Rigidity of an Isometric $SL(3, \mathbb{R})$-Action.

Eli Vanney Roblero Méndez

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Contents

Chapter 1

Introduction

The first results on rigidity theory were a series of works by Selberg, Calabi, Vesentini and Weil in the middle of the last century.

In 1960, A. Selberg discovered that, up to conjugation, the fundamental groups of certain compact locally symmetric spaces are always defined over the algebraic numbers. E. Vesentini, E. Calabi and A. Weil found the following local rigidity theorem in the early sixties.

Theorem 1.1. *Let G be a semisimple Lie group and assume that G is not locally isomorphic to $SL(2,\mathbb{R})$ nor $SL(2,\mathbb{C})$. Let $\Gamma \subset G$ be an irreducible cocompact lattice, then the defining embedding of Γ in G is locally rigid, i.e. any embedding ρ close to the defining embedding is conjugate to the defining embedding by a small element of G.*

In particular, Selberg gives a proof for cocompact lattices in $SL(n,\mathbb{R})$ for $n \geq 3$ in [Sel]. Calabi and Vesentini give a proof when the associated symmetric space G/K is Kähler in [CV] and Calabi gives a proof for $G = SO(1,n)$ when $n \geq 3$ in [Cal]. Weil gives a complete proof of Theorem 1.1 in [We]-[We2].

The proofs of Calabi, Calabi-Vesentini and Weil involve the study of variations of geometric structures on the associated locally symmetric space.

Selberg's proof combines algebraic facts with a study of the dynamic of iteration of matrices. In 1968, G. D. Mostow, motivated by Selberg's work and a "more geometric understanding" of Theorem 1.1 and using tools and ideas from topology, differential geometry, group theory and harmonic analysis, gets a global theorem (Strong Rigidity Theorem, [M]).

Theorem 1.2 (Mostow's Strong Rigidity Theorem)**.** *Let G and G' be as in theorem 1.1. If Γ and Γ' are two irreducible cocompact lattices in G and G', respectively then any isomorphism from Γ to Γ' extends to an isomorphism from G to G'.*

Mostow's work in turn provided inspiration for Margulis and Zimmer to study rigidity properties of higher rank groups.

In 1973, G. A. Margulis classified all finite dimensional linear representations of irreducible lattices in groups of real rank at least 2 ([Mar]).

Theorem 1.3 (Margulis's Superrigidity Theorem)**.** *Let Γ be an irreducible lattice in a connected semisimple Lie group G of real rank at least 2, trivial center, and without compact factors. Suppose $\mathbb{K}$ is a local field. Then any homomorphism π of Γ into a non-compact $\mathbb{K}$-simple group over $\mathbb{K}$ with Zariski dense image either has precompact image or π extends to a homomorphism of the ambient group G.*

In the context of Lie group actions, Zimmer ([Z3]) extended Margulis's Superrigidity to a cocycle superrigidity which has shown to be very useful in the study of actions of semisimple Lie groups without compact factors.

Margulis' theorem, classifying all linear representations, leads to believe that it is possible to classify all homomorphisms to other interesting classes of topological groups.

In the 1986 International Congress of Mathematicians, Robert J. Zimmer presented a program to understand the actions of the groups and its lattices onto other natural classes of groups, such as the group of smooth transformations of compact manifolds. A frequent observation about this theory and homomorphisms is the existence of **strong manifestations of the rigidity theory**.

A big part of current research about dynamics of groups is focused on showing that rigid actions can be classified, up to a smooth "global coordinates change". Zimmer's Program is focus in this direction.

If we assume that G is a connected Lie group, then the structure theory tells us there are two main cases to consider:

- solvable.
- semisimple.

Since there is a classification that provides a list of the semisimple Lie groups, then a case-by-case analysis is possible. Thus, we make emphasis in the case where G is semisimple.

Let G be a connected, non-compact simple Lie group acting isometrically on a connected, analytic manifold M with a finite volume pseudo-Riemannian metric. It has been proved that these actions are rigid and strongly determine the possibilities and properties of the manifold M. The general thesis is that such actions along with some extras conditions, imply that M is an algebraic double coset of the form $K\backslash H/\Gamma$ where H is a semisimple Lie group containing the group G.

Some results have already been obtained by Ólafsson-Quiroga ([OQ]) and Quiroga ([Q2]).

Theorem 1.4 (Quiroga-Barranco, [Q2])**.** *Let G be a connected non-compact simple Lie group with finite center and $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$. If G acts faithfully* *and topologically transitively on a compact manifold M preserving a pseudo-Riemannian metric such that the action is transversely Riemannian, then the G-action on M is ergodic and engaging, and*

$$\widehat{M} \cong K\backslash L/\Gamma$$

where $\widehat{M} \to M$ is a finite covering and L a semisimple Lie group that contains G as a factor.

In the case $G = \widetilde{SO}_0(p,q)$, we have the next result proved by G. Ólafsson and R. Quiroga-Barranco in [OQ].

Theorem 1.5 (Ólafsson-Quiroga, [OQ])**.** *Let M be a connected analytic pseudo-Riemannian manifold. Suppose that M is completely weakly irreducible, has finite volume and admits an analytic and isometric $\widetilde{SO}_0(p,q)$-action with a dense orbit, for some integers $p, q \geq 1$ and $n = p + q \geq 5$. If $\dim(M) = \frac{n(n+1)}{2}$ then there exists*

$$H/\Gamma \to M$$

an analytic finite covering map for H either $\widetilde{SO}_0(p+1,q)$ or $\widetilde{SO}_0(p,q+1)$, and $\Gamma \subset H$ a lattice.

For other similar related works we refer to [B], [Q].

Given the result in Theorem 1.5, it is natural to consider the following questions:

(I) *Can we extend this result to other simple Lie groups and manifolds*?,

(II) *For which integers n, can we obtain a similar result to Theorem 1.5 with the simple Lie group $G = SL(n,\mathbb{R})$*?.

1.1 Main Theorems

In this work, our main goal is to give a partial answer to questions (I) and (II), that we have raised previously.

Recall that if G is a connected non-compact simple Lie group acting isometrically on a connected analytic manifold M with a finite volume pseudo-Riemannian metric then, following Zimmer's program, it has been shown that such action is rigid in the sense of having distinguished properties that restrict the possibilities for M (e.g. Theorems 1.4, 1.5).

In the present work, we obtain results concerning the properties of some analytic connected manifold M when the simple Lie group $\widetilde{SL}(3,\mathbb{R})$ acts isometrically on the manifold preserving a finite volume pseudo-Riemannian metric.

If we denote by $G_{2(2)}$ (in this work) to the simply connected, non-compact simple Lie group such that

$$\mathrm{Lie}(G_{2(2)}) = \mathfrak{g}_{2(2)}.$$

Our main result is the following theorem:

Theorem 1.6. *Let M be a connected analytic pseudo-Riemannian manifold. Suppose that M is complete, has finite volume and admits an analytic and isometric $\widetilde{SL}(3,\mathbb{R})$-action with a dense orbit. If $8 < \dim(M) \leq 14$ then one, and only one, of the following diffeomorphisms is satisfied:*

1) $\widetilde{M} \cong \widetilde{SL}(3,\mathbb{R}) \times \widetilde{N}$, *where $\widetilde{N}$ is a pseudo-Riemannian manifold.*

2) $\widetilde{M} \cong G_{2(2)}$.

3) $\widetilde{M} \cong \mathbb{R}\backslash\widetilde{SL}(4,\mathbb{R})$.

The proof of this result is inspired by the work of G. Ólafsson and R. Quiroga-Barranco in [OQ] which is based on the study of the theory of Killing vector fields that centralize the G-action. The main element, with respect to the centralizer of Killing fields is Proposition 3.7, as already found in [Q], [OQ], [Z2] with different assumptions on the manifold M and the Lie group G.

Proposition 3.7 ensures the existence of a Lie algebra $\mathfrak{g}(x)$, isomorphic to $\mathfrak{g}$ (the Lie algebra of G), of Killing vector fields which vanish at a point x on $\widetilde{M}$, the universal covering of M, with additional properties. The Lie algebra $\mathfrak{g}(x)$ provides T_xM with a $\mathfrak{g}$-module structure which allows us to use representation theory in the tangent space and normal tangent space (Chapter 2) of the orbits to control its behavior (Section 3.2). This $\mathfrak{g}$-module structure allows us to obtain some control of the Lie algebra $\mathcal{H}$ of Killing vector fields which centralize the G-action (Section 3.3). The Lie algebra $\mathcal{H}$ gives rise to an action of a Lie group on the manifold M, which lets us find properties about this manifold (Chapter 4).

With the same hypotheses that Proposition 3.7, Szaro (in [Sza]) shows that the G-action must be locally free and, hence, the orbits define a foliation that we will denote with $\mathcal{O}$. Then, as implication of Proposition 3.7(4) and the analyticity of the elements which take part in this Proposition, we have the following options:

(a) $T_x\mathcal{O}^\perp$ is a trivial $\mathfrak{g}$-module for almost every $x \in S$.

(b) There exists a subset $A \subset S$ of positive measure such that if $x \in A$ then $T_x\mathcal{O}^\perp$ is a non-trivial $\mathfrak{g}$-module.

If case (a) is satisfied this implies that the normal bundle to the orbits, $T\mathcal{O}^\perp$, is integrable and then case (1) in Theorem 1.6 occurs. Furthermore, we can show a specific structure of the manifold M ([Q]).

Theorem 1.7. *With the same hypotheses as in Theorem 1.6, if the bundle $T\mathcal{O}^\perp$ is integrable then there exist:*

i) *an isometric finite covering map $\widehat{M} \to M$ to which the $\widetilde{SL}(3,\mathbb{R})$-action lifts,*

ii) *a simply connected complete pseudo-Riemannian manifold $\widetilde{N}$, and*

iii) *a discrete subgroup* $\Gamma \subset (\widetilde{SL}(3,\mathbb{R}) \times Iso(\widetilde{N}))$,

such that $\widehat{M}$ *is* $SL(3,\mathbb{R})$*-equivariantly isometric to* $(\widetilde{SL}(3,\mathbb{R}) \times \widetilde{N})/\Gamma$.

The previous result is a special case of the results presented in [Q], where the case $T\mathcal{O}^\perp$ integrable is carefully analyzed. When the normal bundle is non-integrable we obtain the second and third case of Theorem 1.6 (subsection 7.2.2).

Theorem 1.8. *With the hypotheses as in Theorem 1.6, if the second case is satisfied then there exist:*

i) *a lattice* $\Gamma \subset G_{2(2)}$, *and*

ii) *an analytic finite covering map* $\varphi : G_{2(2)}/\Gamma \to M$,

such that φ *is* $\widetilde{SL}(3,\mathbb{R})$*-equivariant. Furthermore, we can rescale the metric on* M *along the* $\widetilde{SL}(3,\mathbb{R})$*-orbits and their normal bundle to assume that* φ *is a local isometry for the bi-invariant pseudo-Riemannian metric on* $G_{2(2)}$ *given by the Killing form of its Lie algebra.*

Theorem 1.9. *With the same hypotheses as in Theorem 1.6, if the third case is satisfied then there exists an* $\widetilde{SL}(3,\mathbb{R})$*-equivariant map*

$$\varphi : \mathbb{R}\backslash\widetilde{SL}(4,\mathbb{R}) \to \widetilde{M}.$$

Furthermore, we can rescale the metric on M *along the* $\widetilde{SL}(3,\mathbb{R})$*-orbits and their normal bundle to assume that the composition of the (natural) quotient map*

$$\pi : \widetilde{SL}(4,\mathbb{R}) \to \mathbb{R}\backslash\widetilde{SL}(4,\mathbb{R})$$

with φ, *is a pseudo-Riemannian submersion for the pseudo-Riemannian metric on* $\widetilde{SL}(4,\mathbb{R})$ *given by the Killing form of its Lie algebra.*

In Theorem 1.7 we have assumed that case (a) is satisfied which implies the integrability of $T\mathcal{O}^\perp$. However, the integrability of $T\mathcal{O}^\perp$ does not imply that case (a) holds. Hence, the following result explores the structure of the manifold M when $T\mathcal{O}^\perp$ is integrable and case (b) is satisfied.

Theorem 1.10. *With the same hypotheses as in Theorem 1.7, if we assume the existence of a subset* A *of positive measure such that* $T_x\mathcal{O}^\perp$ *is a non-trivial* $\mathfrak{sl}(3,\mathbb{R})$*-module for all* $x \in A$, *then the manifold* $\widetilde{N}$ *is diffeomorphic to one of the following spaces:*

- $\mathbb{R}^3 \times \mathbb{R}^{3*}$ *as abelian Lie group.*
- $\mathbb{R}^3 \times \mathbb{R}^{3*}$ *as* 2*-step nilpotent Lie group.*
- $SL(3,\mathbb{R})\backslash G_{2(2)}$.
- $(SL(3,\mathbb{R}) \times \mathbb{R})\backslash\widetilde{SL}(4,\mathbb{R})$.

- $SL(4,\mathbb{R})\backslash\widetilde{SO}_0(3,4)$.

On the other hand, if we assume that M is a weakly irreducible manifold in the hypothesis of Theorem 1.6 (similar to Theorem 1.5) then we have the following result.

Theorem 1.11. *Let M be a manifold with the same hypotheses as in Theorem 1.6. Also, assume that M is a weakly irreducible manifold then we have that the only possible conclusions are parts* 2) *and* 3) *of Theorem 1.6.*

Note, that the weakly irreducibility removes the possibility that $T\mathcal{O}^\perp$ is integrable. Hence this work complements the analysis made by R. Quiroga-Barranco in [Q] and contributes to the case-by-case analysis of actions of simple Lie groups.

Chapter 2

Representations, Modules and Subalgebras

2.1 Representations of $\mathfrak{sl}(3,\mathbb{R})$ that preserve a nondegenerate symmetric bilinear form

We are interested in the study of the linear representations of $\mathfrak{sl}(3,\mathbb{R})$ that preserve a non-degenerate symmetric bilinear form. Particularly to describe the representation with minimal dimension that satisfies this property.

2.1.1 Representations of $\mathfrak{sl}(3,\mathbb{R})$ and $\mathfrak{sl}(3,\mathbb{C})$

We start by studying the complex representations of $\mathfrak{sl}(3,\mathbb{C})$ to obtain some results and try the same results to real representations of $\mathfrak{sl}(3,\mathbb{R})$. This is because the complex representations of $\mathfrak{sl}(3,\mathbb{C})$ have some characteristics that we can take advantage of.

It is clear that a real representation of $\mathfrak{g}_0$ gives rise to a complex representation of $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$, but, will it be the same in the other direction? That is, given a complex representation of $\mathfrak{g}$, can we obtain a real representation of $\mathfrak{g}_0$? When $\mathfrak{g}_0 = \mathfrak{sl}(3,\mathbb{R})$ we give partial results that answering those questions.

In this section we will make some observations about the relationship between the irreducible real representations of $\mathfrak{sl}(3,\mathbb{R})$ and the irreducible complex representations of $\mathfrak{sl}(3,\mathbb{C})$. Such results can be found in [On].

Let $\mathfrak{g}_0$ be a real semisimple Lie algebra and V_0 a real vector space, we denote its complexifications by $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ and $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$, respectively.

If V_0 is a real representation of $\mathfrak{g}_0$, i.e., there exists a homomorphism of Lie algebras $\rho : \mathfrak{g}_0 \to \mathfrak{gl}(V_0)$, we have two *complexification* operations related to the real representation $\rho : \mathfrak{g}_0 \to \mathfrak{gl}(V_0)$. First we can extend any $\rho(x)$, for $x \in \mathfrak{g}_0$, to a complex linear operator in V, obtaining a complex representation

$$\rho^{\mathbb{C}} : \mathfrak{g}_0 \to \mathfrak{gl}(V).$$

And we can extend the homomorphism $\rho^{\mathbb{C}} : \mathfrak{g}_0 \to \mathfrak{gl}(V)$ to a homomorphism of complex Lie algebras

$$\rho(\mathbb{C}) : \mathfrak{g} \to \mathfrak{gl}(V),$$

i.e., to a complex representation of $\mathfrak{g}$.

Now, if we begin with a complex representation $\rho : \mathfrak{g}_0 \to \mathfrak{gl}(V)$ of a real Lie algebra $\mathfrak{g}_0$, we may regard ρ as a representation of $\mathfrak{g}_0$ in the real vector space $V_{\mathbb{R}}$, and thus we get a real representation

$$\rho_{\mathbb{R}} : \mathfrak{g}_0 \to \mathfrak{gl}(V_{\mathbb{R}}).$$

This is the *realification* operation. Also, if $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a complex representation of a complex Lie algebra $\mathfrak{g}$, it gives rise to a real representation

$$\rho_{\mathbb{R}} : \mathfrak{g}_{\mathbb{R}} \to \mathfrak{gl}(V_{\mathbb{R}}).$$

Recall that a real (complex) representation is called *irreducible* if the representation space does not contain any non-zero proper real (complex) invariant vector spaces. And the *equivalence* of two real (complex) representations is defined, using real (complex) isomorphisms of representations spaces.

A *complex structure* J in a real vector space V_0 is said to be *invariant* under a real representation ρ of $\mathfrak{g}_0$ in V_0 if $\rho(x)J = J\rho(x)$ for all $x \in \mathfrak{g}_0$. In this case, we may regard ρ as a complex representation of $\mathfrak{g}_0$ in (V_0, J) and the original real representation ρ will be its realification.

Similarly, a *real* (or *quaternion*) *structure* S in a complex vector space V is said to be *invariant* under a complex representation $\rho : \mathfrak{g}_0 \to \mathfrak{gl}(V)$ if $\rho(x)S = S\rho(x)$, $x \in \mathfrak{g}_0$. If a real structure S in V is invariant under ρ, then the real form $V_0 = V^S$ of V is invariant, and the real subrepresentation $\rho_0 : \mathfrak{g}_0 \to \mathfrak{gl}(V_0)$ of ρ satisfies $\rho_0^{\mathbb{C}} = \rho$.

We have then, the first result of Section 8 in [On]

Theorem 2.1 (Th. 1, Section 8, [On]). *Any irreducible real representation $\rho : \mathfrak{g}_0 \to \mathfrak{gl}(V_0)$ of a real Lie algebra $\mathfrak{g}_0$ satisfies precisely one of the following two conditions:*

(I) $\rho^{\mathbb{C}}$ is an irreducible complex representation;

(II) $\rho = \rho'_{\mathbb{R}}$, where ρ' is an irreducible complex representation admitting no invariant real structures.

Conversely, any real representation ρ satisfying (I) or (II) is irreducible.

Hence, all irreducible real representations of a given real Lie algebra $\mathfrak{g}_0$ belongs to two disjoint classes, (I) and (II), characterized by the previous theorem and we get the corollary:

Corollary 2.2. *The Class (I) consists of all irreducible real representations ρ which admit no invariant complex structure. In this case, $\rho^{\mathbb{C}}$ is an irreducible complex representation, admitting an invariant real structure.*

The Class (II) consists of all irreducible real representations ρ which admit an invariant complex structure, i.e., have the form $\rho = \rho'_{\mathbb{R}}$, where ρ' is an irreducible complex representation. In this case, ρ' admits no invariant real structures.

On the other hand, let $\rho : \mathfrak{g}_0 \to \mathfrak{gl}(V)$ be a complex representation of a real Lie algebra $\mathfrak{g}_0$. Then denote by $\bar{\rho} : \mathfrak{g}_0 \to \mathfrak{gl}(\bar{V})$ the *complex conjugate* to the representation ρ. Note that this representation is not necessarily equivalent to ρ but, $(\bar{\rho})_{\mathbb{R}} = \rho_{\mathbb{R}}$.

For this new representation we immediately have the next result

Proposition 2.3 (Proposition 1, Section 8, [On]). *For any complex representation $\rho : \mathfrak{g}_0 \to \mathfrak{gl}(V)$, we have*

$$(\rho_{\mathbb{R}})^{\mathbb{C}} \sim \rho + \bar{\rho}$$

Also, a complex representation of $\mathfrak{g}_0$ is called *self-conjugate* whenever $\rho \sim \bar{\rho}$. The definition of self-conjugate is used to show a result that gives origin to the Cartan Index.

Let $\rho : \mathfrak{g}_0 \to \mathfrak{gl}(V)$ be a self-conjugate irreducible complex representation. We define the *Cartan Index* of ρ as $\epsilon(\rho) = \mathrm{sgn}(c) = \pm 1$, where c is defined by: $S^2 = ce$, where S is an anti-automorphism of V commuting with ρ. This sign is uniquely determined, as shown in [On]. If $\epsilon(\rho) = 1$, then $S_0 = \frac{1}{\sqrt{|c|}} S$ is a real structure in V invariant under ρ. If $\epsilon(\rho) = -1$, then S_0 is a quaternion structure in $V_{\mathbb{R}}$ invariant under ρ, and V admits a structure of a vector space over $\mathbb{H}$, invariant under ρ.

Because any complex representation admitting an invariant real structure is self-conjugate then an irreducible complex representation $\rho : \mathfrak{g}_0 \to \mathfrak{gl}(V)$ admits an invariant real structure if and only if ρ is self-conjugate and its Cartan index is equal to 1.

Now, it is possible to classify the irreducible real representations described in the Theorem 2.1, up to equivalence.

Theorem 2.4. *(i) Two irreducible real representations ρ_1 and ρ_2 of the class I are equivalent if and only if $\rho_1^{\mathbb{C}}$ and $\rho_2^{\mathbb{C}}$ are equivalent.*

(ii) Two irreducible real representations $\rho_1 = (\rho'_1)_{\mathbb{R}}$ and $\rho_2 = (\rho'_2)_{\mathbb{R}}$ of the class II are equivalent if and only if $\rho'_1 \sim \rho'_2$ or $\rho'_1 \sim \overline{\rho'}_2$.

(iii) A representation of the class I cannot be equivalent to one of the class II.

Now, assume that $\mathfrak{g}_0$ is a real semisimple Lie algebra and regard $\mathfrak{g}_0$ as a real form of the complex semisimple Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. In this case [On, Th. 8.3] shows under which conditions a complex representation of a real semisimple Lie algebra $\mathfrak{g}_0$ is self-conjugate,

Theorem 2.5 (Theorem 3, Section 8 of [On]). *Let $\rho_0 : \mathfrak{g}_0 \to \mathfrak{sl}(V)$ be an irreducible complex representation of a real semisimple Lie algebra $\mathfrak{g}_0$, and let $\Lambda, \bar{\Lambda}$ denote the highest weights of the representations $\rho_0, \bar{\rho}_0$, respectively. Then ρ_0 is self-conjugate if and only if $\Lambda = \bar{\Lambda}$.*

Now, let $\mathfrak{g}_0$ be a real form of a complex semisimple Lie algebra $\mathfrak{g}$, if ρ is an irreducible representation of $\mathfrak{g}$ in a complex vector space V and denote $\rho_0 = \rho|_{\mathfrak{g}_0}$. Then ρ_0 determines an irreducible *real representation* of $\mathfrak{g}_0$. With the previous results we distinguish three different cases:

1. The **real case**. If

$$\Lambda = \bar{\Lambda}, \quad \epsilon(\mathfrak{g}_0, \rho_0) = 1 \quad \text{(Cartan index)} \tag{2.1}$$

   then ρ_0 leaves a real form V_0 of V invariant and induces an irreducible representation $\rho_0|_{V_0}$ of $\mathfrak{g}_0$.

2. The **quaternion case**. If

$$\Lambda = \bar{\Lambda}, \quad \epsilon(\mathfrak{g}_0, \rho_0) = -1,$$

   then the realification $(\rho_0)_{\mathbb{R}}$ of ρ_0 acting in $V_{\mathbb{R}}$ is irreducible, and $V_{\mathbb{R}}$ admits a structure of quaternion vector space invariant under $(\rho_0)_{\mathbb{R}}$.

3. The **complex case**. if

$$\Lambda \neq \bar{\Lambda},$$

   then the realification $(\rho_0)_{\mathbb{R}}$ of ρ_0 acting in $V_{\mathbb{R}}$ is irreducible, and ρ_0 admits neither real nor quaternion invariant structures.

Then all irreducible real representations of real semisimple Lie algebras can be obtained in one of these ways. In the real case, we get a bijection between the dominant weights Λ satisfying (2.1) and the irreducible representations $\rho_0|_{V_0}$ of $\mathfrak{g}_0$ regarded up to equivalence.

Remark 2.6. Table 5 (Indices of irreducible representations of simple complex Lie algebras) of [On] shows all the simple real Lie algebras and which case (real, quaternion or complex) is allowed in each of them. Here we note that for the real simple Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ $(n > 1)$ it is always satisfied that $\Lambda = \bar{\Lambda}$ and the Cartan index is always 1, this is, we always have the real case. In particular, that is true for $\mathfrak{g}_0 = \mathfrak{sl}(3, \mathbb{R})$.

2.1.2 Irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$

Because $\mathfrak{sl}(3, \mathbb{R})$ is a semisimple Lie algebra then all of its finite-dimensional linear representations have a decomposition in irreducible representations. Hence, we shall start by studying the irreducible representations of $\mathfrak{sl}(3, \mathbb{R})$.

Since in the previous subsection we have seen that we can study the irreducible $\mathfrak{sl}(3, \mathbb{R})$-modules through the study of irreducible $\mathfrak{sl}(3, \mathbb{C})$-modules. Thus, first we study these irreducible representations.

Recall that

$$\mathfrak{sl}(3, \mathbb{C}) = \{A \in M_{3\times 3}(\mathbb{C}) \mid \text{trace}(A) = 0\}, \tag{2.2}$$

which is a simple Lie algebra of dimension 8.

In [F, p. 224] it is shown that the irreducible representation $\Gamma_{a_1,\ldots,a_{n-1}}$ of $\mathfrak{sl}(n,\mathbb{C})$ with highest weight $(a_1,\ldots,a_{n-1})$, has dimension

$$\dim(\Gamma_{a_1,\ldots,a_{n-1}}) = \prod_{1\le i<j\le n} \frac{a_i+\cdots+a_{j-1}+j-i}{j-i}. \tag{2.3}$$

Particularly, when $n=3$ the irreducible representation Γ_{a_1,a_2} of $\mathfrak{sl}(3,\mathbb{C})$ with highest weight (a_1,a_2), satisfies

$$\dim(\Gamma_{a_1,a_2}) = \frac{(a_1+1)(a_2+1)(a_1+a_2+2)}{2}. \tag{2.4}$$

On other hand, Theorem 13.1 of [F] shows that for each pair of natural numbers, (a_1,a_2), there is a unique (up to isomorphisms) irreducible finite dimensional representation Γ_{a_1,a_2} with highest weight $a_1L_1+a_2(L_1+L_2)$, where $\{L_i\}_{i=1}^3$ is the dual basis of $\{e_i\}_{i=1}^3$. Here, $\{e_1,e_2,e_3\}$ is the standard basis of $\mathbb{C}^3$, for the standard representation of $\mathfrak{sl}(3,\mathbb{C})$ on this vector space.

Therefore, the lower dimensions of the irreducible representations of $\mathfrak{sl}(3,\mathbb{C})$ are:

$$\begin{aligned}
\dim(\Gamma_{0,0}) &= 1\\
\dim(\Gamma_{1,0}) = \dim(\Gamma_{0,1}) &= 3\\
\dim(\Gamma_{2,0}) = \dim(\Gamma_{0,2}) &= 6\\
\dim(\Gamma_{1,1}) &= 8
\end{aligned}$$

Here, if $a_1 \ge b_1$ and $a_2 \ge b_2$, then by (2.4) we have

$$\dim(\Gamma_{a_1,a_2}) \ge \dim(\Gamma_{b_1,b_2})$$

and so, when $|(a_1,a_2)| = a_1+a_2 \ge 3$ then

$$\dim(\Gamma_{a_1,a_2}) > 8.$$

Furthermore, the next result is obtained

Lemma 2.7. *The only (up to isomorphisms) non-trivial representations of $\mathfrak{sl}(n,\mathbb{C})$ with dimension less than or equal to n are $\mathbb{C}^n$ and $\mathbb{C}^{n*}$.*

Proof. From equation (2.3), if $\overline{\omega}_k$ denotes the irreducible $\mathfrak{sl}(3,\mathbb{C})$-module with its highest weight having 1 in the k-th entry and 0 in the other ones, then

$$\dim(\overline{\omega}_k) = \frac{n(n-1)\cdots(n-k+1)}{k!} = \binom{n}{k}. \tag{2.5}$$

From this result we have that

$$\dim(\overline{\omega}_1) = \dim(\overline{\omega}_{n-1}) = n, \tag{2.6}$$

is the smallest dimension of an irreducible non-trivial $\mathfrak{sl}(3,\mathbb{C})$-module.

Because $\overline{\omega}_1 \cong \mathbb{C}^n$ and $\overline{\omega}_{n-1} \cong \mathbb{C}^{n*}$ as $\mathfrak{sl}(3,\mathbb{C})$-modules, the result follows. $\square$

2.1.3 Non-degenerate symmetric bilinear forms

As mentioned at the beginning of this section we are interested in the finite-dimensional linear representations of $\mathfrak{sl}(3,\mathbb{R})$ that preserve a non-degenerate symmetric bilinear form. By the previous section, we will focus on studying the non-trivial representations of $\mathfrak{sl}(3,\mathbb{C})$ that preserve a form with these characteristics.

Let ρ be a non-trivial representation of $\mathfrak{sl}(3,\mathbb{C})$ on the vector space V and suppose that this representation preserves a non-degenerated symmetric bilinear form $\langle \cdot,\cdot \rangle$, that is,

$$\langle \rho(A)x, y\rangle + \langle x, \rho(A)y\rangle = 0 \quad \forall A \in \mathfrak{sl}(3,\mathbb{C}); x, y \in V. \tag{2.7}$$

Therefore,

$$J\rho(A) + \rho(A)^t J = 0, \quad \forall A \in \mathfrak{sl}(3,\mathbb{C}), \tag{2.8}$$

where J is an invertible matrix defined by $\langle \cdot,\cdot \rangle$. Then,

$$\rho(A) = J^{-1}(-\rho(A)^t)J, \quad \forall A \in \mathfrak{sl}(3,\mathbb{C}). \tag{2.9}$$

This last identity shows that a representation of $\mathfrak{sl}(3,\mathbb{C})$ preserving a non-degenerate symmetric bilinear form must be **autodual**.

The autodual property does not hold for all non-trivial representations of $\mathfrak{sl}(3,\mathbb{C})$. This result is stated in the next Lemma

Lemma 2.8. *The minimum dimension of a non-trivial autodual representation of $\mathfrak{sl}(3,\mathbb{C})$ is 6. Moreover, this representation is isomorphic to $\mathbb{C}^3 \oplus \mathbb{C}^{3*}$ and preserves a non-degenerated symmetric bilinear form.*

Proof. By Lemma 2.7, we will focus in representations of $\mathfrak{sl}(3,\mathbb{C})$ on vector spaces with dimension equal or bigger than 3.

DIMENSION 3.

Let $\rho_3 : \mathfrak{sl}(3,\mathbb{C}) \to \mathfrak{gl}(V)$ be a representation with $\dim(V) = 3$. If ρ_3 is irreducible, then Lemma 2.7 shows that $V \cong \mathbb{C}^3$ or $V \cong \mathbb{C}^{3*}$ as $\mathfrak{sl}(3,\mathbb{C})$-module. And from section 13 of [F], these representations are not autodual, then the map ρ_3 does not preserve a non-degenerated symmetric bilinear form.

On the other hand, if the representation $\rho_3 : \mathfrak{sl}(3,\mathbb{C}) \to \mathfrak{gl}(V)$ is reducible, by the same Lemma 2.7, we have that $V \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ as $\mathfrak{sl}(3,\mathbb{C})$-module therefore ρ_3 is a trivial representation.

DIMENSION 4.

Let V be a complex vector space of dimension 4 and suppose there is a representation ρ_4 of $\mathfrak{sl}(3,\mathbb{C})$ on V, due to the non-existence of an irreducible representation with the same dimension (Lemma 2.7), V is a direct sum of irreducible submodules, i.e.

$$V \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \quad \text{or} \quad V \cong \mathbb{C}^3 \oplus \mathbb{C} \quad \text{or} \quad V \cong \mathbb{C}^{3*} \oplus \mathbb{C} \tag{2.10}$$

If $V \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ then the representation ρ_4 is trivial.

Suppose $V \cong \mathbb{C}^3 \oplus \mathbb{C}$, in this case, the representation is non-trivial but, by the properties of representation of $\mathfrak{sl}(3,\mathbb{C})$ on $\mathbb{C}^3$, it is non-autodual.

The same argument can be used to prove that if $V \cong \mathbb{C}^{3*} \oplus \mathbb{C}$ the non-trivial representation ρ_4 is non-autodual.

DIMENSION 5.

The same proof in the case of dimension 4, with the obvious changes, can be used to prove that any non-trivial representation of the complex simple Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ on a vector space of dimension 5 is non-autodual.

DIMENSION 6

Now, focus on representations of $\mathfrak{sl}(3,\mathbb{C})$ on 6-dimensional vector spaces. In this case, from (2.4), we have only (up to automorphisms) two 6-dimensional irreducible representations: $\mathrm{Sym}^2(\mathbb{C}^3)$ and $\mathrm{Sym}^2(\mathbb{C}^{3*})$, corresponding to the highest weight $(2,0)$ and $(0,2)$, respectively. In both cases, from section 13 of [F], these representations are non-autodual.

Let then $\rho_6 : \mathfrak{sl}(3,\mathbb{C}) \to V$ be a reducible representation of $\mathfrak{sl}(3,\mathbb{C})$ with $\dim(V) = 6$. V is isomorphic to a direct sum of irreducible $\mathfrak{sl}(3,\mathbb{C})$-submodules, that is, V is isomorphic (as $\mathfrak{sl}(3,\mathbb{C})$-module) to one of the next vector spaces

$$\begin{aligned}
&\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}\\
&\mathbb{C}^3 \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}\\
&\mathbb{C}^{3*} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}\\
&\mathbb{C}^3 \oplus \mathbb{C}^3\\
&\mathbb{C}^{3*} \oplus \mathbb{C}^{3*}\\
&\mathbb{C}^3 \oplus \mathbb{C}^{3*}
\end{aligned}$$

When $V \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, the representation ρ_6 is trivial.

If V is isomorphic to $\mathbb{C}^3 \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ (or to $\mathbb{C}^{3*} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$), with a similar argument to the case of dimension 4, we can prove that this representation is not autodual.

Now, suppose $V \cong \mathbb{C}^3 \oplus \mathbb{C}^3$. Recall that the direct sum of the representations is given component-wise and that the representation of $\mathfrak{sl}(3,\mathbb{C})$ on $\mathbb{C}^3$ is the standard representation. Therefore, it is clear that V is not an autodual representation. The same argument shows that if $V \cong \mathbb{C}^{3*} \oplus \mathbb{C}^{3*}$ then V is also not an autodual representation.

For the last part of the proof, we will show the existence of a non-degenerate symmetric bilinear form on the vector space $\mathbb{C}^3 \oplus \mathbb{C}^{3*}$ that is preserved by the representation (non-trivial) of $\mathfrak{sl}(3,\mathbb{C})$.

Define the bilinear form $\langle \cdot,\cdot \rangle$ on $\mathbb{C}^3 \oplus \mathbb{C}^{3*}$ as follows: Let $v, v' \in \mathbb{C}^3 \oplus \mathbb{C}^{3*}$ be given. Then there exist unique elements $p, p' \in \mathbb{C}^3$ and $q, q' \in \mathbb{C}^{3*}$ such that $v = (p,q)$ and $v' = (p',q')$. Then $\langle v, v' \rangle$ is defined as

$$\langle v, v' \rangle = \langle (p,q), (p',q') \rangle := q(p') + q'(p) \tag{2.11}$$

where $q(p)$ is the evaluation of the element q in the vector p.

This defines a bilinear form that is non-degenerated and symmetric. Now, let $A \in \mathfrak{sl}(3,\mathbb{C})$ and $v, v' \in \mathbb{C}^3 \oplus \mathbb{C}^{3*}$ be given. Then

$$\begin{aligned}
\langle A\cdot v, v'\rangle + \langle v, A\cdot v'\rangle &= \langle A\cdot(p,q),(p',q')\rangle + \langle (p,q), A\cdot(p',q')\rangle \\
&= \langle (A\cdot p, A\cdot q),(p',q')\rangle + \langle (p,q),(A\cdot p', A\cdot q')\rangle \\
&= (A\cdot q)(p') + q'(A\cdot p) \quad + \quad q(A\cdot p') + (A\cdot q')(p) \\
&= -q(A\cdot p') + q'(A\cdot p) \quad + \quad q(A\cdot p') - q'(A\cdot p) \\
&= 0
\end{aligned}$$

therefore, we have shown that the bilinear form $\langle\cdot,\cdot\rangle$ is preserved by the action of $\mathfrak{sl}(3,\mathbb{C})$. Then, the lemma is proved. □

Now, we use the results from the two previous subsections to show similar results for the real simple Lie algebra $\mathfrak{sl}(n,\mathbb{R})$. Our main interest is the case $n = 3$.

2.1.4 Irreducible representations of $\mathfrak{sl}(3,\mathbb{R})$

Recall that

$$\mathfrak{sl}(n,\mathbb{R}) = \left\{ A \in M_{n\times n}(\mathbb{R}) \middle| \mathrm{trace}(A) = 0 \right\}$$

and a real representation of $\mathfrak{sl}(n,\mathbb{R})$ gives rise to a complex representation $\rho(\mathbb{C})$ of $\mathfrak{sl}(n,\mathbb{C})$.

We immediately have a first result about the irreducible representations of $\mathfrak{sl}(n,\mathbb{R})$.

Lemma 2.9. *The only non-trivial representations of $\mathfrak{sl}(n,\mathbb{R})$ with dimension less than or equal to n are $\mathbb{R}^n$ and $\mathbb{R}^{n*}$.*

Proof. Let $\rho : \mathfrak{sl}(n,\mathbb{R}) \to \mathfrak{gl}(V_0)$ be a non-trivial finite dimensional irreducible real representation of $\mathfrak{sl}(n,\mathbb{R})$, then by Theorem 2.1 and Remark 2.6 we have that $\rho^{\mathbb{C}}$ is a (non-trivial) irreducible complex representation because ρ is an invariant real structure.

Since $\rho^{\mathbb{C}}$ is a non-trivial irreducible representation then, by Table 5 of [On], $\rho(\mathbb{C}) : \mathfrak{sl}(n,\mathbb{C}) \to \mathfrak{gl}(V)$ is a non-trivial irreducible complex representation, where $V = V_0 \otimes \mathbb{C}$. And thus, by Lemma 2.7, $\dim_{\mathbb{R}}(V_0) = \dim_{\mathbb{C}}(V) \geq n$.

Now, letting $\rho : \mathfrak{sl}(n,\mathbb{R}) \to \mathfrak{gl}(V_0)$ be a non-trivial finite dimensional irreducible real representation with $\dim_{\mathbb{R}}(V_0) = n$, then $\rho(\mathbb{C}) : \mathfrak{sl}(n,\mathbb{C}) \to \mathfrak{gl}(V)$ is a non-trivial irreducible complex representation where $\dim_{\mathbb{C}}(V) = n$. By Lemma 2.7 this implies that $V \cong \mathbb{C}^n$ or $V \cong \mathbb{C}^{n*}$ as $\mathfrak{sl}(n,\mathbb{C})$-module.

On the other hand, recall that Table 5 in [On] shows us that (2.1) is always satisfied, then $\rho(\mathbb{C})|_{\mathfrak{sl}(n,\mathbb{R})}$ restricted to V_0 is equal to ρ. In the first case, if $V \cong \mathbb{C}^n$ as $\mathfrak{sl}(n,\mathbb{C})$-module then from Table 2 in [On] we have that $V_0 \cong \mathbb{R}^n$ as $\mathfrak{sl}(n,\mathbb{R})$-module. In a similar way we can prove that if $V \cong \mathbb{C}^{n*}$ as $\mathfrak{sl}(n,\mathbb{C})$-module then $V_0 \cong \mathbb{R}^{n*}$ as $\mathfrak{sl}(n,\mathbb{R})$-module. □

2.1.5 Non-degenerate symmetric bilinear real forms

In Lemma 2.8 we proved that the minimum dimension of a non-trivial representation of $\mathfrak{sl}(3,\mathbb{C})$ being autodual is 6. Now, we wish to prove a similar result in the real case, this is, to show which is the minimum dimension of a non-trivial $\mathfrak{sl}(3,\mathbb{R})$-representation that preserves a non-degenerate symmetric bilinear form.

Let ρ be a non-trivial representation of $\mathfrak{sl}(3,\mathbb{R})$ on the real vector space V_0 and suppose this representation preserves a non-degenerated symmetric bilinear real form $\langle\cdot,\cdot\rangle_0$ in V_0, this is:

$$\langle\rho(A)x, y\rangle_0 + \langle x, \rho(A)y\rangle_0 = 0 \quad \forall A \in \mathfrak{sl}(3,\mathbb{R}); x, y \in V_0. \tag{2.12}$$

We extend this bilinear (real) form to a non-generated symmetric bilinear complex form $\langle\cdot,\cdot\rangle$ on $V = V_0(\mathbb{C})$ such that it is preserved by the non-trivial representation $\rho(\mathbb{C})$, i.e.,

$$\langle\rho(\mathbb{C})(A)x, y\rangle + \langle x, \rho(\mathbb{C})(A)y\rangle = 0 \quad \forall A \in \mathfrak{sl}(3,\mathbb{C}); x, y \in V,$$

then

$$\rho(\mathbb{C})(A) + \rho(\mathbb{C})(A)^t = 0, \quad \forall A \in \mathfrak{sl}(3,\mathbb{C}).$$

This shows that the representation $\rho(\mathbb{C})$ is **autodual**.

Lemma 2.10. *The minimum dimension of a non-trivial real representation of $\mathfrak{sl}(3,\mathbb{R})$ preserving a non-degenerated symmetric bilinear form is* 6*. Furthermore, this representation is isomorphic to $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$. Also the bilinear form has signature* $(3,3)$*.*

Proof. Let $\rho : \mathfrak{sl}(3,\mathbb{R}) \to \mathfrak{gl}(V_0)$ be a non-trivial real representation preserving a non-degenerated symmetric bilinear form $\langle\cdot,\cdot\rangle_0$. By Lemma 2.9, we have that $\dim(V_0) \geq 3$.

Now, suppose that $3 \leq \dim_{\mathbb{R}}(V_0) \leq 6$.

If ρ is an irreducible representation then, by Theorem 2.1, $\rho^{\mathbb{C}}$ is an irreducible complex representation. And from Table 2 in [On], $\rho(\mathbb{C})$ is an irreducible complex representation of $\mathfrak{sl}(3,\mathbb{C})$. But, the proof of Lemma 2.8, shows that this is not possible. Then ρ must be a reducible representation.

On the other hand, the non-trivial complex representation $\rho(\mathbb{C})$ is autodual with the symmetric bilinear form $\langle\cdot,\cdot\rangle$ on V. Then by Lemma 2.8, $\dim_{\mathbb{R}}(V_0) = \dim_{\mathbb{C}}(V) \geq 6$.

The proof that the only non-trivial representation of $\mathfrak{sl}(3,\mathbb{R})$ preserving a non-degenerate symmetric bilinear form is isomorphic to $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ is similar to the complex case in Lemma 2.8: If $v, v' \in \mathbb{R}^3 \oplus \mathbb{R}^{3*}$ then $\langle v, v'\rangle_0$ is defined as

$$\langle v, v'\rangle_0 = \langle (p,q), (p',q')\rangle_0 := q(p') + q'(p) \tag{2.13}$$

where $p, p' \in \mathbb{R}^3$, $q, q' \in \mathbb{R}^{3*}$ and $v = (p,q), v' = (p',q')$.

Let $\{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$, then we can find elements

$$A_{ij} \in \mathfrak{sl}(3,\mathbb{R})$$

such that

$$A_{ij}(e_i) = e_i, \quad A_{ij}(e_j) = e_j \quad \text{for } i \neq j \in \{1, 2, 3\}.$$

Then, the equation

$$\langle A_{ij} \cdot e_i, e_k \rangle_0 + \langle e_i, A_{ij} \cdot e_k \rangle_0 = 0$$

implies that $\langle e_i, e_k \rangle_0 = 0$ for $k = i, j$. From this, the signature of the bilinear form $\langle \cdot, \cdot \rangle_0$ on $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ is $(3, 3)$. □

2.2 Subalgebras and $\mathfrak{sl}(3, \mathbb{R})$-modules of $\mathfrak{so}(3, 3)$.

Since $\mathfrak{sl}(3, \mathbb{R})$ preserves a nondegenerate symmetric bilinear form on $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$, which has signature $(3, 3)$, there exists a Lie algebra homomorphism $\mathfrak{sl}(3, \mathbb{R}) \to \mathfrak{so}(3, 3)$. From the simplicity of $\mathfrak{sl}(3, \mathbb{R})$ such homomorphism is injective. Thus, we conclude that $\mathfrak{so}(3, 3)$ has a structure of $\mathfrak{sl}(3, \mathbb{R})$-module which is non-trivial.

In this section we describe all Lie subalgebras of $\mathfrak{so}(3, 3)$ that are $\mathfrak{sl}(3, \mathbb{R})$-modules with the structure obtained from this injection of $\mathfrak{sl}(3, \mathbb{R})$ to $\mathfrak{so}(3, 3)$.

2.2.1 Injection up to isomorphism.

Since there is an isomorphism between the simple Lie algebras $\mathfrak{so}(3, 3)$ and $\mathfrak{sl}(4, \mathbb{R})$ ([H, p. 519]), we shall work with the decomposition of $\mathfrak{sl}(4, \mathbb{R})$ as $\mathfrak{sl}(3, \mathbb{R})$-modules.

First we know that $\mathfrak{sl}(3, \mathbb{R})$ can be injected into $\mathfrak{sl}(4, \mathbb{R})$. But, we do not know if such injection is unique.

Let

$$\iota : \mathfrak{sl}(3, \mathbb{R}) \hookrightarrow \mathfrak{sl}(4, \mathbb{R}) \tag{2.14}$$

be an injective homomorphism of Lie algebras.

Through the usual representation of $\mathfrak{sl}(4, \mathbb{R})$ on $\mathbb{R}^4$, the homomorphism ι brings a non-trivial representation of $\mathfrak{sl}(3, \mathbb{R})$ on $\mathbb{R}^4$. Then, by the previous section, $\mathbb{R}^4$ is isomorphic as $\mathfrak{sl}(3, \mathbb{R})$-module to either $\mathbb{R}^3 \oplus \mathbb{R}$ or $\mathbb{R}^{3*} \oplus \mathbb{R}$. This implies the existence of elements $\phi_1, \phi_2 \in GL(4, \mathbb{R})$ such that

$$\phi_1^{-1} \cdot \iota(A) \cdot \phi_1 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \forall A \in \mathfrak{sl}(3, \mathbb{R}),$$

in the former case, and

$$\phi_2^{-1} \cdot \iota(A) \cdot \phi_2 = \begin{bmatrix} -A^t & 0 \\ 0 & 0 \end{bmatrix} \quad \forall A \in \mathfrak{sl}(3, \mathbb{R}),$$

in the latter.

2.2.2 Decomposition in submodules, Part I

Assume first that the inclusion map ι of $\mathfrak{sl}(3,\mathbb{R})$ in $\mathfrak{sl}(4,\mathbb{R})$ is given by:

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \forall A \in \mathfrak{sl}(3,\mathbb{R}).$$

The injection ι and the simplicity of $\mathfrak{sl}(3,\mathbb{R})$ produces a decomposition of $\mathfrak{sl}(4,\mathbb{R})$ into a direct sum of irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules.

Note that

$$\mathfrak{sl}(4,\mathbb{R}) \cong \mathfrak{sl}(3,\mathbb{R}) \oplus V, \tag{2.15}$$

where V is a vector subspace of $\mathfrak{sl}(4,\mathbb{R})$ with $dim(V) = 7$.

On the other hand, observe that if $v, w \in \mathbb{R}^3$ and $c \in \mathbb{R}$ then the matrices

$$\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ w^t & 0 \end{bmatrix} \text{and} \begin{bmatrix} cI & 0 \\ 0 & -3c \end{bmatrix} \tag{2.16}$$

belong to $\mathfrak{sl}(4,\mathbb{R})$.

Furthermore, the sets

$$V_1^3 = \left\{ \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \in \mathfrak{sl}(4,\mathbb{R}) \Big| v \in \mathbb{R}^3 \right\},$$

$$V_2^3 = \left\{ \begin{bmatrix} 0 & 0 \\ w^t & 0 \end{bmatrix} \in \mathfrak{sl}(4,\mathbb{R}) \Big| w \in \mathbb{R}^3 \right\}$$

and

$$V^1 = \left\{ \begin{bmatrix} cI & 0 \\ 0 & -3c \end{bmatrix} \in \mathfrak{sl}(4,\mathbb{R}) \Big| c \in \mathbb{R} \right\}$$

are vector subspaces of $\mathfrak{sl}(4,\mathbb{R})$. Note that the intersection of these vector subspaces with $\mathfrak{sl}(3,\mathbb{R})$ and between them is zero.

Since the sum of the dimensions of V_1^3, V_2^3 and V^1 is 7, then

$$\mathfrak{sl}(4,\mathbb{R}) \cong \mathfrak{sl}(3,\mathbb{R}) \oplus V_1^3 \oplus V_2^3 \oplus V^1 \tag{2.17}$$

as a vector space.

If $X_1, X_2 \in \mathfrak{sl}(4,\mathbb{R})$ then there exist unique elements $A_i \in \mathfrak{sl}(3,\mathbb{R})$, $v_i \in \mathbb{R}^3$, $w_i^t \in \mathbb{R}^{3*}$ and $c_i \in \mathbb{R}$ such that

$$X_i = \begin{bmatrix} A_i + c_i I & v_i \\ w_i^t & -3c_i \end{bmatrix}$$

for $i = 1, 2$. Here the bracket product of X_1 and X_2 is given by

$$\begin{aligned} [X_1, X_2] &= \left[\begin{bmatrix} A_1 + c_1 I & v_1 \\ w_1^t & -3c_1 \end{bmatrix}, \begin{bmatrix} A_2 + c_2 I & v_2 \\ w_2^t & -3c_2 \end{bmatrix} \right] \\ &= \begin{bmatrix} A_1A_2 - A_2A_1 + v_1w_2^t - v_2w_1^t & A_1v_2 - A_2v_1 + 4(c_1v_2 - c_2v_1) \\ w_1^tA_2 - w_2^tA_1 + 4(c_2w_1^t - c_1w_2^t) & w_1^tv_2 - w_2^tv_1 \end{bmatrix} \end{aligned} \tag{2.18}$$

Lemma 2.11. *$\mathfrak{sl}(4,\mathbb{R})$ is isomorphic to $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \oplus \mathbb{R}$ as $\mathfrak{sl}(3,\mathbb{R})$-module.*

Proof. Recall that with the inclusion map ι given by

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \forall A \in \mathfrak{sl}(3,\mathbb{R}),$$

we have the decomposition (2.17).

On the other hand, the following are particular cases from for computation in (2.18)

$$\begin{aligned}
\left[\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_2 & 0 \\ 0 & 0 \end{bmatrix}\right] &= \begin{bmatrix} [A, A_2] & 0 \\ 0 & 0 \end{bmatrix} \\
\left[\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right] &= \begin{bmatrix} 0 & A \cdot v \\ 0 & 0 \end{bmatrix} \\
\left[\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \\ w^t & 0 \end{bmatrix}\right] &= \begin{bmatrix} 0 & 0 \\ -(A \cdot w)^t & 0 \end{bmatrix} \\
\left[\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} cI & \\ 0 & -3c \end{bmatrix}\right] &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned} \tag{2.19}$$

where $A, A_2 \in \mathfrak{sl}(3,\mathbb{R})$, $v, w \in \mathbb{R}^3$ and $c \in \mathbb{R}$.

Hence, it is clear that the vector subspaces V_1^3, V_2^3 and V^1 are $\mathfrak{sl}(3,\mathbb{R})$-submodules, and

$$\mathfrak{sl}(4,\mathbb{R}) \cong \mathfrak{sl}(3,\mathbb{R}) \oplus V_1^3 \oplus V_2^3 \oplus V^1$$

as $\mathfrak{sl}(3,\mathbb{R})$-modules.

From (2.19) we observe the following isomorphisms

$$\begin{aligned}
V_1^3 &\cong \mathbb{R}^3 \\
V_2^3 &\cong \mathbb{R}^{3*} \\
V^1 &\cong \mathbb{R}
\end{aligned}$$

as $\mathfrak{sl}(3,\mathbb{R})$-modules. The latter shows that

$$\mathfrak{sl}(4,\mathbb{R}) \cong \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \oplus \mathbb{R}$$

as $\mathfrak{sl}(3,\mathbb{R})$-module. □

Remark 2.12. From representation theory, we recall that the number of summands in a decomposition into a direct sum of irreducible submodules of a simple Lie algebra (in particular $\mathfrak{sl}(3,\mathbb{R})$) is independent of its decomposition and this decomposition is unique up to isomorphisms.

Corollary 2.13. *With the inclusion map ι given by*

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \forall A \in \mathfrak{sl}(3,\mathbb{R}),$$

the decomposition of $\mathfrak{sl}(4,\mathbb{R})$ in irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules is

$$\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \oplus \mathbb{R}$$

2.2.3 Decomposition in submodules, Part II

Now assume that the inclusion map ι of $\mathfrak{sl}(3,\mathbb{R})$ in $\mathfrak{sl}(4,\mathbb{R})$ is given by:

$$A \mapsto \begin{bmatrix} -A^t & 0 \\ 0 & 0 \end{bmatrix} \quad \forall A \in \mathfrak{sl}(3,\mathbb{R}).$$

As in the previous subsection, the injection ι and the semisimplicity of $\mathfrak{sl}(3,\mathbb{R})$ produces a decomposition of $\mathfrak{sl}(4,\mathbb{R})$ into a direct sum of irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules.

Here

$$\mathfrak{sl}(4,\mathbb{R}) \cong \mathfrak{sl}(3,\mathbb{R}) \oplus V', \tag{2.20}$$

where V' is a vector subspace of $\mathfrak{sl}(4,\mathbb{R})$ with $dim(V') = 7$.

On the other hand, if $v, w \in \mathbb{R}^3$ and $c \in \mathbb{R}$ then the matrices

$$\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ w^t & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} cI & 0 \\ 0 & -3c \end{bmatrix} \tag{2.21}$$

belong to $\mathfrak{sl}(4,\mathbb{R})$.

Note that the sets

$$V_1'^3 = \left\{ \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \in \mathfrak{sl}(4,\mathbb{R}) \middle| v \in \mathbb{R}^3 \right\},$$

$$V_2'^3 = \left\{ \begin{bmatrix} 0 & 0 \\ w^t & 0 \end{bmatrix} \in \mathfrak{sl}(4,\mathbb{R}) \middle| w \in \mathbb{R}^3 \right\},$$

$$V'^1 = \left\{ \begin{bmatrix} cI & 0 \\ 0 & -3c \end{bmatrix} \in \mathfrak{sl}(4,\mathbb{R}) \middle| c \in \mathbb{R} \right\}$$

are vector subspaces of $\mathfrak{sl}(4,\mathbb{R})$, and the intersection of these vector subspaces with $\mathfrak{sl}(3,\mathbb{R})$ and between them is zero.

Since $\dim(V_1'^3) + \dim(V_2'^3) + \dim({V'}_1) = 7$, similar to the equation (2.17), we have

$$\mathfrak{sl}(4,\mathbb{R}) \cong \mathfrak{sl}(3,\mathbb{R}) \oplus V_1'^3 \oplus V_2'^3 \oplus V'^1 \tag{2.22}$$

as a vector space.

If $X_1, X_2 \in \mathfrak{sl}(4,\mathbb{R})$ then there exist unique elements $A_i \in \mathfrak{sl}(3,\mathbb{R})$, $v_i \in \mathbb{R}^3$, $w_i^t \in \mathbb{R}^{3*}$ and $c_i \in \mathbb{R}$ such that

$$X_i = \begin{bmatrix} -A_i^t + c_i I & v_i \\ w_i^t & -3c_i \end{bmatrix}$$

for $i = 1, 2$. Here the bracket product of X_1 and X_2 is given by

$$[X_1, X_2] = \left[\begin{bmatrix} -A_1^t + c_1 I & v_1 \\ w_1^t & -3c_1 \end{bmatrix}, \begin{bmatrix} -A_2^t + c_2 I & v_2 \\ w_2^t & -3c_2 \end{bmatrix}\right] \tag{2.23}$$
$$= \begin{bmatrix} \begin{matrix}\big((-A_1^t)(-A_2^t) - (-A_2^t)(-A_1^t) + \\ v_1 w_2^t - v_2 w_1^t\big)\end{matrix} & \begin{matrix}\big(-A_1^t v_2 + \\ A_2^t v_1 + 4(c_1 v_2 - c_2 v_1)\big)\end{matrix} \\ -(A_2 w_1)^t + (A_1 w_2)^t + 4(c_2 w_1^t - c_1 w_2^t) & w_1^t v_2 - w_2^t v_1 \end{bmatrix}$$

Lemma 2.14. *$\mathfrak{sl}(4,\mathbb{R})$ is isomorphic to $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \oplus \mathbb{R}$ as $\mathfrak{sl}(3,\mathbb{R})$-module.*

Proof. With the inclusion map ι given by

$$A \mapsto \begin{bmatrix} -A^t & 0 \\ 0 & 0 \end{bmatrix} \quad \forall A \in \mathfrak{sl}(3,\mathbb{R}),$$

we have the decomposition (2.22).

On the other hand, as particular cases for the computation in (2.23), we have

$$\left[\begin{bmatrix} -A^t & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -A_2^t & 0 \\ 0 & 0 \end{bmatrix}\right] = \begin{bmatrix} [-A^t, -A_2^t] & 0 \\ 0 & 0 \end{bmatrix} \tag{2.24}$$
$$\left[\begin{bmatrix} -A^t & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right] = \begin{bmatrix} 0 & -A^t \cdot v \\ 0 & 0 \end{bmatrix}$$
$$\left[\begin{bmatrix} -A^t & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \\ w^t & 0 \end{bmatrix}\right] = \begin{bmatrix} 0 & 0 \\ (A \cdot w)^t & 0 \end{bmatrix}$$
$$\left[\begin{bmatrix} -A^t & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} cI & \\ 0 & -3c \end{bmatrix}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

where $A, A_2 \in \mathfrak{sl}(3,\mathbb{R})$, $v, w \in \mathbb{R}^3$ and $c \in \mathbb{R}$.

Hence, it is clear that the vector subspaces ${V_1'}^3$, ${V_2'}^3$ and ${V'}^1$ are $\mathfrak{sl}(3,\mathbb{R})$-submodules, and

$$\mathfrak{sl}(4,\mathbb{R}) \cong \mathfrak{sl}(3,\mathbb{R}) \oplus {V_1'}^3 \oplus {V_2'}^3 \oplus {V'}^1$$

as $\mathfrak{sl}(3,\mathbb{R})$-modules.

From (2.24) we observe the isomorphisms

$$\begin{aligned} {V_1'}^3 &\cong \mathbb{R}^{3*} \\ {V_2'}^3 &\cong \mathbb{R}^3 \\ {V'}^1 &\cong \mathbb{R} \end{aligned}$$

of $\mathfrak{sl}(3,\mathbb{R})$-modules. The latter shows that

$$\mathfrak{sl}(4,\mathbb{R}) \cong \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \oplus \mathbb{R}$$

as $\mathfrak{sl}(3,\mathbb{R})$-module. □

Corollary 2.15. *With the inclusion map ι given by*

$$A \mapsto \begin{bmatrix} -A^t & 0 \\ 0 & 0 \end{bmatrix} \quad \forall A \in \mathfrak{sl}(3,\mathbb{R}),$$

the decomposition of $\mathfrak{sl}(4,\mathbb{R})$ in irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules is

$$\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \oplus \mathbb{R}$$

Finally, the previous subsections are summarized in the following theorem

Theorem 2.16. *For any non-trivial homomorphism $\mathfrak{sl}(3,\mathbb{R}) \to \mathfrak{so}(3,3)$ we have that $\mathfrak{so}(3,3) \cong \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \oplus \mathbb{R}$ as $\mathfrak{sl}(3,\mathbb{R})$-module, which is its decomposition in irreducible submodules.*

2.2.4 Submodules which are Lie subalgebras

In the previous subsection we showed the decomposition of $\mathfrak{so}(3,3)$ ($\cong \mathfrak{sl}(4,\mathbb{R})$) in irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules. And because we are interested in all $\mathfrak{sl}(3,\mathbb{R})$-- submodules of $\mathfrak{sl}(4,\mathbb{R})$ which are also subalgebras, we will show now which of these subspaces satisfy both properties.

Since the decomposition

$$\mathfrak{sl}(4,\mathbb{R}) \cong \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \oplus \mathbb{R},$$

in Theorem 2.16, in irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules is unique, up to order, we focus only on these $\mathfrak{sl}(3,\mathbb{R})$-submodules and their sums.

Lemma 2.17. *The irreducible $\mathfrak{sl}(3,\mathbb{R})$-submodules of $\mathfrak{sl}(4,\mathbb{R})$ that are Lie subalgebras are $\mathfrak{sl}(3,\mathbb{R})$, $\mathbb{R}^3$, $\mathbb{R}^{3*}$ and $\mathbb{R}$.*

Proof. From equations (2.18), (2.23) and Lemmas 2.11, 2.14, if $A_1, A_2 \in \mathfrak{sl}(3,\mathbb{R})$, $v_1, v_2 \in \mathbb{R}^3$, $w_1^t, w_2^t \in \mathbb{R}^{3*}$ and $c_1, c_2 \in \mathbb{R}$ then

$$\left[\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_2 & 0 \\ 0 & 0 \end{bmatrix}\right] = \begin{bmatrix} [A_1, A_2] & 0 \\ 0 & 0 \end{bmatrix},$$

$$\left[\begin{bmatrix} 0 & v_1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & v_2 \\ 0 & 0 \end{bmatrix}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\left[\begin{bmatrix} 0 & 0 \\ w_1^t & 0 \end{bmatrix}, \begin{bmatrix} w_2^t & 0 \\ 0 & 0 \end{bmatrix}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\left[\begin{bmatrix} c_1 I & 0 \\ 0 & -3c_1 \end{bmatrix}, \begin{bmatrix} c_2 I & 0 \\ 0 & -3c_2 \end{bmatrix}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that the submodules $\mathbb{R}^3$, $\mathbb{R}^{3*}$ and $\mathbb{R}$ are abelian subalgebras of $\mathfrak{sl}(4,\mathbb{R})$ and $[\mathfrak{sl}(3,\mathbb{R}), \mathfrak{sl}(3,\mathbb{R})] = \mathfrak{sl}(3,\mathbb{R})$. Then every irreducible $\mathfrak{sl}(3,\mathbb{R})$-submodule of $\mathfrak{sl}(4,\mathbb{R})$ is a Lie subalgebra. $\square$

Lemma 2.18. *The Lie subalgebras of $\mathfrak{sl}(4,\mathbb{R})$ that are a direct sum of two irreducible submodules are $\mathfrak{sl}(3,\mathbb{R})\oplus\mathbb{R}^3$, $\mathfrak{sl}(3,\mathbb{R})\oplus\mathbb{R}^{3*}$, $\mathfrak{sl}(3,\mathbb{R})\oplus\mathbb{R}$, $\mathbb{R}^3\oplus\mathbb{R}$ and $\mathbb{R}^{3*}\oplus\mathbb{R}$.*

Proof. Here we work with the decomposition of $\mathfrak{sl}(4,\mathbb{R})$ from Theorem 2.16. By Lemma 2.17 we need only compute the bracket product between different submodules.

Let $A\in\mathfrak{sl}(3,\mathbb{R})$, $v\in\mathbb{R}^3$, $w^t\in\mathbb{R}^{3*}$ and $c\in\mathbb{R}$, then by (2.18) we have

$$\left[\begin{bmatrix}A & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix}0 & v\\ 0 & 0\end{bmatrix}\right]=\begin{bmatrix}0 & A(v)\\ 0 & 0\end{bmatrix}\in\mathfrak{sl}(3,\mathbb{R})\oplus\mathbb{R}^3,$$

$$\left[\begin{bmatrix}A & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\ w^t & 0\end{bmatrix}\right]=\begin{bmatrix}0 & 0\\ -(A(w))^t & 0\end{bmatrix}\in\mathfrak{sl}(3,\mathbb{R})\oplus\mathbb{R}^{3*},$$

$$\left[\begin{bmatrix}A & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix}cI & 0\\ 0 & -3c\end{bmatrix}\right]=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\in\mathfrak{sl}(3,\mathbb{R})\oplus\mathbb{R},$$

$$\left[\begin{bmatrix}0 & v\\ 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\ w^t & 0\end{bmatrix}\right]=\begin{bmatrix}v\cdot w^t & 0\\ 0 & -w^t\cdot v\end{bmatrix}\in\mathfrak{sl}(3,\mathbb{R})\oplus\mathbb{R},$$

$$\left[\begin{bmatrix}0 & v\\ 0 & 0\end{bmatrix},\begin{bmatrix}cI & 0\\ 0 & -3c\end{bmatrix}\right]=\begin{bmatrix}0 & -4cv\\ 0 & 0\end{bmatrix}\in\mathbb{R}^3\oplus\mathbb{R},$$

$$\left[\begin{bmatrix}0 & 0\\ w^t & 0\end{bmatrix},\begin{bmatrix}cI & 0\\ 0 & -3c\end{bmatrix}\right]=\begin{bmatrix}0 & 0\\ 4cw^t & 0\end{bmatrix}\in\mathbb{R}^{3*}\oplus\mathbb{R},$$

where $A(v)$ is the product, or evaluation, of the 3×3 matrix A with column vector v and $w^t\in\mathbb{R}^{3*}$ is the transpose of the column vector $w\in\mathbb{R}^3$.

We observe that only the product $[\mathbb{R}^3,\mathbb{R}^{3*}]$ is not contained in $\mathbb{R}^3\oplus\mathbb{R}^{3*}$.

Finally, we have that the direct sum $\mathbb{R}^3\oplus\mathbb{R}^{3*}$ is the only direct sum of irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules that is not a subalgebra. □

Now, we show which modules are Lie subalgebras containing exactly 3 irreducible submodules.

Lemma 2.19. *The only Lie subalgebras of $\mathfrak{sl}(4,\mathbb{R})$ that are direct sum of three irreducible submodules are $\mathfrak{sl}(3,\mathbb{R})\oplus\mathbb{R}^3\oplus\mathbb{R}$ and $\mathfrak{sl}(3,\mathbb{R})\oplus\mathbb{R}^{3*}\oplus\mathbb{R}$.*

Proof. As in Lemma 2.18, we work with the decomposition of $\mathfrak{sl}(4,\mathbb{R})$ in Theorem 2.16.

By the proof of Lemma 2.18, we have that if $\mathbb{R}^3$ and $\mathbb{R}^{3*}$ belong to a subalgebra, then $\mathfrak{sl}(3,\mathbb{R})$ and $\mathbb{R}$ must as well. Then there is not a subalgebra of $\mathfrak{sl}(4,\mathbb{R})$ with exactly 3 irreducible modules containing simultaneously $\mathbb{R}^3$ and $\mathbb{R}^{3*}$.

Then the only subalgebras with exactly 3 irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules are

$$\mathfrak{sl}(3,\mathbb{R})\oplus\mathbb{R}^3\oplus\mathbb{R}$$
$$\mathfrak{sl}(3,\mathbb{R})\oplus\mathbb{R}^{3*}\oplus\mathbb{R}.$$

□

Now we show, in the following Lemma, the subalgebra containing 4 irreducible submodules of $\mathfrak{sl}(4,\mathbb{R})$, whose proof is immediate.

Lemma 2.20. *The only Lie subalgebra of* $\mathfrak{sl}(4,\mathbb{R})$ *that is a direct sum of four irreducible submodules is* $\mathfrak{sl}(4,\mathbb{R})$*.*

Finally, we exhibit all the subalgebras of $\mathfrak{sl}(4,\mathbb{R})$ which are in turn $\mathfrak{sl}(3,\mathbb{R})$-submodules.

Theorem 2.21. *The subalgebras of* $\mathfrak{sl}(4,\mathbb{R})$ *that are, at the same time,* $\mathfrak{sl}(3,\mathbb{R})$*-submodules with the structure of module induced by the injection of* $\mathfrak{sl}(3,\mathbb{R})$ *into* $\mathfrak{sl}(4,\mathbb{R})$ *are those given in Lemmas 2.17-2.20.*

Proof. The previous Lemmas cover all possible cases of direct sums of $\mathfrak{sl}(3,\mathbb{R})$-submodules that are Lie subalgebras. $\square$

Chapter 3

Action on a Pseudo-Riemannian Manifold

3.1 Automorphisms and Rigid Transformations Groups

In this section we recover some definitions and results from [CQ], [Z1] and [Z2].

3.1.1 On the extension of local Killing fields

In the present and following subsections we consider M to be a connected manifold and ω a geometric structure on M. First, we present some definitions and results about automorphisms of geometric structures.

For any geometric structure ω on M and elements x, y in M, we denote as $\mathrm{Aut}^{\mathrm{loc}}(\omega, x, y)$ the group of germs of automorphisms which are defined in a neighborhood of x and take x to y. We set $\mathrm{Aut}^{\mathrm{loc}}(\omega, x) = \mathrm{Aut}^{\mathrm{loc}}(\omega, x, x)$ for all $x \in M$.

We have a similar definition of automorphisms for the prolongations of the geometric structures

Let $\omega : L^{(k)}(M) \to Q$ be a geometric structure of order k and type Q on M. For every $x, y \in M$ the set $\mathrm{Aut}^{k+r}(\omega, x, y)$ of *infinitesimal automorphisms of ω of order $k+r$ taking x to y* consists of $k+r$-jets of diffeomorphisms in M (denoted as $D^{(k+r)}_{x,y}(M)$) which preserve ω up to order r.

We are interested in the *rigid* geometric structures that are present in many of the manifolds that we are studying, for instance, the pseudo-Riemannian manifolds.

Definition 3.1. Let r be a non-negative integer. A geometric structure ω of order k on M is said to be r**-rigid** if, for every $x \in M$, the canonical projection $\pi_{k+r}^{k+r+1} : \mathrm{Aut}^{k+r+1}(\omega, x) \to \mathrm{Aut}^{k+r}(\omega, x)$ is injective.

Assuming that ω is a rigid structure, we have the following Theorem from [Z2], also proved by Gromov.

Theorem 3.2 (Gromov)**.** .

(a) There is a positive integer k, and an open dense set $U \subset M$ such that for $x, y \in U$, every element of $Aut^{(k)}(\omega, x, y)$ extends uniquely to an element of $Aut^{loc}(\omega, x, y)$. In particular, for $m \in U$, every element of $Aut^{(k)}(\omega, m)$ extends uniquely to an element of $Aut^{loc}(\omega, m)$.

(b) If in addition (M, ω) is compact and analytic, we may take $U = M$.

Recall that a **Killing field** for a geometric structure ω on M is a vector field on M whose local flow acts on M by local automorphisms of ω. The space of Killing fields of a geometric structure ω is denoted by $\mathrm{Kill}(M, \omega)$ and $\mathrm{Kill}^{\mathrm{loc}}(M, \omega, m)$, the space of local Killing fields, will denote the space of germs at m of Killing fields in a neighborhood of m. With this notation we present the next result (theorem 2.2, [Z2]).

Theorem 3.3. *Suppose further that M is analytic and simply connected, and that ω is analytic. Then for every $m \in M$, every element of $Kill^{loc}(\omega, m)$ extends uniquely to an element of $Kill(M, \omega)$.*

Recall that a geometric structure on M is said to be of **algebraic type** if Q is a real algebraic variety and the action of $GL^{(k)}(n)$ is algebraic. For such structures we have the following theorem.

Theorem 3.4 (Gromov)**.** *If ω is a rigid analytic structure of algebraic type, and M is simply connected and compact, then $Aut(M, \omega)$ has finitely many components as does the stabilizer of each point in M.*

The proof of the above theorem shows that there exists a closed submanifold $N \subset M$ which is an orbit of the pseudogroup $\mathrm{Aut}^{\mathrm{loc}}(M, \omega)$ and, therefore, each $m \in M$ has an open neighborhood in N contained in an orbit of the local flow generated by a local Killing field near m.

3.1.2 Transitivity of the centralizer

In this section we assume M to be a compact manifold, ω a rigid unimodular geometric structure of algebraic type (for example, ω a pseudo-Riemannian metric) and G a non-compact simple Lie group acting on M preserving ω.

Recall that a structure ω of type V is called **unimodular** if for each $m \in M$, the $GL^{(r)}(n, \mathbb{R})$-orbit in V determined by $\omega(m)$ has stabilizers whose image in $GL(n, \mathbb{R})$ under the natural projection of jets $GL^{(r)}(n, \mathbb{R}) \to GL(n, \mathbb{R})$ is contained in the group of matrices whose determinant is equal to ± 1.

Define $\widetilde{M}$ to be the universal covering space of M.

Let $\mathfrak{k}$ be the Lie algebra of vector fields on $\widetilde{M}$ that preserve $\widetilde{\omega}$ (the lifting of $\widetilde{\omega}$ to $\widetilde{M}$). By the rigidity of ω, $\dim \mathfrak{k} < \infty$. We have a natural embedding $\mathfrak{g} \to \mathfrak{k}$ defined by the G-action on M (and, the $\widetilde{G}$-action on $\widetilde{M}$). Letting $\mathfrak{h} \subset \mathfrak{n} \subset \mathfrak{k}$ be the subalgebras defined by $\mathfrak{n} = \text{normalizer}_{\mathfrak{k}}(\mathfrak{g})$, and $\mathfrak{h} = \text{centralizer}_{\mathfrak{k}}(\mathfrak{g})$, it is clear that $\mathfrak{g} \subset \mathfrak{n}$ and $\mathfrak{n} = \mathfrak{g} \oplus \mathfrak{h}$ by the simplicity of $\mathfrak{g}$.

With this notation we enunciate the following Lemma, (Lemma 4.1, [Z1]) which was proved by Gromov and reproduced here

Lemma 3.5 (Gromov)**.** $\mathfrak{n}$ *is transitive on an open dense conull set in* $\widetilde{M}$*. In fact, the same is true for* $\mathfrak{h}$*.*

Proof. Let ω' be the structure consisting of the pair whose value at $x \in M$ is $(\omega(x), \mathfrak{g}^{(k)}(x))$, where $\mathfrak{g}^{(k)}(x)$ is the Lie algebra of k-jets of vector fields in $\mathfrak{g}$, where k is sufficiently large. Then G preserves this rigid structure.

By rigidity of the G-action and the argument shown in the proof of the Theorem 3.4 in [Z2], only a portion of this argument is reproduced in Theorem 3.4 (in this case $N \subset M$ is an open conull set because of the G-action and the ergodicity), the 'infinitesimal automorphism' of $(\omega, \mathfrak{g}^{(k)})$ is transitive on an open conull set, and hence so are the local automorphisms. The corresponding vector fields are simply elements of $\mathfrak{n}$.

We have $\mathfrak{n} = \mathfrak{g} \oplus \mathfrak{h}$, and from Corollary 4.3 in [Z2], the $\mathfrak{h}$-orbit actually contains the $\mathfrak{g}$-orbit a.e. Since $\mathfrak{n}$ is transitive on an open conull set, $\mathfrak{h}$ must therefore be so as well. $\square$

3.2 Notations and Definitions

Let G be a connected non-compact simple Lie group with Lie algebra $\mathfrak{g}$ and M a connected finite volume pseudo-Riemannian manifold where G acts isometrically on M with a dense orbit.

Because every isometric G-action on a manifold M with a dense orbit is locally free [Sza], the orbits define a foliation that we will denote with $\mathcal{O}$. Then, for every $x \in M$ there exist $T_x\mathcal{O}^{\perp}$, a vector subspace of T_xM, such that $T_xM = T_x\mathcal{O} \oplus T_x\mathcal{O}^{\perp}$.

We are interested in the case $G = SL(3, \mathbb{R})$ and $8 < \dim(M) \leq 14$. So, $1 < \dim(T_x\mathcal{O}^{\perp}) \leq 6$ for all $x \in M$, in particular.

The geometric structure of pseudo-Riemannian metric on the manifold M will be denoted by σ.

On the other hand, the bundle $T\mathcal{O}$ tangent to the foliation $\mathcal{O}$ is a trivial vector bundle isomorphic to $M \times \mathfrak{g}$, under the isomorphism $M \times \mathfrak{g} \to T\mathcal{O}$ given by $(x, X) \mapsto X^*_x$. This also defines an isomorphism fiber $T_x\mathcal{O}$ with $\mathfrak{g}$. For the rest of these letters for X in the Lie algebra of a group acting on a manifold, we denote by X^* the vector field on the manifold whose one-parameter group of diffeomorphism is given by $(exp(tX))_t$ through the action on the manifold.

Recalling the definition of Killing fields, that will be used in this chapter, $\text{Kill}_0^{\text{loc}}(M, \sigma, x)$ will denote the subspace of $\text{Kill}^{\text{loc}}(M, \sigma, x)$ consisting in vector fields that vanish on x. Unless otherwise indicated, in the rest of this work we will omit the symbol that denotes the structure of pseudo-Riemannian metric. In particular, $\mathrm{Kill}(M) := \mathrm{Kill}(M, \sigma)$.

3.3 Existence of the centralizer

An immediate consequence of the Jacobi identity is the next lemma, proved in [OQ].

Lemma 3.6. *Let N be a pseudo-Riemannian manifold and $x \in N$. Then, the map $\lambda_x : Kill_0(N, x) \to \mathfrak{so}(T_x N)$ given by $\lambda_x(Z)(v) = [Z, V]_x$, where V is any vector field such that $V_x = v$, is a well defined homomorphism of Lie algebras.*

Where $\mathfrak{so}(W)$ denotes the Lie algebra of linear maps on W that are skew-symmetric with respect to a non-degenerate symmetric bilinear form on the vector space W.

Next, we bring to this work the following result that appears in [Q] and [OQ].

Proposition 3.7. *Let G be a connected non-compact simple Lie group acting isometrically and with a dense orbit on a connected finite volume pseudo-Riemannian manifold M. Consider the $\widetilde{G}$-action on $\widetilde{M}$, lifted from the G-action on M. Assume that M and the G-action on M are both analytic. Then, there exists a conull subset $S \subset \widetilde{M}$ such that for every $x \in S$ the following properties are satisfied:*

1. *there is a homomorphism $\rho_x : \mathfrak{g} = \mathfrak{sl}(3, \mathbb{R}) \to Kill(\widetilde{M})$ which is an isomorphism onto its image $\rho_x(\mathfrak{g}) = \mathfrak{g}(x)$.*

2. *$\mathfrak{g}(x) \subset Kill_0(\widetilde{M}, x)$, i.e. every element of $\mathfrak{g}(x)$ vanishes at x.*

3. *For every $X, Y \in \mathfrak{g}$ we have*
$$[\rho_x(X), Y^*] = [X, Y]^* = -[X^*, Y^*].$$
In particular, the elements in $\mathfrak{g}(x)$ and their corresponding local flows preserve both $\mathcal{O}$ and $T\mathcal{O}^\perp$.

4. *The homomorphism of Lie algebras $\lambda_x \circ \rho_x : \mathfrak{g} \to \mathfrak{so}(T_x\widetilde{M})$ induces a $\mathfrak{g}$-module structure on $T_x\widetilde{M}$ for which subspaces $T_x\mathcal{O}$ and $T_x\mathcal{O}^\perp$ are $\mathfrak{g}$-submodules.*

Remark 3.8. The last incise of Proposition 3.7 shows that $T_x\mathcal{O}^\perp$ is, through the map $\lambda_x \circ \rho_x$, a $\mathfrak{g}$-module for every $x \in S$. Hence, we have two possible options for the subset S:

(a) $T_x\mathcal{O}^\perp$ is a trivial $\mathfrak{g}$-module for almost every $x \in S$, or

(b) There exists a subset, $A \subseteq S$, of positive measure such that $T_x\mathcal{O}^\perp$ is a non-trivial $\mathfrak{g}$-module for all $x \in A$.

Here, we can consider the $\mathfrak{g}$-valued 1-form ω on $\widetilde{M}$ which is defined, at every $x \in \widetilde{M}$, by the composition of the projection $T_x\widetilde{M} \to T_x\mathcal{O}$ and the isomorphism of the latter with $\mathfrak{g}$. We can also consider the $\mathfrak{g}$-valued 2-form given by $\Omega = d\omega|_{\wedge^2 T\mathcal{O}^\perp}$.

If part (a) of Remark 3.8 is satisfied then $T\mathcal{O}^\perp$ is integrable. This is a consequence of the following result, whose proof can be found in [Q].

Lemma 3.9. *Let G, M and S be as in Proposition 3.7. If we assume that the G-orbits are non-degenerate, then:*

(1) For every $x \in S$, the maps $\omega_x : T_x\widetilde{M} \to \mathfrak{g}$ and $\Omega_x : \wedge^2 T_x\mathcal{O}^\perp \to \mathfrak{g}$ are both homomorphism of $\mathfrak{g}$-modules, for the $\mathfrak{g}$-module structures from Proposition 3.7.

(2) The normal bundle $T\mathcal{O}^\perp$ is integrable if and only if $\Omega = 0$.

Proof. Let $X \in \mathfrak{g}$ and $x \in S$ be elements fixed but arbitrarily given.

For a Z vector field over $\widetilde{M}$, let $Z^\top$, $Z^\perp$ be its $T\mathcal{O}$ and $T\mathcal{O}^\perp$ components, respectively. Since $\rho_x(X)$ is a Killing field preserving $\mathcal{O}$ and $T\mathcal{O}^\perp$ it follows that:

$$[\rho_x(X), Z]^\top = [\rho_x(X), Z^\top],$$
$$[\rho_x(X), Z]^\perp = [\rho_x(X), Z^\perp].$$

Denote with $\alpha : T_x\mathcal{O} \to \mathfrak{g}$ the inverse map of $X \mapsto X_x^*$. Then we have:

$$\begin{aligned}
\omega_x(X \cdot Z_x) &= \omega_x([\rho_x(X), Z]_x) \\
&= \alpha([\rho_x(X), Z^\top]_x) \\
&= \alpha([\rho_x(X), \omega(Z)^*]_x) \\
&= \alpha([X, \omega(Z)]_x^*) \\
&= [X, \omega_x(Z)] \\
&= X \cdot \omega_x(Z_x),
\end{aligned}$$

thus showing that ω_x is a homomorphism of $\mathfrak{g}$-modules. Here for the second and third identities, the definition of ω is used, and in the fourth identity the formula from Proposition 3.7(3); the rest follows from the definition of the $\mathfrak{g}$-module structures involved.

Next, observe that for every pair of sections Z_1, Z_2 of $T\mathcal{O}^\perp$ we have:

$$\begin{aligned}
\Omega(Z_1 \wedge Z_2) &= Z_1(\omega(Z_2)) - Z_2(\omega(Z_1)) - \omega([Z_1, Z_2]) \\
&= -\omega([Z_1, Z_2]),
\end{aligned}$$

which implies (2).

Now let $u, v \in T_x\mathcal{O}^\perp$ be given and choose U, V sections of $T\mathcal{O}^\perp$ extending them, respectively. Hence, using that ω is a homomorphism of $\mathfrak{g}$-modules, the

Jacobi identity and the above expression relating Ω and ω, we obtain:

$$\begin{aligned}
\Omega_x(X \cdot (u \wedge v)) &= \Omega((X \cdot u) \wedge v) + \Omega(u \wedge (X \cdot v)) \\
&= \Omega([\rho_x(X), U] \wedge V) + \Omega_x(U \wedge [\rho_x(X), V]) \\
&= -\omega_x([[\rho_x(X), U], V]) - \omega_x([U, [\rho_x(X), V]]) \\
&= -\omega_x([\rho_x(X), [U, V]]) \\
&= -\omega_x(X \cdot [U, V]_x) \\
&= -[X, \omega_x([U, V])] \\
&= [X, \Omega_x(U \wedge V)] \\
&= X \cdot \Omega_x(u \wedge v),
\end{aligned}$$

thus showing that Ω_x is a homomorphism of $\mathfrak{g}$-modules. Note that both

$$[\rho_x(X), U] \quad \text{and} \quad [\rho_x(X), V]$$

are sections of $T\mathcal{O}^\perp$. □

Remark 3.10. Let G, M and S be as in Proposition 3.7, suppose also that the G-orbits on M are non-degenerate. From the preceding result and the analyticity of the elements we have two possible cases: *1)* $\Omega \equiv 0$, *and then* $T\mathcal{O}^\perp$ *is integrable* or *2)* $\Omega_x \neq 0$ *for almost all* $x \in \widetilde{M}$.

We will assume in the rest of this work that the G-orbits are non-degenerate with respect to the pseudo-Riemannian metric. So, the $\widetilde{G}$-orbits on $\widetilde{M}$ are non-degenerate as well and we have a direct sum decomposition $T\widetilde{M} = T\mathcal{O} \oplus T\mathcal{O}^\perp$. The non-degeneracy of the orbits is ensured for manifolds with low dimension with respect to the dimension of the Lie group acting on this, (see [Q]).

Lemma 3.11. *Let G be a connected non-compact simple Lie group acting isometrically and with a dense orbit on a connected finite volume pseudo-Riemannian manifold M. If $\dim(M) < 2\dim(G)$, then the bundles $T\mathcal{O}$ and $T\mathcal{O}^\perp$ have fibers that are non-degenerate with respect to the metric on M.*

For the G-action as in Proposition 3.7, we consider $\widetilde{M}$ endowed with the $\widetilde{G}$-action obtained by lifting the G-action on M. Let us denote by $\mathcal{H}$ the Lie subalgebra of $\mathrm{Kill}(\widetilde{M})$ consisting of the fields that centralize the $\widetilde{G}$-action on $\widetilde{M}$. Now, we embed the Lie algebra $\mathfrak{g}$ into $\mathcal{H}$. This result allows us to apply representation theory to the study of $\mathcal{H}$. The next Lemma is proved in [OQ].

Lemma 3.12. *Let S be as in the Proposition 3.7. Then, for every $x \in S$ and for ρ_x given as in the Proposition 3.7, the map $\widehat{\rho}_x : \mathcal{H} \to \mathrm{Kill}(\widetilde{M})$ given by:*

$$\widehat{\rho}_x(X) = \rho_x(X) + X^*,$$

is an injective homomorphism of Lie algebras whose image $\mathcal{G}(x)$ lies in $\mathcal{H}$. In particular, $\widehat{\rho}_x$ induces on $\mathcal{H}$ a $\mathfrak{g}$-module structure such that $\mathcal{G}(x)$ is a submodule isomorphic to $\mathfrak{g}$.

Proof. First, observe that by the identity in Proposition 3.7(3) it is easy to see that the image of $\widehat{\rho}_x$ lies in $\mathcal{H}$.

To prove that $\widehat{\rho}_x$ is a homomorphism of Lie algebras we apply Proposition 3.7(3) as follows for $X, Y \in \mathfrak{g}$:

$$\begin{aligned}
[\widehat{\rho}_x(X), \widehat{\rho}_x(Y)] &= [\rho_x(X) + X^*, \rho_x(Y) + Y^*] \\
&= [\rho_x(X), \rho_x(Y)] + [\rho_x(X), Y^*] + [X^*, \rho_x(Y)] + [X^*, Y^*] \\
&= \rho_x([X,Y]) + [X,Y]^* + [X,Y]^* + [X^*, Y^*] \\
&= \rho_x([X,Y]) + [X,Y]^* \\
&= \widehat{\rho}_x([X,Y])
\end{aligned}$$

For the injectivity of $\widehat{\rho}_x$ we observe that $\widehat{\rho}_x(X) = 0$ implies $X^*_x = (\rho_x(X) + X^*)_x = 0$, which in turns yields $X = 0$ because the G-action is locally free. The last claim is now clear. □

Now we relate the structure of $\mathfrak{g}$-module of $\mathcal{H}$ to that of $T_x\widetilde{M}$.

Lemma 3.13. *Let S be as in Proposition 3.7. Consider $T_x\widetilde{M}$ and $\mathcal{H}$ endowed with the $\mathfrak{g}$-module structures given by Proposition 3.7(4) and Lemma 3.12, respectively. Then, for almost every $x \in S$, the evaluation map:*

$$ev_x : \mathcal{H} \to T_x\widetilde{M}, \quad Z \mapsto Z_x,$$

is a homomorphism of $\mathfrak{g}$-modules that satisfies $ev_x(\mathcal{G}(x)) = T_x\mathcal{O}$. Furthermore, for almost every $x \in S$ we have $ev_x(\mathcal{H}) = T_x\widetilde{M}$.

Proof. For every $x \in S$, if we let $Z \in \mathcal{H}$ and $X \in \mathfrak{g}$ be given, then:

$$\begin{aligned}
ev_x(X \cdot Z) &= [\widehat{\rho}_x(X), Z]_x = [\rho_x(X) + X^*, Z]_x \\
&= [\rho_x(X), Z]_x = X \cdot Z_x = X \cdot ev_x(Z)
\end{aligned}$$

where we have used Lemma 3.6 and the definition of the $\mathfrak{g}$-module structures involved, thus proving the first part. The second part is proved in Lemma 3.5 using the transitivity of $\mathfrak{h}$ on an open dense conull set in M. □

Now, we show some results which relate isometries with Killing fields for complete manifolds. The next result follows from the rigidity of pseudo-Riemannian metrics and appears as Lemma 1.9 from [OQ].

Lemma 3.14. *Let N be an analytic pseudo-Riemannian manifold. Then, every Killing vector field of N, either local or global, is analytic. In particular, the isometry group $Iso(N)$ acts analytically on N.*

From [ONe], on a complete pseudo-Riemannian manifold every global Killing vector field is complete, then we have the following result, that we can find as Proposition 33 on Chapter 9.

Lemma 3.15. *Let N be a complete pseudo-Riemannian manifold and suppose that the action of its isometry group $Iso(N)$ is considered on the left. If $\mathfrak{Iso}(N)$ denotes the Lie algebra of $Iso(N)$, then the map:*

$$\mathfrak{Iso}(N) \to \mathit{Kill}(N), \quad X \to X^*,$$

is an anti-isomorphism of Lie algebras. In particular, $[X, Y]^* = -[X^*, Y^*]$ for every $X, Y \in \mathfrak{Iso}(N)$.

And [OQ] shows that on a complete manifold every Lie algebra of Killing fields can be obtained from an isometric right action.

Lemma 3.16. *Let N be a complete pseudo-Riemannian manifold and H a simply connected Lie group with Lie algebra $\mathfrak{h}$. If $\psi : \mathfrak{h} \to \mathit{Kill}(N)$ is a homomorphism of Lie algebras, then there exists an isometric right H-action $N \times H \to N$ such that $\psi(X) = X^*$, for every $X \in \mathfrak{h}$. Furthermore, if N is analytic, then the H-action is analytic as well.*

Proof. Consider the map $\alpha : \mathfrak{Iso}(N) \to \mathrm{Kill}(N)$ given by $\alpha(Y) = -Y^*$ which is an isomoprhism of Lie algebras by Lemma 3.15. Let $\Psi : H \to \mathrm{Iso}(N)$ be the homomorphism of Lie groups induced by the homomorphism $\alpha^{-1} \circ \psi : \mathfrak{h} \to \mathfrak{Iso}(N)$. This yields a smooth isometric right H-action given by:

$$N \times H \to N, \quad (n, h) \mapsto nh = \Psi(h^{-1})(n).$$

For $X \in \mathfrak{h}$ there is $Y \in \mathfrak{Iso}(N)$ such that $\psi(X) = -\alpha(Y) = Y^*$. Hence for the right H-action one computes X^*at every $p \in N$ as follows:

$$\begin{aligned} X_p^* &= \frac{d}{dt}\Big|_{t=0} p \exp(tX) = \frac{d}{dt}\Big|_{t=0} \Psi(\exp(-tX))(p) \\ &= \frac{d}{dt}\Big|_{t=0} \exp(-t(\alpha^{-1} \circ \psi)(X))p = \frac{d}{dt}\Big|_{t=0} \exp(tY)p \\ &= Y^* = \psi(X)_p, \end{aligned}$$

which proves the first part of the lemma. Note that the first and second to last identities use the definition of Z^* for the right H-action and the left Iso(N)-action, respectively. Finally the last part of our statement follows from the last claim of lemma 3.14. ◻

3.4 Properties of the centralizer

From now on we assume $G = \widetilde{SL}(3, \mathbb{R})$ and $8 < \dim(M) \leq 14$ with the same hypotheses as in Proposition 3.7. We also assume that part (b) of *Remark* 3.8 is satisfied, this is, there exists $A \subseteq S$, of positive measure, such that $T_x\mathcal{O}^\perp$ is a non-trivial $\mathfrak{g}$-module for all $x \in A$.

With these hypotheses we have the following result, similar to Lemma 2.1 in [OQ].

Lemma 3.17. *Let S be as in Proposition 3.7. Consider $T_x\mathcal{O}^\perp$ endowed with the $\mathfrak{sl}(3,\mathbb{R})$-module structure given by Proposition 3.7(4). Then, if $x \in A$ the $\mathfrak{sl}(3,\mathbb{R})$-module $T_x\mathcal{O}^\perp$ is isomorphic to $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ and $\dim(M)=14$. In particular, $\mathfrak{so}(T_x\mathcal{O}^\perp)$ is isomorphic to $\mathfrak{so}(3,3)$ as a Lie algebra and as a $\mathfrak{sl}(3,\mathbb{R})$-module.*

Proof. Let us choose an arbitrary but fixed element $x \in A$.

In the first place, by Lemma 3.11, we have that $T_x\mathcal{O}^\perp$ is a non-degenerate fiber with respect to the metric on $\widetilde{M}$, preserved by the action of $\mathfrak{sl}(3,\mathbb{R})$.

Then, from Proposition 3.7(4), $T_x\mathcal{O}^\perp$ is a vector space with the next properties:

1. $1 < \dim(T_x\mathcal{O}^\perp) \leq 6$, and

2. is a $\mathfrak{sl}(3,\mathbb{R})$-module carrying an invariant inner product.

On the other hand, by our choice of the element x, $T_x\mathcal{O}^\perp$ is a non-trivial $\mathfrak{g}$-module.

Recall, from Lemma 2.10, that $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ is the only non-trivial (up to isomorphism) $\mathfrak{sl}(3,\mathbb{R})$-module with dimension ≤ 6 that has a non-degenerate symmetric bilinear form invariant under the structure of $\mathfrak{sl}(3,\mathbb{R})$-module. Then

$$T_x\mathcal{O}^\perp \simeq \mathbb{R}^3 \oplus \mathbb{R}^{3*}.$$

Thus, we have that $\dim(M) = 14$.

Finally, we observe that the representation of $\mathfrak{sl}(3,\mathbb{R})$ on $T_x\mathcal{O}^\perp$ defines a non-trivial homomorphism $\mathfrak{so}(T_x\mathcal{O}^\perp) \to \mathfrak{so}(3,3)$. Since $\mathfrak{so}(3,3)$ is simple, the latter is injective and so it is an isomorphism. □

Corollary 3.18. *If we assume $T\mathcal{O}^\perp$ is non-integrable in Lemma 3.17 then we can chose the subset A (in Lemma 3.17) such that has total measure.*

Proof. If $T\mathcal{O}^\perp$ is non-integrable then Lemma 3.9 implies that the 2-form Ω is non-zero. This 2-form is analytic and thus it vanishes on a proper analytic subset of $\widetilde{M}$, which is necessarily null. Hence, $\Omega_x \neq 0$ for almost every $x \in S$.

Here, we choose and fix $x \in S$ such that $\Omega_x \neq 0$.

By our choice of $x \in S$, Lemma 3.9(1) implies that the map $\Omega_x : \wedge^2 T_x\mathcal{O}^\perp \to \mathfrak{sl}(3,\mathbb{R})$ is a homomorphism of $\mathfrak{sl}(3,\mathbb{R})$-modules that it is non-trivial. Hence, $T_x\mathcal{O}^\perp$ is a non-trivial $\mathfrak{sl}(3,\mathbb{R})$-module.

Thus, we choose A of the following manner:

$$A := \left\{x \in \widetilde{M} \middle| \Omega_x \neq 0\right\} \cap S.$$

Hence, $A \subseteq S$, and since $\mu(\widetilde{M} \setminus \{x \in \widetilde{M} | \Omega_x \neq 0\}) = \mu(\widetilde{M} \setminus S) = 0$ then $\mu(\widetilde{M} \setminus A) = 0$. □

Corollary 3.19. *There exists a subset of positive measure, S, in $\widetilde{M}$, such that every element of S satisfies Proposition 3.7 and Lemmas 3.13 and 3.17.*

Proof. Let $\widetilde{S} \subset \widetilde{M}$ be the conull subset as in Proposition 3.7.

On the other hand, we have the existence of a subset of positive measure A in $\widetilde{S}$ such that Lemma 3.17 is satisfied for every $x \in A$.

Let S'' be the conull subset in $\widetilde{M}$ of elements which satisfy Lemma 3.13. Define $S := A \cap S''$. Then, since

$$\begin{aligned} A &= ((\widetilde{M} \setminus S'') \cup S'') \cap A \\ &= ((\widetilde{M} \setminus S'') \cap A) \cup (A \cap S'') \end{aligned}$$

and

$$\mu(\widetilde{M} \setminus S'') = 0,$$

we have that $\mu(S) = \mu(A)$.

We note that all the elements in the set S satisfy the properties in Lemmas 3.13 and 3.17. □

Remark 3.20. Since $S \subset \widetilde{S}$ and $\mu(S) = \mu(A)$, we replace from now on, the conull subset of Proposition 3.7 by its subset S as in the previous corollary.

The previous results allow us to obtain a decomposition of the centralizer $\mathcal{H}$ of the $\widetilde{SL}(3,\mathbb{R})$-action into submodules related to the geometric structure on $\widetilde{M}$, similar to Lemma 2.2 in [OQ].

Lemma 3.21. *Let S be as in Corollary 3.19. Then, for every $x \in S$ and for the $\mathfrak{sl}(3,\mathbb{R})$-module structure on $\mathcal{H}$ from Lemma 3.12 there is a decomposition into $\mathfrak{sl}(3,\mathbb{R})$-submodules $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{W}(x)$, satisfying:*

1. *$\mathcal{G}(x) = \widehat{\rho}_x(\mathfrak{sl}(3,\mathbb{R}))$ is a Lie subalgebra of $\mathcal{H}$ isomorphic to $\mathfrak{sl}(3,\mathbb{R})$ and $ev_x(\mathcal{G}(x)) = T_x\mathcal{O}$.*
2. *$\mathcal{H}_0(x) = \ker(ev_x)$ is a Lie subalgebra and a $\mathfrak{sl}(3,\mathbb{R})$-submodule of $\mathcal{H}$, isomorphic to a subset of $\mathfrak{so}(T_x\mathcal{O}^\perp)$.*
3. *$ev_x(\mathcal{W}(x)) = T_x\mathcal{O}^\perp$ and $\mathcal{W}(x)$ is isomorphic to $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ as $\mathfrak{sl}(3,\mathbb{R})$-module.*

In particular, the evaluation map ev_x defines an isomorphism of $\mathfrak{sl}(3,\mathbb{R})$-modules $\mathcal{G}(x) \oplus \mathcal{W}(x) \to T_x\widetilde{M} = T_x\mathcal{O} \oplus T_x\mathcal{O}^\perp$ preserving the summands in that order. The structure of $\mathfrak{so}(T_x\mathcal{O}^\perp)$ as $\mathfrak{sl}(3,\mathbb{R})$-module is obtained via the homomorphism, of Lie algebras, $\lambda_x^\perp$ (Lemma 3.6). Here, $\lambda_x^\perp$ is a homomorphism of $\mathfrak{sl}(3,\mathbb{R})$-modules which is injective in $\mathcal{H}_0(x)$.

Proof. From Corollary 3.19, every element $x \in S$ satisfies Lemma 3.13 and Lemma 3.17. Let us choose and fix one such point $x \in S$. By Lemma 3.12, we conclude that $\mathcal{G}(x) = \widehat{\rho}_x(\mathfrak{sl}(3,\mathbb{R}))$ is a Lie subalgebra isomorphic to $\mathfrak{sl}(3,\mathbb{R})$.

Define $\mathcal{H}_0(x) = \ker(ev_x)$. By Lemma 3.13, it follows that $\mathcal{H}_0(x)$ is an $\mathfrak{sl}(3,\mathbb{R})$-submodule of $\mathcal{H}$. And since $\mathcal{H}_0(x) = \mathcal{H} \cap \mathrm{Kill}_0(\widetilde{M}, x)$, it follows that it is a Lie subalgebra as well.

On the other hand, the elements of $\mathcal{G}(x)$ are of the form $\rho_x(x) + X^*$, with $X \in \mathfrak{sl}(3,\mathbb{R})$, where ρ_x is the Lie algebra homomorphism from Proposition 3.7. Hence, for any such element we have $ev_x(\rho_x(X) + X^*) = X^*_x$; particularly, the condition $ev_x(\rho_x(X)+X^*) = 0$ implies $X = 0$. In other words, $\mathcal{G}(x)\cap\mathcal{H}_0(x) = 0$. Hence, there exists an $\mathfrak{sl}(3,\mathbb{R})$-submodule $\mathcal{W}'(x)$ complementary to $\mathcal{G}(x)\oplus\mathcal{H}_0(x)$ in $\mathcal{H}$. In particular, we have an isomorphism from $\mathcal{G}(x)\oplus\mathcal{W}'(x)$ onto $T_x\widetilde{M}$. So, if we choose $\mathcal{W}(x)$ as the inverse image of $T_x\mathcal{O}$ under this isomorphism, then we have the desired decomposition into $\mathfrak{sl}(3,\mathbb{R})$-submodules.

Now, we see some properties of $\mathcal{H}_0(x)$.

Let $\mathrm{Kill}_0(\widetilde{M}, x, \mathcal{O})$ be the Lie algebra of Killing vector fields on $\widetilde{M}$ which preserve the foliation $\mathcal{O}$ and vanish at x. Every Killing vector field in $\mathrm{Kill}_0(\widetilde{M}, x, \mathcal{O})$ leaves the normal bundle $T_x\mathcal{O}^\perp$ invariant, and the map λ_x from Lemma 3.6 induces a homomorphism of Lie algebras:

$$\lambda_x^\perp : \mathrm{Kill}_0(\widetilde{M}, x, \mathcal{O}) \to \mathfrak{so}(T_x\mathcal{O}^\perp), \quad X \mapsto \lambda_x(X)\,|_{T_x\mathcal{O}^\perp}$$

Observe that $\rho_x(\mathfrak{sl}(3,\mathbb{R}))$ and $\mathcal{H}_0(x)$ lie inside of $\mathrm{Kill}_0(\widetilde{M}, x, \mathcal{O})$.

Claim 1: $\lambda_x^\perp$ is injective when restricted to $\mathfrak{sl}(3,\mathbb{R})(x)$. By Proposition 3.7(4), the vector space $T_x\mathcal{O}^\perp$ has a structure of $\mathfrak{sl}(3,\mathbb{R})$-module induced from the map $\lambda_x^\perp \circ \rho_x$. From the choice of the element $x \in S$ and Lemma 3.13 such module structure is non-trivial. Hence, the map $\lambda_x^\perp \circ \rho_x : \mathfrak{sl}(3,\mathbb{R}) \to \mathfrak{so}(T_x\mathcal{O}^\perp)$ is also non-trivial. As $\mathfrak{sl}(3,\mathbb{R})$ is a simple Lie algebra, the function $\lambda_x^\perp$ restricted to $\mathfrak{sl}(3,\mathbb{R})$ is injective.

Claim 2: $\lambda_x^\perp$ restricted to $\mathcal{H}_0(x)$ is injective. Recall that a Killing vector field is completely determined by its 1-jet at x; this follows from the fact that pseudo-Riemannian metric structures are 1-rigid (see [CQ]). If $Z \in \mathcal{H}_0(x)$ is given, then $Z_x = 0$ and so it is determined by the values of $[Z, V]_x$ for V a vector field on a neighborhood of x. As Z lies in the centralizer of the $SL(3,\mathbb{R})$-action then $[Z, X^*]_x = 0$ for all $X \in \mathfrak{sl}(3,\mathbb{R})$, so $[Z, V]_x = 0$ when $V_x \in T_x\mathcal{O}$. Then, if $[Z, V]_x = 0$ when $V_x \in T_x\mathcal{O}^\perp$ implies $Z = 0$. This shows that $\lambda_x^\perp$ is injective when it is restricted to $\mathcal{H}_0(x)$.

On the other hand, if $X \in \mathfrak{sl}(3,\mathbb{R})$ and $Y \in \mathcal{H}_0(x)$ with the structure of $\mathfrak{sl}(3,\mathbb{R})$-modules of each one, we have that:

$$\begin{aligned}
\lambda_x^\perp(X \cdot Y) &= \lambda_x^\perp([\widehat{\rho}_x(X), Y]) = \lambda_x^\perp([\rho_x(X) + X^*, Y]) \\
&= \lambda_x^\perp([\rho_x(X), Y]) = [\lambda_x^\perp(\rho_x(X)), \lambda_x^\perp(Y)] \\
&= X \cdot \lambda_x^\perp(Y),
\end{aligned}$$

This shows that the map $\lambda_x^\perp$ restricted to $\mathcal{H}_0(x)$ is a homomorphism of $\mathfrak{sl}(3,\mathbb{R})$-modules. □

3.5 Structure with a non-degenerate inner product

We now look at the structure of the inner product in $T_x\mathcal{O}^\perp$, for every element $x \in S$.

Remark 3.22. Let x be an element of S, by Lemma 3.17, we have that

$$\mathfrak{so}(T_x\mathcal{O}^\perp) \cong \mathfrak{so}(3,3),$$

as $\mathfrak{sl}(3,\mathbb{R})$-module. Then $\mathfrak{so}(T_x\mathcal{O}^\perp)$ has a decomposition as a direct sum of irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules. On the other hand, Lemma 3.21 shows that, this decomposition can be given by an isomorphism of $\mathfrak{sl}(3,\mathbb{R})$ onto a subalgebra contained in $\mathfrak{so}(T_x\mathcal{O}^\perp)$.

Recall the following properties of representations of simple Lie algebras:

1. The number of summands in a decomposition of a direct sum of irreducible submodules is independent of its decomposition.

2. Every decomposition into submodules, that are the sum of all the submodules of the same class of isomorphism, is unique except for order.

Then, by the simplicity of $\mathfrak{sl}(3,\mathbb{R})$ and its property of decomposition of modules in direct sum of irreducible submodules, we have from Theorem 2.16 that

$$\mathfrak{so}(T_x\mathcal{O}^\perp) \cong \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \oplus \mathbb{R}.$$

With this result and the properties of $\mathcal{H}_0(x)$ in Lemma 3.21, Theorem 2.21 shows that $\mathcal{H}_0(x)$ is isomorphic (as Lie algebra and $\mathfrak{sl}(3,\mathbb{R})$-module, such property will henceforth denoted as $\simeq$) to one of the next Lie subalgebras of $\mathfrak{sl}(4,\mathbb{R}) \simeq \mathfrak{so}(3,3)$:

$$\begin{array}{rl}
\multicolumn{2}{c}{0} \\
\mathbb{R} & \mathbb{R}^3 \\
\mathbb{R}^{3*} & \mathfrak{sl}(3,\mathbb{R}) \\
\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R} & \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \\
\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^{3*} & \mathbb{R}^3 \oplus \mathbb{R} \\
\mathbb{R}^{3*} \oplus \mathbb{R} & \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R} \\
\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^{3*} \oplus \mathbb{R} & \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \oplus \mathbb{R}
\end{array}$$

Chapter 4

Possibilities of $\mathcal{H}_0(x)$

In the previous chapter we showed all the possible values that $\mathcal{H}_0(x)$ can take. In this chapter we analyze the implications of all these possible cases of $\mathcal{H}_0(x)$. Here, we use the notation of Lemma 3.21.

Remark 4.1. From Lemma 3.21, we have that for every $x \in S$ the map ev_x exhibits an isomorphism of $\mathcal{W}(x)$ onto $T_x\mathcal{O}^\perp$ as $\mathfrak{sl}(3,\mathbb{R})$-modules. Then, from Lemma 3.17, we can choose subspaces $\mathcal{V}(x), \mathcal{V}^*(x) \subset \mathcal{W}(x)$ such that

$$\mathcal{W}(x) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$$

and $\lambda_x^\perp(\mathfrak{sl}(3,\mathbb{R})(x))$ acts on $\mathcal{V}(x)$ and $\mathcal{V}^*(x)$ as $\mathfrak{sl}(3,\mathbb{R})$ acts on $\mathbb{R}^3$ and $\mathbb{R}^{3*}$ as $\mathfrak{sl}(3,\mathbb{R})$-modules, respectively.

By the structure of $\mathcal{H}$ as $\mathfrak{sl}(3,\mathbb{R})$-module and the properties of the map $\lambda_x^\perp$ in Lemma 3.21, we have some properties of the Lie algebra $\mathcal{H}$.

$$\begin{aligned} [\mathcal{G}(x), \mathcal{H}_0(x)] &\subseteq \mathcal{H}_0(x), & (4.1)\\ [\mathcal{G}(x), \mathcal{W}(x)] &= \mathcal{W}(x), & (4.2)\\ [\mathcal{H}_0(x), \mathcal{W}(x)] &\subseteq \mathcal{H}_0(x) \oplus \mathcal{W}(x), & (4.3)\\ [\mathcal{H}_0(x), \mathcal{H}_0(x)] &\subseteq \mathcal{H}_0(x). & (4.4) \end{aligned}$$

In particular, when $\mathcal{H}_0(x)$ is isomorphic to specific algebras, some equalities of the previous equations are satisfied.

Lemma 4.2. *Let S be as in Corollary 3.19. As in Lemma 3.21, if $\mathcal{H}_0(x)$ is isomorphic either to $\mathbb{R}$, $\mathfrak{sl}(3,\mathbb{R})$, $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}$ or $\mathfrak{so}(3,3)$ for some $x \in S$ then*

$$[\mathcal{H}_0(x), \mathcal{W}(x)] = \mathcal{W}(x).$$

Proof. Let x be an element in S such that:

$\mathcal{H}_0(x) \simeq \mathbb{R}$

First, by properties of the Lie bracket we have that

$$[\mathcal{H}_0(x), \mathcal{W}(x)] \cong \mathbb{R} \otimes (\mathbb{R}^3 \oplus \mathbb{R}^{3*})$$

as $\mathfrak{sl}(3,\mathbb{R})$-module. On the other hand, since

$$\mathbb{R} \otimes (\mathbb{R}^3 \oplus \mathbb{R}^{3*}) \cong \mathbb{R}^3 \oplus \mathbb{R}^{3*}$$

as $\mathfrak{sl}(3,\mathbb{R})$-module, hence by equation (4.3) we have

$$[\mathcal{H}_0(x), \mathcal{W}(x)] \cong \mathcal{W}(x).$$

$\mathcal{H}_0(x) \simeq \mathfrak{sl}(3,\mathbb{R})$

Here, by (4.1) and (4.4), $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ is a semisimple Lie algebra isomorphic to $\mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{sl}(3,\mathbb{R})$. Since

$$\dim(\mathcal{H}_0(x)) = \dim(\mathfrak{sl}(3,\mathbb{R})(x)) = 8, \quad \dim(\mathfrak{so}(T_x\mathcal{O}^\perp)) = 15$$

and $\lambda^\perp_{x_0}$ is injective when is restricted to $\mathfrak{sl}(3,\mathbb{R})(x)$ and $\mathcal{H}_0(x)$ then

$$\lambda^\perp_x(\mathfrak{sl}(3,\mathbb{R})(x)) = \lambda^\perp_x(\mathcal{H}_0(x)).$$

From this and the way that we obtain decomposition of $\mathcal{H}$ in a direct sum of irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules (via the homomorphism $\lambda^\perp_x$ and the simple Lie algebra $\mathcal{G}(x)$) we have that

$$[\mathcal{H}_0(x), \mathcal{W}(x)] = \mathcal{W}(x).$$

$\mathcal{H}_0(x) \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}$

With similar arguments of the two previous cases, we have that

$$[\mathcal{H}_0(x), \mathcal{W}(x)] = \mathcal{W}(x).$$

$\mathcal{H}_0(x) \simeq \mathfrak{so}(3,3)$

Recall, the structure of $\mathfrak{sl}(3,\mathbb{R})$-module of $\mathcal{H}$ is given by the subalgebra $\mathcal{G}(x)$. In a similar case, to how is obtained this structure, we can bring to $\mathcal{H}$ a structure of $\mathfrak{so}(3,3)$-module given by the subalgebra $\mathcal{H}_0(x)$, which in this case is a simple Lie algebra.

In Lemma 3.13 we have proved that the map ev_x is a homomorphism of $\mathfrak{sl}(3,\mathbb{R})$-modules. Note that in its proof we only have required

$$\rho_x(\mathfrak{g}) \subset \mathrm{Kill}_0(\widetilde{M}, x, \mathcal{O}).$$

With similar arguments to this proof, the result is the same if we replace the structure of $\mathfrak{sl}(3,\mathbb{R})$-module of $\mathcal{H}$ by its structure of $\mathfrak{so}(3,3)$-module and the algebra $\mathcal{G}(x)$ by the simple Lie algebra $\mathcal{H}_0(x)$.

As consequence of these changes we have the next result

$$[\mathcal{H}_0(x), \mathcal{W}(x)] = \mathcal{W}(x).$$

□

To study all possible cases of $\mathcal{H}_0(x)$ we have divided these, attending the number of irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules that contains.

4.1 $\mathcal{H}_0(x)$ vanishes

Lemma 4.3 ($\mathcal{H}_0(x) = 0$)**.** *Let S be as in Corollary 3.19. With the notation of Lemma 3.21, if $\mathcal{H}_0(x) = 0$ for some $x \in S$, one of the following occurs:*

(1) The radical of $\mathcal{H}$ is $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

(2) $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a simple Lie algebra isomorphic to $\mathfrak{g}_{2(2)}$.

Proof. Let us choose an arbitrary but fixed element $x \in S$ such that $\mathcal{H}_0(x) = 0$, as in Lemma 3.21. In this case, $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

Since $\mathcal{G}(x)$ is a simple Lie algebra, isomorphic to $\mathfrak{sl}(3, \mathbb{R})$, we can choose $\mathfrak{s}$ a Levi factor of $\mathcal{H}$ that contains $\mathcal{G}(x)$. As the structure of $\mathfrak{sl}(3, \mathbb{R})$-module of $\mathcal{H}$ is obtained by the subalgebra $\mathcal{G}(x)$ and $\mathcal{G}(x) \subseteq \mathfrak{s}$, then $\mathfrak{s}$ is a $\mathfrak{sl}(3, \mathbb{R})$-module.

Let W be a $\mathfrak{sl}(3, \mathbb{R})$-submodule of $\mathcal{H}$ such that $\mathfrak{s} = \mathcal{G}(x) \oplus W$.

Since $\mathrm{rad}(\mathcal{H})$ is an ideal of $\mathcal{H}$ this induces the decomposition of $\mathcal{H}$ as a direct sum of $\mathfrak{sl}(3, \mathbb{R})$-modules:

$$\mathcal{H} = \mathcal{G}(x) \oplus W \oplus rad(\mathcal{H})$$

that we can compare with the decomposition of irreducible $\mathfrak{sl}(3, \mathbb{R})$-modules from Lemma 3.21

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

From the properties of representations of Lie algebras and the decomposition $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$, one of the following must occur:

(a) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}^*(x)$.

(b) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}(x)$.

(c) $\mathfrak{s} = \mathcal{G}(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

(d) $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is semisimple.

Next, we analyze all of these possible cases.

Suppose the case (a) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x) \quad \text{and} \quad rad(\mathcal{H}) = \mathcal{V}^*(x).$$

As $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x)$ is a semisimple Lie algebra then $\mathfrak{s}$ is a finite direct product, $\mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$, of simple ideals. Since, every ideal is invariant by $\mathcal{G}(x)$ then these ideals are $\mathfrak{sl}(3, \mathbb{R})$-modules. By properties of representation of $\mathfrak{sl}(3, \mathbb{R})$ and the decomposition of $\mathfrak{s}$ in a direct sum of irreducible $\mathfrak{sl}(3, \mathbb{R})$-modules we have that $k \leq 2$.

If $k = 2$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2$. From (4.2) we have that $[\mathcal{G}(x), \mathcal{V}(x)] \subseteq \mathcal{V}(x)$. On the other hand, since $\mathcal{V}(x)$ is isomorphic to $\mathbb{R}^3$ as $\mathfrak{sl}(3, \mathbb{R})$-module, the bracket operation in $\mathcal{V}(x)$ gives us an isomorphism into the $\mathfrak{sl}(3, \mathbb{R})$-module $\mathbb{R}^{3*}$. This shows that $[\mathcal{V}(x), \mathcal{V}(x)] = 0$. So, we have proved that $\mathcal{V}(x)$ is an ideal of $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x)$. Without loss of generality we assume $\mathfrak{h}_2 = \mathcal{V}(x)$, but this is not possible because then the simple ideal $\mathfrak{h}_2$ would be an abelian ideal. Therefore $k = 1$.

If $k = 1$, $\mathfrak{s}$ is a simple Lie algebra. So, $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x)$ is a real simple Lie algebra with dimension 11. Then, $\mathfrak{s}^{\mathbb{C}}$ is a complex simple Lie algebra with complex dimension 11, that by [H, p. 516] cannot be possible. We have proved that case (a) cannot happen.

Assume case (b) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}^*(x) \quad \text{and} \quad rad(\mathcal{H}) = \mathcal{V}(x).$$

The argument is the same that in case (a), so this case is not possible.

Suppose case (c) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \quad \text{and} \quad rad(\mathcal{H}) = \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

Since $\mathcal{V}(x) \simeq \mathbb{R}^3$ and $\mathcal{V}^*(x) \simeq \mathbb{R}^{3*}$ as $\mathfrak{sl}(3,\mathbb{R})$-modules then the Lie bracket operation on the Lie algebra rad($\mathcal{H}$) produces the next:

$$\begin{aligned} \mathcal{H} \supset [\mathcal{V}(x), \mathcal{V}(x)]\,, &\quad \text{isomorphic to a submodule of} \quad \mathbb{R}^{3*} \\ \mathcal{H} \supset [\mathcal{V}(x), \mathcal{V}^*(x)]\,, &\quad \text{isomorphic to a submodule of} \quad \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R} \\ \mathcal{H} \supset [\mathcal{V}^*(x), \mathcal{V}^*(x)]\,, &\quad \text{isomorphic to a submodule of} \quad \mathbb{R}^3 \end{aligned}$$

So $[\mathcal{V}(x), \mathcal{V}(x)] \subseteq \mathcal{V}^*(x)$, $[\mathcal{V}(x), \mathcal{V}^*(x)] = 0$ and $[\mathcal{V}^*(x), \mathcal{V}^*(x)] \subseteq \mathcal{V}(x)$. Using the solvability of rad($\mathcal{H}$) we have that $[\mathcal{V}(x), \mathcal{V}(x)] = 0$ or $[\mathcal{V}^*(x), \mathcal{V}^*(x)] = 0$.

If case (d) is satisfied:

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \quad \text{is a simple Lie algebra.}$$

Using the same argument as in case (a), $\mathcal{H}$ is direct product of a finite number of simple ideals $\mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$ where every ideal is a $\mathfrak{sl}(3,\mathbb{R})$-module and $k \leq 3$.

If $k = 3$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3$, we can assume, reindexing if necessary, that $\mathfrak{h}_3 = \mathcal{V}^*(x)$ and $\mathfrak{h}_1 \times \mathfrak{h}_2 = \mathcal{G}(x) \oplus \mathcal{V}(x)$. Then $[\mathcal{V}^*(x), \mathcal{V}^*(x)] \subseteq \mathfrak{h}_3$, but as in case (c) we have that $[\mathcal{V}^*(x), \mathcal{V}^*(x)] = 0$, so $\mathfrak{h}_3$ is an abelian Lie algebra and this is not possible. Therefore $k \leq 2$.

If $k = 2$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2$, after decomposing $\mathfrak{h}_1$ and $\mathfrak{h}_2$ as the direct sum of $\mathfrak{sl}(3,\mathbb{R})$-modules, and reindexing if necessary, we can assume that $\mathfrak{h}_1$ is an irreducible $\mathfrak{sl}(3,\mathbb{R})$-module and $\mathfrak{h}_2 = V_1 \oplus V_2$, where V_1 and V_2 are irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules. We can also assume that $\mathcal{V}^*(x) \subset \mathfrak{h}_2$ and $\mathcal{V}(x) \oplus \mathcal{V}^*(x) = \mathfrak{h}_2$, because as in the previous case, it cannot happen that $\mathfrak{h}_1 = \mathcal{V}(x)$. Then $\mathcal{G}(x) \subseteq \mathfrak{h}_1$ and $[\mathcal{G}(x), \mathcal{W}(x)] = [\mathfrak{h}_1, \mathfrak{h}_2] = 0$ that contradicts the equation (4.2).

If $k = 1$, $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a real simple Lie algebra of dimension 14. Then, $\mathcal{H}$ is the realification of a complex simple Lie algebra of dimension 7 or its complexification, $\mathcal{H}^{\mathbb{C}}$, is a complex simple Lie algebra. But, by [H, p. 516], there is not a complex simple Lie algebra of dimension 7. So, $\mathcal{H}^{\mathbb{C}}$ is a complex simple Lie algebra with $\dim_{\mathbb{C}}(\mathcal{H}^{\mathbb{C}}) = 14$. Then, $\mathcal{H}^{\mathbb{C}} \cong \mathfrak{g}_2$. From here, $\mathcal{H}$ is isomorphic to a real form of $\mathfrak{g}_2$.

On the other hand, we recall that $\mathcal{H}$ contains a Lie subalgebra isomorphic to $\mathfrak{sl}(3,\mathbb{R})$, that is simple and non-compact. Then, $\mathcal{H}$ is non-compact. Otherwise, exercise $4(ii)$ in the page 152 of [H], would imply that $\mathfrak{sl}(3,\mathbb{R})$ is compact, which is a contradiction.

Since, by [H, p. 518], there is only a non-compact real form of $\mathfrak{g}_2$, namely $\mathfrak{g}_{2(2)}$. Then

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \simeq \mathfrak{g}_{2(2)}.$$

□

4.2 Subalgebras with one submodule

4.2.1 $\mathcal{H}_0(x) \simeq \mathbb{R}$

Lemma 4.4 ($\mathcal{H}_0(x) \simeq \mathbb{R}$)**.** *Let S be as in Corollary 3.19. With the notation of Lemma 3.21, if $\mathcal{H}_0(x)$ is isomorphic to $\mathbb{R}$ as $\mathfrak{sl}(3,\mathbb{R})$-module for some $x \in S$, one of the following occurs:*

(1) The radical of $\mathcal{H}$ is $\mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ where $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a Lie subalgebra.

(2) $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a simple Lie algebra isomorphic to $\mathfrak{sl}(4,\mathbb{R})$.

Proof. Let us choose an arbitrary but fixed element $x \in S$ such that $\mathcal{H}_0(x) \simeq \mathbb{R}$, as in Lemma 3.21.

We recall that $\mathcal{G}(x)$ is a simple Lie algebra isomorphic to $\mathfrak{sl}(3,\mathbb{R})$. We can choose $\mathfrak{s}$ a Levi factor of $\mathcal{H}$ that contains $\mathcal{G}(x)$. With the $\mathfrak{sl}(3,\mathbb{R})$-module structure of $\mathcal{H}$, defined by the subalgebra $\mathcal{G}(x)$, and as $\mathcal{G}(x) \subset \mathfrak{s}$ then $\mathfrak{s}$ is a $\mathfrak{sl}(3,\mathbb{R})$-submodule of $\mathcal{H}$.

Let W be a $\mathfrak{sl}(3,\mathbb{R})$-submodule of $\mathcal{H}$ such that $\mathfrak{s} = \mathcal{G}(x) \oplus W$.

Since $\mathrm{rad}(\mathcal{H})$ is an ideal, this induces the next decomposition of $\mathcal{H}$ as a direct sum of $\mathfrak{sl}(3,\mathbb{R})$-modules:

$$\mathcal{H} = \mathcal{G}(x) \oplus W \oplus rad(\mathcal{H})$$

that we compare with the decomposition of irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$$

from Lemma 3.21.

By the properties of representations of Lie algebras and the decomposition $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$, one of the following must occur:

(a) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}^*(x)$.

(b) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}(x)$.

(c) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{H}_0(x)$.

(d) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

(e) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}^*(x)$.

(f) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}(x)$.

(g) $\mathfrak{s} = \mathcal{G}(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

(h) $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is semisimple.

Next, we analyze all possible cases.

Suppose that case (a) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \quad \text{and} \quad rad(\mathcal{H}) = \mathcal{V}^*(x).$$

Because $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x)$ is semisimple, then $\mathfrak{s}$ is a direct product of a finite number of simple ideals, $\mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$. Since, every ideal is invariant by $\mathcal{G}(x)$, we have that these ideals are $\mathfrak{sl}(3, \mathbb{R})$-modules.

On the other hand, since $\mathcal{V}(x) \simeq \mathbb{R}^3$ and $[\mathcal{V}(x), \mathcal{V}(x)] \subseteq \mathfrak{s}$. If

$$[\mathcal{V}(x), \mathcal{V}(x)] \neq 0$$

then

$$[\mathcal{V}(x), \mathcal{V}(x)] \simeq \mathbb{R}^{3*}.$$

From here, the projection of $[\mathcal{V}(x), \mathcal{V}(x)]$ in $\mathcal{G}(x)$, $\mathcal{H}_0(x)$ and $\mathcal{V}(x)$ is 0. This implies that $[\mathcal{V}(x), \mathcal{V}(x)] = 0$. Then, $\mathcal{V}(x)$ is an abelian ideal of $\mathfrak{s}$, which is a contradiction. So, case (a) cannot be possible.

Case (b) is not possible and the proof is similar to (a).

Now suppose case (c) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \quad \text{and} \quad rad(\mathcal{H}) = \mathcal{H}_0(x).$$

From Lemma 3.21 we have that

$$0 \neq [\mathcal{H}_0(x), \mathcal{V}(x) \oplus \mathcal{V}^*(x)].$$

Moreover, there is a non-zero element in $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ that belongs to this product. On the other hand, since $\mathrm{rad}(\mathcal{H})$ is an ideal of $\mathcal{H}$, this element is in $\mathrm{rad}(\mathcal{H})$. So, this case cannot occur.

Suppose case (d) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \quad \text{and} \quad rad(\mathcal{H}) = \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

From (4.1) and (4.4), we have that $\mathcal{H}_0(x)$ is an abelian ideal of $\mathfrak{s}$. Which is a contradiction. Then case (d) is not possible.

Suppose, the case (e) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x) \quad \text{and} \quad rad(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}^*(x).$$

As in the first argument in case (a), $\mathcal{V}(x)$ is an abelian ideal of $\mathfrak{s}$. Then, this case is not possible.

Case (f),

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}^*(x) \quad \text{and} \quad rad(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}(x),$$

cannot happen and the proof is similar to that of (e).

Now, we suppose (g) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \quad \text{and} \quad rad(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

From Lemma 4.2 we have that

$$[\mathcal{H}_0(x), \mathcal{V}(x) \oplus \mathcal{V}^*(x)] = \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

On the other hand, since

$$[\mathcal{V}(x) \oplus \mathcal{V}^*(x), \mathcal{V}(x) \oplus \mathcal{V}^*(x)] \subseteq rad(\mathcal{H}),$$

and let $\pi_0 : rad(\mathcal{H}) \to \mathcal{H}_0(x)$ be the projection map on the first component of rad($\mathcal{H}$) then

$$\pi_0([\mathcal{V}(x) \oplus \mathcal{V}^*(x), \mathcal{V}(x) \oplus \mathcal{V}^*(x)]) = 0.$$

Otherwise rad($\mathcal{H}$) will not be solvable. In conclusion, $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a Lie subalgebra of rad($\mathcal{H}$).

If case (h) is satisfied:

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \quad \text{is semisimple.}$$

$\mathcal{H}$ is a finite direct product $\mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$ of simple ideals that are also $\mathfrak{sl}(3, \mathbb{R})$-modules with $k \leq 4$.

If $k = 4$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3 \times \mathfrak{h}_4$. Then $\mathcal{H}_0(x)$ is a simple ideal of $\mathcal{H}$. But, since $\mathcal{H}_0(x) \simeq \mathbb{R}$ is abelian, this is a contradiction. Therefore $k = 4$ cannot be possible and $k \leq 3$.

If $k = 3$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3$. Suppose, reindexing if necessary, that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are irreducible $\mathfrak{sl}(3, \mathbb{R})$-modules and $\mathfrak{h}_3 = V_1 \oplus V_2$, where V_1 and V_2 are also irreducibles. We can assume $\mathcal{H}_0(x) \subset \mathfrak{h}_3$. On the other hand, since $\mathcal{H}_0(x) \simeq \mathbb{R}$ then

$$[\mathcal{H}_0(x), \mathcal{V}(x) \oplus \mathcal{V}^*(x)] = \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

So, $\mathcal{V}(x) \oplus \mathcal{V}^*(x) \subset \mathfrak{h}_3$. But this implies that $\mathfrak{h}_1$ or $\mathfrak{h}_2$ is equal to 0. Therefore, this case is not possible and $k \leq 2$.

If $k = 2$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2$. As in case $k = 3$, we suppose

$$\mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \subset \mathfrak{h}_2.$$

On the other hand, since

$$0 \neq [\mathcal{G}(x), \mathcal{V}(x) \oplus \mathcal{V}^*(x)]$$

then $\mathcal{G}(x) \subset \mathfrak{h}_2$ and $\mathfrak{h}_1 = 0$, that is a contradiction. So, this case cannot be possible and $k = 1$.

If $k = 1$, $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a real simple Lie algebra of dimension 15. Therefore, $(\mathcal{H})^{\mathbb{C}}$ is a complex simple Lie algebra with $\dim_{\mathbb{C}}((\mathcal{H})^{\mathbb{C}}) = 15$. Then, by [H, p. 516], $(\mathcal{H})^{\mathbb{C}} \simeq \mathfrak{sl}(4, \mathbb{C})$. So, $\mathcal{H}$ is isomorphic to a non-compact real form of $(\mathcal{H})^{\mathbb{C}}$.

From [H, Table V], the only non-compact real forms of $\mathfrak{sl}(4, \mathbb{C})$ are $\mathfrak{su}(1, 3)$, $\mathfrak{su}(2, 2)$, $\mathfrak{su}^*(4)$ and $\mathfrak{sl}(4, \mathbb{R})$.

Then, $\mathcal{H}$ is isomorphic to one of the previous Lie algebras. We recall that $\mathcal{H}$ contains a simple Lie subalgebra isomorphic to $\mathfrak{sl}(3, \mathbb{R})$. From here,

$$2 = \operatorname{rank}_{\mathbb{R}}(\mathfrak{sl}(3, \mathbb{R})) \leq \operatorname{rank}_{\mathbb{R}}(\mathcal{H}).$$

By [H, Table V], we have

$$\operatorname{rank}_{\mathbb{R}}(\mathfrak{su}(1, 3)) = \operatorname{rank}_{\mathbb{R}}(\mathfrak{su}^*(4)) = 1.$$

Then, $\mathcal{H}$ cannot be isomorphic to either $\mathfrak{su}(1, 3)$ or $\mathfrak{su}^*(4)$. On the other hand, page 519 of [H] shows $\mathfrak{su}(2, 2) \simeq \mathfrak{so}(4, 2)$. So, if $\mathcal{H} \simeq \mathfrak{su}(2, 2)$ then $\mathfrak{sl}(3, \mathbb{R})$ is isomorphic to a Lie subalgebra of $\mathfrak{so}(4, 2)$. In this case $\mathfrak{sl}(3, \mathbb{R})$ would have a non-trivial representation on a 6-dimensional vector space that preserves a non-degenerate symmetric bilinear form of signature $(4, 2)$. By Lemma 2.10 this cannot be possible. Thus, $\mathcal{H} \simeq \mathfrak{su}(2, 2)$ is not possible. Then

$$\mathcal{H} \simeq \mathfrak{sl}(4, \mathbb{R}).$$

□

4.2.2 $\mathcal{H}_0(x) \simeq \mathbb{R}^3$ or $\mathcal{H}_0(x) \simeq \mathbb{R}^{3*}$

Lemma 4.5 ($\mathcal{H}_0(x) \simeq \mathbb{R}^3$ or $\mathcal{H}_0(x) \simeq \mathbb{R}^{3*}$)**.** *Let S be as in Corollary 3.19. With the notation of Lemma 3.21, if $\mathcal{H}_0(x)$ is isomorphic to $\mathbb{R}^3$ ($\mathbb{R}^{3*}$) as $\mathfrak{sl}(3, \mathbb{R})$-module for some $x \in S$ then the radical of $\mathcal{H}$ is $\mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ where $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a Lie subalgebra.*

Proof. In the first place we will take $\mathcal{H}_0(x) \simeq \mathbb{R}^3$. The proof when $\mathcal{H}_0(x) \simeq \mathbb{R}^{3*}$ is similar.

Let us choose an arbitrary but fixed point $x \in S$ such that $\mathcal{H}_0(x) \simeq \mathbb{R}^3$, as in Lemma 3.21.

We recall that $\mathcal{G}(x)$ is a simple Lie algebra isomorphic to $\mathfrak{sl}(3, \mathbb{R})$. We can choose $\mathfrak{s}$ a Levi factor of $\mathcal{H}$ that contains $\mathcal{G}(x)$. As the structure of $\mathfrak{sl}(3, \mathbb{R})$-module of $\mathcal{H}$ is obtained by the subalgebra $\mathcal{G}(x)$ and $\mathcal{G}(x) \subseteq \mathfrak{s}$, then $\mathfrak{s}$ is a $\mathfrak{sl}(3, \mathbb{R})$-module.

Let W be a $\mathfrak{sl}(3, \mathbb{R})$-submodule of $\mathcal{H}$ such that $\mathfrak{s} = \mathcal{G}(x) \oplus W$.

Since $\operatorname{rad}(\mathcal{H})$ is an ideal, this induce the next decomposition of $\mathcal{H}$ as a direct sum of $\mathfrak{sl}(3, \mathbb{R})$-modules:

$$\mathcal{H} = \mathcal{G}(x) \oplus W \oplus rad(\mathcal{H})$$

that we can compare with the decomposition of irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules from Lemma 3.21

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

Then, by the properties of representation of Lie algebras, one of the following must occur:

(a) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}^*(x)$.

(b) $\mathfrak{s} = \mathcal{G}(x) \oplus V_1 \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = V_2$.

(c) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}(x)$.

(d) $\mathfrak{s} = \mathcal{G}(x) \oplus V_1$ and $\mathrm{rad}(\mathcal{H}) = V_2 \oplus \mathcal{V}^*(x)$.

(e) $\mathfrak{s} = \mathcal{G}(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

(f) $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is semisimple.

where V_1 and V_2 are vector spaces isomorphic to $\mathbb{R}^3$ as $\mathfrak{sl}(3,\mathbb{R})$-modules such that

$$V_1 \oplus V_2 = \mathcal{H}_0(x) \oplus \mathcal{V}(x).$$

In the first place, from the equation (4.3) and since $\mathcal{H}_0(x) \simeq \mathbb{R}^3$ as $\mathfrak{sl}(3,\mathbb{R})$-module, we have that

$$[\mathcal{H}_0(x), \mathcal{V}(x)] = \mathcal{V}^*(x) \quad \text{and} \quad [\mathcal{H}_0(x), \mathcal{V}^*(x)] = 0. \tag{4.5}$$

Next, we analyze all possible cases.

Suppose case (a) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \quad \text{and} \quad rad(\mathcal{H}) = \mathcal{V}^*(x).$$

Since $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x)$ is a semisimple Lie algebra then it is a direct product of a finite number of simple ideals, $\mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$. Since every ideal is invariant under the product by $\mathcal{G}(x)$ we have that the ideals possess a structure of $\mathfrak{sl}(3,\mathbb{R})$-module. Then, by properties of decomposition of $\mathfrak{s}$ as a direct sum of irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules we have $k \leq 3$.

Since $\mathcal{V}(x) \simeq \mathbb{R}^3$, we have that if $[\mathcal{V}(x), \mathcal{V}(x)] \neq 0$ then $[\mathcal{V}(x), \mathcal{V}(x)] \simeq \mathbb{R}^{3*}$ as $\mathfrak{sl}(3,\mathbb{R})$-module. On the other hand, because $\mathcal{G}(x) \simeq \mathfrak{sl}(3,\mathbb{R})$ and $\mathcal{H}_0(x) \simeq \mathcal{V}(x) \simeq \mathbb{R}^3$, this implies that $[\mathcal{V}(x), \mathcal{V}(x)] = 0$. So, $\mathcal{V}(x)$ is an abelian ideal of $\mathfrak{s}$. Which is a contradiction. So, this case cannot be possible.

Now, we suppose case (b) holds:

$$\mathfrak{s} = \mathcal{G}(x) \oplus V_1 \oplus \mathcal{V}^*(x) \quad \text{and} \quad rad(\mathcal{H}) = V_2.$$

Let $\pi^2_{\mathcal{V}} : V_2 \to \mathcal{V}(x)$ be the projection map from the vector space V_2 to $\mathcal{V}(x)$. We note that this map is a homomorphism of $\mathfrak{sl}(3,\mathbb{R})$-modules. If $\pi^2_{\mathcal{V}} = 0$ then $V_2 = \mathcal{H}_0(x)$. In this case by equation (4.5) we have that

$$\mathcal{V}^*(x) = [\mathcal{H}_0(x), \mathcal{V}(x)] = [\mathrm{rad}(\mathcal{H}), \mathcal{V}(x)] \subseteq \mathrm{rad}(\mathcal{H}).$$

That is a contradiction. Then $\pi^2_{\mathcal{V}} \neq 0$ this is, by the irreducibility of $\mathcal{V}(x)$, $\pi^2_{\mathcal{V}}(V_2) = \mathcal{V}(x)$.

In the same way, if $\pi^2_0 : V_2 \to \mathcal{H}_0(x)$ is the projection map from V_2 to $\mathcal{H}_0(x)$ then $\pi^2_0 \neq 0$. Otherwise, $V_2 = \mathcal{V}(x)$. Then, this result and the equation (4.5) will imply

$$\mathcal{V}^*(x) = [\mathcal{H}_0(x), \mathcal{V}(x)] = [\mathcal{H}_0(x), \mathrm{rad}(\mathcal{H})] \subseteq \mathrm{rad}(\mathcal{H}).$$

That is a contradiction. So, $\pi^2_0 \neq 0$, this is, $\pi^2_0(V_2) = \mathcal{H}_0(x)$.

Since $\mathrm{rad}(\mathcal{H})$ is an ideal of $\mathcal{H}$ this implies that $[\mathcal{V}^*(x), \mathrm{rad}(\mathcal{H})] \subset \mathrm{rad}(\mathcal{H})$. In our case, since $\mathrm{rad}(\mathcal{H}) = V_2$ is isomorphic to $\mathbb{R}^3$ then $[\mathcal{V}^*(x), V_2] = 0$.

Let $v \in V_2$ be a non-zero element then $v = \pi^2_0(v) + \pi^2_{\mathcal{V}}(v)$. From here, if $v^* \in \mathcal{V}^*(x)$ then

$$0 = [v^*, v] = [v^*, \pi^2_0(v) + \pi^2_{\mathcal{V}}(v)] = [v^*, \pi^2_0(v)] + [v^*, \pi^2_{\mathcal{V}}(v)].$$

But, from (4.5), $[v^*, \pi^2_0(v)] = 0$ then

$$[v^*, \pi^2_{\mathcal{V}}(v)] = 0.$$

Since this is true for every $v \in V_2$ and $v^* \in \mathcal{V}^*(x)$ we have proved that

$$[\mathcal{V}^*(x), \mathcal{V}(x)] = 0,$$

because $\pi^2_{\mathcal{V}} = \mathcal{V}(x)$. Moreover, since $V_1 \subset \mathcal{H}_0(x) \oplus \mathcal{V}(x)$ then we have that

$$[\mathcal{V}^*(x), V_1] = 0. \tag{4.6}$$

On the other hand, since $\mathfrak{s} = \mathcal{G}(x) \oplus V_1 \oplus \mathcal{V}^*(x)$ is a semisimple Lie algebra then it is a direct product of a finite number of simple ideals, $\mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$. Since every ideal is invariant under the product by $\mathcal{G}(x)$ we have that the ideals possess a structure of $\mathfrak{sl}(3, \mathbb{R})$-module. Then, by properties of decomposition of $\mathfrak{s}$ as a direct sum of irreducible $\mathfrak{sl}(3, \mathbb{R})$-modules we have $k \leq 3$.

If $k = 3$, $\mathfrak{s} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3$, we can assume, reindexing if necessary, that $\mathfrak{h}_3 = \mathcal{V}^*(x)$ and $\mathfrak{h}_1 \times \mathfrak{h}_2 = \mathcal{G}(x) \oplus V_1$. Then $[\mathcal{V}^*(x), \mathcal{V}^*(x)] \subseteq \mathfrak{h}_3$, but $\mathcal{V}^*(x) \simeq \mathbb{R}^{3*}$ as $\mathfrak{sl}(3, \mathbb{R})$-module then $[\mathcal{V}^*(x), \mathcal{V}^*(x)] = 0$, so $\mathfrak{h}_3$ is an abelian Lie algebra and this is not possible. Therefore $k \leq 2$.

If $k = 2$, $\mathfrak{s} = \mathfrak{h}_1 \times \mathfrak{h}_2$, after decomposing $\mathfrak{h}_1$ and $\mathfrak{h}_2$ as the direct sum of $\mathfrak{sl}(3, \mathbb{R})$-modules, and reindexing if necessary, we can assume that $\mathfrak{h}_1$ is an irreducible $\mathfrak{sl}(3, \mathbb{R})$-module and $\mathfrak{h}_2 = W_1 \oplus W_2$, where W_1 and W_2 are irreducible $\mathfrak{sl}(3, \mathbb{R})$-modules. We can also assume that $\mathcal{V}^*(x) \subset \mathfrak{h}_2$. Then, from (4.2) since $[\mathcal{G}(x), \mathcal{W}(x)] = \mathcal{W}(x)$ we have $\mathcal{G}(x) \subset \mathfrak{h}_2$ and $\mathfrak{h}_2 = \mathcal{G}(x) \oplus \mathcal{V}^*(x)$. That, as case (a) in Lemma 4.4, cannot happen. So, this is not possible. From here, $k = 1$.

If $k = 1$, $\mathfrak{s} = \mathcal{G}(x) \oplus V_1 \oplus \mathcal{V}^*(x)$ is a real simple Lie algebra of dimension 14. Since, $V_1 \simeq \mathbb{R}^3$ and $\mathcal{V}^*(x) \simeq \mathbb{R}^{3*}$ as $\mathfrak{sl}(3, \mathbb{R})$-module then

$$[V_1, V_1] \subseteq \mathcal{V}^*(x) \quad \text{and} \quad [\mathcal{V}^*(x), \mathcal{V}^*(x)] \subseteq V_1.$$

from this and (4.6) we have that

$$[V_1 \oplus \mathcal{V}^*(x), V_1 \oplus \mathcal{V}^*(x)] \subseteq V_1 \oplus \mathcal{V}^*(x).$$

Then, using (4.1) and (4.2) we have that $V_1 \oplus \mathcal{V}^*(x)$ is an ideal of $\mathfrak{s}$. That is a contradiction. So, this case cannot happen.

Suppose case (c) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}^*(x) \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}(x).$$

As in case (b), $\mathfrak{s}$ is a finite direct product of simple ideals, $\mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$, with $k \leq 2$. Also, these ideals have a structure of $\mathfrak{sl}(3,\mathbb{R})$-modules, obtained by the product with $\mathcal{G}(x)$.

From (4.1) and (4.4) we have, in this case, that $\mathcal{V}^*(x)$ is an ideal of $\mathfrak{s}$. On the other hand, since $\mathcal{V}^*(x) \simeq \mathbb{R}^{3*}$ then $\mathcal{V}^*(x)$ is abelian, which is a contradiction. So, this case cannot be possible.

Assume case (d) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus V_1 \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = V_2 \oplus \mathcal{V}^*(x).$$

This case cannot happen. The proof of this is similar to case (c) since $V_1 \simeq \mathbb{R}^3$ as $\mathfrak{sl}(3,\mathbb{R})$-module.

Suppose case (e) holds:

$$\mathfrak{s} = \mathcal{G}(x) \quad \text{and} \quad rad(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

From Lemma 3.21, equation (4.5) and the fact that $\mathcal{H}_0(x) \simeq \mathbb{R}^3$ as $\mathfrak{sl}(3,\mathbb{R})$-module then:

$$\begin{aligned}
[\mathcal{H}_0(x), \mathcal{H}_0(x)] &= 0 \\
[\mathcal{H}_0(x), \mathcal{V}(x)] &= \mathcal{V}^*(x) \\
[\mathcal{H}_0(x), \mathcal{V}^*(x)] &= 0.
\end{aligned}$$

On the other hand, we recall that $\mathcal{V}(x) \cong \mathbb{R}^3$ and $\mathcal{V}^*(x) \cong \mathbb{R}^{3*}$ as $\mathfrak{sl}(3,\mathbb{R})$-modules. Then:

$$[\mathcal{V}(x), \mathcal{V}(x)] \subseteq \mathcal{V}^*(x) \tag{4.7}$$

$$[\mathcal{V}(x), \mathcal{V}^*(x)] = 0 \tag{4.8}$$

$$[\mathcal{V}^*(x), \mathcal{V}^*(x)] \subseteq \mathcal{H}_0(x) \oplus \mathcal{V}(x). \tag{4.9}$$

Let π_0, π_1 be the projection maps from $\mathrm{rad}(\mathcal{H})$ on $\mathcal{H}_0(x)$ and $\mathcal{V}(x)$, respectively. We note that these maps are homomorphism of $\mathfrak{sl}(3,\mathbb{R})$-modules.

In the first place, suppose

$$\pi_0([\mathcal{V}^*(x), \mathcal{V}^*(x)]) = \mathcal{H}_0(x). \tag{4.10}$$

By the Jacobi identity, if $v_2, w_2 \in \mathcal{V}^*(x)$ and $v_1 \in \mathcal{V}(x)$, we have

$$\begin{aligned}
0 &= [[v_2, w_2], v_1] + [[w_2, v_1], v_2] + [[v_1, v_2], w_2] \\
&= [[v_2, w_2], v_1] + [0, v_2] + [0, w_2] \\
&= [[v_2, w_2], v_1].
\end{aligned}$$

This shows that $[[v_2, w_2], v_1] = 0$ for every $v_2, w_2 \in \mathcal{V}^*(x)$ and $v_1 \in \mathcal{V}(x)$.

Since $[\mathcal{H}_0(x), \mathcal{V}(x)] = \mathcal{V}^*(x)$, we can choose elements $v_2, w_2 \in \mathcal{V}^*(x)$ and $v_1 \in \mathcal{V}(x)$ such that $[\pi_0([v_2, w_2]), v_1] \neq 0$. Then

$$\begin{aligned} 0 &= [[v_2, w_2], v_1] \\ &= [\pi_0([v_2, w_2]) + \pi_1([v_2, w_2]), v_1] \\ &= [\pi_0([v_2, w_2]), v_1] + [\pi_1([v_2, w_2]), v_1]. \end{aligned}$$

Thus

$$[\pi_1([v_2, w_2]), v_1] = -[\pi_0([v_2, w_2]), v_1] \neq 0.$$

From here

$$\pi_1([\mathcal{V}^*(x), \mathcal{V}^*(x)]) = \mathcal{V}(x) \quad \text{and} \quad [\mathcal{V}(x), \mathcal{V}(x)] = \mathcal{V}^*(x).$$

But this implies, by (4.9) and (4.10) that

$$[\mathcal{V}^*(x), \mathcal{V}^*(x)] = \mathcal{H}_0(x) \oplus \mathcal{V}(x) \quad \text{and} \quad [\mathcal{V}(x), \mathcal{V}(x)] = \mathcal{V}^*(x).$$

Then, $\mathrm{rad}(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is non-solvable, which is a contradiction. From here, (4.10) is not possible. This is, $\pi_0([\mathcal{V}^*(x), \mathcal{V}^*(x)]) = 0$ and

$$[\mathcal{V}(x) \oplus \mathcal{V}^*(x), \mathcal{V}(x) \oplus \mathcal{V}^*(x)] \subset \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

In this case, $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a solvable Lie subalgebra of $\mathrm{rad}(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{W}(x)$.

Now, suppose case (f) is satisfied:

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \quad \text{is semisimple.}$$

Using the same argument of case (a), we have that $\mathcal{H}$ is a finite direct product of simple ideals $\mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$ with $k \leq 4$ and every ideal is a $\mathfrak{sl}(3, \mathbb{R})$-module.

If $k = 4$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3 \times \mathfrak{h}_4$. By properties of representation of simple Lie algebras we can assume, without loss of generality, $\mathfrak{h}_4 = \mathcal{V}^*(x)$. Because $\mathcal{V}^*(x) \simeq \mathbb{R}^{3*}$ as $\mathfrak{sl}(3, \mathbb{R})$-module we have then $[\mathcal{V}^*(x), \mathcal{V}^*(x)] = 0$. So $\mathfrak{h}_4$ is an abelian Lie algebra that cannot be possible, therefore $k \leq 3$.

If $k = 3$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3$. Suppose, reindexing if necessary, that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are irreducible $\mathfrak{sl}(3, \mathbb{R})$-modules and $\mathfrak{h}_3 = W_1 \oplus W_2$ is the direct sum of two irreducible $\mathfrak{sl}(3, \mathbb{R})$-submodules, W_1 and W_2. Then, we can assume $W_2 = \mathcal{V}^*(x)$. As $0 \neq [\mathcal{G}(x), \mathcal{V}(x) \oplus \mathcal{V}^*(x)]$ then $\mathcal{G}(x) \subseteq \mathfrak{h}_3$ and $\mathfrak{h}_3 = \mathcal{G}(x) \oplus \mathcal{V}^*(x)$ is a real simple Lie algebra of dimension 11. This is a contradiction, because there is not real simple Lie algebra of such dimension. So, $k \leq 2$.

If $k = 2$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2$. We can suppose $\mathfrak{h}_2$ is a direct sum of two or more irreducible $\mathfrak{sl}(3, \mathbb{R})$-modules and $\mathcal{V}^*(x) \subset \mathfrak{h}_2$. Let

$$\pi_i : \mathcal{H} \to \mathfrak{h}_i,$$

be the projection map of $\mathcal{H}$ on the i-th element, $i = 1, 2$. We recall that $[\mathcal{H}_0(x), \mathcal{V}(x)] = \mathcal{V}^*(x)$ then

$$0 \neq \pi_2(\mathcal{H}_0(x)), \pi_2(\mathcal{V}(x)).$$

If $\pi_2(\mathcal{H}_0(x)) = \pi_2(\mathcal{V}(x))$ then $\pi_2(\mathcal{V}(x)) \oplus \mathcal{V}^*(x) \subseteq \mathfrak{h}_2$. And if the equality is satisfied, $\dim(\mathfrak{h}_1) = 11$ which is not possible. Therefore $\pi_2(\mathcal{V}(x)) \oplus \mathcal{V}^*(x) \subsetneq \mathfrak{h}_2$ and

$$\mathfrak{h}_2 = \mathcal{G}(x) \oplus \pi_2(\mathcal{V}(x)) \oplus \mathcal{V}^*(x).$$

Then $\mathfrak{h}_1 = \pi_1(\mathcal{H}_0(x)) \simeq \mathbb{R}^3$, that cannot happen. So $\pi_2(\mathcal{V}(x)) \neq \pi_2(\mathcal{H}_0(x))$ and here

$$\mathfrak{h}_2 = \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

But, there is not a real simple Lie algebra of dimension 9 (see [H, p. 518]). So, this case is not possible.

If $k = 1$, $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a real simple Lie algebra of dimension 17. Since there is not a real simple Lie algebra with such dimension ([H]), this finally proves that case (f) cannot be possible.

For the case $\mathcal{H}_0(x) \simeq \mathbb{R}^{3*}$, we only need interchange the factors $\mathcal{V}(x)$ and $\mathcal{V}^*(x)$ in the previous proof.

□

4.2.3 $\mathcal{H}_0(x) \simeq \mathfrak{sl}(3,\mathbb{R})$

Lemma 4.6 ($\mathcal{H}_0(x) \simeq \mathfrak{sl}(3,\mathbb{R})$)**.** *Let S be as in Corollary 3.19. With the notation of Lemma 3.21, if $\mathcal{H}_0(x)$ isomorphic to $\mathfrak{sl}(3,\mathbb{R})$ as $\mathfrak{sl}(3,\mathbb{R})$-module for some $x \in S$, then one of the following occurs:*

(1) The radical of $\mathcal{H}$ is $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ and $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ is the sum of two simple ideals.

(2) $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is the sum of two simple ideals, being one of them $\mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

Proof. Let us choose an arbitrary but fixed element $x \in S$ such that $\mathcal{H}_0(x) = \mathfrak{sl}(3,\mathbb{R})$, as in Lemma 3.21.

From Lemma 3.21, (4.1) and (4.4), $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ is a Lie subalgebra of $\mathcal{H}$ with $\mathcal{H}_0(x)$ ideal.

On the other hand, we have the next sequence

$$0 \to \mathcal{H}_0(x) \to \mathcal{G}(x) \oplus \mathcal{H}_0(x) \to \mathcal{G}(x) \to 0$$

that is exact.

Since $\mathcal{G}(x) \simeq \mathcal{H}_0(x) \simeq \mathfrak{sl}(3,\mathbb{R})$ as $\mathfrak{sl}(3,\mathbb{R})$-module, then $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ is a semisimple Lie algebra. The short exact sequence proves that the complementary ideal to $\mathcal{H}_0(x)$ in $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ is isomorphic to $\mathcal{G}(x)$. So $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ is a Lie algebra isomorphic to $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})$.

We can choose $\mathfrak{s}$, as a Levi factor of $\mathcal{H}$ that contains the semisimple Lie algebra $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$. Then, $\mathfrak{s}$ is a $\mathfrak{sl}(3,\mathbb{R})$-submodule of $\mathcal{H}$.

Let W be a $\mathfrak{sl}(3,\mathbb{R})$-module such that $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus W$.

Since $\mathrm{rad}(\mathcal{H})$ is an ideal in $\mathcal{H}$, this induce the next decomposition of $\mathcal{H}$ in a direct sum of $\mathfrak{sl}(3,\mathbb{R})$-modules:

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus W \oplus rad(\mathcal{H})$$

that we compare with its decomposition in Lemma 3.21

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

By properties of representation of Lie algebras and the decomposition of $\mathcal{H}$, one of the following must occur:

(a) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}^*(x)$.

(b) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}(x)$.

(c) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

(d) $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is semisimple.

Next, we analyze these cases.

Suppose case (a) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \quad \text{and} \quad rad(\mathcal{H}) = \mathcal{V}^*(x).$$

Because $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x)$ is a semisimple Lie algebra then is a direct product of a finite number of simple ideals, $\mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$. Since every ideal is invariant under the product by $\mathcal{G}(x)$, the ideals possess a structure of $\mathfrak{sl}(3,\mathbb{R})$-module. From decomposition of $\mathfrak{s}$ in irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules we have that $k \leq 3$.

Since $\mathcal{V}(x) \simeq \mathbb{R}^3$, if $[\mathcal{V}(x), \mathcal{V}(x)] \neq 0$ then $[\mathcal{V}(x), \mathcal{V}(x)] \simeq \mathbb{R}^{3*}$. Thus, in our case, $[\mathcal{V}(x), \mathcal{V}(x)]$ must have projection zero in $\mathcal{G}(x)$, $\mathcal{H}_0(x)$ and $\mathcal{V}(x)$. This is, $[\mathcal{V}(x), \mathcal{V}(x)] = 0$. So, by Lemma 4.2, $\mathcal{V}(x)$ is an abelian ideal in $\mathfrak{s}$, which is a contradiction. Therefore, this case cannot be possible.

The proof that case (b) cannot be possible is similar to the case (a).

Assume case (c) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \quad \text{and} \quad rad(\mathcal{H}) = \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

We already have proved that $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ is a direct sum of two simple ideals.

Suppose case (d) holds:

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \quad \text{is semisimple.}$$

Then, $\mathcal{H}$ is a direct product of a finite number of simple ideals, $\mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$. These ideals are $\mathfrak{sl}(3,\mathbb{R})$-modules with the product by $\mathcal{G}(x)$, therefore $k \leq 4$.

If $k = 4$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3 \times \mathfrak{h}_4$. We suppose, reindexing if necessary, that $\mathfrak{h}_4 = \mathcal{V}(x)$ (or $\mathcal{V}^*(x)$). Then $[\mathcal{V}(x), \mathcal{V}(x)] = \mathcal{V}(x)$, which is not possible. So, this case cannot be possible and $k \leq 3$.

If $k = 3$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3$. We assume, reindexing if necessary, that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules and $\mathfrak{h}_3 = V_1 \oplus V_2$, direct sum of two irreducibles $\mathfrak{sl}(3,\mathbb{R})$-modules, V_1 and V_2. Then, $\mathcal{V}(x), \mathcal{V}^*(x) \subsetneq \mathfrak{h}_1 \times \mathfrak{h}_2$. So

$$\mathfrak{h}_3 = \mathcal{V}(x) \oplus \mathcal{V}^*(x) \quad \text{and} \quad \mathfrak{h}_1 \times \mathfrak{h}_2 = \mathcal{G}(x) \oplus \mathcal{H}_0(x).$$

Hence

$$[\mathcal{H}_0(x), \mathcal{V}(x) \oplus \mathcal{V}^*(x)] = 0,$$

that by Lemma 3.21 is a contradiction. Thus, this case cannot be possible. Therefore, $k \leq 2$.

If $k = 2$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2$. We assume, reindexing if necessary, that $\mathfrak{h}_2$ is a direct sum of two or more irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules and $\mathcal{V}^*(x) \subset \mathfrak{h}_2$. From here $\mathcal{V}(x) \subset \mathfrak{h}_2$. Otherwise, $\mathcal{V}(x) \subset \mathfrak{h}_1$ and $\mathfrak{h}_2$ would have dimension 11 or 19. That is not possible, (see [H, p. 518]). Then

$$\mathcal{V}(x) \oplus \mathcal{V}^*(x) \subseteq \mathfrak{h}_2.$$

Here, the equality is not satisfied because if $\mathcal{V}(x) \oplus \mathcal{V}^*(x) = \mathfrak{h}_2$ then

$$\mathcal{G}(x) \oplus \mathcal{H}_0(x) = \mathfrak{h}_1 \quad \text{and} \quad [\mathcal{H}_0(x), \mathcal{V}(x) \oplus \mathcal{V}^*(x)] = 0.$$

Which is a contradiction. Therefore, $\mathfrak{h}_2$ is a direct sum of three irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules

$$\mathfrak{h}_2 = V_1 \oplus V_2 \oplus V_3.$$

We assume, $\mathcal{V}(x) \oplus \mathcal{V}^*(x) = V_1 \oplus V_2$. Thus

$$\mathcal{G}(x) \oplus \mathcal{H}_0(x) = \mathfrak{h}_1 \oplus V_3.$$

Since $V_3 \subset \mathfrak{h}_2$ and $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ is a Lie subalgebra then

$$\begin{aligned}
&[V_3, V_3] \subset [\mathfrak{h}_2, \mathfrak{h}_2] \subseteq \mathfrak{h}_2 \\
&[V_3, V_3] \subset [\mathcal{G}(x) \oplus \mathcal{H}_0(x), \mathcal{G}(x) \oplus \mathcal{H}_0(x)] \subset \mathcal{G}(x) \oplus \mathcal{H}_0(x) = \mathfrak{h}_1 \oplus V_3.
\end{aligned}$$

From here, $[V_3, V_3] \subset V_3$ and V_3 itself is a Lie algebra. But

$$[\mathfrak{h}_1, V_3] \subset [\mathfrak{h}_1, \mathfrak{h}_2] = 0.$$

Then $\mathfrak{h}_1 \oplus V_3$ is the decomposition into simple ideals of $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$. Therefore since, by (4.1) and (4.4), $\mathcal{H}_0(x)$ is an ideal on $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$. Thus, $\mathcal{H}_0(x) = \mathfrak{h}_1$ or $\mathcal{H}_0(x) = V_3$. If $\mathcal{H}_0(x) = \mathfrak{h}_1$, then

$$[\mathcal{H}_0(x), \mathcal{V}(x) \oplus \mathcal{V}^*(x)] \subset [\mathfrak{h}_1, \mathfrak{h}_2] = 0$$

which is not possible. Therefore, $\mathcal{H}_0(x) = V_3$ and

$$\mathfrak{h}_2 = \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$$

is a simple Lie algebra that, by case (d) in Lemma 4.3, is isomorphic to $\mathfrak{g}_{2(2)}$.

Finally, the case $k = 1$ cannot be possible because, by [H, p. 518], there is not a real simple Lie algebra of dimension 22. $\square$

4.3 Subalgebras with two irreducible submodules

4.3.1 $\mathcal{H}_0(x) \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3$ or $\mathcal{H}_0(x) \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^{3*}$

Lemma 4.7 ($\mathcal{H}_0(x) \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3$)**.** *Let S be as in Corollary 3.19. With the notation of Lemma 3.21, if $\mathcal{H}_0(x)$ is isomorphic to $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3$ as $\mathfrak{sl}(3,\mathbb{R})$-module for some $x \in S$, then $\mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0$ is a direct sum of two simple ideals and rad$(\mathcal{H}) = \mathcal{V}(x) \oplus \mathcal{V}^*(x) \oplus \mathbb{R}^3_0$, with $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ a Lie subalgebra. Where $\mathfrak{sl}(3,\mathbb{R})_0$ and $\mathbb{R}^3_0$ are $\mathfrak{sl}(3,\mathbb{R})$-submodules of $\mathcal{H}_0(x)$ isomorphic to $\mathfrak{sl}(3,\mathbb{R})$ and $\mathbb{R}^3$, respectively.*

With $\mathcal{H}_0(x) \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^{3*}$ we have a similar result. We only replace $\mathbb{R}^3$ by $\mathbb{R}^{3*}$.

Proof. Let us choose an arbitrary but fixed element $x \in S$ such that

$$\mathcal{H}_0(x) \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3,$$

as in Lemma 3.21.

In the first place, we define $\mathfrak{sl}(3,\mathbb{R})_0$ and $\mathbb{R}^3_0$ the Lie subalgebras of $\mathcal{H}_0(x)$ such that $\mathfrak{sl}(3,\mathbb{R})_0 \simeq \mathfrak{sl}(3,\mathbb{R})$ and $\mathbb{R}^3_0 \simeq \mathbb{R}^3$ via the homomorphism of Lie algebras $\lambda_x^\perp$, from Lemma 3.21.

Since homomorphism $\lambda_x^\perp$ is injective when is restricted to $\mathfrak{sl}(3,\mathbb{R})_0$ and to the Lie algebra $\mathfrak{sl}(3,\mathbb{R})(x)$, then

$$\lambda_x^\perp(\mathfrak{sl}(3,\mathbb{R})(x)) \cap \lambda_x^\perp(\mathfrak{sl}(3,\mathbb{R})_0) = 0 \quad \text{or} \quad \lambda_x^\perp(\mathfrak{sl}(3,\mathbb{R})(x)) = \lambda_x^\perp(\mathfrak{sl}(3,\mathbb{R})_0).$$

This is so because the intersection is a submodule of both. But,

$$\dim(\mathfrak{so}(T_x\mathcal{O}^\perp)) = 15$$

then

$$\lambda_x^\perp(\mathfrak{sl}(3,\mathbb{R})(x)) = \lambda_x^\perp(\mathfrak{sl}(3,\mathbb{R})_0).$$

Since $\mathfrak{sl}(3,\mathbb{R})_0 \subset \mathcal{H}_0(x)$ is a $\mathcal{G}(x)$-module isomorphic to $\mathfrak{sl}(3,\mathbb{R})$ we have $[\mathcal{G}(x), \mathfrak{sl}(3,\mathbb{R})_0] = \mathfrak{sl}(3,\mathbb{R})_0$. From here, we have the next exact sequence

$$0 \to \mathfrak{sl}(3,\mathbb{R})_0 \to \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \to \mathcal{G}(x) \to 0.$$

Since $\mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0$ is a semisimple Lie algebra, the previous short exact sequence shows that the complementary ideal to $\mathfrak{sl}(3,\mathbb{R})_0$ in $\mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0$ is isomorphic to $\mathcal{G}(x)$. So $\mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0$ is a Lie algebra isomorphic to

$$\mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{sl}(3,\mathbb{R}).$$

We choose $\mathfrak{s}$ a Levi factor of $\mathcal{H}$ that contains $\mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0$. With the $\mathfrak{sl}(3,\mathbb{R})$-module structure of $\mathcal{H}$, defined by the subalgebra $\mathcal{G}(x)$, and as $\mathcal{G}(x) \subset \mathfrak{s}$, then $\mathfrak{s}$ is a $\mathfrak{sl}(3,\mathbb{R})$-submodule of $\mathcal{H}$.

Let W be a $\mathfrak{sl}(3,\mathbb{R})$-submodule of $\mathcal{H}$ such that $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus W$.

Since $\mathrm{rad}(\mathcal{H})$ is an ideal, this induces the next decomposition of $\mathcal{H}$ as a direct sum of $\mathfrak{sl}(3,\mathbb{R})$-modules:

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus W \oplus \mathrm{rad}(\mathcal{H})$$

that we compare with the decomposition of irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x),$$

from Lemma 3.21.

By properties of representation of Lie algebras and the decomposition of $\mathcal{H}$ in irreducible modules, one of the following must occur:

(a) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}^3_0 \oplus \mathcal{V}(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}^*(x)$.

(b) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus V_1 \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = V_2$.

(c) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathbb{R}^3_0 \oplus \mathcal{V}(x)$.

(d) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus V_1$ and $\mathrm{rad}(\mathcal{H}) = V_2 \oplus \mathcal{V}^*(x)$.

(e) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0$ and $\mathrm{rad}(\mathcal{H}) = \mathbb{R}^3_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

(f) $\mathcal{H} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}^3_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is semisimple.

Where V_1 and V_2 are vector spaces isomorphic to $\mathbb{R}^3$ as $\mathfrak{sl}(3,\mathbb{R})$-modules such that

$$V_1 \oplus V_2 = \mathbb{R}^3_0 \oplus \mathcal{V}(x).$$

Next, we analyze all these cases.

In the first place, similar to Lemma 4.5 we have, from the equation (4.3) and since $\mathbb{R}^3_0 \simeq \mathbb{R}^3$ as $\mathfrak{sl}(3,\mathbb{R})$-module, that

$$[\mathbb{R}^3_0, \mathcal{V}(x)] = \mathcal{V}^*(x) \quad \text{and} \quad [\mathbb{R}^3_0, \mathcal{V}^*(x)] \subseteq \mathfrak{sl}(3,\mathbb{R})_0. \tag{4.11}$$

Suppose case (a) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}^3_0 \oplus \mathcal{V}(x) \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathcal{V}^*(x).$$

Since $\mathrm{rad}(\mathcal{H}) = \mathcal{V}^*(x)$, from (4.11) we have

$$\mathrm{rad}(\mathcal{H}) = \mathcal{V}^*(x) = [\mathbb{R}^3_0, \mathcal{V}(x)] \subset [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{s}.$$

That is a contradiction. Therefore, this case cannot be possible.

Assume case (b) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus V_1 \oplus \mathcal{V}^*(x) \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = V_2.$$

In the first place, since $\mathrm{rad}(\mathcal{H}) = V_2$ is isomorphic to $\mathbb{R}^3$ as $\mathfrak{sl}(3,\mathbb{R})$-module, we have that

$$[V_2, V_2] = [\mathcal{V}(x), V_2] = [\mathbb{R}^3_0, \mathrm{rad}(\mathcal{H})] = 0. \tag{4.12}$$

On the other hand, by (4.4) and since $\mathcal{H}_0(x) = \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}^3_0$ we have that

$$[\mathbb{R}^3_0, \mathbb{R}^3_0] = 0. \tag{4.13}$$

Let π^2_0 and $\pi^2_{\mathcal{V}}$ be the projection map of V_2 in $\mathbb{R}^3_0$ and $\mathcal{V}(x)$, respectively. In a similar way to case (b) in Lemma 4.5, we can prove that $\pi^2_0(V_2) = \mathbb{R}^3_0$ and $\pi^2_{\mathcal{V}}(V_2) = \mathcal{V}(x)$. Otherwise, (4.12) would contradict the first part of (4.11).

On the other hand, by (4.11), let $v \in \mathbb{R}^3_0$ and $w \in \mathcal{V}(x)$ be such that

$$0 \neq [v, w] \in \mathcal{V}^*(x).$$

Choose $\bar{v}, \bar{w} \in V_2$ elements such that

$$\pi^2_0(\bar{v}) = v \quad \text{and} \quad \pi^2_{\mathcal{V}}(\bar{w}) = w.$$

Then, by (4.12)

$$\begin{aligned}
0 &= [\bar{v}, \bar{w}] \\
&= [\pi^2_0(\bar{v}) + \pi^2_{\mathcal{V}}(\bar{v}), \pi^2_0(\bar{w}) + \pi^2_{\mathcal{V}}(\bar{w})] \\
&= [\pi^2_0(\bar{v}), \pi^2_0(\bar{w})] + [\pi^2_0(\bar{v}), \pi^2_{\mathcal{V}}(\bar{w})] + [\pi^2_{\mathcal{V}}(\bar{v}), \pi^2_0(\bar{w}) + \pi^2_{\mathcal{V}}(\bar{w})] \\
&= 0 + [v, w] + [\pi^2_{\mathcal{V}}(\bar{v}), \bar{w}] \\
&= [v, w] + 0 \\
&= [v, w].
\end{aligned}$$

That is a contradiction. We have used (4.13) and (4.12) in the the fourth and fifth line, respectively. Therefore, case (b) cannot happen.

Suppose case (c) holds:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathcal{V}^*(x) \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathbb{R}^3_0 \oplus \mathcal{V}(x).$$

Since $\mathcal{V}^*(x) \simeq \mathbb{R}^{3*}$ as $\mathfrak{sl}(3,\mathbb{R})$-module then $[\mathcal{V}^*(x), \mathcal{V}^*(x)]$ is isomorphic to a submodule of $\mathbb{R}^3$. From here $[\mathcal{V}^*(x), \mathcal{V}^*(x)]$ has projection zero on $\mathcal{G}(x)$, $\mathfrak{sl}(3,\mathbb{R})_0$ and $\mathcal{V}^*(x)$. So, $\mathcal{V}^*(x)$ is an abelian ideal on $\mathfrak{s}$. That is a contradiction because $\mathfrak{s}$ is a semisimple Lie algebra. From here, this case cannot happen.

Suppose case (d) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus V_1 \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = V_2 \oplus \mathcal{V}^*(x).$$

In a similar way to case (c), since $V_1 \simeq \mathbb{R}^3$ as $\mathfrak{sl}(3,\mathbb{R})$-module, we can prove that V_1 is an abelian ideal on $\mathfrak{s}$. So, this case cannot be possible.

Suppose case (e) holds:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathbb{R}^3_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

At the beginning of this proof we have shown that $\mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0$ is a semisimple Lie algebra isomorphic to $\mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{sl}(3,\mathbb{R})$. On the other hand, since $\mathrm{rad}(\mathcal{H})$ is an ideal in $\mathcal{H}$. Similar to case (e) in Lemma 4.5, we can prove that $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a Lie subalgebra in $\mathrm{rad}(\mathcal{H})$.

Assume case (f) holds:

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}^3_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \quad \text{is semisimple.}$$

Since $\mathcal{H}$ is a semisimple Lie algebra then it is isomorphic to a direct product of a finite number of simple ideals, $\mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$. Being that every ideal is invariant under the Lie bracket by $\mathcal{G}(x)$, then every ideal has the structure of $\mathfrak{sl}(3,\mathbb{R})$-module. So, $k \leq 5$.

If $k = 5$, $\mathcal{H} = \mathfrak{h}_1 \times \cdots \times \mathfrak{h}_5$. We assume, reindexing if necessary, that $\mathfrak{h}_5 = \mathcal{V}^*(x)$. Since $\mathcal{V}^*(x) \cong \mathbb{R}^{3*}$ as $\mathfrak{sl}(3,\mathbb{R})$-module, then

$$[\mathfrak{h}_5, \mathfrak{h}_5] = [\mathcal{V}^*(x), \mathcal{V}^*(x)]$$

is isomorphic to a submodule of $\mathbb{R}^3$. From here, $[\mathcal{V}^*(x), \mathcal{V}^*(x)] = 0$. This is, $\mathfrak{h}_5$ is an abelian algebra, that is a contradiction, because we assumed that $\mathfrak{h}_5$ is a real simple Lie algebra. Therefore $k \leq 4$.

If $k = 4$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3 \times \mathfrak{h}_4$. Since $\mathcal{H}$ is the sum of five irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules, we can assume, reindexing if necessary, that $\mathfrak{h}_1, \mathfrak{h}_2$ and $\mathfrak{h}_3$ are irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules and $\mathfrak{h}_4 = W_1 \oplus W_2$, with W_1 and W_2 irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules, and $W_2 = \mathcal{V}^*(x)$. Let

$$\pi_j : \mathcal{H} \to \mathfrak{h}_j$$

be the projection homomorphism from $\mathcal{H}$ to the ideal $\mathfrak{h}_j$, for $j = 1, 2, 3, 4$. Since $[\mathfrak{sl}(3,\mathbb{R})_0, \mathcal{V}^*(x)] \neq 0$ then $\pi_4(\mathfrak{sl}(3,\mathbb{R})_0) \neq 0$. From here, $\mathfrak{h}_4 = \pi_4(\mathfrak{sl}(3,\mathbb{R})_0) \oplus \mathcal{V}^*(x)$ is a real simple Lie algebra with $\dim(\mathfrak{h}_4) = 11$. That cannot be possible. So $k \leq 3$.

If $k = 3$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3$. We assume $\mathfrak{h}_1$ is an irreducible $\mathfrak{sl}(3,\mathbb{R})$-module and $\mathfrak{h}_3 = U_1 \oplus U_2 \oplus \cdots \oplus U_l$ is a direct sum of two or three irreducibles modules. We can also assume that $\pi_3(\mathfrak{sl}(3,\mathbb{R})_0) \oplus \mathcal{V}^*(x) \subsetneq \mathfrak{h}_3$. On the other hand, being that $[\mathbb{R}^3_0, \mathcal{V}(x)] = \mathcal{V}^*(x)$ then $\pi_3(\mathbb{R}^3_0) \neq 0$ and $\dim(\mathfrak{h}_3) \geq 14$. Thus, $\mathfrak{h}_1$ (or $\mathfrak{h}_2$) is a $\mathfrak{sl}(3,\mathbb{R})$-module of dimension 3. Then, by previous results, this ideal is abelian. Which is a contradiction. So, this case is not possible and $k \leq 2$.

If $k = 2$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2$. As in $k = 3$, we can assume $\pi_2(\mathbb{R}^3_0) \neq 0$ and

$$\pi_2(\mathfrak{sl}(3,\mathbb{R})_0) \oplus \mathcal{V}^*(x) \oplus \pi_3(\mathbb{R}^3_0) \subsetneq \mathfrak{h}_2.$$

Then the only options for the dimension of $\mathfrak{h}_1$ are 11 or 8. By [H, p. 518], there is not a real simple Lie algebra with dimension 11. Then $\dim(\mathfrak{h}_2) = 8$ and $\dim(\mathfrak{h}_1) = 17$. That cannot be possible, because there is not a real simple Lie algebra with dimension 17, so this case cannot happen. Therefore $k = 1$.

If $k = 1$, $\mathcal{H} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}^3_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a real simple Lie algebra with $\dim(\mathcal{H}) = 25$. But, by [H], this case cannot be possible. So, case (f) cannot happen.

$\square$

4.3.2 $\mathcal{H}_0(x) \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}$

Lemma 4.8 ($\mathcal{H}_0(x) \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}$)**.** *Let S be as in Corollary 3.19. With the notation of Lemma 3.21. If $\mathcal{H}_0(x)$ is isomorphic to $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}$ as $\mathfrak{sl}(3,\mathbb{R})$-module for some $x \in S$, then one of the following occurs:*

(1) $\mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0$ is a direct sum of two simple ideals and the radical of $\mathcal{H}$ is $\mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$, where $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a Lie subalgebra.

(2) $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a direct sum of two simple ideals, being one of them $\mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ that is isomorphic to $\mathfrak{sl}(4,\mathbb{R})$.

Where $\mathfrak{sl}(3,\mathbb{R})_0$ and $\mathbb{R}_0$ are $\mathfrak{sl}(3,\mathbb{R})$-submodules of $\mathcal{H}_0(x)$ isomorphic to $\mathfrak{sl}(3,\mathbb{R})$ and $\mathbb{R}$, respectively.

Proof. Let us choose an arbitrary but fixed element $x \in S$ such that

$$\mathcal{H}_0(x) \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R},$$

as in Lemma 3.21.

As in previous case, $\mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0$ is a semisimple Lie algebra isomorphic to the direct product $\mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{sl}(3,\mathbb{R})$.

We choose $\mathfrak{s}$ a Levi factor of $\mathcal{H}$ that contains $\mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0$. Since the $\mathfrak{sl}(3,\mathbb{R})$-module structure of $\mathcal{H}$ is defined by the subalgebra $\mathcal{G}(x)$, and as $\mathcal{G}(x) \subset \mathfrak{s}$, then $\mathfrak{s}$ is a $\mathfrak{sl}(3,\mathbb{R})$-submodule of $\mathcal{H}$.

Let W be a $\mathfrak{sl}(3,\mathbb{R})$-submodule of $\mathcal{H}$ such that $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus W$.

Since $\mathrm{rad}(\mathcal{H})$ is an ideal, this induces the next decomposition of $\mathcal{H}$ as a direct sum of $\mathfrak{sl}(3,\mathbb{R})$-modules:

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus W \oplus \mathrm{rad}(\mathcal{H})$$

that we compare with the decomposition of irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules

$$\begin{aligned} \mathcal{H} &= \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \\ &= \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}_0^3 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x), \end{aligned}$$

from Lemma 3.21.

Then one of the following holds:

(a) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathbb{R}_0$.

(b) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}_0 \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}(x)$.

(c) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}_0 \oplus \mathcal{V}(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}^*(x)$.

(d) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus \mathcal{V}(x)$.

(e) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathcal{V}(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus \mathcal{V}^*(x)$.

(f) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}_0$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

(g) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0$ and $\mathrm{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

(h) $\mathcal{H} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is semisimple.

Next, we analyze these cases.

In the first place, from Lemmas 3.21 and 4.2 we have that

$$[\mathbb{R}_0, \mathcal{V}(x)] = \mathcal{V}(x) \quad \text{and} \quad [\mathbb{R}_0, \mathcal{V}^*(x)] = \mathcal{V}^*(x). \tag{4.14}$$

Suppose case (a) holds:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathbb{R}_0.$$

Since $\mathrm{rad}(\mathcal{H})$ is and ideal on $\mathcal{H}$ then

$$\begin{aligned}\mathcal{V}(x) \oplus \mathcal{V}^*(x) &= [\mathcal{V}(x) \oplus \mathcal{V}^*(x), \mathbb{R}_0]\\ &= [\mathcal{V}(x) \oplus \mathcal{V}^*(x), \mathrm{rad}(\mathcal{H})]\\ &\subset \mathrm{rad}(\mathcal{H})\\ &= \mathbb{R}_0\end{aligned}$$

That is a contradiction. So, this case cannot be possible.

Assume case (b) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}_0 \oplus \mathcal{V}^*(x) \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathcal{V}(x).$$

Since $\mathcal{V}^*(x)$ is isomorphic to $\mathbb{R}^{3*}$ as $\mathfrak{sl}(3,\mathbb{R})$-module, then $[\mathcal{V}^*(x), \mathcal{V}^*(x)]$ is isomorphic to a submodule of $\mathbb{R}^3$. From here, we have that $[\mathcal{V}^*(x), \mathcal{V}^*(x)]$ has projection zero in $\mathcal{G}(x)$, $\mathfrak{sl}(3,\mathbb{R})_0$, $\mathbb{R}_0$ and $\mathcal{V}^*(x)$. This is $\mathcal{V}^*(x)$ is an abelian ideal on $\mathfrak{s}$, that is a contradiction. Thus, this case cannot happen.

Suppose case (c) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}_0 \oplus \mathcal{V}(x) \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathcal{V}^*(x).$$

This case is not possible. The proof is similar to (b), we only need replace $\mathcal{V}(x)$ by $\mathcal{V}^*(x)$.

Assume case (d) holds:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathcal{V}^*(x) \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus \mathcal{V}(x).$$

Similar to case (b). Here, $\mathcal{V}^*(x)$ is an abelian ideal on $\mathfrak{s}$. That is a contradiction. Then this case cannot be possible.

Suppose case (e) holds:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathcal{V}(x) \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus \mathcal{V}^*(x).$$

As in case (b), we can prove that $\mathcal{V}(x)$ is an abelian ideal on $\mathfrak{s}$. Thus, this is a contradiction. And this case cannot happen.

Assume case (f) holds:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}_0 \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

This case cannot be possible. Here, we have that $\mathbb{R}_0$ is an abelian ideal on $\mathfrak{s}$. The proof of this is similar to case (b).

Suppose case (g) holds:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

At first of this proof we have proved that $\mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0$ is isomorphic to a direct sum of two simple Lie algebras. On the other hand, similar to Lemma 4.4, in this case we have that $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a Lie subalgebra of $\mathrm{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

Assume case (h) is satisfied:

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \quad \text{is semisimple.}$$

Then $\mathcal{H}$ is isomorphic to a direct product of a finite number of simple ideals, $\mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$, for some $k \in \mathbb{N}$. Since every ideal is closed with respect to the product of Lie brackets by $\mathcal{G}(x)$ we have a $\mathfrak{sl}(3,\mathbb{R})$-module structure in each ideal. From the decomposition of $\mathcal{H}$ in irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules we obtain $k \leq 5$.

If $k = 5$, $\mathcal{H} = \mathfrak{h}_1 \times \cdots \times \mathfrak{h}_5$. Without loss of generality, reindexing if necessary, we suppose $\mathfrak{h}_5 = \mathcal{V}^*(x)$. But, as in case (b), this cannot be possible. Then, this case cannot happen and $k \leq 4$.

If $k = 4$, $\mathcal{H} = \mathfrak{h}_1 \times \cdots \times \mathfrak{h}_4$. We can assume, reindexing if necessary, that $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3$ are irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules and $\mathfrak{h}_4 = W_1 \oplus W_2$ is a direct sum of two irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules. We can also assume that $\mathcal{V}(x) \subsetneq \mathfrak{h}_4$. Since $[\mathbb{R}_0, \mathcal{V}^*(x)] = \mathcal{V}^*(x)$ then $\mathbb{R}_0 \subsetneq \mathfrak{h}_4$. And

$$\mathfrak{h}_4 = \mathbb{R}_0 \oplus \mathcal{V}^*(x).$$

In this case $\mathcal{V}^*(x)$ is an abelian ideal of $\mathfrak{h}_4$, that cannot happen. So, $k = 4$ is not possible.

If $k = 3$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3$. By previous subcases we can assume that $\mathfrak{h}_3$ is direct sum of two or more irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules and $\mathbb{R}_0 \oplus \mathcal{V}^*(x) \subsetneq \mathfrak{h}_3$. Then, by (4.14), $\mathcal{V}^*(x) \subset \mathfrak{h}_3$. Since $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are nonzero,

$$\mathfrak{h}_3 = \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

By [H], there is no 7-dimensional real simple Lie algebra. Therefore, $k = 3$ cannot be possible.

If $k = 2$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2$. By previous subcases, we can assume that $\mathfrak{h}_2$ is a direct sum of four irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules, $\mathfrak{h}_2 = V_1 \oplus V_2 \oplus V_3 \oplus V_4$, and $\mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \subsetneq \mathfrak{h}_2$. Without loss of generality, we suppose

$$\mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) = V_1 \oplus V_2 \oplus V_3.$$

Then

$$\mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 = \mathfrak{h}_1 \oplus V_4.$$

Similar to case (d) in Lemma 4.6, we can prove that $\mathfrak{sl}(3,\mathbb{R})_0 = V_4$. Thus,

$$\mathfrak{h}_2 = \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

And using the arguments of case (h) in Lemma 4.4, $\mathfrak{h}_2$ is isomorphic to $\mathfrak{sl}(4,\mathbb{R})$.

If $k = 1$, $\mathcal{H} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a simple Lie algebra with $\dim(\mathcal{H}) = 23$. By [H], there is not a real simple Lie algebra with such dimension. So, this cannot be possible. $\square$

4.3.3 $\mathcal{H}_0(x) \simeq \mathbb{R}^3 \oplus \mathbb{R}$ or $\mathcal{H}_0(x) \simeq \mathbb{R}^{3*} \oplus \mathbb{R}$

Lemma 4.9 ($\mathcal{H}_0(x) \simeq \mathbb{R}^3 \oplus \mathbb{R}$)**.** *Let S be as in Corollary 3.19. With the notation of Lemma 3.21, if $\mathcal{H}_0(x)$ is isomorphic to $\mathbb{R}^3 \oplus \mathbb{R}$ as $\mathfrak{sl}(3,\mathbb{R})$-module for some $x \in S$ then the radical of $\mathcal{H}$ is $\mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$. With $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ subalgebra of rad($\mathcal{H}$).*

Proof. Let us choose an arbitrary but fixed element $x \in S$ such that

$$\mathcal{H}_0(x) \simeq \mathbb{R}^3 \oplus \mathbb{R},$$

as in Lemma 3.21.

In the first place we define as $\mathbb{R}^3_0$ and $\mathbb{R}_0$ the Lie subalgebras of $\mathcal{H}_0(x)$ such that

$$\mathcal{H}_0(x) = \mathbb{R}^3_0 \oplus \mathbb{R}_0$$

with $\mathbb{R}^3_0 \simeq \mathbb{R}^3$ and $\mathbb{R}_0 \simeq \mathbb{R}$ via the homomorphism of Lie algebras $\lambda_x^\perp$, from Lemma 3.21.

Since $\mathcal{G}(x) \cong \mathfrak{sl}(3,\mathbb{R})$, then $\mathcal{G}(x)$ is a simple Lie subalgebra.

Let $\mathfrak{s}$ be a Levi factor of $\mathcal{H}$ that contains $\mathcal{G}(x)$. With the $\mathfrak{sl}(3,\mathbb{R})$-module structure of $\mathcal{H}$, defined by the subalgebra $\mathcal{G}(x)$, and as $\mathcal{G}(x) \subset \mathfrak{s}$ then $\mathfrak{s}$ is a $\mathfrak{sl}(3,\mathbb{R})$-submodule of $\mathcal{H}$.

Let W be a $\mathfrak{sl}(3,\mathbb{R})$-submodule of $\mathcal{H}$ such that $\mathfrak{s} = \mathcal{G}(x) \oplus W$.

Since rad($\mathcal{H}$) is an ideal of $\mathcal{H}$, this induces the next decomposition of $\mathcal{H}$ as a direct sum of $\mathfrak{sl}(3,\mathbb{R})$-modules:

$$\mathcal{H} = \mathcal{G}(x) \oplus W \oplus \mathrm{rad}(\mathcal{H})$$

that we compare with the decomposition of Lemma 3.21

$$\begin{aligned} \mathcal{H} &= \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \\ &= \mathcal{G}(x) \oplus \mathbb{R}^3_0 \oplus \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x). \end{aligned}$$

By properties of representation of Lie algebras and the decomposition of $\mathcal{H}$ in irreducible modules, one of the following must occur:

(a) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathbb{R}_0 \oplus \mathbb{R}^3_0 \oplus \mathcal{V}(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}^*(x)$.

(b) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathbb{R}_0 \oplus V_1 \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = V_2$.

(c) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathbb{R}^3_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathbb{R}_0$.

(d) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathbb{R}_0 \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathbb{R}^3_0 \oplus \mathcal{V}(x)$.

(e) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathbb{R}_0 \oplus V_1$ and $\mathrm{rad}(\mathcal{H}) = V_2 \oplus \mathcal{V}^*(x)$.

(f) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathbb{R}^3_0 \oplus \mathcal{V}(x)$ and $\text{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus \mathcal{V}^*(x)$.

(g) $\mathfrak{s} = \mathcal{G}(x) \oplus V_1 \oplus \mathcal{V}^*(x)$ and $\text{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus V_2$.

(h) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}^*(x)$ and $\text{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus \mathbb{R}^3_0 \oplus \mathcal{V}(x)$.

(i) $\mathfrak{s} = \mathcal{G}(x) \oplus V_1$ and $\text{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus V_2 \oplus \mathcal{V}^*(x)$.

(j) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathbb{R}_0$ and $\text{rad}(\mathcal{H}) = \mathbb{R}^3_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

(k) $\mathfrak{s} = \mathcal{G}(x)$ and $\text{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus \mathbb{R}^3_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

(l) $\mathcal{H} = \mathcal{G}(x) \oplus \mathbb{R}^3_0 \oplus \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is semisimple.

Where V_1 and V_2 are vector spaces isomorphic to $\mathbb{R}^3$, as $\mathfrak{sl}(3,\mathbb{R})$-modules, such that

$$V_1 \oplus V_2 = \mathbb{R}^3_0 \oplus \mathcal{V}(x).$$

Next, we analyze these cases.

In the first place, from Lemma 3.21, Lemma 4.2 and by the arguments of Lemma 4.5 we have:

$$[\mathbb{R}^3_0, \mathcal{V}(x)] = \mathcal{V}^*(x) \tag{4.15}$$
$$[\mathbb{R}_0, \mathcal{V}^*(x)] = \mathcal{V}^*(x) \tag{4.16}$$
$$[\mathbb{R}_0, \mathcal{V}(x)] = \mathcal{V}(x) \tag{4.17}$$

Assume case (a) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathbb{R}_0 \oplus \mathbb{R}^3_0 \oplus \mathcal{V}(x) \quad \text{and} \quad \text{rad}(\mathcal{H}) = \mathcal{V}^*(x).$$

Since $\mathfrak{s}$ is a subalgebra and by (4.15) we have

$$\text{rad}(\mathcal{H}) = \mathcal{V}^*(x) = [\mathbb{R}^3_0, \mathcal{V}(x)] \subset \mathfrak{s},$$

that is a contradiction. Then, this case cannot be possible.

Suppose case (b) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathbb{R}_0 \oplus V_1 \oplus \mathcal{V}^*(x) \quad \text{and} \quad \text{rad}(\mathcal{H}) = V_2.$$

Using similar arguments as in case (b) in Lemma 4.7 we can prove that this case cannot happen.

Assume case (c) holds:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathbb{R}^3_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \quad \text{and} \quad \text{rad}(\mathcal{H}) = \mathbb{R}_0.$$

Since $\text{rad}(\mathcal{H})$ is an ideal on $\mathcal{H}$ and by equation (4.16) we have

$$\begin{aligned} \mathcal{V}^*(x) &= [\mathcal{V}^*(x), \mathbb{R}_0] \\ &= [\mathcal{V}^*(x), \text{rad}(\mathcal{H})] \\ &\subset \text{rad}(\mathcal{H}). \end{aligned}$$

That is a contradiction. Hence, this case cannot be possible.

Suppose case (d) holds:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathbb{R}_0 \oplus \mathcal{V}^*(x) \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathbb{R}_0^3 \oplus \mathcal{V}(x).$$

Similar to the previous case and since $\mathbb{R}_0^3, \mathcal{V}(x) \subset \mathrm{rad}(\mathcal{H})$, we have

$$\begin{array}{rcl} \mathcal{V}^*(x) & = & [\mathbb{R}_0^3, \mathcal{V}(x)] \\ & \subset & [\mathrm{rad}(\mathcal{H}), \mathrm{rad}(\mathcal{H})] \\ & = & \mathrm{rad}(\mathcal{H}). \end{array}$$

That is not possible. Then, this case cannot happen.

Assume case (e) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathbb{R}_0 \oplus V_1 \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = V_2 \oplus \mathcal{V}^*(x).$$

We recall that $V_1 \simeq \mathbb{R}^3$, $\mathbb{R}_0 \simeq \mathbb{R}$ and $\mathcal{G}(x) \simeq \mathfrak{sl}(3, \mathbb{R})$ as $\mathfrak{sl}(3, \mathbb{R})$-module. From here,

$$[V_1, V_1]$$

has zero projection in V_1, $\mathbb{R}_0$ and $\mathcal{G}(x)$. Then V_1 is an abelian ideal on $\mathfrak{s}$, that is a contradiction. Therefore, this case cannot be possible.

Suppose case (f) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathbb{R}_0^3 \oplus \mathcal{V}(x) \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus \mathcal{V}^*(x).$$

Since $\mathbb{R}_0 \subset \mathrm{rad}(\mathcal{H})$ and by (4.17) we have

$$\begin{array}{rcl} \mathcal{V}(x) & = & [\mathcal{V}(x), \mathbb{R}_0] \\ & \subset & [\mathcal{V}(x), \mathrm{rad}(\mathcal{H})] \\ & \subset & \mathrm{rad}(\mathcal{H}). \end{array}$$

Which is not possible. Hence, this case cannot happen.

Assume case (g) holds:

$$\mathfrak{s} = \mathcal{G}(x) \oplus V_1 \oplus \mathcal{V}^*(x) \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus V_2.$$

This case cannot be possible, and the proof is similar to the previous one. We only note that $\mathbb{R}_0 \subset \mathrm{rad}(\mathcal{H})$ and $\mathcal{V}^*(x) \subset \mathfrak{s}$, together with the equation (4.16).

Suppose case (h) holds:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}^*(x) \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus \mathbb{R}_0^3 \oplus \mathcal{V}(x).$$

As in case (e), we can prove that $\mathcal{V}^*(x)$ (isomorphic to $\mathbb{R}^{3*}$) is an abelian ideal on $\mathfrak{s}$. Then, this case cannot happen.

Assume case (i) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus V_1 \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus V_2 \oplus \mathcal{V}^*(x).$$

Similar to case (e), and since $V_1 \simeq \mathbb{R}^3$ as $\mathfrak{sl}(3,\mathbb{R})$-module, we can prove that V_1 is an abelian ideal on $\mathfrak{s} = \mathcal{G}(x) \oplus V_1$. Therefore, this case cannot be possible.

Suppose case (j) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathbb{R}_0 \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathbb{R}_0^3 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

Being that $\mathbb{R}_0 \simeq \mathbb{R}$ as $\mathfrak{sl}(3,\mathbb{R})$-module, is clear (in this case) that $\mathbb{R}_0$ is an abelian ideal on $\mathfrak{s}$. Hence, this case cannot happen.

Assume case (k) holds:

$$\mathfrak{s} = \mathcal{G}(x) \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus \mathbb{R}_0^3 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

Similar to case (g) in Lemma 4.4 and case (e) in Lemma 4.5, here, we have that

$$\mathcal{V}(x) \oplus \mathcal{V}^*(x)$$

is a Lie subalgebra of $\mathrm{rad}(\mathcal{H})$, otherwise $\mathrm{rad}(\mathcal{H})$ could not be solvable.

Suppose case (l) holds:

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathbb{R}_0 \oplus \mathbb{R}_0^3 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \quad \text{is semisimple.}$$

Since $\mathcal{H}$ is semisimple then it is isomorphic to a direct product of a finite number of simple ideals, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$. Since every ideal is invariant under the product by $\mathcal{G}(x)$ then every ideal possess a structure of $\mathfrak{sl}(3,\mathbb{R})$-module. By properties of decomposition of $\mathcal{H}$ in irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules we have that $k \leq 5$.

If $k = 5$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_5$. Without loss of generality, reindexing if necessary, we assume $\mathfrak{h}_5 = \mathbb{R}_0$. This is a contradiction. So, this cannot be possible and $k \leq 4$.

If $k = 4$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3 \times \mathfrak{h}_4$. We assume, reindexing if necessary, that $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3$ are irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules and $\mathfrak{h}_4 = W_1 \oplus W_2$ is a direct sum of two irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules with $W_2 = \mathbb{R}_0$. Since

$$[\mathbb{R}_0, \mathcal{V}(x)] = \mathcal{V}(x)$$

then

$$\mathcal{V}(x) \subset \mathfrak{h}_4 \quad \text{and} \quad \mathfrak{h}_4 = \mathcal{V}(x) \oplus \mathbb{R}_0.$$

Being that $\mathcal{V}(x) \cong \mathbb{R}^3$ we have that $\mathcal{V}(x)$ is an ideal of $\mathfrak{h}_4$. This is a contradiction. Thus, this cannot be possible and $k \leq 3$.

If $k = 3$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3$. We can assume, as in previous cases, that $\mathfrak{h}_3$ is direct sum of two or more irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules and $\mathbb{R}_0 \oplus \mathcal{V}(x) \subsetneq \mathfrak{h}_3$. On the other hand, since by equation 4.16, $\mathcal{V}^*(x) \subset \mathfrak{h}_3$. Then

$$\mathfrak{h}_3 = \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

But, by [H, p. 518], there is not a real simple Lie algebra of dimension 7. Hence, this cannot be possible and $k \leq 2$.

If $k = 2$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2$. By previous cases, we can assume that

$$\mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \subsetneq \mathfrak{h}_2$$

On the other hand, since

$$[\mathcal{G}(x), \mathcal{W}(X)] \neq 0$$

then $\mathcal{G}(x) \subset \mathfrak{h}_2$, which implies

$$\mathfrak{h}_2 = \mathcal{G}(x) \oplus \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \quad \text{and} \quad \dim(\mathfrak{h}_1) = 3.$$

But, being that $\mathfrak{h}_1$ is a $\mathfrak{sl}(3,\mathbb{R})$-module, then $[\mathfrak{h}_1, \mathfrak{h}_1] = 0$. That is a contradiction. Hence, this cannot happen.

If $k = 1$, $\mathcal{H} = \mathcal{G}(x) \oplus \mathbb{R}_0^3 \oplus \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a real simple Lie algebra with $\dim(\mathcal{H}) = 18$. By [H, p. 516], there is not a complex simple Lie algebra with dimension equal to 9 or 18. Then, this cannot happen. Therefore, this case cannot be possible.

With $\mathcal{H}_0(x) \simeq \mathbb{R}^{3*} \oplus \mathbb{R}$ we have a similar result, we only need to know that

$$[\mathbb{R}_0^{3*}, \mathcal{V}^*(x)] = \mathcal{V}(x).$$

□

4.4 subalgebras with three submodules

4.4.1 $\mathcal{H}_0(x) \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}$ or $\mathcal{H}_0(x) \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^{3*} \oplus \mathbb{R}$

Lemma 4.10 ($\mathcal{H}_0(x) \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}$)**.** *Let S be as in Corollary 3.19. With the notation of Lemma 3.21, if $\mathcal{H}_0(x)$ is isomorphic to $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}$ as $\mathfrak{sl}(3,\mathbb{R})$-module for some $x \in S$ then $\mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0$ is a direct sum of two simple ideals and rad$(\mathcal{H}) = \mathcal{V}(x) \oplus \mathcal{V}^*(x) \oplus \mathbb{R}_0^3 \oplus \mathbb{R}_0$, with $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ a Lie subalgebra.*

Here, $\mathfrak{sl}(3,\mathbb{R})_0$, $\mathbb{R}_0^3$ and $\mathbb{R}_0$ are $\mathfrak{sl}(3,\mathbb{R})$-submodules of $\mathcal{H}_0(x)$ isomorphic to $\mathfrak{sl}(3,\mathbb{R})$, $\mathbb{R}^3$ and $\mathbb{R}$, respectively.

The same result is obtained with $\mathcal{H}_0(x) \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^{3*} \oplus \mathbb{R}$. We only need to replace the element $\mathbb{R}^3$ by $\mathbb{R}^{3*}$.

Proof. Let us choose an arbitrary but fixed element $x \in S$ such that

$$\mathcal{H}_0(x) \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R},$$

as in Lemma 3.21.

In the first place we define $\mathfrak{sl}(3,\mathbb{R})_0$, $\mathbb{R}_0^3$ and $\mathbb{R}_0$ as the Lie subalgebras of $\mathcal{H}_0(x)$ such that $\mathfrak{sl}(3,\mathbb{R})_0 \simeq \mathfrak{sl}(3,\mathbb{R})$, $\mathbb{R}_0^3 \simeq \mathbb{R}^3$ and $\mathbb{R}_0 \simeq \mathbb{R}$, as $\mathfrak{sl}(3,\mathbb{R})$-modules, via the homomorphism of Lie algebras $\lambda_x^{\perp}$ from Lemma 3.21.

As in Lemma 4.7, we have that $\mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0$ is a semisimple Lie algebra isomorphic to the direct sum

$$\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R}).$$

We choose $\mathfrak{s}$ a Levi factor of $\mathcal{H}$ that contains $\mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0$. Since the $\mathfrak{sl}(3,\mathbb{R})$-module structure of $\mathcal{H}$ is defined by the subalgebra $\mathcal{G}(x)$ and as $\mathcal{G}(x) \subset \mathfrak{s}$ then $\mathfrak{s}$ is a $\mathfrak{sl}(3,\mathbb{R})$-submodule of $\mathcal{H}$.

Let W be a $\mathfrak{sl}(3,\mathbb{R})$-submodule of $\mathcal{H}$ such that $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus W$.

Since $\mathrm{rad}(\mathcal{H})$ is an ideal, this induces the next decomposition of $\mathcal{H}$ as a direct sum of $\mathfrak{sl}(3,\mathbb{R})$-modules:

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus W \oplus \mathrm{rad}(\mathcal{H})$$

that we compare with the decomposition of irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules

$$\begin{aligned}\mathcal{H} &= \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \\ &= \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}^3_0 \oplus \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x),\end{aligned}$$

from Lemma 3.21.

By properties of representation of Lie algebras and the decomposition of $\mathcal{H}$ in irreducible modules, one of the following must occur:

(a) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}_0 \oplus \mathbb{R}^3_0 \oplus \mathcal{V}(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}^*(x)$.

(b) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}_0 \oplus V_1 \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = V_2$.

(c) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}^3_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathbb{R}_0$.

(d) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}_0 \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathbb{R}^3_0 \oplus \mathcal{V}(x)$.

(e) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}_0 \oplus V_1$ and $\mathrm{rad}(\mathcal{H}) = V_2 \oplus \mathcal{V}^*(x)$.

(f) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}^3_0 \oplus \mathcal{V}(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus \mathcal{V}^*(x)$.

(g) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus V_1 \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus V_2$.

(h) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus \mathbb{R}^3_0 \oplus \mathcal{V}(x)$.

(i) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus V_1$ and $\mathrm{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus V_2 \oplus \mathcal{V}^*(x)$.

(j) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}_0$ and $\mathrm{rad}(\mathcal{H}) = \mathbb{R}^3_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

(k) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0$ and $\mathrm{rad}(\mathcal{H}) = \mathbb{R}_0 \oplus \mathbb{R}^3_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

(l) $\mathcal{H} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}^3_0 \oplus \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is semisimple.

Where V_1 and V_2 are vector spaces isomorphic to $\mathbb{R}^3$, as $\mathfrak{sl}(3,\mathbb{R})$-modules, such that

$$V_1 \oplus V_2 = \mathbb{R}^3_0 \oplus \mathcal{V}(x).$$

Next, we analyze these cases.

The proof that **cases (a)-(j) cannot be possible** is similar to their respective cases in Lemma 4.9.

Now, assume case (k) is satisfied:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathbb{R}^3_0 \oplus \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

At the beginning of this proof we have shown that $\mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0$ is isomorphic to a direct product of simple Lie algebras, to know $\mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{sl}(3,\mathbb{R})$. Similar to case (e) in Lemma 4.5 and case (g) in Lemma 4.4, we have that $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a Lie subalgebra of $\mathrm{rad}(\mathcal{H}) = \mathbb{R}^3_0 \oplus \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

Suppose case (l) is satisfied:

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}^3_0 \oplus \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \quad \text{is semisimple.}$$

Since $\mathcal{H}$ is a semisimple Lie algebra then is isomorphic to a finite direct product of simple ideals, $\mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_j$. Being that every ideal is invariant under the Lie bracket by $\mathcal{G}(x)$ then every ideal has a structure of $\mathfrak{sl}(3,\mathbb{R})$-module. From this result and the decomposition of $\mathcal{H}$ in irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules we have that $j \leq 6$.

If $j = 6$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_6$. We assume, reindexing if necessary, that $\mathfrak{h}_6 = \mathcal{V}^*(x)$. On the other hand, $[\mathcal{V}^*(x), \mathcal{V}^*(x)]$ is isomorphic to a $\mathfrak{sl}(3,\mathbb{R})$-submodule of $\mathbb{R}^3$. Then $[\mathfrak{h}_6, \mathfrak{h}_6] = 0$, that is a contradiction. Thus, $k = 6$ cannot be possible. Therefore, $j \leq 5$.

If $j = 5$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_5$. As in $j = 6$, we assume (reindexing if necessary) that $\mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_4$ are irreducible modules and $\mathfrak{h}_5 = V_1 \oplus V_2$ is a sum of two irreducible $\mathfrak{sl}(3,\mathbb{R})$-submodules with $\mathcal{V}^*(x) \subsetneq \mathfrak{h}_5$. Since

$$[\mathbb{R}_0, \mathcal{V}^*(x)] = \mathcal{V}^*(x)$$

then $\mathbb{R}_0 \subset \mathfrak{h}_5$ and $\mathfrak{h}_5 = \mathbb{R}_0 \oplus \mathcal{V}^*(x)$. In this case $\mathcal{V}^*(x)$ is an abelian ideal of $\mathfrak{h}_5$, that is a contradiction. Thus, $k = 5$ cannot happen. Therefore, $j \leq 4$.

If $j = 4$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3 \times \mathfrak{h}_4$. We assume that $\mathfrak{h}_4$ is a direct sum of two or more irreducible $\mathfrak{sl}(3,\mathbb{R})$-submodules, even more, we can suppose that $\mathbb{R}_0 \oplus \mathcal{V}^*(x) \subsetneq \mathfrak{h}_4$. Since

$$[\mathbb{R}_0, \mathcal{V}(x)] = \mathcal{V}(x)$$

then $\mathcal{V}(x) \subset \mathfrak{h}_4$. In this case, $\mathfrak{h}_4 = \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$. By [H] there is not a real simple Lie algebra of dimension 7. Thus $j = 4$ cannot be possible. Therefore, $j \leq 3$.

If $j = 3$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3$. We can assume (as in $j = 4$) that

$$\mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \subsetneq \mathfrak{h}_3.$$

On the other hand, from Lemma 2.18,

$$[\mathbb{R}_0, \mathbb{R}^3_0] = \mathbb{R}^3_0$$

then $\mathfrak{h}_3 = \mathbb{R}^3_0 \oplus \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$. But this implies that

$$\mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 = \mathfrak{h}_1 \times \mathfrak{h}_2,$$

that is a contradiction because $0 \neq [\mathcal{W}(x), \mathcal{G}(x)]$ and

$$[\mathcal{W}(x), \mathcal{G}(x)] \subset [\mathfrak{h}_3, \mathfrak{h}_1 \times \mathfrak{h}_2] = 0.$$

Thus, $j = 3$ cannot happen. Therefore, $j \leq 2$.

If $j = 2$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2$. We assume, without loss of generality, (as in $j = 3$) that

$$\mathbb{R}_0^3 \oplus \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) \subset \mathfrak{h}_2.$$

On the other hand, by Lemma 4.2

$$0 \neq [\mathcal{W}(x), \mathfrak{sl}(3,\mathbb{R})_0]$$

then $\mathfrak{sl}(3,\mathbb{R})_0$ has projection different from zero in $\mathfrak{h}_2$. By the simplicity of $\mathfrak{sl}(3,\mathbb{R})_0$ this projection is isomorphic to $\mathfrak{sl}(3,\mathbb{R})$. Then, $\mathfrak{h}_2$ is a real simple Lie algebra of dimension 18. But this is a contradiction because, by [H, p. 518], there is not a real simple Lie algebra of dimension 18. Thus, $j = 2$ cannot be possible.

If $j = 1$, $\mathcal{H} = \mathcal{G}(x) \oplus \mathfrak{sl}(3,\mathbb{R})_0 \oplus \mathbb{R}_0^3 \oplus \mathbb{R}_0 \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a simple Lie algebra with $\dim(\mathcal{H}) = 26$. But, using the results in [H, p. 518], there is not a real simple Lie algebra of dimension 26. So, $\mathcal{H}$ as simple Lie algebra cannot happen. Therefore, case (l) cannot be possible. □

4.5 subalgebra with four submodules

Lemma 4.11 ($\mathcal{H}_0(x) \simeq \mathfrak{sl}(4,\mathbb{R})$)**.** *Let S be as in Corollary 3.19. With the notation of Lemma 3.21, if $\mathcal{H}_0(x)$ is isomorphic to $\mathfrak{so}(3,3)$ ($\simeq \mathfrak{sl}(4,\mathbb{R})$) as $\mathfrak{sl}(3,\mathbb{R})$-module for some $x \in S$, then one of the following occurs:*

(1) The radical of $\mathcal{H}$ is equal to $\mathcal{W}(x)$ and $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ is a direct sum of two simple ideals.

(2) $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{W}(x)$ is a direct sum of two simple ideals. Being one of them $\mathcal{H}_0(x) \oplus \mathcal{W}(x)$, that is isomorphic to $\mathfrak{so}(3,4)$.

Proof. Let us choose an arbitrary but fixed element $x \in S$ such that

$$\mathcal{H}_0(x) \simeq \mathfrak{sl}(4,\mathbb{R}),$$

as in Lemma 3.21.

Recall, from Lemma 3.21, that $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ is a Lie subalgebra of $\mathcal{H}$. Here, by (4.1) and (4.4), we have

$$[\mathcal{G}(x) \oplus \mathcal{H}_0(x), \mathcal{H}_0(x)] \subseteq \mathcal{H}_0(x).$$

Thus, $\mathcal{H}_0(x)$ is an ideal on $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$.

Therefore, we have the next exact sequence

$$0 \to \mathcal{H}_0(x) \to \mathcal{G}(x) \oplus \mathcal{H}_0(x) \to \mathcal{G}(x) \to 0.$$

Since $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ is a semisimple Lie algebra, the previous short exact sequence shows that the complementary ideal to $\mathcal{H}_0(x)$ in $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ is isomorphic to $\mathcal{G}(x)$. So $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ is a Lie algebra isomorphic to

$$\mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{sl}(4,\mathbb{R}).$$

This is, there is a simple ideal $\mathfrak{h}$ in $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$, isomorphic to $\mathfrak{sl}(3, \mathbb{R})$, such that

$$\mathcal{G}(x) \oplus \mathcal{H}_0(x) = \mathfrak{h} \oplus \mathcal{H}_0(x), \tag{4.18}$$

as a direct sum of simple ideals.

Choose $\mathfrak{s}$ a Levi factor of $\mathcal{H}$ that contains $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$. Since the $(\mathfrak{sl}(3, \mathbb{R}) \times \mathfrak{sl}(4, \mathbb{R}))$-module structure of $\mathcal{H}$ is defined by the subalgebra $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$, and as $\mathcal{G}(x) \oplus \mathcal{H}_0(x) \subset \mathfrak{s}$, then $\mathfrak{s}$ is a $(\mathfrak{sl}(3, \mathbb{R}) \times \mathfrak{sl}(4, \mathbb{R}))$-submodule of $\mathcal{H}$.

Let W be a $(\mathfrak{sl}(3, \mathbb{R}) \times \mathfrak{sl}(4, \mathbb{R}))$-submodule of $\mathcal{H}$ such that

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus W.$$

Since $\mathrm{rad}(\mathcal{H})$ is an ideal, this induces the next decomposition of $\mathcal{H}$ as a direct sum of $(\mathfrak{sl}(3, \mathbb{R}) \times \mathfrak{sl}(4, \mathbb{R}))$-modules:

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus W \oplus \mathrm{rad}(\mathcal{H})$$

that we compare with its decomposition in irreducible $(\mathfrak{sl}(3, \mathbb{R}) \times \mathfrak{sl}(4, \mathbb{R}))$-modules

$$\mathcal{H} = \mathfrak{h} \oplus \mathcal{H}_0(x) \oplus \mathcal{W}(x),$$

from Lemma 3.21 and (4.18).

By properties of representation of Lie algebras and the decomposition of $\mathcal{H}$ in irreducible modules, one of the following must occur:

(a) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{W}(x)$.

(b) $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is semisimple.

Next, we analyze these cases.

Assume case (a) holds:

$$\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \quad \text{and} \quad \mathrm{rad}(\mathcal{H}) = \mathcal{W}(x).$$

We have proved, in the beginning of the proof, that $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ is a direct sum of two simple ideals.

Suppose case (b) holds:

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{W}(x) \quad \text{is a semisimple Lie algebra.}$$

Since $\mathcal{H}$ is a semisimple Lie algebra then is isomorphic to a finite direct product of simple ideals, $\mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$. Here, every ideal is invariant under the Lie bracket by $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ then every ideal has a structure of $(\mathfrak{sl}(3, \mathbb{R}) \times \mathfrak{sl}(4, \mathbb{R}))$-module. From these results and the decomposition of $\mathcal{H}$ in irreducible $(\mathfrak{sl}(3, \mathbb{R}) \times \mathfrak{sl}(4, \mathbb{R}))$-modules we have that $k \leq 3$.

If $k = 3$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3$. Without loss of generality, reindexing if necessary, we can suppose that $\mathfrak{h}_3 = \mathcal{W}(x)$. Then $\mathcal{G}(x) \oplus \mathcal{H}_0(x) = \mathfrak{h}_1 \times \mathfrak{h}_2$. From here,

$$[\mathcal{H}_0(x), \mathcal{W}(x)] \subseteq [\mathfrak{h}_1 \times \mathfrak{h}_2, \mathfrak{h}_3] = 0.$$

That cannot be possible because $\mathcal{H}_0(x) \neq 0$. Therefore, $k \leq 2$.

If $k = 2$, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2$. Assume, without loss of generality and by the previous case, that $\mathfrak{h}_1$ is an irreducible $(\mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{sl}(4,\mathbb{R}))$-module and $\mathfrak{h}_2 = W_1 \oplus W_2$ is a sum of two irreducible $(\mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{sl}(4,\mathbb{R}))$-modules with $\mathcal{W}(x) = W_1$. Thus $\mathcal{G}(x) \oplus \mathcal{H}_0(x) = \mathfrak{h}_1 \oplus W_2$. Since $W_2 \subset \mathfrak{h}_2$ then $[W_2, W_2] \subset \mathfrak{h}_2$. On the other hand

$$[W_2, W_2] \subset [\mathcal{G}(x) \oplus \mathcal{H}_0(x), \mathcal{G}(x) \oplus \mathcal{H}_0(x)] \subset \mathcal{G}(x) \oplus \mathcal{H}_0(x).$$

From here, $[W_2, W_2] \subset W_2$. So, $\mathfrak{h}_1 \oplus W_2$ is a direct sum of two simple ideals of $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$. Since $\mathcal{H}_0(x)$ is an ideal on $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ then

$$\mathcal{H}_0(x) = \mathfrak{h}_1 \quad \text{or} \quad \mathcal{H}_0(x) = W_2.$$

If $\mathcal{H}_0(x) = \mathfrak{h}_1$ then

$$[\mathcal{W}(x), \mathcal{H}_0(x)] \subset [\mathfrak{h}_2, \mathfrak{h}_1] = 0,$$

that cannot be possible. Thus, $\mathcal{H}_0(x) = W_2$ and

$$F = \mathcal{H}_0(x) \oplus \mathcal{W}(x)$$

is a real simple Lie algebra of dimension 21. Then, $F^{\mathbb{C}}$ is a complex simple Lie algebra of $\dim_{\mathbb{C}}(F^{\mathbb{C}}) = 21$. Thus, by [H, p. 516], we have that $F^{\mathbb{C}} \cong \mathfrak{so}(7,\mathbb{C})$ or $F^{\mathbb{C}} \cong \mathfrak{sp}(3,\mathbb{C})$. So, F is isomorphic to one of the next non-compact real simple Lie algebras: $\mathfrak{so}(1,6)$, $\mathfrak{so}(2,5)$, $\mathfrak{so}(3,4)$, $\mathfrak{sp}(3,\mathbb{R})$ or $\mathfrak{sp}(1,2)$.

On the other hand, we recall that if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are semisimple Lie algebras with $\mathfrak{g}_1 \subseteq \mathfrak{g}_2$ then

$$\operatorname{rank}_{\mathbb{R}}(\mathfrak{g}_1) \leq \operatorname{rank}_{\mathbb{R}}(\mathfrak{g}_2).$$

From this result and since $\mathcal{H}_0(x) \simeq \mathfrak{sl}(4,\mathbb{R})$, where $\operatorname{rank}_{\mathbb{R}}(\mathfrak{sl}(4,\mathbb{R})) = 3$, we have that

$$3 \leq \operatorname{rank}_{\mathbb{R}}(F).$$

Then, by [H, p. 518], F can only be isomorphic to $\mathfrak{so}(3,4)$ or to $\mathfrak{sp}(3,\mathbb{R})$. We also recall that $\mathfrak{sp}(3,\mathbb{R})$ preserve a non-degenerate, skew-symmetric bilinear form on a 6-dimensional vector space and $\mathfrak{sl}(4,\mathbb{R}) \cong \mathfrak{so}(3,3)$. From here, we can eliminate the possibility of F isomorphic to $\mathfrak{sp}(3,\mathbb{R})$. Therefore, $F \cong \mathfrak{so}(3,4)$.

If $k = 1$, $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a real simple Lie algebra with $\dim(\mathcal{H}) = 29$. That, by results of [H], cannot be possible. Thus, this subcase cannot happen. $\square$

Chapter 5

The structure of $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ as solvable Lie algebra.

In the previous chapter we have many opportunities to observe the vector space, and $\mathfrak{sl}(3,\mathbb{R})$-module, $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ as a **solvable Lie subalgebra**. In this chapter we describe all the possible structures of $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ as a solvable Lie algebra.

These structures will be determined by the action of the $\mathfrak{sl}(3,\mathbb{R})$-submodules of $\mathfrak{so}(T_x\mathcal{O}^\perp)$.

5.1 The action of $\mathcal{G}(x)$ on $\mathcal{W}(x)$

Choose and fix an element $x \in S$, as in Corollary 3.19, that satisfies Lemma 3.21.

Recall that for $x \in S$, we have the next decomposition of $\mathcal{H}$ in $\mathfrak{sl}(3,\mathbb{R})$-modules

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{W}(x).$$

where $\mathcal{W}(x) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ with $\mathcal{V}(x) \cong \mathbb{R}^3$ and $\mathcal{V}^*(x) \cong \mathbb{R}^{3*}$ as $\mathfrak{sl}(3,\mathbb{R})$-modules.

Let X be an element of the algebra $\mathfrak{sl}(3,\mathbb{R})$ and $Z \in \mathcal{W}(x)$. From Lemma 3.12 we have that $\hat{\rho}_x(X) = \rho_x(X) + X^* \in \mathcal{G}(x)$ and

$$[\hat{\rho}_x(X), Z] \in \mathcal{H}$$

where

$$\begin{aligned}
[\hat{\rho}_x(X), Z] &= [\rho_x(X) + X^*, Z] \\
&= [\rho_x(X), Z] + [X^*, Z] \\
&= [\rho_x(X), Z] + 0 \\
&= [\rho_x(X), Z].
\end{aligned}$$

Since $\rho_x(X)\in\mathfrak{sl}(3,\mathbb{R})(x)$, by definition of the map $\lambda_x^{\perp}$ in Lemma 3.21, then

$$\lambda_x^{\perp}(X)(Z)=[\rho_x(X),Z]=[\hat{\rho}_x(X),Z]\in\mathcal{W}(x). \tag{5.1}$$

Since $\lambda_x^{\perp}$ is a non-zero and injective map when restricted to $\mathfrak{sl}(3,\mathbb{R})(x)$ we have that $\lambda_x^{\perp}(\mathfrak{sl}(3,\mathbb{R})(x))$ is homeomorphic to $\mathfrak{sl}(3,\mathbb{R})$. Then, from Lemma 3.17, we can choose $\mathcal{V}(x)$ and $\mathcal{V}^*(x)$ subspaces of $\mathcal{W}(x)$ such that:

$$[\mathcal{G}(x),\mathcal{V}(x)]\subseteq\mathcal{V}(x)\quad\text{and}\quad[\mathcal{G}(x),\mathcal{V}^*(x)]\subseteq\mathcal{V}^*(x).$$

Note that in this point it is not necessary for $\mathcal{V}(x)\oplus\mathcal{V}^*(x)$ to be an algebra.

5.2 The Lie algebra structure

First, suppose that $\mathcal{W}(x)=\mathcal{V}(x)\oplus\mathcal{V}^*(x)$ is a Lie algebra.

Thus, the Lie bracket restricted to $\mathcal{V}(x)$ gives rise to the following homomorphism of $\mathfrak{sl}(3,\mathbb{R})$-modules:

$$\begin{array}{rcl}[\cdot,\cdot]\big|_{\mathcal{V}(x)}:\mathcal{V}(x)\otimes\mathcal{V}(x) & \to & \mathcal{V}(x)\oplus\mathcal{V}^*(x)\\ v_1\otimes v_2 & \mapsto & [v_1,v_2]\end{array}$$

for every $v_1,v_2\in\mathcal{V}(x)$. On the other hand, if $[\cdot,\cdot]\big|_{\mathcal{V}(x)}$ is nonzero then we have an isomorphism

$$\begin{gathered}\mathcal{V}(x)\otimes\mathcal{V}(x)\to\mathcal{V}(x)\wedge\mathcal{V}(x)\\ v_1\otimes v_2\mapsto v_1\wedge v_2\end{gathered}$$

with $v_1,v_2\in\mathcal{V}(x)$. But $\mathcal{V}(x)\wedge\mathcal{V}(x)\cong\mathbb{R}^{3*}$ as $\mathfrak{sl}(3,\mathbb{R})$-module. From here, if $[\cdot,\cdot]\big|_{\mathcal{V}(x)}\neq 0$ then its image $\mathrm{Im}([\cdot,\cdot]\big|_{\mathcal{V}(x)})$, has projection zero on $\mathcal{V}(x)$. Otherwise, we would have an isomorphism between $\mathcal{V}(x)$ and $\mathbb{R}^{3*}$ as $\mathfrak{sl}(3,\mathbb{R})$-modules, which cannot be possible. So, $[\cdot,\cdot]\big|_{\mathcal{V}(x)}(\mathcal{V}(x)\otimes\mathcal{V}(x))\subseteq\mathcal{V}^*(x)$. From this,

$$[\mathcal{V}(x),\mathcal{V}(x)]\subseteq\mathcal{V}^*(x).$$

In a similar way, we can prove that, if $\mathcal{V}(x)\oplus\mathcal{V}^*(x)$ is a Lie algebra,

$$[\mathcal{V}^*(x),\mathcal{V}^*(x)]\subseteq\mathcal{V}(x).$$

On the other hand, the Lie bracket defines the next homomorphism:

$$\begin{array}{rcl}[\cdot,\cdot]\big|_{\mathcal{V}(x)\otimes\mathcal{V}^*(x)}:\mathcal{V}(x)\otimes\mathcal{V}^*(x) & \to & \mathcal{V}(x)\oplus\mathcal{V}^*(x)\\ v\otimes w & \mapsto & [v,w]\end{array}$$

for every $v\in\mathcal{V}(x)$ and $w\in\mathcal{V}^*(x)$.

If the homomorphism $[\cdot,\cdot]\big|_{\mathcal{V}(x)\otimes\mathcal{V}^*(x)}$ is non-zero then we have an isomorphism between $\mathrm{Im}([\cdot,\cdot]\big|_{\mathcal{V}(x)\otimes\mathcal{V}^*(x)})$ and $\mathcal{V}(x)\wedge\mathcal{V}^*(x)$. But, by [F, Ch. 13],

$$\mathcal{V}(x)\wedge\mathcal{V}^*(x)\cong\mathfrak{sl}(3,\mathbb{R})\oplus\mathbb{R},$$

as $\mathfrak{sl}(3,\mathbb{R})$-module. That cannot happen because $\mathrm{Im}([\cdot,\cdot]|_{\mathcal{V}(x)\otimes\mathcal{V}^*(x)}) \subseteq \mathcal{V}(x) \oplus \mathcal{V}^*(x)$. Then

$$[\mathcal{V}(x), \mathcal{V}^*(x)] = 0. \tag{5.2}$$

In summary, if $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a Lie algebra then

$$[\mathcal{V}(x), \mathcal{V}(x)] \subseteq \mathcal{V}^*(x), \quad [\mathcal{V}^*(x), \mathcal{V}^*(x)] \subseteq \mathcal{V}(x) \quad \text{and} \quad [\mathcal{V}(x), \mathcal{V}^*(x)] = 0. \tag{5.3}$$

If we assume that $\mathcal{W}(x) = V(x) \oplus \mathcal{V}^*(x)$ is a **solvable** Lie algebra, then the condition of solvability of $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ shows that it cannot be possible that

$$[\mathcal{V}(x), \mathcal{V}(x)] = \mathcal{V}^*(x) \quad \text{and} \quad [\mathcal{V}^*(x), \mathcal{V}^*(x)] = \mathcal{V}(x) \tag{5.4}$$

because this would contradict that $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is solvable, therefore

$$[\mathcal{V}(x), \mathcal{V}(x)] = 0 \quad \text{or} \quad [\mathcal{V}^*(x), \mathcal{V}^*(x)] = 0. \tag{5.5}$$

5.2.1 abelian.

If $[\mathcal{V}(x), \mathcal{V}(x)] = 0$ and $[\mathcal{V}^*(x), \mathcal{V}^*(x)] = 0$ then the Lie algebra

$$\mathcal{W}(x) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$$

is abelian.

Next, we analyze the structure of $\mathcal{W}(x)$ when

$$[\mathcal{V}(x), \mathcal{V}(x)] \neq 0 \quad \text{or} \quad [\mathcal{V}^*(x), \mathcal{V}^*(x)] \neq 0.$$

5.2.2 2-Step nilpotent Lie algebra.

First, assume $[\mathcal{V}(x), \mathcal{V}(x)] = \mathcal{V}^*(x)$ and $[\mathcal{V}^*(x), \mathcal{V}^*(x)] = 0$.

In this case, we have $[\mathcal{W}(x), \mathcal{W}(x)] = \mathcal{V}^*(x)$ which implies

$$[\mathcal{W}(x), [\mathcal{W}(x), \mathcal{W}(x)]] = 0, \tag{5.6}$$

that is, $\mathcal{W}(x) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a **2-Step Nilpotent Lie Algebra**.

Next, choose a basis $\{v_1, v_2, v_3\}$ of $\mathcal{V}(x)$ and an element $X \in \mathfrak{sl}(3,\mathbb{R})$ such that, from Lemma 3.21,

$$[\hat{\rho}_x(X), v_i] = [\rho_x(X), v_i] = a_i \cdot v_i$$

where $a_1 = 1$, $a_2 = 2$ and $a_3 = -3$.

Remark 5.1. We can choose non-zero elements $a_i \in \mathbb{R}$, $i = 1, 2, 3$, such that $a_i \neq a_j$ if $i \neq j$, $a_1 + a_2 + a_3 = 0$ and $2a_1 + a_2 \neq 0 \neq a_1 + 2a_2$.

On the other hand, we can find a basis $\{w_1, w_2, w_3\}$ of $\mathcal{V}^*(x)$ satisfying

$$[\hat{\rho}_x(X), w_i] = [\rho_x(X), w_i] = -a_i \cdot w_i$$

for $i = 1, 2, 3$.

In this way, since $[v_1,v_2]\in\mathcal{V}^*(x)$, then there are elements $a_{12}^1,a_{12}^2,a_{12}^3\in\mathbb{R}$ such that

$$[v_1,v_2]=a_{12}^1w_1+a_{12}^2w_2+a_{12}^3w_3.$$

Thus,

$$\begin{aligned}
[\hat\rho_x(X),[v_1,v_2]] &= [\hat\rho_x(X),a_{12}^1w_1+a_{12}^2w_2+a_{12}^3w_3]\\
&= a_{12}^1[\hat\rho_x(X),w_1]+a_{12}^2[\hat\rho_x(X),w_2]+a_{12}^3[\hat\rho_x(X),w_3]\\
&= -a_{12}^1w_1-2a_{12}^2w_2+3a_{12}^3w_3,
\end{aligned}$$

and

$$\begin{aligned}
[[\hat\rho_x(X),v_1],v_2]+[v_1,[\hat\rho_x(X),v_2]] &= [v_1,v_2]+[v_1,2v_2]\\
&= 3[v_1,v_2]\\
&= 3(a_{12}^1w_1+a_{12}^2w_2+a_{12}^3w_3).
\end{aligned}$$

But, by the Jacobi identity

$$[\hat\rho_x(X),[v_1,v_2]]=[[\hat\rho_x(X),v_1],v_2]+[v_1,[\hat\rho_x(X),v_2]].$$

Therefore

$$-a_{12}^1w_1-2a_{12}^2w_2+3a_{12}^3w_3=3(a_{12}^1w_1+a_{12}^2w_2+a_{12}^3w_3),$$

that is,

$$0=4a_{12}^1w_1+5a_{12}^2w_2.$$

From here, $a_{12}^1=a_{12}^2=0$ and $[v_1,v_2]=a_{12}w_3$, for some $a_{12}\in\mathbb{R}$.

Similarly we can prove that

$$[v_1,v_3]=a_{13}w_2\quad\text{and}\quad[v_2,v_3]=a_{23}w_1,\quad\text{with}\quad a_{13},a_{23}\in\mathbb{R}.$$

Since $[\mathcal{V}(x),\mathcal{V}(x)]=\mathcal{V}^*(x)$, then a_{12}, a_{13} and a_{23} are real numbers different from zero.

On the other hand, we can choose an element $Y_1\in\mathfrak{sl}(3,\mathbb{R})$ such that:

$$\begin{aligned}
[\hat\rho_x(Y_1),v_1]=[\rho_x(Y_1),v_1]=&v_1\\
[\hat\rho_x(Y_1),v_2]=[\rho_x(Y_1),v_2]=&v_3\\
[\hat\rho_x(Y_1),v_3]=[\rho_x(Y_1),v_3]=&-v_3,
\end{aligned}$$

and

$$\begin{aligned}
[\hat\rho_x(Y_1),w_1]=[\rho_x(Y_1),w_1]=&-w_1\\
[\hat\rho_x(Y_1),w_2]=[\rho_x(Y_1),w_2]=&0\\
[\hat\rho_x(Y_1),w_3]=[\rho_x(Y_1),w_3]=&-w_2+w_3.
\end{aligned}$$

Remark 5.2. $\lambda_x^{\perp}(Y_1)$ acts on $\{v_1, v_2, v_3\}$ as the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

on the canonical basis of $\mathbb{R}^3$.

Since $[\hat{\rho}_x(Y_1), [v_1, v_2]] = [[\hat{\rho}_x(Y_1), v_1], v_2] + [v_1, [\hat{\rho}_x(Y_1), v_2]]$ then

$$\begin{aligned} [\hat{\rho}_x(Y_1), [v_1, v_2]] &= [\hat{\rho}_x(Y_1), a_{12}w_3] \\ &= a_{12}[\hat{\rho}_x(Y_1), w_3] \\ &= a_{12}(-w_2 + w_3) \\ &= -a_{12}w_2 + a_{12}w_3 \end{aligned}$$

but,

$$\begin{aligned} [[\hat{\rho}_x(Y_1), v_1], v_2] + [v_1, [\hat{\rho}_x(Y_1), v_2]] &= [v_1, v_2] + [v_1, v_3] \\ &= a_{12}w_3 + a_{13}w_2. \end{aligned}$$

From this, $-a_{12}w_2 + a_{12}w_3 = a_{12}w_3 + a_{13}w_2$. Thus $a_{13} = -a_{12}$.

Similarly, with another $Y_2 \in \mathfrak{sl}(3, \mathbb{R})$ acting specifically, we can prove that $a_{23} = a_{12}$. Therefore, if $[\mathcal{V}(x), \mathcal{V}(x)] = \mathcal{V}^*(x)$ and $[\mathcal{V}^*(x), \mathcal{V}^*(x)] = 0$ then

$$\begin{aligned} [v_1, v_2] =& a \cdot w_3 \\ [v_1, v_3] =& - a \cdot w_2 \\ [v_2, v_3] =& a \cdot w_1 \end{aligned}$$

for some $0 \neq a \in \mathbb{R}$.

If instead of the previous case we assume now that $[\mathcal{V}^*(x), \mathcal{V}^*(x)] = \mathcal{V}(x)$ and $[\mathcal{V}(x), \mathcal{V}(x)] = 0$ then $\mathcal{W}(x) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ has, also, a structure of **2-step nilpotent Lie algebra**. Here, with a similar proof as in the previous case, we can find bases $\{\tilde{w}_1, \tilde{w}_2, \tilde{w}_3\}$ and $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ of $\mathcal{V}^*(x)$ and $\mathcal{V}(x)$, respectively, such that

$$\begin{aligned} [\tilde{w}_1, \tilde{w}_2] =& \tilde{a} \cdot \tilde{v}_3 \\ [\tilde{w}_1, \tilde{w}_3] =& - \tilde{a} \cdot \tilde{v}_2 \\ [\tilde{w}_2, \tilde{w}_3] =& \tilde{a} \cdot \tilde{v}_1 \end{aligned}$$

for some $0 \neq \tilde{a} \in \mathbb{R}$.

5.3 Structure with a non-degenerate inner product (II)

Now, we will see how the structure of inner product of $T_x\mathcal{O}^{\perp}$ and the structure of Lie algebra of $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ are related.

5.3.1 $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is an abelian Lie algebra.

Suppose the Lie algebra $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is abelian, i.e. $[v, w] = 0$ for all $v, w \in \mathcal{W}(x)$.

Also, by Lemma 2.10 and Remark 5.1, we can assume the existence of bases $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ of $\mathcal{V}(x)$ and $\mathcal{V}^*(x)$, respectively, such that:

$$\begin{aligned} \langle ev_x(v_i), ev_x(v_j)\rangle &= 0 \\ \langle ev_x(w_i), ev_x(w_j)\rangle &= 0 \\ \langle ev_x(v_i), ev_x(w_j)\rangle &= \delta_{ij} \end{aligned}$$

for all $i, j = 1, 2, 3$.

Here, because of the abelian structure, we have that for $u, v, w \in \mathcal{V}(x) \oplus \mathcal{V}^*(x)$

$$\begin{aligned} \langle ev_x([u,v]), ev_x(w)\rangle + \langle ev_x(v), ev_x([u,w])\rangle &= \langle ev_x(0), ev_x(w)\rangle \\ &\quad + \langle ev_x(v), ev_x(0)\rangle \\ &= \langle 0, ev_x(w)\rangle + \langle ev_x(v), 0\rangle \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

Therefore, when $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is an abelian Lie algebra, the inner product in $T_x\mathcal{O}^{\perp}$ is invariant under the **adjoint map** of this algebra.

5.3.2 $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ as 2-step nilpotent Lie algebra.

Assume, first that $[\mathcal{V}(x), \mathcal{V}(x)] = \mathcal{V}^*(x)$ and $[\mathcal{V}^*(x), \mathcal{V}^*(x)] = 0$. The other case is similar.

By the previous section we can find bases $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ of $\mathcal{V}(x)$ and $\mathcal{V}^*(x)$, respectively, such that:

$$\begin{aligned} [v_1, v_2] &= a \cdot w_3 \\ [v_1, v_3] &= -a \cdot w_2 \\ [v_2, v_3] &= a \cdot w_1 \end{aligned}$$

for some $0 \neq a \in \mathbb{R}$, and

$$\begin{aligned} \langle ev_x(v_i), ev_x(v_j)\rangle &= 0 \\ \langle ev_x(w_i), ev_x(w_j)\rangle &= 0 \\ \langle ev_x(v_i), ev_x(w_j)\rangle &= \delta_{ij} \end{aligned}$$

for all $i, j = 1, 2, 3$.

If $w \in \mathcal{V}^*(x)$, we have proved that $[w, z] = 0$ for all $z \in \mathcal{V}(x) \oplus \mathcal{V}^*(x)$. Then, if $z_1, z_2 \in \mathcal{V}(x) \oplus \mathcal{V}^*(x)$

$$\begin{aligned} \langle ev_x([w,z_1]), ev_x(z_2)\rangle + \langle ev_x(z_2), ev_x([w,z_2])\rangle &= \langle ev_x(0), ev_x(z_2)\rangle \\ &\quad + \langle ev_x(z_2), ev_x(0)\rangle \\ &= \langle 0, ev_x(z_2)\rangle + \langle ev_x(z_2), 0\rangle \\ &= 0. \end{aligned}$$

This is, the inner product in $T_x\mathcal{O}^\perp$ is invariant under the adjoint map restricted to $\mathcal{V}^*(x)$.

Let $0 \neq v \in \mathcal{V}(x)$, we can find an element $z \in \mathcal{V}(x)$ such that $[v, z] \neq 0$. In this case, $[v, z] \in \mathcal{V}^*(x)$. Since $\langle \mathcal{V}^*(x), \mathcal{V}^*(x)\rangle = 0$, then there is $z_2 \in \mathcal{V}(x)$, with $\langle ev_x([w, z]), ev_x(z_2)\rangle \neq 0$. So, if we want study the product in $T_x\mathcal{O}^\perp$, we need only check the result of $\langle ev_x([z_1, z_2]), ev_x(z_3)\rangle$ for elements $z_1, z_2, z_3 \in \mathcal{V}(x)$.

Note that:

$$\begin{aligned}
\langle ev_x([v_1, v_2]), ev_x(v_3)\rangle &= \langle ev_x(a \cdot w_3), ev_x(v_3)\rangle \\
&= \langle a \cdot ev_x(w_3), ev_x(v_3)\rangle \\
&= a\langle ev_x(w_3), ev_x(v_3)\rangle \\
&= a \cdot 1 \\
&= a.
\end{aligned}$$

The same is true for $\langle ev_x([v_3, v_1]), ev_x(v_2)\rangle$ and $\langle ev_x([v_2, v_3]), ev_x(v_1)\rangle$, i.e.,

$$\langle ev_x([v_3, v_1]), ev_x(v_2)\rangle = \langle ev_x([v_2, v_3]), ev_x(v_1)\rangle = a.$$

We also have

$$\begin{aligned}
&\langle ev_x([v_1, v_2]), ev_x(v_3)\rangle + \langle ev_x(v_2), ev_x([v_1, v_3])\rangle \\
&= \langle ev_x(a \cdot w_3), ev_x(v_3)\rangle + \langle ev_x(v_2), ev_x(-a \cdot w_2)\rangle \\
&= \langle a \cdot ev_x(w_3), ev_x(v_3)\rangle + \langle ev_x(v_2), (-a) \cdot ev_x(w_2)\rangle \\
&= a \cdot \langle ev_x(w_3), ev_x(v_3)\rangle + (-a) \cdot \langle ev_x(v_2), ev_x(w_2)\rangle \\
&= a \cdot 1 + (-a) \cdot 1 \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
&\langle ev_x([v_2, v_3]), ev_x(v_1)\rangle + \langle ev_x(v_3), ev_x([v_2, v_1])\rangle \\
&= \langle ev_x(a \cdot w_1), ev_x(v_1)\rangle + \langle ev_x(v_3), ev_x(-a \cdot w_3)\rangle \\
&= \langle a \cdot ev_x(w_1), ev_x(v_1)\rangle + \langle ev_x(v_3), (-a) \cdot ev_x(w_3)\rangle \\
&= a \cdot \langle ev_x(w_1), ev_x(v_1)\rangle + (-a) \cdot \langle ev_x(v_3), ev_x(w_3)\rangle \\
&= a \cdot 1 + (-a) \cdot 1 \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
&\langle ev_x([v_3, v_1]), ev_x(v_2)\rangle + \langle ev_x(v_1), ev_x([v_3, v_2])\rangle \\
&= \langle ev_x(a \cdot w_2), ev_x(v_2)\rangle + \langle ev_x(v_1), ev_x(-a \cdot w_1)\rangle \\
&= \langle a \cdot ev_x(w_2), ev_x(v_2)\rangle + \langle ev_x(v_1), (-a) \cdot ev_x(w_1)\rangle \\
&= a \cdot \langle ev_x(w_2), ev_x(v_2)\rangle + (-a) \cdot \langle ev_x(v_1), ev_x(w_1)\rangle \\
&= a \cdot 1 + (-a) \cdot 1 \\
&= 0
\end{aligned}$$

Hence, the inner product in $T_x\mathcal{O}^\perp$ is *invariant under the adjoint map* in the 2-step nilpotent Lie algebra structure of $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

In [Ov], we can find a deeper study about 2-step nilpotent Lie algebras which admit an *ad*-invariant metric. In particular, shows the existence and uniqueness of a six-dimensional 2-step nilpotent Lie algebra of co-rank zero.

5.3.2.1 Uniqueness

Suppose there are two Lie algebra structures with respect to the assumption $[\mathcal{V}(x), \mathcal{V}(x)] = \mathcal{V}^*(x)$ and $[\mathcal{V}^*(x), \mathcal{V}^*(x)] = 0$.

By the previous section, there exist two non-zero real numbers a and b such that

$$\begin{aligned} [v_1, v_2]_a &= a \cdot w_3 \\ [v_1, v_3]_a &= -a \cdot w_2 \\ [v_2, v_3]_a &= a \cdot w_1 \end{aligned}$$

and

$$\begin{aligned} [v_1, v_2]_b &= b \cdot w_3 \\ [v_1, v_3]_b &= -b \cdot w_2 \\ [v_2, v_3]_b &= b \cdot w_1 \end{aligned}$$

for some bases $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ of $\mathcal{V}(x)$ and $\mathcal{V}^*(x)$, respectively. In the first case we denote the Lie algebra as $\mathcal{W}_a(x)$ and in the second case as $\mathcal{W}_b(x)$.

Suppose also that theses basis satisfy

$$\langle v_i, v_j \rangle = \langle w_i, w_j \rangle = 0 \quad \text{and} \quad \langle v_i, w_j \rangle = \delta_{ij}$$

for $i, j = 1, 2, 3$, where $\langle u_1, u_2 \rangle := \langle ev_x(u_1), ev_x(u_2) \rangle$ for all $u_1, u_2 \in \mathcal{W}_{\{a,b\}}(x)$.

Define the map $T : \mathcal{W}_a(x) \to \mathcal{W}_b(x)$ given by

$$\begin{array}{rcl} T : \mathcal{W}_a(x) & \to & \mathcal{W}_b(x) \\ v_i & \mapsto & \sqrt[3]{\dfrac{a}{b}} v_i \\ w_j & \mapsto & \sqrt[3]{\dfrac{b}{a}} w_j \quad \forall i, j = 1, 2, 3. \end{array}$$

It is clear that this map is an isomorphism of vector spaces. Let's see if it is an isomorphism of Lie algebras. First, note that

$$\begin{array}{rcl} T([v_1, v_2]_a) & = & T(a \cdot w_3) \\ & = & a \cdot T(w_3) \\ & = & a \sqrt[3]{\dfrac{b}{a}} w_3, \end{array}$$

and

$$\begin{aligned}
[T(v_1), T(v_2)]_b &= \left[\sqrt[3]{\frac{a}{b}}v_1, \sqrt[3]{\frac{a}{b}}v_2\right]_b \\
&= \left(\sqrt[3]{\frac{a}{b}}\right)^2 [v_1, v_2]_b \\
&= \left(\sqrt[3]{\frac{a}{b}}\right)^2 b \cdot w_3 \\
&= a\sqrt[3]{\frac{b}{a}}w_3,
\end{aligned}$$

which proves $T([v_1, v_2]_a) = [T(v_1), T(v_2)]_b$. With a similar argument we can prove

$$\begin{aligned}
T([v_1, v_3]_a) &= [T(v_1), T(v_3)]_b \\
T([v_2, v_3]_a) &= [T(v_2), T(v_3)]_b.
\end{aligned}$$

On the other hand, if $w \in \mathcal{V}^*(x)$ and $u \in \mathcal{W}_a(x)$ then

$$\begin{aligned}
T([w, u]_a) &= T(0) \\
&= 0 \\
&= [T(w), T(u)]_b.
\end{aligned}$$

From here, $T : \mathcal{W}_a(x) \to \mathcal{W}_b(x)$ is an isomorphism of Lie algebras.

Now, we look at the behavior of the isomorphism T with respect to the inner product of $T_x\mathcal{O}^\perp$. For this, we need only observe the behavior in the previous bases, that is shown next:

$$\begin{aligned}
\langle T(v_i), T(v_j)\rangle &= \left\langle \sqrt[3]{\frac{a}{b}}v_i, \sqrt[3]{\frac{a}{b}}v_j \right\rangle \\
&= \left(\sqrt[3]{\frac{a}{b}}\right)^2 \langle v_i, v_j\rangle \\
&= \left(\sqrt[3]{\frac{a}{b}}\right)^2 \cdot 0 \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\langle T(w_i), T(w_j)\rangle &= \left\langle \sqrt[3]{\frac{b}{a}}v_i, \sqrt[3]{\frac{b}{a}}w_j \right\rangle \\
&= \left(\sqrt[3]{\frac{b}{a}}\right)^2 \langle w_i, w_j\rangle \\
&= \left(\sqrt[3]{\frac{b}{a}}\right)^2 \cdot 0 \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\langle T(v_i), T(w_j)\rangle &= \left\langle \sqrt[3]{\frac{a}{b}} v_i, \sqrt[3]{\frac{b}{a}} w_j \right\rangle \\
&= \left(\sqrt[3]{\frac{a}{b}}\right)\left(\sqrt[3]{\frac{b}{a}}\right)\langle v_i, w_j\rangle \\
&= 1 \cdot \delta_{ij} \\
&= \delta_{ij}.
\end{aligned}$$

for all $i, j = 1, 2, 3$.

This proves that the isomorphism T is isometric. Then, the two Lie algebras structures are isomorphic and isometric. Therefore, we have uniqueness, of Lie algebra structure, up to isometric isomorphism.

Chapter 6

Structure of the Centralizer.

In this chapter we assume M is a connected analytic pseudo-Riemannian manifold with $\dim(M) = 14$ and finite volume. We also assume that $\widetilde{SL}(3,\mathbb{R})$ acts isometric and analytically on M with a dense orbit, satisfying part b) of Remark 3.8. We study all the possible structures of the Lie algebra $\mathcal{H}$ through the analysis of Lemmas from chapter 3, which are obtained by the different possibilities of $\mathcal{H}_0$. As in the previous chapters, we use the notation in Lemma 3.21.

6.1 Properties of the $SL(3,\mathbb{R})$-action and its centralizer.

6.1.1 Properties of the centralizer.

From Chapter 5 in [OnV] we have the following definitions and results.

Let $\mathfrak{g}$ be a real semisimple Lie algebra, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a **Cartan decomposition** of $\mathfrak{g}$ and $\theta : \mathfrak{g} \to \mathfrak{g}$ a **Cartan involution** corresponding to this decomposition, this is

$$\begin{array}{rcl} \theta : \mathfrak{g} & \to & \mathfrak{g} \\ x + y & \mapsto & x - y \end{array}$$

for all $x \in \mathfrak{k}$ and $y \in \mathfrak{p}$.

Given the previous Cartan decomposition of the real semisimple Lie algebra $\mathfrak{g}$, we call a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ *canonically embedded in* $\mathfrak{g}$ *with respect to the decomposition* $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ if $\theta(\mathfrak{h}) = \mathfrak{h}$, where θ is the automorphism corresponding to the Cartan decomposition, or, equivalently, if

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p}).$$

Any semisimple Lie algebra $\mathfrak{g}$ (over $\mathbb{R}$ or $\mathbb{C}$) can be identified with the linear Lie algebra $ad(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$, over the same field. Therefore we may introduce the notion of an algebraic subalgebra of a semisimple Lie algebra. A subalgebra $\mathfrak{h}$ of a complex semisimple Lie algebra is called *algebraic subalgebra* if $ad(\mathfrak{h})$ is an algebraic linear Lie algebra. A subalgebra $\mathfrak{h}$ of a real semisimple Lie algebra $\mathfrak{g}$ is called *reductive algebraic* if $\mathfrak{h}(\mathbb{C})$ is a reductive algebraic subalgebra of a complex Lie algebra $\mathfrak{g}(\mathbb{C})$.

Here, a complex Lie algebra $\mathfrak{g}$ is called *reductive* if $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1$ where $\mathfrak{z}$ is diagonalizable (that is, $\mathfrak{z}$ is commutative and all its elements are semisimple) and $\mathfrak{g}_1$ is a semisimple ideal of $\mathfrak{g}$. For instance, any semisimple subalgebra of a semisimple Lie algebra (over $\mathbb{C}$ or $\mathbb{R}$) is reductive algebraic.

Theorem 6.1 ([OnV], Th. 4, Section 4.1.1)**.** *Any reductive algebraic subalgebra of a real semisimple Lie algebra $\mathfrak{g}$ is canonically embedded in $\mathfrak{g}$ with respect to a Cartan decomposition.*

With the definitions and results from [OnV] we have the next Lemma.

Lemma 6.2. *Suppose that $\rho : \mathfrak{sl}(3,\mathbb{R}) \to \mathfrak{g}_{2(2)}$ is an injective Lie algebra homomorphism. Then $\mathfrak{s} = \rho(\mathfrak{sl}(3,\mathbb{R}))$ is a subalgebra of $\mathfrak{g}_{2(2)}$ with its centralizer, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$, equal to zero.*

Proof. First, recall that any two Cartan involutions are conjugate. From previous theorems and the semisimplicity of $\mathfrak{sl}(3,\mathbb{R})$ and $\mathfrak{g}_{2(2)}$ we have that the Lie subalgebra $\mathfrak{s}$ is canonically embedded in $\mathfrak{g}_{2(2)}$. Then, every Cartan involution on $\mathfrak{s}$ can be extended to a Cartan involution on $\mathfrak{g}_{2(2)}$.

Let θ be a Cartan involution on $\mathfrak{g}_{2(2)}$ such that $\theta(\mathfrak{s}) = \mathfrak{s}$. The bilinear form $b_\theta(X,Y) = -K(X,\theta(Y))$ is positive definite on $\mathfrak{g}_{2(2)}$ and $\mathfrak{s}$. Let W be the b_θ-orthogonal complement of $\mathfrak{s}$. Then W is a $\mathfrak{s}$-module with $\dim(W) = 6$. Also, $b_\theta|_{W\times W}$ is non-degenerate and $\mathfrak{s}$-invariant. So $\mathfrak{g} = \mathfrak{s} \oplus W$.

Let $Y \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$, then there exist $Y_0 \in \mathfrak{s}$ and $Y_1 \in W$ such that $Y = Y_0 + Y_1$. Since

$$[X,Y] = [X, Y_0 + Y_1] = 0 \quad \forall X \in \mathfrak{s}$$

then

$$[X,Y_0] = [X,Y_1] = 0 \quad \forall X \in \mathfrak{s}$$

that with the semisimplicity of $\mathfrak{sl}(3,\mathbb{R})$ implies $Y_0 = 0$. Thus $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \subseteq W$.

Now, suppose that $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$ is a subalgebra different from zero, then the above equation shows that there exists, at least, a one-dimensional $\mathfrak{sl}(3,\mathbb{R})$-module on W. In this case, by Lemma 2.10, we will have that W is isomorphic, as $\mathfrak{sl}(3,\mathbb{R})$-module, to either

$$L^3 \oplus \mathbb{R}^3, \quad L^3 \oplus \mathbb{R}^{3*} \quad \text{or} \quad L^6$$

where L denotes the trivial representation of $\mathfrak{sl}(3,\mathbb{R})$ on $\mathbb{R}$.

We analyze all these possible cases.

If $W \cong L^3 \oplus \mathbb{R}^3$ then the Lie bracket restricted to W provides us the following homomorphisms of $\mathfrak{sl}(3,\mathbb{R})$-modules:

$$\begin{aligned} [\cdot,\cdot]|_{\wedge^2 L^3} : \wedge^2 L^3 &\to \mathfrak{g}_{2(2)} \\ u \wedge v &\mapsto [u,v], \qquad \forall \quad u \wedge v \in \wedge^2 L^3 \end{aligned}$$

$$\begin{aligned} [\cdot,\cdot]|_{L^3 \otimes \mathbb{R}^3} : L^3 \wedge \mathbb{R}^3 &\to \mathfrak{g}_{2(2)} \\ u \wedge v &\mapsto [u,v], \qquad \forall \quad u \in L^3, v \in \mathbb{R}^3 \end{aligned}$$

$$\begin{aligned} [\cdot,\cdot]|_{\wedge^2 \mathbb{R}^3} : \wedge^2 \mathbb{R}^3 &\to \mathfrak{g}_{2(2)} \\ u \wedge v &\mapsto [u,v], \qquad \forall \quad u \wedge v \in \wedge^2 \mathbb{R}^3 \end{aligned}$$

On the other hand, since

$$\begin{aligned} L^3 \wedge L^3 &\cong L^3 \\ L^3 \wedge \mathbb{R}^3 &\cong \bigoplus_{i=1}^{3} \mathbb{R}^3 \\ \mathbb{R}^3 \wedge \mathbb{R}^3 &\cong \mathbb{R}^{3*} \end{aligned}$$

as $\mathfrak{sl}(3,\mathbb{R})$-modules, then

$$\begin{aligned} &[L^3, L^3] \subset L^3 \subset W \\ &[L^3, \mathbb{R}^3] \subset \mathbb{R}^3 \subset W \\ &[\mathbb{R}^3, \mathbb{R}^3] = 0 \end{aligned}$$

this is, $[W,W] \subset W$, but this would imply that W is an ideal of $\mathfrak{g}_{2(2)}$, which is a contradiction. Then, the case $W \cong L^3 \oplus \mathbb{R}^3$ cannot be possible.

The same result is obtained if we assume that $W \cong L^3 \oplus \mathbb{R}^{3*}$ or W isomorphic to L^6. So, it cannot be possible that $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \neq 0$. $\square$

Lemma 6.3. *Suppose that G is a connected Lie group locally isomorphic to $G_{2(2)}$ and consider $\rho : \widetilde{SL}(3,\mathbb{R}) \to G$ a non trivial homomorphism of Lie groups. Then, the centralizer $Z_G(\rho(\widetilde{SL}(3,\mathbb{R})))$ of $\rho(\widetilde{SL}(3,\mathbb{R}))$ in G contains $Z(G)$ (the center of G) as a finite index subgroup.*

Proof. Let $S = \rho(\widetilde{SL}(3,\mathbb{R}))$ and denote the Lie algebra of S as $\mathfrak{s}$. Since $Z(G) \subseteq Z_G(S)$ and, by the previous Lemma, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) = 0$ then $Z_G(S)$ and $Z(G)$ are discrete. And the proof that $Z_G(S)$ is finite is a consequence of the next Lemma (Lemma 1.1.3.7 of [W]). $\square$

The Lemma below implies that $Z_G(S)$ is contained in the maximal compact subgroup of G with lie algebra $\mathfrak{k}$.

Lemma 6.4 ([W], Lemma 1.1.3.7)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$ with Cartan involution θ; let G be a connected Lie group with Lie algebra $\mathfrak{g}$, K the analytic subgroup of G corresponding to $\mathfrak{k}$; let $\widetilde{Y} = Ad(k\exp(X))Y$ where $\widetilde{Y}, Y \in \mathfrak{g}$, $k \in \mathfrak{k}$ and $X \in \mathfrak{p}$ then, if Y and $\widetilde{Y}$ are eigenvectors of θ, we have $[X, Y] = 0$.*

We prove now the relationship between the Killing form on the simple Lie group $\mathfrak{g}_{2(2)}$ and the $\mathfrak{sl}(3,\mathbb{R})$-invariant bilinear form, both obtained in $\mathfrak{sl}(3,\mathbb{R})$ and $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$. But before we need the following result.

Lemma 6.5. *There is, up to a multiple by a real scalar, exactly one $\mathfrak{sl}(3,\mathbb{R})$-invariant non-degenerate bilinear form on $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$.*

Proof. We have proved that there exists in $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ a $\mathfrak{sl}(3,\mathbb{R})$-invariant non-degenerate bilinear form. This is a consequence of the existence of an isomorphism $\rho : \mathbb{R}^3 \oplus \mathbb{R}^{3*} \to (\mathbb{R}^3 \oplus \mathbb{R}^{3*})^*$ of $\mathfrak{sl}(3,\mathbb{R})$-modules.

Then we have an isomorphism $\rho(\mathbb{C}) : \mathbb{C}^3 \oplus \mathbb{C}^{3*} \to \mathbb{C}^3 \oplus \mathbb{C}^{3*}$ of $\mathfrak{sl}(3,\mathbb{C})$-modules, that by Schur's Lemma, is just the multiple of the identity by a complex number when restricted to $\mathbb{C}^3$ and to another complex number when restricted to $\mathbb{C}^{3*}$. Furthermore, since $\rho(\mathbb{C})$ is the complexification of ρ we have that these numbers are real.

The result follows from the previous arguments and the fact that $\mathbb{R}^3$ and $\mathbb{R}^{3*}$ belong to the nullcone of the inner product. □

Lemma 6.6. *Let $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ be inner products on $\mathfrak{sl}(3,\mathbb{R})$ and $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$, respectively. If we suppose that $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are $\mathfrak{sl}(3,\mathbb{R})$-invariant. Then, there exist $a_1, a_2 \in \mathbb{R}$ such that $a_1\langle\cdot,\cdot\rangle_1 + a_2\langle\cdot,\cdot\rangle_2$ is the Killing form of $\mathfrak{g}_{2(2)}$.*

Proof. Recall that Schur's Lemma implies that in $\mathfrak{g}$, a simple real Lie algebra with a simple complexification, any $\mathfrak{g}$-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$ is unique up to a multiple, that is, the multiple by a scalar of the Killing form.

In particular, we have that $\langle\cdot,\cdot\rangle_1$ is a multiple of the Killing form of $\mathfrak{g}_{2(2)}$ (K) when restricted to $\mathfrak{sl}(3,\mathbb{R})$, this is $\langle X, Y\rangle_1 = c_1 K|_{\mathfrak{sl}(3,\mathbb{R})}(X,Y)$ for all $X, Y \in \mathfrak{sl}(3,\mathbb{R})$ and some non-zero $c_1 \in \mathbb{R}$.

On the other hand, from Lemma 6.5 we have the existence of a non-zero scalar $c_2 \in \mathbb{R}$ such that $\langle\cdot,\cdot\rangle_2 = c_2 K|_{\mathbb{R}^3\oplus\mathbb{R}^{3*}}(\cdot,\cdot)$.

Now, the result is a consequence of the previous arguments and Lemma 6.5. □

6.1.2 Integrability and Weak-irreducibility

In the proof of Lemma 3.17 we assumed that part *b*) of Remark 3.8 is satisfied. This assumption can be obtained if we assume that the bundle $T\mathcal{O}^\perp$, in the manifold M, is non-integrable.

The case when the normal bundle, $T\mathcal{O}^\perp$, is integrable is extensively studied by Quiroga-Barranco in [Q]. The following result is a particular case of Theorem 1.1 in said article, which also appears in [OQ] as Proposition 15.

Proposition 6.7. *Let G be a connected non-compact simple Lie group acting isometrically on a connected complete finite volume pseudo-Riemannian manifold M. If the tangent bundle to the orbits $T\mathcal{O}$ has non-degenerate fibers and the bundle $T\mathcal{O}^\perp$ is integrable, then there is an isometric covering map*

$$\widetilde{G} \times N \to M$$

where the domain has the product metric for a bi-invariant metric on $\widetilde{G}$ and with N a complete pseudo-Riemannian manifold.

This result enables us to find a lower bound for the dimension of the manifold M when the bundle $T\mathcal{O}^\perp$ is non-integrable, which is Proposition 1.6 in [OQ]. Its proof is based in the existence of a conull subset S such that if $x \in S$ then the vector space $T_x\mathcal{O}^\perp$ is a non-trivial $\mathfrak{g}$-module. The following result is based in such Proposition (1.6 in [OQ]) but with the assumption that S has positive measure.

Proposition 6.8. *Let M be a connected analytic pseudo-Riemannian manifold and G a connected non-compact simple Lie group. Suppose that M is complete, satisfies part b) of Remark 3.8, has finite volume and admits an analytic isometric non-transitive G-action with a dense orbit. Then:*

$$\dim(M) \geq \dim(G) + m(\mathfrak{g})$$

where $m(\mathfrak{g})$ is the dimension of the smallest non-trivial representation of $\mathfrak{g}$ (Lie algebra of G) that admits an invariant non-degenerate symmetric bilinear form.

Proof. Suppose $\dim(M) < \dim(G) + m(\mathfrak{g})$. Since $m(\mathfrak{g}) \leq \dim(G)$ (the Killing form defines an inner product), by Lemma 3.11 the bundle $T\mathcal{O}^\perp$ has non-degenerate fibers with dimension $< m(\mathfrak{g})$. Hence, the definition of $m(\mathfrak{g})$ implies that $T_x\mathcal{O}^\perp$ is a trivial $\mathfrak{g}$-module for the structure defined in Proposition 3.7(4). Since this is true for almost every element in S, contradicts parts b) in Remark 3.8. □

Recall that a connected pseudo-Riemannian manifold is **weakly irreducible** if the tangent space at some point has no proper non-degenerate subspaces invariant under the restricted holonomy group at that point.

Remark 6.9. As a consequence of the previous definition and Proposition 6.7, we have that if G is a connected non-compact simple Lie group acting isometrically on a weakly irreducible connected complete finite volume pseudo-Riemannian manifold M such that the tangent bundle to the orbits $T\mathcal{O}$ has non-degenerate fibers then the bundle $T\mathcal{O}^\perp$ is non-integrable.

6.2 The radical of $\mathcal{H}$ is nonzero.

Combining conclusions from the Lemmas in Chapter 4 we have the following result.

Lemma 6.10. *Let S be as in Corollary 3.19. With the notation from Lemma 3.21. If $x \in S$, then one of the following occurs:*

1) $rad(\mathcal{H}) \neq 0$ and $\mathcal{W}(x) = \mathcal{V}(x) \oplus \mathcal{V}^*(x) \subseteq rad(\mathcal{H})$ is a Lie subalgebra,

2) $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{W}(x)$ is a semisimple Lie algebra.

In the first case ($\mathcal{W}(x) \subseteq \mathrm{rad}(\mathcal{H})$) the structure of algebra of $\mathcal{W}(x)$ gives rise to a series of consequences, that we have shown in Chapter 5, such as equations (5.2) and (5.3) that we rewrite here

$$[\mathcal{V}(x), \mathcal{V}^*(x)] = 0, \quad [\mathcal{V}(x), \mathcal{V}(x)] \subseteq \mathcal{V}^*(x) \quad \text{and} \quad [\mathcal{V}^*(x), \mathcal{V}^*(x)] \subseteq \mathcal{V}(x).$$

Remark 6.11. This result together with the solvability of $\mathrm{rad}(\mathcal{H})$ implies that the only cases to consider in part 1) are:

(i) $\mathcal{W}(x)$ is abelian.

(ii) $[\mathcal{V}(x), \mathcal{V}(x)] = \mathcal{V}^*(x)$ and $[\mathcal{V}^*(x), \mathcal{V}^*(x)] = 0$.

(iii) $[\mathcal{V}^*(x), \mathcal{V}^*(x)] = \mathcal{V}(x)$ and $[\mathcal{V}(x), \mathcal{V}(x)] = 0$.

Which appear as equations (5.4) and (5.5) in Chapter 5.

Note, that in the two latter cases we have, by (5.6), that $\mathcal{W}(x)$ satisfies

$$[\mathcal{W}(x), [\mathcal{W}(x), \mathcal{W}(x)]] = 0.$$

Therefore, we have that $\mathcal{W}(x)$ is a **2-step nilpotent Lie algebra**.

In this section, we analyze the subcases that appear in Remark 6.11. In this case we take a suitable fixed element $x_0 \in \widetilde{M}$ which satisfies Lemma 3.21 and such that $\mathrm{rad}(\mathcal{H}) \neq 0$.

From Lemma 6.10 and Remark 6.11, when $\mathcal{W}(x_0) \subseteq \mathrm{rad}(\mathcal{H})$ we have that $\mathcal{W}(x_0)$ is a 2-Step nilpotent or abelian Lie algebra.

6.2.1 $\mathcal{W}(x_0)$ is abelian.

Suppose the Lie subalgebra $\mathcal{W}(x_0) \subseteq \mathrm{rad}(\mathcal{H})$ is abelian.

Here, $\mathcal{G}(x_0) \oplus \mathcal{W}(x_0)$ is a Lie subalgebra of $\mathcal{H}$ with the Lie bracket given as

$$[\mathcal{G}(x_0), \mathcal{W}(x_0)] = \mathcal{W}(x_0) \quad \text{and} \quad [\mathcal{W}(x_0), \mathcal{W}(x_0)] = 0.$$

In this case $\mathcal{G}(x_0) \oplus \mathcal{W}(x_0)$ is a Lie algebra isomorphic to $\mathfrak{sl}(3, \mathbb{R}) \ltimes V$, where

$$V = \mathbb{R}^3 \oplus \mathbb{R}^{3*}$$

with a structure of abelian Lie algebra compatible with its structure as $\mathfrak{sl}(3, \mathbb{R})$-module.

Let us choose

$$\psi : \mathfrak{sl}(3, \mathbb{R}) \ltimes V \to \mathcal{G}(x_0) \oplus \mathcal{W}(x_0)$$

an isomorphism of Lie algebras such that

$$\psi(\mathfrak{sl}(3,\mathbb{R})) = \mathcal{G}(x_0) \quad \text{and} \quad \psi(V) = \mathcal{W}(x_0).$$

Let us denote, also, as $\widetilde{SL}(3,\mathbb{R}) \ltimes V$ the Lie group structure on $\widetilde{SL}(3,\mathbb{R}) \times V$ with the semidirect product given by:

$$(A, v) \cdot (B, w) = (AB, B^{-1}v + w),$$

where we use the representation of $\widetilde{SL}(3,\mathbb{R})$ on the vector space V induced by the representation of $\mathfrak{sl}(3,\mathbb{R})$. Note that the Lie algebra of $\widetilde{SL}(3,\mathbb{R}) \ltimes V$ is $\mathfrak{sl}(3,\mathbb{R}) \ltimes V$. Then, by Lemma 3.16, there is an analytic isometric right action of $\widetilde{SL}(3,\mathbb{R}) \ltimes V$ on $\widetilde{M}$ such that

$$\psi(X) = X^*$$

for every $X \in \mathfrak{sl}(3,\mathbb{R}) \ltimes V$.

Since $\mathcal{H}$ centralizes the left $\widetilde{SL}(3,\mathbb{R})$-action on $\widetilde{M}$ then we have that the right $(\widetilde{SL}(3,\mathbb{R}) \ltimes V)$-action centralizes the left $\widetilde{SL}(3,\mathbb{R})$-action. And, for the same reason, preserves $T\mathcal{O}$ and $T\mathcal{O}^\perp$.

On the other hand, using the right $(\widetilde{SL}(3,\mathbb{R}) \ltimes V)$-action on $\widetilde{M}$, for the element $x_0 \in \widetilde{M}$, previously chosen, we consider the next map:

$$\begin{array}{rcl} f : \widetilde{SL}(3,\mathbb{R}) \ltimes V & \rightarrow & \widetilde{M} \\ h & \mapsto & x_0 h \end{array}$$

for all $h \in \widetilde{SL}(3,\mathbb{R}) \ltimes V$. It is clear, that this action is $(\widetilde{SL}(3,\mathbb{R}) \ltimes V)$-equivariant by the right action on its domain.

If $\bar{e}$ denotes the identity element in the simple Lie group $\widetilde{SL}(3,\mathbb{R})$, then

$$\begin{array}{c} df_{(\bar{e},0)} : \mathfrak{sl}(3,\mathbb{R}) \ltimes V \to \mathcal{G}(x) \oplus \mathcal{W}(x) \to T_{x_0}\widetilde{M} \\ X \mapsto X^* \mapsto X^*_{x_0}. \end{array}$$

Since $\psi(X) = X^*$ for all $X \in \mathfrak{sl}(3,\mathbb{R}) \ltimes V$, then from Lemma 3.21, $df_{(\bar{e},0)}$ maps $\mathfrak{sl}(3,\mathbb{R})$ onto $T_{x_0}\mathcal{O}$ and V onto $T_{x_0}\mathcal{O}^\perp$. Therefore, f is a local diffeomorphism at $(\bar{e}, 0)$.

For every $v_0 \in V$, let R_{v_0} denote the map on $\widetilde{SL}(3,\mathbb{R}) \ltimes V$ and on $\widetilde{M}$ given by the correspondence

$$x \mapsto x(\bar{e}, v_0).$$

Since the subgroup V is abelian, it is not hard to prove the next result

$$\begin{array}{rcl} (dR_{v_0})_{(\bar{e},v)} : T_{(\bar{e},v)}(\widetilde{SL}(3,\mathbb{R}) \ltimes V) & \rightarrow & T_{(\bar{e},v+v_0)}(\widetilde{SL}(3,\mathbb{R}) \ltimes V) \\ (X, Y_v) & \mapsto & (X, Y_{v+v_0}), \end{array}$$

for every $v \in T_{x_0}P$.

Since V is a subgroup of $\widetilde{SL}(3,\mathbb{R}) \ltimes V$ then $R_{v_0}(V) = V$.

Let $P = f(\bar{e} \times V)$, which defines a submanifold of $\widetilde{M}$ in a neighborhood of $x_0 = f(\bar{e}, 0)$. Here, from the previous remarks, we have

$$T_{x_0}P = df_{(\bar{e},0)}\big(T_{(\bar{e},0)}(\bar{e} \times V)\big) = T_{x_0}\mathcal{O}^{\perp}. \tag{6.1}$$

From the equivariant property of the map f and (6.1), if $(\bar{e}, v_0) \in e \times V$ then

$$\begin{aligned}
T_{f(\bar{e},v_0)}P &= df_{(\bar{e},v_0)}\big(T_{(\bar{e},v_0)}(\bar{e} \times V)\big) \\
&= df_{(\bar{e},v_0)}\big((dR_{v_0})_{(\bar{e},0)}(T_{(\bar{e},0)}(\bar{e} \times V))\big) \\
&= d(f \circ R_{v_0})_{(\bar{e},0)}\big(T_{(\bar{e},0)}(\bar{e} \times V)\big) \\
&= d(R_{v_0} \circ f)_{(\bar{e},0)}\big(T_{(\bar{e},0)}(\bar{e} \times V)\big) \\
&= (dR_{v_0})_{f(\bar{e},0)}\big(df_{(\bar{e},0)}T_{(\bar{e},0)}(\bar{e} \times V)\big) \\
&= (dR_{v_0})_{f(\bar{e},0)}T_{x_0}P \\
&= (dR_{v_0})_{f(\bar{e},0)}T_{x_0}\mathcal{O}^{\perp} \\
&= T_{R_{v_0}f(\bar{e},0)}T_{x_0}\mathcal{O}^{\perp} \\
&= T_{f(\bar{e},v_0)}T_{x_0}\mathcal{O}^{\perp}.
\end{aligned}$$

Hence, P is an integral submanifold of $T\mathcal{O}^{\perp}$ passing through $x_0 = f(\bar{e}, 0)$.

On the other hand, from the left $\widetilde{SL}(3,\mathbb{R})$-action on $\widetilde{M}$ we obtain, by restriction to P, a map

$$\begin{aligned}
\phi : \widetilde{SL}(3,\mathbb{R}) \times P &\rightarrow \widetilde{M} \\
(g, x) &\mapsto g \cdot x,
\end{aligned}$$

whose differential at (e, x_0) is given by

$$\begin{aligned}
d\phi_{(e,x_0)} : T_{(e,x_0)}\big(\widetilde{SL}(3,\mathbb{R}) \times P\big) &\rightarrow T_{x_0}\widetilde{M} \\
X + w &\mapsto X^*_{x_0} + w,
\end{aligned}$$

with $X \in \mathfrak{sl}(3,\mathbb{R})$ and $w \in V$. Note that this is an isomorphism. Hence the map ϕ is a diffeomorphism from a neighborhood of (e, x_0) onto a neighborhood of x_0.

Since the left $\widetilde{SL}(3,\mathbb{R})$-action preserves the bundles $T\mathcal{O}$ and $T\mathcal{O}^{\perp}$ then there is an integral manifold of $T\mathcal{O}^{\perp}$ passing through every point in a neighborhood of x_0 in $\widetilde{M}$. Therefore the tensor Ω in Lemma 3.9 vanishes in a neighborhood of x_0. Then, from Remark 3.10, $\Omega \equiv 0$ and $T\mathcal{O}^{\perp}$ is integrable in $\widetilde{M}$.

On the other hand, the local diffeomorphism f induces a metric tensor in the abelian Lie group V which is right-invariant and, by Subsection 5.3.1, $ad(V)$-invariant. Since V is connected, the metric tensor in the abelian Lie group V is $Ad(V)$-invariant. Then, by Proposition 9 of [ONe, p. 304], the geodesics of V starting at 0 are the one-parameter subgroups of V. Therefore we have that V is a complete manifold.

Thus, from the previous arguments we have the following result.

Proposition 6.12. *Let M and $\widetilde{SL}(3,\mathbb{R})$ satisfy the same hypotheses as in Theorem 1.6. If for some $x_0 \in \widetilde{M}$ we have that, in the decomposition of $\mathcal{H}$ (Lemma 3.21), $\mathcal{W}(x_0) \subseteq rad(\mathcal{H})$ is an abelian Lie algebra then $T\mathcal{O}^{\perp}$ is integrable and the pseudo-Riemannian manifold $\widetilde{N}$, in Theorem 1.7, is diffeomorphic to $\mathbb{R}^3 \times \mathbb{R}^{3*}$, an abelian Lie group.*

6.2.2 $[\mathcal{V}(x_0), \mathcal{V}(x_0)] \neq 0$.

In this subsection we assume, by Remark 6.11, that the Lie algebra $\mathcal{W}(x_0)$ satisfies $[\mathcal{W}(x_0), \mathcal{W}(x_0)] \neq 0$. Thus, $\mathcal{W}(x_0)$ is a 2-step nilpotent Lie algebra. Assume first $[\mathcal{V}(x), \mathcal{V}(x)] \neq 0$, this is,

$$[\mathcal{V}(x), \mathcal{V}(x)] = \mathcal{V}^*(x) \quad \text{and} \quad [\mathcal{V}^*(x), \mathcal{V}^*(x)] = 0.$$

Here, $\mathcal{G}(x_0) \oplus \mathcal{W}(x_0)$ is a Lie subalgebra of $\mathcal{H}$ with Lie brackets given by

$$[\mathcal{G}(x_0), \mathcal{W}(x_0)] = \mathcal{W}(x_0) \quad \text{and} \quad [\mathcal{W}(x_0), \mathcal{W}(x_0)] = \mathcal{V}^*(x_0).$$

Then, $\mathcal{G}(x_0) \oplus \mathcal{W}(x_0)$ is a Lie algebra isomorphic to $\mathfrak{sl}(3,\mathbb{R}) \ltimes V_1$ where

$$V_1 = \mathbb{R}^3 \oplus \mathbb{R}^{3*}$$

is a Lie algebra given by $[\mathbb{R}^3, \mathbb{R}^3] = \mathbb{R}^{3*}$, as $\mathfrak{sl}(3,\mathbb{R})$-module.

Let us choose

$$\psi_1 : \mathfrak{sl}(3,\mathbb{R}) \ltimes V_1 \to \mathcal{G}(x_0) \oplus \mathcal{W}(x_0)$$

an isomorphism of Lie algebras such that

$$\psi_1(\mathfrak{sl}(3,\mathbb{R})) = \mathcal{G}(x_0) \quad \text{and} \quad \psi_1(V_1) = \mathcal{W}(x_0).$$

Let $\widetilde{V}_1$ be a simply connected group such that $\mathrm{Lie}(\widetilde{V}_1) = V_1$. We denote by $\widetilde{SL}(3,\mathbb{R}) \ltimes \widetilde{V}_1$ the Lie group structure on $\widetilde{SL}(3,\mathbb{R}) \times \widetilde{V}_1$ with the semidirect product given by:

$$(A, v) \cdot (B, w) = (AB, B^{-1}v \cdot w).$$

Here, we use the representation of $\widetilde{SL}(3,\mathbb{R})$ on $\widetilde{V}_1$ induced by the representation of $\mathfrak{sl}(3,\mathbb{R})$ and the product in $\widetilde{V}_1$. Thus, the Lie algebra of $\widetilde{SL}(3,\mathbb{R}) \ltimes \widetilde{V}_1$ is $\mathfrak{sl}(3,\mathbb{R}) \ltimes V_1$.

By Lemma 3.16, there is an analytic isometric right action of $\widetilde{SL}(3,\mathbb{R}) \ltimes \widetilde{V}_1$ on $\widetilde{M}$ such that

$$\psi(X) = X^*$$

for all $X \in \widetilde{SL}(3,\mathbb{R}) \ltimes \widetilde{V}_1$. Since $\mathcal{H}$ centralizes the left action of $\widetilde{SL}(3,\mathbb{R})$ on $\widetilde{M}$ then the right $(\widetilde{SL}(3,\mathbb{R}) \ltimes \widetilde{V}_1)$-action centralizes the left $\widetilde{SL}(3,\mathbb{R})$-action. And preserves $T\mathcal{O}$ and $T\mathcal{O}^{\perp}$.

With the right $(\widetilde{SL}(3,\mathbb{R}) \ltimes \widetilde{V}_1)$-action on $\widetilde{M}$, for the element $x_0 \in \widetilde{M}$, we consider the next map:

$$\begin{array}{rcl} f_1 : \widetilde{SL}(3,\mathbb{R}) \ltimes \widetilde{V}_1 & \to & \widetilde{M} \\ h & \mapsto & x_0 h \end{array}$$

for every $h \in \widetilde{SL}(3,\mathbb{R}) \ltimes \widetilde{V}_1$. Note that this action is $(\widetilde{SL}(3,\mathbb{R}) \ltimes \widetilde{V}_1)$-equivariant on the right on its domain.

If $\bar{e}$ denotes the element identity in $\widetilde{SL}(3,\mathbb{R})$, we have

$$\begin{array}{rccl} df_{1(\bar{e},0)} : \mathfrak{sl}(3,\mathbb{R}) \ltimes V_1 \to & \mathcal{G}(x_0) \oplus \mathcal{W}(x_0) & \to T_{x_0}\widetilde{M} \\ X \mapsto & X^* & \mapsto X^*_{x_0}. \end{array}$$

Since $\psi(X) = X^*$ for all $X \in \mathfrak{sl}(3,\mathbb{R}) \ltimes V_1$ then, by Lemma 3.21, we have that $df_{1(\bar{e},0)}$ maps $\mathfrak{sl}(3,\mathbb{R})$ onto $T_{x_0}\mathcal{O}$ and V_1 onto $T_{x_0}\mathcal{O}^\perp$. From this, f_1 is a local diffeomorphism at $(\bar{e}, 0)$.

Therefore, considering $P = f_1(\bar{e} \times \widetilde{V}_1)$ and the equivariance of the map f_1, similarly to the abelian case, we can prove that $T\mathcal{O}^\perp$ is integrable in $\widetilde{M}$.

On the other hand, the local diffeomorphism f_1 induces a metric tensor in the 2-step nilpotent Lie group $\widetilde{V}_1$ which is right-invariant and, by Subsection 5.3.2, $ad(V_1)$-invariant. Since $\widetilde{V}_1$ is connected, the metric tensor in $\widetilde{V}_1$ is $Ad(\widetilde{V}_1)$-invariant. Then, by Proposition 9 of [ONe, p. 304], the geodesics of $\widetilde{V}_1$ starting at 0 are the one-parameter subgroups of $\widetilde{V}_1$. Therefore, we have that $\widetilde{V}_1$ is a *complete* Lie group.

From the previous results and the completeness of $\widetilde{V}_1$, we have the following result.

Proposition 6.13. *Let M and $\widetilde{SL}(3,\mathbb{R})$ satisfy the same hypotheses as in Theorem 1.6. If for some $x_0 \in \widetilde{M}$ we have that, in the decomposition of $\mathcal{H}$ (Lemma 3.21), $\mathcal{W}(x_0) \subseteq rad(\mathcal{H})$ is a 2-step nilpotent Lie algebra (Remark 6.11) then $T\mathcal{O}^\perp$ is integrable and the pseudo-Riemannian manifold $\widetilde{N}$, in Theorem 1.7, is diffeomorphic to the 2-step nilpotent Lie group $\mathbb{R}^3 \times \mathbb{R}^{3*}$.*

When $[\mathcal{V}^*(x_0), \mathcal{V}^*(x_0)] \neq 0$, we have a similar result, where the only difference lies in the structure of $\mathbb{R}^3 \times \mathbb{R}^{3*}$ as a 2-step nilpotent Lie group.

6.3 $\mathcal{H}$ is semisimple

In the previous section we analyzed the structure of the manifold $\widetilde{M}$ when, in the decomposition of $\mathcal{H}$, we have that $\mathrm{rad}(\mathcal{H}) \neq 0$. To continue with our analysis, we now consider the case when $\mathrm{rad}(\mathcal{H}) = 0$.

First, choose and fix a suitable element $x_0 \in \widetilde{M}$ such that Lemma 3.21 is satisfied and, in the decomposition of $\mathcal{H}$, $\mathrm{rad}(\mathcal{H}) = 0$. Then, $\mathcal{H}$ is a semisimple Lie algebra.

From the Lemmas in Chapter 4 we have two possible cases for the structure of $\mathcal{H}$: either $\mathcal{H}$ is a simple Lie algebra or it is a direct sum of two simple ideals.

Let us take a look at the latter case.

6.3.1 $\mathcal{H}$ is a direct sum of two simple Lie algebras.

In this case, from the Lemmas in Chapter 4, $\mathcal{H}_0(x_0) \oplus \mathcal{W}(x_0)$ is a simple Lie algebra and an ideal of $\mathcal{H}$.

Since $\mathcal{G}(x_0) \cong \mathfrak{sl}(3,\mathbb{R})$, let $\mathfrak{h}_2$ be a simple Lie algebra such that

$$\psi : \mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{h}_2 \to \mathcal{H} \tag{6.2}$$

is an isomorphism of Lie algebras with $\psi(\mathfrak{h}_2) = \mathcal{H}_0(x_0) \oplus \mathcal{W}(x_0)$.

We now consider the possibilities according to Chapter 4.

6.3.1.1 $\mathcal{H}_0(x_0) \simeq \mathfrak{sl}(3,\mathbb{R})$.

By Lemma 4.6, a possible structure for $\mathcal{H}$ is that $\mathcal{H}_0(x_0) \oplus \mathcal{W}(x_0)$ is a simple ideal of dimension 14, furthermore

$$\mathcal{H}_0(x_0) \oplus \mathcal{W}(x_0) \cong \mathfrak{g}_{2(2)}.$$

If the previous case is satisfied then the isomorphism ψ has as domain $\mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{g}_{2(2)}$ with

$$\psi(\mathfrak{g}_{2(2)}) = \mathcal{H}_0(x_0) \oplus \mathcal{W}(x_0).$$

Let $\widetilde{SL}(3,\mathbb{R})$ and $G_{2(2)}$ be simply connected Lie groups with Lie algebras $\mathfrak{sl}(3,\mathbb{R})$ and $\mathfrak{g}_{2(2)}$, respectively. By Lemma 3.16, there exists an analytic isometric right $(\widetilde{SL}(3,\mathbb{R}) \times G_{2(2)})$-action on $\widetilde{M}$ such that $\psi(X) = X^*$ for all $X \in \mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{g}_{2(2)}$. This right action centralizes the left $\widetilde{SL}(3,\mathbb{R})$-action on $\widetilde{M}$ and preserves the bundles $T\mathcal{O}$ and $T\mathcal{O}^\perp$.

Given the previous right action, we define the map:

$$\begin{array}{rcl} f : \widetilde{SL}(3,\mathbb{R}) \times G_{2(2)} & \to & \widetilde{M} \\ (h_1, h_2) & \mapsto & x_0(h_1, h_2) \end{array}$$

for all $(h_1, h_2) \in (\widetilde{SL}(3,\mathbb{R}) \times G_{2(2)})$. Observe that this map is $(\widetilde{SL}(3,\mathbb{R}) \times G_{2(2)})$-equivariant for the right action on its domain. Also, note that

$$df_{(e_s,e_g)}(X) = X^*_{x_0} = \psi(X)$$

for all $X \in \mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{g}_{2(2)}$, where e_s and e_g are the identity elements in $\widetilde{SL}(3,\mathbb{R})$ and $G_{2(2)}$, respectively. Then, $df_{(e_s,e_g)}$ is surjective and, by Lemma 3.21,

$$\ker(df_{(e_s,e_g)}) = \psi^{-1}(\mathcal{H}_0(x_0)).$$

Here, by our choice of $\mathfrak{g}_{2(2)}$ and Lemma 3.21 we have that $df_{(e_s,e_g)}(\mathfrak{g}_{2(2)}) = T_{x_0}\mathcal{O}^\perp$ and we claim that $df_{(e_s,e_g)}(\mathfrak{sl}(3,\mathbb{R})) = T_{x_0}\mathcal{O}$. Since $\mathcal{G}(x_0)$ and $\psi(\mathfrak{sl}(3,\mathbb{R}))$ are complementary to $\mathcal{H}_0(x_0) \oplus \mathcal{W}(x_0)$ in $\mathcal{H}$ then $\mathcal{G}(x_0) \cong \psi(\mathfrak{sl}(3,\mathbb{R}))$ as $\mathfrak{sl}(3,\mathbb{R})$-module. From the evaluation map, we have that $T_{x_0}\mathcal{O} \subseteq ev_{x_0}(\psi(\mathfrak{sl}(3,\mathbb{R})))$. On the other hand, by properties of the map ev_{x_0} and since $ev_{x_0}(\mathcal{G}(x_0)) = T_{x_0}\mathcal{O}$ then

$$T_{x_0}\mathcal{O} \cong \mathcal{G}(x_0) \quad \text{and} \quad \mathcal{G}(x_0) \ncong T_{x_0}\mathcal{O}^\perp,$$

as $\mathfrak{sl}(3,\mathbb{R})$-modules. Therefore, we have that $df_{(e_s,e_g)}(\mathfrak{sl}(3,\mathbb{R})) = T_{x_0}\mathcal{O}$.

Let H be the connected subgroup of $G_{2(2)}$ such that

$$\mathrm{Lie}(H) = \psi^{-1}(\mathcal{H}_0(x_0)). \tag{6.3}$$

In this case, $\mathrm{Lie}(H)$ is isomorphic to $\mathfrak{sl}(3,\mathbb{R})$. Since $\mathfrak{sl}(3,\mathbb{R})$ is a simple Lie algebra and $G_{2(2)}$ a simply connected Lie group, by exercise 4(ii) in [H, p. 152], H is a closed subgroup of $G_{2(2)}$. Therefore, the map

$$\begin{aligned} \hat{f} : \widetilde{SL}(3,\mathbb{R}) \times H\backslash G_{2(2)} &\to \widetilde{M} && (6.4)\\ (h_1, Hh_2) &\mapsto x_0(h_1,h_2) && (6.5) \end{aligned}$$

is well defined and an analytic $(\widetilde{SL}(3,\mathbb{R}) \times G_{2(2)})$-equivariant map between manifolds. From the properties of $df_{(e_s,e_g)}$, $\hat{f}$ is a local diffeomorphism at (e_s, He_g).

Considering $P = \hat{f}(e_s \times H\backslash G_{2(2)})$ and using the equivariance of $\hat{f}$, we can prove, with similar arguments to those when $\mathcal{W}(x_0)$ is abelian, that $T\mathcal{O}^\perp$ is integrable in $\widetilde{M}$.

Thus, from the previous arguments we have the following result.

Proposition 6.14. *Let M and $\widetilde{SL}(3,\mathbb{R})$ satisfy the same hypotheses as in Theorem 1.6. If for some $x_0 \in \widetilde{M}$ we have that $\mathcal{H}$ (Lemma 3.21) is a semisimple Lie algebra isomorphic to $\mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{g}_{2(2)}$ (Lemma 4.6) then $T\mathcal{O}^\perp$ is integrable and the pseudo-Riemannian manifold $\widetilde{N}$, in Theorem 1.7, contains an open subset diffeomorphic to the quotient space $SL(3,\mathbb{R})\backslash G_{2(2)}$.*

6.3.1.2 $\mathcal{H}_0(x_0) \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}$.

From Lemma 4.8 we have that a possible structure for $\mathcal{H}$ is to be isomorphic to a direct sum of two simple ideals. Being one of them $\mathcal{H}_0(x_0) \oplus \mathcal{W}(x_0)$ with

$$\mathcal{H}_0(x_0) \oplus \mathcal{W}(x_0) \cong \mathfrak{sl}(4,\mathbb{R}).$$

Thus, from (6.2), the isomorphism ψ has domain $\mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{sl}(4,\mathbb{R})$ with

$$\psi(\mathfrak{sl}(4,\mathbb{R})) = \mathcal{H}_0(x_0) \oplus \mathcal{W}(x_0).$$

Let $\widetilde{SL}(3,\mathbb{R})$ and $\widetilde{SL}(4,\mathbb{R})$ be simply connected Lie groups with Lie algebras $\mathfrak{sl}(3,\mathbb{R})$ and $\mathfrak{sl}(4,\mathbb{R})$, respectively. By Lemma 3.16, there exists an analytic isometric right $(\widetilde{SL}(3,\mathbb{R}) \times \widetilde{SL}(4,\mathbb{R}))$-action on $\widetilde{M}$ such that $\psi(X) = X^*$ for all $X \in \mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{sl}(4,\mathbb{R})$. Note that this right action centralizes the left $\widetilde{SL}(3,\mathbb{R})$-action on $\widetilde{M}$ and then preserves the bundles $T\mathcal{O}$ and $T\mathcal{O}^\perp$.

Given the previous right action we define the map

$$\begin{aligned} f : \widetilde{SL}(3,\mathbb{R}) \times \widetilde{SL}(4,\mathbb{R}) &\to \widetilde{M}\\ (h_1, h_2) &\mapsto x_0(h_1,h_2) \end{aligned}$$

for all $(h_1,h_2) \in \widetilde{SL}(3,\mathbb{R}) \times \widetilde{SL}(4,\mathbb{R})$. Observe that this map is $(\widetilde{SL}(3,\mathbb{R}) \times \widetilde{SL}(4,\mathbb{R}))$-equivariant for the right action on its domain. Also note that

$$df_{(e_s,e_l)}(X) = X^*_{x_0} = \psi(X)$$

with $X \in \mathfrak{sl}(3,\mathbb{R})\times\mathfrak{sl}(4,\mathbb{R})$, where e_s and e_l are the identity elements in $\widetilde{SL}(3,\mathbb{R})$ and $\widetilde{SL}(4,\mathbb{R})$, respectively. From here, we have that $df_{(e_s,e_l)}$ is surjective and by Lemma 3.21,

$$\ker(df_{(e_s,e_l)}) = \psi^{-1}(\mathcal{H}_0(x_0)).$$

With the same hypotheses as in (6.2), it is satisfied that $df_e(\mathfrak{sl}(4,\mathbb{R})) = T_{x_0}\mathcal{O}^\perp$ and we claim $df_e(\mathfrak{sl}(3,\mathbb{R})) = T_{x_0}\mathcal{O}$.

Since $\mathcal{G}(x_0)$ and $\psi(\mathfrak{sl}(3,\mathbb{R}))$ are complementary to $\mathcal{H}_0(x_0)\oplus\mathcal{W}(x_0)$ in $\mathcal{H}$ then $\mathcal{G}(x_0) \cong \psi(\mathfrak{sl}(3,\mathbb{R}))$ as $\mathfrak{sl}(3,\mathbb{R})$-module. From the evaluation map we have that $T_{x_0}\mathcal{O} \subseteq ev_{x_0}(\psi(\mathfrak{sl}(3,\mathbb{R})))$. On the other hand, from the properties of the map ev_{x_0} and since $ev_{x_0}(\mathcal{G}(x_0)) = T_{x_0}\mathcal{O}$ then

$$T_{x_0}\mathcal{O} \cong \mathcal{G}(x_0) \quad \text{and} \quad \mathcal{G}(x_0) \ncong T_{x_0}\mathcal{O}^\perp,$$

as $\mathfrak{sl}(3,\mathbb{R})$-modules. Therefore, we have that $df_{(e_s,e_l)}(\mathfrak{sl}(3,\mathbb{R})) = T_{x_0}\mathcal{O}$.

Let H be the connected subgroup of $\widetilde{SL}(4,\mathbb{R})$ such that

$$\mathrm{Lie}(H) = \psi^{-1}(\mathcal{H}_0(x_0)).$$

Since $\mathcal{H}_0(x_0) \oplus \mathcal{W}(x_0) \cong \mathfrak{sl}(4,\mathbb{R})$ then by exercise 4(vi) of [H, p. 152], H is a closed subgroup of $\widetilde{SL}(4,\mathbb{R})$. Thus, the map

$$\begin{array}{rcl} \hat{f} : \widetilde{SL}(3,\mathbb{R}) \times H\backslash\widetilde{SL}(4,\mathbb{R}) & \to & \widetilde{M} \\ (h_1, Hh_2) & \mapsto & x_0(h_1,h_2) \end{array}$$

for all $(h_1, Hh_2) \in \widetilde{SL}(3,\mathbb{R}) \times H\backslash\widetilde{SL}(4,\mathbb{R})$, is a well defined and analytic $(\widetilde{SL}(3,\mathbb{R}) \times \widetilde{SL}(4,\mathbb{R}))$-equivariant map between manifolds. From the properties of $df_{(e_s,e_l)}$, $\hat{f}$ is a local diffeomorphism at (e_s, He_l).

Considering $P = \hat{f}(e_s \times H\backslash\widetilde{SL}(4,\mathbb{R}))$ and using the equivariance of $\hat{f}$, we can prove, with similar arguments as in case $\mathcal{W}(x_0)$ abelian, that $T\mathcal{O}^\perp$ is integrable in $\widetilde{M}$.

Therefore, we have the following result.

Proposition 6.15. *Let M and $\widetilde{SL}(3,\mathbb{R})$ satisfy the same hypotheses as in Theorem 1.6. If for some $x_0 \in \widetilde{M}$ we have that $\mathcal{H}$ (Lemma 3.21) is a semisimple Lie algebra isomorphic to $\mathfrak{sl}(3,\mathbb{R})\times\mathfrak{sl}(4,\mathbb{R})$ (Lemma 4.8) then $T\mathcal{O}^\perp$ is integrable and the pseudo-Riemannian manifold $\widetilde{N}$, in Theorem 1.7, contains an open subset diffeomorphic to the quotient space $(SL(3,\mathbb{R}) \times \mathbb{R})\backslash\widetilde{SL}(4,\mathbb{R})$.*

6.3.1.3 $\mathcal{H}_0(x_0) \simeq \mathfrak{sl}(4,\mathbb{R})$.

By Lemma 4.11, a possible structure of $\mathcal{H}$ is a direct sum of two simple ideals with $\mathcal{H}_0(x_0) \oplus \mathcal{W}(x_0)$ being one of them. Moreover, by that same Lemma, we have

$$\mathcal{H}_0(x_0) \oplus \mathcal{W}(x_0) \cong \mathfrak{so}(3,4).$$

Thus, the isomorphism ψ of (6.2) has domain $\mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{so}(3,4)$ with

$$\psi(\mathfrak{so}(3,4)) = \mathcal{H}_0(x_0) \oplus \mathcal{W}(x_0).$$

Let $\widetilde{SL}(3,\mathbb{R})$ and $\widetilde{SO}_0(3,4)$ be simply connected Lie groups with Lie algebras $\mathfrak{sl}(3,\mathbb{R})$ and $\mathfrak{so}(3,4)$, respectively. By Lemma 3.16, there exists an analytic isometric right $(\widetilde{SL}(3,\mathbb{R}) \times \widetilde{SO}_0(3,4))$-action on $\widetilde{M}$ such that $\psi(X) = X^*$ for all $X \in \mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{so}(3,4)$. This right action centralizes the left $\widetilde{SL}(3,\mathbb{R})$-action on $\widetilde{M}$ and, then, preserves the bundles $T\mathcal{O}$ and $T\mathcal{O}^\perp$.

Given the previous right action we define the map

$$\begin{array}{rcl} f : \widetilde{SL}(3,\mathbb{R}) \times \widetilde{SO}_0(3,4) & \to & \widetilde{M} \\ (h_1, h_2) & \mapsto & x_0(h_1, h_2) \end{array}$$

for all $(h_1, h_2) \in \widetilde{SL}(3,\mathbb{R}) \times \widetilde{SO}_0(3,4)$.

Observe that this map is $(\widetilde{SL}(3,\mathbb{R}) \times \widetilde{SO}_0(3,4))$-equivariant for the right action on its domain. Also note that

$$df_{(e_s,e_o)}(X) = X^*_{x_0} = \psi(X)$$

with $X \in \mathfrak{sl}(3,\mathbb{R})\times\mathfrak{so}(3,4)$, where e_s and e_o are the identity elements in $\widetilde{SL}(3,\mathbb{R})$ and $\widetilde{SO}_0(3,4)$, respectively. Then $df_{(e_s,e_o)}$ is surjective and, by Lemma 3.21,

$$\ker(df_{(e_s,e_o)}) = \psi^{-1}(\mathcal{H}_0(x_0)).$$

Again, from that same Lemma, we have that $df_{(e_s,e_o)}(\mathfrak{so}(3,4)) = T_{x_0}\mathcal{O}^\perp$ and we claim $df_{(e_s,e_o)}(\mathfrak{sl}(3,\mathbb{R})) = T_{x_0}\mathcal{O}$.

Since $\mathcal{G}(x_0)$ and $\psi(\mathfrak{sl}(3,\mathbb{R}))$ are complementary to $\mathcal{H}_0(x_0)\oplus\mathcal{W}(x_0)$ in $\mathcal{H}$ then $\mathcal{G}(x_0) \cong \psi(\mathfrak{sl}(3,\mathbb{R}))$ as $\mathfrak{sl}(3,\mathbb{R})$-module. From the evaluation map we have that $T_{x_0}\mathcal{O} \subseteq ev_{x_0}(\psi(\mathfrak{sl}(3,\mathbb{R})))$. On the other hand, by properties of the map ev_{x_0} and since $ev_{x_0}(\mathcal{G}(x_0)) = T_{x_0}\mathcal{O}$ then

$$T_{x_0}\mathcal{O} \cong \mathcal{G}(x_0) \quad \text{and} \quad \mathcal{G}(x_0) \ncong T_{x_0}\mathcal{O}^\perp,$$

as $\mathfrak{sl}(3,\mathbb{R})$-modules. Therefore, we have that $df_{(e_s,e_o)}(\mathfrak{sl}(3,\mathbb{R})) = T_{x_0}\mathcal{O}$.

Let H be the connected subgroup of $\widetilde{SO}_0(3,4)$ such that

$$\mathrm{Lie}(H) = \psi^{-1}(\mathcal{H}_0(x_0)),$$

which is isomorphic to $\mathfrak{so}(3,3)$ as $\mathfrak{sl}(3,\mathbb{R})$-module. Since $\mathfrak{so}(3,3)$ is a simple Lie algebra and $\widetilde{SO}_0(3,4)$ a simply connected Lie group, by exercise $4(ii)$ in [H, p. 152], H is a closed subgroup of $\widetilde{SO}_0(3,4)$. From this, the map

$$\begin{array}{rcl} \hat{f} : \widetilde{SL}(3,\mathbb{R}) \times H\backslash\widetilde{SO}_0(3,4) & \to & \widetilde{M} \\ (h_1, Hh_2) & \mapsto & x_0(h_1, h_2) \end{array}$$

is a well defined and analytic $(\widetilde{SL}(3,\mathbb{R}) \times \widetilde{SO}_0(3,4))$-equivariant map between manifolds. From the properties of $df_{(e_s,e_o)}$, $\hat{f}$ is a local diffeomorphism at (e_s, He_o).

Considering $P = \hat{f}(e_s \times H\backslash\widetilde{SO}_0(3,4))$ and using the equivariance of $\hat{f}$, we can prove, with similar arguments as in Subsection 6.1.1, that $T\mathcal{O}^{\perp}$ is integrable in $\widetilde{M}$.

Thus, from the previous arguments, we have the following result.

Proposition 6.16. *Let M and $\widetilde{SL}(3,\mathbb{R})$ satisfy the same hypotheses as in Theorem 1.6. If for some $x_0 \in \widetilde{M}$ we have that $\mathcal{H}$ (Lemma 3.21) is a semisimple Lie algebra isomorphic to $\mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{so}(3,4)$ (Lemma 4.11) then $T\mathcal{O}^{\perp}$ is integrable and the pseudo-Riemannian manifold $\widetilde{N}$, in Theorem 1.7, contains an open subset diffeomorphic to the quotient space $SL(4,\mathbb{R})\backslash\widetilde{SO}_0(3,4)$.*

6.3.2 $\mathcal{H}$ is a simple Lie algebra.

From Lemma 4.3 and Lemma 4.4, we have that a possible structure of $\mathcal{H}$, as a Lie algebra, is a simple Lie algebra.

Let us review the options when this possibility occurs.

6.3.2.1 $\mathcal{H}_0(x_0) = 0$.

If $\mathcal{H}_0(x_0) = 0$, then a possibility for the structure of $\mathcal{H}$ is $\mathcal{H} = \mathcal{G}(x_0) \oplus \mathcal{W}(x_0)$ a 14-dimensional simple real Lie algebra. Moreover, by Lemma 4.3, $\mathcal{H} \cong \mathfrak{g}_{2(2)}$. Therefore, we have the next result

Lemma 6.17. *There is an isomorphism*

$$\psi : \mathfrak{g}_{2(2)} = \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3^*} \to \mathcal{H} = \mathcal{G}(x_0) \oplus \mathcal{V}(x_0) \oplus \mathcal{V}^*(x_0) \tag{6.6}$$

of Lie algebras that preserves the summands in that order. In particular we have that ψ is an isomorphism of $\mathfrak{sl}(3,\mathbb{R})$-modules.

Proof. From the previous paragraph we have that $\mathcal{H}$ is isomorphic to $\mathfrak{g}_{2(2)}$, let

$$\psi : \mathfrak{g}_{2(2)} \to \mathcal{H}$$

be an isomorphism of simple Lie algebras.

Then, $\psi^{-1}(\mathcal{G}(x_0)) \subset \mathfrak{g}_{2(2)}$ is a Lie subalgebra isomorphic to $\mathfrak{sl}(3,\mathbb{R})$. We define

$$\widehat{\mathfrak{sl}}(3,\mathbb{R}) = \psi^{-1}(\mathcal{G}(x_0)).$$

This provides $\mathfrak{g}_{2(2)}$ with a structure of $\mathfrak{sl}(3,\mathbb{R})$-module, and consequently a decomposition of $\mathfrak{g}_{2(2)}$ in irreducible $\mathfrak{sl}(3,\mathbb{R})$-submodules. In particular, $\widehat{\mathfrak{sl}}(3,\mathbb{R})$ is an irreducible $\mathfrak{sl}(3,\mathbb{R})$-module isomorphic to $\mathfrak{sl}(3,\mathbb{R})$.

First of all, the proof of Lemma 6.2 shows that the decomposition of $\mathfrak{g}_{2(2)}$ in irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules, given by the Lie subalgebra $\widehat{\mathfrak{sl}}(3,\mathbb{R})$, is

$$\mathfrak{g}_{2(2)} = \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3^*}$$

and then

$$\psi(\mathfrak{sl}(3,\mathbb{R})) \oplus \psi(\mathbb{R}^3) \oplus \psi(\mathbb{R}^{3^*})$$

is a decomposition of $\mathcal{H}$ in irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules.

On the other hand, we have a previous decomposition of $\mathcal{H}$ in irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules

$$\mathcal{H} = \mathcal{G}(x_0) \oplus \mathcal{V}(x_0) \oplus \mathcal{V}^*(x_0).$$

Here, $\psi(\mathfrak{sl}(3,\mathbb{R})) = \mathcal{G}(x_0)$. Let $h \in \mathbb{R}^3$ be an element different from zero, then

$$\psi(h) = v_0 + v_1 + v_2$$

with $v_1 \in \mathcal{G}(x_0)$, $v_2 \in \mathcal{V}(x_0)$ and $v_3 \in \mathcal{V}^*(x_0)$. If $v_1 \neq 0$ then (with the projection map) we have a non-zero homomorphism of $\mathfrak{sl}(3,\mathbb{R})$-modules between $\mathbb{R}^3$ and $\mathfrak{sl}(3,\mathbb{R})$, which cannot be possible, hence $v_1 = 0$. The same argument can be used to show that $v_3 = 0$. Then $\psi(\mathbb{R}^3) = \mathcal{V}(x_0)$. A similar proof shows that $\psi(\mathbb{R}^{3^*}) = \mathcal{V}^*(x_0)$. □

Now, fix an isomorphism of Lie algebras

$$\psi : \mathfrak{g}_{2(2)} \to \mathcal{H}$$

as in Lemma 6.17. Let $G_{2(2)}$ denote a simply connected Lie group such that

$$\mathrm{Lie}(G_{2(2)}) = \mathfrak{g}_{2(2)}.$$

By Lemma 3.16, there exists an analytic isometric right $G_{2(2)}$-action on $\widetilde{M}$ such that

$$\psi(X) = X^*$$

for every $X \in \mathfrak{g}_{2(2)}$. Now, we consider the orbit map

$$\begin{aligned} f : G_{2(2)} &\to \widetilde{M} \\ g &\mapsto x_0 g, \end{aligned}$$

that satisfies

$$df_I(X) = X^*_{x_0} = \psi(X)$$

for every $X \in \mathfrak{g}_{2(2)}$. By our choice of ψ and Lemma 3.21 we have that df_I is an isomorphism that maps $\mathfrak{sl}(3,\mathbb{R})$ onto $T_{x_0}\mathcal{O}$ and $\mathbb{R}^3 \oplus \mathbb{R}^{3^*}$ onto $T_{x_0}\mathcal{O}^\perp$. Since f is $G_{2(2)}$-equivariant for the right action on its domain, then we have an analytic local diffeomorphism.

Lemma 6.18. *Let $\overline{g}$ the metric on $\mathfrak{g}_{2(2)}$ defined as the pullback under df_I of the metric g_{x_0} on $T_{x_0}\widetilde{M}$. Then, $\overline{g}$ is $\mathfrak{sl}(3,\mathbb{R})$-invariant.*

Proof. The proof is similar to Lemma 3.2 in [OQ] which is transcribed here.

By properties of df_I and the isomorphism ψ, such that $\psi(\mathfrak{sl}(3,\mathbb{R})) = \mathcal{G}(x_0)$, we need only prove that the metric on $\mathcal{H}$ defined as the pullback of g_{x_0} to the evaluation map

$$\begin{aligned} &\mathcal{H} \to T_{x_0}\widetilde{M} \\ &X \mapsto ev_{x_0}(X) = X_{x_0} \end{aligned}$$

is $\mathcal{G}(x_0)$-equivariant.

Let $\tilde{g}$ be the metric in $\mathcal{H}$ obtained of this way. Let $X, Y, Z \in \mathcal{H}$ be given with $X \in \mathcal{G}(x_0)$. Then, there exists $X_0 \in \mathfrak{sl}(3,\mathbb{R})$ such that $X = \rho_{x_0}(X_0) + X_0^*$, where ρ_{x_0} is the homomorphism in Proposition 3.7 and X_0^* is the vector field on $\widetilde{M}$ induced by X_0 through the left $\widetilde{SL}(3,\mathbb{R})$-action. Then

$$\begin{aligned} \tilde{g}([X,Y],Z) &= g_{x_0}([X,Y]_{x_0}, Z_{x_0}) \\ &= g([X,Y],Z)|_{x_0} \\ &= g([\rho_{x_0}(X_0) + X_0^*, Y], Z)|_{x_0} \\ &= g([\rho_{x_0}(X_0), Y], Z)|_{x_0} \\ &= \rho_{x_0}(X_0) g(Y,Z)|_{x_0} - g(Y, [\rho_{x_0}(X_0), Z])|_{x_0} \\ &= -g(Y, [\rho_{x_0}(X_0), Z])|_{x_0} \\ &= -g(Y, [\rho_{x_0}(X_0) + X_0^*, Z])|_{x_0} \\ &= -g(Y, [X,Z])|_{x_0} \\ &= -\tilde{g}(Y, [X,Z]). \end{aligned}$$

Where we have used that $\mathcal{H}$ centralizes X_0^* and $\rho(X_0)$ is a Killing vector field for the metric g that vanishes in x_0. Thus, we take $\bar{g}$ as the pullback of $\tilde{g}$ by the isomorphism ψ to obtain the desired result. □

Now, from Lemma 6.18 and Lemma 6.6, we can rescale the metric along the bundles $T\mathcal{O}$ and $T\mathcal{O}^\perp$ in M such that the new metric, which we denote by $\widehat{g}$, on $\widetilde{M}$ satisfies

$$(df_I)^*(\widehat{g}_{x_0}) = K,$$

the Killing form on $\mathfrak{g}_{2(2)}$.

Since the elements of $\mathcal{H} \subset \mathrm{Kill}(\widetilde{M})$ preserve the decomposition of $T\widetilde{M}$ as $T\widetilde{M} = T\mathcal{O} \oplus T\mathcal{O}^\perp$, then $\mathcal{H} \subset \mathrm{Kill}(\widetilde{M}, \widehat{g})$, where $\widehat{g}$ is the new metric rescaled from g as above. Note that $\widehat{g}$ is invariant under both the right $G_{2(2)}$-action and by the left $\widetilde{SL}(3,\mathbb{R})$-action on $\widetilde{M}$, from our hypotheses. Also, observe that our new metric $\widehat{g}$ can be obtained from the lift of a correspondingly rescaled metric $\widehat{g}$ on M.

Now, consider the bi-invariant metric on $G_{2(2)}$ induced by the Killing form K, which we denote by $K^{G_{2(2)}}$. The previous argument and discussion imply that the local diffeomorphism

$$f : (G_{2(2)}, K^{G_{2(2)}}) \to (\widetilde{M}, \widehat{g})$$

is a local isometry. With this property of f, Corollary 20 in [ONe, p. 202], the completeness of $(G_{2(2)}, K^{G_{2(2)}})$ and the simply connectedness of $(\widetilde{M}, \widehat{g})$ imply that f is an isometry.

Corollary 6.19 ([ONe], p. 202, Corollary 20)**.** *Let $\phi : M \to N$ be a local isometry, with N connected. Then M is complete if and only if N is complete and ϕ is a semi-Riemannian covering.*

Then, we have the next result, similar to Lemma 3.3 in [OQ]:

Proposition 6.20. *Let M be a analytic connected finite volume pseudo-Riemannian manifold of dimension 14. If M is complete, admits an analytic and isometric $SL(3,\mathbb{R})$-action with a dense orbit such that $\mathcal{H}$ (Lemma 3.21) is a simple Lie algebra isomorphic to $\mathfrak{g}_{2(2)}$ (Lemma 4.3). Then, there exists an analytic diffeomorphism*

$$f : G_{2(2)} \to \widetilde{M}$$

and an analytic isometric right $G_{2(2)}$-action on $\widetilde{M}$ such that:

(i) On $\widetilde{M}$, the left $\widetilde{SL}(3,\mathbb{R})$-action and the right $G_{2(2)}$-action commute,

(ii) f is $G_{2(2)}$-equivariant for the right $G_{2(2)}$-action on its domain,

(iii) for a pseudo-Riemannian metric $\widehat{g}$ in M obtained by rescaling the original metric on the summands of the decomposition $TM = T\mathcal{O} \oplus T\mathcal{O}^{\perp}$, the map

$$f : (G_{2(2)}, K^{G_{2(2)}}) \to (\widetilde{M}, \widehat{g})$$

is an isometry where $K^{G_{2(2)}}$ is the bi-invariant metric on $G_{2(2)}$ induced from the Killing form of its Lie algebra.

With the previous Lemma, if we consider $G_{2(2)}$ with the bi-invariant pseudo-Riemannian metric K induced by the Killing form of its Lie algebra then we can consider, also, $(G_{2(2)}, K)$ as the isometric universal covering space of $(\widetilde{M}, \widehat{g})$.

Next, we state Lemma 4.5 in [Q2] which will be used later on.

Proposition 6.21 ([Q2], Proposition 4.5)**.** *Let G be a connected non-compact simple Lie group. Then, Iso(G), for a bi-invariant pseudo-Riemannian metric in G, has finitely many components and*

$$Iso(G)_0 = L(G)R(G).$$

From Proposition 4.5 of [Q2] we have that the isometry group $\mathrm{Iso}(G_{2(2)})$ for the pseudo-Riemannian manifold $(G_{2(2)}, K)$ has only a finite number of connected components. Also, the proposition shows that

$$\mathrm{Iso}_0(G_{2(2)}) = L(G_{2(2)})R(G_{2(2)}),$$

where $L(G_{2(2)})$ and $R(G_{2(2)})$ are the subgroups of left and right translations on $G_{2(2)}$, respectively.

Let $\rho : \widetilde{SL}(3,\mathbb{R}) \to \mathrm{Iso}(G_{2(2)})$ be the homomorphism induced by the isometric left $\widetilde{SL}(3,\mathbb{R})$-action on $G_{2(2)}$. Then, from the previous observation the covering

$$G_{2(2)} \times G_{2(2)} \to L(G_{2(2)})R(G_{2(2)})$$

yields the existence of homomorphisms $\rho_1, \rho_2 : \widetilde{SL}(3,\mathbb{R}) \to G_{2(2)}$ such that

$$\rho(g) = L_{\rho_1(g)} \circ R_{\rho_2(g)^{-1}} \qquad \forall g \in \widetilde{SL}(3,\mathbb{R}).$$

On the other hand, by Proposition 6.20, we have

$$\rho(g) \circ R_h = R_h \circ \rho(g)$$

for all $g \in \widetilde{SL}(3,\mathbb{R})$ and $h \in G_{2(2)}$. This implies that $\rho_2(\widetilde{SL}(3,\mathbb{R}))$ is contained in the center of $G_{2(2)}$, from here $R_{\rho_2(g)^{-1}} = L_{\rho_2(g)^{-1}}$ for all $g \in \widetilde{SL}(3,\mathbb{R})$. This is, the $\widetilde{SL}(3,\mathbb{R})$-action on $G_{2(2)}$ is induced by the homomorphism

$$\rho_1 : \widetilde{SL}(3,\mathbb{R}) \to G_{2(2)}$$

and the left action of $G_{2(2)}$ onto itself. Note that by hypotheses, the homomorphism ρ_1 is non-trivial.

From Proposition 6.20 we have that

$$\pi_1(M) \subset \mathrm{Iso}(G_{2(2)})$$

and from previous observations

$$\Gamma_1 = \pi_1(M) \cap \mathrm{Iso}_0(G_{2(2)})$$

is a finite index subgroup of $\pi_1(M)$. So, for every $\gamma \in \Gamma_1$ there exist $h_1, h_2 \in G_{2(2)}$ such that

$$\gamma = L_{h_1} \circ R_{h_2}.$$

Also, since the left $\widetilde{SL}(3,\mathbb{R})$-action on $G_{2(2)}$ is the lift of an action on M then this left $\widetilde{SL}(3,\mathbb{R})$-action commutes with the Γ_1-action. Applying this property to

$$L_{h_1} \circ R_{h_2} = \gamma \in \Gamma_1$$

we obtain that

$$L_{h_1} \circ L_{\rho_1(g)} = L_{\rho_1(g)} \circ L_{h_1}$$

for all $g \in \widetilde{SL}(3,\mathbb{R})$, thus

$$\Gamma_1 \in L(Z(\widetilde{SL}(3,\mathbb{R})))R(G_{2(2)})$$

where $Z(\widetilde{SL}(3,\mathbb{R}))$ is the centralizer of $\rho_1(\widetilde{SL}(3,\mathbb{R}))$ in $G_{2(2)}$. By Lemma 6.3 we have that the center of $G_{2(2)}$ has finite index in $Z(\widetilde{SL}(3,\mathbb{R}))$ and a consequence of this is that $R(G_{2(2)})$ has finite index in $L(\widetilde{SL}(3,\mathbb{R}))R(G_{2(2)})$. In particular

$$\Gamma = \Gamma_1 \cap R(G_{2(2)})$$

is a finite index subgroup of Γ_1 and also of $\pi_1(M)$.

The natural identification of $R(G_{2(2)})$ with $G_{2(2)}$ realizes Γ as a discrete subgroup of $G_{2(2)}$ such that $G_{2(2)}/\Gamma$ is a finite covering space of M.

Now, if

$$\varphi : G_{2(2)}/\Gamma \to M$$

is the corresponding covering map and, for the left $\widetilde{SL}(3,\mathbb{R})$-action on $G_{2(2)}/\Gamma$ given by the homomorphism

$$\rho_1 : \widetilde{SL}(3,\mathbb{R}) \to G_{2(2)},$$

the constructions in the previous paragraphs show that the map φ is $\widetilde{SL}(3,\mathbb{R})$-equivariant. Finally, we note that φ is a local isometry for the metric $\widehat{g}$, on M considered in Proposition 6.20.

Now, we show that Γ is a lattice in $G_{2(2)}$. For the proof of this result it is enough to prove that M has finite volume in the metric $\widehat{g}$. Recall, we are assuming that M has finite volume in its original metric. The following proof is similar to Lemma 3.4 in [OQ].

Lemma 6.22. *If* vol *and* $\mathrm{vol}_{\widehat{g}}$ *denote the volume elements on M for the original metric and the rescaled metric, respectively. Then, there is some constant $C > 0$ such that* $\mathrm{vol}_{\widehat{g}} = C\mathrm{vol}$.

Proof. We consider $(x^1, x^2, \dots, x^{14})$ some coordinate of M in a neighborhood U of a given point such that $(x^1, \dots, x^8)$ defines a set of coordinates of the leaves of the foliation $\mathcal{O}$ in such neighborhood. For the original metric g on M, consider the orthogonal bundle $T\mathcal{O}^\perp$ and a set of 1-forms $\theta^1, \dots, \theta^6$ that define a basis for its dual $(T\mathcal{O}^\perp)^*$ at every point in U. Thus, in U the metric g has an expression of the form:

$$g = \sum_{i,j=1}^{8} l_{ij} dx^i \otimes dx^j + \sum_{i,j=1}^{6} h_{ij} \theta^i \otimes \theta^j.$$

From this and the definition of the volume element as an 14-form, we have:

$$\mathrm{vol} = \sqrt{|\det(l_{ij}) \det(h_{ij})|} dx^1 \wedge \dots \wedge dx^8 \wedge \theta^1 \wedge \dots \wedge \theta^6.$$

On the other hand, since the metric $\widehat{g}$ is obtained by rescaling g along the bundles $T\mathcal{O}$ and $T\mathcal{O}^\perp$, then has an expression of the form:

$$\widehat{g} = \sum_{i,j=1}^{8} C_1 l_{ij} dx^i \otimes dx^j + \sum_{i,j=1}^{6} C_2 h_{ij} \theta^i \otimes \theta^j.$$

for some constants $C_1, C_2 \neq 0$. Therefore, the volume element of $\widehat{g}$ satisfies:

$$\begin{aligned} \mathrm{vol}_{\widehat{g}} &= \sqrt{|\det(C_1 l_{ij}) \det(C_2 h_{ij})|} dx^1 \wedge \dots \wedge dx^8 \wedge \theta^1 \wedge \dots \wedge \theta^6. \\ &= \sqrt{|C_1^8 C_2^6|} \mathrm{vol}. \end{aligned}$$

□

6.3.2.2 $\mathcal{H}_0(x_0) \simeq \mathbb{R}$.

If $\mathcal{H}_0(x_0) = \mathbb{R}$, then a possibility for the structure of $\mathcal{H}$ is $\mathcal{H} = \mathcal{G}(x_0) \oplus \mathcal{H}_0(x_0) \oplus \mathcal{W}(x_0)$ a 15-dimensional simple real Lie algebra. Moreover, by Lemma 4.4, $\mathcal{H} \cong \mathfrak{sl}(4, \mathbb{R}) \cong \mathfrak{so}(3, 3)$. Therefore, we have the following result

Lemma 6.23. *There is an isomorphism*

$$\psi : \mathfrak{sl}(4, \mathbb{R}) = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \to \mathcal{G}(x_0) \oplus \mathcal{H}_0(x_0) \oplus \mathcal{V}(x_0) \oplus \mathcal{V}^*(x_0) = \mathcal{H}$$

of Lie algebras that preserves the summands in that order. In particular, ψ is an isomorphism of $\mathfrak{sl}(3, \mathbb{R})$-modules.

Let us fix an isomorphism of Lie algebras

$$\varphi : \mathfrak{sl}(4, \mathbb{R}) \to \mathcal{H}$$

as in Lemma 6.23. Let $\widetilde{SL}(4, \mathbb{R})$ be a simply connected Lie group such that

$$\mathrm{Lie}(\widetilde{SL}(4, \mathbb{R})) = \mathfrak{sl}(4, \mathbb{R}).$$

By Lemma 3.16, there exists an analytic isometric right $\widetilde{SL}(4, \mathbb{R})$-action on $\widetilde{M}$ such that

$$\varphi(X) = X^*,$$

for every $X \in \mathfrak{sl}(4, \mathbb{R})$. This right action centralizes the left $\widetilde{SL}(3, \mathbb{R})$-action on $\widetilde{M}$ and preserves the bundles $T\mathcal{O}$ and $T\mathcal{O}^\perp$.

Given the previous right action, we define the map

$$\begin{aligned} f : \widetilde{SL}(4, \mathbb{R}) &\to \widetilde{M} \\ g &\mapsto x_0 \cdot g, \end{aligned} \tag{6.7}$$

for all $g \in \widetilde{SL}(4, \mathbb{R})$. Observe that this map is $\widetilde{SL}(4, \mathbb{R})$-equivariant for the right action on its domain and satisfies

$$df_e(X) = X^*_{x_0} = \varphi(X) \tag{6.8}$$

for every $X \in \mathfrak{sl}(4, \mathbb{R})$, where e is the identity element in the group $\widetilde{SL}(4, \mathbb{R})$. Observe that df_e is surjective with

$$\ker(df_e) = \varphi^{-1}(\mathcal{H}_0(x_0)).$$

Let H be a connected subgroup of $\widetilde{SL}(4, \mathbb{R})$ such that

$$\mathrm{Lie}(H) = \varphi^{-1}(\mathcal{H}_0(x_0)). \tag{6.9}$$

Recall that $\mathcal{H}_0(x_0) \simeq \mathbb{R}$ as $\mathfrak{sl}(3, \mathbb{R})$-module.

Note that H is not a compact subgroup and

$$\varphi^{-1}(\mathcal{H}_0(x_0)) \simeq \left\{ \begin{pmatrix} aI & 0 \\ 0 & -3a \end{pmatrix} \in M_{4\times 4}(\mathbb{R}) \middle| a \in \mathbb{R} \right\}. \tag{6.10}$$

Thus, by [H, ex. (vi), pag. 152], H is a closed subgroup.

Since H is a closed subgroup of $\widetilde{SL}(4,\mathbb{R})$, we can define the following map

$$\begin{aligned} \bar{f} : H\backslash\widetilde{SL}(4,\mathbb{R}) &\to \widetilde{M} \\ Hg &\mapsto x_0 \cdot g, \end{aligned} \tag{6.11}$$

for all $Hg = [g] \in H\backslash\widetilde{SL}(4,\mathbb{R})$. Note that

$$T_{[e]}(H\backslash\widetilde{SL}(4,\mathbb{R})) = \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}.$$

By our choice of φ we have that $d\bar{f}_{He}$ is an isomorphism which maps $\mathfrak{sl}(3,\mathbb{R})$ onto $T_{x_0}\mathcal{O}$ and $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ onto $T_{x_0}\mathcal{O}^\perp$. Since $\bar{f}$ is $\widetilde{SL}(4,\mathbb{R})$-equivariant for the right action on its domain, then $\bar{f}$ is an analytic local diffeomorphism at He.

By Lemma 6.23 and the construction of the map $\bar{f}$, it satisfies that

$$d\bar{f}_{[e]} = ev_{x_0} \circ \varphi \tag{6.12}$$

restricted to $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}$.

Now, we obtain information about the metric of the space $H\backslash\widetilde{SL}(4,\mathbb{R})$.

Lemma 6.24. *Let $\bar{g}$ be the metric on $T_{[e]}H\backslash\widetilde{SL}(4,\mathbb{R})$ defined as the pullback under $d\bar{f}_{[e]}$ of the metric g_{x_0} on $T_{x_0}\widetilde{M}$. Then $\bar{g}$ is $\mathfrak{sl}(3,\mathbb{R})$-invariant.*

Proof. By the properties of ev_{x_0} and φ, the map

$$d\bar{f}_{[e]} = ev_{x_0} \circ \varphi$$

is a homomorphism of $\mathfrak{sl}(3,\mathbb{R})$-modules. Where the $\mathfrak{sl}(3,\mathbb{R})$-module structure in $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}$ is given by the subalgebra $\mathfrak{sl}(3,\mathbb{R})$ and in $T_{x_0}\widetilde{M} = T_{x_0}\mathcal{O} \oplus T_{x_0}\mathcal{O}^\perp$ by the subalgebra $\rho_{x_0}(\mathfrak{sl}(3,\mathbb{R})(x_0))$.

Since the metric g in $T_{x_0}\widetilde{M} = T_{x_0}\mathcal{O} \oplus T_{x_0}\mathcal{O}^\perp$ is invariant under the action of $\rho_{x_0}(\mathfrak{sl}(3,\mathbb{R})(x_0))$ then, by properties of the maps ev_{x_0} and φ, the metric $\bar{g}$ in

$$T_{[e]}(H\backslash\widetilde{SL}(4,\mathbb{R})) = \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}.$$

is $\mathfrak{sl}(3,\mathbb{R})$-invariant.

Let $v, w \in T_{He}(H\backslash\widetilde{SL}(4,\mathbb{R}))$ and $X \in \mathfrak{sl}(3,\mathbb{R})$. Since $\hat{\rho}_{x_0}(X) \in \mathfrak{g}(x)$, take

$\bar{X} = \varphi^{-1}(\hat{\rho}_{x_0}(X))$, then

$$\begin{aligned}
\bar{g}([X,v],w) &:= \bar{g}([\bar{X},v],w) \\
&= g_{x_0}(d\bar{f}_{[e]}([\bar{X},v]), d\bar{f}_{[e]}(w)) \\
&= g_{x_0}(ev_{x_0}\circ\varphi[\bar{X},v], ev_{x_0}\circ\varphi(w)) \\
&= g_{x_0}(\varphi[\bar{X},v]_{x_0}, \varphi(w)_{x_0}) \\
&= g(\varphi[\bar{X},v], \varphi(w))|_{x_0} \\
&= g([\varphi(\bar{X}),\varphi(v)], \varphi(w))|_{x_0} \\
&= g([\hat{\rho}_{x_0}(X), V], W)|_{x_0} \\
&= g([\rho_{x_0}(X) + X^*, V], W)|_{x_0} \\
&= g([\rho_{x_0}(X), V], W)|_{x_0} \\
&= \rho_{x_0}(X)(g(V,W))|_{x_0} - g(V, [\rho_{x_0}(X), W])|_{x_0} \\
&= -g(V, [\rho_{x_0}(X), W])|_{x_0} \\
&= -g(V, [\rho_{x_0}(X) + X^*, W])|_{x_0} \\
&= -g(V, [\hat{\rho}_{x_0}(X), W])|_{x_0} \\
&= -g(\varphi(v), [\varphi(\bar{X}), \varphi(w)])|_{x_0} \\
&= -g(\varphi(v), \varphi[\bar{X}, w])|_{x_0} \\
&= -g_{x_0}(d\bar{f}_{[e]}(v), d\bar{f}_{[e]}([\bar{X},w])) \\
&= -\bar{g}(v, [\bar{X},w]) \\
&= -\bar{g}(v, [X,w])
\end{aligned}$$

Where $V = \varphi(v), W = \varphi(w) \in \mathcal{H}$. Recall that $\mathcal{H}$ centralizes X^* and $\rho_{x_0}(X)$ is a Killing field for the metric g in $\widetilde{M}$. □

Next, we analyze the structure of pseudo-Riemannian metric of the analytic manifold $H\backslash\widetilde{SL}(4,\mathbb{R})$.

We want to show the existence of a (semi) pseudo-Riemannian metric on $H\backslash\widetilde{SL}(4,\mathbb{R})$ such that the quotient map

$$\pi : \widetilde{SL}(4,\mathbb{R}) \to H\backslash\widetilde{SL}(4,\mathbb{R})$$

is a pseudo-Riemannian submersion with the pseudo-Riemannian metric on the simple Lie group $\widetilde{SL}(4,\mathbb{R})$ given by the Killing form on its Lie algebra, $\mathfrak{sl}(4,\mathbb{R})$.

Recall the definition of *pseudo-Riemannian submersion* from [ONe, p. 212].

Definition 6.25. A *pseudo-Riemannian submersion* $\pi : M \to B$ is a submersion of Pseudo-Riemannian manifolds such that:

(S1) The fibers $\pi^{-1}(b)$, $b \in B$ are pseudo-Riemannian submanifolds of M.

(S2) $d\pi$ preserves scalar products of vector normal fibers.

Let K^n be the Killing form on $\mathfrak{sl}(n,\mathbb{R})$, $n \geq 2$. Recall that

$$K^n(X,Y) = 2(n)\mathrm{tr}(XY) \tag{6.13}$$

for all $X, Y \in \mathfrak{sl}(n,\mathbb{R})$.

By the decomposition of $\mathfrak{sl}(4,\mathbb{R})$ as a direct sum of irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules we have that $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}$ is contained in $\mathfrak{sl}(4,\mathbb{R})$ in the following manner:

$$\left\{ \begin{bmatrix} A & u \\ v^t & 0 \end{bmatrix} \in M_{4\times 4}(\mathbb{R}) \middle| A \in \mathfrak{sl}(3,\mathbb{R}), u, v \in \mathbb{R}^3 \right\} \tag{6.14}$$

Then, by (6.13) and (6.14) we have that $\mathfrak{sl}(3,\mathbb{R})$ and $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ are non-degenerate subspaces with respect to K^4.

Denote by K_1, K_2 and K the Killing form K^4 restricted to $\mathfrak{sl}(3,\mathbb{R})$, $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ and $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}$, respectively.

Since K^4 is invariant by the adjoint action of $\mathfrak{sl}(4,\mathbb{R})$ then it is clear that K^4 is also invariant under the adjoint action of $\mathfrak{sl}(3,\mathbb{R})$, via its inclusion in $\mathfrak{sl}(4,\mathbb{R})$ in (6.14). Note, by this equation, that K_1, K_2 and K are invariant by the adjoint action of $\mathfrak{sl}(3,\mathbb{R})$.

Remark 6.26. Since $\mathfrak{sl}(3,\mathbb{R})$ is a simple Lie algebra and by the above discussion we have that $K_1 = K^4|_{\mathfrak{sl}(3,\mathbb{R})}$ is a multiple of its Killing form K^3. This is

$$K_1 = c_1 K^3.$$

Note, by (6.14), that $c_1 \neq 0$. Following these observations we have, by Lemma 6.6, that $K_2 = K^4|_{\mathbb{R}^3\oplus\mathbb{R}^{3*}}$ is a multiple of a unique (up to multiples) $\mathfrak{sl}(3,\mathbb{R})$-invariant symmetric bilinear form on $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$. And, by (6.14), this multiple is non-zero.

Looking to prove that π is a pseudo-Riemannian submersion, assume

$$d\pi_0 = d\pi|_{\mathfrak{sl}(3,\mathbb{R})\oplus\mathbb{R}^3\oplus\mathbb{R}^{3*}} : \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \to T_{[e]}(H\backslash \widetilde{SL}(4,\mathbb{R})) \tag{6.15}$$

is a linear isometry.

From [ONe, p. 310] we have the next definition

Definition 6.27. Let G be a Lie group and L a closed subgroup of G. A coset manifold $N = G/H$ is *reductive* if there is an $\mathrm{Ad}(H)$-invariant subspace $\mathfrak{n}$ of $\mathfrak{g} = \mathrm{Lie}(G)$ that is complementary to $\mathfrak{h} = \mathrm{Lie}(H)$ in $\mathfrak{g}$. We call $\mathfrak{n}$ a *Lie subspace* for G/H.

By construction we have that $H\backslash \widetilde{SL}(4,\mathbb{R})$ **is reductive**. By properties of K^4 (Killing form on $\mathfrak{sl}(4,\mathbb{R})$) we have that the metric K is $\mathrm{Ad}(H)$-invariant. This property and the isometry in (6.15) implies the $\widetilde{SL}(4,\mathbb{R})$-invariance on $H\backslash \widetilde{SL}(4,\mathbb{R})$ of the pseudo-Riemannian metric induced by $d\pi_0$. This result appears as Proposition 22 in [ONe, p. 311].

Now, recall the definition of a naturally reductive homogeneous space.

Definition 6.28. A *naturally reductive homogeneous space* is a reductive coset manifold $N = G/H$ furnished with a G-invariant metric such that, for the corresponding scalar product on the Lie subspace $\mathfrak{n}$,

$$\langle [X,Y]_{\mathfrak{n}}, Z\rangle = \langle X, [Y,Z]_{\mathfrak{n}}\rangle \quad \text{for} \quad X, Y, Z \in \mathfrak{n}.$$

Then, by Lemma 6.23 and the properties of the Killing form, $H\backslash\widetilde{SL}(4,\mathbb{R})$ together with the assumption in (6.15), is a **naturally reductive homogeneous space**.

Then, Lemma 24 in [ONe, p. 312] proves that

$$\pi : \widetilde{SL}(4,\mathbb{R}) \to H\backslash\widetilde{SL}(4,\mathbb{R})$$

is a pseudo-Riemannian submersion.

The next result in this book, Proposition 25, proves that naturally reductive homogeneous spaces are complete. From here, our manifold

$$H\backslash\widetilde{SL}(4,\mathbb{R})$$

is complete.

In the next lines, we show that we can rescale the metric g on $\widetilde{M}$ such that

$$\left(d\bar{f}_{[e]}\right)^*(g_{x_0})$$

implies that (6.15) is a linear isometry.

Lemma 6.29. *Let $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ be the inner products on $\mathfrak{sl}(3,\mathbb{R})$ and $\mathbb{R}^3\oplus\mathbb{R}^{3*}$, respectively. Assume that $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are $\mathfrak{sl}(3,\mathbb{R})$-invariant. Then there exist $c_1, c_2 \in \mathbb{R}$ such that*

$$c_1\langle\cdot,\cdot\rangle_1 + c_2\langle\cdot,\cdot\rangle_2,$$

is K, the Killing form of $\mathfrak{sl}(4,\mathbb{R})$ restricted to $\mathfrak{sl}(3,\mathbb{R})\oplus\mathbb{R}^3\oplus\mathbb{R}^{3*}$.

Proof. Recall, Schur's Lemma implies that in $\mathfrak{g}$, a simple real Lie algebra with a simple complexification, any $\mathfrak{g}$-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$ is a multiple by a real scalar of the Killing form.

On the other hand, we have proved in Lemma 6.6 that there is, up to a multiple by a real scalar, a unique $\mathfrak{sl}(3,\mathbb{R})$-invariant non-degenerate bilinear form on $\mathbb{R}^3\oplus\mathbb{R}^{3*}$.

The result follows from previous results. $\square$

Remark 6.30. From Lemma 6.24 and Lemma 6.29 we can rescale the metric along bundles $T\mathcal{O}$ and $T\mathcal{O}^\perp$ in M so that the new metric $\hat{g}$ on $\widetilde{M}$ satisfies

$$\left(d\bar{f}_{[e]}\right)^*(\hat{g}_{x_0}) = K,$$

the Killing form on $\mathfrak{sl}(4,\mathbb{R})$ restricted to $\mathfrak{sl}(3,\mathbb{R})\oplus\mathbb{R}^3\oplus\mathbb{R}^{3*}$.

Remark 6.31. Since the elements of $\mathcal{H}$ preserve the decomposition $TM = T\mathcal{O}\oplus T\mathcal{O}^\perp$, then $\mathcal{H}\subset \mathrm{Kill}(\widetilde{M},\widehat{g})$. Hence, the elements of $\mathcal{H}$ are Killing vector fields for the metric $\widehat{g}$ rescaled as in Remark 6.30. Thus, $\widehat{g}$ is invariant under the right $\widetilde{SL}(4,\mathbb{R})$-action. In the same way, the left $\widetilde{SL}(3,\mathbb{R})$-action on $\widetilde{M}$, from the hypotheses, preserves the rescaled metric $\widehat{g}$. Note, $\widehat{g}$ in $\widetilde{M}$ is the lift of a correspondingly rescaled metric g in M.

Consider the metric K on $H\backslash\widetilde{SL}(4,\mathbb{R})$ induced by the Killing form K^4 on $\mathfrak{sl}(4,\mathbb{R})$. From Remark 6.30 and Remark 6.31 we have that the local diffeomorphism (6.11)

$$\bar{f} : (H\backslash\widetilde{SL}(4,\mathbb{R}), K) \to (\widetilde{M}, \widehat{g})$$

is a local isometry. Then, by [ONe, p. 202], the completeness of $(H\backslash\widetilde{SL}(4,\mathbb{R}), K)$ and the simple completeness of $\widetilde{M}$ imply that $\bar{f}$ is an isometry.

Therefore, we have the following result

Proposition 6.32. *Let M be a connected analytic pseudo-Riemannian manifold. Suppose that M is complete, has finite volume and admits an analytic and isometric $SL(3,\mathbb{R})$-action with a dense orbit such that $\mathcal{H}$ (Lemma 3.21) is a simple Lie algebra isomorphic to $\mathfrak{sl}(4,\mathbb{R})$ (Lemma 4.4). If $\dim(M) = 14$, then there exists an analytic diffeomorphism $\bar{f} : H\backslash\widetilde{SL}(4,\mathbb{R}) \to \widetilde{M}$ and an analytic isometric right $\widetilde{SL}(4,\mathbb{R})$-action on $\widetilde{M}$ such that:*

(1) on $\widetilde{M}$ the left $\widetilde{SL}(3,\mathbb{R})$-action and the right $\widetilde{SL}(4,\mathbb{R})$-action commute with each other,

(2) $\bar{f}$ is $\widetilde{SL}(4,\mathbb{R})$-equivariant for the right $\widetilde{SL}(4,\mathbb{R})$-action on its domain,

(3) for a pseudo-Riemannian metric $\widehat{g}$ in M obtained by rescaling the original metric on the summands of $TM = T\mathcal{O} \oplus T\mathcal{O}^{\perp}$, the map

$$\bar{f} : (H\backslash\widetilde{SL}(4,\mathbb{R}), K) \to (\widetilde{M}, \widehat{g})$$

is an isometry where K is the metric on $H\backslash\widetilde{SL}(4,\mathbb{R})$ which makes of the quotient map π a pseudo-Riemannian submersion.

6.4 Proof of the Main Theorems.

In this section we will assume the hypotheses of Theorem 1.6.

As a consequence of Proposition 3.7 together with the analyticity of the elements, we have Remark 3.8 with only two possible actions for the bundle $T\mathcal{O}^{\perp}$:

(A) $T\mathcal{O}^{\perp}$ is an integrable bundle, or

(B) $T\mathcal{O}^{\perp}$ is a non-integrable bundle.

When case (A) occurs, then we have Theorem 1.7, which is a special case from the main results in [Q], where the integrable normal bundle (from the action) is analyzed.

When $T\mathcal{O}^{\perp}$ is a non-integrable bundle (case (B)), Propositions 6.12-6.16 imply that the centralizer of the action, $\mathcal{H}$, is a simple Lie algebra. From the analysis done in Chapter 4, the possible options for $\mathcal{H}$ are:

$$\mathcal{H} \simeq \mathfrak{g}_{2(2)} \quad \text{or} \quad \mathcal{H} \simeq \mathfrak{sl}(4,\mathbb{R}).$$

If $\mathcal{H} \simeq \mathfrak{g}_{2(2)}$, then the results in Proposition 6.20 and Lemma 6.22 prove Theorem 1.8.

If $\mathcal{H} \simeq \mathfrak{sl}(4, \mathbb{R})$ is satisfied, then Theorem 1.9 is a direct consequence of the conclusions in Proposition 6.32.

On the other hand, if we assume that the bundle $T\mathcal{O}^\perp$ is integrable, Proposition 3.7(4) does not necessarily imply that for almost every element x in the manifold the vector space $T_x\mathcal{O}^\perp$ is a non-trivial $\mathfrak{sl}(3, \mathbb{R})$-module. However, by Remark 3.8, we have two options to explore:

(a) $T_x\mathcal{O}^\perp$ is a trivial $\mathfrak{g}$-module for almost every $x \in S$, or

(b) There exists a subset, $A \subseteq S$, of positive measure such that $T_x\mathcal{O}^\perp$ is a non-trivial $\mathfrak{g}$-module for all $x \in A$.

When case (a) occurs then we have no "control" over the "behavior" of the bundle $T\mathcal{O}^\perp$. We believe it is necessary to develop new "tools" for a complete study of this case.

When case (b) is satisfied we obtain more properties about the structure of the complete pseudo-Riemannian manifold $\widetilde{N}$ in Theorem 1.7. For example, if $\mathcal{W}(x_0)$, in the decomposition of $\mathcal{H}$, is an abelian (2-step nilpotent) Lie algebra then $\widetilde{N}$ is isometric to the abelian (2-step nilpotent) Lie group $\mathbb{R}^3 \times \mathbb{R}^{3*}$, this is the conclusion of Proposition 6.12 (Proposition 6.13).

On the other hand, when $\mathcal{H}$ is isomorphic to a direct sum of two simple Lie algebras, then our manifold $\widetilde{N}$ contains an open subset diffeomorphic to a quotient space of a simple Lie group, this is the result of Propositions 6.14, 6.15 and 6.16.

If we show that these quotient spaces are complete, then we prove that these spaces are isometric to $\widetilde{N}$. This is proved in the following Lemma

Lemma 6.33. *The quotient spaces in Propositions 6.14, 6.15 and 6.16 are complete.*

Proof. First, assume that the hypotheses of Proposition 6.14 are satisfied.

In this case, our manifold $\widetilde{N}$ contains an open subset diffeomorphic to

$$SL(3, \mathbb{R}) \backslash G_{2(2)}.$$

Then, from Lemma 4.6, there exists an $\mathrm{Ad}(SL(3, \mathbb{R}))$-invariant subspace, $\mathfrak{m}$, in $\mathfrak{g}_{2(2)}$ that is complementary to $\mathfrak{sl}(3, \mathbb{R})$ in $\mathfrak{g}_{2(2)}$, that is

$$\mathfrak{g}_{2(2)} = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{m}. \tag{6.16}$$

Hence, by Definition 6.27, the space $SL(3, \mathbb{R}) \backslash G_{2(2)}$ is a reductive coset manifold.

For $X \in \mathfrak{g}_{2(2)}$, denote by $X_{\mathfrak{sl}(3,\mathbb{R})}$ and $X_{\mathfrak{m}}$ the components of X in $\mathfrak{sl}(3, \mathbb{R})$ and $\mathfrak{m}$, respectively.

Now, we induce, through the pullback of the map $\hat{f}$ in (6.4), a scalar product on $\mathfrak{m}$. Following this, we have that $\mathfrak{m}$ and $T_{x_0}\mathcal{O}^\perp$ are linearly isometric.

On the other hand, by the properties of the isometric right $(SL(3,\mathbb{R})\times G_{2(2)})$-action on $\widetilde{M}$ and the map $\hat{f}$, we have that the scalar metric on $\mathfrak{m}$ is $ad(\mathfrak{sl}(3,\mathbb{R}))$-invariant. This previous result is a particular case of Proposition 22 in [ONe, p. 311].

It is not difficult to show that the $ad(\mathfrak{sl}(3,\mathbb{R}))$-invariance on the scalar product in $\mathfrak{m}$ induces that

$$\langle [X,Y]_{\mathfrak{m}}, Z\rangle = \langle X, [Y,Z]_{\mathfrak{m}}\rangle \tag{6.17}$$

for all $X, Y, Z \in \mathfrak{m}$.

Therefore, we have that the quotient space $SL(3,\mathbb{R})\backslash G_{2(2)}$ is a *naturally reductive homogeneous space* (Definition 6.28). Recall that these spaces are characterized because they are *complete*, [ONe, p. 313].

Now, we assume the hypotheses of Proposition 6.15.

Then, our manifold $\widetilde{N}$ contains an open subset diffeomorphic to

$$(SL(3,\mathbb{R})\times\mathbb{R})\backslash\widetilde{SL}(4,\mathbb{R}).$$

From Chapter 1 of [CP], we have that $(SL(3,\mathbb{R})\times\mathbb{R})\backslash\widetilde{SL}(4,\mathbb{R})$ is a pseudo-Riemannian symmetric space. A characteristic of these spaces is that they are *complete*.

Similarly we can prove that the quotient space

$$SL(4,\mathbb{R})\backslash\widetilde{SO}_0(3,4),$$

in Proposition 6.16, is complete. □

The results of Propositions 6.14-6.16 and Lemma 6.33 end up completing the proof of Theorem 1.10.

Note, at this point, that from Theorems 1.7, 1.8 and 1.9 we can conclude Theorem 1.6.

If the property of weak irreducibility is assumed for the manifold M, then Proposition 6.7 and Remark 6.9 imply that $T\mathcal{O}^{\perp}$ is a non-integrable bundle. Then, Propositions 6.12-6.16 evidence that case 1) in Theorem 1.6 is not possible. This is precisely the conclusion of Theorem 1.11.

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