Centro de Investigación en Matemáticas, A.C.

CIMAT

Optimal consumption-investment problems in incomplete financial markets

T E S I S

que para obtener el grado de:

Doctor en Ciencias

(Probabilidad y Estadística)

P r e s e n t a :

Netzahualcóyotl Castañeda Leyva

Director de tesis:

Dr. Daniel Hernández Hernández

Guanajuato, Gto., México, a 15 de agosto de 2003.

CENTRO DE INVESTIGACIÓN EN MATEMÁTICAS

Optimal consumption-investment problems in incomplete financial markets

T E S I S

que para obtener el grado de

Doctor en Ciencias

(Probabilidad y Estadística)

P R E S E N T A :

Netzahualcóyotl Castañeda Leyva

DIRECTOR DE TESIS:

Dr. Daniel Hernández Hernández

S I N O D A L E S :

Dr. Rolando Cavazos Cadena (UAAAN)

Dra. Begoña Fernández Fernández (UNAM)

Dr. Jorge León Vázquez (CINVESTAV)

Dr. Pablo Padilla Longoria (UNAM)

Dra. Ekaterina Todorova Kolkovska (CIMAT)

Guanajuato, Gto., México, a 15 de agosto de 2003.

A mis padres Juan Manuel y Blanca Estela†

A mi tia Concepción Elena

A mi esposa Silvia

A mis hijos Aarón Pavel, Miriam Itzel y Andrea Deyanira

A mis hermanos Juan Manuel, Iván, Deyanira, Yedid y Hernán

Agradecimientos

Al Centro de Investigación en Matemáticas (CIMAT) por la formación y todo el apoyo y gestiones recibidas. Especialmente, a: Dr. Daniel Hernández Hernández, Dr. Víctor Manuel Pérez Abreu Carrión y Mtro. Francisco Mirabal García.

Al Dr. Daniel Hernández Hernández del CIMAT por el entrenamiento y facilidades recibidas, por ser mi director de tesis, de la dedicación y revisión en este trabajo y de la confianza que me tuvo.

A la Universidad Autónoma de Aguascalientes (UAA), por la licencia académica otorgada para el estudio y conclusión del Programa. También por el apoyo y confianza en el transcurso del mismo. En especial a: Dr. Antonio Ávila Storer, M. en C. Q. Rafael Urzúa Macias, M.I.C.H. Netzahualcóyotl López Flores, M. en E. Daniel Eudave Muñoz, M. en S. José Ramiro Alemán López, Dr. Juán Zaragoza Almaraz, M. en P. Patricia Rangel Jiménez, M. en C. María Guadalupe Vera Muñoz y M. en A. Ángel Díaz Palos.

Al Programa de Mejoramiento del Profesorado (PROMEP), por haberme becado en el doctorado, con folio número UAAGS-69.

Al Dr. Rolando Cavazos Cadena, de la Universidad Autónoma Agraria Antonio Narro (UAAAN), por las observaciones dadas a este trabajo y por haber sido mi sinodal y jurado.

A la Dra. Begoña Fernández Fernández, de la Universidad Nacional Autónoma de México (UNAM), por los comentarios a la tesis y por haber sido mi sinodal y jurado.

Al Dr. Jorge León Vázquez, del Centro de Investigación y de Estudios Avanzados (CINVESTAV), por las detalladas correcciones y comentarios a este trabajo y por haber sido mi sinodal y jurado.

Al Dr. Pablo Padilla Longoria, de la UNAM, por los comentarios y refinadas observaciones recibidas a este trabajo y por haber sido mi sinodal y jurado.

A la Dra. Ekaterina Todorova Kolkovska, del CIMAT, por las observaciones a la tesis y por haber sido mi sinodal y jurado.

Por sus comentarios en este trabajo, a los profesores y compañeros: Constantín Tudor, Miguel Ángel Moreles Vázquez, Miguel Ángel Márquez Elías, Carlos Angulo, Armando Domínguez Domínguez, Elifalet López González, Carlos Osvaldo Osuna Castro, Aroldo Pérez Pérez, Leonel Pérez Hernández, Gil Salgado González y José Villa Morales.

A los Doctores y profesores del Programa: Stephanie Dunbar, Maite Fernández Unzueta, Graciela González Farías, John Hasenbein, José Alfredo López Mimbela, Rogelio Ramos Quiroga, José Elías Rodríguez Muñoz, Ekaterina Todorova Kolkovska y Enrique Villa Diharce, por el aprendizaje y comentarios recibidos.

A mis compañeros Jorge Argaes Sosa, Felipe Peraza Garay, Francisco Pérez Arredondo, Luke Akong'o Orawo, José de Jesús Ruíz Gallegos y María Guadalupe Russell Noriega, por sus comentarios e importantes discusiones.

Por la formación y ejemplo, a: Alejandrina Caro Lerma, Francisco Sosa Rivera, Rubén Rocha Moya, Francisco Javier Palomares García, Jorge Ludlow Wiechers, Víctor Manuel Pérez Abreu Carrión y Daniel Hernández Hernández.

A mi familia: Silvia, Aarón Pavel, Miriam Itzel, Andrea Deyanira, Juan Manuel padre, Blanca Estela, Concepción Elena, Carmen, Juan Manuel, Iván, Deyanira, Yedid, Hernán, René y Leonel. También a mis tíos: Serafina, Eusebio, Anselma, Medardo, Bernardino, Silvia, Víctor, Marisa, Georgina, Jesús, Florentino, Guadalupe, Teresa, Consuelo, Jesús, y a Belem, Ramona y Natalia.

Contents

Introduction

Since the fundamental work of Black and Scholes to valuate European options, their model became a cornerstone in the development and study of many problems in mathematical finance. In recent years, different generalizations of this classical model have been studied to explain more precisely the dynamics of the asset prices.

In this sense, it is natural to consider that the coefficients of the model: interest rate, return rate, and volatility, are random or depend on random economic external factors. For instance, it can be a *leader* interest rate; in fact, several contributions show empirical arguments justifying these kinds of models. For example, Fouque, Papanicolaou, and Sircar [FPS00], present a detailed analysis modelling the external factor as a mean reverting Ornstein-Uhlenbeck (O-U) process. See also Davis [Da00], Zariphopoulou [Za01], and Fleming and Hernández-Hernández [FlHe02].

On the other hand, Barndorff-Nielsen and Shephard [BaSp02] propose a model for volatility based on an O-U process with background subordinator (a nonnegative Levy process), which is not a diffusion. They also give a detailed statistical analysis, identifying important volatility effects in the asset prices: heavy tailed of returns, volatility clustering, and skewness to the right in some cases.

The relevance of the diffusion models is not limited only to economical or empirical qualities. They also have proved to be tractable for the solution of important financial problems. For example, we can find explicit solutions of problems in the context of optimal investment (Zariphopoulou [Za01]), optimal consumption process (Fleming and Hernández-Hernández [FlHe02]), and valuation (Davis [Da00]). This feature contrasts with technical constraints or difficulties in implementation of other affine approaches. For instance, in Barndorff-Nielsen and Shephard [BaSp02] the asset prices behave volatility process as a Non-Gaussian O-U process. However, its background subordinator induces a constraint for the trading portfolio proportion: should belong to the interval $[0,1]$. This fact was mentioned by Benth, Karlsen, and Reikvam [BKR03], who gave an explicit solution for the investment problem.

On the other hand, Kramkov and Schachermayer [KrSc99] analyzed investment problem for incomplete markets when the stock prices are driven by semimartingales, and for a wide class of utility functions. They give an existence and uniqueness theorem for the optimal solution, and obtain a dual relationship between the optimal wealth process and the optimal equivalent martingale measure. However in this case, its practical implementation is not included. That is, they do not find an optimal trading strategy.

The goal of this work is to solve the *investor's problem* of maximizing the expected utility of terminal wealth and consumption in some specified time interval $[0,T]$; for $T>0$, as well as to find the *optimal* trading strategy. Here we assume that the investor's financial market is composed by a bank account, a risky asset, and an external correlated factor. The dynamics of the risky asset price and the external factor are diffusion processes where, as it was already mentioned, the external factor affects the coefficients of the model. We deal with two particular utility functions: logarithmic and HARA. We point out that the present market is incomplete, since the external factor is not traded.

There are two general approaches to solve that optimization problem: through stochastic control techniques (classical) and the so called martingale method. The former was used, for instance, by Zariphopoulou [Za01], where an explicit solution for an investment problem with HARA utility was obtained. In the same direction, Fleming and Hernández-Hernández [FlHe02] gave a solution of an optimal consumption problem when the volatility is random.

Another way to solve the problem is using the martingale method. This procedure translates the investor's problem into a convex optimization one, which we called the *primal*. In this context, the primal problem has an associated *dual* problem, which turns out to be a stochastic optimal control problem, where eventually, the control processes belong to the set of equivalent local martingale measures.

The martingale method goes back to the fundamental contribution by Harrison and Pliska [HaPl81], and it has now become a popular approach to study optimal wealth and/or consumption problems. This method is especially powerful when the financial market is incomplete. For instance, in Karatzas and Shreve [KaSr98], and some references therein, a wide class of optimization problems for incomplete markets are studied. In Kramkov and Schachermayer [KrSc99] similar problems when the prices are driven by semimartingales are analyzed. In both references, under suitable conditions, some characterizations of the optimization problem are presented. In particular, they show that the primal and dual problems are equivalent, that is, there is no duality gap. However, explicit optimal solutions are not presented in general, except for logarithmic utility or when the coefficients are deterministic.

We shall solve the investor's problem using a composition of the martingale method and stochastic control techniques. With this goal in mind, we pose the primal and dual problems and state the existence of their solutions, which shall imply the absence of duality gap. When the utility function is HARA, the solution to the dual problem relies on stochastic control techniques, while in the logarithmic case the solution is straightforward.

The thesis is organized as follows. In Chapter 1 the model and the investor's problem is established. Also, is presented a discussion over a wide class of models for the coefficients and, for comparison and motivation, the particular case of constant coefficients is solved. The primal representation of the investor's problem is given in Chapter 2. This result turns out to be important to write down the associated dual problem. The martingale method is also explained and a practical condition for absence of duality gap is given. Furthermore, a relevant relationship between optimal solutions of both problems is obtained. In Chapter 3 closed form solutions when the utility function is logarithmic or HARA are presented. Considering slight changes, analogous ideas shall be used to solve the consumption or investment problems. Furthermore, due to the relationship with the investment problem, the pricing and hedging problems are introduced. Finally, in Chapter 4 the conclusions and a list of related open problems are given.

Chapter 1

Background

In this chapter we shall introduce a factor model for the financial market based on the one presented by Bielecki and Pliska [BiPl99], and state the problem we want to solve. To motivate the discussion, we present the classical model when the coefficients are constant. At the end, some recent models for the coefficients are surveyed.

1.1 The model

Let $\{(W_{1t}, W_{2t}), \mathcal{F}_t\}_{0\leq t\leq T}$ be a standard two-dimensional Brownian motion (BM) in a complete probability space $(\Omega, \mathcal{F}_T, P)$, where $\{\mathcal{F}_t\}_{0\leq t\leq T}$ is the augmentation of the filtration $\{\mathcal{F}_t^{(W_1,W_2)}\}_{0\leq t\leq T}$. Consider a financial market governed by this BM, composed by a bank account, a risky asset, and a correlated external factor, such that, for $t \in [0, T]$:

1. The *bank account* process is given by the equation $S_t^0 \doteq \exp\left(\int_0^t r(Y_u)\, du\right)$; where $r(\cdot)$ is the *interest rate* function.

2. The *asset price* process S is assumed to satisfy the stochastic differential equation (SDE)

$$dS_t = S_t \left[\mu\left(Y_t\right) dt + \sigma\left(Y_t\right) dW_{1t}\right]; \quad \text{with} \quad S_0 = 1, \tag{1.1}$$

where $\mu(\cdot)$ and $\sigma(\cdot)$ are the *return rate* and *volatility* functions, respectively.

3. The dynamics of the *external factor* Y is modelled as a diffusion process solving the SDE

$$dY_t = g\left(Y_t\right) dt + \beta\left(\rho dW_{1t} + \varepsilon dW_{2t}\right); \quad \text{with} \quad Y_0 = y \in \mathbf{R}, \tag{1.2}$$

where $|\rho| \leq 1$, $\varepsilon \circeq \sqrt{1-\rho^2}$, and $\beta \neq 0$. Without loss of generality we take $\beta = 1$.

The parameter ρ is the correlation coefficient between the underlying BM of the asset price W_1 and the BM from the external factor $\check{W} \circeq \rho W_1 + \varepsilon W_2$. When $\rho = \pm 1$, the market is complete. Otherwise, when $|\rho| < 1$ it becomes incomplete, since the external factor cannot be traded. In financial data it is common to find scenarios where the correlation is not perfect but high, that is, $|\rho|$ is near to one. For example, the relationship can be given between two asset prices or between one asset and the stock market index. In this work we consider the general case $|\rho| \leq 1$. On the other hand, this market is free of arbitrage opportunities. Completeness and arbitrage are discussed in the next subsection.

For instance, the factor Y can be a *mean reverting* Ornstein-Uhlenbeck $(O\text{-}U)$ process. In this case, $g(y) = -\alpha_0 (r_0 - y)$; for $y \in \mathbf{R}$ and some constants $\alpha_0, r_0 > 0$.

Assumption 1.

1. $\mu(\cdot)$ and $r(\cdot)$ belong to $C_b^2(\mathbf{R})$.

2. $\sigma(\cdot) \in C_b^2(\mathbf{R})$ and $\sigma(\cdot) > \sigma_0$, for some $\sigma_0 > 0$.

3. $g(\cdot) \in C^1(\mathbf{R})$ such that $g'(\cdot) \in C_b(\mathbf{R})$.

Statements in Assumption 1 imply that the SDE (1.1) and (1.2) have strong solutions. Also, they allows to prove existence of an optimal Markov control process and to get an optimal trading strategy for the investor's problem explained below.

Now, consider a single investor who generates a *wealth* process X, with *initial capital* x, through splitting at each time $t \in [0, T]$ his capital X_t between π_t and $X_t - \pi_t$, where π_t is the net amount allocated in the risky asset. Also, part of his money is used for consumption at some given net rate c_t. Then, at small time interval $[t, t + \Delta t]$, with $\Delta t \geq 0$, the fluctuation of the wealth process is described by the difference equation

$$\Delta X_t = -c_t \Delta t + \frac{X_t - \pi_t}{S_t^0} \Delta S_t^0 + \frac{\pi_t}{S_t} \Delta S_t, \quad \text{with} \quad X_0 = x \geq 0.$$

This discrete dynamics can be approximated by a diffusion. In this sense, the wealth process becomes the solution of the SDE

$$\begin{aligned} dX_t + c_t dt &= (X_t - \pi_t) \frac{dS_t^0}{S_t^0} + \pi_t \frac{dS_t}{S_t} \\ &= (X_t - \pi_t) r(Y_t) dt + \pi_t [\mu(Y_t) dt + \sigma(Y_t) dW_{1t}] \\ &= (r(Y_t) X_t + [\mu(Y_t) - r(Y_t)] \pi_t) dt + \pi_t \sigma(Y_t) dW_{1t}. \end{aligned}$$

The next definition formalizes these concepts.

Definition 2. The real process $\{\pi_t, \mathcal{F}_t\}_{0\leq t\leq T}$ is a trading **portfolio** process if it is progressively measurable and $\int_0^T \pi_u^2 du < \infty$ a.s., whereas $\{c_t, \mathcal{F}_t\}_{0\leq t\leq T}$ is a **consumption** process if it is nonnegative and progressively measurable with $\int_0^T c_t dt < \infty$ a.s. Their associated **wealth** process, denoted by $X^{\pi,c} \circeq X^{x,y,\pi,c}$, is the solution to the integral equation

$$X_t^{\pi,c} + \int_0^t c_u du \circeq x + \int_0^t \left(r\left(Y_u\right) X_u^{\pi,c} + \left[\mu\left(Y_u\right) - r\left(Y_u\right)\right]\pi_u\right) du + \int_0^t \pi_u \sigma\left(Y_u\right) dW_{1u}. \tag{1.3}$$

The trading *strategy* (π, c) is **admissible** if $X^{\pi,c}$ satisfies the *state constraint* $X^{\pi,c} \geq 0$ a.s. The set of such trading strategies is denoted by $\mathcal{A}(x,y)$.

Throughout this work, the initial values $x \in \mathbf{R}_+ \circeq (0,\infty)$, $y \in \mathbf{R}$, and the terminal time $T \in \mathbf{R}_+$ are fixed, unless the opposite is stated.

Finally, given U_1, $U_2 : \mathbf{R}_+ \to \mathbf{R}$ utility functions, we wish to

$$\text{maximize} \quad E\left\{U_1\left(X_T^{\pi,c}\right) + \int_0^T U_2\left(c_t\right) dt\right\} \quad \text{over} \quad (\pi, c) \in \mathcal{A}(x,y), \tag{1.4}$$

as well as to provide an *optimal* trading strategy $(\hat{\pi}, \hat{c})$. This problem will be referred as the *investor's problem.* To obtain the solution of this optimization problem, we will use the martingale approach (see [HaPl81]) and stochastic control techniques. The first step in this direction shall be to obtain a characterization of the family $\mathcal{A}(x,y)$, and then get the *primal* representation of the investor's problem. See Lemma 4 and expression (P) below.

1.1.1 Arbitrage and completeness

In this part we shall show that the market model proposed in this work is incomplete (with the exceptions already mentioned) and free of arbitrage.

We say that the financial market is *free of arbitrage* opportunities if for $x = 0$, $c \equiv 0$, and a trading portfolio π such that $X^{0,y,\pi,0} \geq 0$ a.s., imply that $X^{0,y,\pi,0} \equiv 0$, for all $y \in \mathbf{R}$. It is well known that a financial market is free of arbitrage if and only if the set of *equivalent local martingale measures* $\mathcal{P}(y)$ is non empty, where

$$\mathcal{P}(y) \mathrel{\hat{=}} \left\{ Q \mid P \prec Q \prec P \quad \text{and} \quad \frac{S}{S^0} \quad \text{is a} \quad Q\text{-local martingale} \right\}.$$

It will be shown below that this is true for our model and, in fact, there are many infinite such measures. For a detailed study of arbitrage see [DeSc94].

Now, define the function $\theta : \mathbf{R} \to \mathbf{R}$, as

$$\theta(y) \mathrel{\hat{=}} \frac{\mu(y) - r(y)}{\sigma(y)}; \quad y \in \mathbf{R}.$$

By Assumption 1, the function $\theta(\cdot)$ belongs to $C_b^2(\mathbf{R})$. Let $\mathcal{M}(y)$ be the set of all the progressively measurable processes $\{\nu_t, \mathcal{F}_t\}_{t\in[0,T]}$, with $E\int_0^T \nu_u^2 du < \infty$, such that the local martingale

$$Z_t^\nu \mathrel{\hat{=}} \exp\left(-\int_0^t \left[\theta(Y_u)\, dW_{1u} + \nu_u dW_{2u}\right] - \frac{1}{2}\int_0^t \left[\theta^2(Y_u) + \nu_u^2\right] du \right), \tag{1.5}$$

is a martingale. Note that all the bounded processes belong to $\mathcal{M}(y)$, since $\theta(\cdot)$ is bounded. Define $\mathcal{M} \mathrel{\hat{=}} \bigcap_{y\in\mathbf{R}} \mathcal{M}(y)$. The processes in this class do not depend on y and is large enough for our purposes. For each $\nu \in \mathcal{M}$, a probability measure on $(\Omega, \mathcal{F}_T)$ can be defined as

$$dP^\nu \mathrel{\hat{=}} Z_T^\nu dP. \tag{1.6}$$

Note that

$$P \prec P^{\nu} \prec P \quad \text{and} \quad Z_t^{\nu} \circeq \left.\frac{dP^{\nu}}{dP}\right|_{\mathcal{F}_t}; \quad t \in [0,T].$$

Under the measure P^{ν} the two-dimensional process $\{(W_{1t}^{\nu}, W_{2t}^{\nu}), \mathcal{F}_t\}_{0\le t\le T}$, defined as

$$W_{1t}^{\nu} \circeq W_{1t} + \int_0^t \theta(Y_u)\, du \quad \text{and} \quad W_{2t}^{\nu} \circeq W_{2t} + \int_0^t \nu_u du, \tag{1.7}$$

is also a BM. See Theorem 3.5.1 in [KaSr98]. Moreover, the dynamics of the processes defined above can be written as

$$\begin{aligned} dY_t &= [g(Y_t) - \rho\theta(Y_t) - \varepsilon\nu_t]\, dt + \rho dW_{1t}^{\nu} + \varepsilon dW_{2t}^{\nu}, && (1.8)\\ dZ_t^{\nu} &= Z_t^{\nu}\left(\left[\theta^2(Y_t) + \nu_t^2\right] dt - \theta(Y_t)\, dW_{1t}^{\nu} - \nu_t dW_{2t}^{\nu}\right), && (1.9)\end{aligned}$$

while for the discounted asset price and wealth processes, we have

$$\begin{aligned} d\frac{S_t}{S_t^0} &= \frac{S_t}{S_t^0}\sigma(Y_t)\, dW_{1t}^{\nu}, && (1.10)\\ d\frac{X_t^{\pi,c}}{S_t^0} + \frac{c_t}{S_t^0}dt &= \frac{\pi_t}{S_t^0}\sigma(Y_t)\, dW_{1t}^{\nu}; \quad (\pi, c) \in \mathcal{A}(x,y). && (1.11)\end{aligned}$$

Remark 3. The above imply the following:

1. $\dfrac{S}{S^0}$ is a continuous P^{ν}-martingale, since $\sigma(\cdot)$ is bounded. Hence, $\mathcal{M} \subset \mathcal{P}(y)$, in the sense: $P^{\nu} \in \mathcal{P}(y)$; for $\nu \in \mathcal{M}$. In particular, the market is free of arbitrage opportunities.

2. The discounted process $\dfrac{X^{\pi,c}}{S^0} + \displaystyle\int_0^{\cdot} \frac{c_t}{S_t^0}dt$ is a nonnegative continuous P^{ν}-local martingale and, by Fatou's lemma, it is also a P^{ν}-supermartingale.

Finally, we say that the financial market is *complete* if for each nonnegative $\mathcal{F}_T$-measurable random variable B with $x \circeq E^0 \frac{B}{S_T^0} < \infty$ there exists a trading portfolio

π such that $(\pi,0) \in \mathcal{A}(x,y)$ and $X_T^{x,y,\pi,0} \geq B$ a.s., for all $y \in \mathbf{R}$. Otherwise, we say that the market is incomplete. Note that our market is incomplete, since the external factor cannot be traded. A formal way to see the incompleteness of the market is verifying that $\mathcal{P}(y)$ contains more than one measure Q such that the discounted price process $\frac{S}{S^0}$ is a Q-martingale. But, from Remark 3, this is true for all P^{ν}; $\nu \in \mathcal{M}$. Another way to verify that is using Theorem 1.6.6 in [KaSr98].

1.2 Constant coefficients

In this section we illustrate the classical Merton optimization problem when the coefficients of the model are constant and the utility function is HARA. In this case, the external factor does not affect the market behavior.

The classical way to solve these kinds of problems is through stochastic control techniques. In this sense, it is convenient rewriting the trading strategy in a proportion scale. That is, each $(\pi,c) \in \mathcal{A}(x,y)$ defines $(\bar{\pi}_t,\bar{c}_t) \triangleq \left(\pi_t/X_t^{\pi,c}, c_t/X_t^{\pi,c}\right)$; if $X_t^{\pi,c} > 0$, and $(\bar{\pi}_t,\bar{c}_t) \triangleq (0,0)$, otherwise. For this section, we say that the trading strategy proportion $(\bar{\pi},\bar{c})$ is *admissible* if $\left(\bar{\pi}X^{\pi,c},\bar{c}X^{\pi,c}\right) \in \mathcal{A}(x,y)$. Also we consider the investor's problem in the interval $[t,T]$, with $0 \leq t \leq T$ and $T > 0$ fixed. Thus, the differential form of the wealth process is

$$\begin{aligned} dX_u &= -\bar{c}_u X_u du + r\left(1-\bar{\pi}_u\right) X_u du + \bar{\pi}_u X_u \left(\mu du + \sigma dW_u\right) \\ &= X_u \left(\left[r + (\mu - r)\bar{\pi}_u - \bar{c}_u\right] du + \sigma \bar{\pi}_u dW_u\right). \end{aligned}$$

When the trading strategy $(\bar{\pi},\bar{c})$ is admissible, that is, if $\left(\bar{\pi}X^{\pi,c},\bar{c}X^{\pi,c}\right) \in \mathcal{A}(x,y)$, then

$$X_u^{\pi,c} = x \exp\left(ru + \int_t^u \left[\left(\mu - r - \frac{1}{2}\sigma^2 \right) \bar{\pi}_s - \bar{c}_s \right] ds + \sigma \int_t^u \bar{\pi}_s dW_s \right). \tag{1.12}$$

If

$$U(b) \mathrel{\hat{=}} U_1(b) = U_2(b) = \frac{1}{\gamma} b^\gamma; \quad b > 0, \quad \text{with} \quad \gamma < 1, \quad \gamma \neq 0,$$

the investor's problems is to

$$\text{maximize} \quad J(t, x, \bar{\pi}, \bar{c}) \quad \text{over admissible} \quad (\bar{\pi}, \bar{c}),$$

where

$$J(t, x; \bar{\pi}, \bar{c}) \mathrel{\hat{=}} \frac{1}{\gamma} E\left[\left(X_T^{\pi,c}\right)^\gamma + \int_t^T \left(\bar{c}_u X_u^{\pi,c}\right)^\gamma du \right]; \quad (t, x) \in [0, T] \times \mathbf{R}_+.$$

The associated value function is defined as

$$W(t, x) \mathrel{\hat{=}} \sup_{(\bar{\pi}, \bar{c}) \text{ admissible}} J(t, x, \bar{\pi}, \bar{c}).$$

Using the dynamic programing principle, it can be shown that there is a unique smooth function $w(t, y)$ in $C^{1,2}([0, T] \times \mathbf{R}_+)$ that satisfies the Hamilton Jacobi Bellman (HJB) equation

$$0 = w_t + rxw_x + \sup_{(\bar{\pi}, \bar{c}) \in \mathbf{R} \times \mathbf{R}_+} \left\{ \frac{1}{2}\sigma^2 x^2 w_{xx} \bar{\pi}^2 + \left[(\mu - r)\bar{\pi} - \bar{c} \right] x w_x + \frac{1}{\gamma} x^\gamma \bar{c}^\gamma \right\}, \tag{1.13}$$

with $w(T, x) = \frac{1}{\gamma} x^\gamma$. Furthermore, from the homogeneity of the value function $W(t, x)$ with respect to x, we can assume that

$$w(t, x) =: \frac{1}{\gamma} x^\gamma \omega(t); \quad (t, x) \in [0, T] \times \mathbf{R}_+, \quad \text{with} \quad \omega(T) = 1.$$

Then, the HJB equation (1.13) is equivalent to

$$0=\frac{1}{\gamma}\omega_t+r\omega+\sup_{(\bar{\pi},\bar{c})\in\mathbf{R}\times\mathbf{R}_+}\left\{-\frac{1}{2}\bar{\pi}^2\left(1-\gamma\right)\sigma^2\omega+\left(\mu-r\right)\bar{\pi}\omega-\bar{c}\omega+\frac{1}{\gamma}\bar{c}^\gamma\right\}. \quad (1.14)$$

The maximum within the brackets induces the Markov *policy*

$$\bar{\pi}^*\left(t,x\right)\circeq\frac{\mu-r}{\left(1-\gamma\right)\sigma^2}\quad\text{and}\quad\bar{c}^*\left(t,x\right)\circeq\left[\omega\left(t\right)\right]^{-\frac{1}{1-\gamma}}. \quad (1.15)$$

Substituting these values in (1.14), we have

$$\omega_t+\bar{\gamma}\omega+\left(1-\gamma\right)\omega^{-\frac{\gamma}{1-\gamma}}=0,$$

where $\bar{\gamma}\circeq\gamma\left(r+\frac{1}{2}\left[\frac{\mu-r}{(1-\gamma)\sigma}\right]^2\right)$. Applying the power transformation $\omega=:\bar{\omega}^{1-\gamma}$, the last equation is transformed into the ordinary differential equation

$$\bar{\omega}_t+\bar{\gamma}\bar{\omega}+1=0,\quad\text{with}\quad\bar{\omega}\left(T\right)=1.$$

Therefore

$$\bar{\omega}\left(t\right)=\left(1+\frac{1}{\bar{\gamma}}\right)e^{\bar{\gamma}(T-t)}-\frac{1}{\bar{\gamma}};\quad t\in\left[0,T\right].$$

Thus

$$\omega\left(t\right)=\left[\left(1+\frac{1}{\bar{\gamma}}\right)e^{\bar{\gamma}(T-t)}-\frac{1}{\bar{\gamma}}\right]^{1-\gamma};\quad t\in\left[0,T\right].$$

Hence, from (1.15), the optimal trading strategy is given by

$$\begin{aligned}\widehat{\bar{\pi}}_u &\circeq \bar{\pi}^*\left(u,X_{T-u}^{\hat{\pi},\hat{c}}\right)=\frac{\mu-r}{\left(1-\gamma\right)\sigma^2}\quad\text{and}\\ \widehat{\bar{c}}_u &\circeq \bar{c}^*\left(u,X_{T-u}^{\hat{\pi},\hat{c}}\right)=\left[\omega\left(T-u\right)\right]^{-\frac{1}{1-\gamma}};\quad u\in\left[t,T\right].\end{aligned}$$

Note that the optimal trading portfolio is constant, whereas the optimal consumption process depends on a determinist function of the time u.

1.3 Random coefficients

In this section we explain some diffusion models for the coefficients of the financial market.

For a financial market with a bank account and a risky asset, the general framework of the Black and Scholes diffusion model is

$$\begin{aligned} dS_t^0 &= S_t^0 \tilde{r}_t dt, \\ dS_t &= S_t \left(\tilde{\mu}_t dt + \tilde{\sigma}_t dW_{1t} \right), \end{aligned}$$

where $\tilde{r}$, $\tilde{\mu}$, and $\tilde{\sigma}$ are progessively measurable processes such that $\tilde{\sigma} > 0$. In particular, in this work we assume the form

$$\tilde{r}_t \circeq r\left(Y_t\right), \quad \tilde{\mu}_t \circeq \mu\left(Y_t\right), \quad \text{and} \quad \tilde{\sigma}_t \circeq \sigma\left(Y_t\right),$$

where $r\left(\cdot\right), \mu\left(\cdot\right)$, and $\sigma\left(\cdot\right)$ are smooth functions, and the argument Y is the external correlated factor. For instance, the external factor can be a *leader* interest rate, an exchange rate, or another asset price. In general, Y is an economical process correlated with the asset price, which perturbs the level of the coefficients. Therefore, it is part of the financial market.

On the other hand, it is natural to model the external factor Y as a diffusion. In this sense, consider the integral equation

$$Y_t = y + \int_0^t \tilde{\alpha}_s ds + \int_0^t \tilde{\beta}_s d\tilde{W}_s, \tag{1.16}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are the *coefficients* of Y and $\tilde{W}$ is a BM correlated with W_1.

1.3.1 Interest rate and return rate

The interest rate and the return rate play a similar role or are affected by similar economical phenomena. Then, considering that they can be modelled in a similar way, we only focus on the interest rate. The form in (1.16) allows us a wide class of models for interest rates. The next table resumes a list of some known processes for interest rates. The subscript "0" refers to a constant. The symbol "✓" means that the corresponding model is compatible with the framework proposed in this work, whereas "(✓)" means that it is compatible provided a transformation or truncation argument is made.

Model	$r(y)$	**drift** $\tilde{\alpha}$	**diffusion** $\tilde{\beta}$	**notes**	
Malthus	r_0	0	0		✓
Ho-Lee	y	$\tilde{\alpha}$	β_0	$\tilde{\alpha}$ determinist and bounded	(✓)
Vasicek (mean reverting O-U)	y	$\alpha_0 (r_0 - Y)$	β_0	$\alpha_0, r_0 > 0$	(✓)
Cox-Ingresoll-Ross	y	$\tilde{\alpha}(\tilde{r} - Y)$	$\tilde{\beta}\sqrt{Y}$	$\tilde{\alpha}, \tilde{\beta}, \tilde{r}$ determinists	
Black-Karasinski	e^y	$\tilde{\alpha}(\tilde{r} - Y)$	$\tilde{\beta}$	$\tilde{\alpha}, \tilde{\beta}, \tilde{r}$ determinists	(✓)

The simplest model is when the interest rate is constant: $r(y) = r_0 \geq 0$. In this case, the *money market* relies in the Malthus model

$$S_t^0 = e^{r_0 t}; \quad t \in [0, T]. \tag{1.17}$$

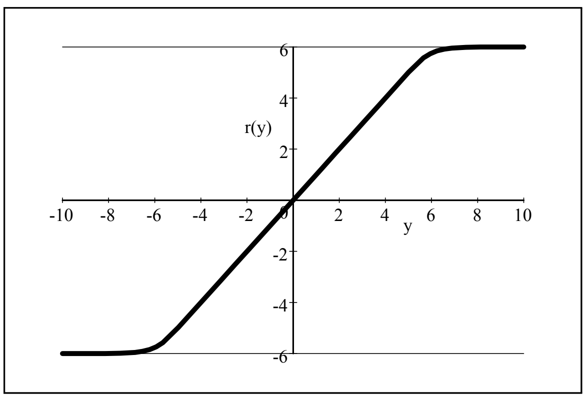
This form is common in the *national production* or *inflation* modelling. The original Black and Scholes model considers constant interest rate.

The Vasicek or mean reverting O-U process usually appears for modelling the interest rates. This Gaussian process allows negative values. Furthermore, in the long return, the interest rate process runs around a value $r_0 > 0$. The model presented in this work is inspired in this one. However, note that the interest rate function $r(y) = y$, is not bounded, unless a truncated smooth version is taken account.

For example, consider the function $r(\cdot)$ in $C_b^2(\mathbf{R})$ defined as

$$r(y) = \begin{cases} y & |y| \leq r_0 \\ [r_0 + \tanh(|y| - r_0)]\,\mathrm{sgn}\, y & |y| \geq r_0 \end{cases},$$

for some fixed $r_0 > 0$. The next figure shows the graph of that function when $r_0 = 5$:



The Cox-Ingresoll-Ross process is a modification of the mean reverting O-U. This process becomes positive if $\tilde{\alpha} \geq \frac{1}{2}\tilde{\beta}^2$. However, it is not compatible with the one presented in this work.

The Black-Karasinski model is similar to the mean reverting O-U process, but in this case the interest rate function is exponential: $r(y) = e^y$. In particular, the interest rate is nonnegative. This model can be considered compatible if an analogous truncation argument used for the mean reverting O-U example is taken.

The Ho-Lee model does not have a mean reverting trend but it is also compatible provided we apply the truncation argument already mentioned. The drift $\tilde{\alpha}$ should be positive if we expect a nonnegative trend.

1.3.2 Volatility

The models for stochastic volatility are similar to those for interest rate, however it is necessary that $\sigma(\cdot) > 0$. Using the general form in (1.16), we present three different models for the external factor Y: lognormal, mean reverting O-U, and Cox-Ingresoll-Ross, which are explained in the next table:

model	**drift** $\tilde{\alpha}$	**diffusion** $\tilde{\beta}$	
lognormal	$\alpha_0 Y$	$\beta_0 Y$	(✓)
mean reverting O-U	$\alpha_0(\sigma_0 - Y)$	β_0	✓
Cox-Ingresoll-Ross	$\alpha_0(\sigma_0 - Y)$	$\beta_0\sqrt{Y}$	

For the lognormal model, we can redefine the external factor as $\tilde{Y} \circeq \log Y$. In this sense, it is like the Ho-Lee model with constant coefficients, and hence, it is compatible with the one proposed in this work.

Considering these three models, the next table shows a list of volatility functions:

Model	$\sigma(y)$	Y process	
Hull-White	$\sqrt{y}$	lognormal	
	y	lognormal	(✓)
Scott	e^y	mean reverting O-U	(✓)
Stein-Stein	$\|y\|$	mean reverting O-U	
Ball-Roma	$\sqrt{y}$	Cox-Ingresoll-Ross	
Heston	$\sqrt{y}$	Cox-Ingresoll-Ross	

Finally, when the market is complete ($\rho = \pm 1$), we can assume that the volatility depends on the level of the asset price such that, with out loss of generality, $Y \equiv S$. In this way, Cox [Co75] suggests the following form for the volatility function:

$$\sigma(y) = \sigma_0 y^\delta; \quad y > 0, \quad \text{with} \quad \sigma_0 > 0, \quad 0 \leq \delta < 1.$$

Levy processes and Fractional Brownian motion

Other interesting models for volatility found in the literature include variants of the Ornstein Uhlenbeck process. For example, in [BaSp02] it is assumed that the volatility is an intrinsic process from the asset price, which is driven by a nonnegative O-U process:

$$d\sigma_t = -\lambda_0 \sigma_t dt + d\tilde{Z}_t, \quad \text{with} \quad \lambda_0 > 0,$$

where $\tilde{Z}$ is a *background* subordinator (a nonnegative Levy process) independent from the underlying BM of the asset price. This model is an important alternative to diffusion models, since it is nonnegative and explains satisfactorily phenomena associated with the volatility, e.g., effect of volatility on heavy tailed returns, volatility clustering, and skewness, in some cases. In [BaSp02] a justification of this model can be found through a detailed empirical study of real financial data. However, the background subordinator $\tilde{Z}$ imposes an economical constraint in the set of admissible trading portfolios. Specifically, the trading portfolio proportion should be in $[0,1]$. Another conceptual difficulty is that the volatility relies in a no directly observable process. In [BKR03], an investment problem applying this model is solved. The optimal trading strategy obtained in that paper relies on a good point of reference for the investor, provided the background process $\tilde{Z}$ is given.

On the other hand, in [CoRe96] and [Hu01] the external factor is modelled as a mean reverting O-U process, where the underlying BM is a *factional* Brownian motion:

$$Y_t = y + \alpha_0 \int_0^t (r_0 - Y_t)ds + \beta_0 W_t^H,$$

where W^H is an independent factional Brownian Motion with *Hurst* parameter $H \in (0,2)$. In particular, for $H = 1$, the fractional BM relies in the classical mean reverting O-U.

We conclude that the proposed framework for the external factor is compatible with a lot of models presented in different references. On the other hand, the mean reverting O-U process is widely used for modelling the coefficients of study; in fact, this example is the most important motivation for this work.

Chapter 2

Primal and dual problems

In this chapter the investor's problem is formulated as a convex optimization problem. The martingale method is explained, and a practical condition for absence of duality gap is given. Furthermore, a relevant relationship between the optimal solutions of both problems is obtained.

The martingale method consists in translating the investor's problem into a convex optimization one, which is called primal. Instead of admissible trading strategies (π, c) the primal problem includes pairs (B, c), where B represents a *final wealth.*

In the context of convex optimization theory, the primal problem has an associated dual problem, whose admissible variables are $\nu \in \mathcal{M}$ and $\lambda > 0$. The next table shows the transformations of the investor's problem under the martingale method:

$$\boxed{\begin{array}{ccc} & & \text{primal problem} \\ & \nearrow & (B,c) \\ \text{investor's problem} & & \downarrow \\ (\pi,c) & \nwarrow & \text{dual problem} \\ & & (\nu,\lambda) \end{array}}$$

We expect that an eventual solution to the dual problem provides the solution of the primal and investor's problems. The dual problem is studied in the next section.

The following lemma allows us to characterize the set of admissible trading strategies $\mathcal{A}(x,y)$, which will be useful to pose the primal problem. It is analogous to Theorem 1 in [Cu97] and Theorem 5.6.2 in [KaSr98]. Some parts are quoted from the above references.

Lemma 4. Let B be a nonnegative $\mathcal{F}_T$-measurable random variable and c a consumption process with

$$\sup_{\nu\in\mathcal{M}} E^{\nu}\left\{\frac{B}{S_T^0}+\int_0^T \frac{c_t}{S_t^0}dt\right\} \le x. \tag{2.1}$$

Then, there exists a trading portfolio π such that $(\pi,c)\in\mathcal{A}(x,y)$ and $X_T^{\pi,c}\ge B$ a.s. Conversely, if $(\pi,c)\in\mathcal{A}(x,y)$, then $B\doteq X_T^{\pi,c}$ satisfies the *budget constraint* (2.1).

Proof. The last part of the lemma is straightforward. From Remark 3, when $\pi\in\mathcal{A}(x,y)$ and $\nu\in\mathcal{M}$, the discounted process $\frac{X^{\pi,c}}{S^0}+\int_0^{\cdot}\frac{c_t}{S_t^0}dt$ is a P^{ν}-supermartingale. Hence

$$E^{\nu}\left[\frac{X_T^{\pi,c}}{S_T^0}+\int_0^T\frac{c_t}{S_t^0}dt\right]\le E^{\nu}X_0^{\pi,c}=x.$$

Now, to show the first part, define the following discounted process:

$$\frac{\check{X}_t}{S_t^0} \triangleq \operatorname*{ess\,sup}_{\nu\in\mathcal{M}} E^\nu\left[\frac{B}{S_T^0} + \int_t^T \frac{c_u}{S_u^0}du \mid \mathcal{F}_t\right]; \quad t\in[0,T]. \tag{2.2}$$

In this case the essential supremum exists, since a non empty family of nonnegative random variables is involved (see Theorem A.3 in [KaSr98]). Note that, by hypothesis,

$$\check{X}_0 = \sup_{\nu\in\mathcal{M}} E^\nu\left\{\frac{B}{S_T^0} + \int_0^T \frac{c_t}{S_t^0}dt\right\} \le x \quad \text{and} \quad \check{X}_T \equiv B. \tag{2.3}$$

It will be shown that $\check{X}$ induces an admissible trading strategy (π, c), such that the associated final wealth $X_T^{\pi,c}$ is greater than or equal to B. First, it will be verified that $\check{X}$ satisfies the dynamic programming equation (DPE):

$$\frac{\check{X}_s}{S_s^0} = \operatorname*{ess\,sup}_{\nu\in\mathcal{M}} E^\nu\left[\frac{\check{X}_t}{S_t^0} + \int_s^t \frac{c_u}{S_u^0}du \mid \mathcal{F}_s\right]; \quad 0\le s\le t\le T. \tag{2.4}$$

Since

$$\begin{aligned} E^\nu\left[\frac{B}{S_T^0} + \int_t^T \frac{c_u}{S_u^0}du \mid \mathcal{F}_s\right] &= E^\nu\left[E^\nu\left(\frac{B}{S_T^0} + \int_t^T \frac{c_u}{S_u^0}du \mid \mathcal{F}_t\right) \mid \mathcal{F}_s\right] \\ &\le E^\nu\left[\frac{\check{X}_t}{S_t^0} \mid \mathcal{F}_s\right], \end{aligned}$$

then

$$\begin{aligned} \frac{\check{X}_s}{S_s^0} &= \operatorname*{ess\,sup}_{\nu\in\mathcal{M}} E^\nu\left[\frac{B}{S_T^0} + \int_s^T \frac{c_u}{S_u^0}du \mid \mathcal{F}_s\right] \\ &= \operatorname*{ess\,sup}_{\nu\in\mathcal{M}} E^\nu\left[\frac{B}{S_T^0} + \int_t^T \frac{c_u}{S_u^0}du + \int_s^t \frac{c_u}{S_u^0}du \mid \mathcal{F}_s\right] \\ &\le \operatorname*{ess\,sup}_{\nu\in\mathcal{M}} E^\nu\left[\frac{\check{X}_t}{S_t^0} + \int_s^t \frac{c_u}{S_u^0}du \mid \mathcal{F}_s\right]. \end{aligned}$$

The reverse inequality is verified next, namely

$$\frac{\check{X}_s}{S_s^0} \ge E^\nu\left[\frac{\check{X}_t}{S_t^0} + \int_s^t \frac{c_u}{S_u^0}du \mid \mathcal{F}_s\right]; \quad \nu\in\mathcal{M}. \tag{2.5}$$

For $\nu \in \mathcal{M}$ and $t \in [0,T]$ fixed, define

$$\mathcal{M}^{\nu}(t) \circeq \{\eta \in \mathcal{M} : \eta \equiv \nu \quad \text{in} \quad [0,t]\}$$

and

$$J_t^{\eta} \circeq E^{\eta}\left[\frac{B}{S_T^0} + \int_t^T \frac{c_u}{S_u^0} du \mid \mathcal{F}_t\right] = E\left[\frac{Z_T^{\eta}}{Z_t^{\eta}}\left(\frac{B}{S_T^0} + \int_t^T \frac{c_u}{S_u^0} du\right) \mid \mathcal{F}_t\right]; \quad \eta \in \mathcal{M}^{\nu}(t).$$

The second equality in the last expression is due to Bayes' formula for conditional expectations. See equation III.3.9 in [JaSh87] or Lemma 3.5.3 in [KaSr91]. On the other hand, since $\frac{Z_T^{\eta}}{Z_t^{\eta}}$ depends only on the values of ν in $[t,T]$ (see equation (1.5)) then

$$\frac{\check{X}_t}{S_t^0} = \sup_{\eta \in \mathcal{M}^{\nu}(t)} J_t^{\eta}.$$

Furthermore

$$\frac{\check{X}_t}{S_t^0} = \lim_{n \to \infty} J_t^{\eta_n}; \tag{2.6}$$

for some increasing sequence $\{J_t^{\eta_n}\}_{n \geq 1}$, with $\eta_n \in \mathcal{M}^{\nu}(t)$. To verify (2.6), it is enough to see that $\{J_t^{\eta}\}_{\eta \in \mathcal{M}^{\nu}(t)}$ is a closed family by pair maximization (see Theorem A.3 in [KaSr98]):

Define

$$\Psi_t \circeq \{J_t^{\eta_1} \geq J_t^{\eta_2}\} \quad \text{and} \quad \eta \circeq 1_{\Psi_t}\eta_1 + 1_{\Psi_t^c}\eta_2; \quad \eta_1, \eta_2 \in \mathcal{M}^{\nu}(t).$$

Note that $\Psi_t \in \mathcal{F}_t$, $\eta \equiv \nu$ in $[0,t]$, and $Z^{\eta} \equiv 1_{\Psi_t} Z^{\eta_1} + 1_{\Psi_t} Z^{\eta_2}$; which is also a martingale. Thus, $\eta \in \mathcal{M}^{\nu}(t)$. Now, let us show that $J_t^{\eta} \equiv J_t^{\eta_1} \vee J_t^{\eta_2}$.

$$\begin{aligned}
J_t^{\eta} &= E^{\eta}\left[\frac{B}{S_T^0} + \int_t^T \frac{c(u)}{S_0(u)}du \mid \mathcal{F}_t\right] \\
&= E\left[\frac{Z_T^{\eta}}{Z_t^{\eta}}\left(\frac{B}{S_T^0} + \int_t^T \frac{c(u)}{S_0(u)}du\right) \mid \mathcal{F}_t\right] \\
&= E\left[\left(1_{\Psi_t}\frac{Z_T^{\eta_1}}{Z_t^{\eta_1}} + 1_{\Psi_t^c}\frac{Z_T^{\eta_2}}{Z_t^{\eta_2}}\right)\left(\frac{B}{S_T^0} + \int_t^T \frac{c(u)}{S_0(u)}du\right) \mid \mathcal{F}_t\right] \\
&= 1_{\Psi_t}E\left[\frac{Z_T^{\eta_1}}{Z_t^{\eta_1}}\left(\frac{B}{S_T^0} + \int_t^T \frac{c(u)}{S_0(u)}du\right) \mid \mathcal{F}_t\right] \\
&\quad + 1_{\Psi_t^c}E\left[\frac{Z_T^{\eta_2}}{Z_t^{\eta_2}}\left(\frac{B}{S_T^0} + \int_t^T \frac{c(u)}{S_0(u)}du\right) \mid \mathcal{F}_t\right] \\
&= 1_{\Psi_t}J_t^{\eta_1} + 1_{\Psi_t^c}J_t^{\eta_2} \\
&= J_t^{\eta_1} \vee J_t^{\eta_2}.
\end{aligned}$$

Hence, from (2.6) and the conditional monotone convergence theorem, inequality (2.5) holds if

$$\frac{\check{X}_s}{S_s^0} \geq E^{\nu}\left[J_t^{\eta_n} + \int_s^t \frac{c_u}{S_u^0}du \mid \mathcal{F}_s\right]; \quad n \geq 1.$$

This follows observing that

$$\begin{aligned}
\frac{\check{X}_s}{S_s^0} &\geq E^{\eta_n}\left[\frac{B}{S_T^0} + \int_s^T \frac{c_u}{S_u^0}du \mid \mathcal{F}_s\right] \\
&= E^{\eta_n}\left[\frac{B}{S_T^0} + \int_s^t \frac{c_u}{S_u^0}du + \int_t^T \frac{c_u}{S_u^0}du \mid \mathcal{F}_s\right] \\
&= E^{\eta_n}\left[E^{\eta_n}\left(\frac{B}{S_T^0} + \int_t^T \frac{c_u}{S_u^0}du \mid \mathcal{F}_t\right) + \int_s^t \frac{c_u}{S_u^0}du \mid \mathcal{F}_s\right] \\
&= E^{\eta_n}\left[J_t^{\eta_n} + \int_s^t \frac{c_u}{S_u^0}du \mid \mathcal{F}_s\right] \\
&= E^{\nu}\left[J_t^{\eta_n} + \int_s^t \frac{c_u}{S_u^0}du \mid \mathcal{F}_s\right].
\end{aligned}$$

Therefore the DPE (2.4) is verified. In particular, this DPE implies that the discounted process $\frac{\check{X}}{S^0} + \int_0^{\cdot} \frac{c_t}{S_t^0}dt$ is a P^{ν}-supermartingale; for each $\nu \in \mathcal{M}$. Using the

Doob-Meyer supermartingale decomposition theorem and local martingale representation theorem, that discounted process can be written as

$$\frac{\check{X}_t}{S_t^0} + \int_0^t \frac{c_s}{S_s^0} ds =: \check{X}_0 + \int_0^t \left(\psi_{1s}^\nu dW_1^\nu + \psi_{2s}^\nu dW_{2s}^\nu\right) - A_t^\nu; \quad t \in [0,T], \tag{2.7}$$

where ψ_1^ν, ψ_2^ν, and A^ν are progressively measurable real processes, such that $\int_0^T ([\psi_{1s}^\nu]^2 + [\psi_{2s}^\nu]^2) ds < \infty$ a.s. and A^ν is predictable increasing with $A_0^\nu \equiv 0$. See Theorem 3.3.9 in [LiSh01] and Problem 3.4.16 in [KaSr91]. Thus, from (2.7), the following identity holds

$$\int_0^t \left(\psi_{1s}^\nu dW_{1s}^\nu + \psi_{2s}^\nu dW_{2s}^\nu\right) - A_t^\nu = \int_0^t \left(\psi_{1s}^0 dW_{1s}^0 + \psi_{2s}^0 dW_{2s}^0\right) - A_t^0.$$

According to expression (1.7), we get

$$\begin{aligned} 0 \;\; = \;\; & \int_0^t \left[\left(\psi_{1s}^\nu - \psi_{1s}^0\right) dW_{1s} + \left(\psi_{2s}^\nu - \psi_{2s}^0\right) dW_{2s}\right] + A_t^0 - A_t^\nu \\ & + \int_0^t \left[\left(\psi_{1s}^\nu - \psi_{1s}^0\right) \theta\left(Y_s\right) + \psi_{2s}^\nu \nu_s\right] ds. \end{aligned}$$

This equation has the form $L + V + \phi \equiv 0$, where L is a continuous local martingale, V is a predictable finite variation process, and ϕ is a continuous process with zero quadratic variation, such that $L_0 = V_0 = \phi_0 = 0$. The above suggests that all those terms should be the zero process. In fact, by Proposition I.4.49.d in [JaSh87], the *covariation* $\langle L, V\rangle$ is identically zero. Thus,

$$0 = \langle \phi, \phi\rangle = \langle L+V, L+V\rangle = \langle L\rangle + \langle V\rangle + 2\langle L, V\rangle.$$

Hence $\langle L\rangle = \langle V\rangle = 0$. This implies

$$\psi_1^\nu \equiv \psi_1^0, \quad \psi_2^\nu \equiv \psi_2^0, \quad \text{and} \quad A^\nu \equiv A^0 + \int_0^\cdot \psi_{2s}^0 \nu_s ds \geq 0.$$

In particular, for constant processes $\nu \equiv v \in \mathbf{R}$, we have

$$A_t^v = A_t^0 + v \int_0^t \psi_{2s}^0 ds; \quad t \in [0, T] \quad \text{a.s.}$$

Define the events

$$\Psi_t^- \doteq \left\{ \int_0^t \psi_{2s}^0 ds < 0 \right\} \quad \text{and} \quad \Psi_t^+ \doteq \left\{ \int_0^t \psi_{2s}^0 ds > 0 \right\},$$

which belong to $\mathcal{F}_t$. Noting that, when $v \to \pm\infty$, $A_t^v \to -\infty$ in $\Psi_t^{\mp}$, one conclude that the unique possibility is $\Psi_t^- \cup \Psi_t^+ = \emptyset$. That is, $A_t^v = A_t^0$ a.s. Hence, $A^v \equiv A^0$ and $\psi_2^0 \equiv 0$, since A^v is *cadlag*. Summarizing:

$$\psi \doteq \psi_1^0 \equiv \psi_1^\nu, \quad \psi_2^\nu \equiv 0, \quad \text{and} \quad A^\nu \equiv A^0; \quad \nu \in \mathcal{M}.$$

Thus, expression (2.7) can be written as

$$\frac{\check{X}_t}{S_t^0} + \int_0^t \frac{c_s}{S_s^0} ds = \check{X}_0 + \int_0^t \psi_s dW_{1s}^\nu - A_t^0, \quad \nu \in \mathcal{M}. \tag{2.8}$$

Now, assume for a moment that the budget constraint (2.1) holds with equality:

$$\check{X}_0 = \sup_{\nu \in \mathcal{M}} E^\nu \left\{ \frac{B}{S_T^0} + \int_0^T \frac{c_t}{S_t^0} dt \right\} = x. \tag{2.9}$$

Next, define the trading portfolio

$$\pi_t \doteq \frac{S_t^0}{\sigma(Y_t)} \psi_t, \quad t \in [0, T]. \tag{2.10}$$

Then, by expressions (1.11), (2.8), and (2.9), $X^{x,y,\pi,c}$ satisfies

$$\begin{aligned} \frac{X_t^{x,y,\pi,c}}{S_t^0} &= x - \int_0^t \frac{c_s}{S_s^0} ds + \int_0^t \frac{\pi_s}{S_s^0} \sigma(Y_s)\, dW_{1s}^\nu \\ &= x - \int_0^t \frac{c_s}{S_s^0} ds + \int_0^t \psi_s dW_{1s}^\nu \\ &= \frac{\check{X}_t}{S_t^0} + A_t^0 \geq \frac{\check{X}_t}{S_t^0} \geq 0 \quad \text{a.s.} \end{aligned}$$

In particular, $X_T^{x,y,\pi,c} \geq \check{X}_T \equiv B$. Otherwise, when $\check{X}_0 < x$, substituting $\check{X}_0$ for x and applying the above arguments to the trading strategy (π, c), but also investing in the bank account the exceeding initial capital $x - \check{X}_0$, we get

$$X^{x,y,\pi,c} \geq X^{\check{X}_0,y,\pi,c} \geq 0 \quad \text{and} \quad X_T^{x,y,\pi,c} \geq B \quad \text{a.s.}$$

■

Thanks to Lemma 4, the investor's problem (1.4) can be written as

$$\text{maximize} \quad E\left\{U_1(B) + \int_0^T U_2(c_t)\,dt\right\} \quad \text{over} \quad (B,c) \in \mathcal{B}(x,y), \tag{P}$$

where

$$\mathcal{B}(x,y) \;\hat{=}\; \Big\{(B,c) \mid B \geq 0 \quad \text{and} \quad \mathcal{F}_T\text{-measurable}, \quad c \quad \text{is a consumption process, and} \quad \sup_{\nu\in\mathcal{M}} E^{\nu}\left[\frac{B}{S_T^0} + \int_0^T \frac{c_t}{S_t^0}dt\right] \leq x\Big\}.$$

This problem will be referred as the **primal** problem, which has the form of a convex optimization problem described in Section 8.6 in [Lu69].

Next theorem suggests the relationship between the trading portfolio π and final wealth B. Its proof is based on arguments given in the proof of the previous lemma. This result is analogous to Theorem 5.8.9 in [KaSr98].

Theorem 5. Let c be a consumption process and $\check{\nu} \in \mathcal{M}$. Then, the following statements are equivalent:

(i)

$$(B,c) \in \mathcal{B}(x,y) \quad \text{and} \quad E^{\check{\nu}}\left[\frac{B}{S_T^0} + \int_0^T \frac{c_t}{S_t^0}dt\right] = x.$$

(ii) $(\pi, c) \in \mathcal{A}(x,y)$, $X_T^{\pi,c} \equiv B$, and $\dfrac{X^{\pi,c}}{S^0} + \displaystyle\int_0^{\cdot} \frac{c_s}{S_s^0} ds$ is a $P^{\check{\nu}}$-martingale with representation

$$\frac{X_t^{\pi,c}}{S_t^0} + \int_0^t \frac{c_s}{S_s^0} ds = x + \int_0^t \psi_s dW_{1s}^{\check{\nu}}; \quad t \in [0,T], \tag{2.11}$$

where ψ is a progressively measurable process with $\int_0^T \psi_u^2 du < \infty$.

Proof. (i) implies (ii). We shall verify that $X^{\pi,c} \equiv \check{X}$, where π is the trading portfolio given in (2.10) and $\check{X}$ is defined in (2.2) satisfying (2.8). From (2.3) and (i), we have

$$\begin{aligned} E^{\check{\nu}}\left[\frac{\check{X}_T}{S_T^0} + \int_0^T \frac{c_t}{S_t^0} dt\right] &= E^{\check{\nu}}\left[\frac{B}{S_T^0} + \int_0^T \frac{c_t}{S_t^0} dt\right] = x = \sup_{\nu \in \mathcal{M}} E^{\nu}\left[\frac{B}{S_T^0} + \int_0^T \frac{c_t}{S_t^0} dt\right] \\ &= \check{X}_0. \end{aligned}$$

Then $\frac{\check{X}}{S^0} + \int_0^{\cdot} \frac{c_s}{S_s^0} ds$ is a $P^{\check{\nu}}$-martingale, since it is a $P^{\check{\nu}}$-supermartingale with constant mean. Thus, from (2.8), $A^0 \equiv 0$, and hence, $X^{\pi,c} \equiv \check{X}$. The rest follows from the martingale representation theorem.

(ii) implies (i). From Remark 3, the discounted process $\frac{X^{\pi,c}}{S^0} + \int_0^{\cdot} \frac{c_s}{S_s^0} ds$ is a P^{ν}-supermartingale; for each $\nu \in \mathcal{M}$. In particular, it is a $P^{\check{\nu}}$-martingale. Hence $(B, c) \in \mathcal{B}(x,y)$ and $E^{\check{\nu}}\left[\frac{B}{S_T^0} + \int_0^T \frac{c_s}{S_s^0} ds\right] = x$, where $B \equiv X_T^{\pi,c}$. ■

Remark 6. In Theorem 5 parts (i) and (ii) are equivalent to the next assertions:

(a) $(\pi, c) \in \mathcal{A}(x,y)$ such that $X_T^{\pi} \equiv B$ and the following DPE holds

$$\begin{aligned} \frac{X_s^{\pi,c}}{S_s^0} &= \operatorname*{ess\,sup}_{\nu \in \mathcal{M}} E^{\nu}\left[\frac{X_t^{\pi,c}}{S_t^0} + \int_s^t \frac{c_u}{S_u^0} du \mid \mathcal{F}_s\right] \\ &= E^{\check{\nu}}\left[\frac{X_t^{\pi,c}}{S_t^0} + \int_s^t \frac{c_u}{S_u^0} du \mid \mathcal{F}_s\right]; \quad 0 \le s \le t \le T. \end{aligned}$$

(b) $(\pi, c) \in \mathcal{A}(x, y)$ such that

$$\begin{aligned}\frac{X_t^{\pi,c}}{S_t^0} &= \operatorname*{ess\,sup}_{\nu\in\mathcal{M}} E^{\nu}\left[\frac{B}{S_T^0} + \int_t^T \frac{c_u}{S_u^0}du \mid \mathcal{F}_t\right] \\ &= E^{\check{\nu}}\left[\frac{B}{S_T^0} + \int_t^T \frac{c_u}{S_u^0}du \mid \mathcal{F}_t\right]; \quad t \in [0, T]. \end{aligned} \tag{2.12}$$

The relationship between the portfolio π and the process ψ in part (ii) of Theorem 5 is given by

$$\pi_t = \frac{S_t^0}{\sigma(Y_t)}\psi_t. \tag{2.13}$$

Finally, we expect that the optimal process $\hat{\nu}$ from the dual problem, defined in (D) below, satisfies parts (a) and (b), and hence statements (i) and (ii) hold too. In the next chapter we will verify this for logarithmic and HARA utility functions.

2.1 Dual problem

In this section we pose the dual problem using techniques from convex analysis. We study the relationship between the optimal expressions of the primal and dual problems. The section is self-contained, taking some basic concepts from [Lu69]. See also Section 3.4 in [KaSr98].

A **utility** function $U : \mathbf{R}_+ \to \mathbf{R}$ is an increasing, concave, and differentiable function. This function captures the investor's attitude with respect to risk.

Assumption 7. For $U(\cdot) = U_1(\cdot), U_2(\cdot)$, the utility functions of the investor's problem (1.4), consider that:

1. $U(\cdot)$ is strictly increasing and strictly concave.
2. $U'(\infty) \doteq \lim_{b\to\infty} U'(b) = 0$ and $U'(0+) \doteq \lim_{b\downarrow 0} U'(b) = \infty$.

From this assumption, it follows that $U'(\cdot)$ is strictly decreasing. On the other hand, the *conjugate* convex function is defined as

$$\tilde{U}(z) \mathrel{\hat{=}} \sup_{b>0} \{U(b) - zb\}; \quad z > 0. \tag{2.14}$$

The function $-\tilde{U}(\cdot)$ is the concave conjugate of $U(\cdot)$ (see Section 7.11 in [Lu69]). From the definition of $\tilde{U}(\cdot)$ and elementary calculus, it follows that

$$\tilde{U}(z) =: U(I(z)) - zI(z); \quad z > 0, \tag{2.15}$$

where $I(\cdot)$ is the inverse function of $U'(\cdot)$. This function is strictly decreasing and holds $I(0+) = \infty$ and $I(\infty) = 0$.

On the other hand, the associated *dual functional* to the primal problem (P) is defined, for $\nu \in \mathcal{M}$ and $\lambda \geq 0$, as follows

$$\begin{aligned} L(\nu, \lambda) &\mathrel{\hat{=}} L(\nu, \lambda; x, y) \\ &\mathrel{\hat{=}} \sup_{B \geq 0, c \geq 0} \left\{ E\left[U_1(B) + \int_0^T U_2(c_t)\, dt \right] - \lambda E^{\nu} \left[\frac{B}{S_T^0} + \int_0^T \frac{c_t}{S_t^0} dt \right] \right\} + \lambda x. \end{aligned}$$

Here the argument "$B \geq 0$, $c \geq 0$" means that B is a nonnegative $\mathcal{F}_T$-measurable random variable and c is a consumption process. This functional with two *Lagrange variables,* λ and ν, is analogous to the one presented in equation (22) in [Cu97], where an optimal consumption problem using the martingale method was studied. Moreover, note that the present definition is a variant of the classical dual functional given in equation (8.6.2) in [Lu69]. The **dual** problem is to

$$\text{minimize} \quad L(\nu, \lambda) \quad \text{over} \quad \nu \in \mathcal{M}, \quad \lambda > 0. \tag{D}$$

Let us get a characterization of the dual functional $L(\nu,\lambda)$. From identity (A.5) in Appendix and (2.14), we have

$$\begin{aligned} & E\left[U_1(B)+\int_0^T U_2(c_t)\,dt\right]-\lambda E^{\nu}\left[\frac{B}{S_T^0}+\int_0^T\frac{c_t}{S_t^0}dt\right] \\ = \;& E\left[U_1(B)-\lambda\frac{Z_T^{\nu}}{S_T^0}B\right]+E\int_0^T\left[U_2(c_t)-\lambda\frac{Z_t^{\nu}}{S_t^0}c_t\right]dt \\ \le \;& E\left[\tilde{U}_1\left(\lambda\frac{Z_T^{\nu}}{S_T^0}\right)+\int_0^T\tilde{U}_2\left(\lambda\frac{Z_t^{\nu}}{S_t^0}\right)dt\right];\quad B\ge 0,\quad c\ge 0. \end{aligned} \tag{2.16}$$

This, together with (2.15), imply that

$$\begin{aligned} L(\nu,\lambda) \;&\le\; E\left[\tilde{U}_1\left(\lambda\frac{Z_T^{\nu}}{S_T^0}\right)+\int_0^T\tilde{U}_2\left(\lambda\frac{Z_t^{\nu}}{S_t^0}\right)dt\right]+\lambda x \\ &=\; E\left[U_1\left(B^{\nu,\lambda}\right)-\lambda\frac{Z_T^{\nu}}{S_T^0}B^{\nu,\lambda}\right]+E\int_0^T\left[U_2\left(c_t^{\nu,\lambda}\right)-\lambda\frac{Z_t^{\nu}}{S_t^0}c_t^{\nu,\lambda}\right]dt+\lambda x \\ &=\; E\left[U_1\left(B^{\nu,\lambda}\right)+\int_0^T U_2\left(c_t^{\nu,\lambda}\right)dt\right]+\lambda\left(x-E^{\nu}\left[\frac{B^{\nu,\lambda}}{S_T^0}+\int_0^T\frac{c_t^{\nu,\lambda}}{S_t^0}dt\right]\right) \\ &\le\; L(\nu,\lambda), \end{aligned}$$

where

$$B^{\nu,\lambda}\circeq I_1\left(\lambda\frac{Z_T^{\nu}}{S_T^0}\right)\quad\text{and}\quad c_t^{\nu,\lambda}\circeq I_2\left(\lambda\frac{Z_t^{\nu}}{S_t^0}\right);\quad t\in[0,T],$$

which are a nonnegative $\mathcal{F}_T$-measurable random variable and a consumption process, respectively. Hence, the dual functional can be written as

$$L(\nu,\lambda)=E\left[\tilde{U}_1\left(\lambda\frac{Z_T^{\nu}}{S_T^0}\right)+\int_0^T\tilde{U}_2\left(\lambda\frac{Z_t^{\nu}}{S_t^0}\right)dt\right]+\lambda x;\quad \nu\in\mathcal{M},\quad \lambda>0. \tag{2.17}$$

This representation of $L(\nu,\lambda)$ is inspired as a natural extension, from complete to incomplete markets, of the results presented in Section 3.6 in [KaSr98].

In this book, the martingale method is implemented building a family of auxiliary complete markets, indexed by $\nu\in\mathcal{P}(y)$. In this sense, the problem is reduced to find the optimal auxiliary market. On the other hand, the dual problem (D) is analogous to the one presented in [Da00] and [KrSc99]. However, their dual functional version is $L(\nu,\lambda)-\lambda x$, which is minimized over ν and then minimize $L(\hat{\nu},\lambda)-\lambda x$ over $\lambda>0$, where $\hat{\nu}$ is the optimal process. In the examples presented in the next chapter, we proceeded in the opposite sense, that is, we find first the optimal variable $\hat{\lambda}$ and then minimize $L(\nu,\hat{\lambda})$.

The martingale method allows us the explicit form of the optimal process $\hat{\nu}$, for logarithmic and HARA utility functions. See sections 3.1 and 3.3.

On the other hand, observe that

$$\sup_{(B,c)\in\mathcal{B}(x,y)} E\left\{U_1(B)+\int_0^T U_2(c_t)\,dt\right\} \leq \inf_{\nu\in\mathcal{M},\lambda>0} L(\nu,\lambda), \tag{2.18}$$

since, from the budget constraint (2.1), we have

$$\begin{aligned}
& \sup_{B\in\mathcal{B}(x,y)} E\left\{U_1(B)+\int_0^T U_2(c_t)\,dt\right\} \\
\leq\ & \sup_{B\in\mathcal{B}(x,y)} \left\{E\left[U_1(B)+\int_0^T U_2(c_t)\,dt\right]-\lambda E^{\nu}\left[\frac{B}{S_T^0}+\int_0^T \frac{c_t}{S_t^0}dt\right]+\lambda x\right\} \\
\leq\ & \sup_{B\geq 0,c\geq 0} \left\{E\left[U_1(B)+\int_0^T U_2(c_t)\,dt\right]-\lambda E^{\nu}\left[\frac{B}{S_T^0}+\int_0^T \frac{c_t}{S_t^0}dt\right]+\lambda x\right\} \\
=\ & L(\nu,\lambda); \quad \nu\in\mathcal{M}, \quad \lambda>0.
\end{aligned}$$

When the equality holds in (2.18), we say that there is no duality *gap*. In the next chapter we will verify, for logarithmic and HARA utility functions, that this holds.

Under a suitable condition, the next proposition shows the relationship between the optimal solutions of the primal (P) and dual (D) problems. Compare with Proposition 6.3.8 in [KaSr98]. Furthermore, in the next chapter, the dual problem is solved for logarithmic and HARA utility functions, and an explicit solution to the investor's problem is given, which are the main contributions of this work. The existence of solution to the dual problem in incomplete markets was considered in Theorem 6.5.1 in [KaSr98].

Proposition 8. Assume that for some $(\hat{\nu}, \hat{\lambda}) \in \mathcal{M} \times \mathbf{R}_{+}$ the pair $(\hat{B}, \hat{c})$, defined as

$$\hat{B} \circeq I_1\left(\hat{\lambda}\frac{Z_T^{\hat{\nu}}}{S_T^0}\right) \quad \text{and} \quad \hat{c}_t \circeq I_2\left(\hat{\lambda}\frac{Z_t^{\hat{\nu}}}{S_t^0}\right); \quad t \in [0,T], \tag{2.19}$$

belongs to $\mathcal{B}(x,y)$ and satisfies

$$E^{\hat{\nu}}\left[\frac{\hat{B}}{S_T^0} + \int_0^T \frac{\hat{c}_t}{S_t^0}dt\right] = x. \tag{2.20}$$

Then, $(\hat{B}, \hat{c})$ is the optimal solution to the primal problem (P), whereas $(\hat{\nu}, \hat{\lambda})$ is the optimal solution to the dual problem (D). In particular, there is no duality gap.

Proof. From (2.17) and (2.15), it follows that

$$\begin{aligned}
\inf_{\nu\in\mathcal{M},\lambda>0} L(\nu,\lambda) &= \inf_{\nu\in\mathcal{M},\lambda>0} E\left\{\tilde{U}_1\left(\lambda\frac{Z_T^{\nu}}{S_T^0}\right) + \int_0^T \tilde{U}_2\left(\lambda\frac{Z_t^{\nu}}{S_t^0}\right)dt + \lambda x\right\} \\
&\leq E\left[\tilde{U}_1\left(\hat{\lambda}\frac{Z_T^{\hat{\nu}}}{S_T^0}\right) + \int_0^T \tilde{U}_2\left(\hat{\lambda}\frac{Z_t^{\hat{\nu}}}{S_t^0}\right)dt\right] + \hat{\lambda}x \\
&= E\left[U_1(\hat{B}) + \int_0^T U_2(\hat{c}_t)\,dt\right] - \hat{\lambda}E\left[\frac{Z_T^{\hat{\nu}}}{S_T^0}\frac{\hat{B}}{S_T^0} + \int_0^T \frac{Z_t^{\hat{\nu}}}{S_t^0}\hat{c}_t dt\right] + \hat{\lambda}x \\
&= E\left[U_1(\hat{B}) + \int_0^T U_2(\hat{c}_t)\,dt\right] - \hat{\lambda}E^{\hat{\nu}}\left[\frac{\hat{B}}{S_T^0} + \int_0^T \frac{\hat{c}_t}{S_t^0}dt\right] + \hat{\lambda}x \\
&= E\left[U_1(\hat{B}) + \int_0^T U_2(\hat{c}_t)\,dt\right] \\
&\leq \sup_{(B,c)\in\mathcal{B}(x,y)} E\left\{U_1(B) + \int_0^T U_2(c_t)\,dt\right\}.
\end{aligned}$$

Due to (2.18), there is no duality gap and $(\hat{B}, \hat{c})$ is the optimal solution to the primal problem (P), whereas $(\hat{\nu}, \hat{\lambda})$ is the optimal solution to the dual problem (D). ∎

Remark 9. The *optimal pair* (2.19) is similar to the one given in equations (6.3.16) and (6.3.17) in [KaSr98], with $\hat{\lambda} = \mathcal{Y}_{\hat{\nu}}(x)$ and $\mathcal{Y}_{\hat{\nu}}(\cdot)$ is the inverse function of $\mathcal{X}_{\hat{\nu}}(\cdot)$, given in equation (6.3.15). An existence result and some characterizations of the solution to the dual problem (D) are also given in Section 6.5 in [KaSr98]. However, except for deterministic coefficients (Section 6.6) and logarithmic case (Example 6.7.2), they do not give the explicit form of the optimal process $\hat{\nu}$.

Remark 10 (Martingale methodology). Based in the results obtained in this chapter, we formulate the elementary steps of the martingale method to get the solution of the investor's problem:

1. Given $U_1(\cdot)$ and $U_2(\cdot)$ utility functions, pose and solve the dual problem. That is, get the optimal solution $(\hat{\nu}, \hat{\lambda}) \in \mathcal{M} \times \mathbf{R}_+$.

2. Verify that the pair $(\hat{B}, \hat{c})$ belongs to $\mathcal{B}(x, y)$ and satisfies (2.20), where $(\hat{B}, \hat{c})$ is defined in (2.19). Then, Proposition 8 and Theorem 5 can be applied.

3. Finally, from (2.11) and (2.13), get the optimal trading portfolio $\hat{\pi}$.

Chapter 3

Results for logarithmic and HARA utility functions

In this chapter the solution to the optimal consumption-investment problem shall be given when the utility function is logarithmic and HARA. Moreover, modifying a little bit the arguments, the solutions for the optimal consumption and optimal investment problems are obtained. Finally, the relationship between the investment problem and pricing and hedging is analyzed.

For the logarithmic and HARA utility functions the martingale method allows us the explicit form of the optimal process $\hat{\nu}$ and the optimal trading strategy $(\hat{\pi}, \hat{c})$; which are the main contributions of this work.

3.1 Logarithmic example

The solution of the logarithmic case exhibits the advantages of the martingale approach, because it turns out to be straightforward.

Suppose that $U(b) \circeq U_1(b) = U_2(b) \circeq \log b$; for $b > 0$. Then

$$I(z) = \frac{1}{z} \quad \text{and} \quad \tilde{U}(z) = -(1 + \log z); \quad z > 0.$$

The dual functional (2.17) has the form

$$\begin{aligned} L(\nu, \lambda) &= -(1+T)(1+\log \lambda) + \lambda x + E\int_0^T r(Y_t) + E\int_0^T \int_0^t r(Y_s)\, ds dt \\ &\quad - E\left[\log Z_T^\nu + \int_0^T \log Z_t^\nu dt\right]. \end{aligned}$$

The optimal value of the variable λ can be obtained by basic calculus arguments, and it is given by $\hat{\lambda} = \frac{1+T}{x}$. This value does not depend on ν. On the other hand, from (1.5), we get

$$-\log Z_t^\nu = \int_0^t [\theta(Y_s)\, dW_{1s} + \nu_s dW_{2s}] + \frac{1}{2}\int_0^t \left[\theta^2(Y_s) + \nu_s^2\right] ds.$$

Then, the dual problem is equivalent to

$$\text{minimize} \quad E\left\{\int_0^T \nu_t^2 dt + \int_0^T \int_0^t \nu_s^2 ds dt\right\}, \quad \text{over} \quad \nu \in \mathcal{M}.$$

Clearly, the optimal solution for this problem is $(\hat{\nu}, \hat{\lambda}) \equiv \left(0, \frac{1+T}{x}\right)$.

Now, as it is suggested in (2.19), we define

$$\begin{aligned} \hat{B} &\circeq I\left(\frac{1+T}{x}\frac{Z_T^0}{S_T^0}\right) = \frac{x}{1+T}\frac{S_T^0}{Z_T^0} \quad \text{and} \\ \hat{c}_t &\circeq I\left(\frac{1+T}{x}\frac{Z_t^0}{S_t^0}\right) = \frac{x}{1+T}\frac{S_t^0}{Z_t^0}; \quad t \in [0, T]. \end{aligned}$$

Next, motivated by (2.12), we define the nonnegative process

$$\frac{\hat{X}_t}{S_t^0} \doteq E^0\left[\frac{\hat{B}}{S_T^0}+\int_t^T \frac{\hat{c}_u}{S_u^0}du \mid \mathcal{F}_t\right]; \quad t\in[0,T]. \tag{3.1}$$

Thus, $\frac{\hat{X}}{S^0}+\int_0^{\cdot}\frac{c_u}{S_u^0}du$ is a P^0-martingale and, from identity (A.5) in Appendix, we have

$$\begin{aligned}
\frac{\hat{X}_t}{S_t^0}+\int_0^t \frac{c_u}{S_u^0}du &= E\left[\frac{Z_T^0}{Z_t^0}\frac{\hat{B}}{S_T^0}+\int_t^T \frac{Z_u^0}{Z_t^0}\frac{\hat{c}_u}{S_u^0}du \mid \mathcal{F}_t\right]+\frac{x}{1+T}\int_0^t \frac{1}{Z_u^0}du \\
&= \frac{x}{1+T}E\left[\frac{1}{Z_t^0}+\int_t^T \frac{1}{Z_t^0}du \mid \mathcal{F}_t\right]+\frac{x}{1+T}\int_0^t \frac{1}{Z_u^0}du \\
&= \frac{x}{1+T}\left[\frac{1+T-t}{Z_t^0}+\int_0^t \frac{1}{Z_u^0}du\right] \\
&= x+\frac{x}{1+T}\int_0^t (1+T-u)\,d\frac{1}{Z_u^0}.
\end{aligned}$$

However, from Ito's formula and expression (1.9), we get

$$d\frac{1}{Z_t^0} = -\frac{dZ_t^0}{\left[Z_t^0\right]^2}+\frac{\left[dZ_t^0\right]^2}{\left[Z_t^0\right]^3} = \frac{\theta(Y_t)}{Z_t^0}dW_{1t}^0.$$

Then

$$\frac{\hat{X}_t}{S_t^0}+\int_0^t \frac{c_u}{S_u^0}du = x+\frac{x}{1+T}\int_0^t (1+T-u)\frac{\theta(Y_u)}{Z_u^0}dW_{1u}^0 =: x+\int_0^t \psi_u dW_{1u}^0,$$

where

$$\psi_t \doteq x\frac{1+T-t}{1+T}\frac{\theta(Y_t)}{Z_t^0}.$$

Note that $\int_0^T \psi_t^2 dt<\infty$, since

$$\int_0^T \left[Z_t^0\right]^{-2}dt = \int_0^T e^{2\int_0^t \theta(Y_u)dW_{1u}+\int_0^t \theta^2(Y_u)du}dt<\infty.$$

Hence, defining

$$\hat{\pi} \doteq \frac{S_t^0}{\sigma(Y_t)}\psi_t,$$

we have $(\hat{\pi}, \hat{c}) \in \mathcal{A}(x, y)$ and $X^{\hat{\pi},\hat{c}} \equiv \hat{X}$. In addition, note that $(\hat{B}, \hat{c}) \in \mathcal{B}(x, y)$, since condition (b) of Remark 6, with $\check{\nu} \equiv 0$, holds.

Summarizing, the optimal trading strategy $(\hat{\pi}, \hat{c})$ is given by

$$\hat{\pi}_t = \pi^*(t, X_t^{\hat{\pi},\hat{c}}, Y_t) \quad \text{and} \quad \hat{c}_t = c^*(t, X_t^{\hat{\pi},\hat{c}}, Y_t); \quad t \in [0, T],$$

where, for $(t, x, y) \in [0, T] \times \mathbf{R}_+ \times \mathbf{R}$:

$$\pi^*(t, x, y) \mathrel{\hat{=}} x\frac{\theta(y)}{\sigma(y)} = x\frac{\mu(y) - r(y)}{\sigma^2(y)} \quad \text{and} \quad c^*(t, x, y) \mathrel{\hat{=}} \frac{x}{1 + T - t}.$$

Note that the optimal trading portfolio does not depend on time, whereas the optimal consumption process depends only on time and the wealth level. This form is analogous to the solution obtained in Example 6.7.2 in [KaSr98], for a market including risky assets that cannot be traded.

3.2 HARA example (primal and dual parts)

In this section the solution of the dual problem for HARA utility function is found. We use the martingale methodology given in Remark 10. This represents the main contribution of this work. In fact, we did not find in the bibliography explicit solutions for the consumption-investment problem in incomplete markets. In particular, we remark that the more difficult case is when the HARA parameter γ is positive.

In order to simplify the presentation, the results are divided in two cases: when $\gamma < 0$ and $0 < \gamma < 1$. The explicit solution for the investor's problem is assigned to the next section.

Here we assume that $U_1(\cdot) = U_2(\cdot) = U(\cdot)$, where $U(\cdot)$ is the hyperbolic absolute risk aversion (HARA) utility function, defined as

$$U(b) \overset{\circ}{=} \frac{1}{\gamma} b^{\gamma}; \quad b > 0, \quad \text{with} \quad \gamma < 1, \quad \gamma \neq 0.$$

Then

$$I(z) = z^{-(1-\alpha)} \quad \text{and} \quad \tilde{U}(z) = -\frac{1}{\alpha} z^{\alpha}; \quad z > 0, \quad \text{where} \quad \alpha \overset{\circ}{=} -\frac{\gamma}{1-\gamma}.$$

Note that $\alpha < 1$, $\alpha \neq 0$, and $\gamma = -\frac{\alpha}{1-\alpha}$.

Thus, the dual functional (2.17) has the form

$$L(\nu, \lambda) = \lambda x - \frac{1}{\alpha} \lambda^{\alpha} E\left[\left(\frac{Z_T^{\nu}}{S_T^0} \right)^{\alpha} + \int_0^T \left(\frac{Z_t^{\nu}}{S_t^0} \right)^{\alpha} dt \right] =: \lambda x - \frac{1}{\alpha} \lambda^{\alpha} \Lambda_{\nu}, \tag{3.2}$$

where

$$\Lambda_{\nu} \overset{\circ}{=} E\left[\left(\frac{Z_T^{\nu}}{S_T^0} \right)^{\alpha} + \int_0^T \left(\frac{Z_t^{\nu}}{S_t^0} \right)^{\alpha} dt \right] > 0.$$

The first two derivatives of $L(\nu, \lambda)$ with respect to λ are

$$L_{\lambda}(\nu, \lambda) = x - \lambda^{\alpha-1} \Lambda_{\nu} \quad \text{and} \quad L_{\lambda\lambda}(\nu, \lambda) = (1-\alpha) \lambda^{\alpha-2} \Lambda_{\nu} > 0.$$

Then, the optimal value of the variable λ is given by

$$\hat{\lambda}(\nu) \overset{\circ}{=} \left(\frac{\Lambda_{\nu}}{x} \right)^{\frac{1}{1-\alpha}}.$$

Substituting this value in (3.2), we get

$$L(\nu, \hat{\lambda}(\nu)) = x^{1-\frac{1}{1-\alpha}} \Lambda_{\nu}^{\frac{1}{1-\alpha}} - \frac{1}{\alpha} x^{-\frac{\alpha}{1-\alpha}} \Lambda_{\nu}^{1+\frac{\alpha}{1-\alpha}} = \frac{1}{\gamma} x^{\gamma} \Lambda_{\nu}^{1-\gamma}; \quad \nu \in \mathcal{M}. \tag{3.3}$$

When $0 < \gamma < 1$ [$\gamma < 0$], minimizing $L(\nu, \hat{\lambda}(\nu))$; over the set of processes ν in $\mathcal{M}$, is equivalent to

$$\text{minimize [maximize]} \quad J(T, y, \nu) \overset{\circ}{=} \Lambda_{\nu}, \quad \text{over} \quad \nu \in \mathcal{M}. \tag{3.4}$$

This is a stochastic control problem, where the control processes belong to $\mathcal{M}$. This will be referred as the *auxiliary* problem and will be solved using dynamic programming techniques as well as analytic arguments. As an additional advantage of the martingale method applied in this case, note that it reduces the investor's problem to one with just a control variable.

On the other hand, observe that

$$\begin{aligned} [Z_t^{\nu}]^{\alpha} &= e^{-\alpha\int_0^t [\theta(Y_s)dW_{1s}+\nu_s dW_{2s}]-\frac{1}{2}\alpha\int_0^t\left[\theta^2(Y_s)+\nu_s^2\right]ds} \\ &= e^{-\alpha\int_0^t [\theta(Y_s)dW_{1s}+\nu_s dW_{2s}]-\frac{1}{2}\alpha^2\int_0^t\left[\theta^2(Y_s)+\nu_s^2\right]ds-\frac{1}{2}\alpha(1-\alpha)\int_0^t\left[\theta^2(Y_s)+\nu_s^2\right]ds} \\ &=: Z_t^{\alpha,\nu}e^{-\frac{1}{2}\alpha(1-\alpha)\int_0^t\left[\theta^2(Y_s)+\nu_s^2\right]ds}; \quad t\in[0,T]\,, \end{aligned} \tag{3.5}$$

where $Z^{\alpha,\nu}$ is defined as Z^{ν} substituting in (1.5) $\alpha\theta(\cdot)$ by $\theta(\cdot)$ and $\alpha\nu$ by ν. That is

$$Z_t^{\alpha,\nu} \circeq \exp\left(-\alpha\int_0^t \left[\theta(Y_u)\,dW_{1u}+\nu_u dW_{2u}\right]-\frac{1}{2}\alpha^2\int_0^t\left[\theta^2(Y_u)+\nu_u^2\right]du\right).$$

Proceeding as in (1.6) and (1.7), we can define the measure $P^{\alpha,\nu}$ in $\mathcal{F}_T$ and the BM $(W_1^{\alpha,\nu}, W_2^{\alpha,\nu})$, respectively. Under the new measure, the dynamics of the external factor Y satisfies

$$dY_t = \left[g(Y_t)-\alpha\rho\theta(Y_t)-\alpha\varepsilon\nu_t\right]dt+\rho dW_{1t}^{\alpha,\nu}+\varepsilon dW_{2t}^{\alpha,\nu}, \quad \text{with} \quad Y_0=y\in\mathbf{R}. \tag{3.6}$$

Compare with representation (1.8). From (3.5) and using identity (A.5) in Appendix, we get

$$
\begin{aligned}
& J(T,y,\nu) \\
= \;& \Lambda_\nu = E\left[\left(\frac{Z_T^\nu}{S_T^0}\right)^\alpha + \int_0^T \left(\frac{Z_t^\nu}{S_t^0}\right)^\alpha dt\right] \\
= \;& E^{\alpha,\nu}\left[e^{-\alpha\int_0^T\left[r(Y_t)+\frac{1}{2}(1-\alpha)\left(\theta^2(Y_t)+\nu_t^2\right)\right]dt} + \int_0^T e^{-\alpha\int_0^t\left[r(Y_s)+\frac{1}{2}(1-\alpha)\left(\theta^2(Y_s)+\nu_s^2\right)\right]ds}dt\right] \\
= \;& : E^{\alpha,\nu}\left[e^{\int_0^T q(Y_t,\nu_t)dt} + \int_0^T e^{\int_0^t q(Y_s,\nu_s)ds}dt\right]; \quad (T,y,\nu)\in\mathbf{R}_+\times\mathbf{R}\times\mathcal{M},
\end{aligned}
$$

where

$$
q(y,v) \mathrel{\mathring{=}} -\alpha\left[r(y)+\frac{1}{2}(1-\alpha)\left(\theta^2(y)+v^2\right)\right]; \quad (y,v)\in\mathbf{R}^2.
$$

For $0<\alpha<1$, the function $q(y,v)$ is bounded from above. Otherwise, when $\alpha<0$, $q(y,v)$ is bounded, provided the control space is compact.

3.2.1 Case $\gamma < 0$

In this case, $0<\alpha<1$ and the *value function* associated with the auxiliary problem (3.4) is

$$
W(T,y) \mathrel{\mathring{=}} \sup_{\nu\in\mathcal{M}} J(T,y,\nu); \quad (T,y)\in\mathbf{R}_+\times\mathbf{R} \quad \text{with} \quad W(0,y)=1. \tag{3.7}
$$

Before writing down the HJB equation associated with this problem, let us give some properties of the value function. Since

$$
\begin{aligned}
& E\left[e^{-\alpha\int_0^T\left(|r|_\infty+\frac{1}{2}(1-\alpha)\left(|\theta|_\infty^2+\nu_u^2\right)\right)du} + \int_0^T e^{-\alpha\int_0^t\left(|r|_\infty+\frac{1}{2}(1-\alpha)\left(|\theta|_\infty^2+\nu_u^2\right)\right)du}dt\right] \\
\leq \;& J(T,y,\nu) \leq 1+T,
\end{aligned} \tag{3.8}
$$

then

$$
0 < K_1 \leq W(T,y) \leq 1+T, \tag{3.9}
$$

where

$$K_1 \doteq (1+T)\, e^{-\alpha\left(|r|_\infty + \frac{1}{2}(1-\alpha)|\theta|_\infty^2\right)T}. \tag{3.10}$$

Note that K_1 does not depend on ν and y.

Now, let us verify that $W(T,\cdot)$ is a Lipschitz function. Since $q(y,v)$ is bounded from above, we can use the dominated convergence theorem to verify the y-differentiability of $J(T,y,\nu)$ (see Theorem 5.5.5 in [Fr75]). In fact, it is easy to see that

$$|J(T,y+h,\nu) - J(T,y,\nu)| \le (1-\alpha)(1+T)\,|\theta|_\infty\,|\theta'|_\infty \int_0^T e^{|g'|_\infty t} dt \times h$$

and

$$\begin{aligned} J_y(T,y,\nu) &= E^{\alpha,\nu}\left[e^{\int_0^T q(Y_t,\nu_t)dt} \int_0^T q_y(Y_t,\nu_t)\frac{\partial}{\partial y}Y_t dt \right. \\ &\quad \left. + \int_0^T e^{\int_0^t q(Y_s,\nu_s)ds} \int_0^t q_y(Y_s,\nu_s)\frac{\partial}{\partial y}Y_s ds dt \right], \end{aligned}$$

where $\frac{\partial}{\partial y}Y_s$ is the solution of the ordinary differential equation

$$d\frac{\partial}{\partial y}Y_s = [g'(Y_s) - \alpha\rho\theta'(Y_s)]\frac{\partial}{\partial y}Y_s ds; \quad \text{with} \quad \frac{\partial}{\partial y}Y_0 = 1,$$

that is,

$$\frac{\partial}{\partial y}Y_s = \exp\left(\int_0^s [g'(Y_u) - \alpha\rho\theta'(Y_u)]\,du\right); \quad \text{in} \quad [0,T].$$

Since

$$\left|\int_0^T q_y(Y_s,\nu_s)\frac{\partial}{\partial y}Y_s ds\right| \le K_2 \doteq \alpha\left(|r'|_\infty + (1-\alpha)\,|\theta|_\infty|\theta'|_\infty\right) T e^{(|g'|_\infty + \alpha|\theta'|_\infty)T},$$

then

$$|J_y(T,y,\nu)| \le K_2(1+T).$$

Observe that K_2 does not depend on ν and y. Hence, $W(T,\cdot)$ is a Lipschitz function with Lipschitz constant $K_2(1+T)$. Finally, from (3.9), we get

$$\frac{|W_y(T,y)|}{W(T,y)} \leq K \doteq \frac{K_2}{K_1}(1+T), \tag{3.11}$$

provided $W_y(T,y)$ is well defined. The estimation (3.11) will be used later in this section.

Now, for a moment let us constrain the set of control processes to those in $\mathcal{M}$ taking values in $[-M,M]$; for a fixed constant $M>0$ given; denote this set as $\mathcal{M}^M$. The corresponding *constrained* value function is denoted by $W^M(T,y)$. Later we shall remove this constraint proving that the value function is independent of M, when it is large enough.

The verification theorem below states that

$$w(T,y) = W^M(T,y);$$

where $w(T,y)$ is the unique smooth function in $C^{1,2}\left(\bar{\mathbf{R}}_+ \times \mathbf{R}\right) \cap C_p\left(\bar{\mathbf{R}}_+ \times \mathbf{R}\right)$ satisfying the associated HJB equation:

$$\begin{aligned} w_T &= 1 + \frac{1}{2}w_{yy} + (g-\alpha\rho\theta)\,w_y - \alpha\left[r + \frac{1}{2}(1-\alpha)\,\theta^2\right]w \\ &\quad + \alpha \sup_{v\in[-M,M]}\left\{-\varepsilon w_y v - \frac{1}{2}(1-\alpha)\,wv^2\right\}, \end{aligned} \tag{3.12}$$

with $w(0,y)=1$. see Theorem IV.4.3 and Remark IV.4.1 in [FlSo93]. As in the last expression, for simplicity, sometimes hereafter, we suppress the arguments of the real functions. For instance, using w instead of $w(T,y)$ and so on.

The HJB equation (3.12) induces a Markov *policy* defined as follows: for $(t,y) \in$

$[0,T] \times \mathbf{R}$:

$$\begin{aligned} \nu^*(t,y) &\mathrel{\mathring{=}} \arg\max_{v\in[-M,M]} \left\{-\varepsilon w_y(t,y)\, v - \frac{1}{2}(1-\alpha)\, w(t,y)\, v^2\right\} \qquad (3.13) \\ &= \begin{cases} -\dfrac{\varepsilon}{1-\alpha}\dfrac{w_y(t,y)}{w(t,y)}, & \dfrac{\varepsilon}{1-\alpha}\dfrac{|w_y(t,y)|}{w(t,y)} \leq M, \quad w(t,y) \neq 0 \\ -M\,\mathrm{sgn}\, w_y(t,y), & \text{otherwise} \end{cases}. \end{aligned}$$

Theorem 11 (Verification). For $M > 0$, let $w(T,y)$ be the unique solution to (3.12). Then:

(i)

$$w(T,y) \geq J(T,y,\nu); \quad (T,y) \in \bar{\mathbf{R}}_+ \times \mathbf{R}, \quad \nu \in \mathcal{M}^M.$$

(ii)

$$w(T,y) = W^M(T,y) = J(T,y,\hat{\nu}),$$

where $\hat{\nu}$ is the Markov control process in $\mathcal{M}^M$, given by

$$\hat{\nu}_t \mathrel{\mathring{=}} \nu^*(T-t, Y_t); \quad t \in [0,T]. \qquad (3.14)$$

In particular, $\hat{\nu}$ is the optimal process for the constrained auxiliary problem relative to (3.7).

Proof. (i) For $v \in [-M,M]$, let $\mathcal{L}^v$ be the functional defined as

$$\mathcal{L}^v f \mathrel{\mathring{=}} f_t + \frac{1}{2} f_{yy} + (g - \alpha\rho\theta - \alpha\varepsilon v)\, f_y; \quad f \in C^{1,2}([0,T] \times \mathbf{R}).$$

In particular, for $f(t,y) \mathrel{\mathring{=}} w(T-t,y)$, we get

$$\begin{aligned} [\mathcal{L}^v + q(y,v)]\, w(T-t,y) &= -w_t + \frac{1}{2} w_{yy} + (g - \alpha\rho\theta)\, w_y - \alpha\left[r + \frac{1}{2}(1-\alpha)\,\theta^2\right] w \\ &\quad + \alpha\left[-\varepsilon w_y v - \frac{1}{2}(1-\alpha)\, w v^2\right]. \end{aligned}$$

Thus, from the HJB equation (3.12), we have

$$[\mathcal{L}^{\nu_t} + q(Y_t, \nu_t)]\, w(T-t, Y_t) \le -1; \quad t \in [0,T], \quad \nu \in \mathcal{M}^M. \tag{3.15}$$

This inequality, together with the Feynman-Kac formula (A.3) in Appendix and a reparameterization in time, imply that

$$\begin{aligned} w(T,y) &= E^{\alpha,\nu}\left[e^{\int_0^T q(Y_u,\nu_u)du} w(0, Y_T) - \right. \\ &\qquad \left. \int_0^T e^{\int_0^t q(Y_u,\nu_u)du} [\mathcal{L}^{\nu_t} + q(Y_t,\nu_t)]\, w(T-t, Y_t) dt \right] \\ &\ge E^{\alpha,\nu}\left[e^{\int_0^T q(Y_u,\nu_u)du} + \int_0^T e^{\int_0^t q(Y_u,\nu_u)du} dt \right] \\ &= J(T,y,\nu). \end{aligned} \tag{3.16}$$

Here we use Corollary 17 in Appendix with

$$\tau \mathrel{\hat{=}} T, \quad s \mathrel{\hat{=}} 0, \quad q_t \mathrel{\hat{=}} q(Y_t, \nu_t), \quad \text{and} \quad \Gamma_t \mathrel{\hat{=}} e^{\int_0^t q(Y_u,\nu_u)du}.$$

For a basic reference, see equation (D.13) in [FlSo93] and, for the reparameterization in time, see Corollary 4.4.5 in [KaSr91].

(ii) Since $w_y(t,y)$ is continuous, then the Markov policy $\nu^*(t,y)$ given in (3.13) becomes a bounded, continuous, and y-locally Lipschitz function (see Proposition 20 in Appendix). That is, $\nu^*(t,y)$ satisfies properties (IV.3.12) in [FlSo93]. Hence $\hat{\nu}$, defined as (3.14), is a Markov control process in $\mathcal{M}^M$. Finally, from the definition of $\nu^*(t,y)$, for $\nu \mathrel{\hat{=}} \hat{\nu}$ inequalities (3.15) and (3.16) become equalities. Therefore, $w(T,y) = W^M(T,y) = J(T,y,\hat{\nu})$. ■

Corollary 12. Let $w(T,y)$ be the unique solution to (3.12) with $M > \dfrac{\varepsilon}{1-\alpha}K$, and let $W(T,y)$ be the unconstrained value function defined in (3.7). Then,

$$W \in C^{1,2}\left(\bar{\mathbf{R}}_+ \times \mathbf{R}\right) \cap C_b^{0,1}\left(\bar{\mathbf{R}}_+ \times \mathbf{R}\right),$$

$$w(T,y) = W(T,y),$$

and $\hat{\nu}$ is the optimal control process, where

$$\hat{\nu}_t \mathrel{\hat{=}} \nu^*(T-t,Y_t) = -\frac{\varepsilon}{1-\alpha}\frac{W_y(T-t,Y_t)}{W(T-t,Y_t)}; \quad t \in [0,T]. \tag{3.17}$$

Furthermore, W solves the partial differential equation (PDE)

$$W_T = 1 + \frac{1}{2}W_{yy} + (g - \alpha\rho\theta)W_y - \alpha\left[r + \frac{1}{2}(1-\alpha)\theta^2\right]W - \frac{1}{2}\gamma\varepsilon^2\frac{W_y^2}{W}, \tag{3.18}$$

with $W(0,y) = 1$.

Proof. For $M > \frac{\varepsilon}{1-\alpha}K$ and from (3.11), we have

$$\frac{\varepsilon}{1-\alpha}\frac{|W_y|}{W} < M \quad \text{and} \quad w(T,y) = W^M(T,y) = W(T,y).$$

Thus, using (3.14) and (3.13), the optimal process is given by

$$\hat{\nu}_t \mathrel{\hat{=}} \nu^*(T-t,Y_t) = -\frac{\varepsilon}{1-\alpha}\frac{W_y(T-t,Y_t)}{W(T-t,Y_t)}; \quad t \in [0,T].$$

Finally, substituting the Markov policy (3.13) in the HJB equation (3.12) we obtain (3.18). ∎

3.2.2 Case $0 < \gamma < 1$

In this case, the *value function* associated with the auxiliary problem (3.4) is defined as: for $(T,y) \in \mathbf{R}_+ \times \mathbf{R}$,

$$W(T,y) \mathrel{\hat{=}} \inf_{\nu \in \mathcal{M}} J(T,y,\nu) = \inf_{\nu \in \mathcal{M}} E^{\alpha,\nu}\left\{e^{\int_0^T q(Y_t,\nu_t)dt} + \int_0^T e^{\int_0^t q(Y_s,\nu_s)ds}dt\right\}; \tag{3.19}$$

Like in the other case, we expect similar conclusions if a suitable bound for the ratio $\frac{W_y(T,y)}{W(T,y)}$ is obtained. However, in this case, we cannot apply the dominated convergence theorem to get estimations of $W_y(T,y)$ independent of M, because $q(y,v)$ is not bounded from above. Instead, we explore qualitative properties of the associated HJB equation written below. First, we give some properties of $W(T,y)$.

Noting that (3.8) holds in the reverse sense, it follows that

$$1+T \leq W(T,y) \leq K_1, \tag{3.20}$$

where K_1 is given by (3.10). Now, we verify that $W(T,y)$ is increasing with respect to T. To get this, it is enough to prove that

$$J(T,y,\nu) \geq J(T-\Delta,y,\nu); \quad 0<\Delta<T, \quad \nu \in \mathcal{M}(T). \tag{3.21}$$

Here we make explicit the dependence of $\mathcal{M}$ on T denoting it as $\mathcal{M}(T)$. Observe that the restriction of $\nu \in \mathcal{M}(T)$ in the interval $[0,T-\Delta]$, and denoted by the same symbol, belongs to $\mathcal{M}(T-\Delta)$. Thus

$$\begin{aligned}
J(T-\Delta,y,\nu) &= E^{\alpha,\nu}_{T-\Delta}\left[e^{\int_0^{T-\Delta} q(Y_t,\nu_t)dt}+\int_0^{T-\Delta} e^{\int_0^t q(Y_s,\nu_s)ds}dt\right] \\
&= E\left(E\left[Z^{\alpha,\nu}_T \mid \mathcal{F}_{T-\Delta}\right]\left[e^{\int_0^{T-\Delta} q(Y_t,\nu_t)dt}+\int_0^{T-\Delta} e^{\int_0^t q(Y_s,\nu_s)ds}dt\right]\right) \\
&= EE\left[Z^{\alpha,\nu}_T\left(e^{\int_0^{T-\Delta} q(Y_t,\nu_t)dt}+\int_0^{T-\Delta} e^{\int_0^t q(Y_s,\nu_s)ds}dt\right) \mid \mathcal{F}_{T-\Delta}\right] \\
&= EZ^{\alpha,\nu}_T\left[e^{\int_0^{T-\Delta} q(Y_t,\nu_t)dt}+\int_0^{T-\Delta} e^{\int_0^t q(Y_s,\nu_s)ds}dt\right] \\
&\leq E^{\alpha,\nu}_T\left[e^{\int_0^{T} q(Y_t,\nu_t)dt}+\int_0^{T} e^{\int_0^t q(Y_s,\nu_s)ds}dt\right] \\
&= J(T,y,\nu).
\end{aligned}$$

The second equality is due to the fact that $Z^{\alpha,\nu}_{T-\Delta}=E\left[Z^{\alpha,\nu}_T \mid \mathcal{F}_{T-\Delta}\right].$

Now, temporarily, consider the bounded control space, that is $\nu \in \mathcal{M}^M$; for $M > 0$ given. In fact, as in the negative HARA parameter case, we will prove below that $W(T, y)$ does not depend on M, for M large enough. Assuming the interval $[-M, M]$ as the control space, the verification theorem below states that

$$w(T, y) = W^M(T, y);$$

where $w(T, y)$ is the unique smooth function (see Theorem IV.4.3 and Remark IV.4.1 in [FlSo93]) in $C^{1,2}\left(\bar{\mathbf{R}}_+ \times \mathbf{R}\right) \cap C_p\left(\bar{\mathbf{R}}_+ \times \mathbf{R}\right)$ satisfying the associated HJB equation

$$\begin{aligned} w_T &= 1 + \frac{1}{2} w_{yy} + (g - \alpha\rho\theta)\, w_y - \alpha \left[r + \frac{1}{2}(1-\alpha)\,\theta^2 \right] w \\ &\quad + \alpha \sup_{v \in [-M, M]} \left\{ -\varepsilon w_y v - \frac{1}{2}(1-\alpha)\, w v^2 \right\}, \end{aligned} \tag{3.22}$$

with $w(0, y) = 1$. Comparing this HJB equation with (3.12), they are similar, but here, $\alpha < 0$.

Theorem 13 (Verification). Given $M > 0$, let $w(T, y)$ be the unique solution to (3.22). Then:

(i)

$$w(T, y) \leq J(T, y, \nu), \quad (T, y) \in \mathbf{R}_+ \times \mathbf{R}, \quad \nu \in \mathcal{M}^M.$$

(ii)

$$w(T, y) = W^M(T, y) = J(T, y, \hat{\nu}),$$

where $\hat{\nu}$ is the Markov control process in $\mathcal{M}^M$ given by (3.14) with $\nu^*(t, y)$ as in (3.13). In particular, $\hat{\nu}$ is the optimal control process for the constrained auxiliary problem relative to (3.19).

Proof. (i) Proceeding as in Theorem 11, from the HJB equation (3.22), we have

$$[\mathcal{L}^{\nu_t} + q(Y_t, \nu_t)]\, w(T-t, Y_t) \geq -1; \quad t \in [0,T], \quad \nu \in \mathcal{M}^M. \tag{3.23}$$

Compare with inequality (3.15). This inequality, together with the Feynman-Kac formula (A.3) in Appendix and a reparameterization in time, imply that

$$\begin{aligned} w(T,y) &= E^{\alpha,\nu}\Bigg[e^{\int_0^T q(Y_u,\nu_u)du} w(0, Y_T) - \\ &\qquad \int_0^T e^{\int_0^t q(Y_u,\nu_u)du} [\mathcal{L}^{\nu_t} + q(Y_t,\nu_t)]\, w(T-t, Y_t)\, dt\Bigg] \\ &\leq E^{\alpha,\nu}\left[e^{\int_0^T q(Y_u,\nu_u)du} + \int_0^T e^{\int_0^t q(Y_u,\nu_u)du} w(T-t, Y_t)\, dt\right] \qquad (3.24) \\ &= J(T,y,\nu); \quad (T,y) \in \mathbf{R}_+ \times \mathbf{R}, \quad \nu \in \mathcal{M}. \end{aligned}$$

Here we use Corollary 17 in Appendix and its subsequent remark.

(ii) Since $w_y(T,y)$ is continuous, the Markov policy $\nu^*(t,y)$ is bounded, continuous, and y-locally Lipschitz. That is, ν^* satisfies properties (IV.3.12) in [FlSo93]. Hence $\hat{\nu}$, defined as (3.14), is a Markov control process in $\mathcal{M}^M$. Finally, from definition of $\nu^*(t,y)$ and for $\nu \mathrel{\hat{=}} \hat{\nu}$, inequalities (3.23) and (3.24) become equalities. Therefore, $w(T,y) = W^M(T,y) = J(T,y,\hat{\nu})$. ■

Theorem 14. Let $W(T,y)$ be the unconstrained value function (3.19). There exists a constant $\tilde{K} > 0$ such that for $M > \dfrac{\varepsilon}{1-\alpha}\tilde{K}$, implies that

$$w(T,y) = W(T,y)$$

and

$$\hat{\nu}_t = \nu^*(T-t, Y_t) = -\frac{\varepsilon}{1-\alpha}\frac{W_y(T-t,Y_t)}{W(T-t,Y_t)}; \quad t \in [0,T], \tag{3.25}$$

is the optimal control process. Furthermore, $W \in C^{1,2}\left(\bar{\mathbf{R}}_+ \times \mathbf{R}\right) \cap C_b^{0,1}\left(\bar{\mathbf{R}}_+ \times \mathbf{R}\right)$ and solves the PDE (3.18) with $w(0,y) = 1$.

Proof. To estimate $W_y(T,y)$ we obtain a bound for $W_T^M(T,y)$ and extract qualitative properties of $W^M(T,y)$ from the HJB equation (3.22), where $w(T,y) = W^M(T,y)$. From (3.21), we know that $W_T^M(T,y) \geq 0$. On the other hand, extend the optimal process $\hat{\nu}$ from the constrained problem in $[0,T]$ (see (3.14) and (3.13)) to the interval $[0, T+\Delta]$, in such a way that it vanishes in $(T, T+\Delta]$, and for simplicity, denote it by the same symbol. This extended process belongs to $\mathcal{M}(T+\Delta)$, since $\theta(\cdot)$ is bounded. Thus,

$$
\begin{aligned}
& W^M(T+\Delta,y) - W^M(T,y) \\
= \; & W^M(T+\Delta,y) - J(T,y,\hat{\nu}) \\
\leq \; & J(T+\Delta,y,\hat{\nu}) - J(T,y,\hat{\nu}) \\
= \; & E_{T+\Delta}^{\alpha,\hat{\nu}}\left[e^{\int_0^{T+\Delta} q(Y_t,\hat{\nu}_t)dt} + \int_0^{T+\Delta} e^{\int_0^t q(Y_s,\hat{\nu}_s)ds} dt\right] \\
& - E_T^{\alpha,\hat{\nu}}\left[e^{\int_0^{T} q(Y_t,\hat{\nu}_t)dt} + \int_0^{T} e^{\int_0^t q(Y_s,\hat{\nu}_s)ds} dt\right] \\
= \; & E_{T+\Delta}^{\alpha,\hat{\nu}}\left[e^{\int_0^{T+\Delta} q(Y_t,\hat{\nu}_t)dt} - e^{\int_0^{T} q(Y_t,\hat{\nu}_t)dt} + \int_0^{T+\Delta} e^{\int_0^t q(Y_s,\hat{\nu}_s)ds} dt - \int_0^{T} e^{\int_0^t q(Y_s,\hat{\nu}_s)ds} dt\right] \\
= \; & E_{T+\Delta}^{\alpha,\hat{\nu}}\left[e^{\int_0^{T} q(Y_t,\hat{\nu}_t)dt}\left(e^{\int_T^{T+\Delta} q(Y_t,0)dt} - 1\right) + e^{\int_0^{T} q(Y_t,\hat{\nu}_t)dt}\int_T^{T+\Delta} e^{\int_T^t q(Y_s,0)ds} dt\right] \\
\leq \; & E_{T+\Delta}^{\alpha,\hat{\nu}} e^{\int_0^{T} q(Y_t,\hat{\nu}_t)dt}\left[e^{-\alpha\left(|r|_\infty + \frac{1}{2}(1-\alpha)|\theta|_\infty^2\right)\Delta} - 1 + \int_T^{T+\Delta} e^{-\alpha\int_T^t \left(|r|_\infty + \frac{1}{2}(1-\alpha)|\theta|_\infty^2\right)ds} dt\right] \\
=: \; & \left[e^{K_3\Delta} - 1 + \int_0^{\Delta} e^{K_3 t} dt\right] E_{T+\Delta}^{\alpha,\hat{\nu}} e^{\int_0^{T} q(Y_t,\hat{\nu}_t)dt}.
\end{aligned}
$$

The last inequality is due to the fact that

$$
q(y,0) \leq K_3 \circeq -\alpha\left(|r|_\infty + \frac{1}{2}(1-\alpha)|\theta|_\infty^2\right).
$$

Observe that K_3 does not depend on M and y. Then, from the dominated convergence theorem, we have

$$0 \leq W_T^M(T,y) \leq (1+K_3)\, W^M(T,y)\,. \tag{3.26}$$

The analysis of the upper and lower bounds of $W_y^M(T,y)$ is presented only for $y \in \mathbf{R}_+$, since the same arguments can be used for $y \in \mathbf{R}_- \mathrel{\circeq} -\mathbf{R}_+$. Define

$$\Phi(p) \mathrel{\circeq} \sup_{v \in [-M,M]} \left\{ -\varepsilon p v - \frac{1}{2}(1-\alpha)\, v^2 \right\}; \quad p \in \mathbf{R}. \tag{3.27}$$

Thus, the HJB equation (3.22) can be written as

$$\begin{aligned} W_T^M &= 1 + \frac{1}{2} W_{yy}^M + (g - \alpha\rho\theta)\, W_y^M - \alpha \left[r + \frac{1}{2}(1-\alpha)\,\theta^2 \right] W^M \\ &\quad + \alpha W^M \Phi\left(\frac{W_y^M}{W^M} \right), \end{aligned} \tag{3.28}$$

with $W^M(0,y) = 1$. On the other hand, from the mean value theorem and expression (3.20): for each $n \geq 1$, there exists $y_n \in [n, 2n]$ such that

$$\left| W_y^M(T, y_n) \right| = \frac{1}{n} \left| W^M(T, 2n) - W^M(T, n) \right| \leq \frac{2}{n} K_1.$$

Therefore, it is enough to analyze the critical points $\tilde{y}$ of $W_y^M(T,\cdot)$, *i.e.,* the points $\tilde{y} > 0$ such that $W_{yy}^M(T,\tilde{y}) = 0$. Now we study three disjoint and exhaustive cases based on the coefficient of W_y^M in the PDE (3.28):

(a) $g(\tilde{y}) - \alpha\rho\theta(\tilde{y}) \leq -1$ and $W_y^M(T,\tilde{y}) > 0$ or

$g(\tilde{y}) - \alpha\rho\theta(\tilde{y}) \geq 1$ and $W_y^M(T,\tilde{y}) < 0$.

(b) $g(\tilde{y}) - \alpha\rho\theta(\tilde{y}) \geq -1$ and $W_y^M(T,\tilde{y}) > 0$.

(c) $g(\tilde{y}) - \alpha\rho\theta(\tilde{y}) \leq 1$ and $W_y^M(T,\tilde{y}) < 0$.

Case (a). Condition (a) is equivalent to

$$|g(\tilde{y}) - \alpha\rho\theta(\tilde{y})| \geq 1 \quad \text{and} \quad [g(\tilde{y}) - \alpha\rho\theta(\tilde{y})] W_y^M(T, \tilde{y}) < 0.$$

Thus, from (3.28), (3.26), and (3.20), we get

$$\begin{aligned} 0 &< -[g(\tilde{y}) - \alpha\rho\theta(\tilde{y})] W_y^M \\ &= -W_T^M + 1 - \alpha\left[r(\tilde{y}) + \frac{1}{2}(1-\alpha)\theta^2(\tilde{y})\right] W^M + \alpha W^M \Phi\left(\frac{W_y^M}{W^M}\right) \\ &\leq 1 - \alpha\left[r(\tilde{y}) + \frac{1}{2}(1-\alpha)\theta^2(\tilde{y})\right] W^M \\ &\leq 1 + K_1 K_3. \end{aligned}$$

That is,

$$\left|(g(\tilde{y}) - \alpha\rho\theta(\tilde{y})) W_y^M(T, \tilde{y})\right| \leq \tilde{K}_1 \triangleq 1 + K_1 K_3,$$

where $\tilde{K}_1$ does not depends on $\tilde{y}$ and M. Thus,

$$\left|W_y^M(T, \tilde{y})\right| \leq \tilde{K}_1. \tag{3.29}$$

To study cases (b) and (c) we use the logarithmic transformation:

$$V(T, y) \triangleq \log W^M(T, y).$$

This transformation has been useful to study several problems in stochastic control; see [FlHe02]. Noting that

$$W_T^M = W^M V_T, \quad W_y^M = W^M V_y, \quad \text{and} \quad W_{yy}^M = W^M\left(V_y^2 + V_{yy}\right),$$

the HJB equation (3.28) can be written as

$$\frac{1}{2}V_y^2 + [g - \alpha\rho\theta] V_y + \frac{1}{2}V_{yy} = V_T - \frac{1}{W^M} + \alpha\left[r + \frac{1}{2}(1-\alpha)\theta^2\right] - \alpha\Phi(V_y). \tag{3.30}$$

From (3.20), the bounds of W_y^M do not depend on M if and only if V_y has the same property. Furthermore, from (3.26) and (3.27), note that

$$V_T \leq 1 + K_3 \quad \text{and} \quad 0 \leq -\alpha\Phi\left(V_y\right) \leq \frac{1}{2}\gamma\varepsilon^2 V_y^2.$$

The previous estimations together with (3.30), imply that

$$\begin{aligned} \frac{1}{2}\left(1-\gamma\varepsilon^2\right)V_y^2 + \left[g - \alpha\rho\theta\right]V_y + \frac{1}{2}V_{yy} &\leq (K_3+1) + \alpha\left[r + \frac{1}{2}(1-\alpha)\theta^2\right] \\ &\leq 1 + 2K_3. \end{aligned}$$

Then,

$$V_y^2 + 2\tilde{g}V_y + \frac{1}{1-\gamma\varepsilon^2}V_{yy} \leq \tilde{\alpha}, \tag{3.31}$$

where

$$\tilde{\alpha} \mathrel{\mathring{=}} \frac{1+2K_3}{1-\gamma\varepsilon^2} > 0 \quad \text{and} \quad \tilde{g}\left(y\right) \mathrel{\mathring{=}} \frac{g\left(y\right) - \alpha\rho\theta\left(y\right)}{1-\gamma\varepsilon^2}; \quad y \in \mathbf{R}.$$

In particular, for any critical point $\tilde{y} > 0$ of $V_y\left(T,\cdot\right)$,

$$V_y^2\left(T,\tilde{y}\right) + 2\tilde{g}\left(\tilde{y}\right)V_y\left(T,\tilde{y}\right) \leq \tilde{\alpha},$$

which is equivalent to

$$\left[V_y\left(T,\tilde{y}\right) + \tilde{g}\left(\tilde{y}\right)\right]^2 \leq \tilde{\alpha} + \tilde{g}^2\left(\tilde{y}\right).$$

Thus,

$$-\tilde{g}\left(\tilde{y}\right) - \sqrt{\tilde{\alpha} + \tilde{g}^2\left(\tilde{y}\right)} \leq V_y\left(T,\tilde{y}\right) \leq -\tilde{g}\left(\tilde{y}\right) + \sqrt{\tilde{\alpha} + \tilde{g}^2\left(\tilde{y}\right)}. \tag{3.32}$$

Case (b). Here $\tilde{g}\left(\tilde{y}\right) \geq -1$ and $V_y\left(T,\tilde{y}\right) > 0$. Since the function $\tilde{h}\left(u\right) \mathrel{\mathring{=}} -u + \sqrt{\tilde{\alpha} + u^2}$; for $u \geq -1$, is bounded, then the right hand side of (3.32) is bounded.

Hence,

$$0 < V_y(T, \tilde{y}) \leq \tilde{K}_2 \mathrel{\hat{=}} 1 + \sqrt{1 + \tilde{\alpha}}. \tag{3.33}$$

The constant $\tilde{K}_2$ does not depends on $\tilde{y}$ and M.

Case (c). Here $\tilde{g}(\tilde{y}) \leq 1$ and $V_y(T, \tilde{y}) < 0$. Analogously, since the function $\tilde{h}(u) \mathrel{\hat{=}} -u - \sqrt{\tilde{\alpha} + u^2}$; for $u \leq 1$, is bounded, then the left hand side of (3.32) is bounded. Hence,

$$-\tilde{K}_2 \leq V_y(T, \tilde{y}) < 0. \tag{3.34}$$

Thus, putting together (3.29), (3.33), and (3.34), we get

$$|V_y(T, \tilde{y})| \leq \tilde{K} \mathrel{\hat{=}} \tilde{K}_1 \vee \tilde{K}_2; \tag{3.35}$$

for all critical points $\tilde{y} > 0$ of $V_y(T, \cdot)$. Now, for $M > \frac{\varepsilon}{1-\alpha}\tilde{K}$ and using (3.35), we have

$$\begin{aligned} \frac{\varepsilon}{1-\alpha}\frac{\left|W_y^M\right|}{W^M} &< M \quad \text{and} \\ w(T, y) &= W^M(T, y) = W(T, y). \end{aligned}$$

Thus, using (3.14) and (3.13), the optimal process is given by

$$\hat{\nu}_t \mathrel{\hat{=}} \nu^*(T - t, Y_t) = -\frac{\varepsilon}{1-\alpha}\frac{W_y(T - t, Y_t)}{W(T - t, Y_t)}; \quad t \in [0, T].$$

Finally, substituting the Markov policy (3.13) in the HJB equation (3.22) we obtain (3.18). ■

3.3 HARA example (investor's problem)

In this section we give the explicit optimal trading strategy for the investor's problem when the utility function is HARA.

With this in mind, we use a result by Dynkin (see Proposition 5.4.2 in [KaSr98]), which turns out to be the key idea to go back from the dual problem to the investor's one and, hence, to get the optimal solutions.

In the last section the optimal process for the associated dual problem was obtained. The process $\hat{\nu}$ is given by (3.17) where $W(T,y)$ satisfies the HJB equation (3.18). According with the methodology described at the beginning of this chapter, we consider

$$\begin{aligned}\hat{B} &\mathrel{\hat{=}} I\left(\hat{\lambda}\frac{Z_T^{\hat{\nu}}}{S_T^0}\right) = \frac{x}{\Lambda_{\hat{\nu}}}\left(\frac{S_T^0}{Z_T^{\hat{\nu}}}\right)^{1-\alpha} \quad \text{and} \\ \hat{c}_t &\mathrel{\hat{=}} I\left(\hat{\lambda}\frac{Z_t^{\hat{\nu}}}{S_t^0}\right) = \frac{x}{\Lambda_{\hat{\nu}}}\left(\frac{S_t^0}{Z_t^{\hat{\nu}}}\right)^{1-\alpha} ; \quad t \in [0,T],\end{aligned}$$

where

$$\begin{aligned}\hat{\lambda} &= \left(\frac{x}{\Lambda_{\hat{\nu}}}\right)^{1-\alpha} \quad \text{and} \\ \Lambda_{\hat{\nu}} &= E\left[\left(\frac{Z_T^{\hat{\nu}}}{S_T^0}\right)^{\alpha} + \int_0^T \left(\frac{Z_t^{\hat{\nu}}}{S_t^0}\right)^{\alpha} dt\right] = W(T,y).\end{aligned}$$

To get the form of the optimal trading portfolio $\hat{\pi}$, let us prove that the process M is a martingale, where

$$M_t \mathrel{\hat{=}} \left(\frac{Z_t^{\hat{\nu}}}{S_t^0}\right)^{\alpha} W(T-t,Y_t) + \int_0^t \left(\frac{Z_u^{\hat{\nu}}}{S_u^0}\right)^{\alpha} du; \quad t \in [0,T].$$

Note first, that $EM_T = M_0$, since

$$\begin{aligned}
M_0 &= W(T,y) = \Lambda_{\hat{\nu}} \quad \text{and} \\
M_T &= \left(\frac{Z_T^{\hat{\nu}}}{S_T^0}\right)^\alpha + \int_0^T \left(\frac{Z_t^{\hat{\nu}}}{S_t^0}\right)^\alpha dt.
\end{aligned}$$

Under the original measure P, we write down the system of SDE in $[0,T]$:

$$\begin{aligned}
dY_t &= g(Y_t)\,dt + \rho dW_{1t} + \varepsilon dW_{2t}; & Y_0 &= y, \\
d\frac{Z_t^{\hat{\nu}}}{S_t^0} &= -\frac{Z_t^{\hat{\nu}}}{S_t^0}\left[r(Y_t)\,dt + \theta(Y_t)\,dW_{1t} + \hat{\nu}_t dW_t^2\right]; & \frac{Z_0^{\hat{\nu}}}{S_0^0} &= z = 1.
\end{aligned}$$

The differential operator of the system is

$$\mathcal{L}f \mathrel{\hat{=}} f_t + gf_y - rzf_z - z(\rho\theta + \varepsilon\nu^*)\,f_{yz} + \frac{1}{2}f_{yy} + \frac{1}{2}z^2\left(\theta^2 + \nu^{*2}\right)f_{zz},$$

for $f \in C^{1,2,2}([0,T] \times \mathbf{R} \times \mathbf{R}_+)$. In particular, when $f(t,y,z) \mathrel{\hat{=}} W(T-t,y)\,z^\alpha$, and using (3.17) and (3.18), we have

$$\begin{aligned}
&\mathcal{L}f(t,y,z) \\
&= z^\alpha\left(-W_t - \alpha\left[r + \frac{1}{2}(1-\alpha)\left(\theta^2 + \nu^{*2}\right)\right]W + \left[g - \alpha(\rho\theta + \varepsilon\nu^*)\right]W_y + \frac{1}{2}W_{yy}\right) \\
&= z^\alpha\left(\frac{1}{2}W_{yy} + \left[g - \alpha\left(\rho\theta - \frac{\varepsilon^2}{1-\alpha}\frac{W_y}{W}\right)\right]W_y\right. \\
&\qquad \left. - W_t - \alpha\left[r + \frac{1}{2}(1-\alpha)\left(\theta^2 + \frac{\varepsilon^2}{(1-\alpha)^2}\frac{W_y^2}{W^2}\right)\right]W\right) \\
&= z^\alpha\left[-W_t + \frac{1}{2}W_{yy} + (g - \alpha\rho\theta)\,W_y - \alpha\left(r + \frac{1}{2}(1-\alpha)\,\theta^2\right)W - \frac{1}{2}\gamma\varepsilon^2\frac{W_y^2}{W}\right] \\
&= -z^\alpha.
\end{aligned}$$

Therefore,

$$\begin{aligned}
M_t &= \left(\frac{Z_t^{\hat{\nu}}}{S_t^0}\right)^\alpha W(T-t,Y_t) + \int_0^t \left(\frac{Z_s^{\hat{\nu}}}{S_s^0}\right)^\alpha ds \\
&= f\left(t, Y_t, \frac{Z_t^{\hat{\nu}}}{S_t^0}\right) - \int_0^t \mathcal{L}\left(s, Y_s, \frac{Z_s^{\hat{\nu}}}{S_s^0}\right) ds.
\end{aligned}$$

Thus, from Proposition 5.4.2 in [KaSr91], M is a nonnegative local martingale. However, this process is a supermartingale with constant mean. Hence, M is a martingale.

Now, motivated by (2.12), define the nonnegative process

$$\frac{\hat{X}_t}{S_t^0} \triangleq E^{\hat{\nu}}\left[\frac{\hat{B}}{S_T^0} + \int_t^T \frac{\hat{c}_u}{S_u^0} du \mid \mathcal{F}_t\right]; \quad t \in [0,T].$$

From Remark 19 in Appendix, we get

$$\begin{aligned}
\frac{\hat{X}_t}{S_t^0} &= E^{\hat{\nu}}\left[\frac{\hat{B}}{S_T^0} + \int_t^T \frac{\hat{c}_u}{S_u^0} du \mid \mathcal{F}_t\right] \\
&= E\left[\frac{Z_T^{\hat{\nu}}}{Z_t^{\hat{\nu}}}\frac{\hat{B}}{S_T^0} + \int_t^T \frac{Z_u^{\hat{\nu}}}{Z_t^{\hat{\nu}}}\frac{\hat{c}_u}{S_u^0} du \mid \mathcal{F}_t\right] \\
&= \frac{x}{\Lambda_{\hat{\nu}} Z_t^{\hat{\nu}}} E\left[\frac{Z_T^{\hat{\nu}}}{S_T^0}\left(\frac{S_T^0}{Z_T^{\hat{\nu}}}\right)^{1-\alpha} + \int_t^T \frac{Z_u^{\hat{\nu}}}{S_u^0}\left(\frac{S_u^0}{Z_u^{\hat{\nu}}}\right)^{1-\alpha} du \mid \mathcal{F}_t\right] \\
&= \frac{x}{W(T,y) Z_t^{\hat{\nu}}} E\left[\left(\frac{Z_T^{\hat{\nu}}}{S_T^0}\right)^{\alpha} + \int_t^T \left(\frac{Z_u^{\hat{\nu}}}{S_u^0}\right)^{\alpha} du \mid \mathcal{F}_t\right] \\
&= \frac{x}{W(T,y) Z_t^{\hat{\nu}}} E\left[M_T - \int_0^t \left(\frac{Z_u^{\hat{\nu}}}{S_u^0}\right)^{\alpha} du \mid \mathcal{F}_t\right] \\
&= \frac{x}{W(T,y) Z_t^{\hat{\nu}}} \left[M_t - \int_0^t \left(\frac{Z_u^{\hat{\nu}}}{S_u^0}\right)^{\alpha} du\right] \\
&= x\frac{\left[Z_t^{\hat{\nu}}\right]^{\alpha-1}}{\left[S_t^0\right]^{\alpha}} \frac{W(T-t,Y_t)}{W(T,y)}.
\end{aligned}$$

However, by Ito's formula

$$\begin{aligned}
d\left[Z^{\hat{\nu}}\right]^{\alpha-1} &= (1-\alpha)\left[Z^{\hat{\nu}}\right]^{\alpha-1}\left[\left(1-\frac{\alpha}{2}\right)\left(\theta^2+\hat{\nu}^2\right)dt + \theta dW_1 + \hat{\nu} dW_2\right], \\
dW &= \left(-W_t + \frac{1}{2}W_{yy} + gW_y\right)dt + W_y\left(\rho dW_1 + \varepsilon dW_2\right), \\
d\left[S^0\right]^{-\alpha} &= -\alpha r\left[S^0\right]^{-\alpha} dt.
\end{aligned}$$

Here W represents the process $W(T-t,Y_t)$; for $t \in [0,T]$. Then, using (3.17) and

(3.18), we get

$$
\begin{aligned}
& d\left\{\frac{\left[Z^{\hat{\nu}}\right]^{\alpha-1}}{\left[S^0\right]^{\alpha}} W\right\} \\
= & \frac{\left[Z^{\hat{\nu}}\right]^{\alpha-1}}{\left[S^0\right]^{\alpha}}\left\{-\alpha r W dt+(1-\alpha) W\left[\left(1-\frac{\alpha}{2}\right)\left(\theta^2+\hat{\nu}^2\right) dt+\theta dW_1+\hat{\nu} dW_2\right]+\right. \\
& \left.\left[-W_t+\frac{1}{2} W_{yy}+g W_y\right] dt+W_y\left(\rho dW_1+\varepsilon dW_2\right)+(1-\alpha)(\rho \theta+\varepsilon \hat{\nu}) W_y dt\right\} \\
= & \frac{\left[Z^{\hat{\nu}}\right]^{\alpha-1}}{\left[S^0\right]^{\alpha}}\left\{\left[(1-\alpha) \theta^2 W+\rho \theta W_y\right] dt+\left[(1-\alpha) \theta W+\rho W_y\right] dW_1+\right. \\
& \left.\left[-W_t+\frac{1}{2} W_{yy}+(g-\alpha \rho \theta) W_y-\alpha\left(r+\frac{1}{2}(1-\alpha) \theta^2\right) W-\frac{1}{2} \gamma \varepsilon^2 \frac{W_y^2}{W}\right] dt\right\} \\
= & \frac{\left[Z^{\hat{\nu}}\right]^{\alpha-1}}{\left[S^0\right]^{\alpha}}\left(-dt+W\left[(1-\alpha) \theta+\rho \frac{W_y}{W}\right]\left[\theta dt+dW_1\right]\right) \\
= & \frac{\left[Z^{\hat{\nu}}\right]^{\alpha-1}}{\left[S^0\right]^{\alpha}}\left(-dt+W\left[(1-\alpha) \theta+\rho \frac{W_y}{W}\right] dW_1^{\hat{\nu}}\right) .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
d \frac{\hat{X}}{S^0}+\frac{\hat{c}}{S^0} dt & = \frac{x}{W(T, y)} d\left(\frac{\left[Z^{\hat{\nu}}\right]^{\alpha-1}}{\left[S^0\right]^{\alpha}} W\right)+\frac{x}{W(T, y)}\left(\frac{S^0}{Z^{\hat{\nu}}}\right)^{1-\alpha} \frac{1}{S^0} dt \\
& = \frac{x}{W(T, y)} \frac{\left[Z^{\hat{\nu}}\right]^{\alpha-1}}{\left[S^0\right]^{\alpha}} W\left[(1-\alpha) \theta+\rho \frac{W_y}{W}\right] dW_1^{\hat{\nu}} \\
& = \frac{\hat{X}}{S^0}\left[(1-\alpha) \theta+\rho \frac{W_y}{W}\right] dW_1^{\hat{\nu}} \\
& =: \frac{\hat{\pi}}{S^0} \sigma dW_1^{\hat{\nu}} ;
\end{aligned}
$$

where

$$
\hat{\pi}_t \circeq \frac{\hat{X}_t}{\sigma\left(Y_t\right)}\left[(1-\alpha) \theta\left(Y_t\right)+\rho \frac{W_y\left(T-t, Y_t\right)}{W\left(T-t, Y_t\right)}\right]=: \pi^*(T-t, \hat{X}_t, Y_t),
$$

with

$$
\pi^*(t, x, y) \circeq \frac{x}{\sigma(y)}\left[(1-\alpha) \theta(y)+\rho \frac{W_y(t, y)}{W(t, y)}\right] ; \quad (t, x, y) \in[0, T] \times \mathbf{R}_{+} \times \mathbf{R}.
$$

Moreover, the optimal consumption process $\hat{c}$ can be written in a feedback form:

$$\begin{aligned}
\hat{c}_t &= \frac{x}{\Lambda_{\hat{\nu}}}\left(\frac{S_t^0}{Z_t^{\hat{\nu}}}\right)^{1-\alpha} \\
&= x\left(\frac{S_t^0}{Z_t^{\hat{\nu}}}\right)^{1-\alpha}\frac{W\left(T-t,Y_t\right)}{W\left(T,y\right)}\frac{1}{W\left(T-t,Y_t\right)} \\
&= \frac{\hat{X}_t}{W\left(T-t,Y_t\right)} \\
&= :c^*\left(T-t,\hat{X}_t,Y_t\right),
\end{aligned}$$

with

$$c^*\left(t,x,y\right) \mathrel{\hat{=}} \frac{x}{W\left(t,y\right)}; \quad \left(t,x,y\right)\in[0,T]\times\mathbf{R}_+\times\mathbf{R}.$$

It follows, from the derivation of $\hat{X}$ and (1.11), that $(\hat{\pi},\hat{c})\in\mathcal{A}\left(x,y\right)$ and $X^{\hat{\pi},\hat{c}}\equiv\hat{X}$. In addition, since $X_T^{\hat{\pi},\hat{c}}\equiv\hat{B}$ and the discounted process $\frac{X^{\hat{\pi},\hat{c}}}{S^0}+\int_0^{\cdot}\frac{\hat{c}_s}{S_s^0}ds$, is a $P^{\hat{\nu}}$-martingale, and applying Theorem 5, it is verified that

$$(\hat{B},\hat{c})\in\mathcal{B}\left(x,y\right) \quad \text{and} \quad E^{\hat{\nu}}\left[\frac{\hat{B}}{S_T^0}+\int_0^T\frac{\hat{c}_s}{S_s^0}ds\right]=x.$$

Thus, from Proposition 8, $\hat{B}$ is the optimal terminal wealth.

In addition, considering the transformation $W\left(T,y\right)=:\left[h\left(T,y\right)\right]^{\delta}$; for some $\delta>0$, then

$$\begin{aligned}
W_T &= \delta h^{\delta-1}h_T, \quad W_y=\delta h^{\delta-1}h_y, \quad \frac{W_y^2}{W}=\delta^2h^{\delta-2}h_y^2, \\
W_{yy} &= \delta h^{\delta-1}h_{yy}+\delta\left(\delta-1\right)h^{\delta-2}h_y^2.
\end{aligned}$$

Thus, using (3.18), it follows that $h\left(T,y\right)$ solves the PDE

$$h_T=\frac{1}{\delta}h^{1-\delta}+\frac{1}{2}h_{yy}+\left(g-\alpha\rho\theta\right)h_y-\frac{\alpha}{\delta}\left[r+\frac{1}{2}\left(1-\alpha\right)\theta^2\right]h+\frac{1}{2}\left(\delta-1-\gamma\delta\varepsilon^2\right)\frac{h_y^2}{h},$$

with initial condition $h(0,y) = 1$. However, observe that when $\delta - 1 - \gamma\delta\varepsilon^2 = 0$, then $\delta \doteq \frac{1}{1-\gamma\varepsilon^2}$ is the unique number in $\mathbf{R}_+$ such that the last nonlinear term in the previous PDE vanishes. Hence, the PDE (3.18) is equivalent to

$$h_T = \frac{1}{\delta}h^{1-\delta} + \frac{1}{2}h_{yy} + (g - \alpha\rho\theta)\, h_y - \frac{\alpha}{\delta}\left[r + \frac{1}{2}(1-\alpha)\,\theta^2\right] h, \quad \text{with} \quad h(0,y) = 1. \tag{3.36}$$

The previous power transformation was used in [Za01] for an optimal investment problem.

On the other hand, since the function $h(T,y)$ belongs to $C^{1,2}\left(\bar{\mathbf{R}}_+ \times \mathbf{R}\right)$, the Feynman-Kac formula can be used to get the following representation

$$\begin{aligned} h(T,y) \quad = \quad & E\left(e^{-\frac{\alpha}{\delta}\int_0^T\left[r(\check{Y}_u)+\frac{1}{2}(1-\alpha)\theta^2(\check{Y}_u)\right]du} + \right. \\ & \left. \frac{1}{\delta}\int_0^T h^{1-\delta}\left(T-u, \check{Y}_u\right) e^{-\frac{\alpha}{\delta}\int_0^u\left[r(\check{Y}_s)+\frac{1}{2}(1-\alpha)\theta^2(\check{Y}_s)\right]ds} du\right), \end{aligned} \tag{3.37}$$

for $(T,y) \in \bar{\mathbf{R}}_+ \times \mathbf{R}$, where $\{\check{Y}_t\}_{t\in[0,T]}$ is the solution to the SDE

$$d\check{Y}_t = \left[g(\check{Y}_t) - \alpha\rho\theta(\check{Y}_t)\right] dt + d\check{W}_t, \quad \text{with} \quad \check{Y}_0 = y \tag{3.38}$$

and $\check{W} = \rho W_1 + \varepsilon W_2$; which is a BM relative to $(\Omega, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$. See Theorem 5.7.6 and Corollary 4.4.5 in [KaSr91].

Summarizing: $h(T,y) = [W(T,y)]^{1/\delta}$ is the unique smooth function in $C^{1,2}$ $\left(\bar{\mathbf{R}}_+ \times \mathbf{R}\right) \cap C_b^{0,1}\left(\bar{\mathbf{R}}_+ \times \mathbf{R}\right)$ solving (3.36), with

$$\delta = \frac{1}{1-\gamma\varepsilon^2} = \frac{1-\alpha}{1-\alpha\rho^2} \quad \text{and} \quad \alpha = -\frac{\gamma}{1-\gamma}.$$

Finally, we give a comparison between the limit behavior of the optimal process when the HARA parameter γ goes to zero and the one for the logarithmic case. From

(3.37), we have

$$
\begin{aligned}
& h_y(T,y) \\
= \; & E\Bigg(-\frac{\alpha}{\delta}e^{-\frac{\alpha}{\delta}\int_0^T\left[r(\check{Y}_u)+\frac{1}{2}(1-\alpha)\theta^2(\check{Y}_u)\right]du}\int_0^T\frac{\partial}{\partial y}\check{Y}_u\left[r'(\check{Y}_u)+(1-\alpha)\,\theta(\check{Y}_u)\theta'(\check{Y}_u)\right]du \\
& +\frac{1-\delta}{\delta}\int_0^T e^{-\frac{\alpha}{\delta}\int_0^u\left[r(\check{Y}_s)+\frac{1}{2}(1-\alpha)\theta^2(\check{Y}_s)\right]ds}\frac{\partial}{\partial y}\check{Y}_u\frac{h_y\left(T-u,\check{Y}_u\right)}{h^{\delta}\left(T-u,\check{Y}_u\right)}du \\
& -\frac{\alpha}{\delta^2}\int_0^T h^{1-\delta}\left(T-u,\check{Y}_u\right)e^{-\frac{\alpha}{\delta}\int_0^u\left[r(\check{Y}_s)+\frac{1}{2}(1-\alpha)\theta^2(\check{Y}_s)\right]ds} \\
& \times\int_0^u\frac{\partial}{\partial y}\check{Y}_s\left[r'(\check{Y}_s)+(1-\alpha)\,\theta(\check{Y}_s)\theta'(\check{Y}_s)\right]ds\,du\Bigg),
\end{aligned}
$$

where

$$\frac{\partial}{\partial y}\check{Y}_t=\exp\left(\int_0^t\left[g'(\check{Y}_u)-\alpha\rho\theta'(\check{Y}_u)\right]du\right);\quad t\in[0,T].$$

Letting $\alpha\to 0$ then $\delta\to 1$, $h(T,y)\to 1$, and $h_y(T,y)\to 0$. That is, from (3.17), the limit process is $\hat{\nu}\equiv 0$, which coincides with the optimal process obtained for the logarithmic utility.

3.4 Investment or consumption problems

The results obtained up to now can be easily adapted to solve the optimal investment or consumption problems, separately. Considering similar models, these problems have been studied in recent contributions. For instance, see [Za01], [CaHe03], and [FlHe02].

Investment case. In this case the following changes are established in the primal problem and its associated dual:

1. The utility function $U_2(\cdot)$ is zero, and define $\tilde{U}_2(\cdot)\circeq I_2(\cdot)\circeq 0$.

2. Take the consumption process as the zero process.

For logarithmic utility, the dual functional is

$$L(\nu,\lambda) = E\int_0^T r(Y_u)\,du - 1 + \lambda x - \log\lambda - E\log Z_T^\nu,$$

whereas the optimal expressions are

$$\begin{aligned}
\hat{\lambda} &= \frac{1}{x}, \quad \hat{\nu} = 0, \\
X_t^{\hat{\pi}} &= x\frac{S_t^0}{Z_t^0}; \quad t \in [0,T], \quad \text{and} \\
\hat{\pi}_t &= x\frac{\theta(Y_t)}{\sigma(Y_t)}\frac{S_t^0}{Z_t^0} = \frac{\mu(Y_t) - r(Y_t)}{\sigma^2(Y_t)}X_t^{\hat{\pi}}.
\end{aligned}$$

For HARA utility:

$$L(\nu,\lambda) = \lambda x - \frac{1}{\alpha}\lambda^\alpha\Lambda_\nu, \quad \text{where} \quad \Lambda_\nu \circeq E\left(\frac{Z_T^\nu}{S_T^0}\right)^\alpha,$$

while the optimal control process is

$$\hat{\nu}_t = -\frac{\varepsilon}{1-\alpha}\frac{W_y(T-t,Y_t)}{W(T-t,Y_t)}; \quad t \in [0,T],$$

where $W(T,y)$ is the corresponding value function, which is the unique solution of the HJB equation:

$$W_T = \frac{1}{2}W_{yy} + (g - \alpha\rho\theta)W_y - \alpha\left[r + \frac{1}{2}(1-\alpha)\theta^2\right]W - \frac{1}{2}\gamma\varepsilon^2\frac{W_y^2}{W},$$

with $W(0,y) = 1$. In addition, if $h(T,y) \circeq [W(T,y)]^{1/\delta}$, with $\delta = \frac{1}{1-\gamma\varepsilon^2}$, the previous equation is equivalent to

$$h_T = \frac{1}{2}h_{yy} + (g - \alpha\rho\theta)h_y - \frac{\alpha}{\delta}\left[r + \frac{1}{2}(1-\alpha)\theta^2\right]h \quad \text{with} \quad h(0,y) = 1.$$

In fact, using the Feynman-Kac formula, its solution has the representation

$$h(T,y) = E\exp\left(-\frac{\alpha}{\delta}\int_0^T \left[r(\check{Y}_t) + \frac{1}{2}(1-\alpha)\theta^2(\check{Y}_t)\right]dt\right); \quad T > 0,$$

where the process $\check{Y}$ is given by the SDE (3.38). Finally, the optimal wealth and trading portfolio processes are

$$\begin{aligned} X_t^{\hat{\pi}} &= x\left(\frac{S_t^0}{Z_t^0}\right)^{\alpha}\frac{W(T-t,Y_t)}{W(T,y)}, \\ \hat{\pi}_t &= \pi^*\left(T-t,X_t^{\hat{\pi}},Y_t\right), \quad \text{where} \\ \pi^*(t,x,y) &= \frac{x}{\sigma(y)}\left[(1-\alpha)\theta(y)+\rho\frac{W_y(t,y)}{W(t,y)}\right]; \quad t\in[0,T]. \end{aligned}$$

For details see [CaHe03].

Remark 15. The above results coincide with Proposition 2.1 in [Za01]. However, in that paper does not provide much details how to get the optimal process. Furthermore, it does not have an explicit form of the optimal wealth process. On the other hand, when the external factor Y depends only on W_1 $(\rho=\pm 1)$, and hence the market is complete, the optimal solution for the dual problem is $\hat{\nu}\equiv 0$. Then,

$$\hat{B}=\frac{x}{\Lambda_0}\left(\frac{Z_T^0}{S_T^0}\right)^{-(1-\alpha)} \quad \text{and} \quad \hat{\lambda}=\left(\frac{\Lambda_0}{x}\right)^{1/(1-\alpha)}.$$

In this sense, the results are similar to Theorem 3.7.6 in [KaSr98].

Consumption case. To study this case, the following changes in the modelling are established in the primal problem and its associated dual:

1. The utility function $U_1(\cdot)$ as well as $\tilde{U}_1(\cdot)$ and $I_1(\cdot)$ are zero.
2. The terminal wealth random variables B are taken as zero.

For logarithmic utility, the dual functional is

$$L(\nu,\lambda)=E\int_0^T\int_0^t r(Y_u)\,du\,dt+\lambda x-T(1+\log\lambda)-E\int_0^T\log Z_t^{\nu}dt,$$

whereas the optimal expressions are

$$\begin{aligned}
\hat{\lambda} &= \frac{T}{x}, \quad \hat{\nu} = 0, \\
X_t^{\hat{\pi},\hat{c}} &= x\frac{T-t}{T}\frac{S_t^0}{Z_t^0}, \\
\hat{\pi}_t &= x\frac{T-t}{T}\frac{\theta(Y_t)}{\sigma(Y_t)}\frac{S_t^0}{Z_t^0} = \frac{\mu(Y_t) - r(Y_t)}{\sigma^2(Y_t)}X_t^{\hat{\pi},\hat{c}}, \\
\hat{c}_t &= \frac{x}{T}\frac{S_t^0}{Z_t^0} = \frac{1}{T-t}X_t^{\hat{\pi},\hat{c}}; \quad t \in [0,T].
\end{aligned}$$

For HARA utility:

$$L(\nu,\lambda) = \lambda x - \frac{1}{\alpha}\lambda^\alpha \Lambda_\nu, \quad \text{where} \quad \Lambda_\nu \circeq E\int_0^T \left(\frac{Z_t^{\hat{\nu}}}{S_t^0}\right)^\alpha dt.$$

The optimal solution to the dual problem is the process

$$\hat{\nu}_t = -\frac{\varepsilon}{1-\alpha}\frac{W_y(T-t,Y_t)}{W(T-t,Y_t)}; \quad t \in [0,T],$$

where $W(T,y)$ is the corresponding value function, which is the unique solution to the PDE

$$W_T = 1 + \frac{1}{2}W_{yy} + (g - \alpha\rho\theta)W_y - \alpha\left[r + \frac{1}{2}(1-\alpha)\theta^2\right]W - \frac{1}{2}\gamma\varepsilon^2\frac{W_y^2}{W},$$

with $W(0,y) = 0$. In addition, if $h(T,y) = [W(T,y)]^{1/\delta}$ with $\delta = \frac{1}{1-\gamma\varepsilon^2}$, then the last equation becomes

$$h_T = \frac{1}{\delta}h^{1-\delta} + \frac{1}{2}h_{yy} + (g-\alpha\rho\theta)h_y - \frac{\alpha}{\delta}\left[r + \frac{1}{2}(1-\alpha)\theta^2\right]h, \quad \text{with} \quad h(0,y) = 0.$$

It has the Feynman-Kac representation

$$h(T,y) = \frac{1}{\delta}E\int_0^T h^{1-\delta}\left(T-u,\check{Y}_u\right)e^{-\frac{\alpha}{\delta}\int_0^u\left[r(\check{Y}_s)+\frac{1}{2}(1-\alpha)\theta^2(\check{Y}_s)\right]ds}du; \quad T > 0.$$

In particular, for $\delta = 1$, then $\rho = \pm 1$ and the market is complete. Finally, the optimal processes are

$$\begin{aligned}
X_t^{\hat{\pi},\hat{c}} &= x\left(\frac{S_t^0}{Z_t^0}\right)^\alpha \frac{W(T-t,Y_t)}{W(T,y)}, \\
\hat{\pi}_t &= \pi^*(T-t, X_t^{\hat{\pi},\hat{c}}, Y_t), \\
\hat{c}_t &= c^*(T-t, X_t^{\hat{\pi},\hat{c}}, Y_t), \quad \text{with} \\
\pi^*(t,x,y) &\circeq \frac{x}{\sigma(y)}\left[(1-\alpha)\,\theta(y) + \rho\frac{W_y(t,y)}{W(t,y)}\right], \\
c^*(t,x,y) &\circeq \frac{x}{W(t,y)}.
\end{aligned}$$

Apparently the form of the optimal trading strategy $(\hat{\pi},\hat{c})$ is similar to the corresponding solutions of the consumption-investment problem. However, the value functions are different. For example, in the former case $W(0,y) = 0$, whereas in the latter case $W(0,y) = 1$.

With a similar model, this optimization problem was studied in [FlHe02]. In that paper a constant interest rate and return rate was considered.

3.5 Pricing and hedging

In this part we describe the approach proposed in [Da00] to valuate European options.

An important question in mathematical finance is how to valuate and hedge the derivatives. When the market is complete, for a given European option H with $E^0 H < \infty$, a *fair* price at time zero is $E^0 \frac{H}{S_T^0}$, where E^0 is the expectation operator with respect to the unique equivalent martingale measure P^0. Furthermore, it is easy to see that there is a *hedging* trading portfolio $\check{\pi}$, such that $X_T^{x,\check{\pi}} \equiv H$. However, in the incomplete case, like the market studied in this work, there are many of such martingale measures. In fact, there is one for each bounded process ν. To answer that question for incomplete markets, is used the *utility approach*. This method is a good alternative to find a unique value price, since it includes the risk attitudes of the investor and of the *writer* (*seller*).

Consider an investor with initial capital $x > 0$, which has two ways to invest his money. The first one is through the market described in the Chapter 1. In this case, the final wealth is $X_T^{x,y,\pi}$, for some admissible trading strategy π. The second one consists in investing in the classical way, just explained, but with initial wealth $x - p$, and also buying an European option $H \circeq H(S_T, Y_T)$, for which he pays to the writer the initial amount p. The *zero marginal rate of substitution* suggests that the fair price p can be determined matching the optimal expected final utilities from both investment strategies:

$$\sup_{\pi \in \mathcal{A}(x,y)} EU\left(X_T^{x,y,\pi}\right) = \sup_{\pi \in \mathcal{A}(x-p,y)} EU\left(X_T^{x-p,y,\pi} + H(S_T, Y_T)\right). \tag{3.39}$$

The price p obtained from the last formula can be interpreted as the fair price when the buyer is indifferent between to buy or not the derivative H. See equations (6.15) in [Da00]. Finally, the right hand side of (3.39) defines a new optimal problem, which reduces to the classical investor's problem when $H \equiv 0$. For instance, in [Da00] the exponential utility is presented when the European option depends only on Y_T, where Y plays the role of a untraded asset.

Thus, using the relationship with the optimal investment problem, pricing and hedging problems can be studied through this approach. See next Chapter.

Chapter 4

Conclusions and open problems

The market model described in this work is a generalization of the Black and Scholes classical model. Within this framework a detailed analysis of the investor's optimization problem was done, obtaining closed form solutions when the utility functions are HARA and logarithmic.

The investor's problem was solved successfully using the martingale method and stochastic control techniques. The formulation of the investor's problem as a convex optimization problem, were developed for a general class of utility functions. The primal problem is solved provided a solution to the dual problem exists, even when the utility functions, from terminal wealth and consumption, are different. In this work we do not present a general existence theorem of solutions to the dual problem (D), because it goes beyond its goal. However a sufficient condition to obtain a solution to the dual and primal problems was given. Moreover, when the utility functions are logarithmic or HARA and equal, we get an explicit solution of both problems. In these cases, an explicit form of the optimal wealth process and the optimal trading strategy were obtained. This is an important contribution for the special case of HARA utility functions.

The dual problem is, roughly speaking, posed on the set of equivalent local martingale measures of P, and it was proved that the optimal solution is an equivalent martingale measures. In other words, $Z^{\hat{\nu}}$ is a martingale. Once this problem was solved, it was possible return successfully to get explicitly the optimal trading portfolio and the optimal wealth process. This contribution also includes the investor's problem for consumption or investment.

The results presented in this work confirm that the martingale approach is a powerful method to solve financial optimization problems. For instance, in the case of logarithmic and HARA utility functions, stand out the reduction of dimensionality in the control problem, as well as the explicit form of the optimal wealth process was obtained.

The contributions from this work are summarized in [CaHe04].

Moreover, it is feasible to apply this approach to other incomplete market models. For example, the case when there are $N \geq 2$ stocks from which L cannot be traded $(1 \leq L < N)$, which still is not solved. However, we cannot ignore that the martingale procedure just translates the original problem into another one. In this sense, the involved stochastic control problem plays an important role.

Finally, these ideas can also be extrapolated to pricing and hedging problems for incomplete markets. For example, assume that $x + p$ is the initial capital of the investor, where x and p are the initial amount allocated in the classical market (a bank account and a risky asset) and the price to paid for European option $H(Y_T)$, respectively. Then, the hedging problem is to

$$\text{maximize} \quad EU\left(X_T^{x,\pi} + H\left(Y_T\right)\right), \quad \text{over} \quad \pi \in \mathcal{A}\left(x, y\right). \tag{4.1}$$

We conjecture that the primal representation of this problem is

$$\text{maximize} \quad E\left\{U\left(B + H\left(Y_T\right)\right)\right\}, \quad \text{over} \quad B \in \mathcal{B}\left(x, y\right),$$

where, in this case,

$$\mathcal{B}\left(x, y\right) \circeq \left\{B \geq 0 \mid B \quad \text{is} \quad \mathcal{F}_T\text{-measurable} \quad \text{and} \quad \sup_{\nu \in \mathcal{M}} E^\nu \frac{B}{S_T^0} \leq x\right\}.$$

This problem remains open.

4.1 Explicit solution

One of the main goals of this work is to give explicit solutions to the consumption-investment problem derived from the dual problem when the utility is HARA. See section 3.3. The solution given here involves the unique smooth solution h of the HJB equation (3.18). A usual alternative to estimate h is through the Feynman-Kac formula (3.37). Writing it again, we have

$$\begin{aligned} h\left(T, y\right) \quad = \quad E & \left[e^{-\frac{\alpha}{\delta}\int_0^T \left[r(\check{Y}_u) + \frac{1}{2}(1-\alpha)\theta^2(\check{Y}_u)\right]du} + \right. \\ & \left. \frac{1}{\delta}\int_0^T h^{1-\delta}\left(T-u, \check{Y}_u\right) e^{-\frac{\alpha}{\delta}\int_0^u \left[r(\check{Y}_s) + \frac{1}{2}(1-\alpha)\theta^2(\check{Y}_s)\right]ds} du\right]. \end{aligned} \tag{4.2}$$

However, the iterative form turns out to be a serious problem. In the bibliographical study we neither find this kind of iterated form, nor a possible solution. We believe that this problem can be solved through numerical techniques. In this sense, we suggest a general algorithm, which involves consecutive substitutions of h on expression

(4.2). Let us explain it:

1. Let $h^{(0)}(T,y) = 1$.

2. Generate the sequence of functions $h^{(1)}$, $h^{(2)}$, $\ldots$; using the iterative formula:

$$\begin{aligned} h^{(n+1)}(T,y) &= E\Bigg(e^{-\frac{\alpha}{\delta}\int_0^T \left[r(\check{Y}_u)+\frac{1}{2}(1-\alpha)\theta^2(\check{Y}_u)\right]du} + \\ &\quad \frac{1}{\delta}\int_0^T \left[h^{(n)}\right]^{1-\delta}\left(T-u,\check{Y}_u\right) e^{-\frac{\alpha}{\delta}\int_0^u \left[r(\check{Y}_s)+\frac{1}{2}(1-\alpha)\theta^2(\check{Y}_s)\right]ds} du \Bigg). \end{aligned}$$

3. To verify the convergence of this sequence, estimate $\left| h^{(n+1)} - h^{(n)} \right|_\infty$.

Appendix

In this part we show some auxiliary results which complement the mathematical development of this work.

Feynman-Kac formula

Here we present the Feynman-Kac formula given in equation (D.13) in [FlSo93], and prove a little extension in Corollary 17 below. We use this formula in the verification Theorems 11 and 13.

Let Y be a diffusion process in $\mathbf{R}^n$ defined in $[s,T]$, with $0 \leq s \leq T$, and satisfying the SDE

$$dY_t = g(t, Y_t)\, dt + h(t, Y_t)\, dW_t; \quad t \in [s,T], \quad \text{with} \quad Y_s = y, \tag{A.1}$$

where W is a standard n-dimensional BM, and $g(t,y)$ and $h(t,y)$ are vector and matrix functions, respectively. Assume that for $l(t,y) = g(t,y),\ h^{(1)}(t,y),\ \ldots,$ $h^{(n)}(t,y)$ (the columns of h) satisfies

$$|l(t,y) - l(t,\check{y})| \leq K|y - \check{y}| \quad \text{and} \quad |l(t,y)|^2 \leq K_1 + K^2|y|^2,$$

for all $y, \check{y} \in \mathbf{R}^n$ and some constants $K, K_1 > 0$. The associated differential operator to the diffusion (A.1) is

$$\mathcal{L}f(t,y) \mathrel{\hat{=}} f_t(t,y) + \frac{1}{2}[\operatorname{tr} hh' f_{yy}](t,y) + g(t,y)\cdot f_y(t,y); \quad f \in C^{1,2}([s,T]\times\mathbf{R}^n),$$

where f_y and f_{yy} denote, respectively, the gradient and Hessian operators.

Let O be a bounded open set in $\mathbf{R}^n$ and denote as $\check{\tau}$ the first exit time of (t, Y_t) from $[s,T]\times O$. Define $\Gamma_t \mathrel{\hat{=}} \exp\left(\int_s^t q_u du\right)$, with q is a bounded from above progessively measurable process. Now, we state the Feynman-Kac theorem.

Theorem 16 (Feynman-Kac). Let $w(t,y)$ a smooth function in $C^{1,2}\left([s,T]\times\bar{O}\right)$ and τ a stopping time with $s \le \tau \le \check{\tau}$. Then, the next equality holds

$$w(s,y) = E\left(\Gamma_\tau w(\tau, Y_\tau) - \int_s^\tau \Gamma_u [\mathcal{L} + q_u] w(u, Y_u)\, du\right), \tag{A.2}$$

Proof. Applying Ito's formula to the product $\Gamma_u w(u, Y_u)$, we have

$$\begin{aligned} d[\Gamma_u w(u,Y_u)] &= w(u,Y_u)\, d\Gamma_u + \Gamma_u dw(u,Y_u) \\ &= \Gamma_u[\mathcal{L}+q_u]\, w(u,Y_u)\, du + \Gamma_u[w_y h](u,Y_u)\cdot dW_u. \end{aligned}$$

Thus

$$\Gamma_\tau w(\tau, Y_\tau) = w(s,y) + \int_s^\tau \Gamma_u[\mathcal{L}+q_u]\, w(u,Y_u)\, du + \int_s^\tau \Gamma_u[w_y h](u,Y_u)\cdot dW_u.$$

On the other hand, since $[s,T]\times\bar{O}$ is a compact set in $\mathbf{R}^{n+1}$, then $w \in C_b^{1,2}\left([s,T]\times\bar{O}\right)$. That is, the last term of the integral equation is a martingale with zero mean. Hence,

$$\begin{aligned} w(s,y) &= E\left(\Gamma_\tau w(\tau,Y_\tau) - \int_s^\tau \Gamma_u[\mathcal{L}+q_u]\, w(u,Y_u)\, du - \int_s^\tau \Gamma_u[w_y h](u,Y_u)\cdot dW_u\right) \\ &= E\left(\Gamma_\tau w(\tau,Y_\tau) - \int_s^\tau \Gamma_u[\mathcal{L}+q_u]\, w(u,Y_u)\, du\right). \end{aligned}$$

■

Corollary 17 (Feynman-Kac II). If $w \in C^{1,2}([s,T] \times O) \cap C_p([s,T] \times O)$ with $[\mathcal{L} + q_t] w(t,y) \leq 0$ and O is an open set of $\mathbf{R}^n$ (not necessarily bounded). Then, for any stopping time τ with $s \leq \tau \leq T$:

$$w(s,y) = E\left(\Gamma_\tau w(\tau, Y_\tau) - \int_s^\tau \Gamma_u [\mathcal{L} + q_u] w(u, Y_u)\, du\right); \tag{A.3}$$

Proof. Applying the arguments of the previous proof to the bounded sets $O_k \circeq O \cap (-k,k)^n$; $k \geq 1$, the identity (A.2) gets the form

$$w(s,y) = E\left(\Gamma_{\tau\wedge\check{\tau}_k} w(\tau \wedge \check{\tau}_k, Y_{\tau\wedge\check{\tau}_k}) - \int_s^{\tau\wedge\check{\tau}_k} \Gamma_u [\mathcal{L} + q_u] w(u, Y_u)\, du\right); \quad k \geq 1, \tag{A.4}$$

where τ is any stopping time with $s \leq \tau \leq T$, and $\check{\tau}_k$ is the first exit time of (t, Y_t) from $[s,T] \times O_k$. Note that the last term from expression (A.4) is nonnegative (including the minus sign), and it is increasing with respect to k. Since $\lim_{k\to\infty} \tau \wedge \check{\tau}_k = \tau \wedge T = \tau$ a.s. and applying the monotone convergence theorem, this term converges to the corresponding term from (A.3). On the other hand, since $w \in C_p([s,T] \times O)$, $|w(\tau \wedge \check{\tau}_k, Y_{\tau\wedge\check{\tau}_k})| \leq K\left(1 + |Y|_\infty^K\right)$, for some constant $K \geq 1$; where $|Y|_\infty \circeq \sup_{t\in[s,T]} |Y_t|_\infty$. From inequality (D.7) in [FlSo93], $E|Y|_\infty < \infty$, and hence, $\{w(\tau \wedge \check{\tau}_k, Y_{\tau\wedge\check{\tau}_k})\}_{k\geq 1}$ is an uniformly integrable family. Therefore,

$$E\Gamma_{\tau\wedge\check{\tau}_k} w(\tau \wedge \check{\tau}_k, Y_{\tau\wedge\check{\tau}_k}) \to E\Gamma_\tau w(\tau, Y_\tau), \quad k \to \infty.$$

Thus, we have (A.3). ■

Remark 18 *In Corollary 17, the hypothesis* $[\mathcal{L} + q_t] w(t,y) \leq 0$, *can be replaced by:* $[\mathcal{L} + q_t] w(t,y) \geq 0$.

Other results

Proposition 19. Let $\nu \in \mathcal{M}$ and c be a nonnegative progressively measurable process with $\int_0^T c_t dt < \infty$ (consumption). Then

$$E^{\nu}\left[\int_t^T \frac{c_u}{S_u^0}du \mid \mathcal{F}_t\right] = E\left[\int_t^T \frac{Z_u^{\nu}}{Z_t^{\nu}}\frac{c_u}{S_u^0}du \mid \mathcal{F}_t\right]; \quad t \in [0,T].$$

In particular

$$E^{\nu}\int_0^T \frac{c_u}{S_u^0}du = E\int_0^T \frac{Z_u^{\nu}}{S_u^0}c_u du. \tag{A.5}$$

The identity (A.5) appears in page 44 in [Cu97].

Proof.

$$\begin{aligned}
E\left[\int_t^T \frac{Z_u^{\nu}}{Z_t^{\nu}}\frac{c_u}{S_u^0}du \mid \mathcal{F}_t\right] &= E\left[\int_t^T \frac{E\left(Z_T^{\nu} \mid \mathcal{F}_u\right)}{Z_t^{\nu}}\frac{c_u}{S_u^0}du \mid \mathcal{F}_t\right] \\
&= E\left[\int_t^T E\left(\frac{Z_T^{\nu}}{Z_t^{\nu}}\frac{c_u}{S_u^0} \mid \mathcal{F}_u\right)du \mid \mathcal{F}_t\right] \\
&= E\left[\int_t^T \frac{Z_T^{\nu}}{Z_t^{\nu}}\frac{c_u}{S_u^0}du \mid \mathcal{F}_t\right] \\
&= E^{\nu}\left[\int_t^T \frac{c_u}{S_u^0}du \mid \mathcal{F}_t\right].
\end{aligned}$$

■

Proposition 20. Let $\phi : \mathbf{R} \to \mathbf{R}$ and $w : \mathbf{R} \to \mathbf{R}$ be real functions, such that $\phi(\cdot)$ is Lipschitz and $w(\cdot)$ is locally Lipschitz. Then, $v(\cdot) \doteq \phi(w(\cdot))$ is locally Lipschitz.

Proof. For each $N > 0$, there exist constants L and L_N such that, holds

$$\begin{aligned}
|\phi(y) - \phi(\tilde{y})| &\leq L|y - \tilde{y}|; \quad y, \tilde{y} \in \mathbf{R}, \\
|w(y) - w(\tilde{y})| &\leq L_N|y - \tilde{y}|; \quad y, \tilde{y} \in [-N, N].
\end{aligned}$$

Thus, for $y, \tilde{y} \in [-N, N]$, it follows

$$|v(y) - v(\tilde{y})| \leq |\phi(w(y)) - \phi(w(\tilde{y}))| \leq L|w(y) - w(\tilde{y})| \leq LL_N |y - \tilde{y}|.$$

■

This result was used in the proof of verification Theorems 11 and 13. In this case

$$\phi(y) \doteq \begin{cases} -\dfrac{\varepsilon}{1-\alpha} y, & \dfrac{\varepsilon}{1-\alpha} |y| \leq M \\ -M \operatorname{sgn} y, & \text{otherwise} \end{cases} ; \quad M > 0,$$

and $\phi(\cdot)$ is Lipschitz, with Lipschitz constant $L = \dfrac{\varepsilon}{1-\alpha}$.

Symbol index

General

$\prec$	absolutely continuous
a.s.	almost sure with respect to the original measure P
BM	standard Brownian motion in $\mathbf{R}$, $\mathbf{R}^2$, or $\mathbf{R}^n$
$C^k(\mathbf{R})$	class of real functions $f : \mathbf{R} \to \mathbf{R}$, with continuous k derivative $f^{(k)}$; $k \geq 0$
$C^{k,l}\left(\mathbf{R}^2\right)$[1]	class of functions $f : \mathbf{R}^2 \to \mathbf{R}$, with $f(\cdot, y) \in C^k(\mathbf{R})$ and $f(x, \cdot) \in C^l(\mathbf{R})$, $(x, y) \in \mathbf{R}^2$, $k, l \geq 0$
$C_b^k(\mathbf{R})$[1]	class of functions $f \in C^k(\mathbf{R})$, with $f^{(j)}$ bounded; $0 \leq j \leq k$
$C_p^k(\mathbf{R})$[1]	class of functions $f \in C^k(\mathbf{R})$, such that $f^{(j)}$ is *polynomial growing*; $0 \leq j \leq k$
DPE	dynamic programing equation
$\circeq$	a definition
$=:$	an implicit definition
$\equiv$	identically or a.s. equal (applies for processes and random variables)
$\operatorname{ess\,sup} \mathcal{D}$	essential supremum of a class of nonnegative random variables $\mathcal{D}$
HJB	Hamilton Jacobi Bellman
$\inf D$	infimum of a subset $D \subset \mathbf{R}$
$\lvert\cdot\rvert_\infty$	supremum norm in $C_b(\mathbf{R})$
PDE	partial differential equation (determinist)
$\mathbf{R}_+ \circeq (0, \infty)$, $\quad \bar{\mathbf{R}}_+ \circeq [0, \infty)$, $\quad \mathbf{R}_- \circeq -\mathbf{R}_+$	
SDE	stochastic differential equation
$\sup D$	supremum of a subset $D \subset \mathbf{R}$

Finance

[1] Similar notation applies for other Euclidean spaces

REFERENCES

BaSp02 Barndorff-Nielsen, O. and N. Shephard. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics, *Journal of the Royal Statistical Society* **B 63** (2001), 167-241.
xi, xii, 14, 15

BKR03 Benth, F.E., K.H. Karlsen, and K. Reikvam. Merton's trading portfolio optimization problem in a Black & Scholes market with non-Gaussian stochastic volatility of Ornstein-Uhlenbeck type, *Mathematical Finance* **13** (2003), 215-244.
xii, 15

BiPl99 Bielecki, T.R. and S.R. Pliska. Risk Sensitive dynamic asset management, *Applied Mathematics and Optimization* **39** (1999), 337-360.
1

CaHe03 Castañeda-Leyva, N. and D. Hernández-Hernández. Optimal investment in incomplete financial markets with stochastic volatility, *Contemporary Mathematics*, **336** (2003).
59, 61

CaHe04 Castañeda-Leyva, N. and D. Hernández-Hernández. Optimal consumption-investment in incomplete financial markets with stochastic coefficients, *Preprint* (2004).
66

CoRe96 Comte, F. and R. Renault. Long memory in continuous-time stochastic volatility models, *Mathematical Finance* **8** (1996), 291-323.
15

Co75 Cox, J.C. Notes on option pricing I: constant elasticity of variance diffusions, *Working paper*, Stanford University, 1975.
14

Cu97 Cuoco, D. Optimal consumption and equilibrium prices with trading portfolio constraints and stochastic income, *Journal of Economic Theory* **72** (1997), 33-73.
18, 27, 72

Da00 Davis, M. Optimal hedging with basis risk, *Preprint* (2000).
xi, xii, 29, 63, 64, 64

DeSc94 Delbaen, F. and W. Schachermayer. A general version of the fundamental theorem of asset pricing, *Mathematische Annalen* **300** (1994), 463-520.
5

FlHe02 Fleming, W.H. and D. Hernández-Hernández. An optimal consumption model with stochastic volatility, *Finance and Stochastics* **7** (2003), 245-262.
xi, xii, xiii, 50, 59, 63

FlSo93 Fleming, W.H. and H.M. Soner. *Controlled Markov processes and viscosity solutions,* Springer New York, 1993.
41, 43, 43, 46, 47, 69, 71

FPS00 Fouque, J.-P., G. Papanicolaou, and K.R. Sircar. *Derivatives in financial markets with stochastic volatility,* Cambridge University Press, 2000.
xi

Fr75 Friedman, A. *Stochastic Differential Equations and Applications*, Academic Press, New York, 1975.
40

HaPl81 Harrison, J.M. and S.R. Pliska. Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and their Applications* **11** (1981), 215-260.
xiii, 4

Hu01 Hu, Y. Option Pricing in a market where the volatility is driven by a factional Brownian motion, *Preprint* (2001).
15

JaSh87 Jacod, J. and A.N. Shiryaev. *Limit Theorems for Stochastic Processes,* Springer Berlin, 1987.
20, 22

KaSr91 Karatzas, I. and S.E. Shreve. *Brownian Motion and Stochastic Calculus*, Springer New York, 1991.
20, 22, 43, 55, 58

KaSr98 Karatzas, I. and S.E. Shreve. *Methods of Mathematical Finance,* Springer New York, 1998.
xiii, 6, 7, 18, 19, 20, 24, 26, 28, 29, 30, 31, 31, 36, 53, 61

KrSc99 Kramkov, D. and W. Schachermayer. The Asymptotic elasticity of utility functions and optimal investments in incomplete markets. *The Annals of Applied Probability* **9** (1999), 904-950.
xii, xiii, 29

LiSh01 Lipster, R.S. and A.N. Shiryaev. *Statistics of Random Processes,* Springer Berlin, 2001.
22

Lu69 Luenberger, D.G. *Optimization by Vector Space Methods*, Wiley New York, 1969.
24, 26, 27, 27

Za01 Zariphopoulou, T. A solution approach to valuation with unhedgeable risks, *Finance and Stochastics* **5** (2001), 61-82.
xi, xii, xii, 58, 59, 61