CIMAT

Centro de Investigación en
Matemáticas A.C.

Virus Spread on Networks with Long-Range Interactions

T H E S I S

To obtain the title of

Master of Science

with speciality in

Applied Mathematics

P R E S E N T :

Elkin Wbeimar Ramírez

Thesis advisor:

Dr. Marcos Aurelio Capistrán Ocampo

GUANAJUATO, GTO. AUGUST 2014

Centro de Investigación en Matemáticas A.C.

CIMAT

Virus Spread on Networks with Long-Range Interactions

T H E S I S

To obtain the title of

Master of Science

with speciality in

Applied Mathematics

P R E S E N T :

Elkin Wbeimar Ramírez

Thesis Committee Members:

Dr. José Ignacio Barradas Bribiesca
(President)

Dra. Mayra Núñez López
(Secretary)

Dr. Marcos Aurelio Capistrán Ocampo
(Thesis advisor)

GUANAJUATO, GTO. AUGUST 2014

To my dear family

Acknowledgments

I would like to express my gratitude to my thesis advisor Dr. Marcos Capistrán for his useful comments, remarks and advice through the learning process of this master thesis. Furthermore, I would like to thank CONACyT for supporting me with a scholarship. Also, I would like to thank CIMAT for giving me the opportunity of an academic and cultural interchange. I would like to thank my loved ones and friends, who have supported me throughout this entire process.

Abstract

In this thesis we study a SIS epidemic model on complex networks that takes into account non-random long-range interactions. In particular, we examine numerically and theoretically the validity of the following conjecture: *hub-vaccination is not necessarily the optimal vaccination strategy*. In the first part we conduct a series of numerical experiments that support our claim. Later, we study theoretically our conjecture on a star graph. Finally, we conclude that indeed, it is possible to have regimes where hubs are not determinant to support the spread of an infection on a complex network.

Introduction

Epidemiology can be defined as *"the study of the distribution and determinants of health-related states or events in specified populations, and the application of this study to control health problems"*. When populations can be represented as a graph, different models arise to study the behavior of viruses or diseases inside them [1, 6, 10, 12, 13]. In this text, a variant of the SIS model presented by *Piet Van Mieghem* [12, 13] will be studied taking into account long-range interactions. In [10], Estrada introduces a measure of this interaction. Here, the Mieghem and Estrada models are mixed to propose a "new model". Imagine a graph that represents people in a town and its links of friendship, and a hypothetical case of an infected person within a closed place, such as a bus, bank, etc. It is apparent that the possibility exists that this person infects others who are in the same place, without being friends or relatives, i.e. they do not need to be directly related in the graph. This situation is not considered in Mieghem's model, because there are only interactions with people directly related in the graph.

Here, vaccination strategies are presented, taking into account some centrality measures given in graph theory, which are compared in order to examine a non-intuitive conjecture: *hub-vaccination is not necessarily the optimal vaccination strategy.* We offer theoretical conditions where the conjecture holds. These conditions are given in particular graphs: Barabási-Albert graphs, Erdös-Renyi graphs, random geometric graphs and star graphs.

Contents

Chapter 1

Motivation

Epidemics dynamics on complex networks can be counterintuitive. There is strong dependence on vertex distribution and centrality. Therefore, non-random long range interactions may play an important role. Unpublished research by Capistran [4] shows the following example: Consider a stochastic, discrete time, SIS epidemic model on a Barabási-Albert complex network with 100 nodes. Figures 1.1 and 1.2 show the average of 250 realizations the SIS model as a function of the graph conductivity, e.g. the strength of the long-range interactions. In both figures, we can see that if conductivity is near 0.14, the final size of infected nodes when they are vaccinated to the largest degree nodes is larger than when they are vaccinated randomly. The purpose of the present work is to study vaccination on a mean field approximation of the stochastic SIS model defined over a random network. The mean field equation was introduced by Piet Van Mieghem [12, 13]. This work, under the same assumptions given by Piet Van Mieghem, we have modified the mean field equation in order to take account non-random long-range interactions.

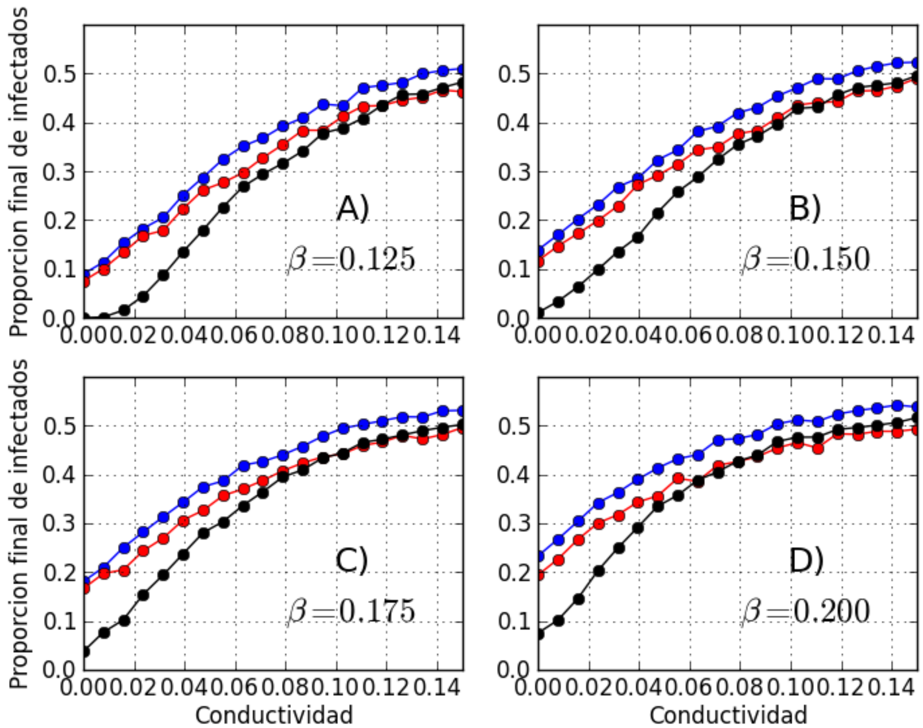


Figure 1.1: In blue, the SIS model without vaccination. In red, the SIS model with random vaccination. In black, the SIS model with largest degree vaccination. In the graphic appears the final size of infected nodes against conductivity. $\delta = 0.8$. 4% of vaccinated nodes.

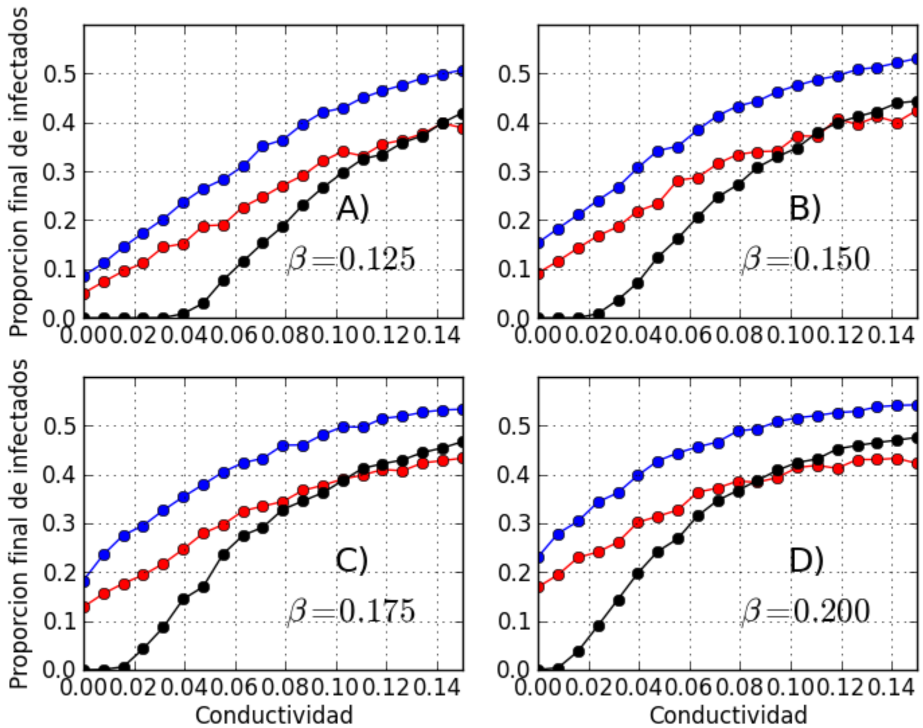


Figure 1.2: In blue, the SIS model without vaccination. In red, the SIS model with random vaccination. In black, the SIS model with largest degree vaccination. In the graphic appears the final size of infected nodes against conductivity. $\delta = 0.8$. 10% of vaccinated nodes.

Chapter 2

Preliminaries

For the sake of making this work self-contained. In this chapter we include some theoretical results. These are about probability [16, 18] and graph theory [2, 3, 7, 8, 9, 14, 15] principally.

2.1 Probability theory topics

Definition 2.1.1. Let Ω be a set, and 2^Ω represent its power set. Then a subset $\mathcal{A} \subset 2^\Omega$ is called *$\sigma-$algebra* if it satisfies the following properties.

1. $\Omega \in \mathcal{A}$.

2. $\mathcal{A}$ is closed under complementation: if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.

3. $\mathcal{A}$ is closed under countable unions: if $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$, then $\cup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

Definition 2.1.2. A *probability space* is a tripe $(\Omega, \mathcal{A}, \mathcal{P})$, where Ω is a set, $\mathcal{A}$ is a $\sigma-$algebra and $\mathcal{P} : \mathcal{A} \to [0, 1]$ is a function such that:

1. $\mathcal{P}(\Omega) = 1$.

2. If $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ is a countable collection of pairwise disjoint sets, then $\mathcal{P}\left(\uplus_{n\in\mathbb{N}} A_n\right) = \uplus_{n\in\mathbb{N}} \mathcal{P}(A_n)$, where $\uplus$ represents the disjoint union.

Definition 2.1.3. Events $\{A_n\}_{n\in\mathbb{N}}$ are *independent* if

$$\mathcal{P}(A_{n_1} \cap \cdots \cap A_{n_k}) = \mathcal{P}(A_{n_1}) \cap \cdots \cap \mathcal{P}(A_{n_k}),$$

for every finite set of distinct indices $n_1, \ldots, n_k$.

Definition 2.1.4. Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space. A *random variable* is a function $X : \Omega \to \mathbb{R}$ such that,

$$X^{-1}\left((-\infty, x]\right) \in \mathcal{A}, \quad \forall x \in \mathbb{R}.$$

Definition 2.1.5. Let X be a random variable. It is called *Bernoulli random variable* if X can only take two values 0 or 1 according to,

$$\mathcal{P}(X = j) = \begin{cases} p & \text{if} \quad j = 1, \\ q = 1 - p & \text{if} \quad j = 0. \end{cases}$$

For example, if A is a event and 1_A is the indicator function from A, then 1_A is a Bernoulli random variable with parameter $p = \mathrm{E}[1_A] = \mathcal{P}(A)$. As other example, if we flip a fair coin. Let $X =$ number of heads. Then X is a Bernoulli random variable with $p = \frac{1}{2}$.

Definition 2.1.6. A *stochastic process* is a family of random variables denoted as $\{X(t), t \in T\}$, where t is a parameter running over a suitable index set T.

Stochastic processes for which $T = [0, \infty)$ are particularly important for applications. There are two important types of processes, Poisson process and Markov process.

Definition 2.1.7. A *Poisson process* with parameter or rate $\lambda > 0$ is an integer-valued, continuous time stochastic process $\{X(t), t \geq 0\}$ satisfying

1. $X(0) = 0$.
2. For all $t_0 = 0 < t_1 < \cdots < t_n$, the increments $X(t_1)-X(t_0), X(t_2)-X(t_1), \ldots, X(t_n)-X(t_{n-1})$ are independent random variables.
3. For $t \geq 0, s > 0$ and non-negative integers k, the increments have to Poisson distribution
$$\mathcal{P}\left(X(t+s) - X(s) = k\right) = \frac{(\lambda t)^k \mathrm{e}^{-\lambda t}}{k!}.$$

Definition 2.1.8. A *Markov process* $\{X(t), t \in T\}$, is a stochastic process with the property that, given the value of $X(t)$, the values of $X(s)$, $s \in T$ and $s > t$ are not influenced by values of $X(u)$ with $u \in T$ and $u < t$.

If sample space is discrete, the Markov process is called *Markov chain.* For a continuous-time Markov chain $\{X(t), t \geq 0\}$ with N states, the Markov chain's definition can be written as

$$\mathcal{P}[X(t+\tau) = j | X(\tau) = i, X(u) = x(u), 0 \leq u < \tau] = \mathcal{P}[X(t+\tau) = j | X(\tau) = i],$$

This reflects the fact that the future state at time $t + \tau$ only depends on the current state at time τ. The transition probability is

$$P_{ij}(t) = \mathcal{P}[X(t+\tau) = j | X(\tau) = i] = \mathcal{P}[X(t) = j | X(0) = i]. \tag{2.1}$$

If we defined a state vector $s(t)$ as $s(t) = (s_1(t), \ldots, s_N(t))$, where $s_k(t) = \mathcal{P}[X(t) = k]$, $s(t)$ obeys (page 158, [14]),

$$s(t+\tau) = s(\tau) P(t), \tag{2.2}$$

where $P(t) = [P_{ij}]$. Immediately, it follows from (2.2) that

$$\begin{aligned} s(t+u+\tau) &= s(\tau)P(t+u) \\ &= s(\tau+u)P(t) = s(\tau)P(u)P(t) \\ &= s(\tau+t)P(u) = s(\tau)P(t)P(u), \end{aligned}$$

such that, for all $t, u \geq 0$, the $N \times N$ transition probability matrix $P(t)$ satisfies

$$P(t+u) = P(u)P(t) = P(t)P(u) \tag{2.3}$$

This fundamental relation is called the Chapman-Kolmogorov equation. Also, for any state i (page 159, [14]),

$$\sum_{j=1}^{N} P_{ij}(t) = 1, \tag{2.4}$$

for continuous-time Markov chains, it is convenient to postulate the initial condition of the transition probability matrix

$$P(0) = I, \tag{2.5}$$

where $P(0) = \lim_{t\to 0} P(t)$. Here, we have one result,

Lemma 2.1.1. *The transition probability matrix $P(t)$ is continuous for all $t \geq 0$.*

The proof can be found in [14], page 180. If $P(t)$ is a differential function, then the matrix,

$$\lim_{h\to 0} \frac{P(h) - I}{h} = P'(0) = Q, \tag{2.6}$$

is called the *infinitesimal generator* of the continuous-time Markov process. From (2.4),

$$\sum_{j=1, j\neq i}^{N} P_{ij}(h) = 1 - P_{ii}(h), \tag{2.7}$$

and, dividing both sides by h and letting h approach zero, we find for each i with the definition of Q that

$$\sum_{j=1, j\neq i}^{N} q_{ij} = -q_{ij} \geq 0. \tag{2.8}$$

Lemma 2.1.2. *Given the infinitesimal generator Q, the transition probability matrix $P(t)$ is differentiable for all $t \geq 0$,*

$$\begin{aligned} P'(t) &= P(t)Q \\ &= QP(t) \end{aligned} \tag{2.9}$$

The equations in (2.9) are called the forward and backward equations respectively. (Proof in [14], page 182.) Suppose we are interested in the probabilities $s_k = \mathcal{P}[X(t) = k]$ of finding the system in state k at time t. Each component of the state vector $s(t)$ is determined by (2.2) as

$$s_k(t+h) = \sum_{j=1}^{N} s_j(t) P_{jk}(h),$$

from which

$$\frac{s_k(t+h) - s_k(t)}{h} = s_k \frac{P_{kk}(h) - 1}{h} + \sum_{j=1, j\neq k}^{N} s_j(t) \frac{P_{jk}(h)}{h}.$$

In the limit $h \to 0$, we find with $q_{jk} = \lim_{h\to 0} \frac{P_{jk}(h)}{h}$ and $q_k = \lim_{h\to 0} \frac{1-P_{kk}(h)}{h}$, the differential equation for $s_k(t)$,

$$s'_k(t) = -q_k s_k(t) + \sum_{j=1, j\neq k}^{N} q_{jk} s_j(t), \tag{2.10}$$

which, together with the initial condition $s_k(0)$, completely determines the probability $s_k(t)$ that the Markov process is in state k at time t.

2.2 Graph theory topics

Definition 2.2.1. A simple *graph (network)* is the pair $G = (V, E)$, where $V$is a *finite set of points* and E is a symmetric and antireflexive relation on V. In a *directed graph (network)* the relation$E$ is non-symmetric. The elements of V are the *vertices* (or *nodes*, or *points*) of G. The elements of E are its *links* (*edges* or *lines*).

From now on, to assume that all graphs are simple.

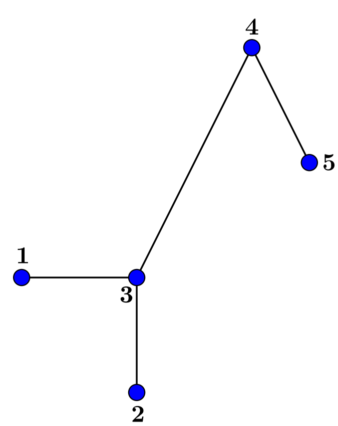


Figure 2.1: The graph on $V = \{1, 2, 3, 4, 5\}$ with edge set $E = \{\{1, 3\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$. Thus, $G = (V, E)$.

Definition 2.2.2. Let $G = (V, E)$ be a (non-empty) graph. The set of neighbors of a vertex $v \in V$ in G is denoted by $B_G(v)$, or briefly $N(v)$. The *degree* $d_G(v) = d(v)$ of a vertex v is the number $|E(v)|$.

For G in the figure 2.1, $d(1) = d(2) = d(5) = 1,\ d(4) = 2$ and $d(3) = 3$.

Definition 2.2.3. A *walk* of leng l is any sequence of (not necessarily different) nodes $v_1, v_2, \ldots, v_l, v_{l+1}$ such that for each $i = 1, 2, \ldots, l$ there is a link from v_i to v_{i+1}. A *path* of length l is a walk of length l in which all nodes (an all the links) are distinct. A *cycle* is a closed walk in which all the links and all the nodes (except the first and last) are distinct.

Definition 2.2.4. A non-empty graph G is called *connected* if any two of its vertices are linked by a path in G.

From now on, to assume that all graphs are connected.

Definition 2.2.5. A graph $H = (V', E')$ is subgraph of a network $G = (V, E)$ if and only if $V' \subseteq V$ and $E' \subseteq E$.

Definition 2.2.6. Let $G = (V, E)$ be a (non-empty) graph. A subgraph $H(C, E)$ induced by the set $C \subseteq V$ is a k-*core* or a core of order k if and only if the degree of every node $v \in C$ induced in H is greater or equal than k, and H is the maximum connected subgraph with this property. The $k-$core with highest index is called *main core.*

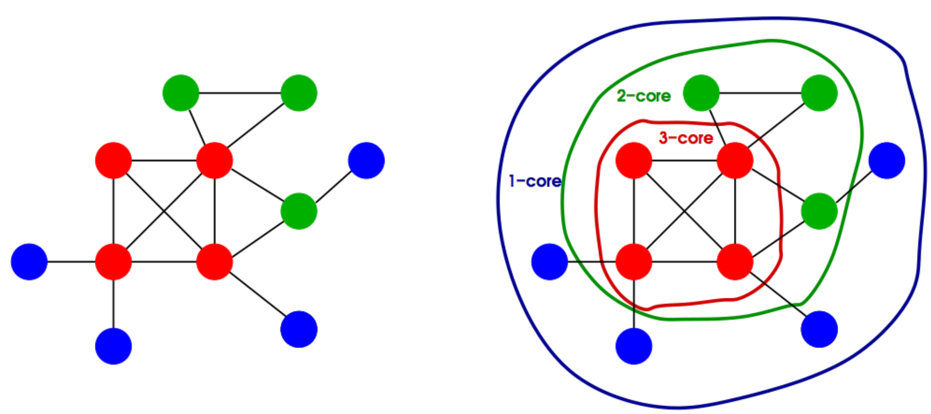


Figure 2.2: Example of $k-$core.

2.2.1 Some kinds of graphs

Definition 2.2.7. A graph with N nodes is a *Complete Graph*, denoted K_N, if every pair of nodes is connected by a link. That is, there are $\frac{n(n-1)}{2}$ links.

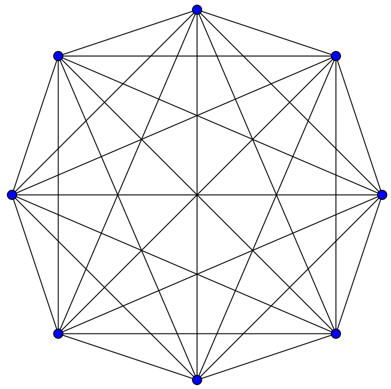


Figure 2.3: A complete graph. K_8.

Definition 2.2.8. A *Tree* of N nodes is a graph for which the following statements are equivalent:

- It is connected and has no cycles.
- It has $n - 1$ links and no cycle.
- It is connected and has $n - 1$ links.
- It is connected and become disconnected by removing any link.
- Any pair of nodes is connected by exactly one path.
- It has no cycles, but the addition of any new link creates a cycle.

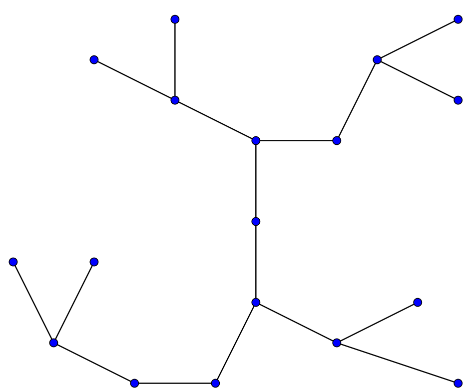


Figure 2.4: A tree.

Definition 2.2.9. Let $r \geq 2$ be a integer. A graph $G = (V, E)$ is called *r-partite* if V admits a partition into r classes such that every edge has its ends in different classes: vertices in the same partition class must no be linked. Instead of “2-partite” one usually says bipartite. An r-partite graph in which every two vertices from different partition classes are linked is called *complete multipartite* graphs. The complete r-partite graph $\overline{K^{n_1}} * \cdots * \overline{K^{n_r}}$ is denoted by $K_{n_1,\cdots,n_r}$; if $n_1 = \cdots = n_r = s$, it is possible abbreviate this to N_s^r. Thus, K_s^r is complete r-partite graph in which every partition class contains exactly s vertices. Graphs of the form K_1^n are called *stars*.

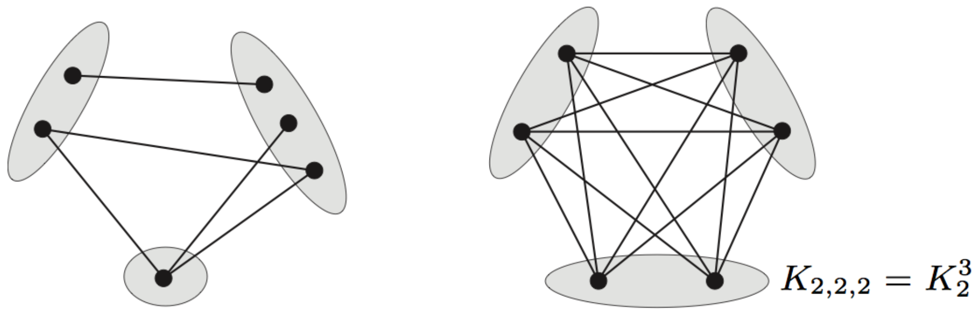


Figure 2.5: Two 3-partite graphs. The right graph is a complete 3-partite graph. Figure taken from [7].

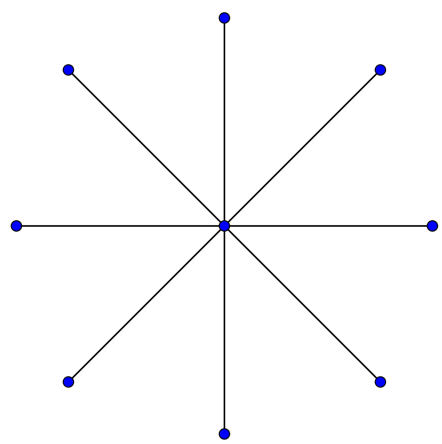


Figure 2.6: A star graph. K_1^8.

Definition 2.2.10. The *adjacency matrix* $A = (a_{ij})_{n\times n}$ of a graph G is defined by

$$\begin{cases} w_{ij} & \text{if the node } i \text{ is linked with the node, } j \\ 0 & \text{otherwise,} \end{cases}$$

where w_{ij} is the weight of the link $\{i, j\} \in E$. If this values is no specified, then $w_{ij} = 1$.

For the graph in the figure 2.1, the adjacency matrix is

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

2.2.2 Some centrality measures

Definition 2.2.11. *Closeness centrality* of a node i ($C(i)$) is the reciprocal of the sum of the shortest path distances from i to all $N-1$ other nodes. Since the sum of distance depends on the number of nodes in the graph, closeness is normalized by the sum of minimum possible distances $N-1$. In other words, *Closeness centrality* is the inverse of the average distance to all other nodes.

$$C(i) = \frac{N-1}{\sum_{j=1, j\neq i}^{N} d(i,j)},$$

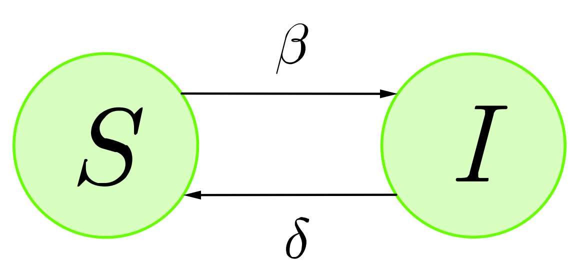
where $d(i,j)$ is the shortest-path distance between i and j and N is the number of nodes in the graph.

Definition 2.2.12. *Betweenness centrality* of a node i ($c_B(i)$) is the sum of the fraction of all-pairs shortest paths that pass through i :

$$c_B(i) = \sum_{s,t \in V} \frac{\sigma(s,t|i)}{\sigma(s,t)},$$

where V is the set of nodes, $\sigma(s,t)$ is the number of shortest (s,t)-paths and $\sigma(s,t|i)$ is the number of those paths passing through some node i other than s,t. If $s=t$, $\sigma(s,t)=1$ and if $i \in \{s,t\}$ then $\sigma(s,t|i)=0$.

2.3 SIS model



In this model, individuals are either susceptible (S) or infected (I). We assume that people meet and make contacts sufficient to result in the spread of the disease entirely at random with a per-individual rate β, meaning that each individual has, on average, β contacts with randomly chosen others per unit time. Also, infected individuals recover at some constant average rate δ. The disease is transmitted only when an infected person has contact with susceptible one. Likewise, infected individuals move back to into the susceptible state upon recovery. If the total population consists of N people, then the average probability of a person you meet at random being susceptible is $\frac{S}{N}$, and hence an infected person has contact with an average of $\beta\frac{S}{N}$ susceptible per unit time. Thus, the differential equations for this model are

$$\begin{aligned}\frac{dS}{dt} &= \delta I - \frac{\beta}{N}SI,\\ \frac{dI}{dt} &= \frac{\beta}{N}SI - \delta I.\end{aligned}$$

Letting $\bar{S} = \frac{S}{N}$ and $\bar{I} = \frac{I}{N}$, we get

$$\begin{aligned}\frac{d\bar{S}}{dt} &= \delta\bar{I} - \beta\bar{S}\bar{I},\\ \frac{dI}{dt} &= \beta\bar{S}\bar{I} - \delta\bar{I},\end{aligned}$$

due to $S + I = N$, then $\bar{S} + \bar{I} = 1$. Using this,

$$\frac{d\bar{I}}{dt} = (\beta - \delta - \beta\bar{I})\bar{I},$$

which is a logistic equation. $\bar{I}(t) = 1 - \frac{\delta}{\beta}$ is called *endemic disease state.* For simplicity of writing, $\bar{S} = S$ and $\bar{I} = I$. Our interest is to know R_0 because this value give us information about the disease. It is known by survival function that R_0 is the continuous sum of the product between $b(t)$ and $F(t)$, where $b(t)$ is the average number of newly infected individuals an infectious individual produces per unit time when infected total time t, and $F(t)$ is the probability that a newly infected individual remains infectious for at least time t. By β's definition $b(t) = \beta t$. Now, for $F(t)$, the probability of recovering in any interval $[t, t + dt]$ is $\delta\, dt$, thus the probability of not doing so is $1 - \delta\, dt$. Then, the probability that the individual is still infected after a total time t is given by

$$\lim_{dt\to 0}(1 - \delta\, dt)^{\frac{t}{dt}},$$

and this limit is equal to $\mathrm{e}^{-\delta t}$. Hence, the probability that the individual remains infected this long and then recovers in the interval $[t, t + dt]$ is $F(t) = \delta\mathrm{e}^{-\delta t}\, dt$. Therefore,

$$R_0 = \int_0^\infty \beta\delta t\, \mathrm{e}^{-\delta t}\, dt,$$

thus

$$R_0 = \frac{\beta}{\delta}. \tag{2.11}$$

Chapter 3

Theoretical Framework

In this chapter we describe van Mieghem and Estrada models. Then we discuss how we combine them to obtain a mean field equation of a stochastic SIS model that takes into account non-random long range interactions. The Estrada model will be used in the rest of this work to study the outcome of several vaccination strategies in terms of the measure centralities.

3.1 Piet Van Mieghem's Model

The model considers the virus spread in an undirected graph $G(N, L)$, characterized by its adjacency matrix A [3, 9, 11]. The following hypotheses are assumed:

- The state of the node i is given by a Bernoulli random variable $X_i \in \{0, 1\}$, where $X_i = 0$ for a healthy node and $X_i = 1$ for a infected node.
- A node i at time t can be in just one of the two states, infected with probability $v_i(t) = \Pr[X_i(t) = 1]$ or healthy with probability $1 - v_i(t)$.
-
- The curing process per node i is a Poisson process with rate δ, and the infection rate per link is a Poisson process with rate β.
- All involved Poisson processes are independent.
- The effective infection rate is defined as $R_0 = \frac{\beta}{\delta}$.
- The initial infection condition is known per node i. It is $v_i(0)$.

Applying Markov theory, the infinitesimal generator $Q_i(t)$ of this two states Markov chain is,

$$Q_i(t) = \begin{bmatrix} -q_{1,i}(t) & q_{1,i}(t) \\ q_{2,i}(t) & -q_{2,i}(t) \end{bmatrix},$$

with $q_{2,i}(t) = \delta$ is the curing rate and $q_{1,i}(t) = \beta \sum_{j=1}^{N} a_{ij} 1_{X_j(t)=1}$, where $a_{i,j}$ is the component (i,j) of the adjacency matrix and $1_{X_j(t)=1}$ is the indicator function. $q_{1,i}(t)$ is equal to the sum over all infection rates of infected neighbors of node i. The coupling of node i to the rest of the network is described by an infection rate $q_{1,i}$ that is a random variable. The mean field approximation consists of replacing $q_{1,i}$ by its average $\mathrm{E}[q_{1,i}]$, this is a real number and allows immediate application of continuous-Markov theory.

$$\mathrm{E}[q_{1,i}] = \mathrm{E}\left[\beta \sum_{j=1}^{N} a_{ij} 1_{X_j(t)=1}\right],$$

using the linearity and that $\mathrm{E}\left[1_X\right] = \Pr\left[X\right]$,

$$\mathrm{E}[q_{1,i}] = \beta \sum_{j=1}^{N} a_{ij} \Pr\left[X_j(t)=1\right] = \beta \sum_{j=1}^{N} a_{ij} v_i(t),$$

which results in an effective infinitesimal generator,

$$\overline{Q_i(t)} = \begin{bmatrix} -\mathrm{E}[q_{1,i}] & \mathrm{E}[q_{1,i}] \\ \mathrm{E}[q_{2,i}] & -\mathrm{E}[q_{2,i}] \end{bmatrix} = \begin{bmatrix} -\mathrm{E}[q_{1,i}] & \mathrm{E}[q_{1,i}] \\ \delta & -\delta \end{bmatrix}.$$

The Markov differential equation [14] for state $X_i = 1$ turns out,

$$\frac{dv_i(t)}{dt} = \beta \sum_{j=1}^{N} a_{ij} v_j(t) - v_i(t) \left(\sum_{j=1}^{N} a_{ij} v_j(t) + \delta \right),$$

i.e. the next differential equations system is obeyed,

$$\begin{aligned} \frac{dv_1(t)}{dt} &= \beta \sum_{j=1}^{N} a_{1j} v_j(t) - v_1(t) \left(\sum_{j=1}^{N} a_{1j} v_j(t) + \delta, \right), \\ &\vdots \\ \frac{dv_N(t)}{dt} &= \beta \sum_{j=1}^{N} a_{Nj} v_j(t) - v_N(t) \left(\sum_{j=1}^{N} a_{Nj} v_j(t) + \delta, \right) \end{aligned} \tag{3.1}$$

Written (3.1) in matrix equation,

$$\frac{dV(t)}{dt} = \beta A V(t) - \mathrm{diag}(v_i(t))(\beta A V(t) + \delta u), \tag{3.2}$$

where $V(t) = [v_1(t) \ldots v_N(t)]^T$, $u = [1, \ldots 1]^T$ and $\mathrm{diag}(v_i(t))$ is the diagonal matrix with elements $v_1(t), \ldots, v_N(t)$. Now, rewriting (3.2) using $V(t) = \mathrm{diag}(v_i(t))u$, the following is obtained

$$\frac{dV(t)}{dt} = (\beta A - \delta I)\, V(t) - \beta\, \mathrm{diag}\,(v_i)\, AV(t).$$

3.2 Modification of the Mieghem's Model

In [10], a generalization of the adjacency matrix of a graph is introduced by Estrada, taking into account long-range interactions among nodes non directly related. Following Estrada's idea, we consider a modification of Mieghem's model, changing the adjacency matrix for the matrix presented by Estrada. This new matrix is $M = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } \; i \sim j, \\ d_{ij}\, r^{d_{ij}-1} & \text{if } \; i \neq j \text{ and } i \nsim j, \\ 0 & \text{if } \; i = j, \end{cases} \tag{3.3}$$

$r \in [0, 1]$ is the conductivity and d_{ij} is the long of the shortest path between the node i and the node j.

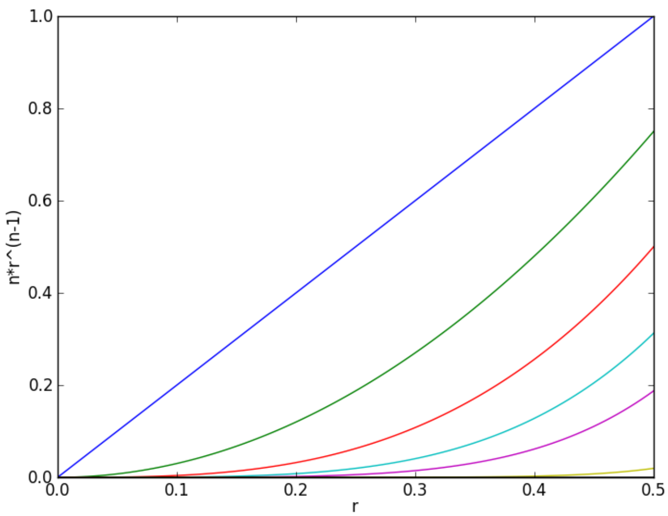


Figure 3.1: This graphic represents the behavior of the expression $n\, r^{n-1}$ with $n = 2$ (blue), 3 (green), 4 (red), 5 (light blue), 6 (purple), 10 (gold).

Note that m_{ij} gets closer to 0 as d_{ij} gets larger.

3.2.1 Model

We consider the virus spread in an undirected complete weithed graph $G(N, L)$. Weights are assigned to the new links according to model (3.3). This new graph is called $\widetilde{G}(N, L)$. (See figure 3.2.1). The adjacency matrix of $\widetilde{G}(N, L)$ is M. Notice that if $r = 0$, then M is the adjacency matrix of $G(N, L)$. Under the same assumptions of the Mieghem's model and a similar procedure, the next system is obtained,

$$\frac{dV(t)}{dt} = (\beta M - \delta I)\, V(t) - \beta\, \mathrm{diag}\,(v_i)\, MV(t), \tag{3.4}$$

where $V(t) = [v_1(t) \dots v_N(t)]^T$, M is like (3.3) and $\mathrm{diag}(v_i(t))$ is the diagonal matrix with elements $v_1(t), \dots, v_N(t)$.

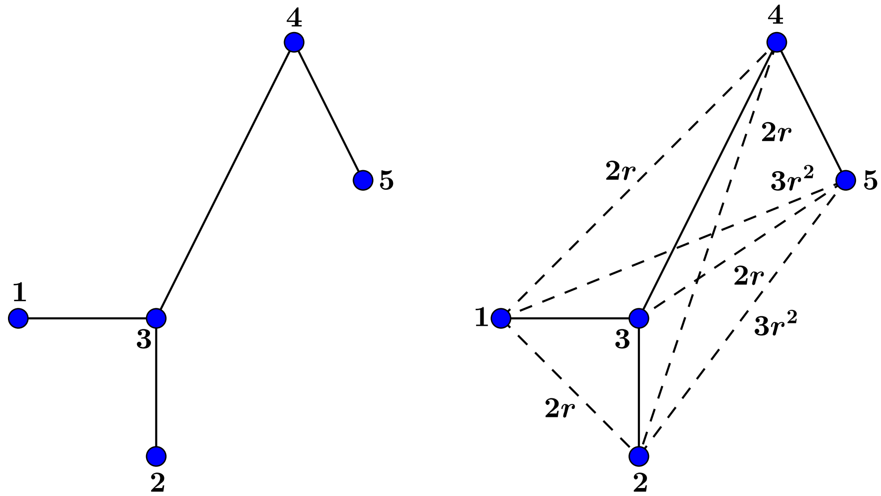


Figure 3.2: A graph with $N = 5$ nodes and its completeness following (3.3).

Chapter 4

Numerical Results

In this chapter we study the numerical solution from (3.4). We consider three kinds of networks: small world, scale free and random geometric. As small world and scale free networks were taken Erdös-Renyi and Barabási-Albert networks, respectively. Next, we examine some vaccination strategies taking into account some centrality measures in graphs like closeness, betweenness and largest degree. The solution is presented on graphs with 100 nodes and recovery rate $\delta = 1$. In the vaccination part, 10% of the nodes were vaccinated.

Figures 4.1, 4.2 and 4.3 show the average of the entries of the vector solution from (3.4). This average is represented in a color graph. One can see that if r and β are "large", the average of the probabilities is close to 1, this mean that the probability of that each node is infected is close to 1. If the infection rate β is sufficiently small, the probability of that each node is infected is close to 0.

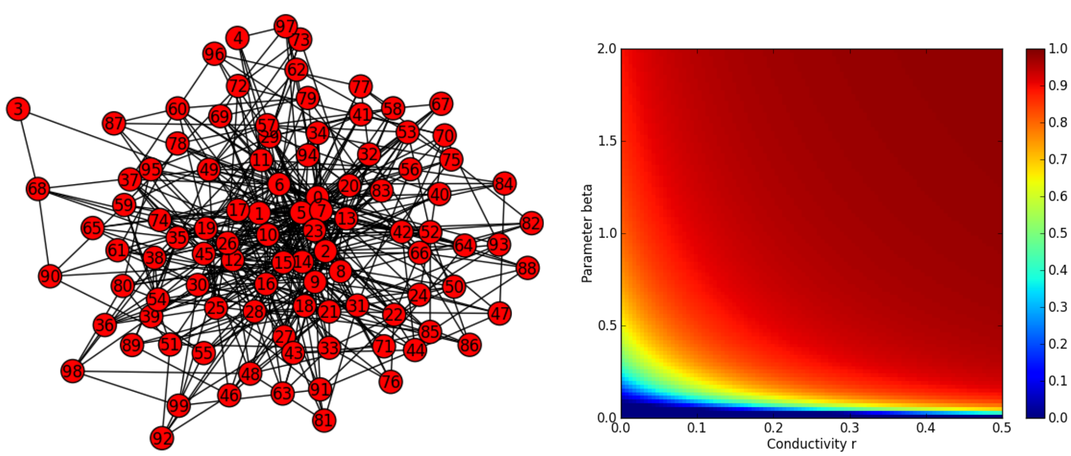


Figure 4.1: In the left, a Barabási-Albert network with 100 nodes. In the right, average of final size of the the solution from (3.4) in the previous graph at $t = 100$, $r \in [0, 0.5]$, $\beta \in [0, 2]$ and $\delta = 1$.

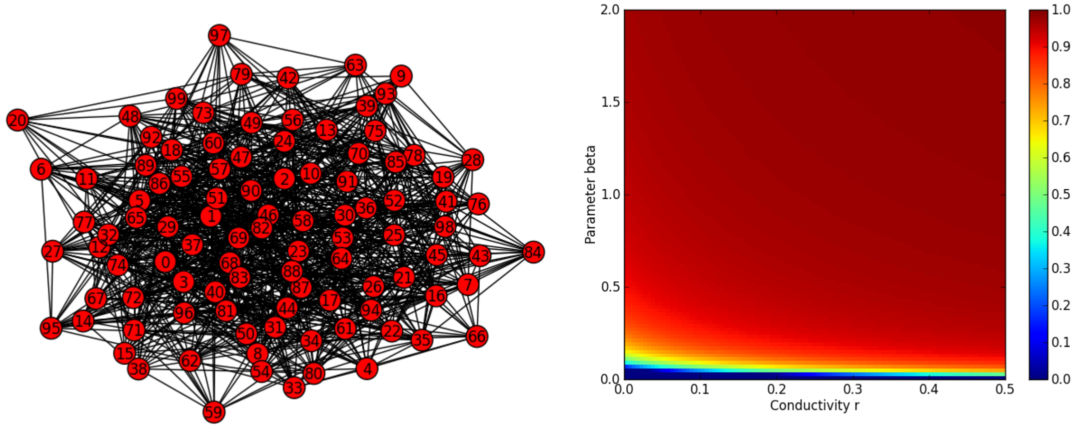


Figure 4.2: In the left, a Erdös-Renyi network with 100 nodes. In the right, average of final size of the solution from (3.4) in the previous graph at $t = 100$, $r \in [0, 0.5]$, $\beta \in [0, 2]$ and $\delta = 1$.

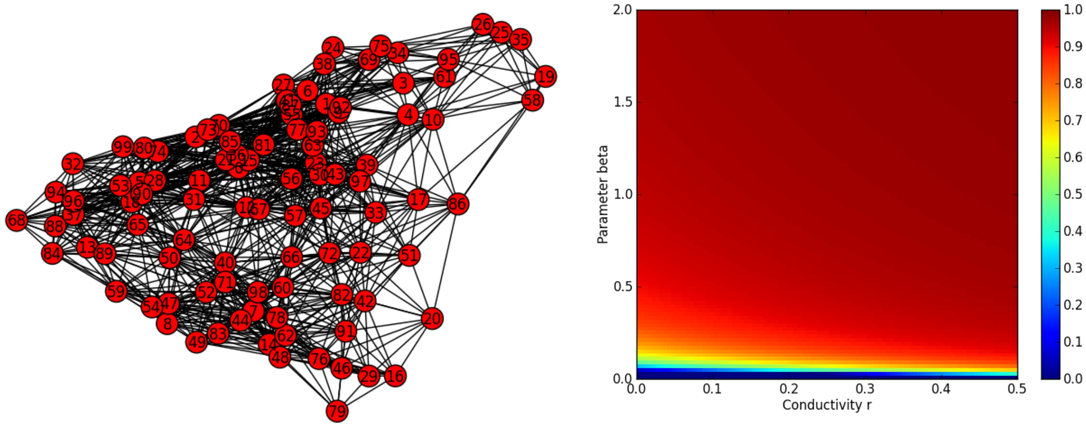


Figure 4.3: In the left, a random geometric network with 100 nodes. In the right, average of final size of the solution from (3.4) in the previous graph at $t = 100$, $r \in [0, 0.5]$, $\beta \in [0, 2]$ and $\delta = 1$.

4.1 Vaccination

When there is an infection in the environment, a natural question arises: how can we fight with that? In many cases vaccination is the answer. But, if we can see the population as a graph, how can we vaccinate it? Intuitively, the answer is to vaccinate hubs. In 2012, Claudio Castellano and Romualdo Pastor-Satorras in [5] showed that hubs in some cases are not determinant in virus spread on networks, they said that virus spread depends on the main k-core of the network [5, 17].

From now on, the vaccinate notion mean: if the node i is vaccinated, we will put zeros in the ith row and column of the adjacency matrix of the associated graph.

We shall compare the solutions from (3.4), considering several vaccination strategies. These strategies are given by some centrality measures, like largest degree, betweenness and closeness. Also, random vaccination will be studied. All these in order to verify our conjecture, *hub-vaccination is not necessarily the optimal vaccination strategy.*

From now on, when "size of the solution" appears, it must be understood as "average of the entries of the vector solution".

4.1.1 Vaccination on Barabási-Albert Network

In this part, we consider the solution of (3.4) in a fixed Barabási-Albert network (see figure 4.1). This solution will be through three vaccination strategies. The results are shown next,

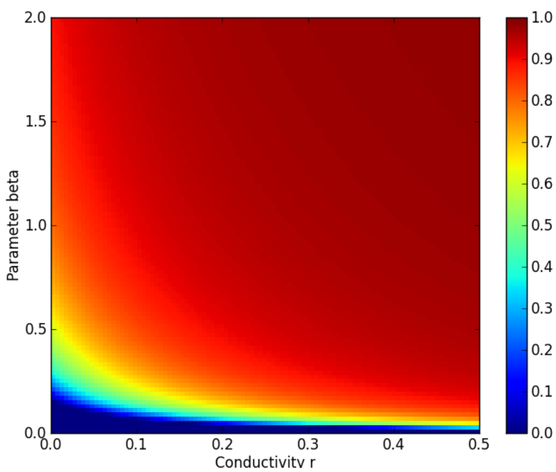


Figure 4.4: This graphic represents the size of the solution from (3.4) with largest degree vaccination on the Barabási-Albert network 4.1, at $t = 100$, $r \in [0, 0.5]$, $\beta \in [0, 2]$, $\delta = 1$ and a size $N = 100$.

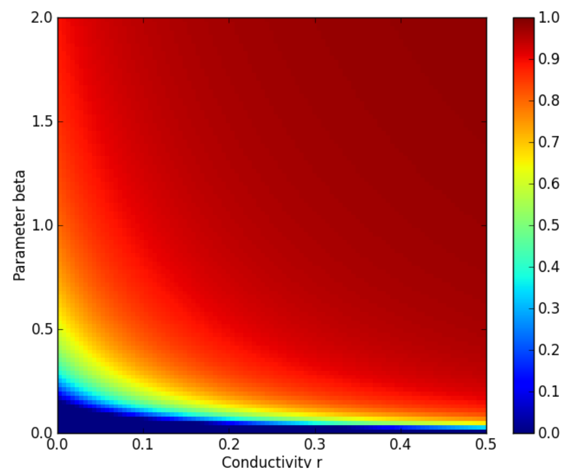


Figure 4.5: This graphic represents the size of the solution from (3.4) with betweenness vaccination on the Barabási-Albert network 4.1, at $t = 100$, $r \in [0, 0.5]$, $\beta \in [0, 2]$, $\delta = 1$ and a size $N = 100$.

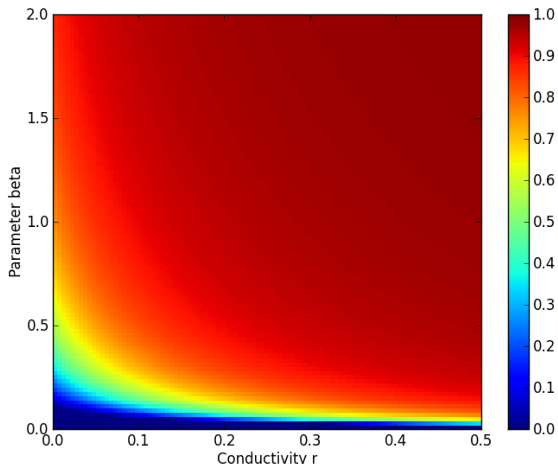


Figure 4.6: This graphic represents the size of the solution from (3.4) with random vaccination on the Barabási-Albert network 4.1, at $t = 100$, $r \in [0, 0.5]$, $\beta \in [0, 2]$, $\delta = 1$ and a size $N = 100$.

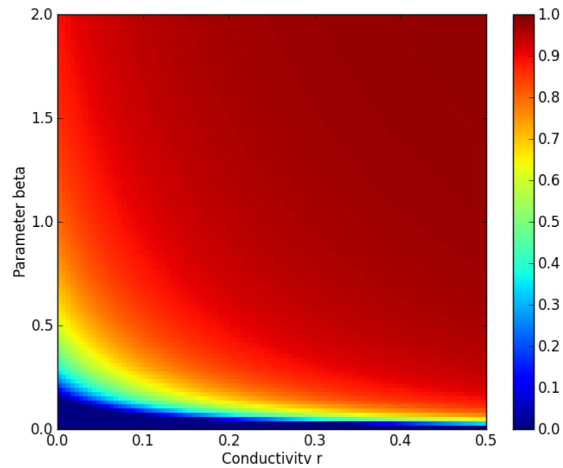


Figure 4.7: This graphic represents the size of the solution from (3.4) with closeness vaccination on the Barabási-Albert network 4.1, at $t = 100$, $r \in [0, 0.5]$, $\beta \in [0, 2]$, $\delta = 1$ and a size $N = 100$.

We can see that the figures 4.4, 4.5, 4.6, 4.7 are similar, so we will study the subtraction between them. This in order to identify regions wherever our hypothesis is true in this kind of network.

Convention 4.1.1. From now on, when sign of "anything" appears in any graphic, we will understand that the blue region the sign is positive, and the dark red region the sign is negative.

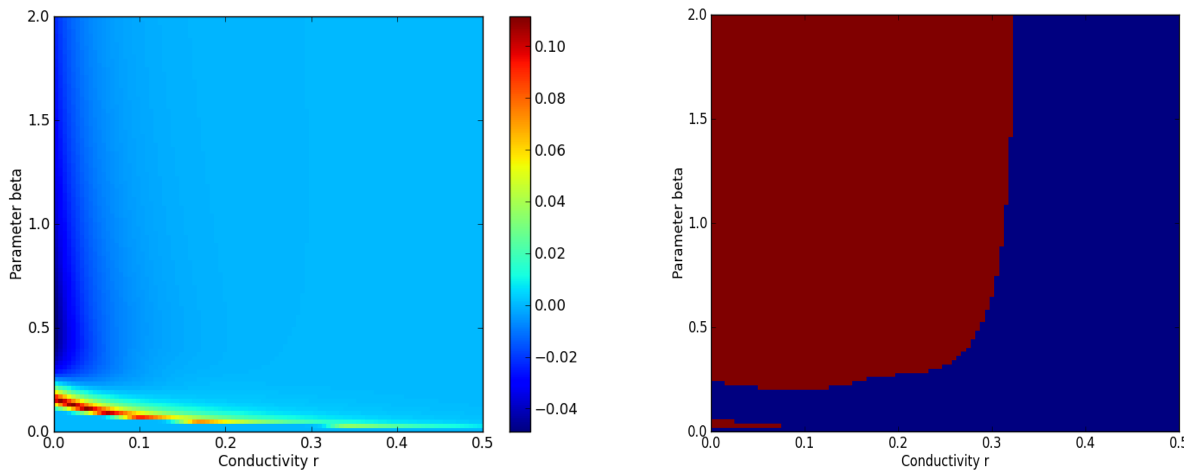


Figure 4.8: The first graphic represents the subtraction between the solution from (3.4), with random and largest degree vaccination (in that order). The sign of the subtraction appears in the second graphic.

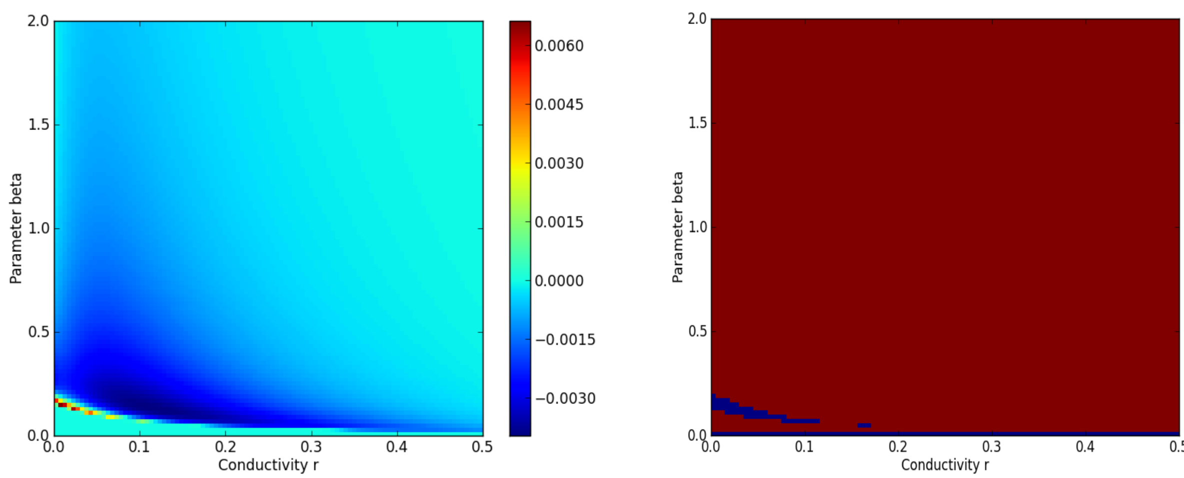


Figure 4.9: The first graphic represents the subtraction between the solution from (3.4), with closeness and largest degree vaccination (in that order). The sign of the subtraction appears in the second graphic.

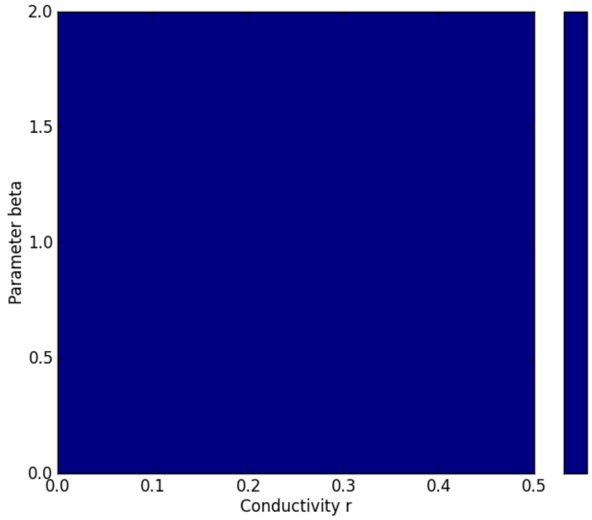


Figure 4.10: This graphic represents the subtraction between the solution from (3.4), with betweenness and largest degree vaccination (in that order). Here, the subtraction is zero for all values of β and r.

Figures 4.8 and 4.9 show that there are regions wherever our conjecture is true: *hub-vaccination is not necessarily the optimal vaccination strategy*. If r in $[0, 0.33]$ and a appropriate value of β in figure 4.8, it is a better idea to vaccinate randomly than to the nodes of largest degree on the network 4.1. In figure 4.9 shows that almost always it is better the strategy closeness vaccination over largest degree vaccination on the network 4.1.

In the figure 4.10 we can see that the difference is zero, because in the network 4.1 betweenness centrality is equal to degree centrality. In this network, the nodes with major betweenness and degree centrality are 5, 0, 7, 6, 2, 16, 9, 15, 14, 13.

Figures 4.8 and 4.9 are important because our non-intuitive conjecture is verified on Barabási-Albert network 4.1.

4.1.2 Vaccination on Erdös-Renyi Network

In this part, we consider the solution of (3.4) in a fixed Erdös-Renyi network (see figure 4.2). This solution will be through examined over three vaccination strategies. The results are shown next,

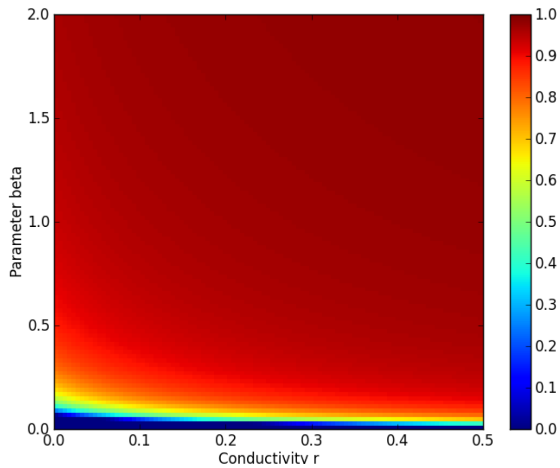


Figure 4.11: This graphic represents the size of the solution from (3.4) with largest degree vaccination on the Erdös-Renyi network 4.2, at $t = 100$, $r \in [0, 0.5]$, $\beta \in [0, 2]$, $\delta = 1$ and a size $N = 100$.

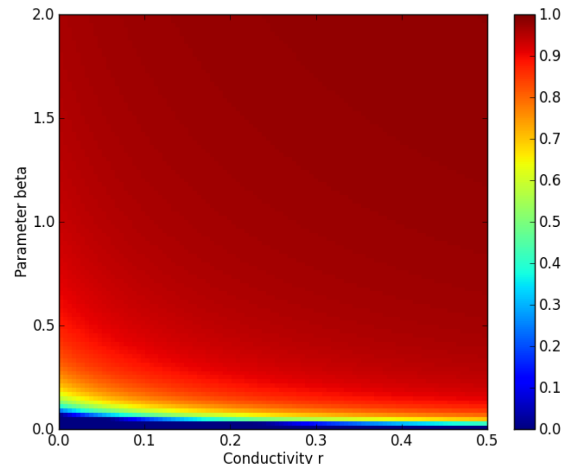


Figure 4.12: This graphic represents the size of the solution from (3.4) with betweenness vaccination on the Erdös-Renyi network 4.2, at $t = 100$, $r \in [0, 0.5]$, $\beta \in [0, 2]$, $\delta = 1$ and a size $N = 100$.

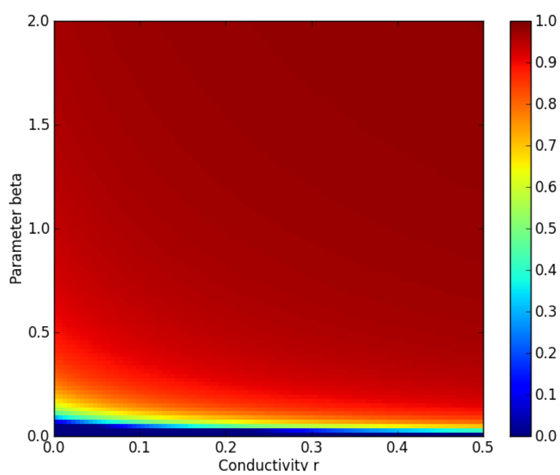


Figure 4.13: This graphic represents the size of the solution from (3.4) with random vaccination on the Erdös-Renyi network 4.2, at $t = 100$, $r \in [0, 0.5]$, $\beta \in [0, 2]$, $\delta = 1$ and a size $N = 100$.

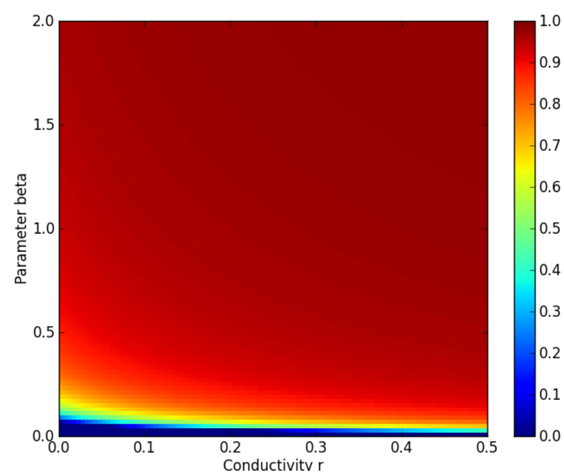


Figure 4.14: This graphic represents the size of the solution from (3.4) with closeness vaccination on the Erdös-Renyi network 4.2, at $t = 100$, $r \in [0, 0.5]$, $\beta \in [0, 2]$, $\delta = 1$ and a size $N = 100$.

As in the last strategy, the figures 4.11, 4.12, 4.13, 4.14 are similar, so we will study the subtraction between them. This in order to identify regions wherever our hypothesis is true in this kind of network.

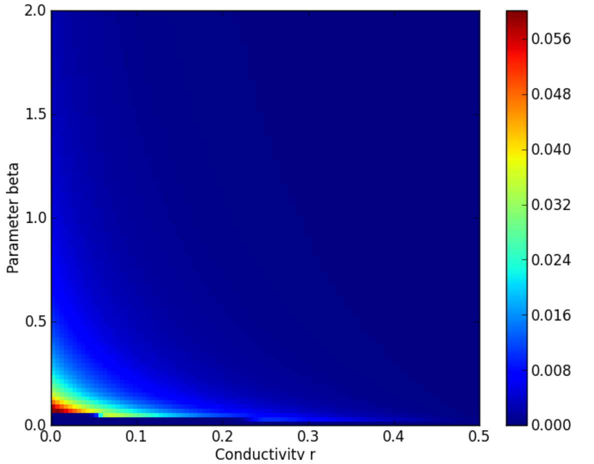


Figure 4.15: This graphic represents the subtraction between the sizes of the solution from (3.4), with random and largest degree vaccination (in that order). Here, the subtraction is positive for all values of β and r.

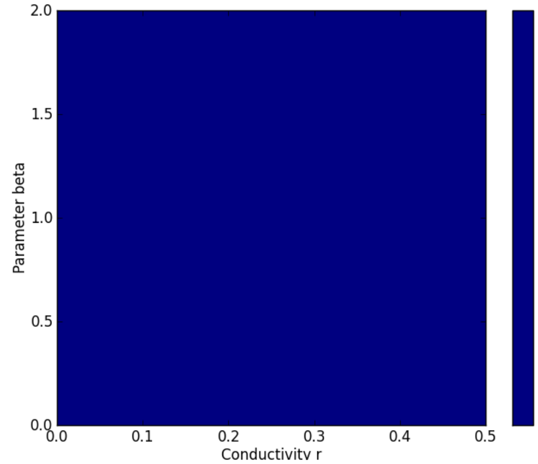


Figure 4.16: This graphic represents the subtraction between the sizes of the solution from (3.4), with closeness and largest degree vaccination (in that order). Here, the subtraction is zero for all values of β and r.

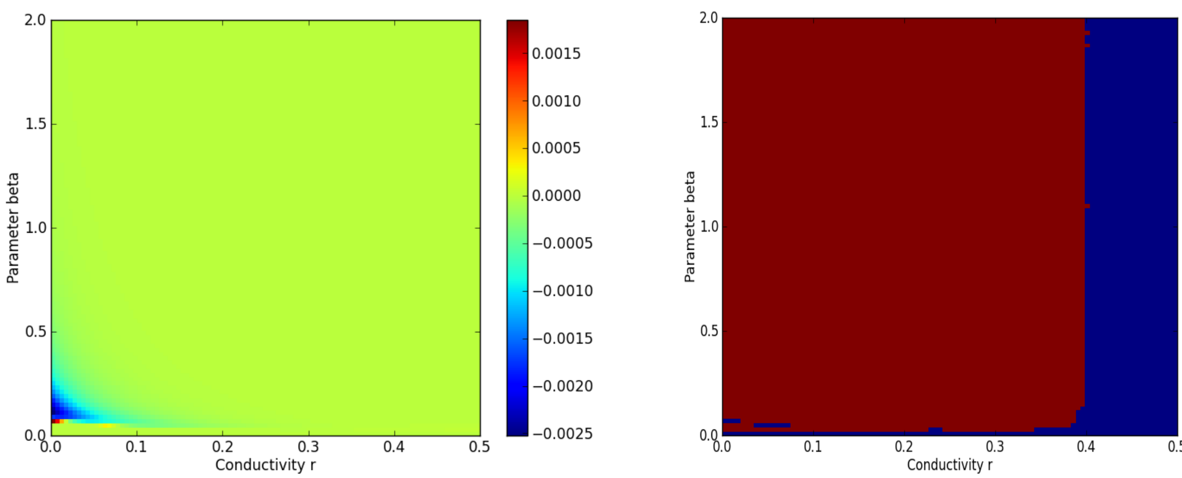


Figure 4.17: The first graphic represents the subtraction between the sizes of the solution from (3.4), with betweenness and largest degree vaccination (in that order). The sign of the subtraction appears in the second graphic.

Figures 4.17 shows that there is a region wherever our conjecture is verified. For values less to 0.4 of r and almost all β in $[0, 2]$, it is a better strategy to vaccinate the nodes with major betweenness centrality than the nodes with largest degree on the network 4.2.

In the figure 4.15, we can not verify our conjecture, because the subtraction is always positive i.e. to vaccinate the nodes of largest degree is a better strategy than to vaccinate randomly.

In the figure 4.16 we can see that the difference is zero, because in the network 4.2 closeness centrality is equal to degree centrality. In this network, the nodes with major closeness and degree centrality are 44, 58, 1, 30, 89, 5, 57, 68, 82, 29.

Figure 4.17 is important because our non-intuitive conjecture is verified on Erdös-Renyi network 4.2.

4.1.3 Vaccination on Random Geometric Network

In this part, we consider the solution of (3.4) in a fixed Random Geometric graph (see figure 4.3). As before, the solution will be computed for three vaccination strategies. The results are shown next,

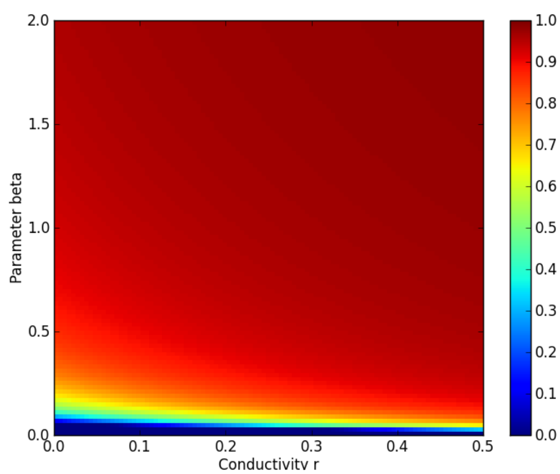


Figure 4.18: This graphic represents the size of the solution from (3.4) with largest degree vaccination on the Random Geometric network 4.3, at $t = 100$, $r \in [0, 0.5]$, $\beta \in [0, 0.08]$, $\delta = 0.1$ and a size $N = 100$.

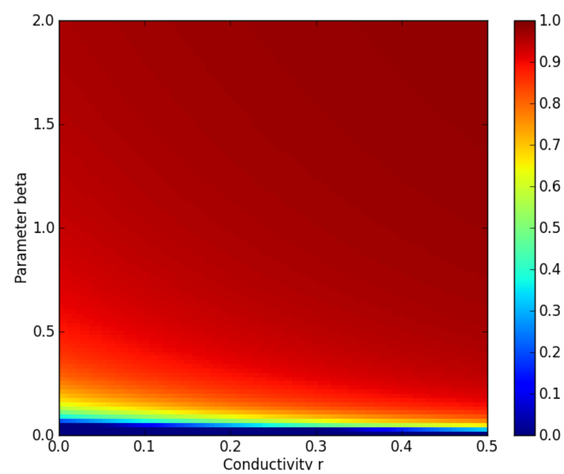


Figure 4.19: This graphic represents the size of the solution from (3.4) with betweenness vaccination on the Random Geometric network 4.3, at $t = 100$, $r \in [0, 0.5]$, $\beta \in [0, 2]$, $\delta = 1$ and a size $N = 100$.

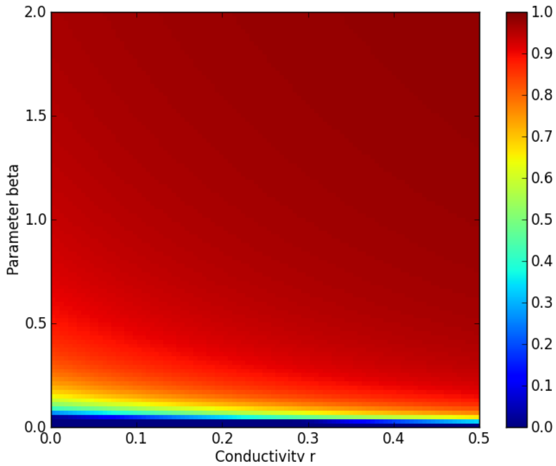


Figure 4.20: This graphic represents the size of the solution from (3.4) with random vaccination on the Random Geometric network 4.3, at $t = 100$, $r \in [0, 0.5]$, $\beta \in [0, 2]$, $\delta = 1$ and a size $N = 100$.

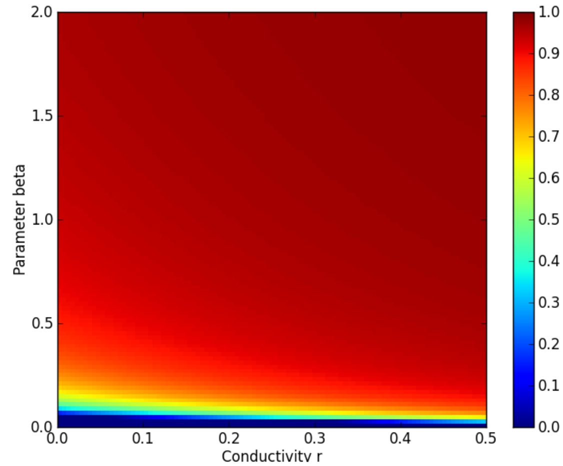


Figure 4.21: This graphic represents the size of the solution from (3.4) with closeness vaccination on the Random Geometric network 4.3, at $t = 100$, $r \in [0, 0.5]$, $\beta \in [0, 2]$, $\delta = 1$ and a size $N = 100$.

As in the both last two strategies, the figures 4.18, 4.19, 4.20, 4.21 are similar, so we will study the subtraction between them. This in order to identify regions wherever our hypothesis is true in this kind of network.

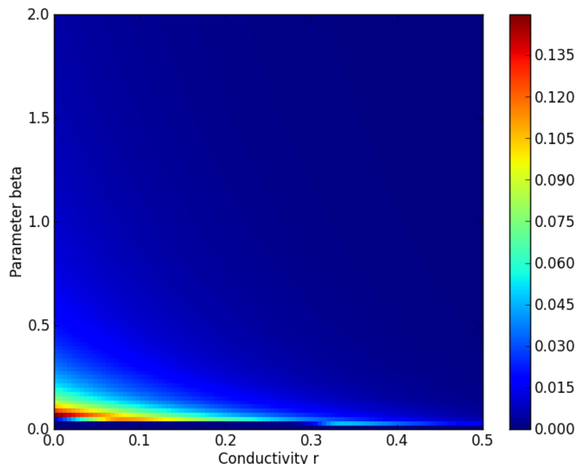


Figure 4.22: This graphic represents the subtraction between the sizes of the solution from (3.4), with random and largest degree vaccination (in that order).

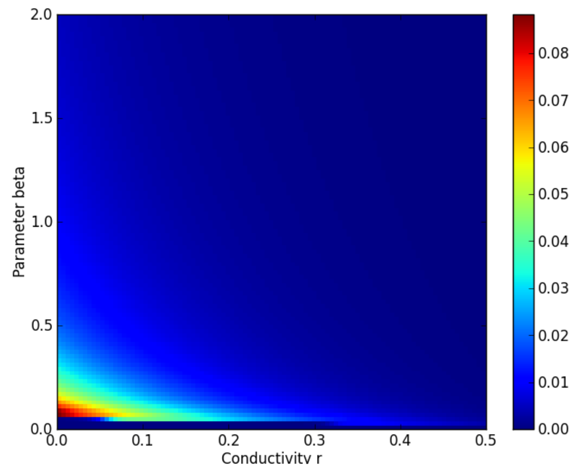


Figure 4.23: This graphic represents the subtraction between the sizes of the solution from (3.4), with betweenness and largest degree vaccination (in that order).

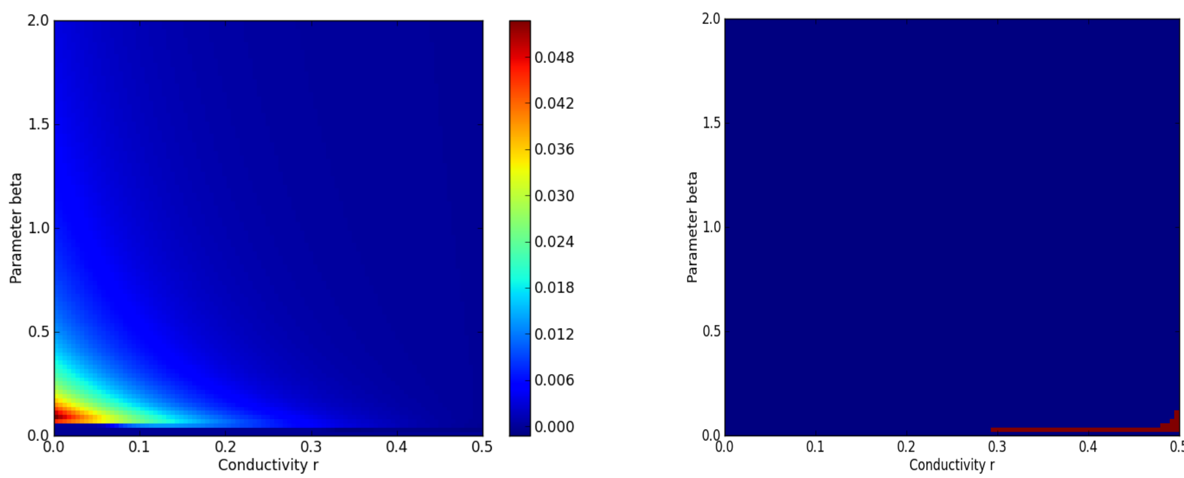


Figure 4.24: The first graphic represents the subtraction between the sizes of the solution from (3.4), with closeness and largest degree vaccination (in that order). The sign of the subtraction appears in the second graphic.

In the figures 4.22 and 4.23, the subtraction is always positive. Therefore, it is not possible to verify our conjecture: *hub-vaccination is not necessarily the optimal vaccination strategy.*

Figure 4.24 shows that our conjecture is true. For β "small" and r in $[0.3, 0.5]$, it is a better strategy to vaccinate the nodes with major closeness centrality than the nodes with largest degree on network 4.3.

It is remarkable that with closeness vaccination, both the Barabási-Albert network 4.1 and the Random Geometric network 4.3 verify our conjecutre close to $r = 0.5$. So, can we say something if $r \in [0.5, 1]$? The answer is that for some values of r is true too. Figures 4.25 and 4.26 offer further evidence.

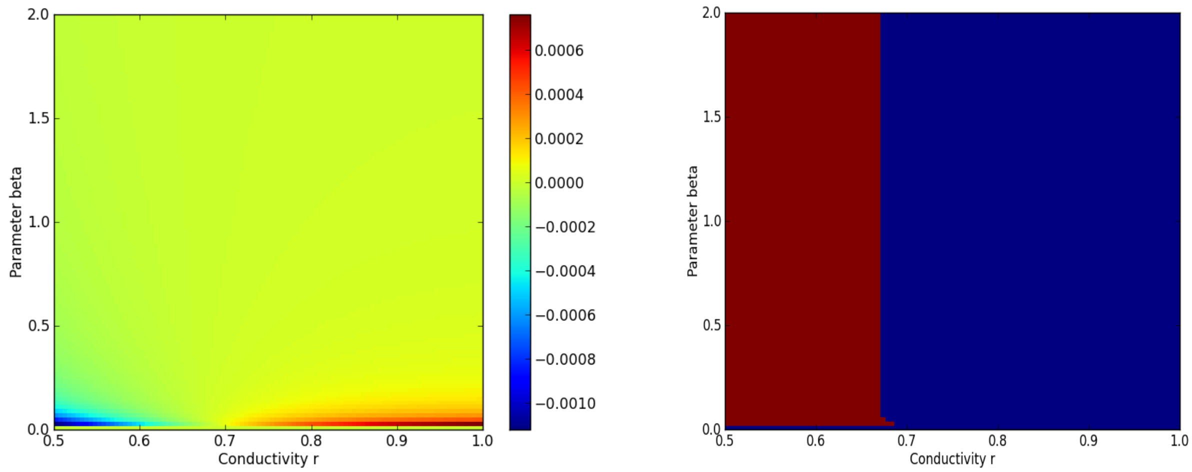


Figure 4.25: The first graphic represents the subtraction between the sizes of the solution from (3.4) on 4.1, with closeness and largest vaccination (in that order), $r \in [0.5, 1]$. The sign of the subtraction appears in the second graphic.

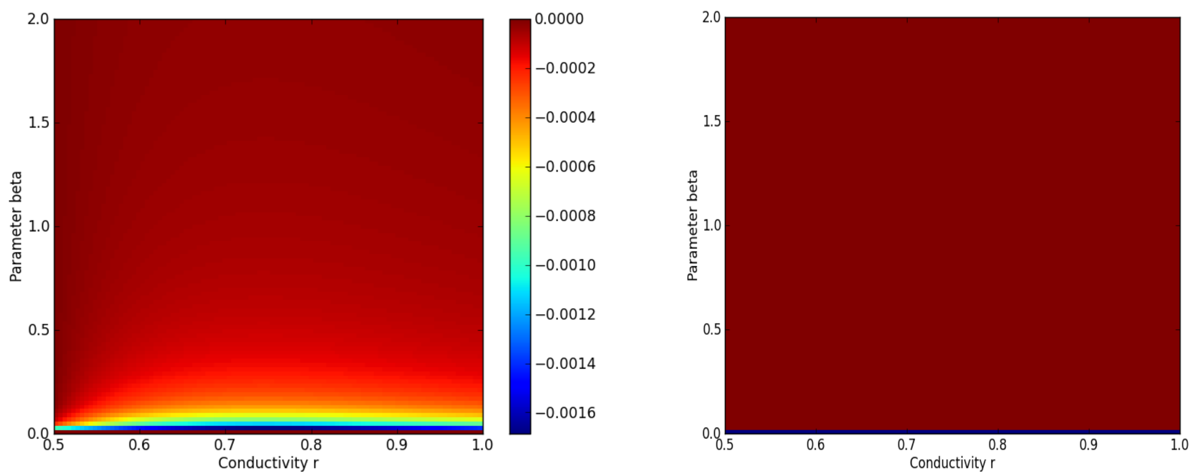


Figure 4.26: The first graphic represents the subtraction between the sizes of the solution from (3.4) on 4.3, with closeness and largest vaccination (in that order), $r \in [0.5, 1]$. The sign of the subtraction appears in the second graphic.

In this section, we have seen that for $N = 100$ and $\delta = 1$, there are values for r and β, which we verify our conjecture: *hub-vaccination is not necessarily the optimal vaccination strategy*. It was presented with three kinds of networks. Next, a natural question appears, will it be true for every Barabási-Albert, Erdös-Renyi and random geometric networks? Unfortunately the answer to this question is NO. For example, in the case of a Barabási-Albert network.If we study the next graph, to vaccinate the nodes of largest degree is the best strategy over any other strategy studied.

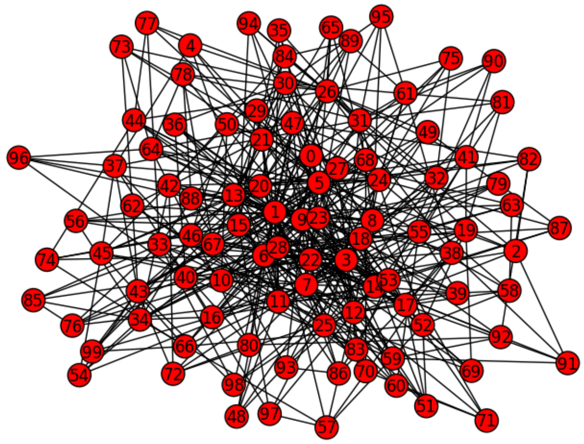


Figure 4.27: Barabási-Albert network with 100 nodes.

The respective figures of the subtractions between the strategies are,

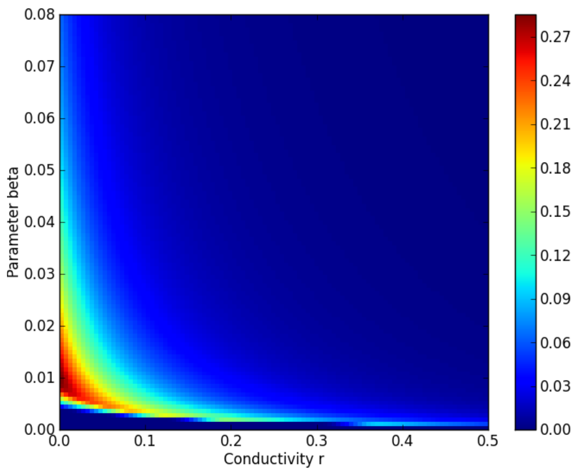


Figure 4.28: This graphic represents the subtraction between the solution from (3.4) on the network shown in the figure 4.27, with random and largest degree vaccination (in that order).

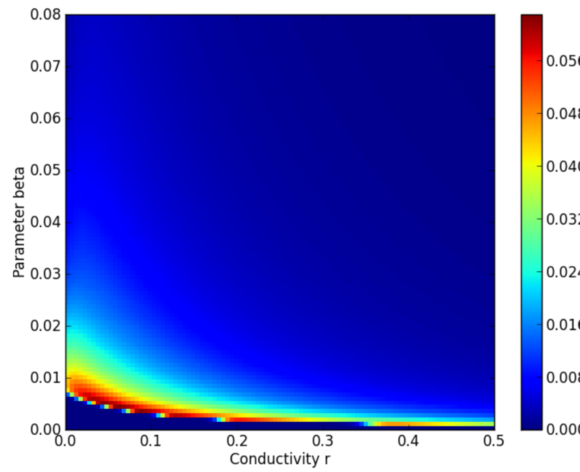


Figure 4.29: This graphic represents the subtraction between the solution from (3.4) on the network shown in the figure 4.27, with closeness and largest degree vaccination (in that order).

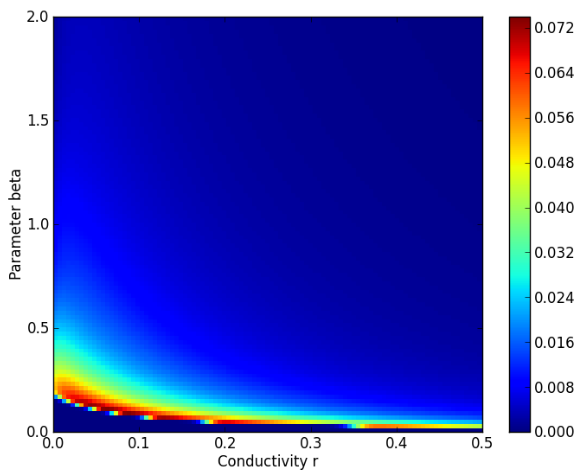


Figure 4.30: This graphic represents the subtraction between the solution from (3.4) on the network shown in the figure 4.27, with betweenness and largest degree vaccination (in that order).

The figures 4.28, 4.29 and 4.30 confirms the above statement, because the differences are always positive. Now, we know that the result is not general. But, is it frequent? to have an idea, we generated a set of randomly 100 Barabási-Albert networks, and we look in which of them our conjecture is true. The figures 4.31, 4.32 and 4.33 show in which Barabási-Albert networks is better to vaccinate with a strategy different vaccination of nodes with largest degree.

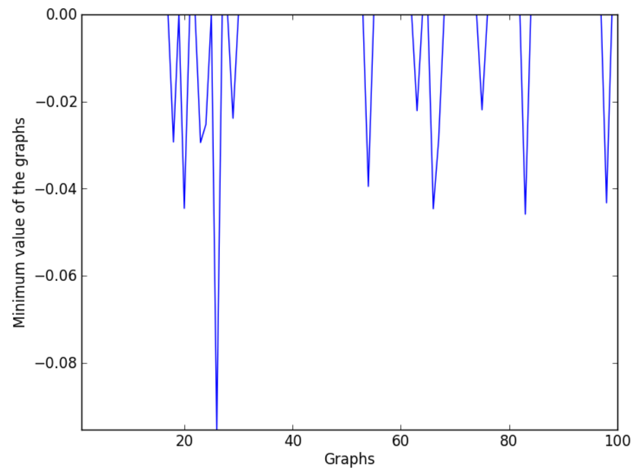


Figure 4.31: This graphic represents the minimum of the subtraction of 100 Barabási-Albert networks, between the solution from (3.4), with random and largest degree vaccination (in that order).

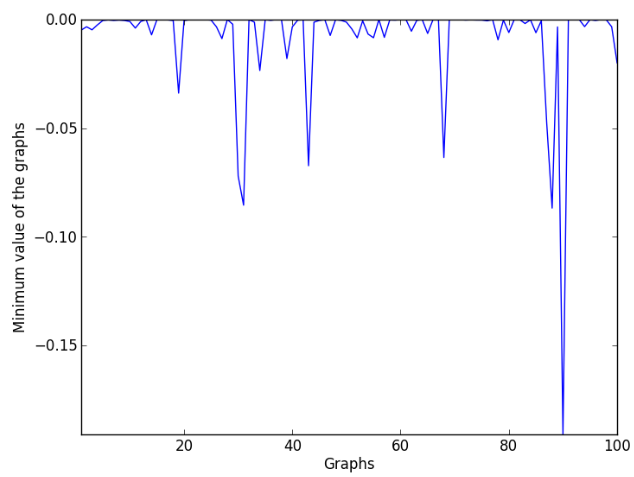


Figure 4.32: This graphic represents the minimum of the subtraction of 100 Barabási-Albert networks, between the solution from (3.4), with closeness and largest degree vaccination (in that order).

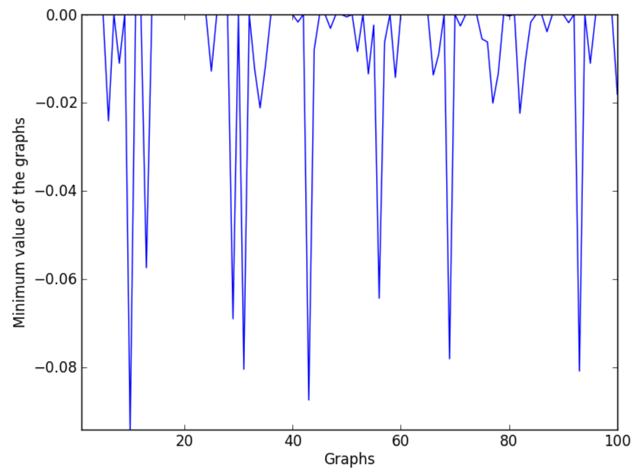


Figure 4.33: This graphic represents the minimum of the subtraction of 100 Barabási-Albert networks, between the solution from (3.4), with betweenness and largest degree vaccination (in that order).

For the others kinds of networks, see appendix A.

Chapter 5

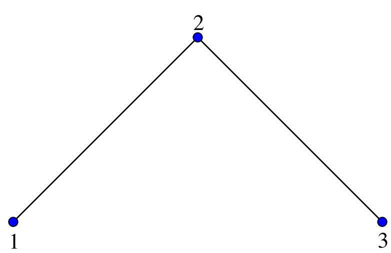
Vaccination of a Star Graph with N Nodes

In Chapter 4, we verified numerically the conjecture: *hub-vaccination is not necessarily the optimal vaccination strategy*, for some conditions from some random graphs. In this chapter, we will study theoretically this conjecture in a particular graph, a star graph (see figure 2.6). But, why did we choose a star graph? Because hubs in star graphs determine the communication of the whole graph.

The purpose of this chapter, it is to compare the steady states of a star graph with N nodes, using two vaccination strategies; randomly and largest degree vaccination. Cases $N \in \{1, 2\}$ are too simple and we skip their discussion.

5.1 Star graph with $N = 3$ nodes

5.1.1 Random vaccination



The matrix (3.3) associated to a star graph with 3 nodes is

$$M = \begin{bmatrix} 0 & 1 & 2r \\ 1 & 0 & 1 \\ 2r & 1 & 0 \end{bmatrix} \tag{5.1}$$

Choosing randomly one node different to 2 (because this is the hub), for example, the node 1 and this node is vaccinated, the matrix (5.1) would change to

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

System 3.4 would be,

$$\begin{aligned} v_1'(t) &= -\delta v_1(t), \\ v_2'(t) &= \beta(1 - v_2(t))v_3(t) - \delta v_2(t), \\ v_3'(t) &= \beta(1 - v_3(t))v_2(t) - \delta v_3(t). \end{aligned}$$

This system is uncoupled. The solution to the first equation is

$$v_1(t) = C\mathrm{e}^{-\delta t},$$

where C is a constant that depends on the initial conditions. For the other two equations, doing $v_2 = v$ and $v_3 = w$, we get the next equations,

$$\begin{aligned} v' &= \beta(1 - v)w - \delta v, \\ w' &= \beta(1 - w)v - \delta w. \end{aligned}$$

Simplifying the above system doing $\tau = \delta t$, we obtain

$$\begin{aligned} v' &= R_0(1 - v)w - v, \\ w' &= R_0(1 - w)v - w, \end{aligned} \tag{5.2}$$

where $R_0 = \frac{\beta}{\delta}$. Observe that $v = w$, because $v' - R_0 w + v = -vw = w' - R_0 v + w$. Thus

$$v' - R_0 w + v = w' - R_0 v + w$$

and

$$v' + R_0 v + v = w' + R_0 w + w. \tag{5.3}$$

As (5.3) is true for every v and w, then $v = w$. Thus, there is one single equation,

$$v' = R_0(1 - v)v - v,$$

i.e.

$$v' = (R_0(1 - v) - 1)v. \tag{5.4}$$

Equation (5.4) is a logistic equation. If $R_0 = 1$ then

$$v' = -v^2,$$

whose solution is given by

$$v(t) = \frac{1}{t + C_1}, \tag{5.5}$$

where C_1 is a constant that depends on the initial conditions. As $v(0) = \frac{1}{C_1}$ must be positive, because v represents a probability, we have that $C_1 > 0$. In the figure 5.1 shows the graph of the function (5.5).

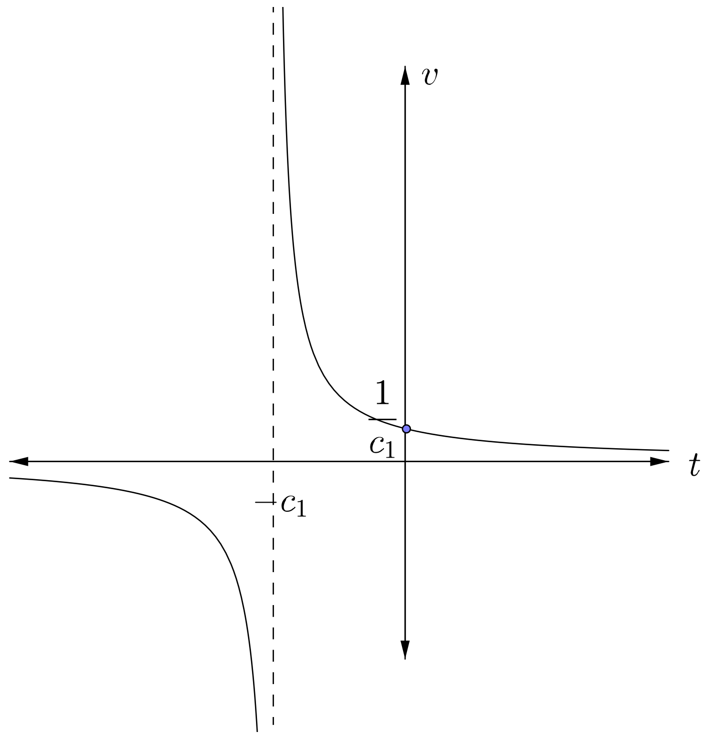


Figure 5.1: Sketch of the function (5.5) when $R_0 = 1$.

If $0 < R_0 < 1$, the sketch of the solution from (5.4) is shown in the figure 5.2.

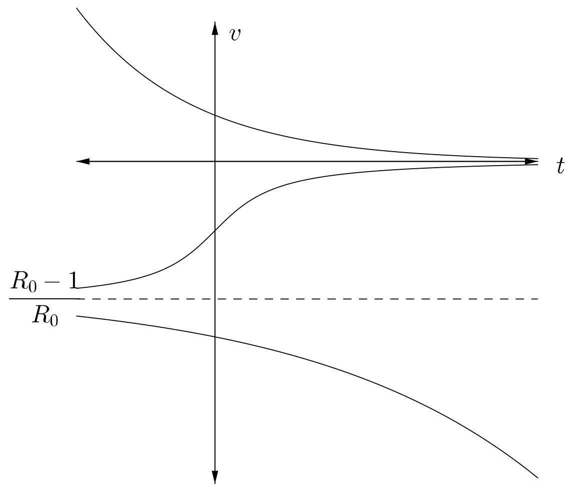


Figure 5.2: Sketch of the solution from (5.4) when $0 < R_0 < 1$.

If $1 < R_0$, the sketch of the solution from (5.4) is shown in the figure 5.3.

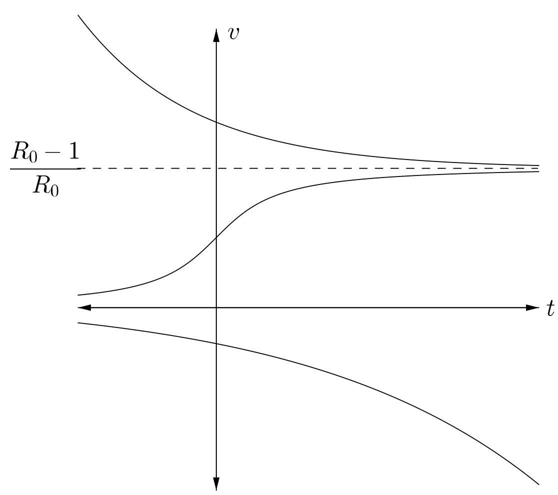


Figure 5.3: Sketch of the solution from (5.4) when $1 < R_0$.

As summary of results,

Conditions	Equilibriums from (5.4)	Estability
$0 < R_0 < 1$	$v = 0$ $v = \frac{R_0 - 1}{R_0}$	Stable Unstable
$R_0 = 1$	One steady state, $v = 0$	Stable
$R_0 > 1$	$v = 0$ $v = \frac{R_0 - 1}{R_0}$	Unstable Stable

5.1.2 Largest degree vaccination

Now, if the hub is vaccinated i.e. the node 2. The matrix (5.1) would change to

$$M_2 = \begin{bmatrix} 0 & 0 & 2r \\ 0 & 0 & 0 \\ 2r & 0 & 0 \end{bmatrix}$$

and the system to solve is,

$$\begin{aligned} v_1'(t) &= 2r\beta v_3(t)(1 - v_1(t)) - \delta v_1(t), \\ v_2'(t) &= -\delta v_2(t), \\ v_3'(t) &= 2r\beta v_1(t)(1 - v_3(t)) - \delta v_3(t). \end{aligned}$$

This system is uncoupled. The solution to the second equation is

$$v_2(t) = C_2 \mathrm{e}^{-\delta t},$$

where C_2 is a constant that depends on the initial conditions. For the other two equations, doing $v_1 = v, v_3 = w$ and simplifying the system doing $\tau = \delta t$, we obtain

$$\begin{aligned} v' &= 2rR_0 w(1 - v) - v, \\ w' &= 2rR_0 v(1 - w) - w. \end{aligned} \tag{5.6}$$

Under the same above argument. Just one equation is obtained,

$$v' = (2rR_0(1 - v) - 1)v, \tag{5.7}$$

Equation (5.7) is a logistic equation again. The equilibria are $v = 0$ and $v = \frac{2rR_0 - 1}{2rR_0}$, as long as $r \neq 0$.

Conditions	Equilibriums from (5.7)	Estability
$r \neq 0$ and $0 < R_0 < \frac{1}{2r}$	$v = 0$ $v = \frac{2rR_0-1}{2rR_0}$	Stable Unstable
$r \neq 0$ and $R_0 > \frac{1}{2r}$	$v = 0$ $v = \frac{2rR_0-1}{2rR_0}$	Unstable Stable
$r \neq 0$ and $R_0 = \frac{1}{2r}$	One steady state, $v = 0$	Stable
$r = 0$	One steady state, $v = 0$	Stable

Now, comparing the stable steady states of both strategies, we can give a answer to the next question. Is random vaccination better than vaccinating largest degree node? The nonzero steady states from (5.4) and (5.7) are $\hat{v} = \frac{R_0-1}{R_0}$ for random vaccination, and $\overline{v} = \frac{2rR_0-1}{2rR_0}$ for largest degree vaccination. $\hat{v}$ and $\overline{v}$ are stable if $R_0 > 1$ and if $R_0 > \frac{1}{2r}$, respectively. Thus, assuming that $r < 0.5$ and defining the next function,

$$f(R_0, r) = \frac{R_0 - 1}{R_0} - \frac{2rR_0 - 1}{2rR_0},$$

the difference between the nonzero steady states in both strategies. If $f(R, r)$ is negative, this mean that to vaccinate randomly is better that to vaccinate the largest degree node. $f(R, r) < 0$ implies that $r > 0.5$. But to compare the steady state is necessary that $r < 0.5$. Thus, this hypothesis is not true for star graph with $N = 3$ nodes. Now, will the difference be always positive for all star graph with N nodes?

5.2 Star Graph with N Nodes ($N \geq 4$)

5.2.1 Random Vaccination

Following the same idea of case $N = 3$, vaccinating randomly, the general system of differential equations for a star graph with $N \geq 4$ nodes is,

$$v' = \beta\,(N-2)\,(1-v)\,w - \delta v,$$
$$w' = \beta\,(1-w)\,(v + 2r\,(N-3)\,w) - \delta w.$$

If we set

$$\overline{\theta}_1 = \beta(N-2),\ \overline{\theta}_2 = -2r\beta(N-3),$$

the following system of differential equations is obtained

$$v' = \overline{\theta}_1 w - \overline{\theta}_1 wv - \delta v, \tag{5.8a}$$
$$w' = \overline{\theta}_2 w^2 - (\overline{\theta}_2 + \delta)w - \beta vw + \beta v. \tag{5.8b}$$

System (5.8) can be further simplified if we rescale the time by $\tau = \delta t$. Thus obtaining

$$v' = \theta_1 w - \theta_1 wv - v, \tag{5.9a}$$
$$w' = \theta_2 w^2 - (\theta_2 + 1)w - R_0 vw + R_0 v, \tag{5.9b}$$

or

$$\begin{pmatrix} v' \\ w' \end{pmatrix} = f(w, v, \delta, \beta, r, N), \tag{5.10}$$

where $f(w, v, \delta, \beta, r, N) = (\theta_1 w - \theta_1 wv - v, \theta_2 w^2 - (\theta_2 - 1)w - R_0 vw + R_0 v)$ and

$$R_0 = \frac{\beta}{\delta},\ \theta_1 = R_0(N-2),\ \theta_2 = -2rR_0(N-3). \tag{5.11}$$

Next, we shall study the stationary solutions of (5.9) and their stability. If we let

$$\theta_1 w - \theta_1 wv - v = 0, \tag{5.12a}$$
$$\theta_2 w^2 - (\theta_2 + 1)w - R_0 vw + R_0 v = 0, \tag{5.12b}$$

from equation (5.12a), we obtain

$$v = \frac{\theta_1 w}{\theta_1 w + 1} = \frac{R_0(N-2)w}{R_0(N-2)w + 1}, \tag{5.13}$$

now, using (5.13) in the second equation from (5.12),

$$\theta_2 w^2 - (\theta_2 + 1)w - R_0 \left(\frac{\theta_1 w}{\theta_1 w + 1} \right) w + R_0 \left(\frac{\theta_1 w}{\theta_1 w + 1} \right) = 0.$$

This implies that,

$$w = 0, \quad \text{or} \quad 1 + \frac{\theta_1 w + 1}{\theta_2(\theta_1 w + 1) - R_0 \theta_1} = w. \tag{5.14}$$

Thus, the steady states are $w = 0$ or fixed points of $\phi(w)$, where

$$\phi(w) = 1 + \frac{\theta_1 w + 1}{\theta_2(\theta_1 w + 1) - R_0 \theta_1}.$$

The next points allow us to study the behavior of the function ϕ.

1. If $r = \frac{1 - R_0^2(N-2)}{2R_0(N-3)}$, then $w = 0$ is a fixed point of $\phi(w)$. Moreover, we have that

$$\phi(0) \begin{cases} > 0 \quad \text{if} \quad r > \frac{1-R_0^2(N-2)}{2R_0(N-3)}, \\ = 0 \quad \text{if} \quad r = \frac{1-R_0^2(N-2)}{2R_0(N-3)}, \\ < 0 \quad \text{if} \quad r < \frac{1-R_0^2(N-2)}{2R_0(N-3)}, \end{cases}$$

where $\phi(0) = 1 + \frac{1}{\theta_2 - R_0\theta_1}$.

2. $\phi(w)$ is strictly decreasing. If we consider the first derivative

$$\phi'(w) = -\frac{R_0\theta_1}{(\theta_2(\theta_1 w + 1) - R_0\theta_1)^2},$$

as $R_0\theta_1 > 0$ and $(\theta_2(\theta_1 w + 1) - R_0\theta_1)^2 \geq 0$, we get that $\phi'(w) < 0$, $w \in \mathbb{R}$.

3. The second derivative

$$\phi''(w) = \frac{2R_0\theta_1^2\theta_2}{(\theta_2(\theta_1 w + 1) - R_0\theta_1)^3},$$

implies

$$\phi''(w) \begin{cases} > 0 \quad \text{if} \quad w > \frac{R_0\theta_1 - \theta_2}{\theta_1\theta_2}, \\ < 0 \quad \text{if} \quad w < \frac{R_0\theta_1 - \theta_2}{\theta_1\theta_2}. \end{cases}$$

Thus, ϕ is concave up if $w \in \left(\frac{R_0\theta_1-\theta_2}{\theta_1\theta_2}, \infty\right)$, and concave down if $w \in \left(-\infty, \frac{R_0\theta_1-\theta_2}{\theta_1\theta_2}\right)$

4. For $\phi_\infty = \lim_{w\to\infty} \phi(w)$,

$$\phi_\infty = 1 + \frac{1}{\theta_2}.$$

We have that

$$\phi_\infty \begin{cases} > 0 \quad \text{if} \quad r > \frac{1}{2R_0(N-3)} \\ = 0 \quad \text{if} \quad r = \frac{1}{2R_0(N-3)} \\ < 0 \quad \text{if} \quad r < \frac{1}{2R_0(N-3)} \end{cases}$$

and $\phi(0) > \phi_\infty$.

i. Figure 5.4 shows the graph of function ϕ when $\phi(0) = 0$ and $\phi_\infty < 0$. In this case, there are two steady states of the system (5.9), $(v^*, w^*) = (0, 0)$ and $(\overline{v}, \overline{w})$, where $\overline{v}$ and v^* are $v(w)$ from the equation (5.13) evaluated at $w = \overline{w}$ and $w = w^*$, respectively. Since w represents a probability, the case $\overline{w} < 0$ will not be considered.

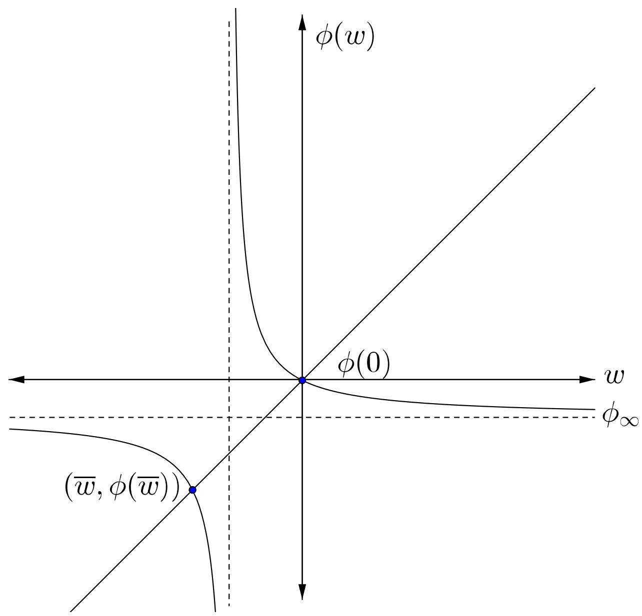


Figure 5.4: Sketch of the function ϕ when $\phi(0) = 0$ and $\phi_\infty < 0$.

Thus, using (5.13), there is just one steady state for the system (5.9), $(v, w) = (0, 0)$. To meet the stability of this steady state, we need to understand the linear system associated from (5.10),

$$\begin{bmatrix} v' \\ w' \end{bmatrix} = J_f(0,0) \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -1 & \theta_1 \\ R_0 & -(\theta_2 + 1) \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}, \tag{5.15}$$

where $J_f(0, 0)$ is the Jacobian matrix of the function f from (5.10), evaluated at $v = w = 0$. The eigenvalues of $J_f(0, 0)$ are

$$\lambda_1 = \frac{-\theta_2 - 2 + \sqrt{\theta_2^2 + 4R_0\theta_1}}{2} \text{ and } \lambda_2 = \frac{-\theta_2 - 2 - \sqrt{\theta_2^2 + 4R_0\theta_1}}{2},$$

or

$$\begin{aligned} \lambda_1 &= rR_0(N-3) - 1 + \sqrt{r^2R_0^2(N-3)^2 + R_0^2(N-2)}, \\ \lambda_2 &= rR_0(N-3) - 1 - \sqrt{r^2R_0^2(N-3)^2 + R_0^2(N-2)}. \end{aligned} \tag{5.16}$$

Taking into account that

$$rR_0(N-3) - 1 \leq rR_0(N-3) = \sqrt{r^2R_0^2(N-3)^2} \leq \sqrt{r^2R_0^2(N-3)^2 + R_0^2(N-2)},$$

then,

1. If $r = \frac{1}{R_0(N-3)}$ then $\lambda_1 > 0$ and $\lambda_2 < 0$, so that $v = w = 0$ is an unstable state.
2. If $r > \frac{1}{R_0(N-3)}$ then $\lambda_1 > 0$ and $\lambda_2 < 0$, so that $v = w = 0$ is an unstable state.
3. If $r < \frac{1}{R_0(N-3)}$ then $\lambda_1 > 0$ and $\lambda_2 < 0$, so that $v = w = 0$ is an unstable state.

Hence, in any case $(v, w) = (0, 0)$ is an unstable equilibrium.

ii. Figure 5.5 shows the graph of function ϕ when $\phi(0)$, $\phi_\infty > 0$. In this case, there are three steady states of the system (5.9), $(\overline{v}, \overline{w})$, $(0,0)$, and (v^*, w^*). Since w represents a probability, the case $\overline{w} < 0$ will not be considered. Here, $0 < \phi_\infty < \phi(w^*) < \phi(0)$ i.e. $0 < \phi_\infty < w^* < \phi(0)$.

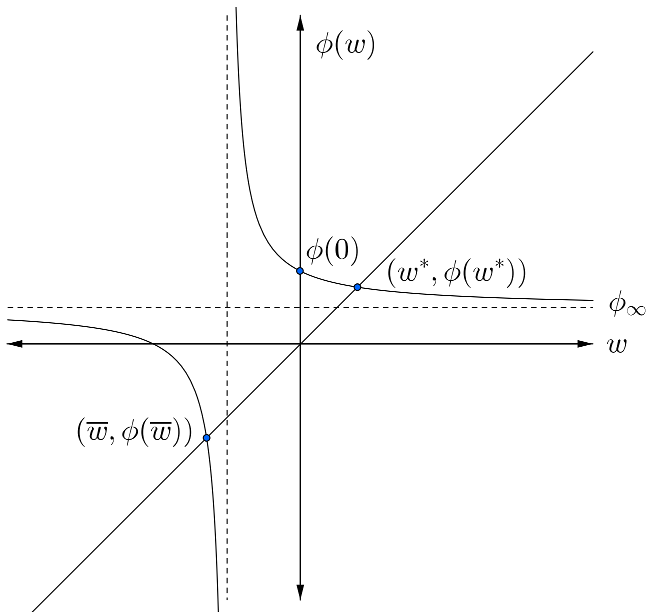


Figure 5.5: Sketch of the function ϕ when $\phi(0),\ \phi_\infty > 0$.

From case (i), $(0,0)$ is unstable. Now, for (v^*, w^*) the Jacobian matrix is

$$J_f(v^*, w^*) = \begin{bmatrix} -\theta_1 w^* - 1 & \theta_1 - \theta_1 v^* \\ -R_0 w^* + R_0 & 2\theta_2 w^* - (\theta_2 + 1) - R_0 v^* \end{bmatrix}$$

The characteristic polynomial associated to the matrix $J_f(v^*, w^*)$ is

$$P(\lambda) = \lambda^2 + B\lambda + C,$$

where $B = -\mathrm{tr}(J_f(v^*, w^*))$ and $C = \det(J_f(v^*, w^*))$. The eigenvalues of $J_f(v^*, w^*)$ are,

$$\lambda^+ = \frac{-B + \sqrt{B^2 - 4C}}{2} \text{ and } \lambda^- = \frac{-B - \sqrt{B^2 - 4C}}{2}, \tag{5.17}$$

it follows that,

$$\begin{cases} \text{If } C>0 \text{ and } B^2-4C<0, \text{ then } \begin{cases} \text{If } B>0, \text{ then } \mathrm{Re}(\lambda^+),\ \mathrm{Re}(\lambda^-)<0, \\ \text{If } B=0, \text{ then } \mathrm{Re}(\lambda^+)=\mathrm{Re}(\lambda^-)=0, \\ \text{If, } B<0, \text{ then } \mathrm{Re}(\lambda^+),\ \mathrm{Re}(\lambda^-)>0, \end{cases} \\ \\ \text{If } C>0 \text{ and } B^2-4C>0, \text{ then } \begin{cases} \text{If } B>0, \text{ then } \lambda^+,\ \lambda^-<0, \\ \text{If } B=0, \text{ then } \lambda^+>0,\ \lambda^-<0 \\ \text{If } B<0, \text{ then } \lambda^+,\ \lambda^->0, \end{cases} \\ \\ \text{If } C>0 \text{ and } B^2-4C=0, \text{ then } \begin{cases} \text{If } B>0, \text{ then } \lambda^+=\lambda^-<0, \\ \text{If } B=0, \text{ then } \lambda^+=\lambda^-=0, \\ \text{If } B<0, \text{ then } \lambda^+,\ \lambda^->0, \end{cases} \\ \\ \text{If } C=0, \text{ then } \begin{cases} \text{If } B<0, \text{ then } \lambda^+>0,\ \lambda^-=0, \\ \text{If } B=0, \text{ then } \lambda^+=\lambda^-=0, \\ \text{If } B>0, \text{ then } \lambda^+=0,\ \lambda^-<0, \end{cases} \\ \\ \text{If } C<0, \text{ then } \lambda^+>0 \text{ and } \lambda^-<0. \end{cases} \tag{5.18}$$

As we want (v^*, w^*) is a stable state for (5.9); looking the last conditions for C and B, we need that $C > 0$ and $B > 0$. Thus for C, we have that

$$\begin{aligned} C &= R_0(v^* - \theta_1 + \theta_1 w^* + \theta_1 v^*) + \theta_1 w^* - \theta_1\theta_2 w^* + 1 + \theta_2(1 - 2w^*) \\ &= R_0\Psi_1 + \Psi_2, \end{aligned}$$

where $\Psi_1 = v^* - \theta_1 + \theta_1 w^* + \theta_1 v^*$ and $\Psi_2 = \theta_1 w^* - \theta_1\theta_2 w^* + 1 + \theta_2(1 - 2w^*)$.

$$\Psi_1 \begin{cases} >0 & \text{if} \quad w^* \in (-\infty, \Delta_2) \cup (\Delta_1, \infty), \\ =0 & \text{if} \quad w^* = \Delta_1 \text{ or } w^* = \Delta_2, \\ <0 & \text{if} \quad w^* \in (\Delta_2,\ \Delta_1), \end{cases} \tag{5.19}$$

where

$$\Delta_1 = \frac{-1+\sqrt{1+\theta_1}}{\theta_1},\ \ \Delta_2 = \frac{-1-\sqrt{1+\theta_1}}{\theta_1},$$

and

$$\Psi_2 \begin{cases} >0 & \text{if} \quad w \in (\overline{\Delta_1}, \infty), \\ =0 & \text{if} \quad w = \overline{\Delta_1}, \\ <0 & \text{if} \quad w \in (-\infty,\ \overline{\Delta_1}). \end{cases} \tag{5.20}$$

where

$$\overline{\Delta_1} = -\frac{\theta_2 + 1}{\theta_1 - \theta_1\theta_2 - 2\theta_2}. \tag{5.21}$$

The sign of $\overline{\Delta_1}$ is

$$\overline{\Delta_1}\begin{cases} > 0 & \text{if} \quad r > \frac{1}{2R_0(N-3)}, \\ \\ = 0 & \text{if} \quad r = \frac{1}{2R_0(N-3)}, \\ \\ < 0 & \text{if} \quad r < \frac{1}{2R_0(N-3)}. \end{cases}$$

One clear condition over Ψ_1 and Ψ_2 to get $C > 0$ is that $\Psi_1,\ \Psi_2 > 0$. As $\phi(0),\ \phi_\infty > 0,\ 0 \leq w \leq 1,\ 0 \leq r \leq 1$ and $R_0 \geq 0$ then, C is positive if

$$\begin{cases} w^* \in (\Delta_1, 1] \cap \left(\overline{\Delta_1}, 1\right] \cap [0,1], \\ \\ r \in \left(\frac{1}{2R_0(N-3)}, \infty\right) \cap [0,1]. \end{cases} \tag{5.22}$$

Now for B, due to (5.18) it is necessary that $B > 0$. Then

$$B = \theta_1 w^* + R_0 v^* - 2\theta_2 w^* + \theta_2 + 2,$$

taking into account that $\phi(0),\ \phi_\infty > 0,\ 0 \leq w \leq 1,\ 0 \leq r \leq 1$ and $R_0 \geq 0$, B is positive if

$$\begin{cases} w^* < \frac{1}{2} \\ \\ r \in [0,1] \cap \left[0, -\frac{\theta_1 w^* + R_0 v^* + 2}{2R_0(N-3)(2w^*-1)}\right), \end{cases} \tag{5.23}$$

or

$$w^* \geq \frac{1}{2}. \tag{5.24}$$

Therefore, (v^*, w^*) is a stable equilibrium for (5.9b) if

$$\begin{cases} w^* \in \left[0, \frac{1}{2}\right) \cap (\Delta_1, 1] \cap \left(\overline{\Delta_1}, 1\right], \\ \\ r \in [0,1] \cap \left[0, -\frac{\theta_1 w^* + R_0 v^* + 2}{2R_0(N-3)(2w^*-1)}\right) \cap \left(\frac{1}{2R_0(N-3)}, \infty\right), \end{cases}$$

or

$$\begin{cases} w^* \in \left[\frac{1}{2}, 1\right] \cap (\Delta_1, 1] \cap \left(\overline{\Delta_1}, 1\right], \\ \\ r \in [0,1] \cap \left(\frac{1}{2R_0(N-3)}, \infty\right). \end{cases}$$

As $-\frac{\theta_1 w^* + R_0 v^* + 2}{2R_0(N-3)(2w^*-1)} \geq \frac{1}{2R_0(N-3)}$ while $w^* \in \left[0, \frac{1}{2}\right)$, then (v^*, w^*) is a stable equilibrium in (5.9b) if

$$\begin{cases} w^* \in \left[0, \frac{1}{2}\right) \cap (\Delta_1, 1] \cap \left(\overline{\Delta_1}, 1\right], \\ r \in [0,1] \cap \left(\frac{1}{2R_0(N-3)}, -\frac{\theta_1 w^* + R_0 v^* + 2}{2R_0(N-3)(2w^*-1)}\right), \end{cases} \tag{5.25}$$

or

$$\begin{cases} w^* \in \left[\frac{1}{2}, 1\right] \cap (\Delta_1, 1] \cap \left(\overline{\Delta_1}, 1\right], \\ r \in [0,1] \cap \left(\frac{1}{2R_0(N-3)}, \infty\right). \end{cases} \tag{5.26}$$

iii. Figure 5.6 shows the graph of function ϕ when $\phi_\infty = 0$.In this case, there are three steady states of the system (5.9b), $(\overline{v}, \overline{w}), (0,0), (v^*, w^*)$. For the same above mentioned reasons, $(\overline{v}, \overline{w})$ will not be considered. Here, $0 = \phi_\infty \leq \phi(w^*) \leq \phi_0$ i.e. $0 \leq w^* \leq \phi(0)$.

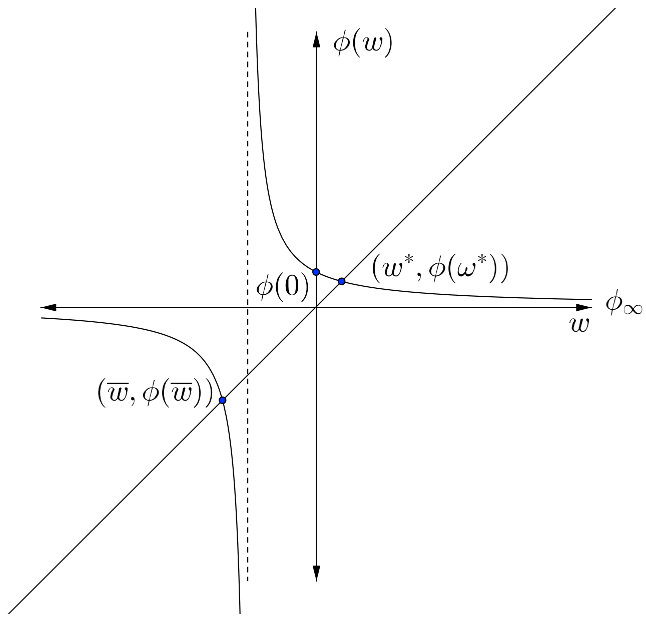


Figure 5.6: Sketch of the function ϕ when $\phi(0) > 0$ and $\phi_\infty = 0$.

The analysis of stability of the points $(0,0)$ and (v^*, w^*) is analogue to case (ii). $(0,0)$ is a unstable equilibrium by case (i). Taking into account that our condition in this case is

$$r = \frac{1}{2R_0(N-3)},$$

then C is positive if

$$w^* \in [0,1) \cap (\Delta_1, 1], \tag{5.27}$$

because $\overline{\Delta_1} = 0$. Also,

$$B = \theta_1 w^* + R_0 v^* - 2\theta_2 w^* + \theta_2 + 2 = \theta_1 w^* + R_0 v^* + 2w^* + 1 > 0,$$

Hence, (v^*, w^*) is a stable equilibrium from (5.9b) if (5.27) is verified.

iv. Figure 5.7 shows the graph of function ϕ when $\phi(0) > 0$ and $\phi_\infty < 0$. In this case, there are three steady states of the system (5.9), $(\overline{v}, \overline{w}), (0,0), (v^*, w^*)$. For the same above mentioned reasons, $\overline{w} < 0$ will not be considered. Here, $\phi_\infty < 0 \leq \phi(w^*) \leq \phi(0)$.

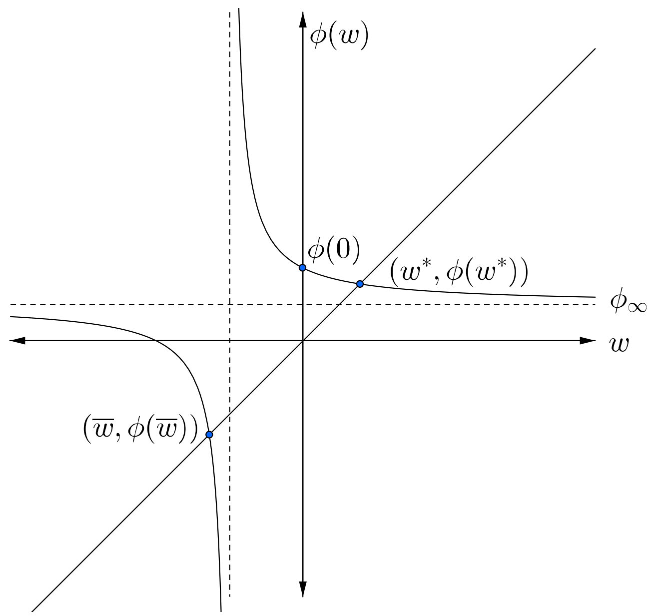


Figure 5.7: Sketch of the function ϕ when $\phi(0) > 0$ and $\phi_\infty < 0$.

The analysis of stability of the points $(0,0)$ and (v^*, w^*) is analogue to case (ii). $(0,0)$ is unstable equilibrium by case (i). In case (ii), C is positive if expressions (5.19), (5.20) are verified. Thus, taking into account that our condition in this case is

$$r \in \left(\frac{1 - R_0^2(N-2)}{2R_0(N-3)}, \infty\right) \cap \left(-\infty, \frac{1}{2R_0(N-3)}\right).$$

As $0 \le r \le 1$ and $\frac{1-R_0^2(N-2)}{2R_0(N-3)} < \frac{1}{2R_0(N-3)}$, then

$$r \in \left(\frac{1 - R_0^2(N-2)}{2R_0(N-3)}, \frac{1}{2R_0(N-3)}\right) \cap [0,1],$$

Then, $\overline{\Delta_1} < 0$. Hence, C is positive if

$$\begin{cases} w^* \in (\Delta_1, 1], \\ r \in \left(\frac{1-R_0^2(N-2)}{2R_0(N-3)}, \frac{1}{2R_0(N-3)}\right) \cap [0,1]. \end{cases}$$

In case (ii), B is positive in the expressions (5.23) and (5.24). Hence, (v^*, w^*) is a stable equilibrium in (5.9b) in this case if

$$\begin{cases} w^* \in \left[0, \frac{1}{2}\right) \cap (\Delta_1, 1], \\ r \in [0,1] \cap \left[0, -\frac{\theta_1 w^* + R_0 v^* + 2}{2R_0(N-3)(2w^*-1)}\right) \cap \left(\frac{1-R_0^2(N-2)}{2R_0(N-3)}, \frac{1}{2R_0(N-3)}\right), \end{cases}$$

or

$$\begin{cases} w^* \in \left[\frac{1}{2}, 1\right] \cap [(_1, 1], \\ r \in [0,1] \cap \left(\frac{1-R_0^2(N-2)}{2R_0(N-3)}, \frac{1}{2R_0(N-3)}\right). \end{cases}$$

As $-\frac{\theta_1 w^* + R_0 v^* + 2}{2R_0(N-3)(2w^*-1)} \geq \frac{1}{2R_0(N-3)}$ while $w^* \in \left[0, \frac{1}{2}\right)$, then (v^*, w^*) is a stable equilibrium in (5.9b) in this case if

$$\begin{cases} w^* \in \left[0, \frac{1}{2}\right) \cap (\Delta_1, 1], \\ r \in [0,1] \cap \left(\frac{1-R_0^2(N-2)}{2R_0(N-3)}, \frac{1}{2R_0(N-3)}\right), \end{cases} \tag{5.28}$$

or

$$\begin{cases} w^* \in \left[\frac{1}{2}, 1\right] \cap (\Delta_1, 1], \\ r \in [0,1] \cap \left(\frac{1-R_0^2(N-2)}{2R_0(N-3)}, \frac{1}{2R_0(N-3)}\right). \end{cases} \tag{5.29}$$

v. Figure 5.8 shows the graph of function ϕ when $\phi(0),\ \phi_\infty < 0$. In this case, there are three steady states of the system (5.9), $(\overline{v}, \overline{w}), (0,0), (v^*, w^*)$. Since w represents a probability, and $\overline{w},\ w^* < 0$, then these will not be considered. With respect to $(0,0)$, by case (i) it is known that this is a unstable equilibrium.

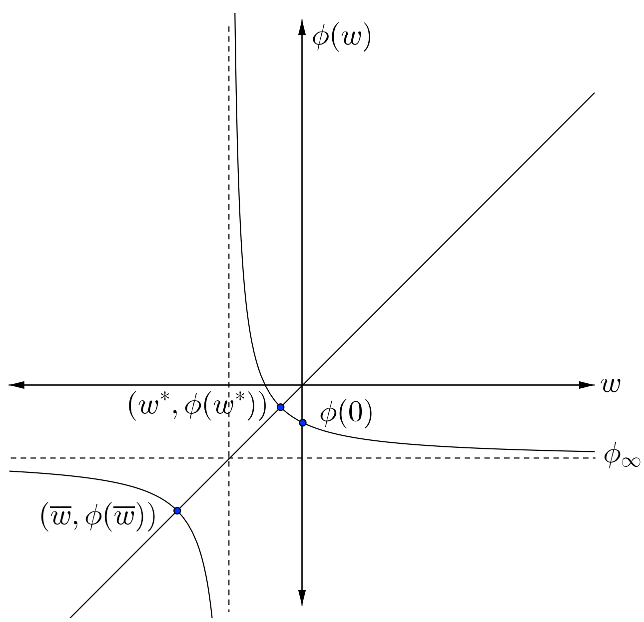


Figure 5.8: Sketch of the function ϕ when $\phi(0),\ \phi_\infty < 0$.

So far, we know conditions in which equilibria of the system (5.9b) are stable or unstable. Now, we can begin to give a expression about them, specially to the stable equilibria. Using the right expression from (5.14), the steady states are,

$$w = \frac{-\left(\theta_1\theta_2 + \theta_1(1+R_0) - \theta_2\right) \pm \sqrt{\left(\theta_1\theta_2 + \theta_1(1+R_0) - \theta_2\right)^2 - 4(-\theta_1\theta_2)\left(\theta_2 - R_0\theta_1 + 1\right)}}{-2\theta_1\theta_2}. \tag{5.30}$$

Changing the notation, if we suppose that

$$\begin{aligned} \xi_1 &= -\theta_1\theta_2, \\ \xi_2 &= -\left(\theta_1\theta_2 + \theta_1(1+R_0) - \theta_2\right), \\ \xi_3 &= \theta_2 - R_0\theta_1 + 1, \end{aligned} \tag{5.31}$$

then, expression (5.30) can be expressed as

$$w_1 = \frac{\xi_2 + \sqrt{\xi_2^2 - 4\xi_1\xi_3}}{2\xi_1} \text{ and } w_2 = \frac{\xi_2 - \sqrt{\xi_2^2 - 4\xi_1\xi_3}}{2\xi_1}. \tag{5.32}$$

Here

$$\begin{cases} \text{If } \xi_1\xi_3 < 0 \text{ then } w_1 > 0 \text{ and } w_2 < 0, \\ \text{If } \xi_1\xi_3 > 0 \wedge \xi_2^2 - 4\xi_1\xi_3 \geq 0 \text{ then } \begin{cases} \text{if } \xi_2 > 0 \text{ then } w_1, w_2 > 0, \\ \text{if } \xi_2 < 0 \text{ then } w_1, w_2 < 0. \end{cases} \end{cases}$$

The next table will be helpful as summary of this part

Conditions	Physically admissible steady states	Where are they stable?
$\phi(0) = 0$ and $\phi_\infty < 0$	$(0,0)$	Unstable
$\phi(0) > 0$ and $\phi_\infty > 0$	$(w^*, \phi(w^*))$	$\begin{cases} w^* \in \left[0, \frac{1}{2}\right) \cap (\Delta_1, 1] \cap \left(\overline{\Delta_1}, 1\right], \\ r \in [0,1] \cap \left(\frac{1}{2R_0(N-3)}, -\frac{\theta_1 w^* + R_0 v^* + 2}{2R_0(N-3)(2w^*-1)}\right), \end{cases}$ $\begin{cases} w^* \in \left[\frac{1}{2}, 1\right] \cap (\Delta_1, 1] \cap \left(\overline{\Delta_1}, 1\right], \\ r \in [0,1] \cap \left(\frac{1}{2R_0(N-3)}, \infty\right). \end{cases}$
	$(0,0)$	Unstable
$\phi(0) > 0$ and $\phi_\infty = 0$	$(w^*, \phi(w^*))$	$w^* \in [0,1) \cap (\Delta_1, 1]$.
	$(0,0)$	Unstable
$\phi(0) > 0$ and $\phi_\infty < 0$	$(w^*, \phi(w^*))$	$\begin{cases} w^* \in \left[0, \frac{1}{2}\right) \cap (\Delta_1, 1], \\ r \in [0,1] \cap \left(\frac{1-R_0^2(N-2)}{2R_0(N-3)}, \frac{1}{2R_0(N-3)}\right), \end{cases}$ $\begin{cases} w^* \in \left[\frac{1}{2}, 1\right] \cap (\Delta_1, 1], \\ r \in [0,1] \cap \left(\frac{1-R_0^2(N-2)}{2R_0(N-3)}, \frac{1}{2R_0(N-3)}\right). \end{cases}$
	$(0,0)$	Unstable
$\phi(0) < 0$ and $\phi_\infty < 0$	-	-

5.2.2 Largest Degree Node Vaccination

Following the same idea of case $N = 3$, vaccinating the largest degree node, the general differential equation for a star graph with $N \geq 4$ nodes is

$$w' = 2r\beta\,(N-2)\,(1-w)\,w - \delta w.$$

If we set, $\overline{\theta}_1 = \beta(N-2)$, the following differential equation is obtained

$$w' = w(-2r\overline{\theta}_1 w + (2r\overline{\theta}_1 - \delta)),$$

this equation can be simplified doing $\tau = \delta t$, rescaling the time. Thus we obtain

$$w' = w(-2r\theta_1 w + (2r\theta_1 - 1)), \tag{5.33}$$

where, $\theta_1 = R_0(N-2)$. The steady states from (5.33) are

$$w_1^* = 0, \quad w_2^* = \frac{2r\theta_1 - 1}{2r\theta_1}.$$

The analytic solution from (5.33) is

$$w(t) = \frac{(2r\theta_1 - 1)\mathrm{e}^{2r\theta_1 t + c_1}}{2r\theta_1 \mathrm{e}^{2r\theta_1 t + c_1} - \mathrm{e}^{2r\theta_1 c_1 + t}}, \tag{5.34}$$

where c_1 is a constant that depends on the initial conditions.

5.3 Comparison of the Steady States

Our goal is to give conditions over the parameters R_0, r and N, for which vaccinating the node of largest degree in a star graph is not better than vaccinate randomly.

To establish such conditions, it is necessary compare the steady states w_1^* from (5.32) with w_2^* from (5.2.2). To do that, we are going to study the sign of the difference $w_1^* - w_2^*$. If equation (5.35) is negative, we will find the conditions wherever our conjecture is true in a star graph, i.e. to vaccinate the node of largest degree is not the optimal strategy, due to that the stationary state vaccinating the node of largest degree is larger than the stationary state vaccinating randomly. Now, let us see where the next subtraction is negative

$$\begin{aligned} w_1^* - w_2^* &= \frac{\xi_2 + \sqrt{\xi_2^2 - 4\xi_1\xi_3}}{2\xi_1} - \frac{2r\theta_1 - 1}{2r\theta_1} \\ &= \frac{\xi_2 + K + \sqrt{\xi_2^2 - 4\xi_1\xi_3}}{2\xi_1}, \end{aligned} \tag{5.35}$$

where

$$K = 2\theta_1\theta_2 + 2\theta_1 - 2R_0.$$

Equation (5.35) is negative if

$$\sqrt{\xi_2^2 - 4\xi_1\xi_3} < -\xi_2 - K,$$

i.e.

$$-4\xi_1\xi_3 < 2\xi_2 K + K^2,$$

developing the above expression, then

$$0 < 4R_0^2\theta_1 + 4R_0 - 4R_0\theta_1^2 - 4R_0\theta_1\theta_2 - 4R_0\theta_1 - 4R_0\theta_2,$$

thus,

$$\begin{aligned} r &\geq \frac{\theta_1^2 + \theta_1(1-R_0) - 1}{2R_0(N-3)(\theta_1+1)} \\ &= \frac{\theta_1^2 + \theta_1(1-R_0) - 1}{2(\theta_1^2 + \theta_1(1-R_0) - R_0)} \\ &= \frac{1}{2} + \frac{R_0 - 1}{2(\theta_1^2 + \theta_1(1-R_0) - R_0)}, \end{aligned} \tag{5.36}$$

here, there are two relevant cases,

1. If $R_0 = 1$, then $r \geq \frac{1}{2}$.
2. If $0 < R_0 < 1$, then
   (a) Given that $1 + \frac{R_0 - 1}{\theta_1^2 + \theta_1(1-R_0) - R_0} < 1$, then, if $r \geq \frac{1}{2}$ then, condition (5.36) is verified.
   (b) As $\Sigma = \theta_1^2 + \theta_1(1-R_0) - R_0 = (\theta_1 - R_0)(\theta_1 - 1)$, then $\Sigma \to 0$ when $R_0 \to 0$. Thus, condition (5.36) is verified for all r positive while R_0 is enough small, because
   $$\frac{1}{2} + \frac{R_0 - 1}{2(\theta_1^2 + \theta_1(1-R_0) - R_0)} \to -\infty, \text{ while } R_0 \to 0.$$

Conclusions

We verify the conjecture in the three kinds of studied networks, *hub-vaccination is not necessarily the optimal vaccination strategy*, with a important hypothesis in the behavior of a virus on networks, long-range interactions. We compare two to two strategies and this is that we got:

- For the Barabási-Albert network studied 4.1, the strategies random and closeness vaccination are a better strategy than largest degree vaccination, for some region of β Vs r and $\delta = 1$. In this network, betweenness vaccination is equal to largest degree vaccination.

- For Erdös-Renyi network studied 4.2, the strategy betweenness vaccination is a better strategy than largest degree vaccination, for some region of β Vs r and $\delta = 1$. Closeness vaccination is equal to largest degree vaccination.

- For the Random Geometric network studied 4.3, the strategy closeness vaccination is a better strategy than largest degree vaccination, for some region of β Vs r and $\delta = 1$.

Also, we could see that the conjecture was not general i.e. there are a networks such that largest degree vaccination is the optimal vaccination strategy.

In search of a theoretical proof of the numerical results found in the chapter 4, the conjecture was studied in a particular graph, a star graph. We found that if next inequality for the conductivity is verified,

$$2r \geq \frac{\theta_1^2 + \theta_1(1 - R_0) - 1}{\theta_1^2 + \theta_1(1 - R_0) - R_0},$$

where $\theta_1 = R_0$, $R_0 = \frac{\beta}{\delta}$, then the conjecture is true i.e. largest degree vaccination is not the optimal vaccination strategy on a star graph. My advice if this inequality is verified is random vaccination over largest degree vaccination on a star graph.

As future work could be study the conjecture on other kinds of graphs, and to analyze if there is some relation with spectral properties of the networks.

Appendix A

In the end of chapter 4 was shown what frequent in Barabási-Albert networks, we can find that our conjecture is true. In this appendix is shown what frequent is true our conjecture in Erdös-Renyi and Random Geometric networks. For Erdös-Renyi networks, we have

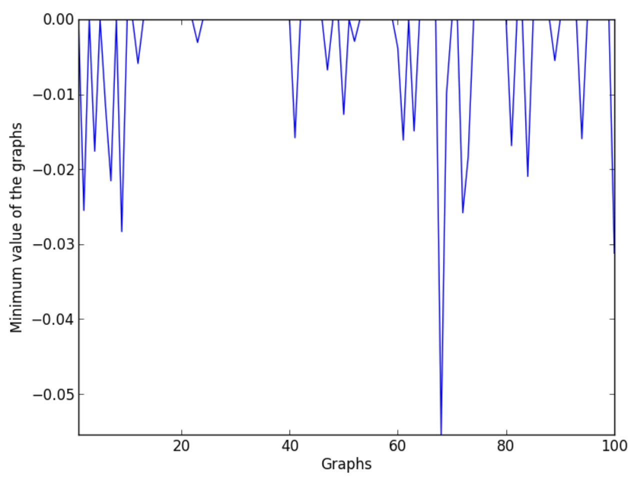


Figure A.1: This graphic represents the minimum of the subtraction of 100 Erdös-Renyi networks, between the solution from (3.4), with random and largest degree vaccination (in that order).

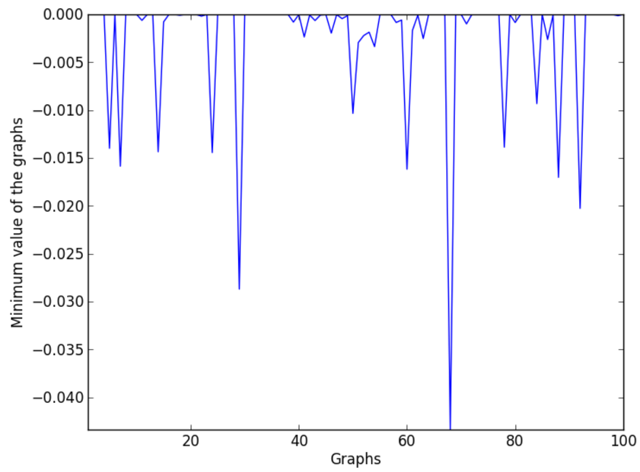


Figure A.2: This graphic represents the minimum of the subtraction of 100 Erdös-Renyi networks, between the solution from (3.4), with closeness and largest degree vaccination (in that order).

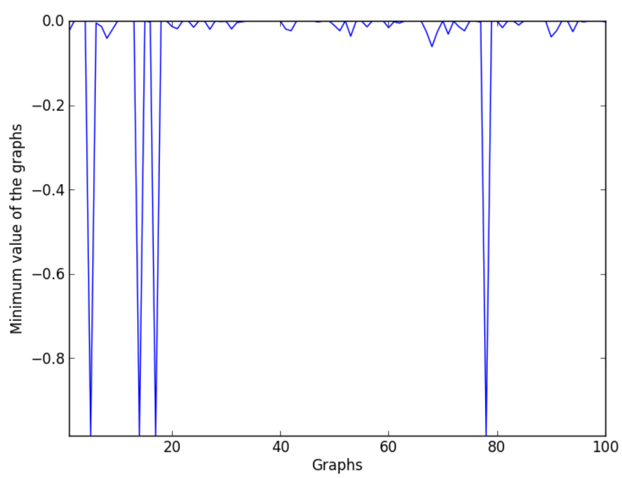


Figure A.3: This graphic represents the minimum of the subtraction of 100 Erdös-Renyi networks, between the solution from (3.4), with betweenness and largest degree vaccination (in that order).

For Random Geometric networks,

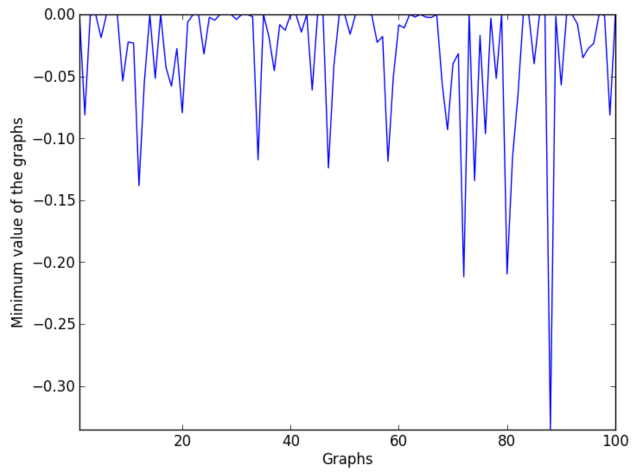


Figure A.4: This graphic represents the minimum of the subtraction of 100 Random Geometric networks, between the solution from (3.4), with random and largest degree vaccination (in that order).

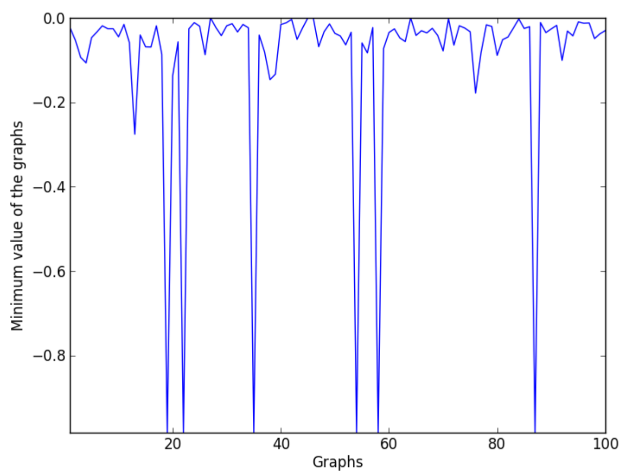


Figure A.5: This graphic represents the minimum of the subtraction of 100 Random Geometric networks, between the solution from (3.4), with closeness and largest degree vaccination (in that order).

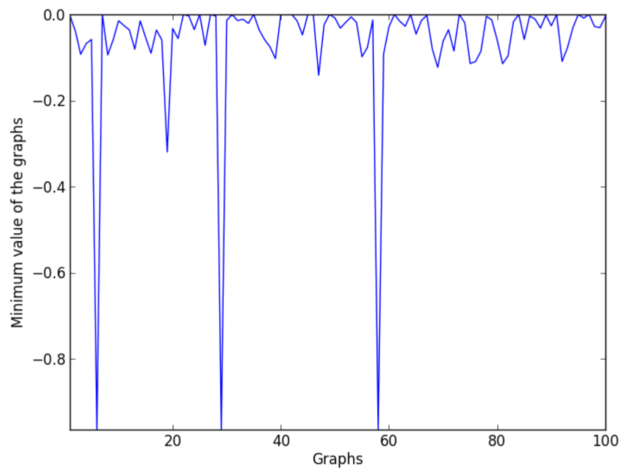


Figure A.6: This graphic represents the minimum of the subtraction of 100 Random Geometric networks, between the solution from (3.4), with betweenness and largest degree vaccination (in that order).

Bibliography

[1] Arenas A., S. Gomez, J. Borge-Holthoefer, Y. Moreno, and S. Meloni. Discrete-time Markov chain approach to contact-based disease spreading in complex networks. *Review*, 2010.

[2] J. Alvarez-Hamelin, L. Dall'Asta, A. Barrat, and Vespignani. K-core descomposition of internet graphs: hierarchies, self-similarity and measurement biases. *American Institute of Mathematical Sciences*, 2008.

[3] A. Barrat, M. Barthélemy, and A. Vespignani. *Dynamical processes on complex networks*. Cambridge, University Press, 2000.

[4] M. Capistrán. Algunos comentarios sobre el modelo SIS en redes. *Notes*, 2013.

[5] C. Castellano and R. Pastor-Satorras. Competing activation mechanisms in epidemics on networks. *Scientific Reports*, 2012.

[6] D. Chakrabarti, Y. Wang, C. Wang, J. Leskovec, and C. Faloutsos. Epidemic threshold in real networks. *ACM Transactions on Information and System Security*, 2008.

[7] R. Diestel. *Graph Theory*. Springer-Verlag, 2008.

[8] R. Durrett. *Random graph dynamics*. Cambridge, 2007.

[9] E. Estrada. *The structure of complex networks, theory and applications*. Oxford, University Press, 2012.

[10] E. Estrada, F. Kalala-Mutombo, and A. Valverde-Colmeiro. Epidemic spreading in networks with nonrandom long-range interactions. *Physical Review*, 2011.

[11] P. Mieghem. *Graph Spectra*. Cambridge, University Press, 2011.

[12] P. Mieghem. The N-interwined SIS epidemics network model. *Springer*, 2011.

[13] P. Mieghem, J. Omic, and R. Kooij. Virus spread in networks. *IEEE Xplore*, 2011.

[14] P. Van Mieghem. *Performance analysis of communications systems and networks*. Cambrige, University Press, 2006.

[15] Newman. *Networks, an introduction.* OXFORD, University Press, 2010.

[16] Joaquín Ortega Sánchez. *Modelos Estocásticos I.* Class Notes, 2013.

[17] S. B. Sheidman. Network structure and minimum degree. *Social Networks*, 1983.

[18] Howard M. Taylor. and Samuel Karlin. *An introduction to stochastic modeling.* Academic Press, 3 edition, 1998.