 + A_1\partial_t^{-1} \tag{10.3}$$

where A_0, A_1 are square matrices of size μ and A_1 is a diagonal matrix. It holds (cf. [SA5], [H1]) that, in such a basis the K. Saito higher residue form [S1] takes the form

$$K_f(\eta_i, \eta_j) = \pm\delta_{\kappa(i)j}.\partial_t^{-n-1}, \tag{10.4}$$

where δ is the Kronecker delta and κ is an involution of the set $\{1, ..., \mu\}$ as an index set of a specific basis of $\mathcal{G}$ namely $\{\eta_i\}_{i=1}^{\mu}$. The extension of a PVMHS's may be explained by the solvability of the Poincarè-Birkhoff problem associated to the Gauss-Manin systems. One step in solving the Poincarè-Birkhoff problem for the Gauss-Manin system of f is to glue different lattices in the Gauss-Manin vector bundle to obtain a vector bundles over $\mathbb{C}P(1)$.

Example 10.1.2. The equation (10.2) completely explains how to do conjugation on the elementary sections of the Deligne extension. Specifically

$$\psi_\alpha^{-1}(\overline{t^\alpha(\log t)^l A_{\alpha,l}}) = \begin{cases} \psi_{1-\alpha}^{-1}(t^{1-\alpha}(\log t)^{\nu-l}\bar{A}_{1-\alpha,\nu-l}), & \alpha \in]0,1[\\ \psi_0^{-1}((\log t)^l \bar{A}_{0,\nu-l}), & \alpha = 0 \end{cases}$$

where ν is the size of the corresponding Jordan block. Regarding the map Φ defined in 8.5, the conjugation on Ω_f must satisfy similar relations. That is the conjugate of an element in $Gr_V^{\alpha} Gr_l^W \Omega_f$ is either in $Gr_V^{1-\alpha} Gr_{\nu-l}^W \Omega_f$ or $Gr_V^0 Gr_{\nu-l}^W \Omega_f$, in respective cases, such that the corresponding sections of vanishing cohomology satisfy the above.

10.2 Higher residues

K. Saito [S1], [S2] introduced the concept of higher residue pairings $K_F^{(k)}$, $k = 0, 1, 2, ...$ which are defined on the relative de Rham cohomology module $\mathcal{H}_F^{(k)}$ of the family F and take values in the ring $\mathcal{O}_T$ of holomorphic functions on the parameter space. The introductory material in this section is taken from [S1].

For a holomorphic isolated singularity germ over the disc take a representative of the Milnor fibration $f : X \to T$. In the following we have replaced the germ f by a universal unfolding of it. A universal unfolding of f parametrized by S is by definition, a map $F : Z \subset X \times S \to \mathbb{C}$ such that $F(0) = f$. We briefly review the machinery of K. Saito.
Let

$$\Omega_f := \Omega_{X/T}^{n+1} / df \wedge \Omega_{X/T}^n.$$

Definition 10.2.1. (E. Brieskorn) [B], [S1]

$$\mathcal{H}^{(0)} = \mathcal{H}_f^{(0)} := f_*(\Omega_{X/T}^{n+1} / df \wedge d\Omega_{X/T}^{n-1})$$
$$\mathcal{H}^{(-1)} = \mathcal{H}_f^{(-1)} := f_*(\Omega_{X/T}^n / df \wedge \Omega_{X/T}^{n-1} + d\Omega_{X/T}^{n-1}).$$

There is an exact sequence

$$0 \to \mathcal{H}^{(-1)} \xrightarrow{dt} \mathcal{H}^{(0)} \to f_* \Omega_f \to 0. \tag{10.5}$$

We regard $\mathcal{H}^{(-1)}$ as a sub-module of $\mathcal{H}^{(0)}$ by this exact sequence. There exists a natural operation of Gauss-Manin connection,

$$\nabla = \partial_t : \mathcal{H}^{(-1)} \to \mathcal{H}^{(0)}. \tag{10.6}$$

One obtains a decreasing filtration on $\mathcal{H}^{(0)}$ by

$$\mathcal{H}^{(-k-1)} := \{\omega \in \mathcal{H}^{(-1)} : \partial_t \ \omega \in \mathcal{H}^{(-k)}\}, \qquad k \geq 1$$
$$0 \to \mathcal{H}^{(-k-1)} \xrightarrow{df} \mathcal{H}^{(-k)} \to f_* \Omega_f \to 0$$
$$\nabla : \mathcal{H}^{(-k-1)} \to \mathcal{H}_f^{(-k)}.$$

We define

$$\widehat{\mathcal{H}^{(0)}} := \lim_{\leftarrow} \frac{\mathcal{H}_f^{(0)}}{\mathcal{H}_f^{(-k)}}$$

where we have

$$\bigcap_{k=0}^{\infty} \mathcal{H}_f^{(-k)} = 0. \tag{10.7}$$

K. Saito similarly considers the dual flat vector bundle $(\check{\mathcal{H}}^{(0)}, \check{\nabla})$ with the dual connection, and writes

$$0 \to f_* \Omega_f \to \check{\mathcal{H}}^{(0)} \xrightarrow{df} \check{\mathcal{H}}^{(1)} \to 0 \tag{10.8}$$

$$\check{\nabla} : \check{\mathcal{H}}^{(0)} \to \check{\mathcal{H}}^{(1)} \tag{10.9}$$

and regards $\mathcal{H}^{(1)}$ as a quotient of $\mathcal{H}^{(0)}$ via this sequence. We then obtain

$$0 \to f_* \Omega_f \to \check{\mathcal{H}}^{(k)} \to \check{\mathcal{H}}^{(k+1)} \to 0$$
$$\check{\nabla} : \check{\mathcal{H}}^{(k)} \to \check{\mathcal{H}}^{(k+1)}.$$

By local duality for residue pairing ([S1], [G3] page 659 and 693); we have $\mathcal{O}_T$-bilinear maps

$$\mathcal{H}^{(0)} \times \check{\mathcal{H}}^{(0)} \to \mathcal{O}_T$$
$$\mathcal{H}^{(-1)} \times \check{\mathcal{H}}^{(1)} \to \mathcal{O}_T$$

which induces an infinite sequence of $\mathcal{O}_T$-dualities

$$\mathcal{H}^{(k)} \times \check{\mathcal{H}}^{(k)} \to \mathcal{O}_T.$$

Using $k = 0$ in former exact sequences one obtains the following exact sequence

$$0 \to \mathcal{H}^{(-1)} \to \mathcal{H}^{(0)} \xrightarrow{\chi} \check{\mathcal{H}}^{(0)} \to \check{\mathcal{H}}^{(1)} \to 0 \tag{10.10}$$

where χ is given by the correspondence $\phi dx \to \phi/(\partial f/\partial x_0 ... \partial f/\partial x_n)$.

Then $f_*\Omega_f$ becomes a self dual module, on which the bilinear form,

$$Res = Res_{f,0} = Res_{X/T} : f_*\Omega_f \times f_*\Omega_f \to \mathcal{O}_T$$
$$\phi dx \times \psi dx \to Res_{f,0}(\frac{\phi\psi dx}{(\partial f/\partial x_0 ... \partial f/\partial x_n)})$$

is well-defined.

On the other hand, lets consider the set of formal Laurent series in ∂_t^{-1} with coefficients in $\Omega^\bullet_{X/T}$,

$$\Omega := \Omega^\bullet = \Omega_{X/T}[[\partial_t^{-1}]][\partial_t] := \{\sum_{k \leq k_0} \omega_k \partial_t^k : k_0 \in \mathbb{Z},\ \omega_k \in \Omega_{X/T}\}. \tag{10.11}$$

$\Omega^\bullet$ has an increasing filtration

$$F^k\Omega^\bullet := \{\omega \in \Omega^\bullet,\ \omega = \sum_{m \leq k},\ \omega_k \in \Omega^\bullet_{X/T}\}. \tag{10.12}$$

The wedge product and the exterior derivative $d_{X/T}$ of Poincarè complex $\Omega^\bullet_{X/T}$ naturally extend to $\Omega^\bullet$ by formally requiring that these operations commute with ∂_t^{-1}.

Lets define $\hat{d} : \Omega^\bullet \to \Omega^{\bullet+1},\ \hat{d} = \partial_t^{-1} d_{X/T} - df \wedge (.)$. Then

$$\hat{d} \circ \hat{d} = 0 \tag{10.13}$$

$$\hat{d} F^k\Omega^\bullet \subset F^k\Omega^{\bullet+1}. \tag{10.14}$$

Proposition 10.2.2. *[S1] Consider the natural homomorphisms;*

$$\mathcal{H} \to R^{n+1} f_*(\Omega, \hat{d})$$

Then there exists natural ∂_t^{-1}-equivariant isomorphisms

$$\hat{\alpha}_k : \widehat{\mathcal{H}^{(-k)}} \cong R^{n+1} f_*(F^{-k}\Omega, \hat{d}), \qquad k \geq 1.$$

Thus, the relation between $R^{n+1} f_* \Omega^\bullet$ and the former construction becomes of the form

$$\hat{\alpha} : \widehat{\mathcal{H}^{(0)}} \cong R^{n+1} f_*(F^0\Omega, \hat{d}). \tag{10.15}$$

K. Saito [S1], [S2] in this way defines a bilinear map

$$K = K_f : R^{n+1} f_*(\Omega, \hat{d}) \times R^{n+1} f_*(\Omega, \hat{d}) \to \mathcal{O}[[\partial_t^{-1}]][\partial_t] \tag{10.16}$$

which induces higher residue pairings

$$K^{(k)} = K_f^{(k)} : \mathcal{H}^{(0)} \times \mathcal{H}^{(0)} \to \mathcal{O}_T,$$

for $k \in \mathbb{Z}$.

Definition 10.2.3. (K. Saito) [S1] Denote by $K = K_f$ the $\mathcal{O}_T$-bilinear map

$$R^{n+1}(f_*\Omega, \hat{d}) \times R^{n+1}(f_*\Omega, \hat{d}) \to \mathcal{O}_T[[\partial_t^{-1}]][\partial_t]$$

which is induced by

$$K_1(\omega, \zeta) = Res_{X/T}[\beta(\omega).\bar{\zeta}]$$

where $\beta(\omega) = \sum \partial_t^k(\omega)\partial_t^{-k}$ and

$$\zeta = \sum P_k \partial_t^k \Leftrightarrow \bar{\zeta} = \sum (-1)^k P_k \partial_t^k.$$

Proposition 10.2.4. *(K. Saito) [S1] $K = K_f$ has the following properties, ($\omega_i \in R^{n+1} f_* \Omega^\bullet$),*

(1) $\phi K(\omega_1, \omega_2) = K(\phi\omega_1, \omega_2) = K(\omega_1, \bar{\phi}\omega_2), \qquad \phi \in \mathcal{O}_T[[\partial_t^{-1}]][\partial_t]$

(2) $K(\omega_1, \omega_2) = \overline{K(\omega_2, \omega_1)}$

(3) $\partial_t K(\omega_1, \omega_2) = K(\partial_t \omega_1, \omega_2) + K(\omega_1, \partial_t \omega_2)$.

Definition 10.2.5. If we expand K in a Laurent series in ∂_t^{-1}

$$K(\omega_1, \omega_2) = \sum_k K(\omega_1, \omega_2) \partial_t^{(-k)},$$

we get an infinite sequence of $\mathcal{O}_T$-bilinear forms

$$K^{(k)} = K_f^{(k)} : R^{n+1}(f_*\Omega) \times R^{n+1}(f_*\Omega) \to \mathcal{O}_T, \qquad k \in \mathbb{Z}.$$

Theorem 10.2.6. *(K. Saito) [S1]* $K^{(k)}$*'s have the following properties;*

1) $K^{(k)}$ *is symmetric for even* k *and skew-symmetric for odd* k*.*

2) $K^{(k+1)}(\omega_1, \omega_2) = K^{(k)}(\partial_t \omega_1, \omega_2) = -K^{(k)}(\omega_1, \partial_t \omega_2)$

3) $\partial K^{(k)}(\omega_1, \omega_2) = K^{(k)}(\partial_t \omega_1, \omega_2) + K^{(k)}(\omega_1, \partial_t \omega_2)$

4) $K^{(k)}(t\omega_1, \omega_2) + K^{(k)}(\omega_1, t\omega_2) = (n+k) K^{(k-1)}(\omega_1, \omega_2)$

5) $K^{(0)}$ *induces the zero map on* $R^{n+1}(F^{-1} f_*\Omega) \times R^{n+1}(F^0 f_*\Omega)$ *so that the induced bilinear map on*

$$(f_*\Omega_f) \times (f_*\Omega_f) (\cong gr^0 R^{n+1}(f_*\Omega) \times gr^0 R^{n+1}(f_*\Omega))$$

coincides with $Res_{f,0}$*.*

Theorems 8.6.1 and 8.7.1 explain the relation between the form of K. Saito K_f and the polarization form on $H^n(X_\infty, \mathbb{C})$ in accordance to property 5). The form K_f defines a conjugation functor C_X, satisfying squares,

$$\begin{array}{ccc} V^{\alpha} & \xrightarrow[\cong]{C_X^{\alpha}} & V^{-\alpha^{\vee}} \\ {\scriptstyle DR_{X,\lambda}} \downarrow & & \downarrow {\scriptstyle DR_{X,-\lambda}} \\ H_{-\lambda} & \xrightarrow[c_X^{-\lambda}]{\cong} & H_{\lambda}^{\vee} \end{array} \qquad C_X^{\alpha}(\omega) = K_f(\omega, \ ) \qquad (10.17)$$

where V stands for the V-filtration, $\lambda = e^{-2\pi i \alpha}$ and DR is the solution functor. Define

$$PGr_l^W V^\alpha := \ker(t\partial_t - \alpha)^{l+1} : Gr_l^W V^\alpha \to Gr_{-l-2}^W V^\alpha.$$

We will also obtain a set of positive definite bilinear maps,

$$K_{f,l} \circ (id \otimes (t\partial_t - \alpha)^l) : PGr_l^W V^\alpha \otimes_{\mathbb{C}} PGr_l^W V^\alpha \to \mathbb{C}[t, t^{-1}]. \qquad (10.18)$$

Remark 10.2.7. We have included this section in order to realize the close interactions between Higher residues introduced in 6.5 with the polarization form on one side and Grothendieck residue on the other side. A reasonable question is the quantity that Theorem 8.6.1 intersects Theorem 10.2.6 or the brief in 6.5. First of all there is a positivity criteria established in 8.6.1 and 8.7.6 of the polarization form can not be deduced from 10.2.6. The inter-relation of K. Saito form with polarizations of the fibers is explained in the next section. However, these relations are extremely complicated to deduct the second Riemann-Hodge bilinear relations. Mathematically the fact that two non-degenerate bilinear form are reductions of a global form in different charts is not enough to establish they are equal. It seems that this process according to the properties listed in 6.5 involves some analysis of residues of the Gauss-Manin connection with respect to different lattices in the Gauss-Manin system.

10.3 Generalizations

In this last section we embedd our former machinery in a more modern language, that of D-modules. The D-modules we consider are all equipped with two filtrations F and W, and their underlying local system is a perverse sheaf having mixed Hodge structure. In the literature such D-modules are also called mixed Hodge modules. An example of this is the Gauss-Manin system defined in 4.1 or 6.1. However, the concept of a mixed Hodge module is more general, in the way that they can be defined along a stratification of the ambient manifold, by inductive extensions, beginning from a pure polarized variation of Hodge structure. The solution sheaf is an intersection cohomology complex in this case. Intersection cohomology complexes are the basic building blocks of perverse sheaves.

Suppose M is a D_X-module. The sheaf $\mathrm{Hom}_D(M, \mathcal{O}_X)$ is called the solution module of M. The derived functors $\mathcal{R}\mathrm{Hom}_D(M, \mathcal{O}_X)$ are called higher solution modules of M, [AR], [PS]. The Riemann-Hilbert correspondence, [AR], [PS] asserts that

$$\mathcal{R}Hom_D(M, \mathcal{O}_X) : D^b_{rh}(X) \to D^b(X, \mathbb{C})$$

is an equivalence of categories, where D_{rh} means regular holonomic D-modules. Holonomicity may be thought as a finitness assumption for the solution sheaf of the D-module. A sheaf in $D^b(X, \mathbb{C})$ is called *perverse* if it is isomorphic to $\mathcal{R}\mathrm{Hom}_D(M, \mathcal{O}_X)$ or the solution module of some regular holonomic M, [AR].

Definition 10.3.1. A variation of mixed Hodge structure over the punctured disc D^* is admissible if

- The pure variations $Gr_m^W(L)$ are polarizable.
- There exists a limit Hodge filtration $F_{\lim}$ compatible with the one on $Gr_m^W(L)$ constructed by Schmid.
- There exists a so called relative monodromy filtration U on $(E = L_t, W)$ with respect to the logarithm N of the unipotent part of the monodromy. This means that $NU_k \subset U_{k-2}$ and U induces the monodromy filtration on $Gr_k^W(E)$.

The concept of admissibility is defined similarly in general and not only over the disc, [AR], [P2]. This assumption is crucial in the mixed case, [PS], [SAI5], [P2]. It should be understood as the condition in order that the VMHS can be extended on the degenerate points. We assume this condition through the remainder of the text.

Example 10.3.2. A basic example is given by a fibration $f : X \to \Delta$ with $D = f^{-1}(0)$ a normal crossing divisor. It leads to the following diagram

$$\begin{array}{ccccccc} X_\infty & \longrightarrow & U & \longrightarrow & X & \longleftarrow & E \\ {\scriptstyle f_\infty}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow \\ H & \xrightarrow{e} & \Delta^* & \longrightarrow & \Delta & \longleftarrow & 0 \end{array} \tag{10.19}$$

namely Specialization diagram, where the monodromies are assumed to be quasi-unipotent. $X_\infty = X \times_{\Delta^*} H$ is called the canonical fiber, [PS].
Suppose $f : X \to \mathbb{C}$ is a non-constant function on a complex manifold X, with $f^{-1}(0)$ possibly a degenerate fiber. The nearby cycle functor applied to $F \in D^b(X)$ is

$$\Psi_f F = i^* \mathbb{R}p_* p^* F$$

where $p : H \times_{\mathbb{C}} X \to X$, $i : X_0 = f^{-1}(0) \hookrightarrow X$, and H is the universal cover of $\mathbb{C}^*$. The vanishing cycle functor is the mapping cone of the adjunction morphism $i^{-1}F \to \Psi_f F$. Thus, we have a diagram

$$\begin{array}{ccccccc} i^*F & \longrightarrow & \psi_* F & \xrightarrow{can} & \phi_* F & \longrightarrow & i^*F[-1] \\ \downarrow & & \downarrow{\scriptstyle T-I} & & \downarrow{\scriptstyle var} & & \downarrow \\ 0 & \longrightarrow & \psi_* F & \xrightarrow{=} & \psi_* F & \longrightarrow & 0 \end{array} \quad . \tag{10.20}$$

Assume $\mathbb{Q}_X[n+1]$ is a perverse sheaf (in particular $\dim(X) = n+1$). This is satisfied if X is a local complete intersection. Denote $\psi_f\mathbb{Q}_X,\ \phi_f\mathbb{Q}_X$ the nearby and vanishing cycle complexes on $X_0 = f^{-1}(0)$. It is known that $\psi_f\mathbb{Q}_X[n],\ \phi_f\mathbb{Q}_X[n]$ are perverse.
Then

$$\psi_{f,\lambda}\mathbb{Q}_X = \ker(T_s - \lambda), \qquad \phi_{f,1}\mathbb{Q}_X = \ker(T_s - id)$$

and $\phi_{f,\lambda} = \psi_{f,\lambda}$ for $\lambda \neq 1$. We know that

$$H^j(F_x, \mathbb{Q})_\lambda = H^j(\psi_{f,\lambda}\mathbb{Q}_X), \qquad \tilde{H}^j(F_x, \mathbb{Q})_\lambda = H^j(\phi_{f,\lambda}\mathbb{Q}_X).$$

Its relation with monodromy is reflected in the Wang sequence

$$\to H^j(L_x \setminus X_0) \to H^j(F_x)_1 \xrightarrow{N} H^j(F_x)_1(1) \to H^{j+1}(L_x \setminus X_0) \to ...$$

In order to explain the V-filtration, consider the following example. Let $X = \mathbb{C}$ with coordinate t and $Y = 0$. Fix a rational number $r \in (-1, 0)$, and let $M = \mathcal{O}_{\mathbb{C}}[t^{-1}]t^r$, with ∂_t acting on the left in the usual way. For each $\alpha \in \mathbb{Q}$ define $V_\alpha M \subset M$ to be the $\mathbb{C}$-span of $\{t^{n+r} | n \in \mathbb{Z}, n + r > -\alpha\}$. The following properties are easy to check,

- The filtration is exhaustive and left continuous: $\cup V_\alpha M = M$, and $V_{\alpha+\epsilon} = V_\alpha M$, for $0 < \epsilon << 1$
- Each $V_\alpha M$ is stable under $t^i \partial_t^j$ if $i > j$.
- $\partial_t V_\alpha M \subset V_{\alpha+1} M$, and $t.V_\alpha M \subset V_{\alpha-1}$.
- The associated graded

$$Gr_\alpha^V M = V_\alpha / V_{\alpha-\epsilon} = \begin{cases} \mathbb{C}t^{-\alpha} & \text{if } \alpha \in r + \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

  is an eigenspace of $t\partial_t$ with eigenvalue $-\alpha$.

The last item implies that the set of indices that $V_\alpha M$, jumps is discrete.

The above construction is generalized to define the V-filtration for a regular holonomic D-module on X that is quasi-unipotent along a closed sub-variety Y. It is indexed by $\mathbb{Q}$. If Y is smooth, then for such a D-module, there always exists a unique filtration satisfying similar properties as listed above, called the V-filtration along Y, [PS] page 350. Then t would be replaced by the ideal sheaf of $Y \hookrightarrow X$. In case Y is not smooth this construction can be done using embedding by graph.
In the normal crossing case obtained by blow up (or some compactification) in a locus of an isolated singularity of the divisor, by choosing f to be a

defining equation of the local divisor, we reach to a situation similar to the Theorem 8.6.1.

We explain a method of descent on extension and specialization of duality for D-modules, originally belonging to C. Sabbah and M. Saito cf.[SA4], [SAI3]. It can also be applied to K. Saito higher residue pairing. Assume $X = Z \times \mathbb{C}$, where Z a complex manifold identified with $Z = Z \times 0$, and let M be a holonomic D_X-module. For $p \in \mathbb{N}$, set,

$$M_{\alpha,p} := \bigoplus_{k=0}^{p} M[t^{-1}] \otimes e_{\alpha,k} \tag{10.21}$$

with $e_{\alpha,k} = 0$ for $k < 0$ and $e_{\alpha,k} = t^{\alpha}(\log t)^k/k!$ otherwise.
We have natural maps

$$M_{\alpha,p} \overset{a_{p,p+1}}{\hookrightarrow} M_{\alpha,p+1}, \qquad \textstyle\sum_{k=0}^{p} m_{\alpha,k} \otimes e_{\alpha,k} \mapsto \sum_{k=0}^{p} m_{\alpha,k} \otimes e_{\alpha,k}$$

$$M_{\alpha,p+1} \overset{b_{p+1,p}}{\to} M_{\alpha,p+1}, \qquad \textstyle\sum_{k=0}^{p} m_{\alpha,k} \otimes e_{\alpha,k} \mapsto \sum_{k=0}^{p} m_{\alpha,k+1} \otimes e_{\alpha,k}$$

$$N = a_{p-1,p} \circ b_{p,p-1}, \qquad m \otimes e_{\alpha,k} \to m \otimes e_{\alpha,k-1}.$$

Then

$$Gr_{-1}^{V} M_{\alpha,p} \cong \bigoplus_{k=0}^{p} Gr_{\alpha}^{V} M \otimes e_{\alpha,k}.$$

Define the maps,

$$Gr_{\alpha}^{V} M \to GR_{-1}^{V} M_{\alpha,p}, \qquad m_0 \mapsto \textstyle\bigoplus_{k=0}^{p} (t\partial_t - \alpha)^k m_0 \otimes e_{\alpha,k}$$

$$Gr_{-1}^{V} M_{\alpha,p} \to Gr_{\alpha}^{V} M, \qquad \textstyle\sum_{k=0}^{p} m_k \otimes e_{\alpha,k} \mapsto \sum_{k=0}^{p} (t\partial_t - \alpha)^k m_{p-k}.$$

For p large enough (actually when $(t\partial_t - \alpha)^p = 0$ by 10.21) they induce isomorphisms;

$$Coker(t\partial_t) \cong Gr_{\alpha}^{V} M \cong \ker(t\partial_t).$$

The limit is called moderate nearby cycle module, denoted $\psi_{t,\lambda}^{mod} M$. The case of the moderate vanishing cycle module $\phi_{t,1}^{mod}$ is done in a similar way, by considering the inductive system $M \to M_{-1,p}$ instead of the single module $M_{\alpha,p}$, and the action of N is the endomorphism $t\partial_t$ on $Gr_0^V M$ (see [SA4] sec. 4). Then, we have,

$$Can = -\partial_t : Gr_{-1}^V M \leftrightarrows Gr_0^V M : t = Var.$$

which are isomorphisms, [SA4]. Let

$$S : M \otimes M \to \mathbb{C}[[t, t^{-1}]]$$

be a duality. Write

$$\psi_t S : \psi_t M \otimes \psi_t M \to Db_{\mathbb{C}}^{mod(0)}$$
$$\psi_t S(\sum_{k=0}^{p} \mu_k \otimes e_{\alpha,k}\ ,\ \sum_{l=0}^{p} m_l \otimes e_{\alpha,l}) = \sum_{k+l=p} (\mu_k, m_l) e_{\alpha,k} \overline{e_{\alpha,l}}$$

as the formal extension of the bilinear form S according to the above procedure and where $Db_{\mathbb{C}}^{mod(0)}$ is the ring of C^∞ distributions with moderate growth in dimension 1. These distributions naturally receive a doubly indexed V-filtration w.r.t the coordinates t and $\bar{t}$. $Db_{\mathbb{C}}^{mod(0)}$ is the set of elements of the form, [SA4]

$$\sum_{\alpha,p} \mathbb{C}\{t\}[t^{-1}]\mathbb{C}\{\bar{t}\}[\bar{t}^{-1}](\log |t|)^p$$

which is a $D_{\mathbb{C}} \otimes D_{\bar{\mathbb{C}}}$-module in the obvious way. Then, for $-1 \leq \alpha < 0$ we obtain the induced forms,

$$\psi_\lambda S : Gr_\alpha^V M \otimes_{\mathbb{C}} Gr_\alpha^V M \to \mathbb{C}, \qquad \phi_1 S : Gr_0^V M \otimes_{\mathbb{C}} Gr_0^V M \to \mathbb{C} \quad (10.22)$$

with properties;

$$\psi_\lambda S(N\bullet, \bullet) = \psi_\lambda S(\bullet, N\bullet), \qquad \phi_1 S(N\bullet, \bullet) = \phi_1 S(\bullet, N\bullet)$$

which says N is an infinitesimal isometry of the descendants. We will also obtain a set of positive definite bilinear maps,

$$\psi_{\lambda,l}S \otimes (id \otimes N^l) : PGr_l^W Gr_\alpha^V M \otimes_{\mathbb{C}} PGr_l^W Gr_\alpha^V M \to \mathbb{C}. \qquad (10.23)$$

The form S is non-degenerate in a neighbourhood of Z iff all the forms $P\psi_{\lambda,l}S$ are non-degenerate. A similar statement is true for hermitian or polarization forms. The graded pairings $\psi_\lambda S,\ -1 \leq \alpha < 0$ are given by the formal residue of the form S at $t = \alpha$ and $t = 0$ respectively for $\psi_\lambda S$.

$$\psi_\lambda S = \langle \bullet, \bullet \rangle : Gr_\alpha^V M \otimes_{\mathbb{C}} Gr_\alpha^V M \stackrel{\langle \bullet, \bullet \rangle}{\to} \mathbb{C}$$

is given as the composition of a Poincarè pairing followed by residue map,

$$\psi_\lambda S \langle \sum_{l=0}^{p} m_l \otimes e_{\alpha,l}, \overline{\sum_{l=0}^{p} m_l \otimes e_{\alpha,l}} \rangle = *.\ Res_{s=\alpha} \langle \tilde{S}, |t|^{2s} dt \wedge d\bar{t} \rangle, \qquad * \neq 0,\ \alpha \neq 0$$

for $\alpha \neq 0$. The formula for $\phi_1 S$ is similar

$$\phi_1 S(\bullet, \bar{\bullet}) = *.\ Res_{t=-1} \langle \tilde{S}, |t|^{2s} \mathcal{F}_{loc} dt \wedge d\bar{t} \rangle, \qquad * \neq 0$$

.

where $\tilde{S}$ is the formal extension of S, and $\mathcal{F}_{loc}$ is the local Fourier transform, cf. [SA4] sec. 4 . We have proved the following.

Theorem 10.3.3. *[SA4] Assume $(\mathcal{G}, F, W, H, S)$ is a polarized MHM (hence regular holonomic) with quasi-unipotent underlying variation of mixed Hodge structure H, defined on a Zariski dense open subset $U = X \setminus Z$ of an algebraic manifold X, where Z is a smooth projective hypersurface. Then, the Gauss-Manin system $\mathcal{G}$ has a smooth extension to all of X and the extended MHM is also polarized. The polarization on the fibers can be described by residues of the Mellin transform of a formal extension of the polarization S over the* *elementary sections, by the two formulas*

$$\psi_\lambda S\langle\sum_{l=0}^{p} m_l \otimes e_{\alpha,l}, \overline{\sum_{l=0}^{p} m_l \otimes e_{\alpha,l}}\rangle = *. \ Res_{s=\alpha}\langle\tilde{S}, |t|^{2s} dt \wedge d\bar{t}\rangle, \qquad * \neq 0, \ \alpha \neq 0$$

$$\phi_1 S(\bullet, \bar{\bullet}) = *. \ Res_{t=-1}\langle\tilde{S}, |t|^{2s}\mathcal{F}_{loc} dt \wedge d\bar{t}\rangle, \qquad * \neq 0$$

.

The theorem can be stated for admissible variation of PMHS in a similar way. Thus we conclude that the polarization of the MHS on the fibers of the local system satisfies similar relations. In other words Theorem 10.3.3 shows how to extract the bilinear form induced on the fibers in PVMHS from a flat pairing on the disc. Summarizing with 8.6.1, 8.7.1, 9.3.6 and 10.3.3 we obtain the following result stated in the introduction as Theorem 1.0.9.

Theorem 10.3.4. *Assume $(\mathcal{G}, F, W, H)$ is a polarized MHM with underlying admissible variation of mixed Hodge structure H, defined on a Zariski dense open subset U of an algebraic manifold X. Assume $X \setminus U = D$ is a normal crossing divisor defined by a holomorphic germ f. Then the extended MHM is polarized and in a neighborhood of D, the polarization of the extension of H is given either by the modified Grothendieck residue associated to the holomorphic germ f defining the normal crossing divisor as in 8.6.1 or the usual residues of moderate extension of polarization as in Theorem 10.3.3. Moreover, the Hodge filtration on the extended fibers are opposite to the limit Hodge filtration on H. These Hodge filtrations pair together to constitute a polarized complex variation of HS.*

Proof. When we are locally dealing with an isolated singularity of the normal crossing divisor the polarization is given by the modified residue as in 8.6.1 and 8.7.1, with f the local defining equation of the divisor. The other case is when dealing with a higher dimensional locus on the divisor that is a smooth submanifold, then we are in a situation as in 10.3.3. The oppositeness of filtrations is a consequence of 9.2.6, and the discussion in 8.5. □

The result on the family of Jacobians in section 9.3 can be extended as follows. Let $\mathcal{H}$ be a variation of Hodge structure. We are interested to family of intermediate Jacobians

$$J(H_s) = H_{s,\mathbb{Z}(p)} \setminus H_{s,\mathbb{C}}/F^p H_{s,\mathbb{C}}$$

$$J(\mathcal{H}) = \bigcup_{s \in S^*} J(H_s)$$

associated to such VMHS, called a Neron model of $\mathcal{H}$ (here we have assumed the weight is 2p-1). The sections of the bundle $J(\mathcal{H})$ are called Normal functions. Again, like 9.3 we have the equality

$$J(\mathcal{H}) = Ext^1_{PVHS}(\mathbb{Z}(p), H_{\mathbb{C}}) \tag{10.24}$$

where the right hand side is the extension in the category of admissible polarized variation of Hodge structures, [SAI7].

Theorem 10.3.5. *The limit of the Poincarè product on the canonical fibers of the Neron model of a degenerate admissible variation of Hodge structure $\mathcal{H}$ is given by the modification of the residue pairing or induced by the residues as in 10.3.3. The extension describes the limit Jacobians as the Jacobians of the Opposite Hodge filtration on $\mathcal{H}$.*

Proof. The same diagrams as (9.14) and (9.15) are also valid in this case, subject to the condition that one only works with sections of $J(\mathcal{H})$ that are quasi-horizontal, i.e that have a lift to a flat section on $\mathcal{H}$. □

Extensions of normal functions is one of the most important questions in Hodge theory. Their infinitesimal invariants, i.e those properties related to the Gauss-Manin connection are one of the active research areas related to Hodge conjecture.

Example 10.3.6. We give an example of a degenerating Neron model for Jacobian bundles, to provide some picture of the construction, and leave more details for further studies. The example is taken from [SCHN], page

52 and belongs to M. Saito. Lets remark that there exists different notions of extensions for Jacobian bundles. In this example we only describe its construction over a Deligne extension. The minimal extension process is left to the reader as above. Let $H_{\mathbb{Z}} = \mathbb{Z}^4$, with $\mathbb{R}$-split Hodge structure given by $I^{1,-1} \oplus I^{-1,1} \oplus I^{0,2} \oplus I^{2,0}$, and S be given by

$$Q = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and nilpotent operator

$$N_1 = N_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\omega \in \mathbb{C}$ have $Im(\omega) \neq 0$. If the mixed Hodge structure be split over $\mathbb{Z}$, we may set

$$I^{1,-1} = \mathbb{C}\begin{pmatrix} 0 \\ 0 \\ 1 \\ \omega \end{pmatrix}, \qquad I^{1,-1} = \mathbb{C}\begin{pmatrix} 0 \\ 0 \\ 1 \\ \bar{\omega} \end{pmatrix}, \qquad I^{1,-1} = \mathbb{C}\begin{pmatrix} 1 \\ \omega \\ 0 \\ 0 \end{pmatrix}, \qquad I^{1,-1} = \mathbb{C}\begin{pmatrix} 1 \\ \bar{\omega} \\ 0 \\ 0 \end{pmatrix}.$$

These data define an $\mathbb{R}$-split nilpotent orbit on $(\Delta^*)^2$, by the rule $(z_1, z_2) \to e^{z_1 N_1 + z_2 N_2} F$, where F is given by $I^{p,q}$. it is a pull back of a nilpotent orbit on $(\Delta^*)^2$ by the map $(z_1, z_2) \mapsto z_1 z_2$. F^0 on the Deligne extension is spanned by

$$e_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \omega \end{pmatrix}, \qquad e_1 = \frac{1}{s_1}\begin{pmatrix} 1 \\ \omega \\ 0 \\ 0 \end{pmatrix}, \qquad e_1 = \frac{1}{s_2}\begin{pmatrix} 1 \\ \omega \\ 0 \\ 0 \end{pmatrix}.$$

It has a presentation as

$$\mathcal{O} \xrightarrow{\begin{pmatrix} 0 \\ -s_1 \\ s_2 \end{pmatrix}} \mathcal{O}^3 \to F^0 \to 0.$$

Thus, F^0 is the subset of $\Delta^2 \times \mathbb{C}^3$ given by the equation $s_1 v_1 = s_2 v_2$, using the coordinate $(s_1, s_2, v_1, v_2, v_3)$. Therefore, the Jacobian bundle T, is a bundle of rank 2 outsite the origin and has fiber $\mathbb{C}^3$ over 0. Lets look at the embedding of $T_{\mathbb{Z}}$. If $h \in \mathbb{Z}^4$ is any integral vector, one has

$$S(e_0, e^{z_1 N_1 + z_2 N_2} h) = (z_1 + z_2)(h_3 + h_4\omega) - (h_1 + h_2\omega)$$
$$S(e_j, e^{z_1 N_1 + z_2 N_2} h) = -(h_3 + h_4\omega)/s_j, \qquad j = 1, 2.$$

Then the closure of $T_{\mathbb{Z}}$ is given by

$$(e^{2\pi i z_1}, e^{2\pi i z_2}, (z_1 + z_2)(h_3 + h_4\omega) - (h_1 + h_2\omega), -\frac{(h_3 + h_4\omega)}{e^{2\pi i z_1}}, -\frac{(h_3 + h_4\omega)}{e^{2\pi i z_2}}).$$

Then the Jacobian bundle over $(\Delta^*)^2$ consists of usual intermediate Jacobians. However over 0 is $J \times \mathbb{C}^2$, and over remaining points with $s_1 s_2 = 0$ is $J \times \mathbb{C}$, where $J = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\omega$ (see the reference for more details).

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